SUFFICIENTLY ABUNDANT NUMBERS ARE PSEUDOPERFECT

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Abstract. There exists an absolute constant C such that if n is any positive integer with $\sigma(n) \geq Cn$, then n is a sum of distinct proper divisors. This answers a question of Benkoski and Erdős.

Following Sierpiński [7], we say a number n is *pseudoperfect* if it can be written as a sum of distinct proper divisors of n. We call $\sigma(n)/n$, where $\sigma(n)$ is the sum of the divisors of n, the *abundance index* of n. Benkoski and Erdős conjectured [1] that if the abundance index of n is sufficiently large, then n must be pseudoperfect. In this paper, we prove this conjecture.

Following a general strategy suggested by Tao and Bloom [3], as exemplified in work of Croot [5], Bloom [2], and Conlon et al. [4], we use a probabilistic method to show that a suitably weighted random selection process of proper divisors of n produces n as a sum with positive probability. The main technical difficulty stems from the possibility that the prime factors of n may accumulate on very different scales. We use a greedy-but-not-too-greedy algorithm to approach n from below closely enough that the divisors on one scale are suitable for "terminal guidance."

Following Friedman [6], we find it convenient to recast the problem in terms of Egyptian fractions, representing 1 as a sum of reciprocals of divisors of n. Schematically, our approach can be decomposed into stages as follows:

1. We divide prime factors of n into dyadic intervals, discarding intervals whose prime density is too low.
2. We block the dyadic intervals into ranges of intervals such there are no very large gaps within any block.
3. We use a greedy algorithm to identify the first block on which the sum of reciprocals reaches 1 with sufficient slack.
4. We then pull back, if necessary, so that the next block of reciprocals add up to (ideally) twice the distance from the current sum to 1.
5. Finally, we use a circle method argument to show that the next block of reciprocals can precisely fill the gap.

The letter p will always refer to a prime. We say n is N*-smooth* if all its prime factors are $\leq N$ and N*-rough* if all its prime factors are $\geq N$. We say something is true for almost all elements of a set S if the number of exceptions has size $o(|S|)$. Finally, we write $A \cdot B = \{ab : a \in A, b \in B\}$.

We use standard asymptotic notation with the additions $x \sim y$, which means $x \in [y, 2y)$, and the equivalent statements $x \lesssim_\alpha y$ and $y \gtrsim_\alpha x$, which, for a parameter $\alpha > 1$ that will be specified prior to this usage, mean there exists an absolute constant C such that $x < \alpha^C \cdot y$.

We now state the main theorem.

Theorem 1. *For every $\varepsilon > 0$ there exists an integer L such that if n is an integer with*

$$\sigma(n)/n > 2 + \varepsilon$$

and it has no prime factors less than L, then n is pseudoperfect.

Because any positive integer multiple of a pseudoperfect number is pseudoperfect, we may assume that $\sigma(n)/n = O(1)$. We may also assume that n is squarefree since if $\sigma(n)/n > 2+\varepsilon$, the squarefree part of n also has abundance index bounded away from 2 if

$$\sum_{p\geq L}\frac{1}{p^2}=o(\varepsilon).$$

(We will assume that L is sufficiently large in terms of ε that all such claims are true.) We will also assume ε is small, specifically that $\varepsilon<1$ and that

$$\left(1+\frac{\varepsilon}{8}\right)\left(1-\frac{\varepsilon}{10}\left(1+\frac{\varepsilon}{8}\right)\right)>1+\frac{\varepsilon}{100}. \tag{1}$$

Define

$$w(A)=\sum_{1<a\in A}\frac{1}{a},$$

$\mathrm{Div}(N)$ to be the set of divisors of N, and $\mathrm{Div}^*(N)$ the set of divisors greater than 1. For N squarefree and L-rough,

$$\log(1+w(\mathrm{Div}^*(N)))=\sum_{p|N}\log\frac{p+1}{p}=\sum_{p|N}\frac{1}{p}+O\Big(\frac{1}{L}\Big). \tag{2}$$

Let $x_1,\dots,x_K$ be a maximal subsequence of the powers of 2 such that $x_{i+1}\geq 2x_i$, and

$$|P_i|>x_i/(\log x_i)^2$$

where

$$P_i=\{p\sim x_i : p\mid n\}.$$

Note that this sequence is non-empty since

$$\sum_{\substack{p|n\\ \nexists\, i:p\sim x_i}}\frac{1}{p}\ll\sum_{j\geq\lfloor\log L\rfloor}\frac{1}{j^2}\ll\frac{1}{\log L}, \tag{3}$$

which implies by (2) that

$$\sum_{\substack{1<d|n\\ \exists p\notin P:p|d}}\frac{1}{d}\leq\frac{\sigma(n)}{n}\sum_{\substack{1<d|n\\ \forall p|d,\,p\notin P}}\frac{1}{d}\ll\frac{1}{\log L}=o(\varepsilon), \tag{4}$$

setting P to be the union of the P_i.

