MAT1856/APM466 Assignment 1

Lars Kutschinski, Student #: 1010254350

February, 2024

Fundamental Questions - 25 points

1.
   (a) Issuing bonds allows the goverment to fund projects, while printing more money will just lead to inflation.

   (b) In a country experiencing economic growth, the long-term part of the yield curve flattens as investors, anticipating an economic slowdown and lower future interest rates, increase their demand for long-term bonds, driving their yields down.

   (c) Quantitative easing is when the central bank buys a lot of bonds to put more money into the economy and make it easier for people to borrow and spend; since the COVID-19 pandemic started, the US Federal Reserve has been doing this a lot more to help and boost the economy.

2. We chose the bonds "CAD 2.25 Mar 24", "CAD 1.5 Sep 24", "CAD 1.25 March 25", "CAD 0.5 Sep 25", "CAD 0.25 Mar 26", "CAD 1 Sep 26", "CAD 1.25 Mar 27", "CAD 2.75 Sep 27", "CAD 3.5 Mar 28", "CAD 3.25 Sep 28", "CAD 4 Mar 29". The reason for choosing these bonds is because the maturity dates are at equispaced time points across the 0-5 year time period.

3. The eigenvalues associated with the covariance matrix of those stochastic processes tell us the characteristics of the yield curve. The eigenvalues tell you how much variance can be explained by its associated eigenvector. The largest eigenvalue represents the largest variance, and the second largest eigenvalue corresponds to the second largest variance, etc. The largest variance comes from a parallel shift in the curve, the second largest variance comes from a tilt of the curve, and the third largest variance comes from the flexing of the curve. Eigenvectors can help us identify the shifts, tilts, flexing and so on. For example for a yield curve we usually have that the first eigenvector has all components positive (parallel level shift), the second eigenvector has the first half of the components positive and the second half negative (slope tilt), the third eigenvector has the first third of the components positive, second third negative, and the last third positive (flexing).

Empirical Questions - 75 points

4.
   (a) We calculated the ytm for the ten selected bonds in question 2 and plotted the 5-year yield curve in Figure 1.

   (b) We use bootstrapping to calculate the spot rates. Note that for two consecutive bonds, if their maturity dates have over half year time difference, use linear estimation. Below is the pseudo code (Figure 4b), and the plot (Figure 2) for spot rates. We notice that the spot rate curves are very close to ytm curves only with some minor differences.

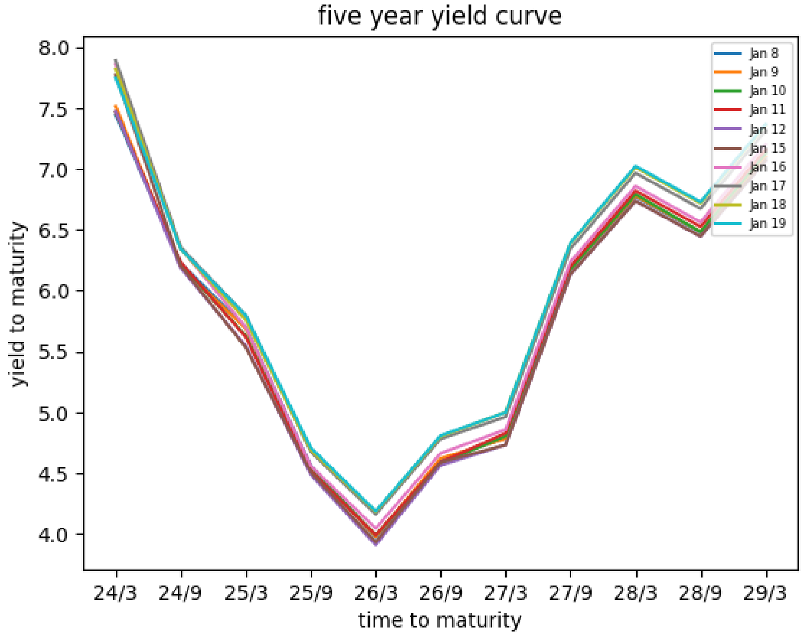

Figure 1: 4(a)

Algorithm *calculate_spot_rate*(*bonds*)

