The complexity of FXL

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Motivation

The XL and FXL algorithms are among the state-of-the-art polynomial system solvers

They currently even hold some record computations

Furthermore, for random MQ systems, their complexity can be analyzed, as opposed to some more involved algorithms

A recap of XL

Multivariate systems

We consider polynomials

$$f_1, \dots, f_m \in \mathcal{R} = \mathbb{F}_q[x_1, \dots, x_n]$$

and assume the corresponding systems have a single solution, $(a_1, \dots, a_n)$, so that

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For example:

$$\begin{aligned} \langle\ 8z^2 + 7x + 3y + 7z + 7,& \\ 6yz + 5z^2 + 7x + 7,& \\ 8xz + 6yz + 7y + 8z + 6,& \\ 4x^2 + 5xz + 6yz + 7x + 2\ \rangle& \end{aligned} \quad = \quad \langle\ x - 3,\ y - 1,\ z - 4\ \rangle$$

How to get the solution?

We "just" need to find $g_{ij} \in \mathcal{R}$ $(= \mathbb{F}_q[x_1, \ldots, x_n])$ so that

$$
\begin{gathered}
g_{11} \cdot f_1 + \ldots + g_{1m} \cdot f_m = x_1 - a_1 \\
\vdots \\
g_{n1} \cdot f_1 + \ldots + g_{nm} \cdot f_m = x_n - a_n
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In practice, there are many such g_{ij}. However, we can limit the degree of $g_{ij} \cdot f_j$ so that there is still a solution

The XL strategy

(1) Determine the degree d_{solv} for which such g_{ij} should exist

(2) Consider the Macaulay matrix which has as rows all f_i multiplied by all monomials g up to a certain degree

(3) Find a right kernel element of this matrix

Analyzing the operating degree

Monomials

First, let us define R_d to be the monomials of $\mathcal{R}$ of degree d

$$R_d = \left\{ \prod_{j=1}^{d} x_{i_j} \mid 1 \leq i_1, \ldots, i_d \leq n \right\} \quad \text{for } d \geq 0$$

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For example:

$$R_0 = \{1\}$$
$$R_1 = \{x_1, \ldots, x_n\}$$
$$R_2 = \{x_1^2, x_1 x_2, \ldots, x_{n-1} x_n, x_n^2\}$$

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Furthermore we define: $R_{\leq d} = R_d \cup \cdots \cup R_0$.

Pictorially

$R_0 = \{1\}$	$R_1 = \{x_i\}_i$	$R_2 = \{x_i x_j\}_{ij}$	R_3	R_4	$\cdots$

$\deg_X$

A quadratic equation is then a linear combination of the union of the blue boxes $R_{\leq 2}$.

Macaulay rows

Then, let us define the rows of the Macaulay matrix

$$I_{\leq d} = \{uf_i \mid u \in R, 1 \leq i \leq m, \deg(uf_i) \leq d\}$$

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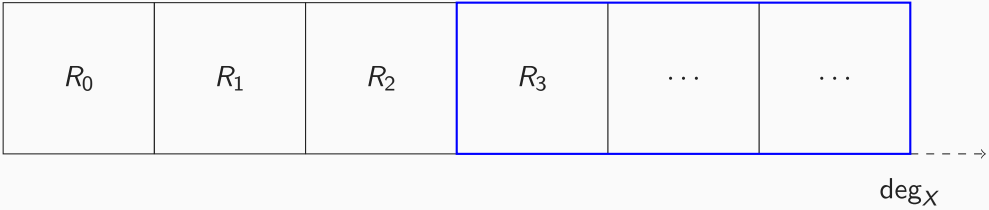
Now we can describe these using $R_{\leq d}$

$$I_{\leq d} = f_1 \cdot R_{\leq d - \deg(f_1)} + \ldots + f_m \cdot R_{\leq d - \deg(f_m)}$$

Pictorially again



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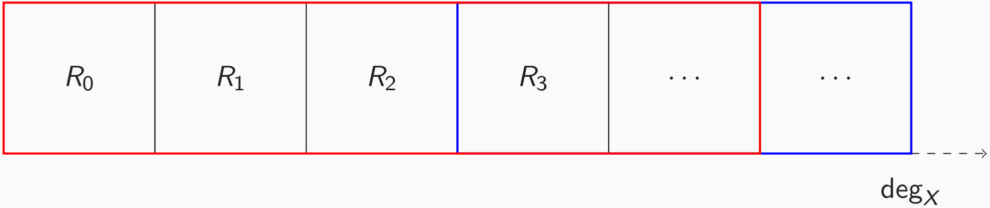

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For all systems, the $I_{\leq 4}$ polynomials are linear combinations of the union of the red boxes.

