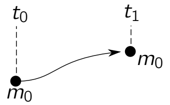
Smoothed Particle Hydrodynamics (SPH)

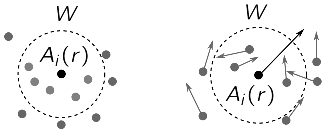
What is SPH?

„A mesh-free method for the discretization of functions and partial differential operators.“ [1]

- Lagrangian approach



- Discretization of functions A_i and fields A_i based on a kernel W interpolation.



- Approximation of differential operators by exact derivative of the kernel function [2].

$$A(r) \approx \int A(r')W(r - r', h)\mathrm{d}r' \tag{1}$$

$$\nabla A(r) \approx \int A(r')\nabla W(r - r', h)\mathrm{d}r' \tag{2}$$

SPH discretization

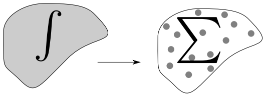
- Interpolating through a convolution integral [2]:

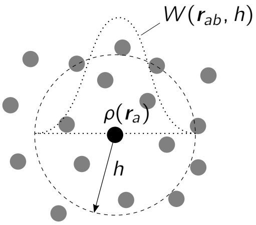
$$(A * W)(r_i) = \int \frac{A(r')}{\rho(r')} W(r_i - r', h) \underbrace{\rho(r')\mathrm{d}r'}_{\mathrm{d}m'} \tag{3}$$

$$\approx \sum_{b \in \mathcal{N}} A_b \frac{m_b}{\rho(r_b)} W(\underbrace{r_a - r_b}_{r_{ab}}, h) \tag{4}$$

- The density is computed using a weighted summation over neighboring particles:

$$\rho(r_a) = \sum_{b \in \mathcal{N}} m_b W(r_{ab}, h) \tag{5}$$





Fluid Structure Interaction (FSI)

Interaction of fluid particle *a* with sets of other particles containing the neighboring fluid, boundary and structure particles:

$$
\begin{aligned}
\frac{\mathrm{D}\boldsymbol{v}_a}{\mathrm{D}t} = &- \sum_{b \in \mathcal{N}_{\text{fluid}}} m_b \left(\frac{p_b}{\rho_b^2} + \frac{p_a}{\rho_a^2} + \underbrace{\Pi_{ab}}_{\text{viscosity}} \right) \nabla_a W_{ab} \\
&- \sum_{b \in \mathcal{N}_{\text{boundary}}} m_b \left(\frac{p_b}{\rho_b^2} + \frac{p_a}{\rho_a^2} + \Pi_{ab} \right) \nabla_a W_{ab} \\
&- \sum_{b \in \mathcal{N}_{\text{structure}}} m_b \left(\frac{p_b}{\rho_b^2} + \frac{p_a}{\rho_a^2} + \Pi_{ab} \right) \nabla_a W_{ab} + \boldsymbol{g}
\end{aligned}
$$

[3]

Fluid Structure Interaction (FSI)

Interaction of structure particle *a* with sets of other particles containing the neighboring fluid and structure particles:

$$
\frac{\mathrm{D}\boldsymbol{v}_a}{\mathrm{D}t} = -\frac{m_a}{m_{0a}} \sum_{b \in \mathcal{N}_{\text{fluid}}} m_b \left(\frac{p_b}{\rho_b^2} + \frac{p_a}{\rho_a^2} + \Pi_{ab} \right) \nabla_a W_{ab}
$$

$$
+ \sum_{b \in \mathcal{N}_{\text{structure}}} m_{0b} \left(\frac{\boldsymbol{P}_a \boldsymbol{L}_{0a}}{\rho_{0a}^2} + \frac{\boldsymbol{P}_b \boldsymbol{L}_{0b}}{\rho_{0b}^2} \right) \nabla_{0a} W(\boldsymbol{X}_{ab}) + \underbrace{\frac{\boldsymbol{f}_a^{PF}}{m_{0a}}}_{\text{optional penalty force}} + \boldsymbol{g}
$$

where m_a and ρ_a refer to the hydrodynamic mass and density of a structure particle. Whereas m_{0a} and ρ_{0a} refer to the material mass and density of structure particle *a*. And $\boldsymbol{P}$ is the first Piola-Kirchhoff (PK1) stress tensor.

