

②

$$Y = X\beta^* + \varepsilon, \quad \varepsilon \sim \mathcal{N}(0, \sigma^2) \Rightarrow E[\varepsilon] = 0, \text{Cov}[\varepsilon] = \sigma^2 \mathbf{1}$$

$$\hat{\beta}_{OLS} = (X^T X)^{-1} X^T Y = (X^T X)^{-1} X^T (X\beta^* + \varepsilon) = \beta^* + (X^T X)^{-1} X^T \varepsilon$$

$$\bullet E[Y] = X\beta^*$$

$$\begin{aligned} \bullet \text{Cov}[\hat{\beta}_{OLS}] &= \text{Cov}[\beta^* + (X^T X)^{-1} X^T \varepsilon] = \text{Cov}[(X^T X)^{-1} X^T \varepsilon] \\ &= (X^T X)^{-1} X^T \text{Cov}[\varepsilon] (X^T X)^{-1} X^T \\ &= \sigma^2 (X^T X)^{-1} X^T X (X^T X)^{-1} \\ &= \sigma^2 (X^T X)^{-1} = \sigma^2 S^{-1} \end{aligned}$$

$$\bullet \hat{\beta}_T = (X^T X + \tau \mathbf{1})^{-1} X^T Y$$

$$\begin{aligned} \bullet E[\hat{\beta}_T] &= E[(X^T X + \tau \mathbf{1})^{-1} X^T Y] = (X^T X + \tau \mathbf{1})^{-1} X^T E[Y] \\ &= (X^T X + \tau \mathbf{1})^{-1} X^T X \beta^* \\ &= S_T^{-1} S \beta^* \end{aligned}$$

$$\begin{aligned} \bullet \text{Cov}[\hat{\beta}_T] &= \text{Cov}[(X^T X + \tau \mathbf{1})^{-1} X^T Y] = \text{Cov}[(X^T X + \tau \mathbf{1})^{-1} X^T X \underbrace{(X^T X)^{-1} X^T Y}_{= \hat{\beta}_{OLS}}] \\ &= \text{Cov}[S_T^{-1} S \hat{\beta}_{OLS}] = S_T^{-1} S \text{Cov}[\hat{\beta}_{OLS}] (S_T^{-1} S)^T \\ &\quad S, S_T \text{ symm.} \downarrow \\ &= S_T^{-1} S (\sigma^2 S^{-1}) S S_T^{-1} \\ &= \sigma^2 S_T^{-1} S S_T^{-1} \end{aligned}$$