TTK4115 – Linear System Theory

Helicopter lab preparation

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1 Introduction

This document contains derivations and other preparatory work for the helicopter-lab. All tasks must be done to to complete the lab. Your time at the lab is limited and the preparations must be done ahead of time to save time.

In the preparation for each lab-day you will be asked to make a test-plan which you will use on your lab-day to experiment with the system. The goal of the experimentation is to test out the theory in practice and identify potential deviations. Your test-plan should consist of a plan on how to do the experimentation, which tuning values that should be tested, and a hypothesis of what you think will happen. Try to make tests that will show different types of behavior and that challenges the system and the theory. You don't need to specify numerical values, a qualitative description of the type of values you want to test out is sufficient. If you realize that you should test other things once you are at the lab, then you are free to revise the test-plan. The final report should contain your revised test-plan.

You will be evaluated based on the quality/creativity of your experimentation and the discussion of your results. Good reports challenge the hypothesis and discuss why they were right or wrong. Wrong hypotheses will not count negatively as long as they are discussed properly and you show a good understanding of the outcome of the experiments.

2 Tasks

2.1 Part I – Monovariable control

In this part you will apply a PD-controller to a linearized model of a physical helicopter in order to control the pitch. You will develop the linearized equations of motion for the system, implement the controller using state-feedback, compute the transfer function and experiment with pole placements. You will attempt to fly the helicopter using a joystick both without and with the PD-controller implemented.

The helicopter is modeled as three point-masses: two point-masses represent the two motors that are connected to the propellers, and one point mass represents the counterweight. The model of the helicopter is depicted in Fig. 2.1. The cubes in the figure represent the point masses whilst the cylinders represent the helicopter joints. The rotations of the joints are defined in the following manner: p denotes the pitch angle of the helicopter head, e denotes its elevation angle, and λ denotes the travel angle of the helicopter. In Fig. 2.1, all joint angles are zero. This means that $p = 0$ if the helicopter head is horizontal, and that $e = 0$ if the arm between the elevation axis and the helicopter head is horizontal.

The propeller forces of the front and back propeller are given by F_f and F_b, respectively. It is assumed that there is a linear relation between the voltages V_f and V_b supplied to the motors and the forces generated by the propellers:

$$F_f = K_f V_f \tag{2.1a}$$

$$F_b = K_f V_b \tag{2.1b}$$

Here, K_f is a motor force constant. The two propellers are placed symmetrically about the pitch axis. The sum of the forces $F_b + F_f$ developed by the two propellers determines the net lift of the helicopter. The difference between the forces $F_b - F_f$ is proportional to the torque about the pitch axis.

The gravitational forces for the front and the back motor are denoted by $F_{g,f}$ and $F_{g,b}$, while the gravitational force of the counterweight is denoted by $F_{g,c}$. Note that the gravitational forces always point in a vertical direction, whilst the direction of the propeller forces is dependent on the joint angles.

The masses of the front and the back motors are assumed to be equal and given by m_p. The mass of the counterweight is given by m_c. The distance from the elevation axis to the head of the helicopter is given by l_h, whilst the distance from the elevation axis to the counterweight is given by l_c. Because the two propellers are placed symmetrically about the pitch axis, the distance from the pitch axis to both motors is the same and is given by l_p. The masses and distances are depicted in Fig. 2.2. Other forces, such as friction and centripetal forces, are neglected.

Note: The actual values for the masses and distances differ from helicopter to helicopter and are therefore not given here. These values can be found in an initialization file that can be downloaded from "Blackboard". Due to the mass disparities, some helicopters may be more docile than others. The motor force constant K_f also differs quite a bit from helicopter to helicopter. This constant is to be determined experimentally at the lab.

