MAC 2312 Calculus w/ Analytic Geometry II

COURSE SYLLABUS (Rev. 08/11)

GENERAL INFORMATION

Section Number: 74472

Meeting Times: MW 6:00 PM to 8:15 PM

Location: SSCI 225

Credits: 5

Term: Fall 2015

INSTRUCTOR INFORMATION

Instructor: Mr. Thomas Carty

Office: SMPF 222 - E

Office Hours:	Monday and Wednesday:	12:00 PM to 12:30 PM
		1:45 PM to 2:30 PM
		5:00 PM to 6:00 PM
	Tuesday and Thursday:	9:45 AM to 10:00 AM (Room SSCI 225)
		11:40 PM to 12:30 PM
		1:45 PM to 2:30 PM

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or

www.hccfl.edu→HCC Directory→Employee's Last Name: "carty"→Click to Search→Homepage

Course Description

MAC 2312 is a continuation of MAC 2311. It focuses on differentiation and integration of trigonometric, logarithmic, exponential and hyperbolic functions, integration techniques, indeterminate forms and improper integrals, sequences and series, polar coordinates, conics, rotation of axes and parametric equations.

Prerequisite: MAC 2311 with a minimum grade of C.

- The assignments must be ready to hand in at the beginning of the class.
- **Paper torn out of a spiral notebook will not be accepted.**

If doing only the assigned problems is not enough to comfortably comprehend the topic, students are encouraged to do more than the assigned problems. Please ask for help when needed, in and out of class.

Final Exam: The final exam will be comprehensive.

Grading Procedures

Calculation of Final Grade

Tests:	60%	90% on each test gives 54pts of 60
Quizzes:	15%	+ 15pts } = 79 pts of 100
Homework:	10%	+ 10 pts }
Final Exam:	15%	need at least a 73 on final

Grading Scale

A	100%	to	90%
B	89%	to	80%
C	79%	to	70%
D	69%	to	60%
F	0%	to	59%

Course Policies

Attendance Policy

Attendance will be taken for this course. Each student is required to sign the Attendance Sheet located at the front of the class. This is the record used to validate attendance matters. If you arrive late to class, be sure to sign the Attendance Sheet before leaving class. If you are unable to attend a class, you are required to email or text the instructor at the referenced email address/cell phone number on the front page of this syllabus.

- Students receiving financial aid are advised to discuss with a Financial Aid Advisor the impact of not attending class on their financial aid or veterans benefits.

Attending all classes and being on time are critical to your success. Copies of the Lesson Presentations will be available on the web page if you miss class or need to review the notes. However, there is no substitute for being in class to learn the material, to participate in class, and to be able to ask questions when they arise. Please commit to attending all classes.

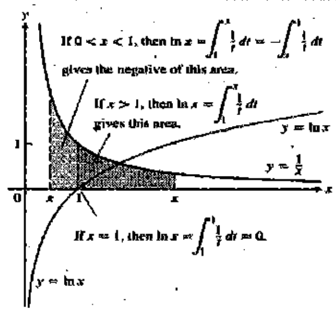
MAC 2312 – Calculus w/Analytical Geometry II
Pacing Schedule and Homework Assignments
Instructor: Mr. Thomas Carty
Fall 2015 – 16 Weeks – MW
6:00 PM to 8:15 PM (74472)

Date	Topics	Homework
08/17 M L1	Introduction, General Information, Homework Integration by Substitution 7.1 Logarithm Defined as an Integral	7.1: 1, 3, 5, 10, 21, 25, 31, 37, 47, 52
08/19 W L2	7.2 Exponential Change and Separable Differential Equations 7.3 Hyperbolic Functions	7.2: 9, 13, 17, 19, 23, 25, 43 7.3: 1, 7, 13, 17, 27, 35, 37, 39, 41, 43, 67, 71
08/24 M L3	7.4 Relative Rates of Growth 8.2 Integration by Parts	7.4: 1, 3, 7, 9, 17 8.2: 3, 9, 13, 19, 23, 27, 31, 33, 37, 39, 55, 63, 65
08/26 W L4	8.3 Trigonometric Integrals	8.3: 1, 3, 7, 11, 17, 19, 23, 27, 33, 35, 37, 41, 51, 57
08/31 M L5	Review for Test 1	Study for Test
09/02 W	**Test 1**	**DUE: Hmwk. from 7.1 to 7.4**
09/07 M L6	8.4 Trigonometric Substitutions	8.4: 3, 5, 7, 9, 11, 13, 35, 37, 43
09/09 W	*No Classes – Labor Day*	*Enjoy the day!*
09/14 M L7	8.5 Integration of Rational Function by Partial Fractions	8.5: 3, 7, 11, 15, 19, 21, 27, 33, 37, 39, 41, 51
09/16 W L8	8.6 Integral Tables and Computer Algebra Systems 8.7 Numerical Integration	8.6: 3, 9, 13, 17, 31, 34, 41, 47 8.7: 1, 3, 5, 7, 9
09/21 M L9	8.8 Improper Integrals	8.8: 3, 7, 15, 21, 25, 35, 39, 43
09/23 W L10	10.1 Sequences 10.2 Infinite Series	10.1: 5, 9, 11, 15, 17, 19, 23, 31, 39, 45, 53, 63, 92 10.2: 9, 13, 19, 21, 24, 27, 31, 35, 38, 49, 53, 63, 83
09/28 M L11	Review for Test 2	Study for Test
09/30 W	**Test 2**	**DUE: Hmwk. from 8.1 to 8.7**
10/05 M L12	10.3 The Integral Test 10.4 Comparison Test	10.3: 1, 5, 9, 11, 14, 17, 23, 27, 29, 33, 35 10.4: 1, 3, 5, 9, 13, 17, 23, 25, 27, 30, 41
10/07 W L13	10.5 Absolute Convergence; The Ratio and Root Tests 10.6 Alternating Series and Conditional Convergence	10.5: 3, 5, 7, 9, 11, 19, 26, 27, 28, 35, 39 10.6: 1, 5, 7, 11, 17, 21, 29, 39
10/12 M L14	10.7 Power Series	10.7: 1, 3, 7, 13, 19, 55

Date	Topics	Homework
10/14 W L15	10.8 Taylor and Maclaurin Series 10.9 Convergence of Taylor Series	**10.8:** 1, 5, 25, 29 **10.9:** 1, 5, 11, 13, 21, 29
10/19 M L16	10.10 The Binomial Series and Applications of Taylor Series	**10.10:** 1, 5, 11, 41, 45, 67, 69
10/21 W L17	11.1 Parameterizations of Plane Curves	**11.1:** 1, 5, 9, 11, 17, 22, 25
10/26 M L18	Review for Test 3	Study for Test
10/28 W	**Test 3**	**DUE: Hmwk. from 10.1 to 10.10**
11/02 M L19	11.2 Calculus with Parametric Curves	**11.2:** 3, 5, 9, 15, 21, 23, 25, 27, 31, 33
11/04 W L20	11.3 Polar Coordinates	**11.3:** 1, 5, 7, 13, 17, 23, 29, 33, 35, 39, 43, 45, 49, 51, 53, 57, 59, 61, 65
11/09 M L21	11.4 Graphing in Polar Coordinates 11.5 Area and Lengths in Polar Coordinates	**11.4:** 1, 5, 9, 13, 17, 21, 25, 29 **11.5:** 1, 5, 9, 13, 17, 21, 25
11/11 W	*No Classes – Veterans' Day*	*To all of our veterans:* *Thank you for your service!!*
11/16 M L22	11.6 Conic Sections	**11.6:** 11, 12, 19, 23, 29, 31, 43, 57, 59, 65
11/18 W L23	11.7 Conics in Polar Coordinates	**11.7:** 29, 31, 33, 35, 37, 39, 41, 43
11/23 M L24	Review for Test 4	Study for Test
11/25 W	**Test 4**	**DUE: Hmwk. from 11.1 to 11.6**
11/26-29	*Thanksgiving – Holiday break*	
11/30 M L25	Review for Final Exam	
12/07 M	**Final Exam – 6:00 PM to 7:50 PM**	**Comprehensive**
	Enjoy the winter break!	

DEFINITION The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$



Properties of the Natural Logarithm Function

1. $\ln bx = \ln b + \ln x$
2. $\ln \frac{b}{x} = \ln b - \ln x$
3. $\ln \frac{1}{x} = -\ln x$
4. $\ln x^r = r \ln x$

The Derivative of the Natural Logarithm Function

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0$$

If u is a differentiable function that is never zero,

$$\int \frac{1}{u}\,du = \ln |u| + C.$$

If u is a differentiable function that is never zero,

$$\int \frac{1}{u}\,du = \ln|u| + C.$$

Additional Integration Rules

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan u\,du = \ln|\sec u| + C \qquad \int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

The Exponential Function (e^u)

If u is any differentiable function of x, then

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

$$\int e^u\,du = e^u + C.$$

The General Exponential Function (a^u)

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}.$$

The integral equivalent of this last result is

$$\int a^u \, du = \frac{a^u}{\ln a} + C.$$

Derivatives and Integrals using $log_a u$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \frac{du}{dx}$$

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:* $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:* $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Reciprocal Rule:* $\log_a \frac{1}{y} = -\log_a y$
4. *Power Rule:* $\log_a x^y = y \log_a x$

Review

$$\int (2x^3+3)^2 5x^2\,dx$$

$$\cancel{\left[\frac{1}{2}(2x^3+3)\cdot 5x^2 + (2x^3+3)^2\cdot\left(\frac{5}{2}x\right)\right]\left[\frac{2}{3}x^2\right.}$$

$u = 2x^3+3$

$du = 6x^2\,dx$

$$> \frac{5}{6}\int u^2\,dx = \frac{5}{6}\cdot\frac{u^3}{3}+C$$

$$\boxed{\frac{5}{18}(2x^3+3)^3+C}$$

$$\int x\sec^2(4x^2)\,dx$$

$u = 4x^2$

$du = 8x\,dx$

$$\frac{1}{8}\int 8x\sec^2(4x^2)\,dx = \frac{1}{8}\int \sec^2(u)\,d\cancel{x}$$

$$= \frac{1}{8}\tan u + C = \boxed{\frac{1}{8}\tan(4x^2)+C}$$

$$\int \frac{x^2}{\sqrt[4]{x^3+2}}\,dx$$

$u = x^3+2$

$du = 3x^2\,dx$

$$= \frac{1}{3}\int \frac{3x^2\,dx}{\sqrt[4]{x^3+2}}$$

$$\frac{1}{3}\int \frac{1}{(u)^{\frac{1}{4}}}\,du = \frac{1}{3}\int u^{-\frac{1}{4}}\,du$$

$$= \frac{1}{3}\left(\frac{4}{3}u^{3/4}\right)+c = \boxed{\frac{4}{9}\left(u^{3/4}\right)+c}$$

$$\int \sin^3(2x)\cos(2x)\,dx$$

$u = \sin(2x)$

$du = \cos(2x)\,2dx$

$$= \frac{1}{2}\int u^3\,du = \frac{1}{2}\left(\frac{u^4}{4}\right)+c$$

$$\frac{1}{8}u^4+c = \boxed{\frac{1}{8}\sin^4(2x)+c}$$

7.1 Logarithm Defined as Integral

Natural Log $\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0$

Review:

1. $\ln bx = \ln b + \ln x$
2. $\ln \frac{b}{x} = \ln b - \ln x$
3. $\ln \frac{1}{x} = -\ln x$ > because $\overset{0}{\ln 1} - \ln x = -\ln x$
4. $\ln x^r = r \ln x$

Derivative of Natural Log

$$\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}$$

$$\int \frac{1}{u}\,du = \ln|u| + C$$

Integration Rules:

$$\int \tan u \, du = \ln|\sec u| + c$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \csc u \, du = -\ln|\csc u + \cot u| + c$$

Examples

① $\int \frac{x}{x^2-3}\,dx$

$U = x^2-3$
$du = 2x\,dx$

$= \frac{1}{2}\int \frac{2x\,dx}{x^2-3} = \frac{1}{2}\int \frac{1}{U}\,du$

$\frac{1}{2}\ln|U| + c = \ln|U|^{\frac{1}{2}} + c$

$= \boxed{\ln|x^2-3|^{\frac{1}{2}} + c}$

② $\int \frac{\sec x \tan x}{\sec x - 1}\,dx$

$U = \sec x - 1$
$dU = \sec x \tan x\,dx$

$= \int \frac{1}{U}\,du = \ln|U| + c$

$= \boxed{\ln|\sec x - 1| + c}$

7.1 Exponential Function

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

$$\int e^u \, du = e^u + c$$

$$\int x^2 e^{x^3} \, dx$$

$u = x^3$

$du = 3x^2 dx$

$$= \frac{1}{3} \int e^{x^3} 3x^2 dx = \frac{1}{3} \int e^u du$$

$$= e^u + c = \boxed{e^{x^3} + c}$$

$$\int \frac{2e^{2x}}{1 + e^{2x}} \, dx$$

$u = 1 + e^{2x}$

$du = e^{2x} \cdot 2dx$

$$= \int \frac{1}{1+e^{2x}} \cdot 2e^{2x} dx = \int \frac{1}{u} du$$

$$= \ln|u| + c = \boxed{\ln|1+e^{2x}| + c}$$

General Exponential (a^u)

$$\frac{d}{dx} = a^u = a^u \ln a \frac{du}{dx}$$

$$\int a^u du = \frac{a^u}{\ln a} + C$$

Ex:

① $f(x) = 3^{2x}$

$F'(x) = \boxed{3^{2x} \cdot \ln 3 \cdot 2} =$ Cant mult 2 & 3

② $y = 7^{2x-1}$

$\boxed{y' = 7^{2x-1} \cdot \ln 7 \cdot 2}$

③ $y = t^2 \cdot 2^t$

$y' = 2t \cdot 2^t + t^2 \cdot (2^t \cdot \ln 2)$

$y' = t \cdot 2^t (t \ln 2 + 2)$

(4) $\int 5^x \, dx = \boxed{\frac{5^x}{\ln 5} + C}$

(5) $\int 2^{\sin x} \cos x \, dx$

$u = \sin x$
$du = \cos x \, dx$

$= \int 2^u \, du = \frac{2^u}{\ln 2} + C = \boxed{\frac{2^{\sin x}}{\ln 2} + C}$

Derivatives + Integrals of $\log_a U$

$$\frac{d}{dx}\left(\log_a U\right) = \frac{1}{\ln a} \cdot \frac{1}{U} \cdot \frac{dx}{dU}$$

Ex:

① $\log_8 9 = \boxed{\dfrac{\ln 9}{\ln 8}}$ $\quad \dfrac{\ln \text{ of argument}}{\ln \text{ of base}}$

② $y = \log_4 (5x+1)$

$$y' = \frac{1}{\ln 4} \cdot \frac{1}{5x+1} \cdot 5 = \frac{5}{\ln 4} \cdot \frac{1}{5x+1} = \boxed{\frac{5}{(\ln 4)(5x+1)}}$$

③ $f(x) = \log_3 (x^3 - 3x)$

$$f'(x) = \frac{1}{\ln 3} \cdot \frac{1}{x^3-3x} \cdot (3x^2-3) = \boxed{\frac{3x^2-3}{\ln 3 (x^3-3x)}}$$

④ $g(x) = \log_3 (x^2+7)^3$

$$g'(x) = \frac{1}{\ln 3} \cdot \frac{1}{(x^2+7)^3} \cdot 3(x^2+7)^2 \cdot (2x)$$

$$g'(x) = \frac{3(x^2+7)^2 \cdot 2x}{\ln 3 \, (x^2+7)^3} = \boxed{\frac{6x}{\ln 3 (x^2+7)}}$$

OR

$$g'(x) = 3 \log_3 (x^2+7) = 3 \cdot \frac{1}{\ln 3} \cdot \frac{1}{x^2+7} \cdot 2x = \boxed{\frac{6x}{\ln 3 (x^2+7)}}$$

(5) $\int \left(\frac{1}{3}\right)^{\tan x} \sec^2 x \, dx$

$a = \frac{1}{3}$ $\quad u = \tan x$

$du = \sec^2 x \, dx$

$$\int a^u \, du = \frac{a^u}{\ln a} + C = \boxed{\frac{\frac{1}{3}^{\tan x}}{\ln \frac{1}{3}} + C}$$

