Where did my RAM go? Using algebraic cryptanalysis in practice (modelling exercises)

Lars Ran

Monika Trimoska

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TU/e

Algebraic cryptanalysis

A type of cryptanalytic methods where the problem of finding the secret key (or any attack goal) is reduced to the problem of finding a solution to a nonlinear multivariate polynomial system of equations.

Exercise 1: Trivium

Trivium

Initialisation:

$(s_1, \ldots, s_{93}) \leftarrow (K_1, \ldots, K_{80}, 0, \ldots, 0)$

$(s_{94}, \ldots, s_{177}) \leftarrow (IV_1, \ldots, IV_{80}, 0, \ldots, 0)$

$(s_{178}, \ldots, s_{288}) \leftarrow (0, \ldots, 0, 1, 1, 1)$

Algorithm 8.1 Trivium's iterative function for keystream generation.

Input: The number of bits to be generated, denoted Z.
Output: Keystream vector z.

for $i = 1$ to Z **do**
  $t_1 \leftarrow s_{66} + s_{93}$
  $t_2 \leftarrow s_{162} + s_{177}$
  $t_3 \leftarrow s_{243} + s_{288}$
  $z_i \leftarrow t_1 + t_2 + t_3$
  $t_1 \leftarrow t_1 + s_{91} \cdot s_{92} + s_{171}$
  $t_2 \leftarrow t_2 + s_{175} \cdot s_{176} + s_{264}$
  $t_3 \leftarrow t_3 + s_{286} \cdot s_{287} + s_{69}$
  $(s_1, s_2 \ldots, s_{93}) \leftarrow (t_3, s_1, \ldots, s_{92})$
  $(s_{94}, s_{95} \ldots, s_{177}) \leftarrow (t_1, s_{94}, \ldots, s_{176})$
  $(s_{178}, s_{179} \ldots, s_{288}) \leftarrow (t_2, s_{178}, \ldots, s_{287})$
end for

Iterate for 1155 rounds without producing any output

Trivium

Keystream generation:

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end for

Exercise 2: MQ-Sign

The trapdoor construction

The trapdoor construction

- Central map:
  $f : (x_1, \ldots, x_n) \in \mathbb{F}_q^n \to \left(f^{(1)}(x_1, \ldots, x_n), \ldots, f^{(m)}(x_1, \ldots, x_n)\right) \in \mathbb{F}_q^m$

- Two bijective linear (or affine) transformations:
  $\mathbf{S} \in \mathrm{GL}_n(\mathbb{F}_q)$ and $\mathbf{T} \in \mathrm{GL}_m(\mathbb{F}_q)$

- Public map:
  $p = \mathbf{T} \circ f \circ \mathbf{S}$

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Main idea:

- The central map has a structure such that it is easy to find preimages: it is easy (polynomial time) to compute $f^{-1}(\mathbf{x})$ for a target vector $\mathbf{x}$.

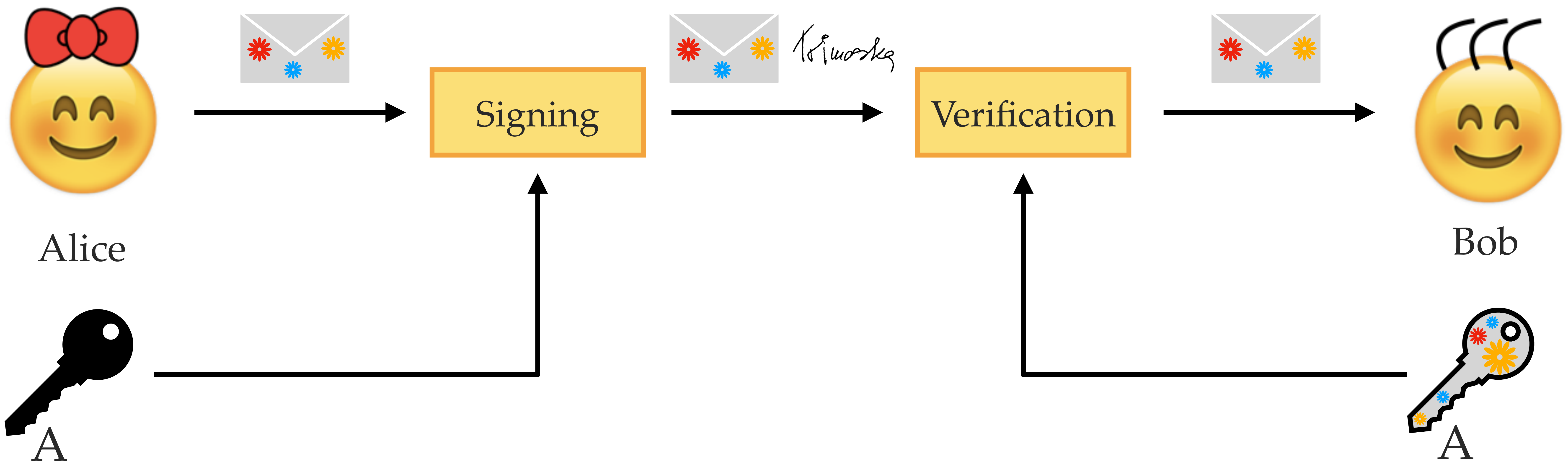
- The linear transformations hide the structure of the central map.

The trapdoor construction

$$
\begin{tikzcd}
\mathbf{w} \in \mathbb{F}_q^m \arrow[r, "\mathbf{T}^{-1}"] & \mathbf{x} \in \mathbb{F}_q^m \arrow[r, "f^{-1}"] & \mathbf{y} \in \mathbb{F}_q^n \arrow[r, "\mathbf{S}^{-1}"] & \mathbf{z} \in \mathbb{F}_q^n \arrow[lll, bend left, "p"]
\end{tikzcd}
$$

General workflow

The trapdoor construction



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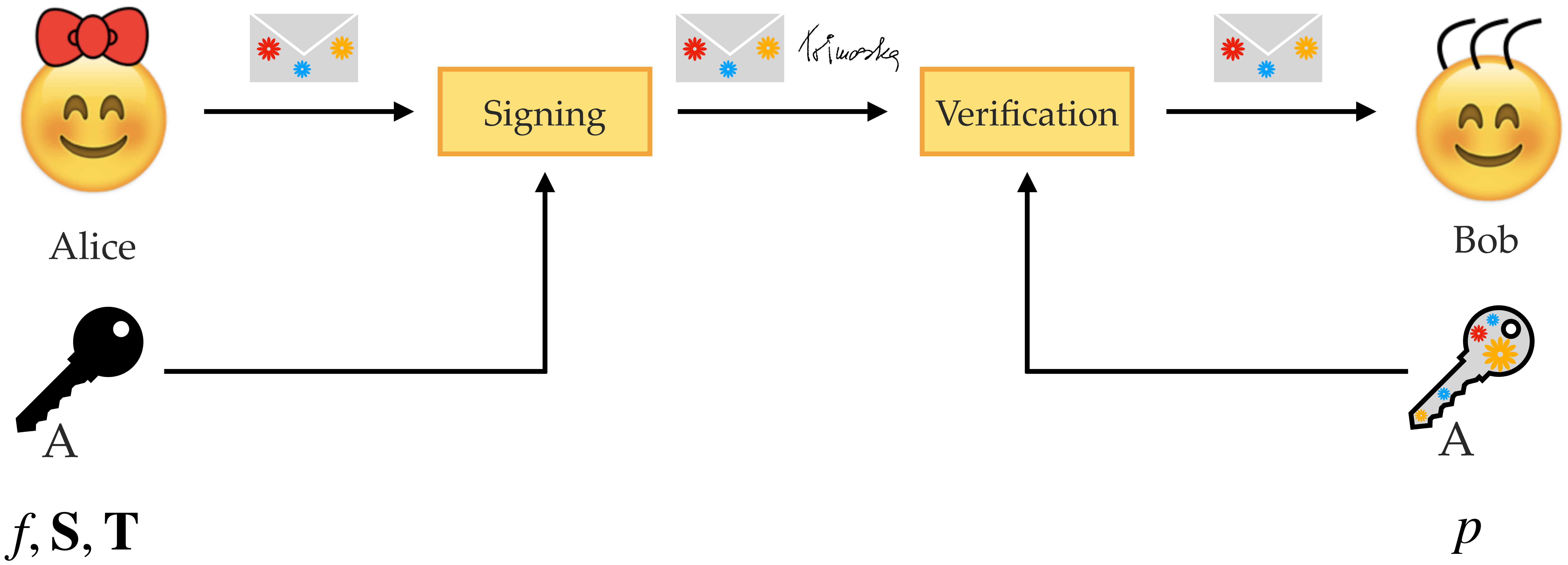

