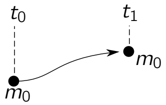
Smoothed Particle Hydrodynamics (SPH)

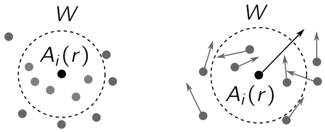
What is SPH?

„A mesh-free method for the discretization of functions and partial differential operators.“ [1]

- Lagrangian approach



- Discretization of functions A_i and fields A_i based on a kernel W interpolation.



- Approximation of differential operators by exact derivative of the kernel function [2].

$$A(r) \approx \int A(r')W(r - r', h)\mathrm{d}r' \tag{1}$$

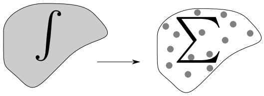
$$\nabla A(r) \approx \int A(r')\nabla W(r - r', h)\mathrm{d}r' \tag{2}$$

SPH discretization

- Interpolating through a convolution integral [2]:

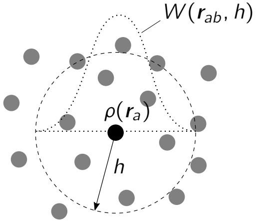
$$(A * W)(r_i) = \int \frac{A(r')}{\rho(r')} W(r_i - r', h) \underbrace{\rho(r')\mathrm{d}r'}_{\mathrm{d}m'} \tag{3}$$

$$\approx \sum_{b \in \mathcal{N}} A_b \frac{m_b}{\rho(r_b)} W(\underbrace{r_a - r_b}_{r_{ab}}, h) \tag{4}$$



- The density is computed using a weighted summation over neighboring particles:

$$\rho(r_a) = \sum_{b \in \mathcal{N}} m_b W(r_{ab}, h) \tag{5}$$



Backup

SPH discretization (fluid)

HLRS

The governing equations in a Lagrangian framework read as

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\boldsymbol{F}^{(\nu)} + \boldsymbol{g}, \tag{6}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\boldsymbol{v}, \tag{7}$$

$$\frac{\mathrm{D}\boldsymbol{r}}{\mathrm{D}t} = \boldsymbol{v}, \tag{8}$$

$$p = F(\rho)\ . \tag{9}$$

Using Monaghans [3] SPH interpolation rules, eq. (6) and (7) become

$$\frac{\mathrm{D}\boldsymbol{v}_a}{\mathrm{D}t} = -\sum_{b\in\mathcal{N}} m_b\left(\frac{p_b}{\rho_b^2} + \frac{p_a}{\rho_a^2} + \underbrace{\Pi_{ab}}_{\text{viscosity}}\right)\nabla_a W_{ab} + \boldsymbol{g}, \tag{10}$$

$$\frac{\mathrm{D}\rho_a}{\mathrm{D}t} = \sum_{b\in\mathcal{N}} m_b(v_a - v_b)\cdot\nabla_a W_{ab}\ . \tag{11}$$

The equation of state to describe the relationship between pressure and density of water up to high pressures is given as

$$p = \frac{\rho_0 c^2}{\gamma}\left(\left(\frac{\rho}{\rho_0}\right)^{\gamma} - 1\right) + p_0 - p_{\text{background}} \tag{12}$$

SPH discretization (solid)

The conservation equation of linear momentum is given by

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{1}{\rho}\nabla_0 \boldsymbol{P} + \boldsymbol{g} \,, \tag{13}$$

where $\boldsymbol{P}$ is the first Piola-Kirchhoff (PK1) stress tensor.

The discretized version of this equation is given by O'Connor et al [4]:

$$\frac{\mathrm{D}\boldsymbol{v}_a}{\mathrm{D}t} = \sum_{b\in\mathcal{N}} m_{0b}\left(\frac{\boldsymbol{P}_a\boldsymbol{L}_{0a}}{\rho_{0a}^2} + \frac{\boldsymbol{P}_b\boldsymbol{L}_{0b}}{\rho_{0b}^2}\right)\nabla_{0a}W(\boldsymbol{X}_{ab}) + \underbrace{\frac{\boldsymbol{f}_a^{PF}}{m_{0a}}}_{\text{optional penalty force}} + \boldsymbol{g}, \tag{14}$$

with the inverse correction (renormalisation) matrix

$$\boldsymbol{L}_{0a} := \left(\sum_{b\in\mathcal{N}} \frac{m_{0b}}{\rho_{0b}}\nabla_{0a}W(\boldsymbol{X}_{ab})\boldsymbol{X}_{ab}^T\right)^{-1} \in^{d\times d} . \tag{15}$$

The difference in the initial (reference) configuration is denoted as $\boldsymbol{X}_{ab} = \boldsymbol{X}_a - \boldsymbol{X}_b$.

PK1 stress tensor

For the computation of the PK1 stress tensor, the deformation gradient $\boldsymbol{J}$ is computed per particle as

$$\boldsymbol{J}_a = \sum_b \frac{m_{0b}}{\rho_{0b}} \boldsymbol{x}_{ba} (\boldsymbol{L}_{0a} \nabla_{0a} W(\boldsymbol{X}_{ab}))^T \tag{16}$$

with $1 \leq i, j \leq d$. From the deformation gradient, the Green-Lagrange strain

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{J}^T \boldsymbol{J} - \boldsymbol{I}) \tag{17}$$

and the second Piola-Kirchhoff stress tensor

$$\boldsymbol{S} = \lambda \operatorname{tr}(\boldsymbol{E}) \boldsymbol{I} + 2\mu \boldsymbol{E} \tag{18}$$

are computed to obtain the PK1 stress tensor as $\boldsymbol{P} = \boldsymbol{J}\boldsymbol{S}$. Here,

$$\mu = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{19}$$

are the Lamé coefficients, where E is the Young's modulus and ν is the Poisson ratio.