---

procedure *calculate_spot_rate*(*bonds*)
    assign empty list as spot_rate_list
    for i in length(bonds): **do**
        if i is 0 **then**
            $Y_X := -\frac{1}{X} \ln \frac{P}{C+F}$
            spot_rate_list.append(Y_X)
        time_difference := bonds[i].maturity_date - bonds[i-1].maturity_date
        if i > 0 **then**
            if time_difference > 0.5 year **then** solve for Y_X according to the equation
                $PV = \sum_{n=1}^{N-1} Ce^{-Y_{t_n} t_n} + Ce^{-\frac{Y_{t_{N-1}}+Y_X}{2} \cdot t_N} + Fe^{-Y_X X}$
                spot_rate_list.append(Y_X)
            if time_difference $\leqslant$ 0.5 year **then** solve for Y_X according to the equation
                $PV = \sum_{n=1}^{N} Ce^{-Y_{t_n} t_n} + Fe^{-Y_X X}$
                spot_rate_list.append($\frac{t_{t_{N-1}}+Y_X}{2}$)
                spot_rate_list.append(Y_X)
    return spot_rate_list

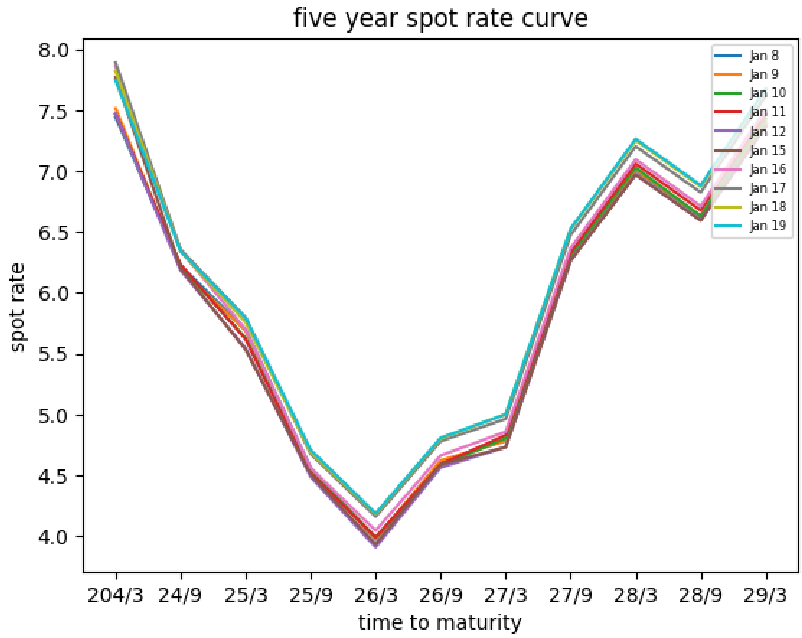

Figure 2: 4(b)

Algorithm *calculate_forward_rate*

```
procedure calculate_forward_rate(spot_rates_in_march)
    assign empty list as forward_rates_list
    r_01 := spot_rates_in_march[1]
    j := 2
    for spot_rate in spot_rates_in_march[2:] do
        forward_rate := (spot_rate * j - r_01) / (j-1)
        forward_rate_list.append(forward_rate)
        j := j + 1
    return forward_rate_list
```

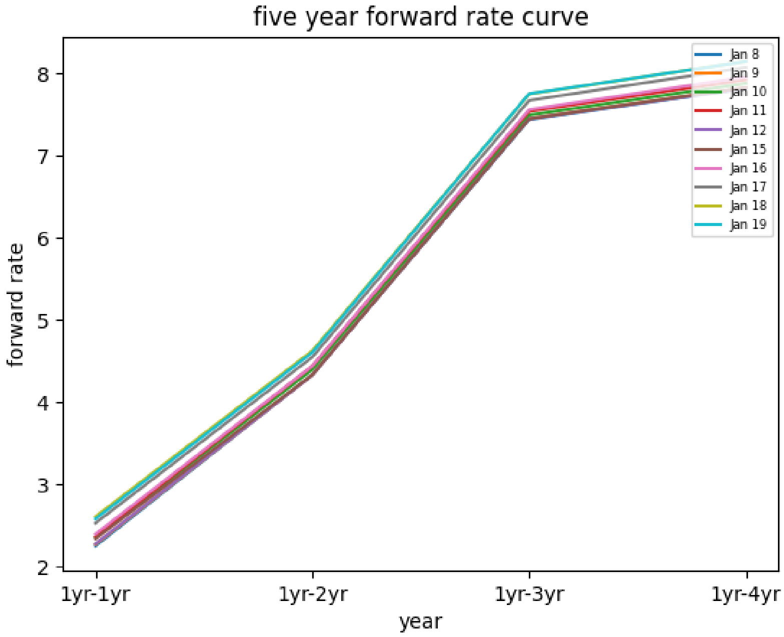

Figure 3: 4(c)

(c) In the figure above (Figure 4c) we give the pseudo code for calculating the forward rates using the spot rates from (b).