(6) $\int \frac{\log_2 x}{x} dx = \int \frac{\ln x}{\ln 2} \cdot \frac{1}{x} dx$

$a = 2$

$$\int \frac{\ln x}{x \ln 2} dx = \int \underbrace{\frac{1}{\ln 2}}_{\text{constant}} \cdot \frac{\ln x}{x} dx$$

$$\frac{1}{\ln 2} \int \frac{\ln x}{x} dx = \frac{1}{\ln 2} \int u \, du$$

$U = \ln x$

$dU = \frac{1}{x} du$

$$= \frac{1}{\ln 2} \cdot \frac{U^2}{2} + C = \boxed{\frac{(\ln x)^2}{2 \ln 2} + C}$$

$ds = \tan 2x \, dx \qquad (0,2) \rightarrow S(0) = 2$

$\int 1 \, ds = \int \tan 2x \, dx =$

$S = \int \frac{\sin 2x}{\cos 2x} dx$

$U = \cos 2x$
$du = -\sin 2x \, dx \cdot 2$
$dU = -2 \sin 2x$

$> \quad S = -\frac{1}{2} \int \frac{-2 \sin 2x \, dx}{\cos 2x}$

$S = -\frac{1}{2} \int \frac{1}{U} dU = -\frac{1}{2} \ln|U| + C$

$S = -\frac{1}{2} \ln|\cos 2x| + C$

$2 = -\frac{1}{2} \ln|\cos 2(0)| + C$

$2 = -\frac{1}{2} \ln|\cos 0| + C$

$2 = -\frac{1}{2} \ln|1| + C$

$2 = -\frac{1}{2}(0) + C$

$\boxed{2 = C}$

Separable Differential Equations

$$\frac{dy}{dx} = \frac{2x}{e^{2y}}$$

rewrite

$$dy = \frac{2x}{e^{2y}}dx$$

Separate y's & x's

$$e^{2y}dy = 2xdx$$

$$\int e^{2y}dy = \int 2xdx$$

$$u = 2y \qquad du = 2dy$$

$$\frac{1}{2}\int 2e^{2y}dy = \int 2xdx$$

$$\frac{1}{2}\int e^{u}du = \int 2xdx$$

$$\frac{1}{2}e^{u} = \frac{2x^2}{2} + C$$

$$\frac{1}{2}e^{2y} = x^2 + C$$

$$e^{2y} = 2x^2 + 2C$$

$$e^{2y} = 2x^2 + C$$

$$\ln(e^{2y}) = \ln(2x^2 + C)$$

$2y \ln e = \ln(2x^2 + c) \rightarrow \ln e = 1$

$$2y = \ln(2x^2 + C)$$

$$y = \frac{1}{2}\ln(2x^2 + C)$$

Separable Differential Equations

$$secx\frac{dy}{dx} = e^{y+sinx}$$

$$secxdy = e^{y}e^{sinx}dx$$

$$\frac{1}{e^{y}}dy = \frac{e^{sinx}}{secx}dx$$

$$e^{-y}dy = cosxe^{sinx}dx$$

$$\int e^{-y}dy = \int cosxe^{sinx}dx$$

$$u = -y \qquad u = sinx$$
$$du = -dy \qquad du = cosx\,dx$$

$$-\int -e^{-y}dy = \int cosxe^{sinx}dx$$

$$-\int e^{u}du = \int e^{u}du$$

$$-e^{u} = e^{u} + C$$

$$-e^{-y} = e^{sinx} + C$$

$$e^{-y} = -e^{sinx} + C$$

$$\ln(e^{-y}) = \ln(-e^{sinx} + C)$$

$$-y = \ln(-e^{sinx} + C)$$

$$y = -\ln(-e^{sinx} + C)$$

→ Worked out nxt page

Uninhibited Exponential Growth

$$A(t) = A_0 e^{kt}$$

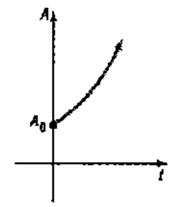
$A(t) = number\ or\ weight\ after\ time$

$A_0 = initial\ number\ or\ weight$

$k = rate\ of\ growth;\ the\ \textbf{positive}\ constant\ (k > 0)$

$t = time\ passed$

↳ positive K



(a) $A(t) = A_0 e^{kt}, k > 0$

Uninhibited Exponential Decay

$$A(t) = A_0 e^{kt}$$

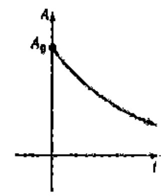
$A(t) = number\ or\ weight\ after\ time$

$A_0 = initial\ number\ or\ weight$

$k = the\ rate\ of\ decay;\ the\ \textbf{negative}\ constant\ (k < 0)$

$t = time\ passed$

↳ neg K



(b) $A(t) = A_0 e^{kt}, k < 0$

Example

The population of the United States was approximately 227 million in 1980 and 282 million in 2000. Estimate the population in the year 2010.

$$A(t) = A_0 e^{kt}$$

end year

Find k → Start year

$$282 = 227e^{k(2000-1980)}$$

$$\frac{282}{227} = e^{20k}$$

$$ln\frac{282}{227} = 20k$$

$$k = \frac{ln\frac{282}{227}}{20} = 0.010848$$

now...
what is the population A(t)
in 2010?

2010

$$A(t) = 227e^{0.010848\,(2010-1980)}$$

$$A(t) = 314.3\ million$$

$From\ 2010\ Census{:}\ 308.7\ million$

Ex ①

$$\sec x \frac{dy}{dx} = e^{y+\sin x}$$

$$\sec x \, dy = e^{y+\sin x} \, dx$$

$$\sec x \, dy = e^{y} \cdot e^{\sin x} \, dx$$

$$\frac{1}{e^{y}} dy = \frac{e^{\sin x}}{\sec x} dx$$

$$e^{-y} dy = \frac{e^{\sin x}}{\frac{1}{\cos x}} dx = e^{-y} dy = \cos x \, e^{\sin x} dx$$

$$\int e^{-y} dy = \int \cos x \, e^{\sin x} dx$$

$U = -y$ $\qquad$ $V = \sin x$

$du = -dy$ $\qquad$ $dV = \cos x \, dx$

$$-\int -e^{-y} dy = \int \cos x \, e^{\sin x} dx \rightarrow -\int e^{u} du = \int e^{v} dv$$

$$= -e^{u} = e^{v} + C \rightarrow -e^{-y} = e^{\sin x} + C$$

$$e^{-y} = -e^{\sin x} + C \longrightarrow \ln(e^{-y}) = \ln(-e^{\sin x} + c)$$

$$-y = \ln(-e^{\sin x} + c) = \boxed{y = -\ln(-e^{\sin x} + c)}$$

$$\frac{dy}{dx} = 6y^2x \longrightarrow dy = 6y^2x\,dx$$

$$= \frac{dy}{y^2} = 6x\,dx = \int \frac{1}{y^2}\,dy = \int 6x\,dx$$

$$\int y^{-2}\,dy = \int 6x\,dx \longrightarrow -y^{-1} = 3x^2 + C$$

$$= -\frac{1}{y} = 3x^2 + C = -y = \frac{1}{3x^2+C}$$

$$\boxed{y = \frac{-1}{3x^2+C}}$$

Example

A radioactive material has a half-life of 700 years. If there were ten grams initially, how much would remain after 300 years? When will the material weigh 7.5 grams?

pick a number for weight

$$A(t) = A_0e^{kt}$$

Find k

half it

$$1 = 2e^{k(700)}$$
$$0.5 = e^{700k}$$
$$ln0.5 = 700k$$
$$k = \frac{ln0.5}{700}$$
$$k = -0.000990$$

300 years

$$A(t) = 10e^{-0.00099\,(300)}$$
$$A(t) = 7.43\ grams$$

or

$$A(t) = 10e^{\frac{ln0.5}{700}(300)}$$
$$A(t) = 7.43\ grams$$

7.5 grams

$$7.5 = 10e^{-0.00099\,t}$$
$$0.75 = e^{-0.00099\,t}$$
$$ln0.75 = -0.00099t$$
$$t = 290.6\ years$$

or

$$7.5 = 10e^{\frac{ln0.5}{700}t}$$
$$0.75 = e^{\frac{ln0.5}{700}t}$$
$$ln0.75 = \frac{ln0.5}{700}t$$
$$t = 290.5\ years$$

Newton's Law of Cooling

$$H - H_s = (H_0 - H_S)e^{kt}$$

H = temperature of a heated object at any given time

H_S = constant temperature of the surrounding medium = Room Temp

H_0 = initial temperature of the heated object

k = negative cooling constant ($k < 0$)

t = time passed

Newton's Law of Cooling $H - H_s = (H_0 - H_S)e^{kt}$

Example

A pizza pan is removed at 3:00 PM from an oven whose temperature is fixed at 450° F into a room that is a constant 70° F. After 5 minutes, the pizza pan is at 300° F. How long will it take for the pan to cool to 135° F?

Find k pick H=300 because we have time

$$300 - 70 = (450 - 70)e^{k5}$$
$$230 = 380e^{k5}$$
$$\frac{230}{380} = e^{k5}$$
$$ln\frac{23}{38} = 5k$$
$$k = \frac{ln\frac{23}{38}}{5} = -0.100418$$

t @ 135° F

$$135 - 70 = (450 - 70)e^{-0.100418t}$$
$$65 = 380e^{-0.100418t}$$
$$\frac{65}{380} = e^{-0.100418t}$$
$$ln\frac{13}{75} = -0.100418t$$
$$t = 17.45\ minutes$$

Hyperbolic functions are generated using the two exponential functions: e^x and e^{-x}.

The hyperbolic sine function is defined by:

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

more formulas

The hyperbolic cosine function is defined by:

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

Hyperbolic functions have similar properties to the trigonometric functions.

Trigonometric functions are referred to as the circular functions as many of their identities are based on $x^2 + y^2 = 1$. base of Trig

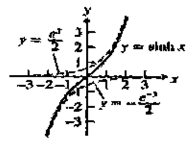
Hyperbolic functions were derived from the hyperbola equation of $x^2 - y^2 = 1$. base of hyperbolas

Uses:

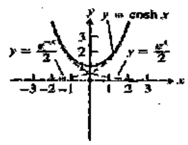
Catenary cables	Specific and general relativity
Magnetic polarization	Voltage input/output for transistors and amplifiers

Hyperbolic Functions

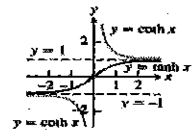
TABLE 7.3 The six basic hyperbolic functions



(a)
Hyperbolic sine:
$\sinh x = \frac{e^x - e^{-x}}{2}$

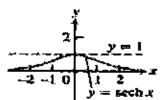


(b)
Hyperbolic cosine:
$\cosh x = \frac{e^x + e^{-x}}{2}$

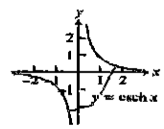


(c)
Hyperbolic tangent:
$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent:
$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$



(d)
Hyperbolic secant:
$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$



(e)
Hyperbolic cosecant:
$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

Hyperbolic Functions

TABLE 7.4 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Hyperbolic Functions

TABLE 7.5 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.6 Integral formulas for hyperbolic functions

$$\int \sinh u\, du = \cosh u + C$$

$$\int \cosh u\, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

Inverse Hyperbolic Functions

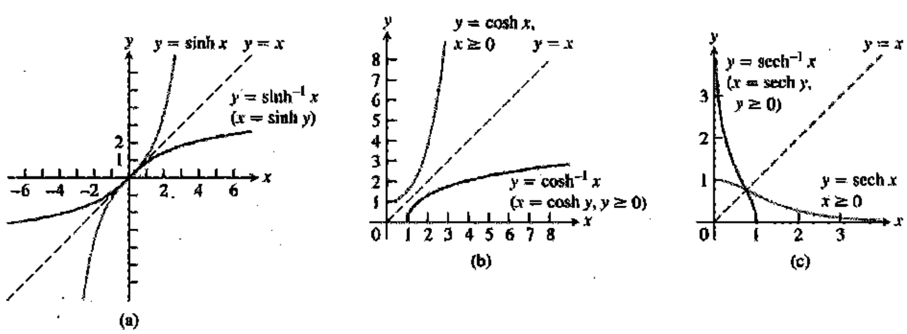


FIGURE 7.5 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

Examples:

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \qquad \cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$$

$$= \quad d(\sinh x) = \tfrac{1}{2}\left(e^{x} - e^{-x}(-1)\right)$$

$$= d(\sinh x) = \frac{e^{x} + e^{-x}}{2} = \cosh x$$

$$d(\sinh x) = \cosh x$$

Inverse Hyperbolic Functions

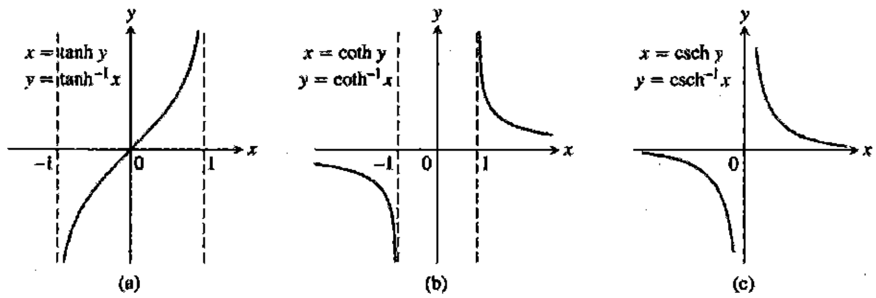


FIGURE 7.6 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

Inverse Hyperbolic Functions

TABLE 7.7 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1}\frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1}\frac{1}{x}$$

$\cos^2 + \sin^2 = 1$

$1 + \tan^2 = \sec^2$

$\cot^2 + 1 = \csc^2$

Inverse Hyperbolic Functions

TABLE 7.8 Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \qquad u \neq 0$$

Inverse Hyperbolic Functions

TABLE 7.9 Integrals leading to inverse hyperbolic functions

1. $$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

2. $$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

3. $$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C, & u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & u^2 > a^2 \end{cases}$$

4. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

5. $$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$$

Examples Hyperbolic

① $y = \frac{1}{2}\ \text{Sinh}(2x+1)$

$$y' = \frac{1}{2}\ \text{Cosh}(2x+1) \cdot 2$$

$$\boxed{y' = \text{Cosh}(2x+1)}$$

② $y = \ln(\text{Cosh } z)$

$$y' = \frac{1}{\text{Cosh} z} \cdot \text{Sinh} z = \frac{\text{Sinh } z}{\text{Cosh } z} = \boxed{\text{Tanh } z}$$

③ $y = (\text{csch } \theta)(1 - \ln(\text{csch } \theta))$

$$y' = \begin{array}{c} (-\text{csch}\theta \coth\theta)(1 - \ln(\text{csch}\theta)) \\ + \\ (\text{csch}\theta)\left(-\frac{1}{\text{csch}\theta} \cdot (-\text{csch}\theta \coth\theta)\right) \end{array}$$

$$y' = (-\text{csch}\theta \coth\theta)(1 - \ln(\text{csch}\theta)) + (\text{csch}\theta)(\coth\theta)$$

$$y' = (-\text{csch}\theta \coth\theta)(1 - \ln(\text{csch}\theta)) + (\text{csch}\theta \coth\theta)$$

$$y' = -\text{csch}\theta \coth\theta + \text{csch}\theta \coth\theta + \ln(\text{csch}\theta)\text{csch}\theta \coth\theta$$

$$\boxed{y' = +\ln(\text{csch}\theta) \cdot \text{csch}\theta \coth\theta}$$

(4) $y = (1 - t^2)\ \coth^{-1} t$

$y' = -2t(\coth^{-1} t) + (1-t^2)\left(\frac{1}{1-t^2}\right)$

$y' = -2t \coth^{-1} t + \cancel{\frac{1}{1-t^2}} - \cancel{\frac{t^2}{1-t^2}}$

$y' = -2t \coth^{-1} t$

(5) $\int \sinh \frac{x}{5}\, dx =$

$u = \frac{1}{5}x$

$du = \frac{1}{5}$

$5 \int \sinh u\, du$

$= 5 \cosh u + C = \boxed{5 \cosh \frac{1}{5}x + C}$

⑥ $$\int \frac{\cosh \frac{\theta}{\sqrt{3}}}{\sinh \frac{\theta}{\sqrt{3}}}\, d\theta \longrightarrow \sqrt{3}\int \frac{\cosh(\frac{\theta}{\sqrt{3}})\frac{1}{\sqrt{3}}\, d\theta}{\sinh \frac{\theta}{\sqrt{3}}}$$