[3]

Problem - Pressure field of structure and boundary

HLRS

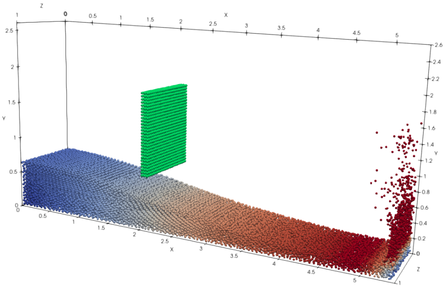
`TrixiParticles.jl` provides five options to compute the hydrodynamic boundary (structure) density and pressure. The one which usually yields the best results is described in [4]. The pressure of a boundary (structure) particle is obtained by extrapolating the pressure of the fluid:

$$p_{\text{solid}} = \frac{\sum_{\text{fluid}} (p_{\text{fluid}} + \rho_{\text{fluid}} (\boldsymbol{g} - \boldsymbol{a}_{\text{solid}}) \cdot \boldsymbol{x}_{\text{solid-fluid}}) W(\boldsymbol{x}_{\text{solid-fluid}})}{\sum_{\text{fluid}} W(\boldsymbol{x}_{\text{solid-fluid}})}, \tag{6}$$

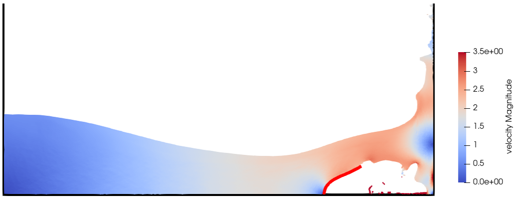
The hydrodynamic density for boundary and structure is then obtained by the inverse equation of state.

Present results

Dam break

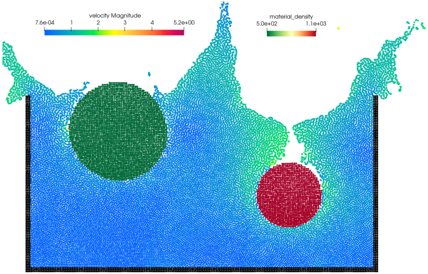


Free surface dam break with opening gate.

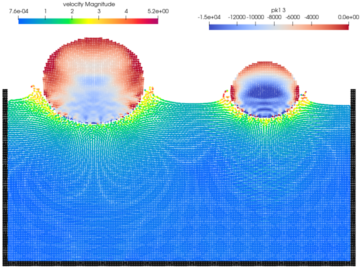


Dam break impacting an elastic plate.

Falling spheres

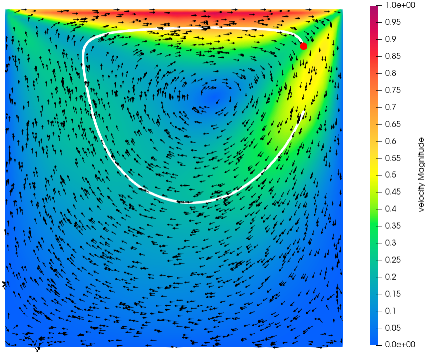


Two elastic spheres ($\rho_{\text{left}} = 500$, $\rho_{\text{right}} = 1100$) falling into a tank of water ($\rho_{\text{water}} = 1000$).

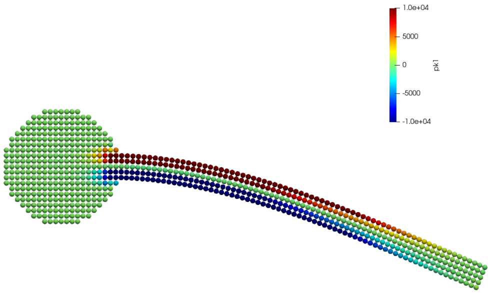


Pk1 stress tensor of elastic spheres.