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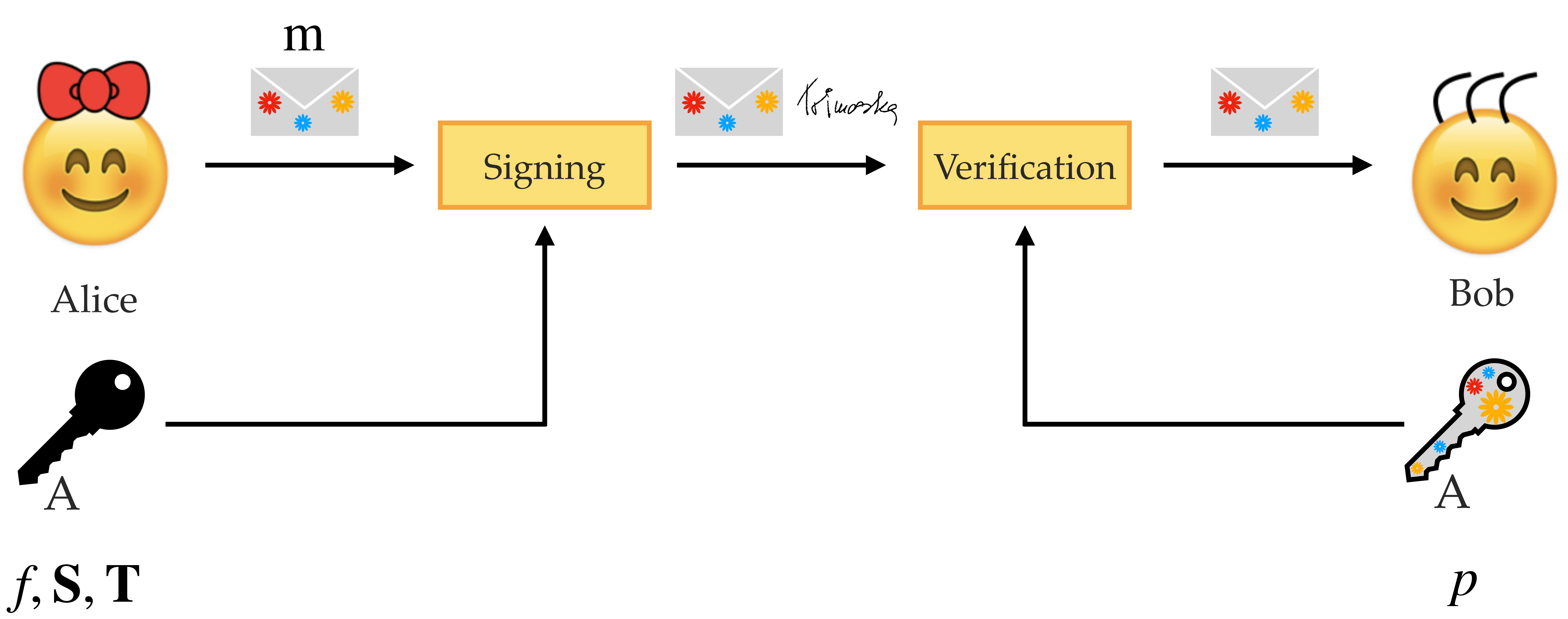

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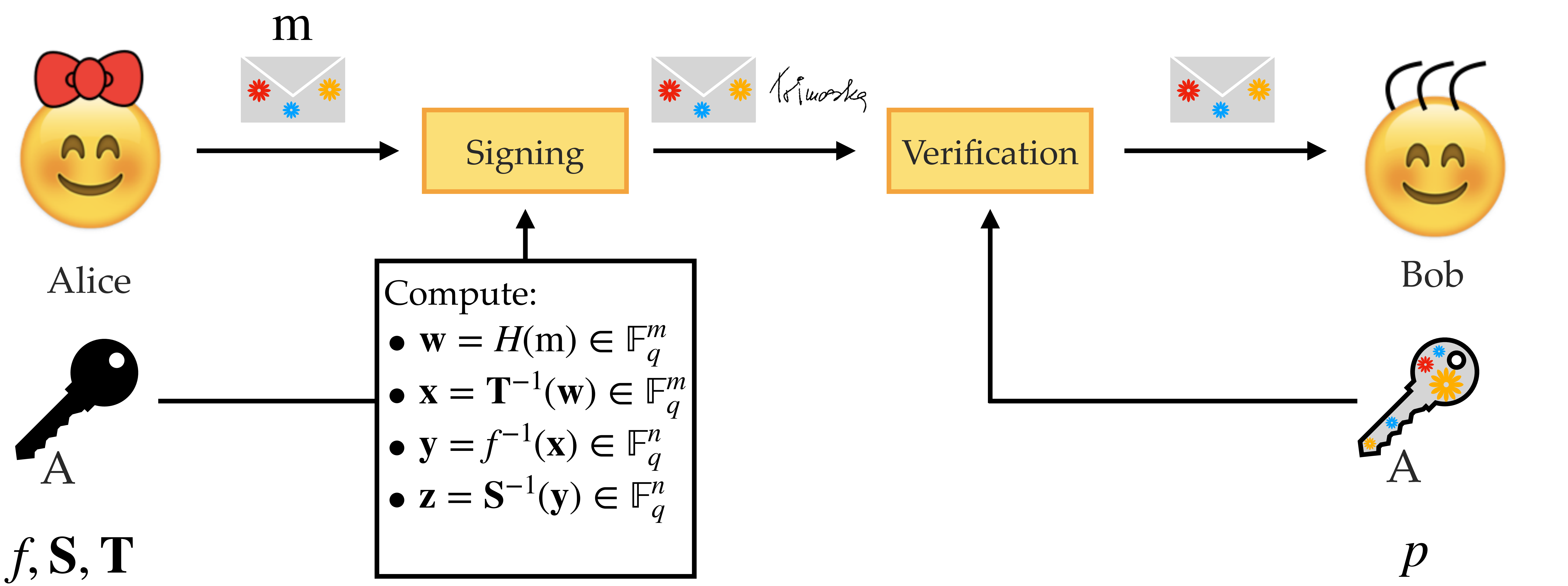

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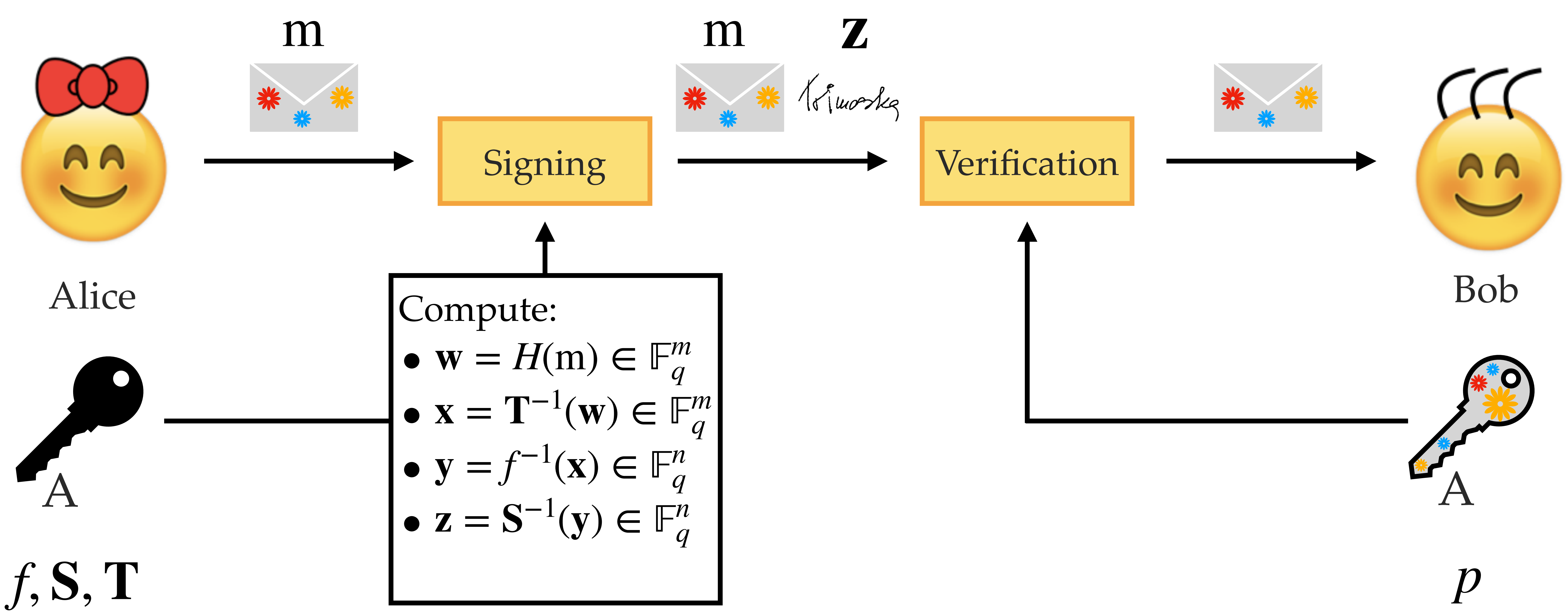