$U = \sinh \frac{\theta}{\sqrt{3}}$

$dU = \cosh \frac{\theta}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)d\theta$

$$\sqrt{3}\int \frac{1}{U}\, dU$$

$$= \sqrt{3}\left(\ln|U| + C\right) = \boxed{\sqrt{3}\left(\ln\left|\sinh \frac{\theta}{\sqrt{3}}\right| + C\right)}$$

⑦ $$\int \frac{6\, dx}{\sqrt{1+9x^2}}$$

$U^2 = 9x^2 \quad U = 3x$

$dU = 3\,dx \quad a = 1$

$$2\int \frac{\frac{1}{2}\cdot 6}{\sqrt{1+9x^2}}\, dx = 2\int \frac{1}{\sqrt{1+9x^2}}\cdot 3\,dx$$

$$2\int \frac{1}{\sqrt{a^2+U^2}}\, dU = 2\sinh^{-1}\left(\frac{U}{a}\right) + C$$

$$= \boxed{2\sinh^{-1}\left(\frac{3x}{1}\right) + C}$$

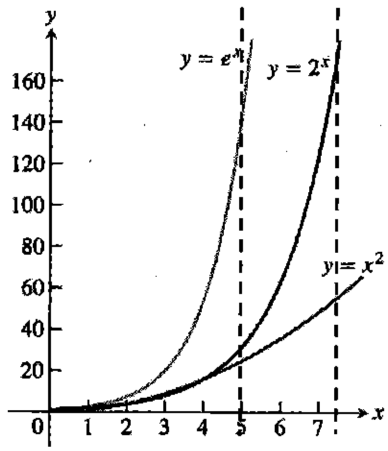


FIGURE 7.7 The graphs of e^x, 2^x, and x^2.

e^x *grows faster than* 2^x

or

2^x *grows slower than* e^x

2^x *grows faster than* x^2

or

x^2 *grows slower than* 2^x

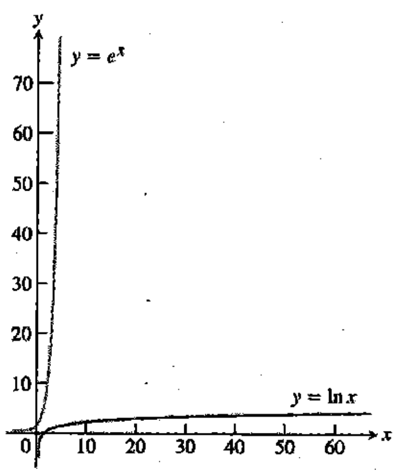


FIGURE 7.8 Scale drawings of the graphs of e^x and $\ln x$.

e^x *grows faster than* $\ln(x)$

or

$\ln(x)$ *grows slower than* e^x

DEFINITION Rates of Growth as $x \to \infty$

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. **f grows faster than g** as $x \to \infty$ if
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$
or, equivalently, if
$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$
We also say that g **grows slower than** f as $x \to \infty$.
2. f and g **grow at the same rate** as $x \to \infty$ if
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$
where L is finite and positive.

Little-oh notation

DEFINITION A function f is **of smaller order than** g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $\underbrace{f = o(g)}_{\text{same}}$ ("f is little-oh of g").

As $x \to \infty$, $f(x)$ grows strickly slower than $g(x)$

Big-oh notation

DEFINITION Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive integer M for which
$$\frac{f(x)}{g(x)} \le M,$$
for x sufficiently large. We indicate this by writing $\underbrace{f = O(g)}$ ("f is big-oh of g").

As $x \to \infty$, $f(x)$ grows no faster than $g(x)$

grows @ the same rate

7.4 Examples

① $$\lim_{x\to\infty} \frac{e^x}{\left(\frac{5}{2}\right)^x} = \lim_{x\to\infty} \frac{e^\infty}{\left(\frac{5}{2}\right)^\infty} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \left(\frac{2e}{5}\right)^x = \boxed{\infty}$$

$\therefore$ $\boxed{e^x \text{ grows faster than } \left(\frac{5}{2}\right)^x}$

② Compare x^2 to $(1.1)^x$

$$\lim_{x\to\infty} \frac{x^2}{(1.1)^x} = \frac{\infty^2}{(1.1)^\infty} = \frac{\infty}{\infty}$$

L'hopitals $$\lim_{x\to\infty} \frac{2x}{(1.1)^x \cdot \ln(1.1)} = \frac{\infty}{\infty}$$

again $$\lim_{x\to\infty} = \frac{2}{(1.1)^x \cdot \ln(1.1) \cdot \ln(1.1)} = \frac{2}{\infty} = \boxed{0}$$

$\therefore$ x^2 grows slower than $(1.1)^x$

(3) Compare $\ln(x)$ to $\ln(2x+5)$

$$\lim_{x\to\infty} \frac{\ln(x)}{\ln(2x+5)} = \frac{\infty}{\infty}$$

L'hopital

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{(2x+5)}\cdot 2} = \frac{(2x+5)}{2x} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{2x+5}{2x} = \boxed{1}$$ b/c degrees of exponents the same

$\therefore$ $\ln x$ and $\ln(2x+5)$ grows at the same rate

(4) Prove $\frac{1}{x+3} = O\left(\frac{1}{x}\right)$

$\therefore$ $f(x)$ is big-oh of $g(x)$

$$\lim_{x\to\infty} \frac{\frac{1}{x+3}}{\frac{1}{x}} = \frac{x}{x+3} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{x}{x+3} = \frac{1}{1} = 1$$

$\therefore$ grows at the same rate

(5) Prove $\frac{1}{x} - \frac{1}{x^2} = o\left(\frac{1}{x}\right)$

$$\lim_{x \to \infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{\frac{1}{x}} = \frac{x}{x} - \frac{x}{x^2} = 1 - \frac{1}{x}$$

$$\lim_{x \to \infty} 1 - \frac{1}{x} = 1 - 0 = \boxed{1}$$

$\therefore$ $\frac{1}{x} - \frac{1}{x^2}$ and $\frac{1}{x}$ grows at the same rate is false

Prove $x \ln x = o(x^2)$ f(x) is little-oh of g(x)

$$\lim_{x \to \infty} \frac{x \ln x}{x^2} = \frac{\ln x}{x} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{\ln x}{x} = \frac{\frac{1}{x}}{1} = \frac{1}{x}$$

$$\lim_{x \to \infty} \frac{1}{x} = \boxed{0}$$

$\therefore$ $x \ln x$ grows slower than x^2
$x \ln x = o(x^2)$ is true

(ii) Prove $\frac{1}{x} - \frac{1}{x^2} = o\left(\frac{1}{x}\right)$

Prove $\frac{d}{dx} x = o(x^n)$

Cant Integrate these

$$\int x^2 \ln x \, dx \qquad \int \ln x \, dx \qquad \int \tan^{-1} x \, dx$$

$$\int f(x)\, g'(x)\, dx$$

To integrate these integrals, Integration by Parts is required.

$$D_x(f(x)\cdot g(x)) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

$$\int D_x(f(x)\cdot g(x)) = \int f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

$$f(x)\cdot g(x) = \int f(x)\cdot g'(x) + \int g(x)\cdot f'(x)$$

Let: $u = f(x)$ $\qquad v = g(x)$

$du = f'(x)$ $\qquad dv = g'(x)$

$$f(x)\cdot g(x) = \int f(x)\cdot g'(x) + \int g(x)\cdot f'(x)$$

Let: $u = f(x)$ $\qquad v = g(x)$

$du = f'(x)$ $\qquad dv = g'(x)$

$$u\cdot v = \int u\cdot dv + \int v\cdot du$$

$$u\cdot v - \int v\cdot du = \int u\cdot dv$$

$\int u\cdot dv = u\cdot v - \int v\cdot du$ For indefinite integrals Learn this one.

$\int_a^b u\cdot dv = u\cdot v\big|_a^b - \int_a^b v\cdot du$ For definite integrals

Choosing u

<u>Lipet</u>

L: log fxns

i: inverse fxns

p: polynomials

E: exponentials

t: trig fxns

Ex: $\int u \cdot dv = u \cdot v - \int v \cdot du$

$\int x \cos x \, dx$ = use Lipet "e" before "t"

$\therefore \quad u = x \xrightarrow{\text{times}} dv = \cos x \, dx$

$du = dx \quad v = \sin x$

$\xleftarrow{\text{minus integral } v \cdot du}$

$\therefore \int x \cos x \, dx = x \cdot \sin x - \int \sin x \, dx$

$= x \sin x - (-\cos x) \, dx$

$\underline{\int x \cos x \, dx = x \sin x + \cos x + C}$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

8.1 Examples

① $\int x \ln x \, dx$ $\qquad u = \ln x \qquad dv = x\,dx$

$\qquad du = \frac{1}{x}dx \qquad v = \frac{x^2}{2}$

minus integral

$$\int x \ln x \, dx = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$\int x \ln x \, dx = \boxed{\frac{x^2}{2}\ln x - \left(\frac{x^2}{4}\right) + c}$$

② $\int x e^x \, dx$ $\qquad u = x \qquad dv = e^x dx$

$\qquad du = dx \qquad v = e^x$

$$= x \cdot e^x - \int e^x dx$$

$$\boxed{\int x e^x dx = xe^x - e^x + c}$$

③ $\int z^3 \ln z \, dz$ $\quad U = \ln z \quad dv = z^3 dz$

$du = \frac{1}{z} dz \quad V = \frac{1}{4} z^4$

$= \ln z \cdot \frac{1}{4} z^4 - \int \frac{1}{4} z^4 \cdot \frac{1}{z} dz$

$= \boxed{\frac{1}{4} z^4 \ln z - \frac{1}{16} z^4 + c}$

④ $\int x \csc^2 x \, dx$ $\quad U = x \quad dv = \csc^2 x \, dx$

$du = dx \quad V = -\cot x$

$= -x \cot x - \int -\cot x \, dx$

$= -x \cot x + \int \frac{\cos x}{\sin x} dx$ $\quad U = \sin x$, $du = \cos x \, dx$

$-x \cot x + \int \frac{1}{u} du =$

$-x \cot x + \ln |u| + c$

$\boxed{-x \cot x + \ln |\sin x| + c}$

(5) $\int x^2 e^x \, dx$

$$u = x^2 \qquad dv = e^x dx$$
$$du = 2x\,dx \qquad v = e^x$$

$$= x^2 e^x - \int e^x \cdot 2x \, dx \qquad \begin{array}{ll} u = 2x & dv = e^x dx \\ du = 2dx & v = e^x \end{array}$$

$$= 2xe^x - \int e^x 2dx \rightarrow 2xe^x - 2e^x$$

$$= x^2 e^x - (2xe^x - 2e^x) \rightarrow x^2 e^x - 2xe^x + 2e^x + C$$

or

$\int x^2 e^x \, dx$

$$\begin{array}{ccc} \text{derive} & & \text{integrate} \\ x^2 & + & e^x \\ 2x & - & e^x \\ 2 & + & e^x \\ 0 & & e^x \end{array}$$

$$\therefore \boxed{x^2 e^x - 2xe^x + 2e^x + C}$$

(6) $\int e^x \sin x \, dx$

$$u = e^x \qquad dv = \sin x \, dx$$
$$du = e^x \, dx \qquad v = -\cos x$$

$$e^x \cdot (-\cos x) - \int -\cos x \cdot e^x \, dx$$

$$= e^x \cdot (-\cos x) + \int \cos x \, e^x \, dx$$

$$u = e^x \qquad dv = \cos x \, dx$$
$$du = e^x \, dx \qquad v = \sin x$$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x + C$$

$$\int e^x \sin x \, dx = \frac{-e^x \cos x + e^x \sin x + C}{2}$$

Techniques of Integration

Trigonometric Integrals

A. $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$

1) If n is odd:
   a) Factor out a sinx or a cosx.
   b) The remaining part will be raised to an even power.
   c) Use the appropriate identity as listed below.
   $$\sin^2\theta = 1-\cos^2\theta$$
   $$\cos^2\theta = 1-\sin^2\theta$$
   d) Replace the even power with appropriate identity, foil, and distribute the factor.
   e) Use "u du" substitution where needed.
2) If n is even:
   a) Use the half-angle identities as listed below.
   $$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$
   $$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$
   b) Foil and integrate. (You may need the half-angle identities again.)

B. $\int \sin^m x\cos^n x\,dx$

1) One of the exponents (m or n) is odd.
   a) Factor a trig functions from the one with the odd power.
   b) The remaining factor should be substituted the appropriate identity from below.
   $$\sin^2\theta = 1-\cos^2\theta$$
   $$\cos^2\theta = 1-\sin^2\theta$$
   c) Foil (if necessary), distribute, and integrate.

C. $\int \tan^m x\,dx$ or $\int \sec^m x\,dx$

1) $\tan^2 x = \sec^2 x - 1$
2) $\sec^2 x = \tan^2 x + 1$

D. Other Identities

1) $\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$
2) $\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$
3) $\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$

8.3 Examples

① $\int \sin^2 x \, dx =$ use half angle identity

$$\int \frac{1-\cos 2x}{2} dx \qquad \begin{aligned} u &= 2x \\ du &= 2dx \end{aligned}$$

$$\frac{1}{2} \cdot \frac{1}{2} \int 1 - \cos 2x \; 2dx = \frac{1}{4} \int 1 - \cos u \, du$$

$$\frac{1}{4} \int 1 dx - \int \cos u \, du = \boxed{\frac{x}{2} - \frac{1}{4} \sin 2x + C}$$

② $\int \cos^3 x \, dx =$ odd exp factor out trig

$$\int \cos^2 x \cos x \, dx = \text{Identity}$$

$$\int (1 - \sin^2 x) \cos x \, dx = \int \cos x - \sin^2 x \cos x \, dx$$

$$= \int \cos x \, dx - \int \sin^2 x \cos x \, dx \qquad \begin{aligned} u &= \sin x \\ du &= \cos x \end{aligned}$$

$$= -\sin x + C - \int u^2 du = -\sin x + C - \frac{u^3}{3}$$

$$\boxed{= -\sin x - \frac{1}{3} \sin^3 x + C}$$

(3) $\int \cos^2 x \, dx =$ Use half angle identity

$= \int \frac{1+\cos 2x}{2} = \frac{1}{2}\int 1+\cos 2x \, dx$ $\quad U = 2x$, $dU = 2dx$

$= \frac{1}{2}\int 1 \, dx + \frac{1}{4}\int \cos 2x \cdot 2dx$

$= \boxed{\frac{1}{2}x + \frac{1}{4}\sin 2x + C}$

(4) $\int \sin^3 x \cos^4 x \, dx$ factor out odd trig

$\int \sin^2 x \cos^4 x \sin x \, dx$ change $\sin^2 x$

$\int (1-\cos^2 x)\cos^4 x \sin x \, dx$ distribute cos

$\int (\cos^4 x - \cos^6 x) \sin x \, dx$

$\int \cos^4 x \sin x \, dx - \int \cos^6 x \sin x \, dx$

$U = \cos x$

$du = -\sin x\,dx$

$$-\int \cos^4 x \cdot (-\sin x\,dx) - \left(-\int \cos^6 x \cdot (-\sin x\,dx)\right)$$

$$-\int u^4\,du + \int u^6\,du = -\frac{u^5}{5} + \frac{u^7}{7} + C$$

$$\boxed{-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C}$$

5 $$\int \cos^3(3\theta)\sin^{-2}(3\theta)\,d\theta = \int \cos^2(3\theta)\sin^{-2}(3\theta)\cos(3\theta)$$

$$= \int (1-\sin^2(3\theta))\cdot\sin^{-2}(3\theta)\cos(3\theta)\,d\theta$$

$$= \int (\sin^{-2}(3\theta)-1)\cdot\cos(3\theta)\,d\theta$$

$$= \int \sin^{-2}(3\theta)\cos(3\theta)\,d\theta - \int \cos(3\theta)\,d\theta$$