Others



Lid driven cavity

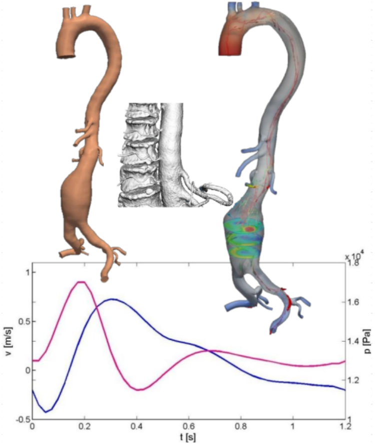


Oscillating beam

Outlook

Abdominal Aortic Aneurysm (AAA) - Current Modelling Technique

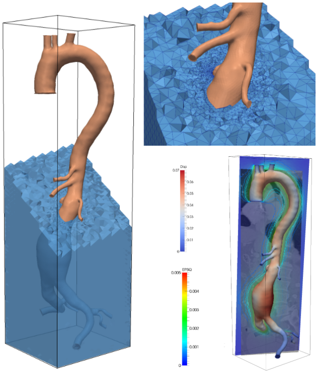
- Modelling from Aortic-Valve to Iliac-Bifurcation
- Modelling based on computer tomography (CT) or magnet resonance imaging (MRI)
- Rigid walls (only fluid domain modelled)
- Transient boundary conditions derived from literature
- Newtonian fluid model
- Transient simulation
- Simulation with commercial and open source CFD packages.



[5]

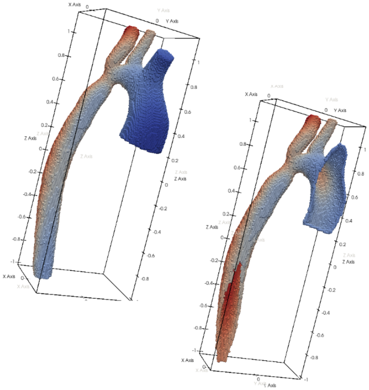
Plans - Modelling FSI and cracking

Modelling of the FSI with non-Newtonian fluid (blood) and also modelling the crack propagation of the aorta due to AAA.



Mesh-based approach - large scale problem

[5]



Particle-based approach - to be done with `TrixiParticles.jl`

Backup

SPH discretization (fluid)

The governing equations in a Lagrangian framework read as

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\boldsymbol{F}^{(\nu)} + \boldsymbol{g}, \tag{7}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\boldsymbol{v}, \tag{8}$$

$$\frac{\mathrm{D}\boldsymbol{r}}{\mathrm{D}t} = \boldsymbol{v}, \tag{9}$$

$$p = F(\rho)\,. \tag{10}$$

Using Monaghans [6] SPH interpolation rules, eq. (7) and (8) become

$$\frac{\mathrm{D}\boldsymbol{v}_a}{\mathrm{D}t} = -\sum_{b\in\mathcal{N}} m_b\left(\frac{p_b}{\rho_b^2} + \frac{p_a}{\rho_a^2} + \underbrace{\Pi_{ab}}_{\text{viscosity}}\right)\nabla_a W_{ab} + \boldsymbol{g}, \tag{11}$$

$$\frac{\mathrm{D}\rho_a}{\mathrm{D}t} = \sum_{b\in\mathcal{N}} m_b(v_a - v_b)\cdot\nabla_a W_{ab}\,. \tag{12}$$

The equation of state to describe the relationship between pressure and density of water up to high pressures is given as

$$p = \frac{\rho_0 c^2}{\gamma}\left(\left(\frac{\rho}{\rho_0}\right)^{\gamma} - 1\right) + p_0 - p_{\text{background}} \tag{13}$$

SPH discretization (solid)

The conservation equation of linear momentum is given by

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{1}{\rho}\nabla_0 \boldsymbol{P} + \boldsymbol{g} \,, \tag{14}$$

where $\boldsymbol{P}$ is the first Piola-Kirchhoff (PK1) stress tensor.