Let $N_i=\prod_{p\in P_i}p$ and $J_i=[x_i^8,e^{x_i/\log^2 x_i}]$. Define an equivalence relation on $\{1,\dots,K\}$ taking the transitive closure of the relation which identifies i and j when $J_i\cap J_j\neq\emptyset$. Note that when $J_i\cap J_j=\emptyset$ with $j>i$, we have $x_i\ll\log^{1+\varepsilon}x_j$. There exist $a_1,\dots,a_r$ with $a_1=1$ and $a_r=K+1$ giving equivalence classes $S_j=[a_j,a_{j+1})$.

Let $M_j=\prod_{i\in S_j}N_i$ and

$$M_{j,\leq i^*}=\prod_{\substack{i\in S_j\\ i\leq i^*}}N_i.$$

Let $B_j=\mathrm{Div}(M_{\leq j})$ where $M_{\leq j}=\prod_{j'\leq j}M_{j'}$. Note that the sets $B_{j-1}\cdot\mathrm{Div}^*(M_j)$ for $1\leq j<r$ partition $\mathrm{Div}^*(M_{r-1})$. By (4),

$$w(\mathrm{Div}^*(M_{\leq r-1}))>1+\frac{\varepsilon}{2}. \tag{5}$$

Different values of j correspond to exponentially separated scales on which n has many prime factors. We want to apply the circle method to the divisors of n, but these vastly different scales will not combine well. Therefore we begin with a greedy process.

We initialize $A_0 = \{1\}$. Beginning with $j = 1$ we form A_j according to the following rules. If

$$w(B_{j-1} \cdot \mathrm{Div}^*(M_j)) \geq \left(1 + \frac{\varepsilon}{8}\right)(1 - w(A_{j-1}))$$

(which we think of as meaning the scales up to j not only reach 1 but do so decisively) we set $A = A_{j-1}$, $B = B_{j-1}$, and $j_0 = j$ and stop. On the other hand, if

$$w(B_{j-1} \cdot \mathrm{Div}^*(M_j)) < \left(1 + \frac{\varepsilon}{8}\right)(1 - w(A_{j-1})) \tag{6}$$

then we start by partitioning the elements of $B_{j-1} \cdot \mathrm{Div}^*(M_j)$ into dyadic intervals and letting E_j be obtained by choosing one element from each dyadic interval for which this is possible. We then set $A_j = A_{j-1} \cup C_j$, where C_j is a maximal subset of $B_{j-1} \cdot \mathrm{Div}^*(M_j)$ such that $w(C_j) < 1 - w(A_{j-1})$. If $w(E_j) < 1 - w(A_{j-1})$, we choose C_j so that it contains E_j. In fact, this is always possible since

$$w(E_j) \ll \frac{1}{x_{a_j}} = o(w(B_{j-1} \cdot \mathrm{Div}^*(M_j))) \ll 1 - w(A_{j-1}),$$

using (6). Note that if

$$w(A_{j-1}) + w(B_{j-1} \cdot \mathrm{Div}^*(M_j)) < 1,$$

then we take C_j to be all of $B_{j-1} \cdot \mathrm{Div}^*(M_j)$.

We claim that this process must terminate. The smallest elements of $B_{j-1} \cdot \mathrm{Div}^*(M_j)$ are all primes between x_{a_j} and $2x_{a_j}$ which divide n. Therefore, no summand in

$$\sum_{a \in B_{j-1} \cdot \mathrm{Div}^*(M_j)} \frac{1}{a}$$

is greater than δ times the sum for any absolute constant $\delta > 0$ when L is sufficiently large. When $C_j \neq B_{j-1} \cdot \mathrm{Div}^*(M_j)$, it follows that

$$w(C_j) = (1 - o(1))(1 - w(A_{j-1})), \tag{7}$$

so

$$1 - w(A_j) = o(1 - w(A_{j-1})) \tag{8}$$

and

$$w((B_{j-1} \cdot \mathrm{Div}^*(M_j)) \setminus C_j) \leq (1 + \frac{\varepsilon}{8} - 1 + o(1))(1 - w(A_{j-1})) \leq \frac{\varepsilon}{8} + o(1), \tag{9}$$

subtracting (7) from (6). Thus, if the process does not terminate, then $\sum_j w(C_j) < 1$ so remembering (5),

$$\sum_j w((B_{j-1} \cdot \mathrm{Div}^*(M_j)) \setminus C_j) = w(\mathrm{Div}^*(M_{\leq r-1})) - \sum_j w(C_j) \geq \frac{\varepsilon}{2}. \tag{10}$$