$U = \sin 3\theta$

$du = 3\cos 3\theta$

$V = 3\theta$

$dV = 3d\theta$

$$= \frac{1}{3}\int \sin^{-2}(3\theta)\cdot 3\cos(3\theta)\,d\theta - \frac{1}{3}\int \cos(3\theta)\cdot 3\,d\theta$$

$$= \frac{1}{3}\int u^{-2}\,du - \frac{1}{3}\int \cos(V)\,dV$$

$$= -\frac{1}{3} U^{-1} - \frac{1}{3}\sin(V) + C$$

$$= \boxed{-\frac{1}{3}\sin^{-1}(3\theta) - \frac{1}{3}\sin(3\theta) + C}$$

(6) $\int \sqrt{1-\cos 2x}\, dx =$ $\quad \sin^2 x = \frac{1-\cos 2x}{2}$

$$\sqrt{2}\sin x = \sqrt{1-\cos 2x}$$

$$\sqrt{2}\int \sin x\, dx = \boxed{-\sqrt{2}\cos x + C}$$

if $(x-3)(x+3) = x^2-9$

$(x-3)^{1/2}(x+3)^{1/2} = (x-9)^{1/2}$

$$\int \sqrt{1+\sin x}\,dx$$

$$\int \sqrt{1+\sin x}\cdot\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}} = \int \frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}}$$

$$= \int \frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin x}}\,dx = \int \frac{\cos x}{\sqrt{1-\sin x}}\,dx$$

$$= -\int -(1-\sin x)^{-1/2}\cos x\,dx$$

$u = 1-\sin x$

$du = -\cos x\,dx$

$$= -\int u^{-1/2}\,du = -\frac{u^{1/2}}{\frac{1}{2}} + C$$

$$= -2(1-\sin x)^{1/2} + C$$

$$-2\sqrt{1-\sin x} + C$$

(7) $\int \sec^6 x \, dx$

$\int \sec^2 x \sec^2 x \sec^2 x \, dx$

$\int (\tan^2 x + 1)(\tan^2 x + 1) \sec^2 dx$

$\int (\tan^4 x + 2\tan^2 x + 1) \sec^2 x \, dx$

$\int \tan^4 x \sec^2 x \, dx + 2\int \tan^2 x \sec^2 x \, dx + \int \sec^2 x \, dx$

$u = \tan x$ $\quad$ $u = \tan x$

$du = \sec^2 x \, dx$ $\quad$ $du = \sec^2 x \, dx$

$\int u^4 du + 2\int u^2 du + \int \sec^2 x \, dx$

$= \frac{1}{5}u^5 + \frac{2}{3}u^3 + \tan x + C$

$= \boxed{\frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C}$

⑧ $\int \cos x \cos 7x \, dx$

$$\cos x \cos 7x = \frac{1}{2}\left(\cancel{\cos(1-7)x + \cos(1+7)x}\right)$$

$$\cos x \cos 7x = \frac{1}{2}\cos(-6x) + \frac{1}{2}\cos(8x)$$

$$\frac{1}{2}\cdot\frac{-1}{6}\int \cos(-6x)\,dx\cdot(-6) + \frac{1}{2}\cdot\frac{1}{8}\int \cos(8x)\,dx\cdot(8)$$

$u = -6x$
$du = -6dx$

$u = 8x$
$du = 8dx$

$$\frac{1}{2}\sin(-6x)\cdot\left(\frac{-1}{6}\right) + \frac{1}{2}\cos(8x)\cdot\left(\frac{1}{8}\right)$$

$$\boxed{-\frac{1}{12}\sin(-6x) + \frac{1}{16}\cos(8x) + C}$$

(9) $\int \sec^3 x \tan^3 x \, dx$

$$\int \sec^2 x \tan^2 x \sec x \tan x \, dx$$

$$\int \sec^2 x \left(\sec^2 x - 1\right) \sec x \tan x \, dx$$

$$\int \left(\sec^4 x - \sec^2 x\right) \sec x \tan x \, dx$$

$$\int \sec^4 x \sec x \tan x \, dx - \int \sec^2 x \sec x \tan x \, dx$$

$u = \sec x$ $u = \sec x$

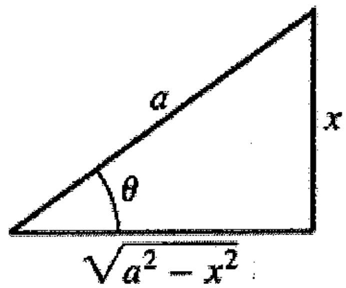
$du = \sec x \tan x \, dx$ $du = \sec x \tan x \, dx$

$$\int u^4 \, du - \int u^2 \, du$$

$$= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C$$

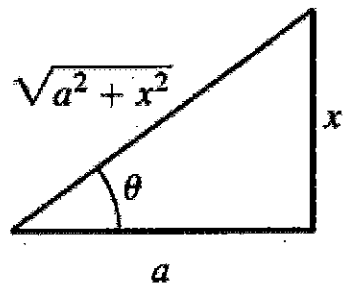
$$\boxed{= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C}$$

Section 8.4 – Trigonometric Substitutions



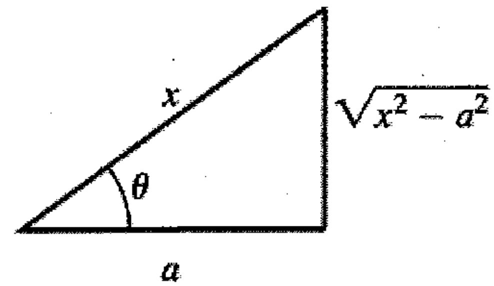
$x = a \sin\theta$

$\sqrt{a^2 - x^2} = a|\cos\theta|$



$x = a \tan\theta$

$\sqrt{a^2 + x^2} = a|\sec\theta|$



$x = a \sec\theta$

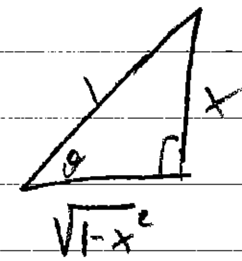
$\sqrt{x^2 - a^2} = a|\tan\theta|$

Radical	Substitution	Restriction on θ
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$0 \le \theta \le \pi, \quad \theta \ne \frac{\pi}{2}$

8.4 Trig Substitution

Examples:

① $\int \frac{1}{\sqrt{1-x^2}}\,dx$



$a^2=1 \quad a=1$

$\sin\theta = \frac{x}{1}$

$x=\sin\theta$

$\theta=\sin^{-1}x$

$x=\sin\theta$

$dx=\cos\theta\,d\theta$

$$\int \frac{1}{\sqrt{1-(\sin\theta)^2}}\cdot\cos\theta\,d\theta$$

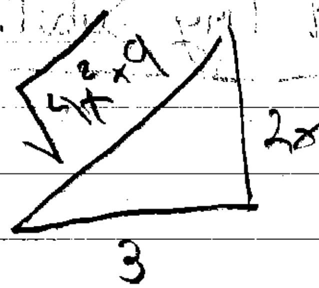
$$\int \frac{1}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\,d\theta = \int \frac{1}{\sqrt{\cos^2\theta}}\cos\theta\,d\theta$$

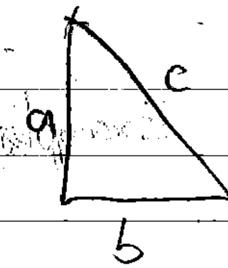
$$= \int \frac{1}{\cancel{\cos\theta}}\cdot\cancel{\cos\theta}\,d\theta = \int d\theta = \theta + C$$

$\theta = \sin^{-1}x$

$$\boxed{\sin^{-1}x + C}$$

(2) $\int \frac{-5}{4x^2+9}\,dx$





$a^2=9$ $\qquad$ $a^2+b^2=c^2$

$4x^2 \to 2x \quad a \to 3$

$\tan\theta = \frac{2x}{3} \qquad \theta = \tan^{-1}\left(\frac{2x}{3}\right)$

$\frac{3}{2}\tan\theta = x$

$\frac{3}{2}\sec^2\theta\, d\theta = dx$

$-5\int \frac{1}{4\left(\frac{3}{2}\tan\theta\right)^2+9}\cdot\frac{3}{2}\sec^2\theta\, d\theta$

$-5\int \frac{1}{4\cdot\frac{9}{4}\tan^2\theta+9}\cdot\frac{3}{2}\sec^2\theta\, d\theta$

$-5\int \frac{1}{9\tan^2\theta+9}\cdot\frac{3}{2}\sec^2\theta\, d\theta$

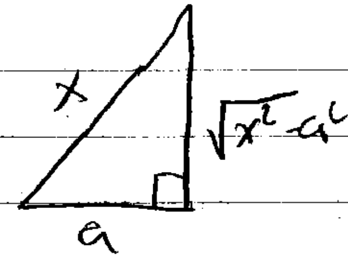
$$5\int \frac{}{9(\tan^2\theta+1)}\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$$-5\int \frac{1}{3\sec^2\theta}\,\frac{1}{2}\sec^2\theta\,d\theta$$

$$-\frac{5}{6}\int \frac{\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}\,d\theta \longrightarrow -\frac{5}{6}\int d\theta$$

$$-\frac{5}{6}\theta + C \longrightarrow \boxed{-\frac{5}{6}\tan^{-1}\left(\frac{2x}{3}\right)+C}$$

(3) $$\int \frac{\sqrt{x^2-4}}{x}\,dx$$



$a^2 = 4$

$a = 2$

$x = a\sec\theta \qquad \theta = \sec^{-1}\left(\frac{x}{2}\right)$

$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{(2\sec\theta)^2-4}}{\cancel{2\sec\theta}} \cdot \left(\cancel{2\sec\theta}\tan\theta\,d\theta\right)$$

$$\int \sqrt{4\sec^2\theta-4} \cdot \tan\theta\,d\theta$$

$$\int \sqrt{4(\sec^2\theta-1)} \cdot \tan\theta\,d\theta$$

$$= \int 2\sqrt{\sec^2\theta-1} \cdot \tan\theta\,d\theta$$

$$= \int 2\sqrt{\tan^2\theta} \cdot \tan\theta\,d\theta$$

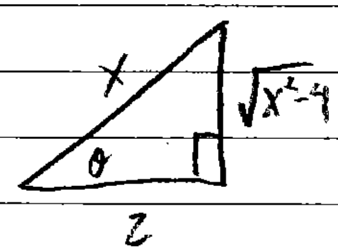
$$\int 2(\tan\theta)\cdot\tan\theta\, d\theta$$

$$2\int \tan^2\theta\, d\theta = 2\int \sec^2\theta - 1\, d\theta$$

$$2\int \sec^2\theta\, d\theta - 2\int d\theta = 2\tan\theta - 2\theta + C$$

$$\boxed{2\tan\left(\sec^{-1}\left(\frac{x}{2}\right)\right) - 2\sec^{-1}\frac{x}{2} + C}$$

You can also use



$$\tan\theta = \frac{\sqrt{x^2-4}}{2}$$

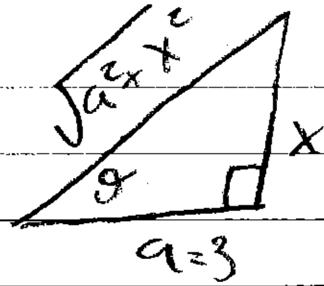
$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2-4}}{2}\right)$$

$$2\tan\theta - 2\theta + C$$

$$\boxed{= 2\left(\frac{\sqrt{x^2-4}}{2}\right) - 2\left(\tan^{-1}\left(\frac{\sqrt{x^2-4}}{2}\right) + C\right.}$$

$\cos\theta = \frac{3}{\sqrt{9+x^2}} \quad \sec\theta = \frac{\sqrt{9+x^2}}{3}$

(4) $\int \frac{1}{\sqrt{9+x^2}}\,dx$



$a^2 = 9 \quad a = 3$

$x = a\tan\theta$

$x = 3\tan\theta \qquad \theta = \tan^{-1}\left(\frac{x}{3}\right)$

$dx = 3\sec^2\theta\,d\theta$

$$\int \frac{1}{\sqrt{9+(3\tan\theta)^2}} \cdot 3\sec^2\theta\,d\theta$$

$$\int \frac{1}{3\sqrt{\tan^2\theta+1}} \cdot 3\sec^2\theta\,d\theta$$

$$\int \frac{1}{3\sqrt{\sec^2\theta}} \cdot 3\sec^2\theta\,d\theta$$

$$\int \frac{1}{\sec\theta} \cdot \sec^2\theta\,d\theta \longrightarrow \int \sec\theta\,d\theta$$

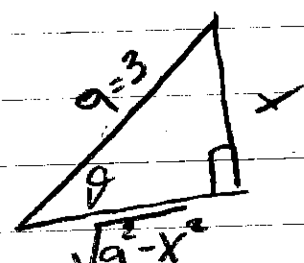
$$\int \frac{\sqrt{9+x^2}}{3}\,d\theta \longrightarrow \int \frac{3+x}{3}\,d\theta$$

$$\int \frac{3}{3} + \frac{x}{3}\, d\theta \rightarrow \int 1 + \frac{x}{3}\, d\theta \rightarrow \int d\theta + \int \frac{x}{3}\, d\theta$$

Correct Way   Use Integral Table #95

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

5 $$\int \frac{1}{x^2\sqrt{9-x^2}}\, dx$$



$a^2 = 9$

$a = 3$

$x = a \sin\theta$

$x = 3\sin\theta$ $\theta = \sin^{-1}\left(\frac{x}{3}\right)$

$dx = 3\cos\theta \, d\theta$

$x^2 = 9\sin^2\theta$

$$\int \frac{1}{9\sin^2\theta\sqrt{9-9\sin^2\theta}} \cdot 3\cos\theta \, d\theta$$

$$\int \frac{3\cos\theta \, d\theta}{9\sin^2\theta \cdot 3\sqrt{1-\sin^2\theta}}$$

$$\int \frac{3\cos\theta \, d\theta}{27\sin^2\theta\sqrt{\cos^2\theta}} \rightarrow \frac{1}{9}\int \frac{\cos\theta \, d\theta}{\sin^2\theta \cdot \cos\theta}$$

$$\frac{1}{9}\int \frac{1 \, d\theta}{\sin^2\theta} = \frac{1}{9}\int \csc^2\theta \, d\theta$$

$$= \frac{1}{9} - \cot\theta + C$$

$$-\frac{1}{9}\cot\left(\sin^{-1}\left(\frac{x}{3}\right)\right) + C$$

8.5 Fraction Decomposition

$$\int\left(\frac{2}{x+3}+\frac{3}{x-3}\right)dx = \int\frac{2}{x+3}\,dx+\int\frac{3}{x-3}\,dx$$

$$\int\frac{5x+3}{x^2-9}\,dx$$ is a little harder but there is a tech

find the common denominator for
$x^2-9 = (x+3)(x-3)$

Set placeholder coefficients for the numerator of each factor + solve
use A + B

$$\frac{5x+3}{x^2-9}=\frac{A}{(x+3)}\underset{\text{always plus}}{+}\frac{B}{(x-3)}$$

$$\frac{5x+3\,\cancel{(x^2-9)}}{\cancel{x^2-9}}=\frac{A\cancel{(x+3)}(x-3)}{\cancel{(x+3)}}+\frac{B(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$5x+3 = A(x-3)+B(x+3)$$

Solve for A by letting $x=-3$

$$5(-3)+3 = A(-3-3)+B\underbrace{(-3+3)}_{0}$$

$$-12 = -6A \qquad \boxed{A=2}$$

Solve for B by letting $x=3$

$5(3)+3 = A(\cancel{3-3}) + B(3+3)$

$18=6B \quad B=3$

So ...

$$\frac{5x+3}{x^2-9} = \frac{2}{x+3} + \frac{3}{x-3} \longleftarrow$$ We can easily integrate

* Reminder on Long Division

$$\int \frac{x^5+2x^3-x+1}{x^3-4x} =$$

$$\begin{array}{r} x^2+6 \\ x^3-4x \overline{\big) x^5+2x^3-x+1} \\ -\left(x^5-4x^3\right) \\ \hline 6x^3-x \\ -\left(6x^3-24x\right) \\ \hline 23x+1 \end{array}$$ Remainder

$$\int \left(x^2+6+\frac{23x+1}{x^3-4x}\right)dx$$

$$=\int x^2 dx + \int 6dx + \int \frac{23x+1}{x^3-4x} \longrightarrow$$ Break Apart