What about d_{solv}?

Now d_{solv} is exactly the degree for which the Macaulay matrix has a right kernel of dimension 1.

The degree d Macaulay matrix $\mathcal{M}(f_1, \ldots, f_m)$ of a system is the matrix where:

- Its rows are labeled by the products $u \cdot f_i$ (the vectors that span $I_{\leq d}$)
- Its columns are labeled by monomials r in $R_{\leq d}$ (sorted in graded reverse order*)
- Its coefficients are the coefficient of r in $u \cdot f_i$

Macaulay matrices example $d = 3$

$$
\begin{array}{c c}
 & \begin{array}{cccccc} x_1^3 & x_1^2 x_2 & \dots & x_n & 1 \end{array} \\
\begin{array}{c} 1 \cdot f_1 \\ 1 \cdot f_2 \\ \vdots \\ x_1 \cdot f_1 \\ x_1 \cdot f_2 \\ \vdots \\ x_n \cdot f_m \end{array} &
\begin{pmatrix}
0 & 0 & \dots & 5 & 9 \\
0 & 0 & \dots & 7 & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
3 & 1 & \dots & 0 & 0 \\
4 & 8 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 6 & 0
\end{pmatrix}
\end{array}
$$

Our hope is, that after row-reduction, we find linear polynomials

Can we force this?

If there are enough linear independent rows in the matrix, we surely obtain a unique solution

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The amount of columns is given by:

$$\dim(R_{\leq d}) = \binom{n+d}{d}$$

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So we can easily determine d_{solv} right? Not yet...

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The rows that we generate might contain linear dependencies, called syzygies

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Not only that, but even syzygies between syzygies can appear!

When we account for these syzygies, and no other syzygies appear up to d_{solv} we can predict d_{solv}

Note that at that point syzygies must appear!

Counting the linear independent equations

When we correctly account for these syzygies and no other syzygies appear, we obtain the following number of linear independent rows

$$\sum_{i=1}^{n} (-1)^{i+1} \binom{m}{i} \binom{n+d-2i}{d-2i}$$

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Then d_{solv} is exactly the lowest d for which the alternating sum above is greater than the number of columns $\binom{n+d}{d}$, i.e.

$$\sum_{i=0}^{n}(-1)^{i+1}\binom{m}{i}\binom{n+d-2i}{d-2i} \leq 0$$

Lets plug in some numbers!

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1	12	58	124	-5	-623

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Let us consider a random quadratic system with 11 variables and 20 equations

We consider the numbers $\dim(R_{\leq d})$

1	12	78	364	1365	4368

$\deg_X$

The complexity

The cost of reducing a matrix

Recall, that we have a matrix of size $m \cdot \binom{n+d-2}{d-2} \times \binom{n+d}{d}$ that we want to reduce.
This has a complexity of

$$\mathcal{O}\left(\binom{n+d-2}{d-2}\binom{n+d}{d}^2\right) \text{ field operations}$$

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It turns out that we can randomly remove rows to get a square matrix of the same rank, for a cost of

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The parameter d is of a large influence in determining the complexity!

Because d has such a large influence, it would be worthwhile to find ways to lower it.

One such way is by guessing variables, essentially reducing n.

This has a cost, but might result in a good tradeoff.

$$C_{FXL}(q, n, m) = \min_{0 \leq k \leq n} q^n C_{XL}(q, n-k, m)$$

Sparse linear algebra instead

The matrix that we construct is generally really sparse. I.e. it has a lot of zero entries. We can use algorithms optimized for such systems such as Wiedemann. They have cost

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Here ρ is the density of the matrix and N the size of the square matrix.

For MQ we have a density of $\binom{n+2}{2}$.

Hilbert series and semi-regularity

Semi-regularity

In our computation of the number of linear independent rows, we assumed that there are no additional syzygies.

This assumption is called the semi-regularity assumption of random polynomial systems. For random systems, this is believed to be true (but not yet proven!).

For structured systems, this is often very much not the case. If we cannot correctly predict the solving degree, we can not correctly predict the complexity!

We described the rank of the Macaulay matrix using an alternating sum constraint.

In the literature, this is often done with the Hilbert series

$$\frac{(1-t^2)^m}{(1-t)^{n+1}} = \sum_{d \geq 0} \sum_i (-1)^i \binom{m}{i} \binom{n+d-2i}{d-2i} \cdot t^d$$

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Semi-regularity now claims that the polynomial system exactly follows the Hilbert series.

What did we do?

We explored the theory of Macaulay matrices for XL.

We computed the solving degree for semi-regular polynomial systems.

We computed the complexity for XL and FXL.

Thanks for listening!