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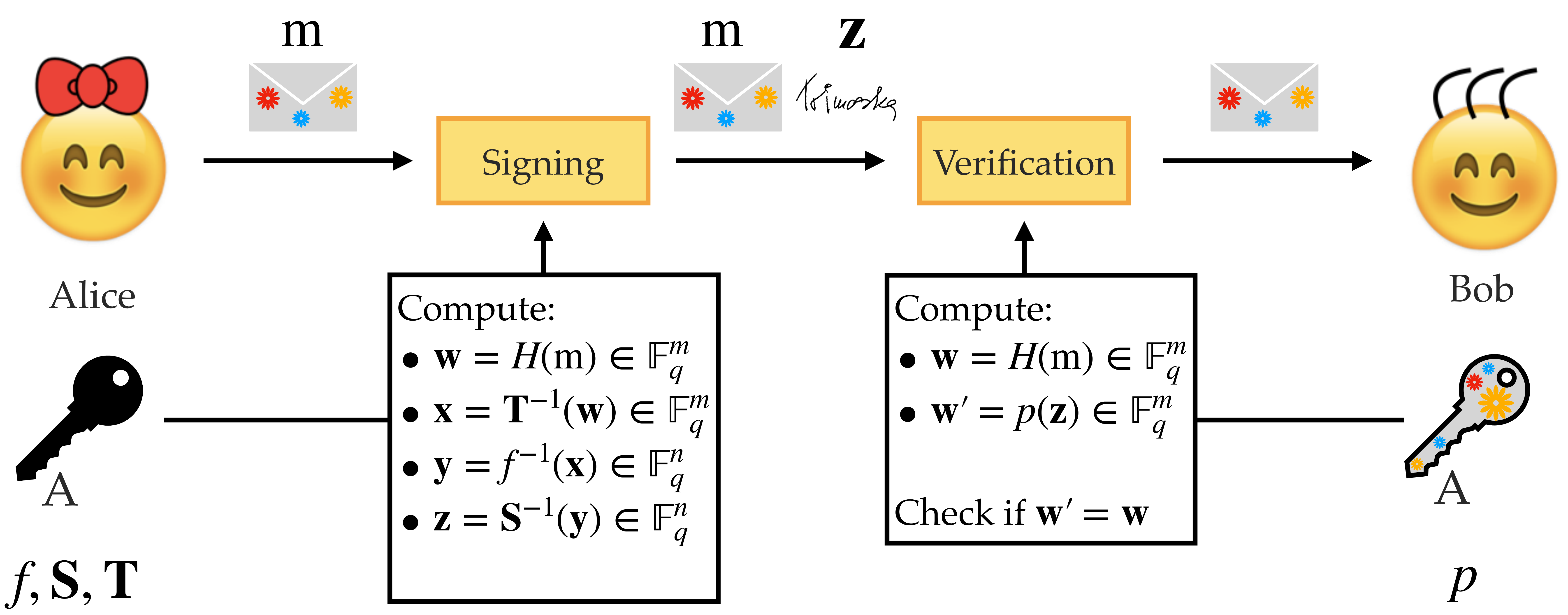

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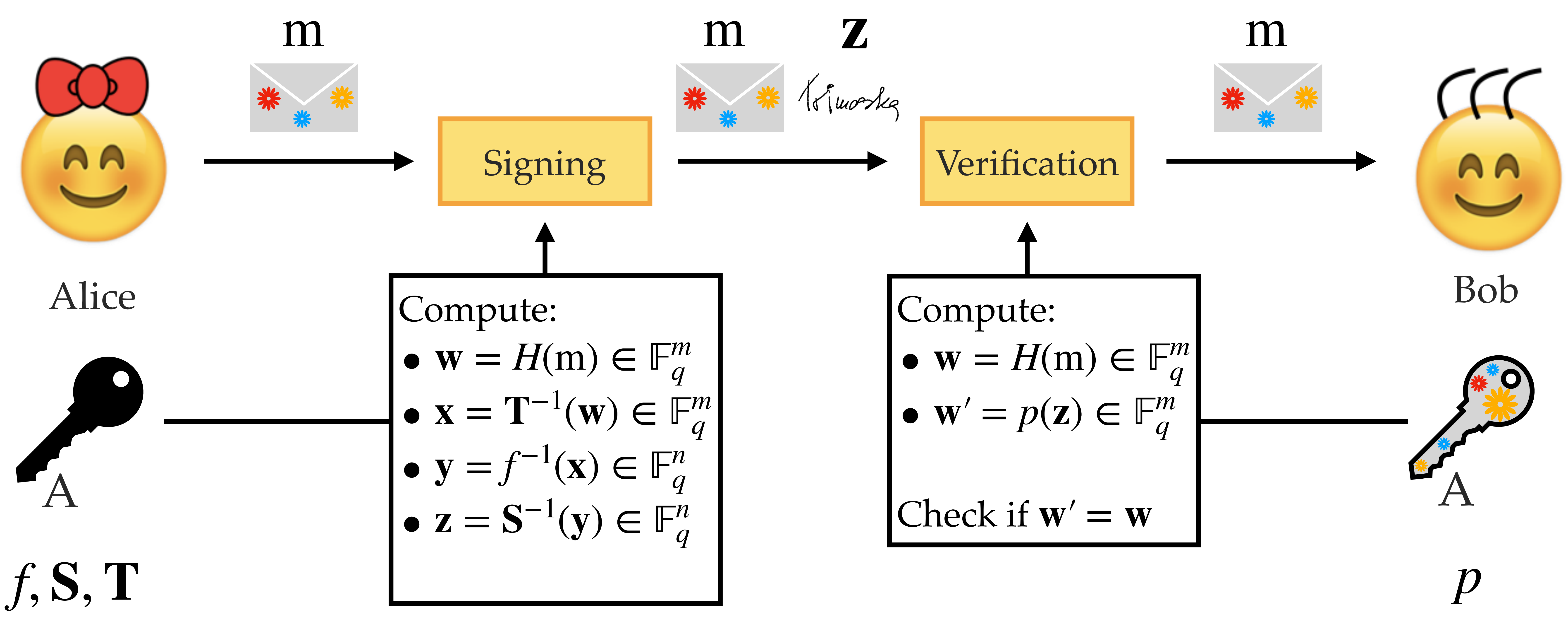

The UOV central map

Unbalanced Oil and Vinegar [Kipnis, Patarin, Goubin, '99]

$$f^{(k)}(x_1, \ldots, x_n) = \sum_{i \in V, j \in V} \gamma_{ij}^{(k)} x_i x_j + \sum_{i \in V, j \in O} \gamma_{ij}^{(k)} x_i x_j + \sum_{i=1}^{n} \beta_i^{(k)} x_i + \alpha^{(k)}$$

Index set of vinegar variables: $V = \{1, \ldots, v\}$

Index set of oil variables: $O = \{v+1, \ldots, n\}$

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The central map is constructed in such a way that enumerating all of the vinegar variables leaves us with a linear system in the oil variables (oil does not mix with oil).

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- The central map is constructed in such a way that enumerating all of the vinegar variables leaves us with a linear system in the oil variables (oil does not mix with oil).
- Everything is as described in the previous slides, except that we do not have a linear transformation on the output: $\mathbf{T} = \mathbf{I}$.

Matrix representation of quadratic forms

Quadratic form: $f(\mathbf{x}) = \sum \gamma_{ij} x_i x_j$

$$\begin{array}{ccc} \mathbf{x}^\top & \mathbf{F} & \mathbf{x} \\ \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} & \begin{bmatrix} \gamma_{1,1} & \frac{\gamma_{1,2}}{2} & \frac{\gamma_{1,3}}{2} & \frac{\gamma_{1,4}}{2} \\ \frac{\gamma_{2,1}}{2} & \gamma_{2,2} & \frac{\gamma_{2,3}}{2} & \frac{\gamma_{2,4}}{2} \\ \frac{\gamma_{3,1}}{2} & \frac{\gamma_{3,2}}{2} & \gamma_{3,3} & \frac{\gamma_{3,4}}{2} \\ \frac{\gamma_{4,1}}{2} & \frac{\gamma_{4,2}}{2} & \frac{\gamma_{4,3}}{2} & \gamma_{4,4} \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \end{array}$$

so with $\mathbf{x} = (x_1, \ldots, x_n)$, we get $\mathbf{x}^\top \mathbf{F} \mathbf{x}$.