Techniques of Integration
Partial Fractions – Lesson Problems

Non-repeating Linear Factors

1) $\frac{5x+3}{x^2-9} = \; = \frac{A}{x+3} + \frac{B}{x-3} = (x+3)(x-3)$

$5x+3 = A(x-3) + B(x+3)$

$x=3 \quad 5(3)+3 = A(\cancel{3-3})^0 + B(3+3)$

$= 18 = 6B \quad \boxed{B=3}$

for A $x=-3 = 5(-3)+3 = A(-3-3)+B(\cancel{-3+3})^0$

$-12=-6A \quad \boxed{A=2}$

$$\boxed{\frac{5x+3}{x^2-9} = \frac{2}{x+3} + \frac{3}{x-3}}$$

2) $\int \frac{3x-1}{x^2-x-6}dx \quad (x-3)(x+2)$

$\frac{3x-1}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2} \quad \therefore \; 3x-1 = A(x+2) + B(x-3)$

for B let $x=-2 \rightarrow 3(-2)-1 = A(\cancel{-2+2})^0 + B(-2-3) \rightarrow -7 = -5B \quad B = \frac{7}{5}$

for A let $x=3 \rightarrow 3(3)-1 = A(3+2) + B(\cancel{3-3})^0 \rightarrow 8 = 5A \rightarrow A = \frac{8}{5}$

$$\int \frac{8/5}{x-3} + \int \frac{7/5}{x+2} \longrightarrow \frac{8}{5}\int \frac{1}{x-3}dx + \frac{7}{5}\int \frac{1}{x+2}dx \rightarrow \boxed{\frac{8}{5}\ln(x-3) + \frac{7}{5}\ln(x+2) + C}$$

$u = x-3 \quad u = x+2$

3) $\int \frac{x+4}{x^2+x}dx \quad x(x+1)$

$\frac{x+4}{x^2+x} = \frac{A}{x} + \frac{B}{x+1} \quad \therefore \quad x+4 = A(x+1) + B(x)$

for A let $x=0 \rightarrow 0+4 = A(0+1) + \cancel{B(0)}^0 \rightarrow \boxed{4=A}$

for B let $x=-1 \rightarrow -1+4 = A(\cancel{-1+1})^0 + B(-1) \rightarrow 3=-B \rightarrow \boxed{B=-3}$

$$\int \frac{4}{x}dx + \int \frac{-3}{x+1}dx = \boxed{4\ln|x| - 3\ln|x+1| + C}$$

Non-repeating Linear Factors

4) $\int \frac{5x+3}{x^3-2x^2-3x}dx$ $x(x^2-2x-3) = x(x-3)(x+1)$

$$\frac{5x+3}{x^3-2x^2-3x} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+1)}$$

$$5x+3 = A(x-3)(x+1) + B(x)(x+1) + C(x)(x-3)$$

$$x=0 \to 5(0)+3 = A(0-3)(0+1) + \cancel{B(0)(0+1)} + \cancel{C(0)(0-3)} = A=-1$$

$$x=-1 \to 5(-1)+3 = \cancel{A(-1-3)(-1+1)} + \cancel{B(-1)(-1+1)} + C(-1)(-1-3) = C=-2$$

$$x=+3 \to 5(3)+3 = \cancel{A(3-3)(3+1)} + B(3)(3+1) + \cancel{C(3)(3-3)} = B=\frac{18}{12}=\frac{3}{2}$$

$$\int \frac{-1}{x}dx + \int \frac{3}{2}\frac{1}{(x-3)}dx + \int \frac{-2}{(x+1)}dx = \boxed{-\ln|x| + \frac{3}{2}\ln|x-3| - 2\ln(x+1) + C}$$

5) $\int \frac{7x^2+2x-3}{(2x-1)(3x+2)(x-3)}dx$

Repeating Linear Factors

6) $\int \frac{x}{x^2-6x+9}\,dx$ $\quad (x-3)(x-3) = (x-3)^2$

$$\frac{x(x-3)^2}{(x-3)^2} = \frac{A(x-3)^2}{(x-3)} + \frac{B(x-3)^2}{(x-3)^2} = \quad x = A(x-3) + B$$

$$x = +3 \rightarrow \quad 3 = A(\cancel{3-3})^0 + B \qquad B=3$$

$$x=0 \rightarrow \quad 0 = A(0-3)+3 \rightarrow -3A = -3 \rightarrow A=1$$

$$\int \frac{1}{(x-3)}dx + \int \frac{3}{(x-3)^2} = \ln|x-3| + 3\int \frac{1}{u^2}du = \ln|x-3| + 3\int u^{-2}$$

$$\ln|x-3| + 3(-u^{-1}) = \boxed{\ln|x-3| - \frac{3}{x-3} + C}$$

7) $\int \frac{3x^2-8x+13}{(x+3)(x-1)^2}\,dx$ $\quad (x+3)(x-1)^2$

$$\frac{3x^2-8x+13}{(x+3)(x-1)^2} = \frac{A}{(x+3)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$3x^2-8x+13 = A(x-1)^2 + B(x+3)(x-1) + C(x+3)$$

$$x=-3 \rightarrow 3(-3)^2-8(-3)+13 = A(-3-1)^2 + B(\cancel{-3+3})^0(-3-1) + C(\cancel{-3+3})^0$$

$$\rightarrow 27+24+13 = A\,16 \qquad \boxed{A=4}$$

$$x=+1 \rightarrow 3(1)^2-8(1)+13 = A(\cancel{1-1})^2 + B(1+3)(\cancel{1-1})^0 + C(1+3) = \boxed{C=2}$$

$$x=0 \rightarrow 3(0)^2-8(0)+13 = 4(0-1)^2 + B(0+3)(0-1) + 2(0+3) = \quad \boxed{B=-1}$$

$$\int \frac{4}{x+3} + \int \frac{-1}{x-1} + \int \frac{2}{(x-1)^2} = 4\int \frac{1}{x+3} - \int \frac{1}{x-1} + 2\int \frac{1}{(x-1)^2}$$

$$4\ln|x+3| - \ln|x-1| - \frac{2}{x-1}$$

Repeating Linear Factors

8) $\int \frac{x^2-3}{x^3+2x^2}dx$

9) $\int \frac{5x+7}{x^2+4x+4}dx$ $(x+2)(x+2)$

Repeating Linear Factors

10) $\int \frac{3x^2 - 21x + 32}{x^3 - 8x^2 + 16x}dx$

11) $\int \frac{x^2 + 19x + 10}{2x^4 + 5x^3}dx$

Quadratic and Repeating Quadratic Factors

12) $\int \frac{6x^2-3x+1}{4x+1)(x^2+1)}dx$ $= \frac{6x^2-3x+1}{(4x+1)(x^2+1)} = \frac{A}{4x+1} + \frac{Bx+C}{x^2+1}$

$6x^2-3x+1 = A(x^2+1) + (Bx+C)(4x+1)$

$x = -\frac{1}{4} \rightarrow A(-\frac{1}{4}^2+1) + (B\cancel{-\frac{1}{4}+C})\cancel{(4(-\frac{1}{4})+1)} = \frac{3}{8} + \frac{3}{4} + 1 = A(\frac{1}{16}+1)$

$\frac{17}{16}A = \frac{17}{8} \rightarrow A = \frac{16}{8} \rightarrow \underline{A=2}$

$x=0 \rightarrow 6(0)^2-3(0)+1 = 2(0^2+1)+(B0+C)(4(0)+1)$

$1 = 2 + C(1) \rightarrow \underline{C=-1}$

$x=1 \rightarrow 6(1)^2-3(1)+1 = 2(1^2+1)+(B(1)+(-1))(4(1)+1)$

$4 = 4 + (B-1)(4+1) \rightarrow 4 = 4+5B-5$

$5B = 4-4+5 \quad \underline{B=1}$

$\int\frac{2}{4x+1} + \int\frac{x-1}{x^2+1}$

$\int\frac{2}{4x+1} + \int\frac{x}{x^2+1} + \int\frac{-1}{x^2+1}$

$\frac{1}{2}\int\frac{1}{u} + \frac{1}{2}\int\frac{1}{u} - \int\frac{1}{x^2+1}$

$\frac{1}{2}\ln|4x+1| + \frac{1}{2}\ln|x^2+1| - \tan^{-1}x$

13) $\int \frac{x^3+1}{x^2+2x}dx$

Long Division

$$\begin{array}{r} x-2 \\ x^2+2x \overline{) x^3+1} \\ -x^3 \mp 2x^2 \\ \hline -2x^2+1 \\ \pm 2x^2 \pm 4x \\ \hline +4x+1 \end{array}$$ Remainder

$\int x-2+\frac{(+4x+1)}{x^2+2x}$

$\int x\,dx - \int 2\,dx + \int\frac{+4x+1}{x^2+2x} \rightarrow x(x+2)$

$\frac{+4x+1}{x^2+2x} = \frac{A}{x} + \frac{B}{(x+2)}$

$4x+1 = A(x+2)+Bx$

$x=0 \rightarrow 4(0)+1 = A(0+2)+0$

$A = \frac{1}{2}$

$x=-2 \rightarrow 4(-2)+1 = A(-2+2)+B(-2)$

$\rightarrow -7 = -2B \quad B = \frac{7}{2}$

$\int x\,dx - \int 2\,dx + \frac{1}{2}\int\frac{1}{x}dx + \frac{7}{2}\int\frac{1}{x+2}dx$

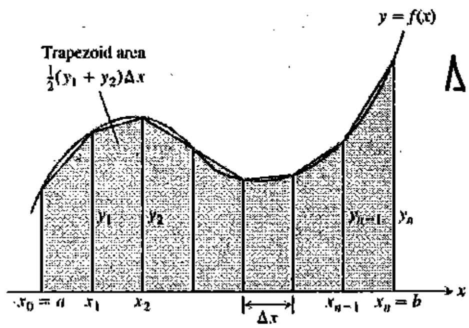
$= \boxed{\frac{1}{2}x^2 - 2x + \frac{1}{2}\ln|x| + \frac{7}{2}\ln|x+2| + C}$

Long Division

14) $\int \frac{x^3+2}{x^2-1}\,dx$

15) $\int \frac{x^4+8x^2+8}{x^3-4x}\,dx$

Trapezoidal Rule



$$\Delta x \cdot \left(\frac{y_0+y_1}{2}\right) + \Delta x \cdot \left(\frac{y_1+y_2}{2}\right)$$

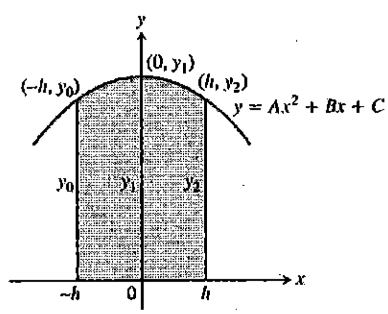
Area Approximation:

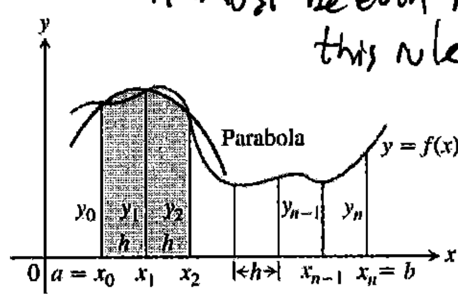
$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_3) + \cdots + 2(f(x_{n-1}) + f(x_n))$$

Where $\Delta x = \frac{b-a}{n}$ and n represents the number of trapezoids.

Simpson's Rule

n must be even for this rule





Area Approximation:

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

Where $\Delta x = \frac{b-a}{n}$ and n represents an even number of partitions.

Error Estimation

THEOREM 1—Error Estimates in the Trapezoidal and Simpson's Rules If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}. \qquad \text{Trapezoidal Rule}$$

b + a = upper | lower limit
n = # of traps

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}. \qquad \text{Simpson's Rule}$$

8.7 Trapezoid & Simpsons Rule

$$\int_{-1}^{1} (t^3+1)\,dt \quad n=4 \quad = \quad \longleftarrow \text{Trap}$$

$$\Delta X = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{1}{2}$$

$X =$ from -1 to 1 in ΔX increments

$$X = -1,\ -\tfrac{1}{2},\ 0,\ \tfrac{1}{2},\ 1$$

$$\frac{\Delta x}{2}\left(f(x_0)+2f(x_1)\ldots\right)$$

$$\frac{\frac{1}{2}}{2}\left(0+2(7/8)+2(1)+2(9/8)+2\right)$$

$$1/4(8) = \boxed{2}$$

Error for Trap

$$= E_T \leq \frac{m(b-a)^3}{12n^2}$$

find $f''(x)$ to get m

$f''(x) = 6t$

$f(-1) = -6$ & $f''(1) = 6$

$m = 6$

$$E_t \leq \frac{6(1+1)^3}{12(4)^2} = \frac{48}{192}$$

$$E_T \leq .25$$

Simpson rule

$$\int_{-1}^{1} (t^3+1)\,dt \qquad n=4 \qquad \Delta X = \frac{1}{2}$$

$X = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$

$$\frac{1/2}{3}\left(0 + 4(.875) + 2(1) + 4(1.125) + 2\right)$$

$$\frac{1}{6}(12) = \boxed{2}$$

Simpson Error

$$E_s \leq \frac{M(b-a)^5}{180\,n^4}$$

M = get the 4th derivative

$f''''(X) = 0$

$$E_s \leq \frac{0(1+1)^5}{180(4)^4} = 0$$

$$\int_0^1 \sin \pi t \, dt$$

$n = 4$

$\Delta x = \frac{1-0}{4} = \frac{1}{4}$

$X = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

$$\frac{1/4}{2}\left(0 + 2\left(\frac{\sqrt{2}}{2}\right) + 2(1) + 2\left(\frac{\sqrt{2}}{2}\right) + 0\right)$$

$$\frac{1}{8}(4.8284) = \boxed{.60355}$$

$$E_T \leq \frac{M(b-a)^3}{12n^2}$$

$f'(x) = \pi \cos(\pi t)$

$f''(x) = -\pi^2 \sin(\pi t)$

$f''(1) = -\pi^2 \sin(\pi 1)$

$f''(1) = 0 \quad f''(0) = 0$

$f'''(x) = -\pi^3 \cos \pi t$

$0 = -\pi^3 \cos \pi t$

$\cos^{-1} 0 = \pi t$

$\pi/2 = \pi t$

$t = 1/2 \quad \boxed{f''(\tfrac{1}{2}) = \pi^2 = M}$

$$E_T = \frac{\pi^2(1-0)^3}{12(4)^2}$$

$$\boxed{E_T = 0.052}$$

$$\int_0^1 \sin \pi t \, dt \qquad \Delta x = 1/4$$
$$x = 0, 1/4, 1/2, 3/4, 1$$

$$\frac{1/4}{3}\left(0 + 4\left(\frac{\sqrt{2}}{2}\right) + 2(1) + 4\left(\frac{\sqrt{2}}{2}\right) + 0\right)$$

$$\frac{1}{12}(7.6569) = \boxed{.63807}$$

$$E_s \leq \frac{m(b-a)^5}{180n^4}$$

$$E_s \leq \frac{\pi^4(1-0)^5}{180(4)^4}$$

$$\boxed{E_s \leq .0021}$$

$$f''(x) = -\pi^2 \sin(\pi t)$$
$$f'''(x) = -\pi^3 \cos(\pi t)$$
$$f''''(x) = \pi^4 \sin(\pi t)$$
$$f''''(1) = \pi^4 \sin(\pi 1)$$
$$f'''''(x) = \pi^5 \cos(\pi t)$$
$$0 = \pi^5 \cos \pi t$$
$$\frac{0}{\pi^5} = \cos \pi t$$
$$0 = \cos \pi t$$
$$\cos^{-1}(0) = \pi t$$
$$\frac{\pi}{2} = \pi t$$
$$t = 1/2$$
$$f''''(1/2) = \pi^4 \sin\left(\frac{\pi}{2}\right)$$
$$= \pi^4 \cdot 1$$
$$f''''(1/2) = \pi^4 = \text{Max}$$

8.6 Integral Tables

① $\displaystyle\int \frac{1}{x\sqrt{x+4}}\,dx =$ T-2 #29 (a)

$a=1 \quad b=4$

$$\boxed{\frac{1}{\sqrt{4}} \ln \left| \frac{\sqrt{x+4} - \sqrt{4}}{\sqrt{x+4} + \sqrt{4}} \right| + C}$$

② $\displaystyle\int x(7x+5)^{3/2}\,dx$ T-1 #22

$a=7 \quad b=5 \quad n=3/2$

$$= \frac{(7x+5)^{5/2}}{7^2}\left[\frac{7x+5}{3/2+2} - \frac{5}{3/2+1}\right] + C$$

$$= \frac{(7x+5)^{5/2}}{49}\left[\frac{2(7x+5)}{7} - \frac{2(5)}{5}\right] + C$$

$$\boxed{= \frac{(7x+5)^{5/2}}{49}\left[\frac{2(7x+5)}{7} - 2\right] + C}$$

③ $\displaystyle\int \frac{1}{x^2\sqrt{4x-9}}\,dx$ T-2 #81

$a=4 \quad b=-9$

$\displaystyle -\frac{\sqrt{4x-9}}{-9x} - \frac{4}{2(-9)} \int \frac{1}{x\sqrt{4x-9}}\,dx + C$ #29b

④ $\displaystyle\int \frac{1}{y\sqrt{3+(\ln y)^2}}$ $\quad u=\ln y$, $\quad du=\frac{1}{y}\,dy$

$\displaystyle\int \frac{1}{\sqrt{3+u^2}}\,du$ $\quad a^2=3$, $\quad a=\sqrt{3}$

$\displaystyle = \sinh^{-1}\left(\frac{u}{\sqrt{3}}\right) + C = \sinh^{-1}\left(\frac{\ln y}{\sqrt{3}}\right) + C$

$\displaystyle = \boxed{\ln\left((\ln y) + \sqrt{3+(\ln y)^2}\right) + C}$

⑤ $\int 8\cos^4(2\pi t)\,dt = 8\int \cos^4(2\pi t)\,dt$ T-3 # 68

$a = 2\pi$

$n = 4$

$$8\left[\frac{\cos^3(2\pi t)\cdot\sin(2\pi t)}{(4)(2\pi)} + \frac{3}{4}\int \cos^2(2\pi t)\,dt\right]$$

$$\frac{\cos^3(2\pi t)\sin(2\pi t)}{\pi} + 6\int \cos^2(2\pi t)\,dt$$ T-3 # 66

$$\frac{\cos^3(2\pi t)\sin(2\pi t)}{\pi} + 6\left(\frac{t}{2} + \frac{\sin 2(2\pi t)}{8\pi}\right)$$

$$\frac{\cos^3(2\pi t)\sin(2\pi t)}{\pi} + 3t + \frac{3\sin 2(2\pi t)}{4\pi}$$

$$\boxed{\frac{\cos^3(2\pi t)\sin(2\pi t)}{\pi} + 3t + \frac{3}{4\pi}\sin 4\pi t + C}$$

Infinite Limits

If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)dx = \lim_{b \to \infty} \int_a^{b} f(x)dx$$

when replacing, we can pick any value

and

If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_a^{b} f(x)dx$$

If the limit is finite, then the integral converges to that value. = it approaches that area value ever so slightly as we get closer to infinity

If the limit is infinite, then the integral will diverge.