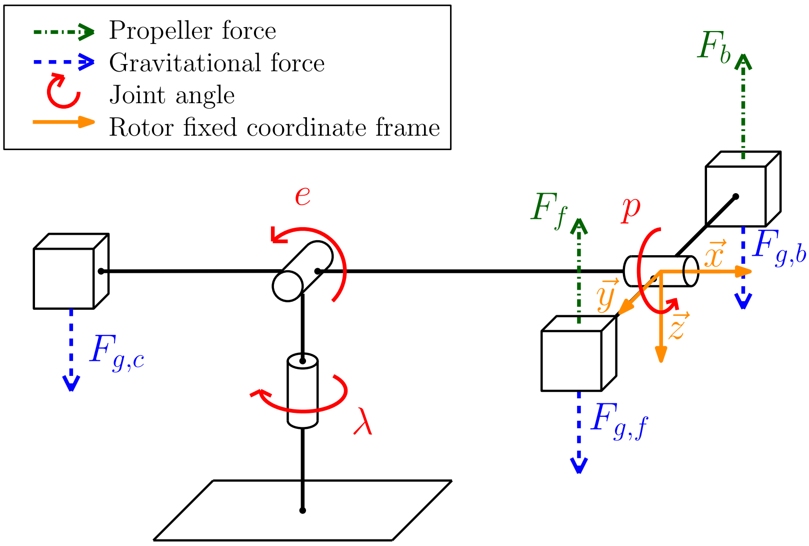

Figure 2.1: Forces and angles

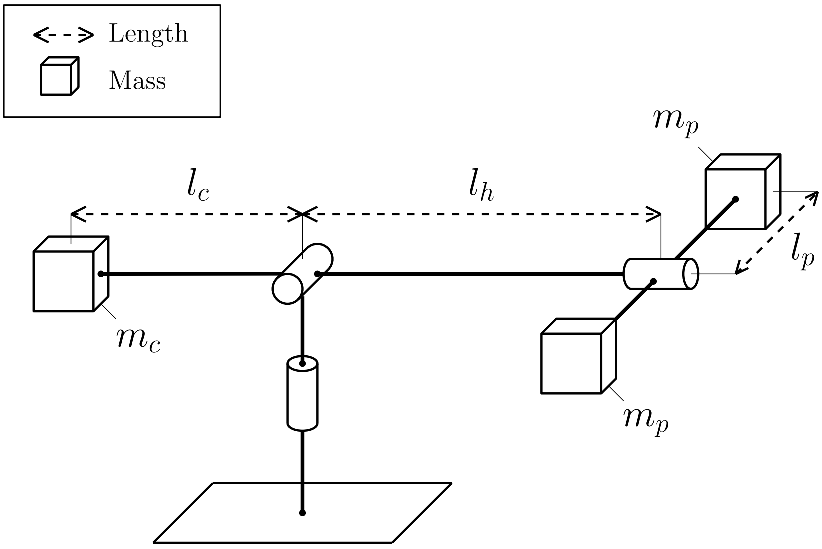

Figure 2.2: Masses and distances

2.1.1 Problem 1 - Equations of motion

Compute the equations of motion (differential equations) for the pitch angle p, the elevation angle e, and the travel angle λ. Let the moments of inertia about the pitch, elevation, and the travel axes be denoted by J_p, J_e, and J_λ, respectively. Show that the equations of motion can be stated in the form

$$J_p\ddot{p} = L_1 V_d \tag{2.2a}$$
$$J_e\ddot{e} = L_2\cos(e) + L_3 V_s\cos(p) \tag{2.2b}$$
$$J_\lambda\ddot{\lambda} = L_4 V_s\cos(e)\sin(p) \tag{2.2c}$$

where L_i, $i = 1, 2, 3, 4$ are constants, and where the sum of the motor voltages V_s and the difference of the motor voltages V_d are given by

$$V_s = V_b + V_f \tag{2.3a}$$
$$V_d = V_b - V_f \tag{2.3b}$$

Hint: $J_\theta\ddot{\theta} = \Sigma\tau = \Sigma Fr$, where J is the moment of inertia, $\ddot{\theta}$ is a rotational acceleration $(\ddot{p}, \ddot{e}, \ddot{\lambda})$, $\Sigma\tau$ is the net torque or the sum of forces times a position vector from the point about which the torque is being measured to the point where the force is applied.