Key generation

In matrix representation

$$\mathbf{P}^{(k)} = \mathbf{S}^{\top}\mathbf{F}^{(k)}\mathbf{S}, \text{ for all } k \in \{1, \ldots, m\}.$$

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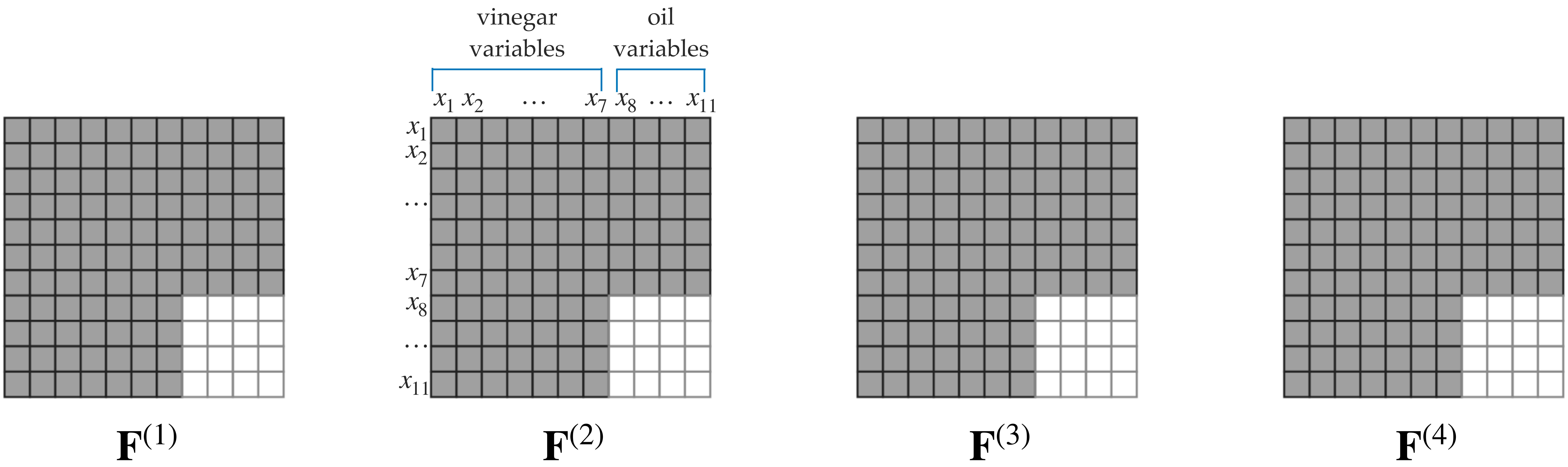
Spoilers ahead !

The UOV central map

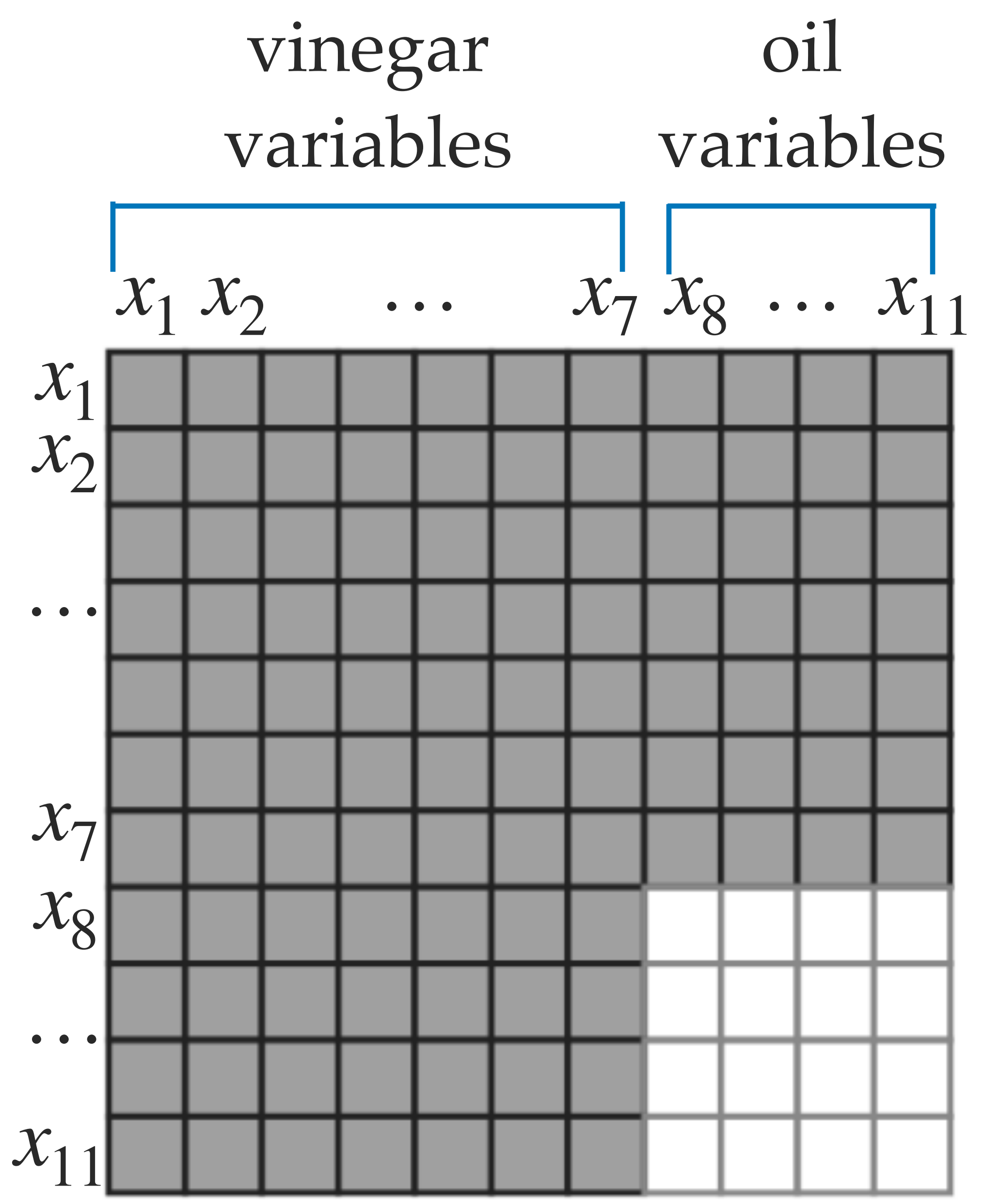
Toy example: $v = 7, m = 4$



*Grayed areas represent the entries that are possibly nonzero; blank areas denote the zero entries;

MQ-Sign

Variants with additional structure to the vinegar-vinegar or/and the vinegar-oil part, with the goal to reduce the size of the secret key.



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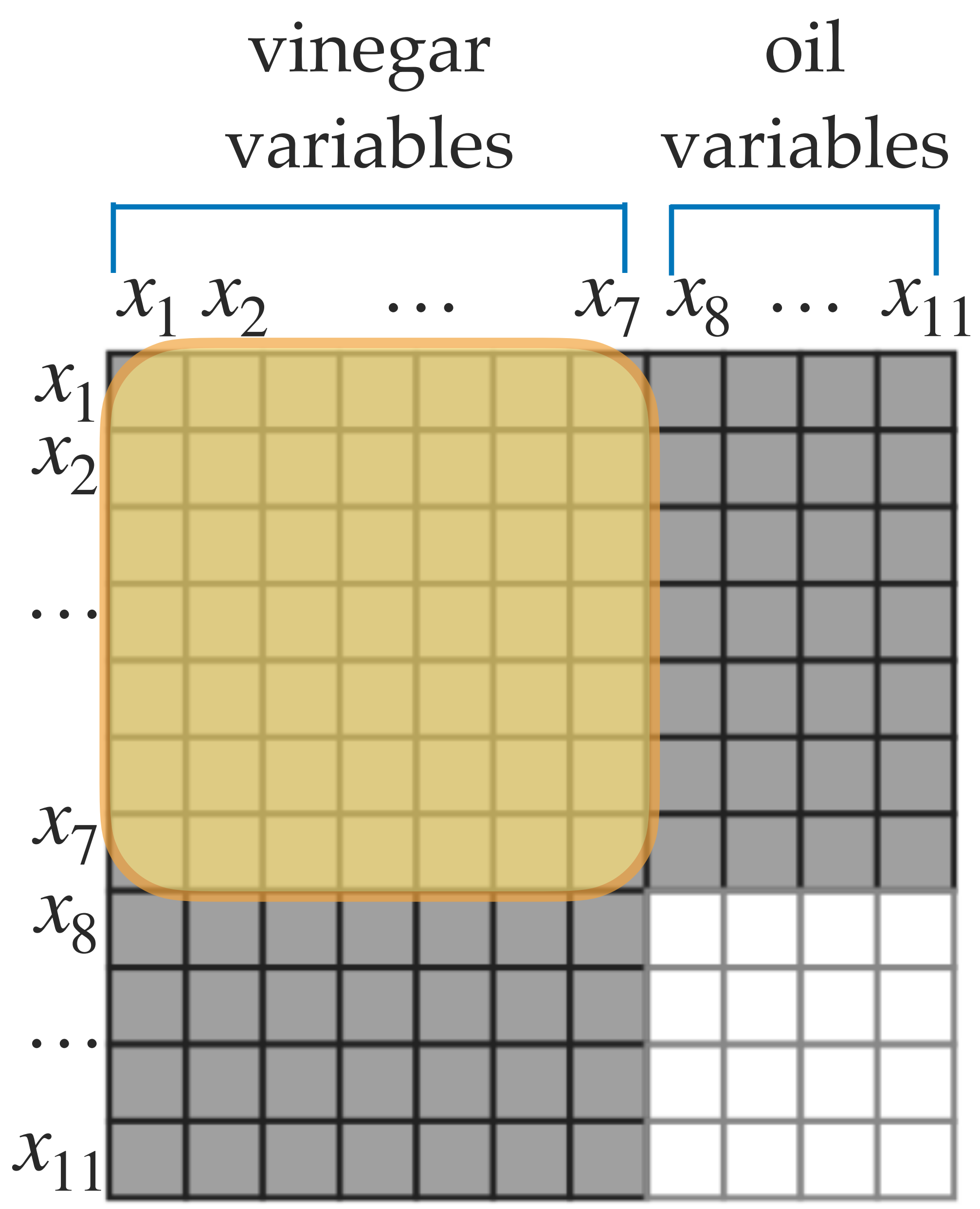