Infinite Limits

If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx = \lim_{t \to -\infty} \int_t^{c} f(x)dx + \lim_{t \to \infty} \int_c^{t} f(x)dx$$

If the both limits are finite, then the integral converges to that value.

If the one of the limits is infinite, then the integral will diverge.

Infinite Integrands

If $f(x)$is continuous on $[a, b)$, and $\lim_{x \to b^-} f(x) = \pm\infty$ then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

If the limit is finite, then the integral converges to that value.

If the limit is infinite, then the integral will diverge.

If $f(x)$is continuous on $(a, b]$, and $\lim_{x \to a^+} f(x) = \pm\infty$ then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

If the limit is finite, then the integral converges to that value.

If the limit is infinite, then the integral will diverge.

Infinite Integrands

If f(x)is continuous on $[a, b]$, but is discontinuous at a point c, where c is in the interval, and $\lim_{x \to c} f(x) = \pm\infty$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{t \to c^-} \int_a^t f(x)dx + \lim_{t \to c^+} \int_t^b f(x)dx$$

If the both limits are finite, then the integral converges to that value.

If the one of the limits is infinite, then the integral will diverge.

8.8 Examples

① $\displaystyle\int_{-\infty}^{1} x e^{-x^2}\,dx = \lim_{t\to-\infty}\int_{t}^{-1} x e^{-x^2}\,dx \qquad u = -\cancel{x}^2 \quad du = -2x\,dx$

$$\lim_{t\to-\infty} -\frac{1}{2}\int_{t}^{-1} -2x e^{-x^2}\,dx = -\frac{1}{2}\int_{t}^{-1} e^{u} = -\frac{1}{2}e^{u}\Big|_{t}^{-1}$$

$$= -\frac{1}{2}e^{-x^2}\Big|_{t}^{-1}$$

$$= \lim_{t\to-\infty}\left(-\frac{1}{2}e^{-(-1)^2} - -\frac{1}{2}e^{-t^2}\right)$$

$$= \lim_{t\to-\infty}\left(-\frac{1}{2}e^{-1} + \frac{1}{2}e^{-t^2}\right)$$

$$= \lim_{t\to-\infty}\left(-\frac{1}{2e} + \frac{1}{2e^{t^2}}\right) = \left(-\frac{1}{2e} + \underbrace{\frac{1}{2(\text{large Value})}}_{0}\right)$$

$$= \left(-\frac{1}{2e} + 0\right) = \boxed{-\frac{1}{2e}}$$

(2) $\displaystyle\int_0^\infty \sin x\,dx = \lim_{t\to\infty}\int_0^t \sin x\,dx$

$= -\cos x \Big|_0^t$

$= \displaystyle\lim_{t\to\infty}\left(-\cos t + \cos(0)\right)$

$\displaystyle\lim_{t\to\infty}\left(-\cos t + 1\right) = \begin{cases} (-(-1)+1) = 2 \\ (-(1)+1) = 0 \end{cases}$ oscillates

$\boxed{\begin{array}{l} -\cos t + 1 \text{ varies from } 0 \text{ to } 2 \\ \text{not a finite value} \\ \therefore \text{ integral diverges} \end{array}}$

③ $$\int_{-\infty}^{\infty} \frac{2}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{2}{1+x^2}\,dx + \int_{0}^{\infty} \frac{2}{1+x^2}\,dx$$

$$\lim_{t\to-\infty} \int_{t}^{0} \frac{2}{1+x^2}\,dx = \lim_{t\to-\infty} 2\int_{t}^{0} \frac{1}{1+x^2}\,dx$$

$$= 2\,\mathrm{Tan}^{-1} x \Big|_{t}^{0} = \lim_{t\to-\infty} \left(2\,\mathrm{Tan}^{-1}(0) - 2\,\mathrm{Tan}^{-1}(t)\right)$$

$$\lim_{t\to-\infty} \left(0 - 2\,\mathrm{Tan}^{-1}(t)\right) = \left(-2\left(-\frac{\pi}{2}\right)\right)$$

$$= \boxed{\pi}$$

Plus

$$\lim_{t\to\infty} \int_{0}^{t} \frac{2}{1+x^2}\,dx = 2\,\mathrm{Tan}^{-1} x \Big|_{0}^{t}$$

$$= \lim_{t\to\infty} \left(2\,\mathrm{Tan}\, t - 2\,\mathrm{Tan}^{-1}(0)\right)$$

$$\left(2\left(\frac{\pi}{2}\right) + 0\right) = \boxed{\pi}$$

$$\pi + \pi = \boxed{2\pi}$$

finite value ∴ converges to 2π

(4) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ integrand is infinite @ $x=1$

$\lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x \Big|_0^t$

$\lim_{t\to 1^-} \left(\sin^{-1}(t) - \sin^{-1}(0)\right)$

$\lim_{t\to 1^-} \left(\sin^{-1}t - 0\right) = \boxed{\frac{\pi}{2}}$ finite limit $\therefore$ converges @ $\frac{\pi}{2}$

(5) $\int_2^3 \frac{1}{(x-2)^{4/3}}$ integrand is infinite @ $x=2$

$\lim_{t\to 2^+} \int_t^3 \frac{1}{(x-2)^{4/3}}\,dx$ $U = x-2$, $dU = dx$ $= \lim_{t\to 2^+} \int^3 U^{-4/3}$

$\lim_{t\to 2^+} -\frac{3}{U^{1/3}} \Big|_t^3 = \lim_{t\to 2^+} \frac{-3}{(x-2)^{1/3}} \Big|_t^3$

$= -3\left(\frac{1}{(3-2)^{1/3}} - \frac{1}{(t-2)^{1/3}}\right) = \lim_{t\to 2^+} -3\left(1 - \frac{1}{(t-2)^{1/3}}\right)$

$-3(1-\infty) \to \boxed{\infty}$ divergent

(6) $\int_{-2}^{1} \frac{1}{x^2}dx$ Integrand is infinite at $x=0$

$$\lim_{t\to 0^-} \int_{-2}^{t} x^{-2}dx + \lim_{t\to 0^+} \int_{t}^{1} x^{-2}dx$$

$$-\frac{1}{x}\Big|_{-2}^{t} + \left[-\frac{1}{x}\Big|_{t}^{1}\right]$$

$$\lim_{t\to 0^-} -\left(\frac{1}{t} - \frac{1}{(-2)}\right) + \lim_{t\to 0^+} -\left[\frac{1}{1} - \frac{1}{t}\right]$$

$$-\left(\infty + \frac{1}{2}\right) - (1 - \infty) = \boxed{\infty}$$ divergent

[illegible]

Section 10.1 – Sequences

Sequences

Definition - A function whose domain is the set of all positive integers.

Finite Sequence - finite number of values or elements

x-inputs will be whole numbers to give these y-outputs $\left\{1, \frac{2}{3}, 6, \frac{7}{8}\right\}$ $\{2,4,6,8,10\}$

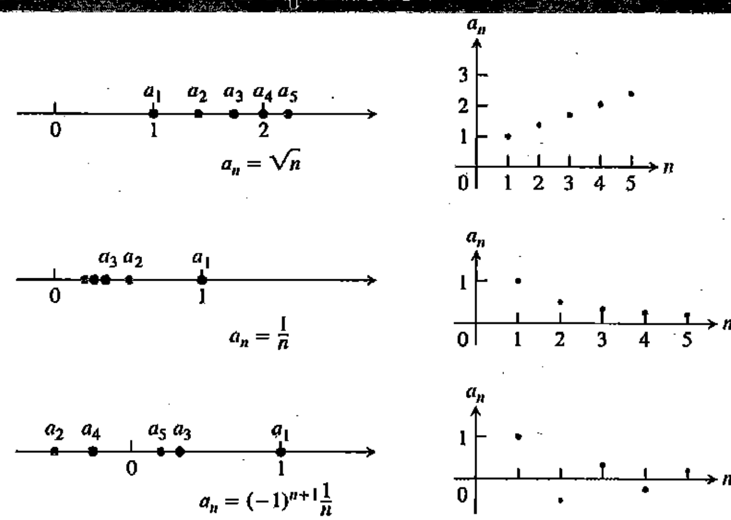
Infinite Sequence - infinite number of values or elements

$\{4,7,8,13,\ldots\}$ $\{1,3,5,7,9\ldots\}$

Notation - a_n or $\{b_n\}$

Section 10.1 – Sequences

Definition - A function whose domain is the set of all positive integers.



Section 10.1 - Sequences

Three Types of Sequences

Specified - enough information is given to find a pattern

$\{1,4,7,10,13,\ldots\}$ $\{2,5,11,23,47,\ldots\}$

+3 double add 1

Explicit Formula

$$a_n = 3n - 2, \qquad n \geq 1$$

Recursion Formula

$$b_n = b_{n-1} + 3, \qquad n \geq 2, \qquad b_1 = 1$$

Section 10.1 - Sequences

Definitions

If a sequence has a limit that exists, then it is convergent and it converges to the limit value.

If a sequence has a limit that does not exist, then it is divergent.

Theorems

Given $a_n = f(x)$, then $\lim_{x\to\infty} f(x) = L$ implies $\lim_{x\to\infty} a_n = L$.

Given $a_n = f(x)$, then $f(x) \to L$ implies $a_n \to L$.

If the $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{x\to\infty} a_n = 0$.

If the $|a_n| \to 0$, then $a_n \to 0$.

Section 10.1 – Sequences

THEOREM 2—The Sandwich Theorem for Sequences Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Section 10.1 – Sequences

Lowest ~~Largest~~ Value   Largest Value

EXAMPLE 4 Since $1/n \to 0$, we know that

(a) $\dfrac{\cos n}{n} \to 0$ because $-\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n};$

is   NO

(b) $\dfrac{1}{2^n} \to 0$ because $\boxed{0} \le \boxed{\dfrac{1}{2^n}} \le \boxed{\dfrac{1}{n}};$ going to zero

Zero   choice

(c) $(-1)^n \dfrac{1}{n} \to 0$ because $-\dfrac{1}{n} \le (-1)^n \dfrac{1}{n} \le \dfrac{1}{n}.$

Section 10.1 – Sequences

THEOREM 5 The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

ex: $3! = 3\cdot 2\cdot 1 = 6$

Section 10.1 – Sequences

THEOREM 1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k\cdot b_n) = k\cdot B$ (any number k)
4. *Product Rule:* $\lim_{n\to\infty}(a_n\cdot b_n) = A\cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

Examples:

① $a_n = \dfrac{n^2}{n+1}$ $a_1 = \dfrac{1^2}{1+1} = \boxed{\dfrac{1}{2}}$

$a_2 = \dfrac{2^2}{2+1} = \boxed{\dfrac{4}{3}}$ $a_3 = \dfrac{3^2}{3+1} = \dfrac{9}{4}$ = $a_4 = \dfrac{16}{5}$

$a_5 = \dfrac{25}{6}$

$\lim\limits_{n\to\infty} \dfrac{n^2}{n+1} = \lim\limits_{n\to\infty} \dfrac{2n}{1} = \boxed{\infty}$ divergent

② $a_n = \dfrac{3n}{2n+4} = \dfrac{3}{6} = \boxed{\dfrac{1}{2}}$ $a_2 = \dfrac{6}{8} = \boxed{\dfrac{3}{4}}$ $a_3 = \boxed{\dfrac{9}{10}}$

$a_4 = \dfrac{12}{12} = \boxed{1}$ $a_5 = \boxed{\dfrac{15}{14}}$

$\lim\limits_{n\to\infty} \dfrac{3n}{2n+4} \to \lim\limits_{n\to\infty} \boxed{\dfrac{3}{2}}$ converges

$\boxed{\text{2nd}} > \boxed{\text{STAT}}$ Seq$(X^2/(X+1), X, 1, 5)$

$\boxed{\text{OPS \#5}}$

Calculator Hint

(3) $\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \right\}$ $a_n = \frac{1}{2^n}$

(4) $\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots \right\}$ = $a_n = \frac{1}{2n}$

(5) $\left\{ \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}, \ldots \right\}$ $a_n = \frac{n+2}{(n+3)^2}$

(6) $a_n = (-1)^n \frac{n}{n+2}$

$$\lim_{n \to \infty} (-1)^n \cdot \lim_{n \to \infty} \frac{n}{n+2} = \boxed{1}$$

$\hookrightarrow \pm 1 \cdot 1$

$= \boxed{\pm 1}$ = diverges

(7) $b_n = \frac{e^{2n}}{n^2 + 3n - 1}$

$$\lim_{n \to \infty} e^{2n} = \infty \quad \lim_{n \to \infty} n^2 + 3n - 1 = \boxed{2}$$

$\hookrightarrow \boxed{\infty}$ Divergent

See Examples

Previous Page

Section 10.2 – Infinite Series

DEFINITIONS Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum.** If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Section 10.2 – Infinite Series

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots ar^{n-1} + ar^n$$

A Geometric Series will converge to $\boxed{S_n = \frac{a}{1-r}}$ provided that $|r| < 1$.

$\boxed{\text{If } |r| \geq 1,}$ then the series will diverge.

THEOREM 7 If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

$$\sum_{n=1}^{\infty} \left(\frac{1}{7}\right)^n = \frac{1}{7} + \frac{1}{7} \cdot \overbrace{\frac{1}{7}}^{r} + \frac{1}{7}\overbrace{\left(\frac{1}{7}\right)^2}^{r^2} + \frac{1}{7}\overbrace{\left(\frac{1}{7}\right)^3}^{r^3} + \cdots \quad a = \frac{1}{7} \quad r = \frac{1}{7} < 1 \quad conv.$$

$$S = \frac{\frac{1}{7}}{1 - \frac{1}{7}} = \frac{\frac{1}{7}}{\frac{6}{7}} = \boxed{\frac{1}{6}}$$

Section 10.2 – Infinite Series

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ $\overbrace{\text{fails to exist}}^{\infty}$ or $\overbrace{\text{is different from zero.}}^{\# \text{ not } = 0}$

$\sum_{n=1}^{\infty} n^2$ $\quad \lim_{n\to\infty} (n)^2 = \boxed{\infty}$ The limit does not exist, $\boxed{\text{therefore it diverges.}}$

$\sum_{n=1}^{\infty} \frac{n+1}{n}$ $\quad \lim_{n\to\infty} \frac{n+1}{n} = 1$ The limit $\boxed{\text{does not equal 0}}$ therefore $\boxed{\text{it diverges.}}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad \lim_{n\to\infty} \frac{1}{n} = \boxed{0}$ The limit equals 0, therefore the n^{th} – Term Test for Divergence cannot be used.