2.1.2 Problem 2 - Linearization

We will now linearize the system (2.2a)-(2.2c) around the equilibrium point shown in Fig. 2.1. Note that a constant voltage, $V_{s,0}$, is required to keep the helicopter stationary at this equilibrium point. As we want to work with origo as our linearization point, do the following variable transformation: $\tilde{V}_s = V_s - V_{s,0}$. $V_{s,0}$ should be chosen such that $\tilde{V}_s = 0$ results in $e = 0$ being an equilibrium point of the system.

Assume that the moments of inertia are constant and given by

$$J_p = 2m_p l_p^2 \tag{2.4a}$$
$$J_e = m_c l_c^2 + 2m_p l_h^2 \tag{2.4b}$$
$$J_\lambda = m_c l_c^2 + 2m_p(l_h^2 + l_p^2) \tag{2.4c}$$

Show that the linearized equations of motion can be written as

$$\ddot{p} = K_1 V_d \tag{2.5a}$$
$$\ddot{e} = K_2\tilde{V}_s \tag{2.5b}$$
$$\ddot{\lambda} = K_3 p \tag{2.5c}$$

where K_i, $i = 1, 2, 3$ are constants and are necessary to complete the lab assignments.

Hint: The equation for the lineraization of a function $f(x, y)$ at a point $p(a, b)$ is:

$$f(x,y) \approx f(a,b) + \frac{\partial f(x,y)}{\partial x}\bigg|_{a,b}(x-a) + \frac{\partial f(x,y)}{\partial y}\bigg|_{a,b}(y-b) \tag{2.6}$$

2.1.3 Problem 3 - PD Control

A PD controller is to be implemented to control the pitch angle, p. This controller is given as

$$V_d = K_{pp}(p_c - p) - K_{pd}\dot{p} \tag{2.7}$$

where p_c is the desired reference for the pitch angle p given by a joystick. Substitute (2.7) in the equation of motion for the pitch angle in (2.5a). Apply the Laplace transform to the resulting differential equation to find the transfer function $G(s) = \frac{p(s)}{p_c(s)}$. Note that the derivative component of the controller equation is usually given by the derivative of the error of the state (deviation from the reference). However, due to rapid changes in the reference, the resulting dirac impulse would make the implementation in our set up behave poorly. Hence, this term is approximated and works ok in the range $0 < (p_c - p) << 1$.

2.1.4 Problem 4 - Pole Placement

Poles of a transfer function are the frequencies for which the value of the denominator in the transfer function becomes zero. The values of the poles of a system determine whether the system is stable, and how well the system performs. Control systems, in the most simple sense, can be designed simply by assigning specific values to the poles of the system.

We wish to experiment with different configurations of pole values for the closed loop pitch-system. Develop an expression for K_{pp} and K_{pd} as a function of the desired poles λ_1 and λ_2, i.e. the values that make the denominator, $D_{tf}(s)$ in the transfer function $G(s) = \frac{N_{tf}(s)}{D_{tf}(s)}$, satisfy $D_{tf}(\lambda_{1,2}) = 0$.

Now we want to formulate a hypothesis of how different choices of pole-values will affect the physical helicopter. How should the response of the helicopter be in theory? Is the response of the physical helicopter as we expect it to be in theory? What are the differences between theory and practice here, and why? After formulating your hypothesis, make a test-plan for different values for the poles λ_1 and λ_2 that are to be tested during your assigned lab time-slot. The test-plan should test out different types of behavior that can be achieved by placing the two poles in different places in the complex plane. It is encouraged to make tests that challenge the theory.

Hint: Some keywords you might want to consider for your hypothesis: stable, unstable, saturation, damped, oscillations, etc...

2.2 Part II – Multivariable control

In this part, you will apply the LQR optimal controller covered in lecture week 4 (week 37). LQR is not covered in the Chen or Brown books, but some information in addition to lecture slides can be found in the LQR note (available on Blackboard).