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The vinegar-vinegar part

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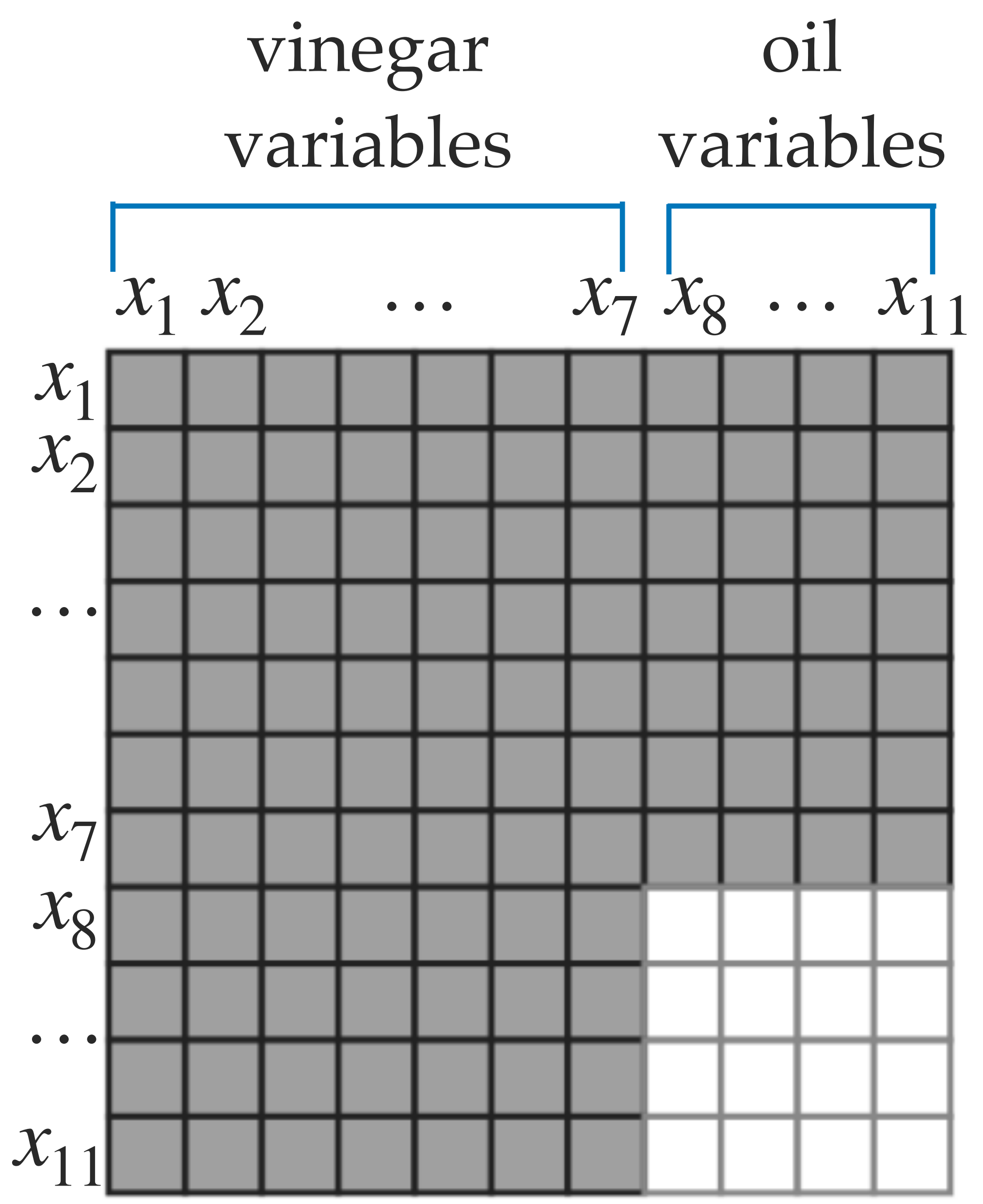

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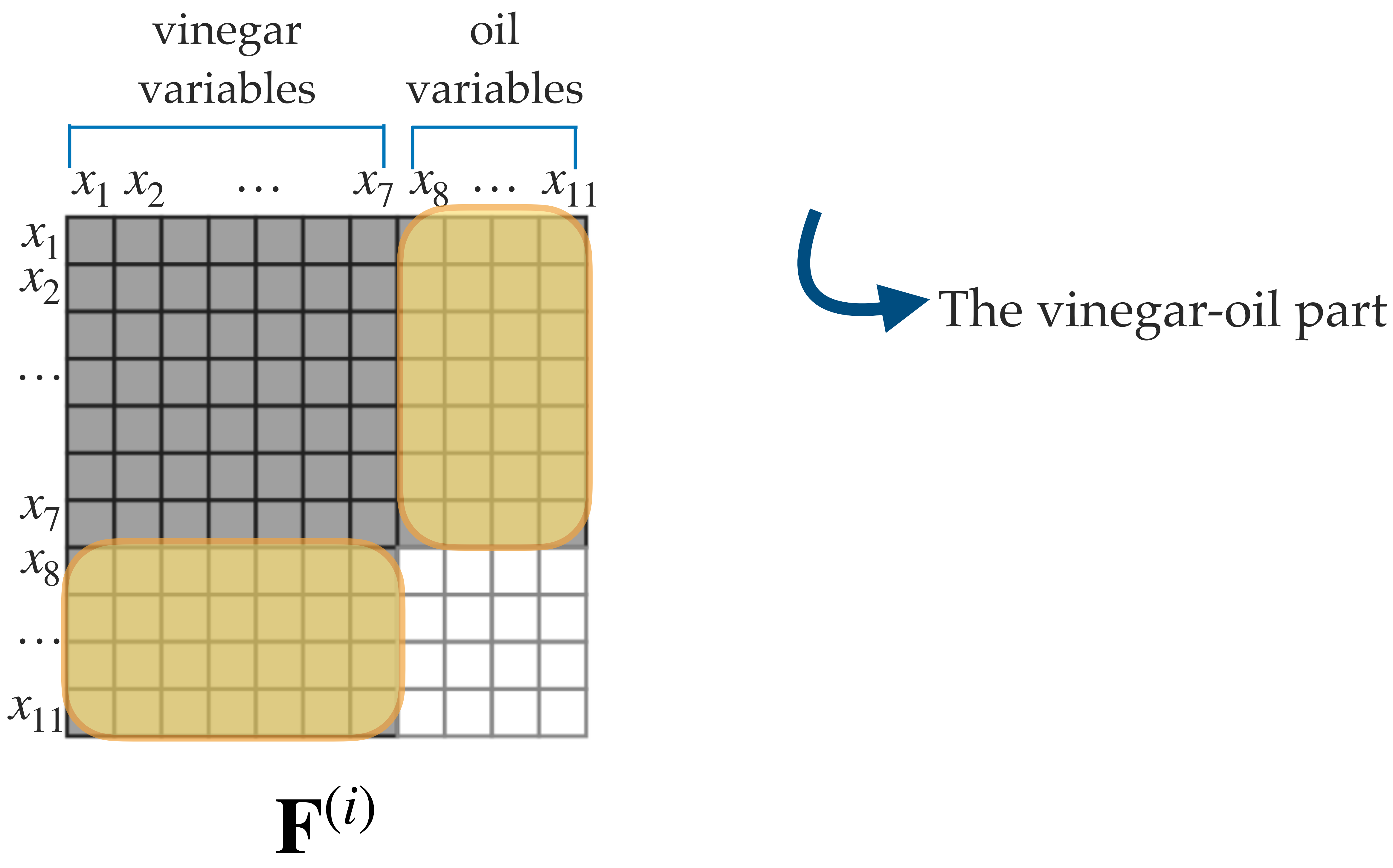