= pick another test

Examples

n^{th} term test for Div.

(1) $\sum_{n=1}^{\infty} \frac{n-1}{n+2}$ $\lim_{n\to\infty} \frac{n-1}{n+2} = \frac{\infty}{\infty} = \lim_{n\to\infty} 1 = \boxed{1}$

$\boxed{\text{Divergent}}$

(2) $\sum_{n=1}^{\infty} \frac{3}{n}$ $\lim_{n\to\infty} \frac{3}{n} = \boxed{0}$ n^{th} term test for Div did not work

Geometric Test

(3) $\sum_{K=1}^{\infty} 5\left(\frac{1}{2}\right)^K = \sum_{n=1}^{\infty} ar^{n-1} = S = \frac{a}{1-r}$

$$\sum_{K=1}^{\infty} 5\left(\tfrac{1}{2}\right)^K = 5\frac{1}{2} + 5\frac{1}{2}\cdot\frac{1}{2} + 5\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 \ldots$$

$$S = \frac{\frac{5}{2}}{1-\frac{1}{2}} = \frac{\frac{5}{2}}{\frac{1}{2}} = \boxed{5}$$ $\boxed{\text{Converges}}$

(4) $\sum_{n=2}^{\infty}\left(\frac{1}{n}-\frac{1}{n-1}\right) = \underbrace{\cancel{\frac{1}{2}} - 1}_{2} + \underbrace{\cancel{\frac{1}{3}} - \cancel{\frac{1}{2}}}_{3} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{3}} + \frac{1}{5} - \cancel{\frac{1}{4}}$

$\rightarrow \boxed{-1}$ Converges because we get a sum

Section 10.2 – Infinite Series

THEOREM 8 If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k\sum a_n = kA$ (any number k).

multiply by anything that isnt zero on divergent still equals div.

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

Section 10.2 – Infinite Series

Telescoping Series (collapsing series)

$$\sum_{n=1}^{\infty} a_n - a_{n+1}$$

A Telescoping Series $\boxed{\text{will converge.}}$

$$\sum_{n=2}^{\infty}\left(\frac{1}{n} - \frac{1}{n-1}\right) = \frac{1}{2} - \frac{1}{1} + \frac{1}{3} - \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + \cdots = -1$$

Section 10.3 – The Integral Test

Positive Series:

A series whose elements are non-negative.

Conditions

* **The Integral Test:** 1. 2. 3. terms getting smaller

Let f be a continuous, positive and decreasing function and $a_n = f(x)$ for all positive integers, then $\sum_{n=1}^{\infty} a_n$ converges iff the improper integral $\int_1^{\infty} f(x)dx$ converges

If the improper integral diverges, then the series diverges.

Section 10.3 – The Integral Test

Example: Continuous, positive & getting smaller use Int test

$$\sum_{n=1}^{\infty} \frac{1}{n+3} \qquad \int_1^{\infty} \frac{1}{x+3}\,dx \qquad \lim_{b\to\infty} \int_1^{b} \frac{1}{x+3}\,dx \qquad u = x+3 \quad du = dx \qquad \lim_{b\to\infty} \int_1^{b} \frac{1}{u}\,du$$

$$\lim_{b\to\infty} \ln|u|\,\Big|_1^b \qquad \lim_{b\to\infty} \ln|x+3|\,\Big|_1^b \qquad \lim_{b\to\infty} \left[\ln(b+3) - \ln(1+3)\right] \qquad \infty - \ln 4 \qquad \therefore \textit{divergent}$$

Section 10.3 – The Integral Test

Positive Series

* **p-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **p is a constant**

If p is > 1, then the series converges.
If p is ≤ 1, then the series diverges.

int test for proof = $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\int_1^{\infty} \frac{1}{x^p}\,dx$ $\lim_{b\to\infty} \int_1^{b} \frac{1}{x^p}\,dx$ $\lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$ $\frac{1}{1-p} \lim_{b\to\infty} \left[\frac{1}{x^{p-1}}\right]_1^b$

$$\frac{1}{1-p} \lim_{b\to\infty} \left[\frac{1}{b^{p-1}} - 1\right]$$

Section 10.3 – The Integral Test

Positive Series

Examples:

P series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ $p-series$ $p = 4 > 1$ **converges**

$\sum_{n=1}^{\infty} n^{-\frac{1}{2}}$ $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ $p-series$ $p = \frac{1}{2} \le 1$ **diverges**

Examples 10.3

① $\int_1^{\infty} \frac{1}{x+3}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x+3} =$ $u = x+3$, $du = dx$

$= \lim_{b\to\infty} \int_1^b \frac{1}{u}\,du = \lim_{b\to\infty} \ln|x+3| \Big|_1^b$

$= \lim_{b\to\infty} \ln|b+3| - \ln|4| = \boxed{\infty - \ln 4 = \text{Divergent}}$

② $\sum_{n=1}^{\infty} n^{-\frac{1}{2}}$ p-series, $p = \frac{1}{2} < 1 = \boxed{\text{Diverges}}$

③ $\sum_{n=1}^{\infty} \frac{1}{n}$ p-series, $p = 1 \le 1$ $\boxed{\text{Diverges}}$

④ $\sum_{k=1}^{\infty} \frac{-2}{\sqrt{k+2}} = -2\sum_{k=1}^{\infty} \frac{1}{(k+2)^{1/2}}$ Cont. pos. decreas

$= -2\int_1^{\infty} \frac{1}{(k+2)^{1/2}} = -2 \lim_{b\to\infty} \int_1^b \frac{1}{(k+2)^{1/2}}\,dk$ $u = k+2$, $du = dk$

$-2 \lim_{b\to\infty} \int_1^b \frac{1}{u^{1/2}}\,du = -2 \lim_{b\to\infty} 2(k+2)^{1/2} \Big|_1^b$

$\lim_{b\to\infty} -4\left[(b+2)^{1/2} - (1-2)^{1/2}\right] = \cdot 4 \cdot \infty = \boxed{\text{Diverges}}$

$$\sum_{n=1}^{\infty} \frac{3}{(4+3n)^{7/6}} = 3\sum_{n=1}^{\infty} \frac{1}{(4+3n)^{7/6}}$$ Cont, pos, dec. use int test

$$= 3\int_1^{\infty} \frac{1\,dn}{(4+3n)^{7/6}} = 3\lim_{b\to\infty} \int_1^{b} \frac{1\,dn}{(4+3n)^{7/6}} =$$

$$U = 4+3n$$
$$du = 3du$$

$$\lim_{b\to\infty} \int_1^{b} \frac{3dn}{(4+3n)^{7/6}} = \lim_{b\to\infty} \int_1^{b} \frac{1\,du}{U^{7/6}} = \int_1^{b} U^{-7/6}$$

$$= -6\,U^{-1/6}\Big|_1^{b} = \lim_{b\to\infty} -6\left[(4+3n)^{-1/6}\right]\Big|_1^{b}$$

$$\lim_{b\to\infty} -6\left[(4+3b)^{-1/6} - (4+3)^{-1/6}\right]$$

$$-6\left[0 - \frac{1}{(7)^{1/6}}\right] = \frac{6}{7^{1/6}}$$

Converges therefore Series Converges

④ $$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+\frac{1}{2^n}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum\frac{1}{2^n}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}$$ p-series, $p=2>1$ converges

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2+\dots \quad \boxed{r=\frac{1}{2}} \quad a=\frac{1}{2}$$

$$S_n=\frac{a}{1-r}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{\frac{1}{2}}{\frac{1}{2}}=\frac{1}{2}\cdot 2=1$$ converges since $|r|\geq 1$

Section 10.3 – The Integral Test

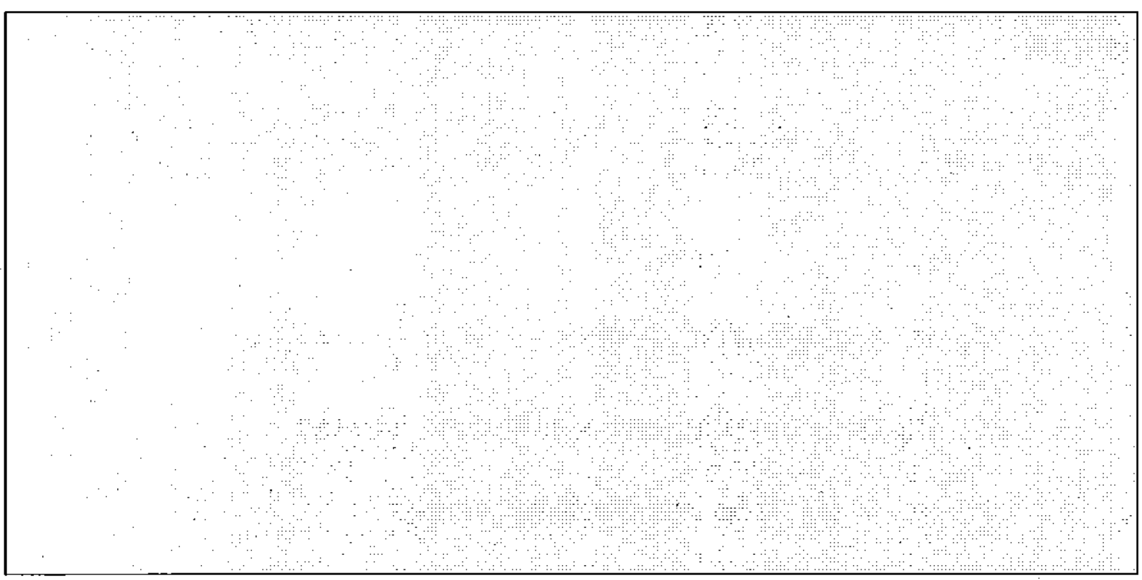
Positive Series

Harmonic Series

$$\sum_{n=1}^{\infty}\frac{1}{n} \qquad p-series \qquad p = 1 \leq 1 \qquad \textbf{diverges}$$

Harmonic Series $\sum_{n=1}^{\infty}\frac{1}{n}$ = Diverges

Section 10.3 – The Integral Test



Section 10.4 – Comparison Tests

Positive Series

* **Comparison Test (Ordinary Comparison Test)**

Let $\sum_{n=1}^{\infty} U_n$ be a positive series. Start analyses where n becomes positive

If $\sum_{n=1}^{\infty} V_n$ is a positive convergent series and $0 \le U_n \le V_n$, then the series $\sum_{n=1}^{\infty} U_n$ converges.

If $\sum_{n=1}^{\infty} V_n$ is a positive divergent series and $0 \le V_n \le U_n$, then the series $\sum_{n=1}^{\infty} U_n$ diverges.

Section 10.4 – Comparison Tests

Positive Series

Examples:

$$\sum_{n=1}^{\infty} \frac{4}{3^n + 1} \qquad U_n = \frac{4}{3^n + 1} \qquad V_n = \frac{4}{3^n} = \frac{4}{3} + \frac{4}{3} \cdot \frac{1}{3} + \frac{4}{3}\left(\frac{1}{3}\right)^2 + \cdots$$

$$U_n \le V_n$$

V_n is a geometric series. $\quad a = \frac{4}{3}, r = \frac{1}{3} < 1$

V_n is a convergent geometric series.

$\therefore U_n$ is a convergent series.

Section 10.4 – Comparison Tests

Better Test!

Positive Series

* **Limit Comparison Test**

Let $\sum_{n=1}^{\infty} U_n$ be a positive series and U_n is a rational expression.

Choose V_n to be a positive series. $\boxed{\text{If } \lim_{b\to\infty} \frac{U_n}{V_n} = L, \text{ then}}$ $\frac{\text{original}}{\text{chosen}} = L$

1. $\sum_{n=1}^{\infty} U_n$ will converge or diverge depending on $\sum_{n=1}^{\infty} V_n$ provided $0 < L < \infty$.
2. $\sum_{n=1}^{\infty} U_n$ will converge if $\sum_{n=1}^{\infty} V_n$ converges provided $L = 0$.
3. $\sum_{n=1}^{\infty} U_n$ will diverge if $\sum_{n=1}^{\infty} V_n$ diverges provided $L = \infty$.

Section 10.4 – Comparison Tests

Positive Series

Examples:

$$\sum_{n=1}^{\infty} \frac{1}{(n^2+2)^{1/3}} \qquad V_n = \frac{1}{(n^2)^{1/3}} = \frac{1}{n^{2/3}}$$

V_n is a p-series.

$p = \frac{2}{3} < 1 \quad \therefore \sum_{n=1}^{\infty} V_n$ is div.

$$\lim_{n\to\infty} \frac{\frac{1}{(n^2+2)^{1/3}}}{\frac{1}{n^{2/3}}} \qquad \lim_{n\to\infty} \frac{n^{2/3}}{(n^2+2)^{1/3}} \qquad \lim_{n\to\infty} \frac{(n^2)^{1/3}}{(n^2+2)^{1/3}} \qquad \lim_{n\to\infty} \left(\frac{n^2}{n^2+2}\right)^{1/3}$$

$$\lim_{n\to\infty} \left(\frac{n^2}{n^2+2}\right)^{1/3} = 1$$

$L = 1 \quad 0 < 1 < \infty$, $\sum_{n=1}^{\infty} V_n$ is div.

$\therefore \sum_{n=1}^{\infty} U_n$ is div.

10.4 Examples

① $\sum_{n=26}^{\infty} \frac{1}{\sqrt{n}-5} = \sum_{n=26}^{\infty} \frac{1}{n^{1/2}-5}$ $\quad U_n = \frac{1}{n^{1/2}-5}$

$V_n = \frac{1}{n^{1/2}}$

$\sum_{n=26}^{\infty} \frac{1}{n^{1/2}}$ p-Series, $p = 1/2 \leq 1$ Diverges

$$\boxed{V_n \leq U_n \text{ \& } V_n \text{ Diverges } \therefore U_n \text{ Diverges}}$$

Limit test

$\sum_{n=26}^{\infty} \frac{1}{\sqrt{n}-5}$ $\quad U_n = \frac{1}{\sqrt{n}-5}$

$V_n = \frac{1}{n^{1/2}}$

$\sum_{n=26}^{\infty} \frac{1}{n^{1/2}}$ p-series, $p = \frac{1}{2} \leq 1 \therefore$ Series Diverges

$$\lim_{n \to \infty} \frac{\frac{1}{n^{1/2}-5}}{\frac{1}{n^{1/2}}} = \frac{n^{1/2}}{n^{1/2}-5} = \lim_{n \to \infty} \frac{n^{1/2}}{n^{1/2}-5} = 1$$

$$\boxed{L = 1, \quad 0 < 1 < \infty \quad V_n \text{ Diverges } \therefore U_n \text{ Diverges}}$$

Comparison Test

(2) $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ $U_n = \frac{1}{2+3^n}$ $V_n = \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\left(\frac{1}{3}\right)^2 \ldots$ Geometric Series

$a = 1/3$ $|r| \leq 1$ Converges

$r = 1/3$

$U_n \leq V_n$ & V_n Converges $\therefore$ U_n Converges

Limit test

$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ $U_n = \frac{1}{2+3^n}$ $V_n = \frac{1}{3^n}$

$\sum_{n=1}^{\infty} \frac{1}{3^n}$ Geometric Series $a = \frac{1}{3}$, $r = \frac{1}{3}$ $|r| \leq 1$ Converges

$$\lim_{n\to\infty} \frac{\frac{1}{2+3^n}}{\frac{1}{3^n}} = \frac{3^n}{2+3^n} = \lim_{n\to} \frac{3^n}{3^n} = 1$$

$L = 1$, $0 < 1 < \infty$

V_n Conv $\therefore$ U_n Conv.

(3) $\sum_{n=1}^{\infty} \frac{1}{3n^2-4n+5}$ $\qquad U_n = \frac{1}{3n^2-4n+5}$

$V_n = \frac{1}{3n^2} = \frac{1}{3} \cdot \frac{1}{n^2}$

p-series, $p=2>1$ converge

$$\lim_{n\to\infty} \frac{\frac{1}{3n^2-4n+5}}{\frac{1}{3n^2}} = \lim_{n\to\infty} \frac{3n^2}{3n^2-4n+5} = \boxed{1}$$

$L=1$, $0<1<\infty$ V_n Conv. ∴ U_n Conv.