5. **YTM**: The covariance matrix for daily log-returns yield is given by (see code for calculation):

$$\begin{bmatrix} 7.67469838e-05 & 3.66186224e-05 & 4.22828583e-05 & 5.19946110e-05 & 5.66914073e-05 \\ 3.66186224e-05 & 9.85219229e-05 & 9.05480317e-05 & 7.22000457e-05 & 7.13640494e-05 \\ 4.22828583e-05 & 9.05480317e-05 & 1.08180303e-04 & 7.99725810e-05 & 8.17642762e-05 \\ 5.19946110e-05 & 7.22000457e-05 & 7.99725810e-05 & 7.76638792e-05 & 8.12751919e-05 \\ 5.66914073e-05 & 7.13640494e-05 & 8.17642762e-05 & 8.12751919e-05 & 8.63136523e-05 \end{bmatrix}$$

forward rates: The covariance matrix for forward rates is given by (see code for calculation):

$$\begin{bmatrix} 7.68764452e-04 & 3.90566016e-04 & 1.93579403e-04 & 2.05418905e-04 \\ 3.90566016e-04 & 2.53753793e-04 & 1.32430581e-04 & 1.37836540e-04 \\ 1.93579403e-04 & 1.32430581e-04 & 7.15138626e-05 & 7.35601440e-05 \\ 2.05418905e-04 & 1.37836540e-04 & 7.35601440e-05 & 7.65333604e-05 \\ 5.66914073e-05 & 7.13640494e-05 & 8.17642762e-05 & 8.12751919e-05 \end{bmatrix}$$

6. **YTM**: The eigenvalues (and respective eigenvectors) of the ytm covariance matrix are:

- λ_1 = 3.62722040e-04, v_1 =(0.31048501 0.81448297 0.43335636 -0.22884966 -0.00683012)
- λ_2 = 5.75609791e-05, v_2 =(0.46809282 -0.42248306 0.66533438 0.38854391 0.09360003)
- λ_3 = 1.61625450e-05, v_3 =(0.51058971 -0.34652654 -0.12488589 -0.77691575 -0.0047886)
- λ_4 = 1.06924649e-05, v_4 =(0.45200915 0.09028822 -0.33865596 0.31589307 -0.75700692)
- λ_5 = 2.88712357e-07, v_5 =(0.46848057 0.17289579 -0.48912992 0.30540947 0.64661424)

Forward Rates: The eigenvalues (and respective eigenvectors) of the forward rates covariance matrix are:

- λ_1 = 1.09998486e-03, v_1 =(0.82409058 0.56266402 0.06491001 -0.00840307)
- λ_2 = 6.86381159e-05, v_2 =(0.45619584 -0.59299879 -0.66013098 -0.06681973)
- λ_3 = 1.58989466e-06, v_3 =(0.2312663 -0.41751042 0.59987198 -0.64214839)
- λ_4 = 3.52596360e-07, v_4 =(0.24346657 -0.39679553 0.4474007 0.76361637)

The first eigenvalue is the largest and thus contributes to the most variance and the corresponding eigenvector dictates the direction of the variance.

References and GitHub Link to Code

References

[**1**] Principal Component Analysis of yield curve change. Available: https://quant.stackexchange.com/questions/36844/principal-component-analysis-of-yield-curve-change

[**2**] What is percentage of variance in PCA?. Available: https://stats.stackexchange.com/questions/31908/what-is-percentage-of-variance-in-pca

[**3**] Continuous Compound Interest. Available: https://www.investopedia.com/articles/07/continuously_compound.asp

[**4**] Bond data from: https://markets.businessinsider.com/bonds/finder?borrower=71&maturity=shortterm&yield=&bondtype=2%2c3%2c4%2c16&coupon=&currency=184&rating=&country=19 and https://markets.businessinsider.com/bonds/finder?borrower=71&maturity=midterm&yield=&bondtype=2%2c3%2c4%2c16&coupon=&currency=184&rating=&country=19

GitHub Link to Code:
https://github.com/LarsKut/MathFinance