Label the elements in $\{j : w(A_{j-1}) + w(B_{j-1} \cdot \mathrm{Div}^*(M_j)) \geq 1\}$ as $j_1 < j_2 < \cdots$. Then

$$\frac{\varepsilon}{2} \leq \sum_k w((B_{j_k-1} \cdot \mathrm{Div}^*(M_{j_k})) \setminus C_{j_k}) < 2w((B_{j_1-1} \cdot \mathrm{Div}^*(M_{j_1})) \setminus C_{j_1}) \leq \frac{\varepsilon}{4} + o(1)$$

by (10), (8), and (9) respectively. This gives the desired contradiction.

Thus, we may assume that

$$w(B \cdot \mathrm{Div}^*(M_{j_0})) \geq (1 + \varepsilon/8)(1 - w(A)).$$

We would like to use the circle method to find a subset of the summands of the left hand side which sum to $1 - w(A)$. The difficulty is that $1 - w(A)$ may be very small at this point. We happily embraced this possibility before so that we could avoid the situation in which

$$w(A) + w(B \cdot \mathrm{Div}^*(M_{j_0})) - 1$$

was very small. It is time to take a step backward from the greedy method and remove some elements of A.

Let i_0 be the smallest index such that

$$w(D) \geq \Big(1+\frac{\varepsilon}{8}\Big)(1-w(A)). \tag{11}$$

Let $D = B \cdot \mathrm{Div}^*(M_{j_0,\leq i_0})$. Note that $\sum_{p|N_{i_0}} 1/p = o(1)$. It follows from this and the fact that

$$w\Big(B \cdot \mathrm{Div}^*(M_{j_0,\leq i_0-1})\Big) < \Big(1+\frac{\varepsilon}{8}\Big)(1-w(A))$$

that

$$w(D) < 2 - w(A). \tag{12}$$

Lemma 2. *There exists a subset D_0 of A such that setting*

$$\alpha = \frac{w(D)}{1-w(D_0)},$$

we have

$$1 + \frac{\varepsilon}{100} \leq \alpha \leq \log^{1+\varepsilon} x_{a_{j_0}}.$$

Proof. If $w(D) \geq 1+\varepsilon/8$, it is clear by (12) that we can take $D_0 = A$, and this will result in $\alpha < \frac{2}{\varepsilon/8} = 16/\varepsilon$. In the other direction, by (11), $\alpha \geq 1 + \frac{\varepsilon}{8}$. This takes care of the $j_0 = 1$ case.

Now assume $j_0 > 1$. If $w(D) \ll 1 - w(A)$, then we can take $D_0 = A$ and $1 + \frac{\varepsilon}{8} \leq \alpha \ll 1$. Otherwise,

$$1/w(D) = o(1/(1-w(A))) \leq M_{\leq j_0 - 1}.$$

Since every prime factor of this product has size at most $2x_{a_{j_0}-1}$, there exists some $d \in B$ such that $1/w(D)$ is between $\varepsilon d/(40 x_{a_{j_0}-1})$ and $\varepsilon d/20$. At the cost of supposing only that $1/w(D)$ is between $\varepsilon d/(80 x_{a_{j_0}-1})$ and $\varepsilon d/10$, we may assume that

$$d \in \bigcup_{j<j_0} E_j \subset A.$$

Remove d from A to form D_0. Then

$$1 - w(D_0) = 1 - w(A) + \frac{1}{d} > \frac{1}{d} \gg \frac{w(D)}{x_{a_{j_0}-1}}.$$

Rearranging, $\alpha \ll x_{a_{j_0}-1} = o(\log^{1+\varepsilon} x_{a_{j_0}})$ as claimed. At the same time, we still have

$$\alpha \geq \frac{w(D)}{1-w(A)+\varepsilon w(D)/10}.$$

Note that if $a/b \geq 1 + \varepsilon/8$, then

$$\frac{a}{b+\varepsilon a/10} = \frac{a(b-\varepsilon a/10)}{b^2 - \varepsilon^2 a^2/100} > \frac{a}{b}\big(1 - \frac{\varepsilon a}{10b}\big) \geq 1 + \frac{\varepsilon}{100}$$

by (1), unless $a/b \geq 5/\varepsilon$, in which case $a/(b+\varepsilon a/10) \geq 10/3$. Plugging in $a = w(D)$ and $b = 1 - w(A)$, we conclude that $\alpha \geq 1 + \frac{\varepsilon}{100}$.

