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Exercise 4.2

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$$
\begin{aligned}
E_Z\{\log p(\mathbf{X},\mathbf{Z}|\boldsymbol{\theta})\} &= E_Z\{\log[p(\mathbf{X}|\mathbf{Z},\boldsymbol{\theta})p(\mathbf{Z}|\boldsymbol{\theta})]\} \\
&= E_Z\left\{\log\left[p(\mathbf{z}_1|\boldsymbol{\theta})p(\mathbf{x}_1|\mathbf{z}_1,\boldsymbol{\theta})\prod_{t=2}^{T}p(\mathbf{z}_t|\mathbf{z}_{t-1},\boldsymbol{\theta})p(\mathbf{x}_t|\mathbf{z}_t,\boldsymbol{\theta})\right]\right\} \\
&= E_Z\left\{-\frac{1}{2}\left[T\log|\boldsymbol{\Sigma}| + (\mathbf{z}_1-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\mathbf{z}_1-\boldsymbol{\mu}_0)\right.\right. \\
&\quad + \sum_{t=2}^{T}(\mathbf{z}_t-\mathbf{A}\mathbf{z}_{t-1})^T\boldsymbol{\Sigma}^{-1}(\mathbf{z}_t-\mathbf{A}\mathbf{z}_{t-1}) \\
&\quad \left.\left. + T\log|\boldsymbol{\Gamma}| + \sum_{t=1}^{T}(\mathbf{x}_t-\mathbf{B}\mathbf{z}_t)^T\boldsymbol{\Gamma}^{-1}(\mathbf{x}_t-\mathbf{B}\mathbf{z}_t)\right] + \text{const.}\right\}
\end{aligned}
\tag{7.54}
$$

$$
\begin{aligned}
&\mathbb{E}_z[\log p(x|z,\theta)] \\
&= \mathbb{E}_z\Big[-\tfrac{1}{2}\Big[T\log|\Sigma| + (z_1-\mu_0)^T\Sigma^{-1}(z_1-\mu_0) \\
&\qquad + \sum_{t=2}^{T}(z_t-Az_{t-1})^T\Sigma^{-1}(z_t-Az_{t-1}) + \text{const}\Big]\Big] \\
&= -\tfrac{1}{2}\Big[T\log|\Sigma| + \mathbb{E}_z[z_1^T\Sigma^{-1}z_1] - \mathbb{E}_z[z_1^T\Sigma^{-1}\mu_0] - \mathbb{E}_z[\mu_0^T\Sigma^{-1}z_1] + \mathbb{E}_z[\mu_0^T\Sigma^{-1}\mu_0] \\
&\qquad + \sum_{t=2}^{T}\Big(\mathbb{E}_z[z_t^T\Sigma^{-1}z_t] - \mathbb{E}_z[z_t^T\Sigma^{-1}Az_{t-1}] - \mathbb{E}_z[z_{t-1}^TA^T\Sigma^{-1}z_t] + \mathbb{E}_z[z_{t-1}^TA^T\Sigma^{-1}Az_{t-1}]\Big)\Big] \\
&= -\tfrac{1}{2}\Big[T\log|\Sigma| + \mathbb{E}_z[z_1^T\Sigma^{-1}z_1] - \mathbb{E}_z[z_1^T]\Sigma^{-1}\mu_0 - \mu_0^T\Sigma^{-1}\mathbb{E}_z[z_1] + \mu_0^T\Sigma^{-1}\mu_0 \\
&\qquad + \sum_{t=2}^{T}\Big(\mathbb{E}_z[z_t^T\Sigma^{-1}z_t] - \mathbb{E}_z[z_t^T\Sigma^{-1}Az_{t-1}] - \mathbb{E}_z[z_{t-1}^TA^T\Sigma^{-1}z_t] + \mathbb{E}_z[z_{t-1}^TA^T\Sigma^{-1}Az_{t-1}]\Big)\Big]
\end{aligned}
$$

$$\Big(x^TAy = tr(Ayx^T) \Rightarrow \mathbb{E}_z[z^T\Sigma^{-1}z] = tr(\Sigma^{-1}\mathbb{E}_z[zz^T]), \dots\Big)$$

$$
\begin{aligned}
&= -\tfrac{1}{2}\Big[T\log|\Sigma| + tr(\Sigma^{-1}\mathbb{E}_z[z_1z_1^T]) - \mathbb{E}_z[z_1^T]\Sigma^{-1}\mu_0 - \mu_0^T\Sigma^{-1}\mathbb{E}_z[z_1] + \mu_0^T\Sigma^{-1}\mu_0 \\
&\qquad + \sum_{t=2}^{T}\Big(tr(\Sigma^{-1}\mathbb{E}_z[z_tz_t^T]) - tr(\Sigma^{-1}A\,\mathbb{E}_z[z_{t-1}z_t^T]) \\
&\qquad - tr(A^T\Sigma^{-1}\mathbb{E}_z[z_tz_{t-1}^T]) + tr(A^T\Sigma^{-1}A\,\mathbb{E}_z[z_{t-1}z_{t-1}^T])\Big)\Big]
\end{aligned}
$$

$$\Big[\;\cdot\ \partial_x tr(X\cdot A) \overset{(100)}{=} A^T \qquad \cdot\ \partial_x tr(AXB) \overset{(101)}{=} A^TB^T \qquad \cdot\ \partial_x a^TX^{-1}b \overset{(61)}{=} -X^{-T}ab^TX^{-T} \qquad \cdot\ \partial_x tr(AX^{-1}B) \overset{(124)}{=} -X^{-T}A^TB^TX^{-T}\;\Big]$$

