(2) $Y = X\beta^* + \varepsilon, \quad \varepsilon \sim \mathcal{N}(0, \sigma^2) \Rightarrow \mathbb{E}[\varepsilon] = 0, \ \mathrm{Cov}[\varepsilon] = \sigma^2 \mathbb{1}$

$\hat{\beta}_{OLS} = (X^T X)^{-1} X^T Y = (X^T X)^{-1} X^T (X\beta^* + \varepsilon) = \beta^* + (X^T X)^{-1} X^T \varepsilon$

- $\mathbb{E}[Y] = X\beta^*$
- $\mathrm{Cov}[\hat{\beta}_{OLS}] = \mathrm{Cov}[\beta^* + (X^T X)^{-1} X^T \varepsilon] = \mathrm{Cov}[(X^T X)^{-1} X^T \varepsilon]$

$$= (X^T X)^{-1} X^T \,\mathrm{Cov}[\varepsilon] \left((X^T X)^{-1} X^T\right)^T$$

$$= \sigma^2 (X^T X)^{-1} X^T X (X^T X)^{-1}$$

$$= \sigma^2 (X^T X)^{-1} = \sigma^2 S^{-1}$$

- $\hat{\beta}_\tau = (X^T X + \tau \mathbb{1})^{-1} X^T Y$
- $\mathbb{E}[\hat{\beta}_\tau] = \mathbb{E}[(X^T X + \tau \mathbb{1})^{-1} X^T Y] = (X^T X + \tau \mathbb{1})^{-1} X^T \mathbb{E}[Y]$

$$= (X^T X + \tau \mathbb{1})^{-1} X^T X \beta^*$$

$$= S_\tau^{-1} S \beta^*$$

- $\mathrm{Cov}[\hat{\beta}_\tau] = \mathrm{Cov}[(X^T X + \tau \mathbb{1})^{-1} X^T Y] = \mathrm{Cov}[(X^T X + \tau \mathbb{1})^{-1} \overbrace{X^T X}^{= S} \underbrace{(X^T X)^{-1} X^T Y}_{= \hat{\beta}_{OLS}}]$

$$= \mathrm{Cov}[S_\tau^{-1} S \hat{\beta}_{OLS}] = S_\tau^{-1} S \,\mathrm{Cov}[\hat{\beta}_{OLS}] (S_\tau^{-1} S)^T$$

S, S_τ symm.

$$= S_\tau^{-1} S (\sigma^2 S^{-1}) S S_\tau^{-1}$$

$$= \sigma^2 S_\tau^{-1} S S_\tau^{-1}$$