A multivariable controller can be tuned based on pole placement, but the relation between the poles and the controller's performance may not be very intuitive. LQR allows you instead to tune the controller based on a cost function that punishes state deviations and inputs. By setting up the Q and R matrices you can prioritize which state deviations you would like to reduce the most, and how strong inputs you want the controller to use. Note that when using LQR you still have a linear state feedback controller, it is just the tuning of the feedback matrix K that is done through LQR optimization.

You will also implement integral action (also covered lectures in week 4) as part of the LQR controller, and see in practice how that affects the feedback controlled system.

2.2.1 Problem 1 - State space formulation

Put the system of equations given by the relations for pitch and elevation in (2.5a)-(2.5b) in a state-space formulation of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \tag{2.8}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices of suitable dimension. The state vector and the input vector are defined by

$$\boldsymbol{x} = \begin{bmatrix} p \\ \dot{p} \\ \dot{e} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{u} = \begin{bmatrix} \tilde{V}_s \\ V_\mathrm{d} \end{bmatrix} \tag{2.9}$$

2.2.2 Problem 2 - Controllability

Examine the controllability of the system.

Hint: $A^2B = A \cdot (AB)$. You can use Matlab to verify your answers, check the Matlab/Simulink tips & tricks note on Blackboard.

2.2.3 Problem 3 - Feedback and feedforward

We aim to track the reference $\mathbf{r} = [p_c,\ \dot{e}_c]^T$ for the pitch angle p and elevation rate $\dot{e}$, which will be given by the joystick output.

Consider a state-feedback controller with reference-feed-forward of the following form:

$$\boldsymbol{u} = \boldsymbol{F}\boldsymbol{r} - \boldsymbol{K}\boldsymbol{x} \tag{2.10}$$

where

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix} \tag{2.11}$$

Derive an expression for matrix $\boldsymbol{F}$ as a function of $k_{11}\, k_{12}\, k_{13}\, k_{21}\, k_{22}\, k_{23}$ such that (in theory) $\lim_{t\to\infty} p(t) = p_c$ and $\lim_{t\to\infty} \dot{e}(t) = \dot{e}_c$ for fixed values of p_c and $\dot{e}_c$.

Hint: Insert the state-feedback controller with reference-feed-forward, $\boldsymbol{u}$, into the state-space equation $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$. How does $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ look in the limit $t \to \infty$?

2.2.4 Problem 4 - Linear Quadratic Regulator (LQR)

In the first lab-assignment we chose the values in the state-feedback matrix directly through manual placement of poles. Now, the $\boldsymbol{K}$ matrix will be chosen such that the following cost function is minimized

$$J = \int_0^\infty \left(\boldsymbol{x}^\intercal(t)\boldsymbol{Q}_{\text{LQR}}\boldsymbol{x}(t) + \boldsymbol{u}^\intercal(t)\boldsymbol{R}_{\text{LQR}}\boldsymbol{u}(t)\right) dt \tag{2.12}$$

$\boldsymbol{Q}_{\text{LQR}}$ and $\boldsymbol{R}_{\text{LQR}}$ are weighting matrices. For simplicity, let the weighting matrices $\boldsymbol{Q}_{\text{LQR}}$ and $\boldsymbol{R}_{\text{LQR}}$ be diagonal.

Now we want to formulate a hypothesis of how different choices of $\boldsymbol{Q}_{\text{LQR}}$ and $\boldsymbol{R}_{\text{LQR}}$ affect the physical helicopter. How should the response of the helicopter be in theory? Is the response of the physical helicopter as we expect it to be in theory? What are the differences between theory and practice here, and why? Make a test-plan for different choices in values on the diagonal $\boldsymbol{Q}_{\text{LQR}}$ and $\boldsymbol{R}_{\text{LQR}}$ that are to be tested during your assigned lab time-slot.