□

This concludes the preliminary stage of the argument. The advantage of approaching 1 within a specific block rather than proceeding one dyadic interval at a time is that we may reserve judgment on which divisors from previous dyadic intervals should be used and simply throw everything within the block into the circle method. We now prove a general theorem which will give what we need to finish the proof of Theorem 1. To begin with we define the following condition:

Hypothesis 3 ($\mathcal{D} \subset \mathbb{N}$, $y \in \mathbb{N}$, $0 < \beta \leq 1$)**.** *For every integer $h \in [y/2, y^{\lceil 10/\beta \rceil - 2}]$,*

$$\sum_{d \in \mathcal{D}} h_d^2/d^2 > y^{1/4},$$

where h_d is the distance between h and $d\mathbb{Z}$,

This is the condition needed for the medium arcs in the following circle method argument.

Theorem 4. *Let $\ell/k \in (0,1]$, $\epsilon > 0$ and $\beta > 0$ be constants, and $\mathcal{L}$ be an integer sufficiently large in terms of ϵ and β. Let $y_1, \dots, y_s$ be a sequence of positive integers greater than $\mathcal{L}$ such that for every $i < s$, $y_i \leq y_{i+1}/2$ and*

$$\left[y_i^{\lceil 4/\beta \rceil - 5/4}, e^{y_i/\log^2 y_i}\right] \cap \left[y_{i+1}^{\lceil 4/\beta \rceil - 5/4}, e^{y_{i+1}/\log^2 y_{i+1}}\right] \neq \emptyset.$$

Let $\mathcal{Q}_i$ be a set of integers between y_i and $2y_i$ such that the elements of $\mathcal{Q} := \bigcup_i \mathcal{Q}_i$ are pairwise coprime. Let $\mathcal{B}$ be a set including 1 of integers dividing $(y_1^2)!$ which are coprime to every element of $\mathcal{Q}$. Let $E \subset \mathcal{B}$ be a subset which contains at the very least all elements of $\mathcal{B}$ less than y_1^2. Let

$$\mathcal{D} = \Big(\mathcal{B} \cdot \operatorname{Div}(\prod_{q \in \mathcal{Q}} q)\Big) \setminus E.$$

Let m be the least common multiple of $\mathcal{D}$. Assume that $k \mid m$. Let $\alpha = w(\mathcal{D})(\ell/k)^{-1}$. If

$$1 + \frac{\epsilon}{100} \leq \alpha \leq \log^{1+\epsilon} y_1,$$

$$\frac{y_i^\beta}{\log^2 y_i} \ll |\mathcal{Q}_i| \leq y_i^\beta$$

for all i, and Hypothesis 3 holds for $y = y_1$, then there exist at least two subsets $\mathcal{D}'$ of $\mathcal{D}$ such that $w(\mathcal{D}') \equiv \ell/k \pmod 1$.

Proof. Let $F(\mathcal{D})$ be the number of solutions to

$$\sum_{d \in \mathcal{D}'} \frac{1}{d} \equiv \frac{\ell}{k} \pmod 1, \tag{13}$$

where $\mathcal{D}' \subset \mathcal{D}$, and let $F_\alpha(\mathcal{D})$ be the weighted number of solutions giving $\mathcal{D}'$ satisfying (13) weight $(\alpha - 1)^{|\mathcal{D}| - |\mathcal{D}'|}$.