- $\partial_\Sigma \log|\Sigma| = \frac{1}{|\Sigma|}\partial_\Sigma|\Sigma| = \frac{1}{|\Sigma|}\cdot|\Sigma|\Sigma^{-T} = \Sigma^{-T}$
- $\partial_\Sigma tr(\Sigma^{-1}\mathbb{E}_z[z_1z_1^T]) = -\Sigma^{-T}\mathbb{E}_z[z_1z_1^T]^T\Sigma^{-T}$
- $\partial_\Sigma \mathbb{E}_z[z_1^T]\Sigma^{-1}\mu_0 = -\Sigma^{-T}\mathbb{E}_z[z_1]\mu_0^T\Sigma^{-T}$
- $\partial_\Sigma \mu_0^T\Sigma^{-1}\mathbb{E}_z[z_1] = -\Sigma^{-T}\mu_0\mathbb{E}_z[z_1^T]\Sigma^{-T}$
- $\partial_\Sigma \mu_0^T\Sigma^{-1}\mu_0 = -\Sigma^{-T}\mu_0\mu_0^T\Sigma^{-T}$
- $\partial_\Sigma tr(\Sigma^{-1}\mathbb{E}_z[z_tz_t^T]) = -\Sigma^{-T}\mathbb{E}_z[z_tz_t^T]^T\Sigma^{-T}$
- $\partial_\Sigma tr(\Sigma^{-1}A\,\mathbb{E}_z[z_{t-1}z_t^T]) = -\Sigma^{-T}(A\mathbb{E}_z[z_{t-1}z_t^T])^T\Sigma^{-T}$
- $\partial_\Sigma tr(A^T\Sigma^{-1}\mathbb{E}_z[z_tz_{t-1}^T]) = -\Sigma^{-T}A\,\mathbb{E}_z[z_tz_{t-1}^T]^T\Sigma^{-T}$
- $\partial_\Sigma tr(A^T\Sigma^{-1}A\,\mathbb{E}_z[z_{t-1}z_{t-1}^T]) = -\Sigma^{-T}A(A\mathbb{E}_z[z_{t-1}z_{t-1}^T])^T\Sigma^{-T}$

$$\Rightarrow\ 0 \overset{!}{=} \partial_\Sigma\,\mathbb{E}_z[\log p(x|z,\theta)]$$

$\Sigma = \Sigma^T$, $\Sigma^{-T} = \Sigma^{-1}$

$$
\begin{aligned}
&= T\Sigma^{-1} - \Sigma^{-1}\mathbb{E}[z_1z_1^T]\Sigma^{-1} + \Sigma^{-1}\mathbb{E}_z[z_1]\mu_0^T\Sigma^{-1} + \Sigma^{-1}\mu_0\mathbb{E}_z[z_1^T]\Sigma^{-1} - \Sigma^{-1}\mu_0\mu_0^T\Sigma^{-1} \\
&\quad + \sum_{t=2}^{T}\Big(-\Sigma^{-1}\mathbb{E}_z[z_tz_t^T]\Sigma^{-1} + \Sigma^{-1}\mathbb{E}_z[z_{t-1}z_t^T]^TA^T\Sigma^{-1} + \Sigma^{-1}A\,\mathbb{E}_z[z_tz_{t-1}^T]^T\Sigma^{-1} - \Sigma^{-1}A(A\mathbb{E}_z[z_{t-1}z_{t-1}^T])^T\Sigma^{-1}\Big)
\end{aligned}
$$

$\Sigma\cdot, \cdot\Sigma$

$$
\begin{aligned}
\Rightarrow\ 0 &= \Sigma T - \mathbb{E}_z[z_1z_1^T] + \mathbb{E}_z[z_1]\mu_0^T + \mu_0\mathbb{E}_z[z_1^T] - \mu_0\mu_0^T \\
&\quad - \sum_{t=2}^{T}\mathbb{E}_z[z_tz_t^T] + \sum_{t=2}^{T}\mathbb{E}_z[z_tz_{t-1}^T]A^T + A\cdot\sum_{t=2}^{T}\mathbb{E}_z[z_tz_{t-1}^T]^T - A\cdot\sum_{t=2}^{T}\mathbb{E}_z[z_{t-1}z_{t-1}^T]\cdot A^T
\end{aligned}
$$

combine

$$\sum_{t=2}^{T}\mathbb{E}_z[z_tz_{t-1}^T] = A\cdot\sum_{t=2}^{T}\mathbb{E}_z[z_{t-1}z_{t-1}^T] \qquad \text{cancel} \qquad = \sum_{t=2}^{T}\mathbb{E}_z[z_tz_{t-1}^T]$$

combine $\sum_{t=2}^{T} \mathbb{E}_z[z_t z_{t-1}^T] = A \cdot \sum_{t=2}^{T} \mathbb{E}_z[z_{t-1} z_{t-1}^T]$ cancel

$$= \Sigma T + \mathbb{E}_z[z_1]\mu_0^T + \mu_0 \mathbb{E}_z[z_1^T] - \mu_0\mu_0^T - \sum_{t=1}^{T} \mathbb{E}_z[z_t z_t^T] + A \cdot \sum_{t=2}^{T} \mathbb{E}_z[z_t z_{t-1}^T]^T$$

$$\Rightarrow \Sigma = \frac{1}{T}\left[\sum_{t=1}^{T} \mathbb{E}_z[z_t z_t^T] - A \cdot \sum_{t=2}^{T} \mathbb{E}_z[z_t z_{t-1}^T]^T - \left(\mathbb{E}_z[z_1]\mu_0^T + \left(\mathbb{E}_z[z_1]\mu_0^T\right)^T\right) + \mu_0\mu_0^T\right]$$