2.2.5 Problem 5 - Integral action

We want to include an integral effect for the elevation rate and pitch angle in our controller. We do this by introducing two additional states γ and ζ, for which the differential equations are given by

$$\begin{aligned} \dot{\gamma} &= p_c - p \\ \dot{\zeta} &= \dot{e}_c - \dot{e} \end{aligned} \tag{2.13}$$

The augmented state vector shown below should be used.

$$\boldsymbol{x} = \begin{bmatrix} p \\ \dot{p} \\ \dot{e} \\ \gamma \\ \zeta \end{bmatrix} \tag{2.14}$$

The system should take on the following form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{G}\boldsymbol{r} \tag{2.15}$$

$$\boldsymbol{u} = \boldsymbol{F}\boldsymbol{r} - \boldsymbol{K}\boldsymbol{x} \tag{2.16}$$

Find the augmented $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{G}$ matrices.

As in the previous task, you will make a hypothesis on how the choices of the diagonal values in $\boldsymbol{Q}_{\text{LQR}}$ and $\boldsymbol{R}_{\text{LQR}}$ will affect the response of the helicopter. Focus on how the introduction of γ and ζ affect the behavior. Additionally, you should hypothesize how the introduction of integral effect affect our choice of $\boldsymbol{F}$. Make a test-plan for the different values in the different variables you want to test during your assigned lab time-slot.

2.3 Part III – Luenberger observer

Observability of LTI systems is covered in lecture week 5 (week 38), and in Section 6.3 of the Chen book. State estimation and feedback from estimated states is covered in lecture week 6 (week 39), and in Sections 8.4-8.8 of the Chen book.

For an observable LTI system, a Luenberger observer can be used to estimate the state values based on the system matrices and the measured outputs. If it is not possible to directly measure all states, a state estimator such as the Luenberger observer is required to estimate the full state vector. When the measurements are noisy, such as the IMU measurements for the helicopter, an observer is particularly useful, since a well tuned observer can provide more accurate estimates of the state values than what we get from the measurements directly.

In the previous tasks we have used the encoder measurements as a direct measurement of the state which we have used for feedback. Unfortunately encoders require the helicopter to be stuck to ground such that the angles of the joints can be measured. As we aspire towards controlling a free-flying helicopter one day, we have to use a different technology than encoders to measure the state of the system. An inertial measurement unit (IMU) called MPU-9250 produced by InvenSense will instead be used. This is a cheap IMU that is often used in hobby projects and light-weight drones.

The IMU is mounted in the middle of the bar connecting the rotors. The IMU consists of an accelerometer and a gyroscope that will measure acceleration and rotations in its local frame that rotates when the helicopter rotates, this frame is shown in orange in figure Fig. 2.1. This frame is defined such that the $\vec{y}$ axis always aligns with the bar between the rotors, and the $\vec{x}$ axis always aligns with the arm out to the rotors, even if the helicopter is rotated.

The gyroscope measures the rotational velocity in units rad/s, whilst the accelerometer measures the proper acceleration in units $\mathrm{m/s^2}$. In contrast to regular acceleration which measures the second derivative of linear position, proper acceleration measures the acceleration in relation to free-fall. When the system is in free-fall the proper acceleration evaluates to $0\mathrm{m/s^2}$. When the system is laying stationary on a table it will be affected by the normal force of the table pushing up, and will therefore measure $9.81\mathrm{m/s^2}$ in the upwards direction. This information will be used to indirectly measure the orientation of the IMU.

2.3.1 Problem 1 - Extended state-space formulation

Consider the following state-space formulation of the system

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned} \tag{2.17}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices. The state vector and the output vector should be

$$\mathbf{x} = \begin{bmatrix} p \\ \dot{p} \\ e \\ \dot{e} \\ \dot{\lambda} \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} \tilde{V}_s \\ V_d \end{bmatrix} \tag{2.18}$$

Use the linearized equations of motion (2.5a)-(2.5c) to evaluate $\boldsymbol{A}$ and $\boldsymbol{B}$ in this extended state-space formulation.