Setting $e(x) = e^{2\pi i x}$, by orthogonality of characters on $\mathbb{Z}/m\mathbb{Z}$,

$$\begin{aligned}
F_\alpha(\mathcal{D}) &= \sum_{\mathcal{D}' \subset \mathcal{D}} \mathbb{1}_{w(\mathcal{D}') \equiv \ell/k \bmod 1} (\alpha - 1)^{|\mathcal{D}| - |\mathcal{D}'|} \\
&= \sum_{\mathcal{D}' \subset \mathcal{D}} \frac{1}{m} \sum_{-m/2 < h \leq m/2} e(h(w(\mathcal{D}') - \ell/k))(\alpha - 1)^{|\mathcal{D}| - |\mathcal{D}'|} \\
&= \frac{1}{m} \sum_{-m/2 < h \leq m/2} e(-h\ell/k) \prod_{d \in \mathcal{D}} \Big(\alpha - 1 + e(h/d)\Big).
\end{aligned}$$

Moreover,

$$F(\mathcal{D}) \geq \frac{F_\alpha(\mathcal{D})}{\max(1, \alpha - 1)^{|\mathcal{D}|}}.$$

We write

$$F_\alpha(\mathcal{D}) = \frac{\alpha^{|\mathcal{D}|}}{m} \sum_{-m/2 < h \leq m/2} e\big(h(w(\mathcal{D})/\alpha - \ell/k)\big) f(h) = \frac{\alpha^{|\mathcal{D}|}}{m} \sum_{-m/2 < h \leq m/2} f(h)$$

where

$$f(h)=\prod_{d\in\mathcal{D}}\left(\frac{(\alpha-1)e(-h/d\alpha)+e\left(\frac{(\alpha-1)h}{d\alpha}\right)}{\alpha}\right).$$

It is clear that $h=0$ contributes $\alpha^{|\mathcal{D}|}/m$ to $F_\alpha(\mathcal{D})$. We now need to show the other components of the sum do not have much of an effect.

Let $t=\lceil 10/\beta\rceil$. Fix $i\le s$. Let $h_d\in(-d/2,d/2]$ denote the residue of h mod d. Suppose it is not true that $|h_d|\le y_i^{t-2}$ for almost all d which can be expressed as a product of t elements of $\mathcal{Q}_i$. Then

$$|h_d|/d\gg y_i^{-2}$$

for $\gg y_i^{10}/\log^{2t}y_i$ values of $d\in\mathcal{D}$. Note that when $d/|h_d|$ is greater than some fixed power of α,

$$1-\left|\frac{(\alpha-1)e(-h/d\alpha)+e\big((\alpha-1)h/d\alpha\big)}{\alpha}\right|\gtrsim_\alpha\frac{h_d^2}{d^2}.$$

Thus for such h, $|f(h)|<\exp(-y_i^5)$ for large y_i. We will use this method of bounding $f(h)$ by bounding $|h_d|/d$ repeatedly.

If $|h_d|\le y_i^{t-2}$ for almost all considered d, there exist $q_1,\dots,q_{t-1}\in\mathcal{Q}_i$ such that for almost all $q_t\in\mathcal{Q}_i$, setting $d=q_1\cdots q_t$, $|h_d|\le y_i^{t-2}$, i.e. h is within y_i^{t-2} of a multiple of $q_1\cdots q_t$, which we call M. We see that M must be divisible by almost all elements of $\mathcal{Q}_i$, since $q_1\cdots q_{t-1}>y_i^{t-2}$.

Fix $q_t\in\mathcal{Q}\cap(1,2y_i)$. Either $q_t\mid M$ or for almost all $q_1,\dots,q_{t-1}\in\mathcal{Q}_i$, setting $d=q_1\cdots q_t$, $|h_d|/d\gg 1/q_t$, implying that

$$|f(h)|<\exp(-y_i^5).$$

Thus either h is within y_i^{t-2} of a multiple of

$$R_i:=\prod_{\substack{q\in\mathcal{Q}\\ q\le 2y_i}}q,$$

or its contribution is exponentially small. In particular,

$$\sum_{|h|\in[y_i^{t-2},R_i/2]}|f(h)|<\exp(-y_i^5/2)$$

because

$$R_i\le(y_1^2)!\prod_{i'\le i}(2y_{i'})^{y_{i'}^\beta}=(2y_i)^{O(y_i^2)}.$$

Since these intervals of summation overlap as we vary i,

$$\sum_{y_1^{t-2}\le|h|\le m/2}|f(h)|<2\exp(-y_1^5/2).$$

Now consider the case that $0<|h|\le y_1^{1-5\beta/12}$. For any d,

$$\left|\mathrm{Im}\frac{(\alpha-1)e(-h/d\alpha)+e((\alpha-1)h/d\alpha)}{\alpha}\right|\lesssim_\alpha\frac{h^3}{d^3},$$