MQ-Sign (round 1)

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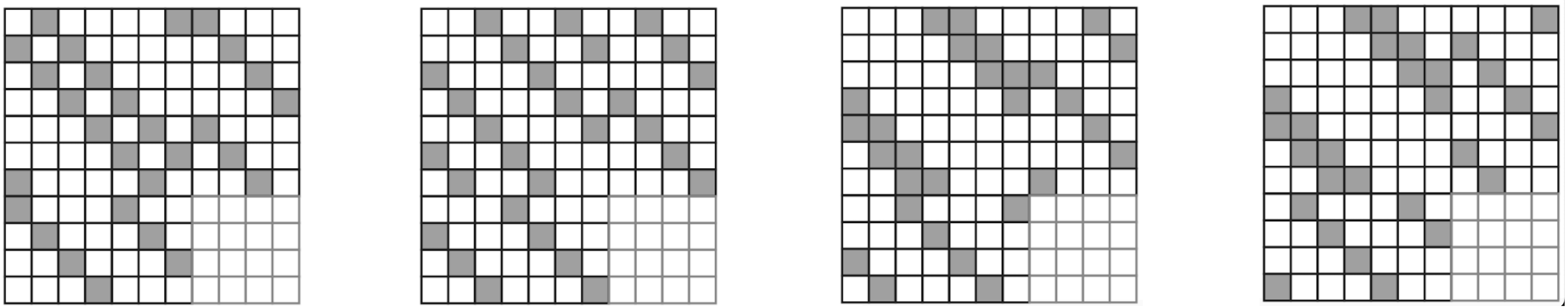

$\mathbf{F}^{(1)}$ $\mathbf{F}^{(2)}$ $\mathbf{F}^{(3)}$ $\mathbf{F}^{(4)}$

Equivalent secret keys

For any instance of a UOV secret key $(f', \mathbf{S}')$, there exists an equivalent secret key $(f, \mathbf{S})$ with

$$\mathbf{S} = \begin{pmatrix} \mathbf{I}_{v\times v} & \mathbf{S}_1 \\ \mathbf{0}_{m\times v} & \mathbf{I}_{m\times m} \end{pmatrix}.$$

- A key of this *equivalent keys* form is used for efficiency (fewer entries in $\mathbf{S}$).

Equivalent secret keys optimisation

Key generation $\mathbf{P} = \mathbf{S}^\top \mathbf{F} \mathbf{S}$

$$\begin{pmatrix} \mathbf{P}_1^{(k)} & \mathbf{P}_2^{(k)} \\ 0 & \mathbf{P}_4^{(k)} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{S}_1^\top & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{F}_1^{(k)} & \mathbf{F}_2^{(k)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{S}_1 \\ 0 & \mathbf{I} \end{pmatrix}$$

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$$\begin{pmatrix} \mathbf{P}_1^{(k)} & \mathbf{P}_2^{(k)} \\ 0 & \mathbf{P}_4^{(k)} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1^{(k)} & (\mathbf{F}_1^{(k)} + \mathbf{F}_1^{(k)\top})\mathbf{S}_1 + \mathbf{F}_2^{(k)} \\ 0 & \mathsf{Upper}(\mathbf{S}_1^\top \mathbf{F}_1^{(k)} \mathbf{S}_1 + \mathbf{S}_1^\top \mathbf{F}_2^{(k)}) \end{pmatrix}$$

Recovering the central map

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We obtain constraints:

$$\mathbf{P}_1^{(k)} = \mathbf{F}_1^{(k)}$$

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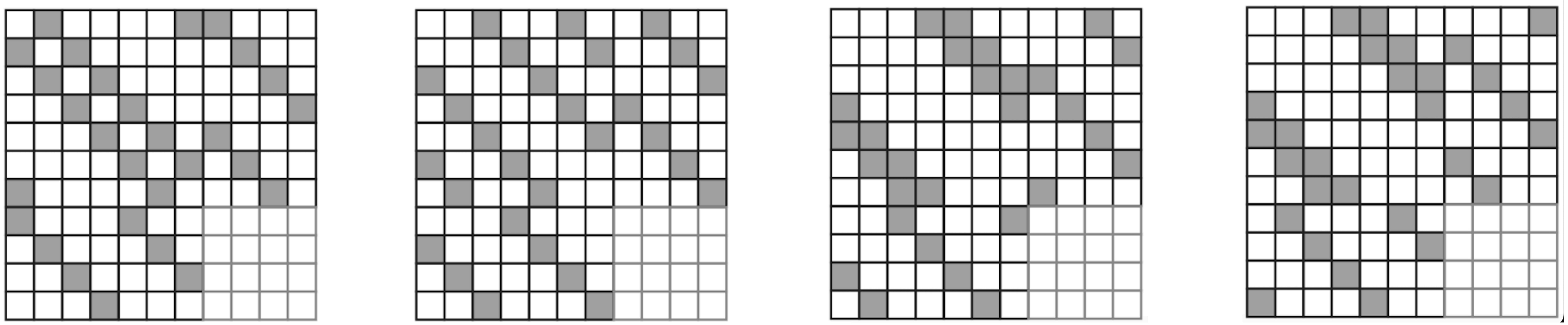
Recall the structure of the central map



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for $mv(m-1)$ entries

huge probability to obtain vm linearly independent equations

Complexity of solving the system column-by-column:

$$\mathcal{O}(mv^\omega)$$

Exercise 3:
Matrix Code Equivalence

Matrix (rank-metric) codes

Matrix code

A matrix code $\mathscr{C}$ over $\mathbb{F}_q$ is a k-dimensional $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^{m\times n}$.

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Basis of a matrix code

The basis of a matrix code $\mathscr{C}$ is given by the k-tuple $(\mathbf{C}^{(1)}, \ldots, \mathbf{C}^{(k)})$.

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Rank metric

For $\mathbf{C} \in \mathbb{F}_q^{m\times n}$, the rank weight of $\mathbf{C}$ is given by the rank of $\mathbf{C}$, aka.

$$\mathrm{wt}(\mathbf{C}) = \mathrm{rk}(\mathbf{C}).$$

Matrix (rank-metric) codes

Example. $q = 13, \quad m = 4, \quad n = 6, \quad k = 5$

$$\mathbf{C} = \lambda_1 \cdot \begin{pmatrix} 2 & 8 & 10 & 4 & 5 & 7 \\ 1 & 11 & 7 & 9 & 6 & 12 \\ 3 & 0 & 13 & 5 & 4 & 8 \\ 9 & 6 & 3 & 2 & 10 & 11 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 12 & 0 & 4 & 11 & 9 & 3 \\ 5 & 6 & 8 & 13 & 2 & 1 \\ 10 & 7 & 3 & 9 & 4 & 6 \\ 2 & 5 & 11 & 8 & 1 & 10 \end{pmatrix} + \lambda_3 \cdot \begin{pmatrix} 5 & 2 & 9 & 11 & 4 & 8 \\ 3 & 7 & 1 & 10 & 12 & 0 \\ 6 & 9 & 2 & 13 & 11 & 8 \\ 1 & 5 & 6 & 3 & 10 & 7 \end{pmatrix} + \lambda_4 \cdot \begin{pmatrix} 9 & 4 & 6 & 1 & 13 & 2 \\ 8 & 0 & 5 & 12 & 6 & 11 \\ 3 & 7 & 10 & 9 & 4 & 5 \\ 2 & 8 & 11 & 3 & 7 & 1 \end{pmatrix} + \lambda_5 \cdot \begin{pmatrix} 7 & 10 & 4 & 6 & 8 & 3 \\ 1 & 5 & 2 & 11 & 9 & 0 \\ 13 & 7 & 6 & 4 & 12 & 2 \\ 8 & 3 & 1 & 9 & 5 & 10 \end{pmatrix} \qquad \lambda_i \in \mathbb{F}_q$$

Matrix code equivalence

Isometry

An isometry (for our purposes) between two codes $\mathscr{C}$ and $\mathscr{D}$ is a linear map $\mu : \mathscr{C} \to \mathscr{D}$ that preserves the metric.

Matrix code equivalence

Isometry

An isometry (for our purposes) between two codes $\mathscr{C}$ and $\mathscr{D}$ is a linear map $\mu : \mathscr{C} \to \mathscr{D}$ that preserves the metric.