Hint: Note that this state vector is not identitcal to the state-vector for the LQR controller in the previous task. Change the names of your matrices in your Matlab program accordingly. We are allowed to estimate states even though we are not controlling them.

2.3.2 Problem 2 - Observability

Assume we are free to measure any of the states in $\mathbf{x}$. Which are the minimum set of states that if measured makes the system (2.18) observable? You may use the Matlab $obsv(A, C)$ command.

Include a discussion in the report on which states that need to be measure to make the system observable, and why we need to measure exactly these states.

2.3.3 Problem 3 - Angle measurement

In the previous task you should have identified that it is not sufficient to only use rate measurements from the gyroscope. Additionally, some angle measurements are needed. The IMU does not have a sensor that let us measure its angle. Instead we will utilize the fact that the accelerometer measures the force counteracting gravity when the helicopter is stationary. This force will always point straight upwards with a magnitude equal to the gravitational constant g.

Assume that the only force affecting the IMU is the normal force pointing upwards with a constant magnitude. Derive an equation for pitch and elevation given the acceleartion $[a_x, a_y, a_z]$ measured by the IMU. Show that the equations can be written as:

$$p = \arctan\left(\frac{a_y}{a_z}\right) \tag{2.19a}$$

$$e = \arctan\left(\frac{a_x}{\sqrt{a_y^2 + a_z^2}}\right) \tag{2.19b}$$

Hint: Draw a sketch of the helicopter along different axes showing the trigonometric relation between the angles p and e and the accelerations a_x, a_y, a_z. It may be beneficial to start by looking at the elevation angle and keeping $p = 0$. You can also start with the answer and work your way backwards.

2.3.4 Problem 4 - State estimator

The following linear observer will be used to estimate the state of the system.

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{2.20}$$

Similar to the monovariable control in lab assignment 1 we will manually determine the observer gain, L, using pole placement. We can use the *place* function in Matlab: $L = place(\mathbf{A}', \mathbf{C}', \mathbf{p}).'$, where $\mathbf{p}$ is a vector of poles. Note that $\mathbf{A}$, $\mathbf{C}$, and the resulting L are transposed in the Matlab function when placing poles for observers. Experiment with different C matrices to verify the theoretical relationship between observability and placing of estimator poles.

When flying the helicopter we want to use all the available measurements from the IMU. Hence, we need to change the C matrix accordingly. As in the previous task, you should make a hypothesis on how different choices of pole-values affect the estimated states. Make a test-plan that will be carried out during your assigned lab time-slot.

Hint: Some keywords you might want to consider for your hypothesis: fast estimator, slow estimator, noisy measurements.

2.4 Part IV – Kalman filter

The Kalman filter is an observer whose estimator feedback gain matrix is computed to minimize the variance of the estimation error. Where a Luenberger observer is tuned based on pole placement (or other approaches), the Kalman filter is tuned based on measurement uncertainty and our assumptions of process noise leading to model uncertainty. The Kalman filter is covered in lecture weeks 11 and 12 (weeks 44 and 45), and in Chapter 4 of the Brown book.

In this part, the Kalman filter will be explored as an alternative to the Luenberger observer for the helicopter. Since the model has uncertainty, and the measurements are noisy, it makes sense to model the system as a stochastic system. You need to estimate the measurement noise (R_d), and make assumptions about the process noise (Q_d), and the Kalman filter will provide the optimal feedback gain for the estimator based on these assumptions.