The discretized version of this equation is given by O'Connor et al [3]:

$$\frac{\mathrm{D}\boldsymbol{v}_a}{\mathrm{D}t} = \sum_{b\in\mathcal{N}} m_{0b} \left(\frac{\boldsymbol{P}_a \boldsymbol{L}_{0a}}{\rho_{0a}^2} + \frac{\boldsymbol{P}_b \boldsymbol{L}_{0b}}{\rho_{0b}^2} \right) \nabla_{0a} W(\boldsymbol{X}_{ab}) + \underbrace{\frac{\boldsymbol{f}_a^{PF}}{m_{0a}}}_{\text{optional penalty force}} + \boldsymbol{g}, \tag{15}$$

with the inverse correction (renormalisation) matrix

$$\boldsymbol{L}_{0a} := \left(\sum_{b\in\mathcal{N}} \frac{m_{0b}}{\rho_{0b}} \nabla_{0a} W(\boldsymbol{X}_{ab}) \boldsymbol{X}_{ab}^T \right)^{-1} \in^{d\times d} \,. \tag{16}$$

The difference in the initial (reference) configuration is denoted as $\boldsymbol{X}_{ab} = \boldsymbol{X}_a - \boldsymbol{X}_b$.

PK1 stress tensor

For the computation of the PK1 stress tensor, the deformation gradient $\boldsymbol{J}$ is computed per particle as

$$\boldsymbol{J}_a = \sum_b \frac{m_{0b}}{\rho_{0b}} \boldsymbol{x}_{ba} (\boldsymbol{L}_{0a} \nabla_{0a} W(\boldsymbol{X}_{ab}))^T \tag{17}$$

with $1 \leq i, j \leq d$. From the deformation gradient, the Green-Lagrange strain

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{J}^T \boldsymbol{J} - \boldsymbol{I}) \tag{18}$$

and the second Piola-Kirchhoff stress tensor

$$\boldsymbol{S} = \lambda \operatorname{tr}(\boldsymbol{E}) \boldsymbol{I} + 2\mu \boldsymbol{E} \tag{19}$$

are computed to obtain the PK1 stress tensor as $\boldsymbol{P} = \boldsymbol{J}\boldsymbol{S}$. Here,

$$\mu = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{20}$$

are the Lamé coefficients, where E is the Young's modulus and ν is the Poisson ratio.

Literatur

Literatur I

[1] D Koschier und J Bender. „Smoothed Particle Hydrodynamics Tutorial“. In: (2019). Available at https://interactivecomputergraphics.github.io/SPH-Tutorial/.

[2] DJ Price. „Smoothed particle hydrodynamics and magnetohydrodynamics“. In: *Journal of Computational Physics* 231.3 (Feb. 2012), S. 759–794. DOI: 10.1016/j.jcp.2010.12.011.

[3] J O'Connor und BD Rogers. „A fluid–structure interaction model for free-surface flows and flexible structures using smoothed particle hydrodynamics on a GPU“. In: *Journal of Fluids and Structures* 104 (Juli 2021), S. 103312. DOI: 10.1016/j.jfluidstructs.2021.103312.

[4] S Adami, X Hu und N Adams. „A generalized wall boundary condition for smoothed particle hydrodynamics“. In: *Journal of Computational Physics* 231.21 (Aug. 2012), S. 7057–7075. DOI: 10.1016/j.jcp.2012.05.005.

[5] N Patel und U Küster. „Semi-Automatic Segmentation and Analysis of Vascular Structures in CT Data“. In: *Sustained Simulation Performance 2015*. Springer International Publishing, 2015, S. 205–218. DOI: 10.1007/978-3-319-20340-9_17.

[6] JJ Monaghan. „Smoothed particle hydrodynamics“. In: *Reports on Progress in Physics* 68.8 (Juli 2005), S. 1703–1759. DOI: 10.1088/0034-4885/68/8/r01. URL: https://doi.org/10.1088/0034-4885/68/8/r01.