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- Multiply a codeword on the left by $\mathbf{A} \in \mathrm{GL}_m$

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- Multiply a codeword on the left by $\mathbf{A} \in \mathrm{GL}_m$ ✓
- Take the transposition of a codeword (only when $m = n$, does not make the equivalence problem harder) ✓

Matrix code equivalence

The Matrix Code Equivalence (MCE) problem

Input: Two k-dimensional matrix codes $\mathscr{C}, \mathscr{D} \subset \mathbb{F}_q^{m\times n}$ for two matrix codes $\mathscr{C}$ and $\mathscr{D}$.
Question: Find - if any - a map $(\mathbf{A}, \mathbf{B})$, where $\mathbf{A} \in \mathrm{GL}_m(\mathbb{F}_q)$ and $\mathbf{B} \in \mathrm{GL}_n(\mathbb{F}_q)$ such that for all $\mathbf{C} \in \mathscr{C}$, it holds that $\mathbf{ACB} \in \mathscr{D}$.

Matrix code equivalence

The MCE problem in matrix form

Let $(\mathbf{C}^{(1)}, \ldots, \mathbf{C}^{(k)})$ be a basis of code $\mathscr{C}$ and let $(\mathbf{D}^{(1)}, \ldots, \mathbf{D}^{(k)})$ be a basis of code $\mathscr{D}$. Find $\mathbf{A} \in \mathrm{GL}_m(\mathbb{F}_q)$, $\mathbf{B} \in \mathrm{GL}_n(\mathbb{F}_q)$ and $\mathbf{T} \in \mathrm{GL}_k(\mathbb{F}_q)$ such that

$$\mathbf{D}^{(i)} = \sum_{1 \le j \le k} t_{j,i} \mathbf{A} \mathbf{C}^{(j)} \mathbf{B}, \quad \forall 1 \le i \le k$$

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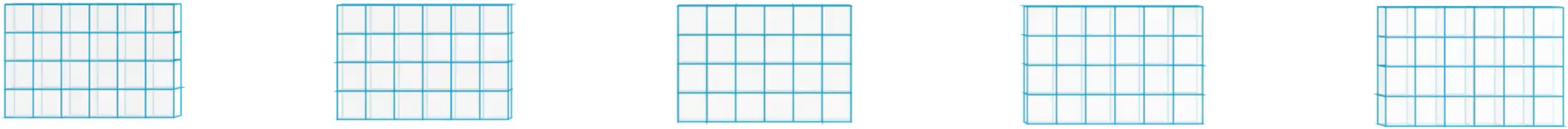
change of basis

From matrix codes to 3-tensors

We can think of a matrix code as a 3-tensor over $\mathbb{F}_q$.



$\mathbf{C}_1$ $\mathbf{C}_2$ $\mathbf{C}_3$ $\mathbf{C}_4$ $\mathbf{C}_5$

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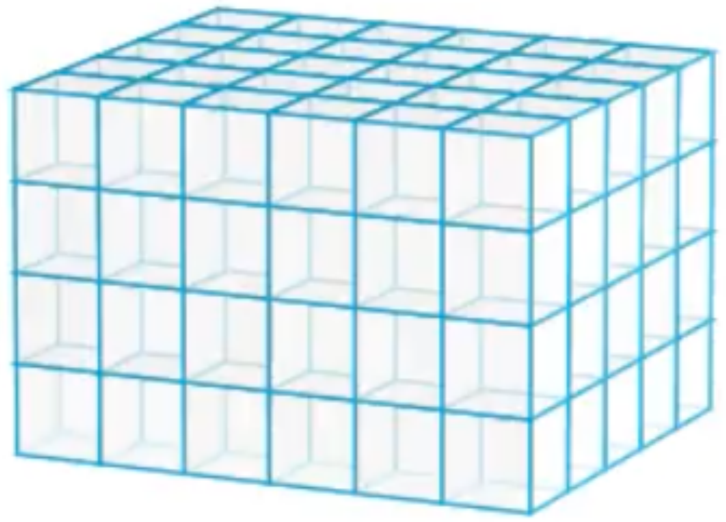
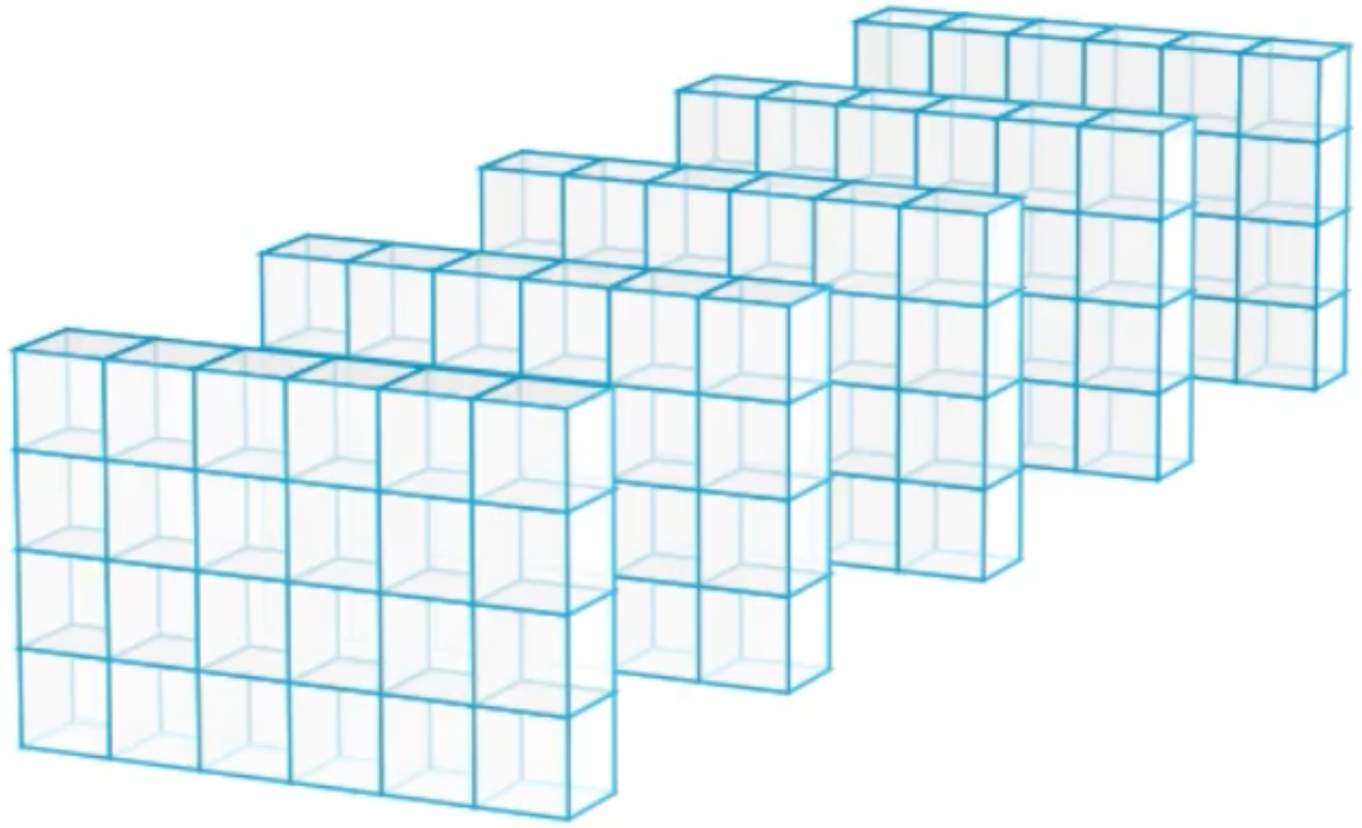
$$\mathcal{C} \subseteq \mathbb{F}_q^{m \times n \times k}$$



From matrix codes to 3-tensors

Viewed as a 3-tensor, we can see $\mathscr{C}$ from three directions

- a k-dimensional code in $\mathbb{F}_q^{m\times n}$
- an m-dimensional code in $\mathbb{F}_q^{n\times k}$
- an n-dimensional code in $\mathbb{F}_q^{m\times k}$



