# Ex03

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$\omega \in \mathbb{R}^D,\quad x_i \in \mathbb{R}^D,\ y_i \in \{\pm 1\},\ i \in \{1,\dots,N\},\quad N_1 = N_{-1} = N/2$

**3.1**

$$\frac{\partial}{\partial b} \sum_{i=1}^{N} (\omega^T x_i + b - y_i)^2 = 0$$

$$\Rightarrow 0 = \sum_{i=1}^{N} \partial_b (\omega^T x_i + b - y_i)^2 = 2 \sum_{i=1}^{N} (\omega^T x_i + b - y_i)$$

$$\Rightarrow \sum_{i=1}^{N} b = -\sum_{i=1}^{N} \omega^T x_i - y_i$$

$$\sum_{i=1}^{N} y_i = \sum_{i:y_i=1} y_i + \sum_{i:y_i=-1} y_i = N_1 - N_{-1} = 0$$

$$\Rightarrow \hat{b} = \frac{1}{N} \sum_{i=1}^{N} y_i - \omega^T x_i = -\frac{1}{N} \sum \omega^T x_i$$

**Our own comment for 3.1: Our solution is identical to the sample solution, although we also included the middle steps.**

**3.2**

$$\partial_\omega (\omega^T x) = x, \quad a \cdot (b^T \cdot c) = (a \cdot c^T) \cdot b$$

$$\partial_\omega \sum_{i=1}^{N} (\omega^T x_i + \hat{b} - y_i)^2 = 0$$

$$\Rightarrow 0 = \sum_{i=1}^{N} 2 (\omega^T x_i + \hat{b} - y_i)\, x_i$$

$$\Rightarrow 0 = \sum x_i \cdot (\omega^T x_i) + \sum_j \hat{b} x_j - \sum y_i \cdot x_i$$

$$\sum y_i x_i = \sum_{i:y_i=1} y_i x_i + \sum_{i:y_i=-1} y_i x_i = N_1 \mu_1 - N_{-1}\mu_{-1} = \frac{N}{2}(\mu_1 - \mu_{-1})$$

$$= \sum (x_i \cdot x_i^T) \cdot \omega - \frac{1}{N} \sum_j \Big(\sum_i \omega^T x_i\Big) x_j - \frac{N}{2}(\mu_1 - \mu_{-1})$$

$$\Rightarrow \frac{\mu_1 - \mu_{-1}}{2} = \frac{1}{N} \sum (x_i \cdot x_i^T)\,\omega - \frac{1}{N^2} \sum_{j,i} (x_j \cdot x_i^T)\, \omega$$

$$= \Big[\frac{1}{N} \sum_i x_i x_i^T - \frac{1}{N^2} \sum_i \Big(\sum_j x_j\Big) \cdot x_i^T\Big] \cdot \omega$$

$$= \Big[\frac{1}{N} \sum_i x_i x_i^T - \frac{1}{N^2} \sum_i (N_1 \mu_1 + N_{-1} \mu_{-1}) \cdot x_i^T\Big] \cdot \omega$$

$$= \Big[\frac{1}{N} \sum_i x_i x_i^T - \frac{1}{2N} (\mu_1 + \mu_{-1}) \sum_i x_i^T\Big] \cdot \omega$$

$$= \Big[\frac{1}{N} \sum_i x_i x_i^T - \frac{1}{4} (\mu_1 + \mu_{-1})(\mu_1 + \mu_{-1})^T\Big] \cdot \omega$$

$$= \Big[\frac{1}{N} \sum_i x_i x_i^T + \frac{1}{4} (\mu_1 - \mu_{-1})(\mu_1 - \mu_{-1})^T - \frac{1}{2}(\mu_1 \mu_1^T + \mu_{-1}\mu_{-1}^T)\Big] \cdot \omega$$

$$= \Big[\frac{1}{N} \sum_i x_i x_i^T - \frac{1}{2}(\mu_1 \mu_1^T + \mu_{-1}\mu_{-1}^T) + \frac{1}{4} S_B\Big] \cdot \omega$$

$$= \Big[S_\omega + \frac{1}{4} S_B\Big] \cdot \omega$$

$$S_\omega = \frac{1}{N} \sum_i (x_i - \mu_{y_i})(x_i - \mu_{y_i})^T = \frac{1}{N} \sum_i x_i x_i^T - x_i \mu_{y_i}^T - \mu_{y_i} x_i^T + \mu_{y_i} \mu_{y_i}^T$$

$$= \frac{1}{N} \sum_i x_i x_i^T + \frac{1}{N} \Big[\sum_{i:y_i=1} \mu_1 \mu_1^T - x_i \mu_1^T - \mu_1 x_i^T\Big] + \frac{1}{N} \Big[\sum_{i:y_i=-1} \mu_{-1}\mu_{-1}^T - x_i \mu_{-1}^T - \mu_{-1} x_i^T\Big]$$

$$= \frac{1}{N} \sum_i x_i x_i^T + \frac{1}{N} \Big[\frac{N}{2} \mu_1 \mu_1^T - \frac{N}{2} \mu_1 \mu_1^T - \frac{N}{2} \mu_1 \mu_1^T\Big] + \frac{1}{N} \Big[\frac{N}{2} \mu_{-1}\mu_{-1}^T - \frac{N}{2} \mu_{-1}\mu_{-1}^T - \mu_{-1}\mu_{-1}^T\Big]$$

$$= \frac{1}{N} \sum_i x_i x_i^T - \frac{1}{2} (\mu_1 \mu_1 + \mu_{-1} \mu_{-1}^T)$$

**Our own comment for 3.2: As expected by the mathematical nature of this exercise, this part is also equivalent the sample solution. However our steps differ considerably from the sample solution in some orderings, but of course this doesn't influence the end result, both are okay.**

**3.3**

Define $c := (\mu_1 - \mu_{-1})^T \hat{\omega}$

$$\Rightarrow S_B \hat{\omega} = (\mu_1 - \mu_{-1})(\mu_1 - \mu_{-1})^T \hat{\omega} = c\,(\mu_1 - \mu_{-1})$$

$$S_\omega \hat{\omega} + \frac{1}{4} S_B \hat{\omega} = S_\omega \hat{\omega} + \frac{c}{4} (\mu_1 - \mu_{-1}) = \frac{1}{2}(\mu_1 - \mu_{-1})$$

$$\Rightarrow S_\omega \hat{\omega} = \Big(c' + \frac{1}{2}\Big)(\mu_1 - \mu_{-1})$$

$$\Rightarrow \hat{\omega} = \Big(c' + \frac{1}{2}\Big) S_\omega^{-1} (\mu_{-1} - \mu_{-1}) = \tau\, S_\omega^{-1} (\mu_1 - \mu_{-1})$$

$\tau := c' + \frac{1}{2}$

**Our own comment for 3.3: Here we didn't include 1/4 as a part of our constant in the beginning (as it was done in the sample solution), but of course that's not a problem and the solutions are mathematically equivalent up to the definition Tau' = c' + 1/2.**