Let

$$\zeta_{\mathcal{Q}}(\sigma)=\prod_{q\in\mathcal{Q}}(1+q^{-\sigma})=\frac{\sum_{d\in\mathcal{D}}d^{-\sigma}+\sum_{d\in E}d^{-\sigma}}{\sum_{b\in\mathcal{B}}b^{-\sigma}}.$$

Then

$$\log\zeta_{\mathcal{Q}}(3)\le\sum_{q\in\mathcal{Q}}q^{-3}=\sum_i\sum_{q\in\mathcal{Q}_i}q^{-3}\le\sum_i y_i^{\beta-3}\le 2y_1^{\beta-3}$$

so

$$\zeta_{\mathcal{Q}}(3)-1\ll y_1^{\beta-3}.$$

Consequently,

$$|\arg f(h)| \lesssim_\alpha h^3 \sum_{d\in\mathcal{D}} d^{-3} \le h^3\Big(\sum_{b\in\mathcal{B}} b^{-3}e^{2y_1^{\beta-3}} - \sum_{d\in E} d^{-3}\Big) \ll h^3\Big(\sum_{b\in\mathcal{B}\cap[y_1^2,\infty)} b^{-3} + \sum_{b\in\mathcal{B}} b^{-3}y_1^{\beta-3}\Big)$$
$$\ll h^3 y_1^{\beta-3}\sum_{b\in\mathcal{B}} b^{-3} + h^3 y_1^{-4} \ll y_1^{-\beta/4},$$

implying that $\operatorname{Re} f(h) > 0$.

Suppose $y_1^{1-5\beta/12} \le |h| \le y_1/2$. Then for all $q \in \mathcal{Q}_1$, $|h_q|/q \gg y_1^{-5\beta/12}$. Therefore, the total contribution from this range is at most $\exp(-y_1^{\beta/12})$. The remaining range of h-values is covered by Hypothesis 3.

Putting it all together,

$$F(\mathcal{D}) \ge \frac{\alpha^{|\mathcal{D}|}}{m(\max(1,\alpha-1))^{|\mathcal{D}|}}(1-o(1)) > 1,$$

using the fact that $|\mathcal{D}| > 2^{y_s/\log^3 y_s}$ and

$$\frac{\alpha}{\max(1,\alpha-1)} > 1 + \frac{1}{\log^{1+\epsilon} y_1}.$$

□

We claim Hypothesis 3 holds for $\mathcal{D} = B\cdot \operatorname{Div}^*(M_{j_0,\le i_0})$, $y = x_{a_{j_0}}$, and $\beta = 1$. Indeed, if h is fixed, $h + C$ can have $O(1)$ factors in $P_{a_{j_0}}$ when $|C| \le x_{a_{j_0}}^{2/3}$. Consequently, almost all primes in $P_{a_{j_0}}$ will not have a multiple within $x_{a_{j_0}}^{2/3}$ of h, from which the hypothesis follows.

We can therefore apply Theorem 4 to $\mathcal{B} = B$, $\mathcal{Q}_i = P_{i+a_{j_0}-1}$ for $i = 1,\dots,i_0 - a_{j_0} + 1$,

$$\mathcal{D} = B\cdot \operatorname{Div}^*(M_{j_0,\le i_0}),$$

$\beta = 1$, $\epsilon = \varepsilon$, $\mathcal{L} = L$, and $\ell/k = 1 - w(D_0)$. Let D' be a non-empty subset of D with

$$w(D') \equiv 1 - w(D_0) \pmod 1. \tag{14}$$

By (12),

$$w(D') \le w(D) < 2 - w(A) \le 2 - w(D_0),$$

so in fact the two sides of (14) are equal. Thus

$$n = \sum_{\substack{d\in D'\cup D_0\\ d>1}} \frac{n}{d}$$

as elements of D' must have a prime factor $\ge y_{a_{j_0}}$, distinguishing them from elements of D_0. This concludes the proof of Theorem 1.

Corollary 5. *There exists an absolute constant C such that if n is any positive integer with $\sigma(n) \ge Cn$, then n is a sum of distinct proper divisors.*

Proof. Fix $\varepsilon = 1/10$, so (1) holds. Let L the corresponding value guaranteed by Theorem 1. If n is not pseudoperfect, then no factor of n is pseudoperfect. Let m be the L-rough part of n. Then

$$\frac{\sigma(n)}{n} \le \frac{\sigma(m)}{m}\prod_{p<L}\frac{p}{p-1} \ll 1$$

if m is not pseudoperfect. □

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