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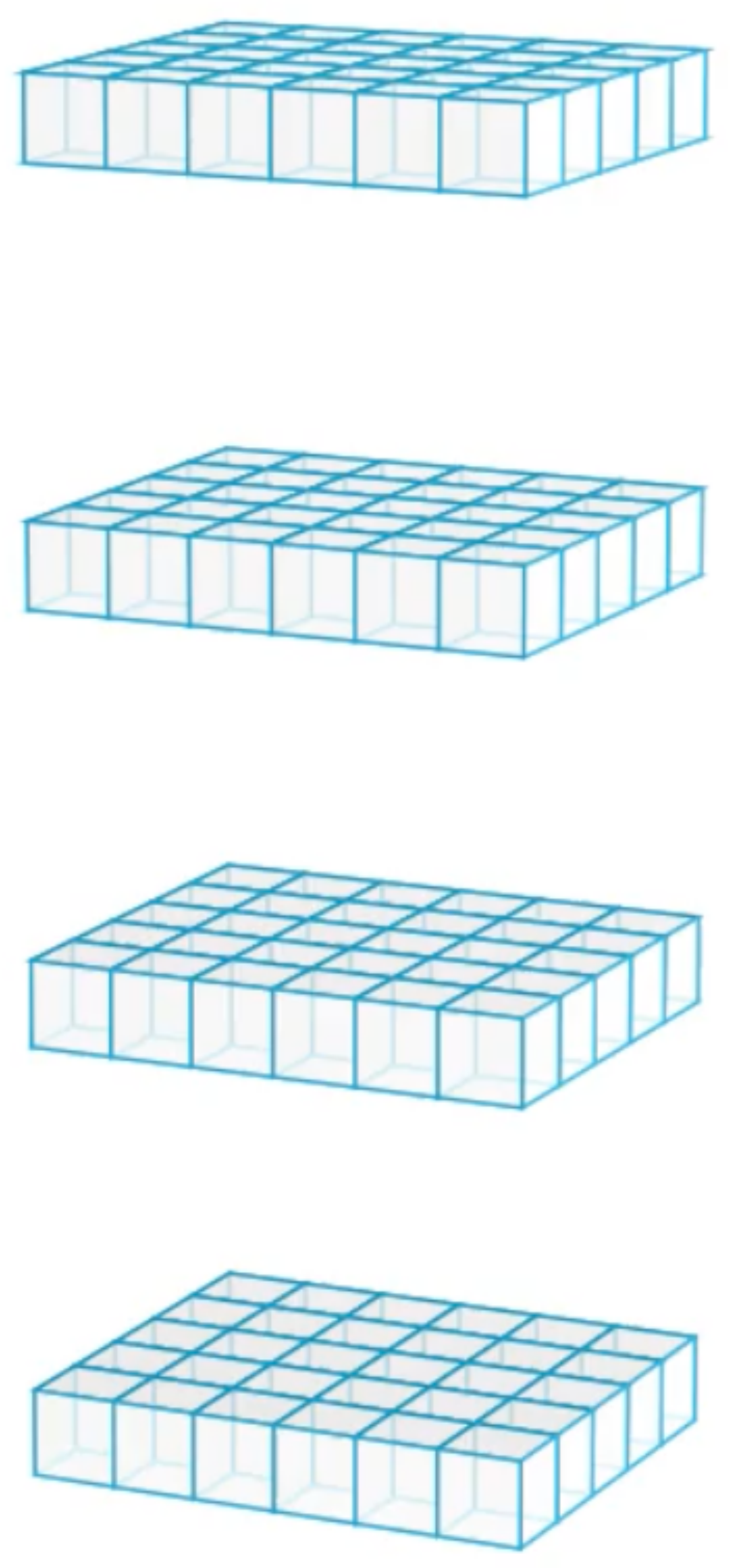

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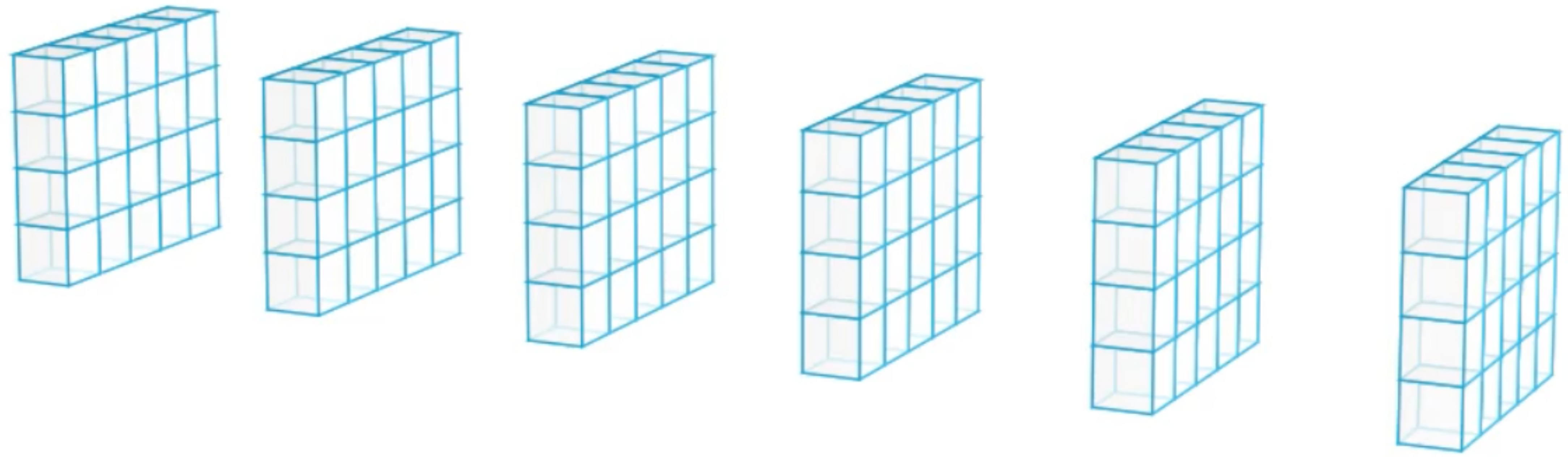

Spoilers ahead !

Direct algebraic attack

The MCE problem in matrix form

Let $(\mathbf{C}^{(1)}, \ldots, \mathbf{C}^{(k)})$ be a basis of code $\mathscr{C}$ and let $(\mathbf{D}^{(1)}, \ldots, \mathbf{D}^{(k)})$ be a basis of code $\mathscr{D}$. Find $\mathbf{A} \in \mathrm{GL}_m(\mathbb{F}_q)$, $\mathbf{B} \in \mathrm{GL}_n(\mathbb{F}_q)$ and $\mathbf{T} \in \mathrm{GL}_k(\mathbb{F}_q)$ such that

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$$\mathbf{D}^{(i)} = \sum_{1 \leq j \leq k} t_{j,i} \mathbf{A}\mathbf{C}^{(j)}\mathbf{B}, \quad \forall 1 \leq i \leq k$$

Alternatively, this gives a better modelling:

$$\sum_{1 \leq j \leq k} t_{j,i} \mathbf{D}^{(j)} = \mathbf{A}\mathbf{C}^{(i)}\mathbf{B}, \quad \forall 1 \leq i \leq k$$

Exercise 4: Matrix Code Equivalence with hints

Collision

We have a collision when we know a codeword $\mathbf{C}$ in $\mathscr{C}$ that maps to a codeword $\mathbf{D}$ in $\mathscr{D}$.

$$\mathbf{D} = \mathbf{ACB}$$

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$$\mathbf{D} = \mathbf{A}\mathbf{C}\mathbf{B}$$

Recall how we can represent codewords with their coordinate vectors

$$\mathbf{C} = \lambda_1 \cdot \begin{pmatrix} 2 & 8 & 10 & 4 & 5 & 7 \\ 1 & 11 & 7 & 9 & 6 & 12 \\ 3 & 0 & 13 & 5 & 4 & 8 \\ 9 & 6 & 3 & 2 & 10 & 11 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 12 & 0 & 4 & 11 & 9 & 3 \\ 5 & 6 & 8 & 13 & 2 & 1 \\ 10 & 7 & 3 & 9 & 4 & 6 \\ 2 & 5 & 11 & 8 & 1 & 10 \end{pmatrix} + \lambda_3 \cdot \begin{pmatrix} 5 & 2 & 9 & 11 & 4 & 8 \\ 3 & 7 & 1 & 10 & 12 & 0 \\ 6 & 9 & 2 & 13 & 11 & 8 \\ 1 & 5 & 6 & 3 & 10 & 7 \end{pmatrix} + \lambda_4 \cdot \begin{pmatrix} 9 & 4 & 6 & 1 & 13 & 2 \\ 8 & 0 & 5 & 12 & 6 & 11 \\ 3 & 7 & 10 & 9 & 4 & 5 \\ 2 & 8 & 11 & 3 & 7 & 1 \end{pmatrix} + \lambda_5 \cdot \begin{pmatrix} 7 & 10 & 4 & 6 & 8 & 3 \\ 1 & 5 & 2 & 11 & 9 & 0 \\ 13 & 7 & 6 & 4 & 12 & 2 \\ 8 & 3 & 1 & 9 & 5 & 10 \end{pmatrix} \quad \lambda_i \in \mathbb{F}_q$$

$$(q = 13, \quad m = 4, \quad n = 6, \quad k = 5)$$

Spoilers ahead !

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We have a collision when we know a codeword $\mathbf{C}$ in $\mathscr{C}$ that maps to a codeword $\mathbf{D}$ in $\mathscr{D}$.

$$\mathbf{D} = \mathbf{A}\mathbf{C}\mathbf{B}$$

We can then infer linear constraints from

$$\mathbf{A}^{-1}\mathbf{D} = \mathbf{C}\mathbf{B}$$