We wish to use a discrete-time Kalman Filter to estimate the state of the system. The Kalman Filter can be described by the equations derived in homework assignment 6:

Correction with new data:

$$\boldsymbol{L}[k] = \bar{\boldsymbol{P}}[k]\boldsymbol{C}_\mathrm{d}^\top(\boldsymbol{C}_\mathrm{d}\bar{\boldsymbol{P}}[k]\boldsymbol{C}_\mathrm{d}^\top + \boldsymbol{R}_\mathrm{d})^{-1} \tag{2.21a}$$

$$\hat{\boldsymbol{x}}[k] = \bar{\boldsymbol{x}}[k] + \boldsymbol{L}[k](\boldsymbol{y}[k] - \boldsymbol{C}_\mathrm{d}\bar{\boldsymbol{x}}[k]) \tag{2.21b}$$

$$\hat{\boldsymbol{P}}[k] = (\boldsymbol{I} - \boldsymbol{L}[k]\boldsymbol{C}_\mathrm{d})\bar{\boldsymbol{P}}[k](\boldsymbol{I} - \boldsymbol{L}[k]\boldsymbol{C}_\mathrm{d})^\top + \boldsymbol{L}[k]\boldsymbol{R}_\mathrm{d}\boldsymbol{L}^\intercal[k] \tag{2.21c}$$

Predicting ahead:

$$\bar{\boldsymbol{x}}[k+1] = \boldsymbol{A}_\mathrm{d}\hat{\boldsymbol{x}}[k] + \boldsymbol{B}_\mathrm{d}\boldsymbol{u}[k] \tag{2.21d}$$

$$\bar{\boldsymbol{P}}[k+1] = \boldsymbol{A}_\mathrm{d}\hat{\boldsymbol{P}}[k]\boldsymbol{A}_\mathrm{d}^\intercal + \boldsymbol{Q}_\mathrm{d} \tag{2.21e}$$

2.4.1 Problem 1 - Discretization

The following discrete-time model will be used to describe the system.

$$\boldsymbol{x}[k+1] = \boldsymbol{A}_\mathrm{d}\boldsymbol{x}[k] + \boldsymbol{B}_\mathrm{d}\boldsymbol{u}[k] + \boldsymbol{w}_\mathrm{d}[k] \tag{2.22a}$$

$$\boldsymbol{y}[k] = \boldsymbol{C}_\mathrm{d}\boldsymbol{x}[k] + \boldsymbol{v}_\mathrm{d}[k] \tag{2.22b}$$

$$\boldsymbol{w}_\mathrm{d} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_\mathrm{d}), \qquad \boldsymbol{v}_\mathrm{d} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}_\mathrm{d}) \tag{2.22c}$$

The covariance of the stochastic disturbance $\boldsymbol{Q}_\mathrm{d}$ serves as a tuning variable and can be left undetermined. The measurement covariance $\boldsymbol{R}_\mathrm{d}$ will be experimentally estimated in the lab.

For all of the subsequent tasks we will use the full state-space model with $\boldsymbol{x} = [p \quad \dot{p} \quad e \quad \dot{e} \quad \lambda \quad \dot{\lambda}]$. Discretize the continuous-time system with the same time-step as Simulink to obtain $\boldsymbol{A}_\mathrm{d}$, $\boldsymbol{B}_\mathrm{d}$ and $\boldsymbol{C}_\mathrm{d}$. The provided Simulink diagram operates in fixed-step mode with a time-step of 0.002s. You may employ the `c2d` function in Matlab.

2.4.2 Problem 2 - experimentation

As in the previous task, you should make a hypothesis on how different choices of values along the diagonal of $\boldsymbol{Q}_d$ will affect the estimation of the different states. Make a test-plan of different values that will be carried out during your assigned lab time-slot. Particularly interesting limits to test are $\boldsymbol{Q}_\mathrm{d}$ equal to 0 or infinity.

2.4.3 Bonus tasks:

You don't have to do these tasks, but it makes for a nice discussion in your report.

1. Find $\boldsymbol{Q}_d$ by using Van Loan's method.
2. In the previous lab-day you may have noticed a constant bias between the IMU and the encoders. This bias is in reality slowly varying which can make it change from one lab-day to the next. To avoid having to evaluate new biases every time we are at the lab we wish to estimate their values with the Kalmanfilter.

   Include a new state in your state vector $\mathbf{x}$ for each bias you wish to estimate. Update your $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\boldsymbol{Q}_\mathrm{d}$ matrix. Evaluate the observability of the new system.