Section 10.5 – The Ratio and Root Tests

The Ratio Test:

Let $\sum_{n=1}^{\infty} a_n$ be a positive series and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$ → Greek letter row?

If $\rho < 1$, then the series converges.
If $\rho > 1$, then the series diverges.
If $\rho = 1$, then the test is inconclusive.

The Ratio Test is a good test to use when the series contains sequences such as $n!, r^n$, and n^n.

Section 10.5 – The Ratio and Root Tests

Ratio Test and the Geometric Series

$$\sum_{n=1}^{\infty} a_n = a_n + a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho \quad \frac{a_{n+1}}{a_n} = \rho$$

$$\sum_{n=1}^{\infty} a_n = a_n + a_n\,\rho + a_n\rho^2 + a_n\rho^3 + \cdots$$

$$a_{n+1} = a_n\rho$$
$$a_{n+2} = a_{n+1}\rho = a_n\rho\rho = a_n\rho^2$$
$$a_{n+3} = a_{n+2}\rho = a_n\rho\rho\rho = a_n\rho^3$$
$$a_{n+4} = a_{n+3}\rho = a_n\rho\rho\rho\rho = a_n\rho^4$$

Geometric Series

$a = a_n \qquad r = \rho$

$\rho < 1$ convergent series
$\rho \geq 1$ divergent series

$$\lim_{x\to\infty} \sqrt{x^3} = \sqrt{\lim_{x\to\infty} x^3}$$

Section 10.5 – The Ratio and Root Tests

The Root Test:

Let $\sum_{n=1}^{\infty} a_n$ **be a positive series and** $\lim_{n\to\infty} |\sqrt[n]{a_n}| = \rho.$

If $\rho < 1$, **then the series converges.**

If $\rho > 1$, **then the series diverges.**

If $\rho = 1$, **then the test is inconclusive.**

The Root Test is a good test to use when the series contains sequences using exponential expressions.

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Section 10.5 – The Ratio and Root Tests

10.5 Examples

(1) $\sum_{n=1}^{\infty} \frac{2^n}{n!} = \quad a_n = \frac{2^n}{n!} \quad a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right| = \lim_{n\to\infty} \frac{2^{n+1} \cdot n!}{2^n \cdot (n+1)!}$$

$$\lim_{n\to\infty} \frac{2^n \cdot 2^1 \quad n!}{2^n (n+1)} = \lim_{n\to\infty} \frac{2^n \cdot 2\, n!}{2^n (n+1)\, n!}$$

$$\lim_{n\to\infty} = \frac{2}{n+1} = 0$$

$$5! = 5 + 4! = (4+1)! = (4+1)\,4!$$

$p = 0 < 1 \therefore$ Converges

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^{20}}$ $\quad a_n = \frac{2^n}{n^{20}} \quad a_{n+1} = \frac{2^{n+1}}{(n+1)^{20}}$

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)^{20}}}{\frac{2^n}{n^{20}}} \right| = \lim_{n\to\infty} \frac{2^{n+1} \cdot n^{20}}{2^n (n+1)^{20}}$$

$$\lim_{n\to\infty} \frac{2^n \cdot 2^1 \cdot n^{20}}{2^n (n+1)^{20}} = \lim_{n\to\infty} \frac{2 \cdot n^{20}}{(n+1)^{20}} = 2$$

largest $n = n^{20}$ so get coeff.

$p = 2 > 1 \therefore$ Diverges

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n!}$ $\quad a_n = \frac{n^4}{n!} \quad a_{n+1} = \frac{(n+1)^4}{(n+1)!}$

$$\lim_{n\to\infty} \frac{\frac{(n+1)^4}{(n+1)!}}{\frac{n^4}{n!}} = \lim_{n\to\infty} \frac{n!\,(n+1)^4}{n^4 (n+1)!}$$

$$\lim_{n\to\infty} \frac{n!\,(n+1)^4}{n^4 (n+1) n!} = \lim_{n\to\infty} \frac{(n+1)^4}{n^4 (n+1)}$$

$$\lim_{n\to\infty} \frac{(n+1)^3}{n^4} = \boxed{0}$$

$p = 0 < 1$ Series Converge

Root Test

(4) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$ $\lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{n^n}} = \lim_{n\to\infty} \sqrt[n]{\frac{(e^2)^n}{n^n}}$

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{e^2}{n}\right)^n} = \lim_{n\to\infty} \frac{e^2}{n} = 0$

$\rho = 0 < 1 \therefore$ Conv

(5) $\sum_{n=2}^{\infty} \left(\frac{2n+1}{n-1}\right)^n = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+1}{n-1}\right)^n}$

$= \lim_{n\to\infty} \frac{2n+1}{n-1} = \boxed{2}$ $\rho = 2 > 1 \therefore$ Div.

(6) $$\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n]{(\ln)^n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\ln(n)} = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0$$

$\rho = 0 < 1$ Conv.

Section 10.6 – Alternating Series: Absolute and Conditional Convergence

Alternating Series:

A series whose terms are alternately positive and negative.

The general forms are as follows:

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

$$\sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$$

$(-1)^n$ or $(-1)^{n+1}$ = alternators

Section 10.6 – Alternating Series: Absolute and Conditional Convergence

Alternating Series Test

* Conditions

If:

1) $a_n > 0$ positive
2) $a_{n+1} \le a_n$ decreasing don't worry signs
3) $\lim\limits_{n\to\infty} a_n = 0,$

then the alternating series is convergent.

Example:

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n} = -\frac{1}{1}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots$$

1) $a_n > 0$

2) $a_{n+1} \le a_n$

3) $\lim\limits_{n\to\infty}\frac{1}{n} = 0$

$\therefore \sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ **convergent.**

* **Alternating Harmonic Series**

Section 10.6 – Alternating Series: Absolute and Conditional Convergence

Example:

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n} = -\frac{1}{1}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots$$

1) $a_n > 0$ 2) $a_{n+1} \le a_n$ 3) $\lim_{n\to\infty}\frac{1}{n} = 0$

$$\therefore \sum_{n=1}^{\infty}(-1)^n\frac{1}{n} \quad \text{convergent}$$

Alternating Harmonic Series

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n}\right| =$$

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

Harmonic Series

$\therefore$ divergent

$$\therefore \sum_{n=1}^{\infty}(-1)^n\frac{1}{n} \quad \text{converges conditionally}$$

10.6 Examples Alternating Series

(1) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{3n+1}$

1) $a_n > 0$ ✓
2) $a_{n+1} \leq a_n$ ✓
3) $\lim_{n\to\infty} a_n = 0$ ✓

∴ Series Converge

(2) Absolute Convergence

$$\sum_{n=1}^{\infty} \frac{n^2}{e^n} = \frac{(n+1)^2}{e^{n+1}} = a_{n+1}$$

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \lim_{n\to\infty} \frac{e^n (n+1)^2}{e^{n+1} n^2} = \frac{e^n (n+1)^2}{e^n e^1 \cdot n^2}$$

$$\lim_{n\to\infty} \frac{(n+1)^2}{en^2} = \frac{1}{e} \quad \boxed{p = \frac{1}{e} < 1 \text{ Converges}}$$

This absolutely converges

(3) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ 1) $a_n > 0$ ✓
2) $a_n > a_{n+1}$ ✓
Converges! 3) $\lim_{n\to\infty} a_n = 0$ ✓

Absolute Convergence test with Limit Test

$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ $V_n = \frac{1}{n}$ $\sum_{n=1}^{\infty} \frac{1}{n} =$ Div Harmonic

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \frac{n^2}{n^2+1} = \boxed{1}$$

$\boxed{L=1 \quad 0 < 1 < \infty \quad V_n \text{ is Div} \therefore U_n \text{ is Div}}$

Conditional Convergence

④ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}$

Absolute test for Convergence

Root test

$\lim_{n \to \infty} \sqrt[n]{\frac{n}{2^n}} = \lim_{n \to \infty} \frac{\sqrt[n]{n}}{\sqrt[n]{2^n}}$

$\lim_{n \to \infty} \frac{1}{2}$ $\boxed{\rho = \frac{1}{2} < 1 \quad \text{Converges}}$

Absolute Convergence

Section 10.7 – Power Series

Previous series have consisted of constants.

$$\sum_{n=1}^{\infty} a_n \rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n} \qquad \sum_{n=1}^{\infty} \frac{10^n}{(n+1)!} \qquad \sum_{n=1}^{\infty} (-1)^n \frac{ln(n)}{n - ln(n)}$$

Another type of series will include the variable *x*.

$$\sum_{n=1}^{\infty} a_n(x) \rightarrow \quad \sum_{n=1}^{\infty} \frac{x}{n} \qquad \sum_{n=1}^{\infty} \frac{10^n x^n}{(n+1)!} \qquad \sum_{n=1}^{\infty} (-1)^n \frac{ln(n)(x-4)^n}{n - ln(n)}$$

Section 10.7 – Power Series

A Power Series is of the form

constant → variable

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1\, x + c_2 x^2 + c_3\, x^3 + \cdots$$

where x is a variable and c_n represents the coefficients.

The sum of the series is the function

$$f(x) = c_0 + c_1\, x + c_2 x^2 + c_3\, x^3 + \cdots$$

where its domain is the set of all x for which the series converges.

Section 10.7 - Power Series

A more general form of the power series is of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where x is a variable, c_n represents the coefficients and a is a number.

This form is referred as:

a power series in (x - a) or

a power series centered at a.

Section 10.7 - Power Series

Definitions

Interval of Convergence: The interval of x values where the series converges.

Radius of Convergence (R): Half the length of the interval of convergence.

interval of conv: -3 — 3 (R) = 3 left or right

* There are only three ways for a power series to converge.

1) The series only converges at $x = a$. $(R = 0)$
2) The series converges for all x values. $(R = \infty)$
3) The series converges for some interval of x.

$(|x-a| < R) \Rightarrow$

$\boxed{(a - R < x < a + R)}$

The end values of the interval must be tested for convergence.

The use of the Ratio test is recommended when finding the radius of convergence and the interval of convergence.

Section 10.7 - Power Series

Example: Find the Radius of Convergence and the Interval of Convergence for the following power series

include alternator

❶ $$\sum_{n=0}^{\infty}\frac{(-1)^n n(x+3)^n}{4^n}$$ **Ratio Test** $$c_{n+1}=\frac{(-1)^{n+1}(n+1)(x+3)^{n+1}}{4^{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)(x+3)^{n+1}}{4^{n+1}}\div\frac{(-1)^n n(x+3)^n}{4^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^n}(-1)(n+1)\cancel{(x+3)^n}(x+3)}{\cancel{4^n}(4)}\cdot\frac{\cancel{4^n}}{\cancel{(-1)^n}n\cancel{(x+3)^n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{(-1)(n+1)(x+3)}{4n}\right|=\frac{1}{4}|x+3|$$

Section 10.7 - Power Series

❶ $$\sum_{n=0}^{\infty}\frac{(-1)^n n(x+3)^n}{4^n}\quad\lim_{n\to\infty}\left|\frac{(-1)(n+1)(x+3)}{4n}\right|=\frac{1}{4}|x+3|$$

Ratio Test: Convergence for $L<1$

$$\frac{1}{4}|x+3|<1$$

$$|x+3|<4 = |x-a|<R$$

Radius of Convergence

$$R=4$$

Interval of Convergence:

$$-4<x+3<4$$

$$\boxed{-7<x<1}$$

End points need to be tested.

end points are -7 & 1

*plug these back in to original series →

Section 10.7 - Power Series

❶ $$\sum_{n=0}^{\infty} \frac{(-1)^n n(x+3)^n}{4^n}$$

$$\boxed{x = -7}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n n(-7+3)^n}{4^n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n n(-4)^n}{4^n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n n(-1)^n \cancel{(4)^n}}{\cancel{4^n}}$$

$$\sum_{n=0}^{\infty} n$$

n^{th} *term test for divergence*

$$\lim_{n\to\infty} n = \infty \neq 0$$

divergent at $x = -7$

-7 *cannot be included in the interval of convergence*

Section 10.7 - Power Series

❶ $$\sum_{n=0}^{\infty} \frac{(-1)^n n(x+3)^n}{4^n}$$

$$x = 1$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n n(1+3)^n}{4^n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n n(4)^n}{4^n}$$

$$\sum_{n=0}^{\infty} (-1)^n n$$

$$\sum_{n=0}^{\infty} (-1)^n n$$

n^{th} *term test for divergence*

$$\lim_{n\to\infty} (-1)^n n = DNE \neq 0$$

divergent at $x = 1$

1 *cannot be included in the interval of convergence*

Therefore, the interval of convergence is:

$$-7 < x < 1$$

Section 10.7 - Power Series

Example: Find the Radius of Convergence and the Interval of Convergence for the following power series

❷ $$\sum_{n=0}^{\infty} \frac{2^n(4x-8)^n}{n}$$ Ratio Test $$c_{n+1} = \frac{2^{n+1}(4x-8)^{n+1}}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{2^{n+1}(4x-8)^{n+1}}{n+1} \div \frac{2^n(4x-8)^n}{n}\right|$$

$$\lim_{n\to\infty}\left|\frac{2^n 2(4x-8)^n(4x-8)}{n+1} \cdot \frac{n}{2^n(4x-8)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{2n(4x-8)}{n+1}\right| = 2|4x-8|$$

Section 10.7 - Power Series

❷ $$\sum_{n=1}^{\infty} \frac{2^n(4x-8)^n}{n}$$ $$\lim_{n\to\infty}\left|\frac{2n(4x-8)}{n+1}\right| = 2|4x-8|$$

Ratio Test: Convergence for $L < 1$

$$2|4x-8| < 1$$
$$2|4(x-2)| < 1$$
$$8|x-2| < 1$$
$$|x-2| < \frac{1}{8}$$

Radius of Convergence

$$R = \frac{1}{8}$$

Interval of Convergence:

$$-\frac{1}{8} < x-2 < \frac{1}{8}$$
$$\frac{15}{8} < x < \frac{17}{8}$$

End points need to be tested.

Section 10.7 - Power Series

❷ $$\sum_{n=1}^{\infty} \frac{2^n(4x-8)^n}{n}$$

$$x = \frac{15}{8}$$

$$\sum_{n=1}^{\infty} \frac{2^n\left(4\left(\frac{15}{8}\right)-8\right)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{2^n\left(\frac{15}{2}-8\right)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{2^n(-1)^n}{n \cdot 2^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Alternating harmonic series is convergent

$$\therefore \textit{ convergent at } x = \frac{15}{8}$$

$\frac{15}{8}$ ***can be included in the interval of convergence***

Section 10.7 - Power Series

❷ $$\sum_{n=1}^{\infty} \frac{2^n(4x-8)^n}{n}$$

$$x = \frac{17}{8}$$

$$\sum_{n=1}^{\infty} \frac{2^n\left(4\left(\frac{17}{8}\right)-8\right)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{2^n\left(\frac{17}{2}-8\right)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{2^n}{n \cdot 2^n} \rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$$

Harmonic series is divergent

$$\therefore \textit{ divergent at } x = \frac{17}{8}$$

$\frac{17}{8}$ ***cannot be included in the interval of convergence***

Therefore, the interval of convergence is:

$$\frac{15}{8} \le x < \frac{17}{8}$$

Section 10.7 - Power Series

Example: Find the Radius of Convergence and the Interval of Convergence for the following power series

❸ $\displaystyle\sum_{n=0}^{\infty} n!\,(2x+1)^n$ Ratio Test $c_{n+1} = (n+1)!\,(2x+1)^{n+1}$

$$\lim_{n\to\infty}\left|\frac{(n+1)!\,(2x+1)^{n+1}}{n!\,(2x+1)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{n!\,(n+1)(2x+1)^n(2x+1)}{n!\,(2x+1)^n}\right|$$

$$\lim_{n\to\infty}|(n+1)(2x+1)| = \infty > 1$$

The series will converge at one point.

The limit is zero at

$$x = -\frac{1}{2}$$

Radius of Convergence: $R = 0$

The interval of convergence is:

$$x = -\frac{1}{2}$$

Section 10.7 - Power Series

Example: Find the Radius of Convergence and the Interval of Convergence for the following power series

❹ $\displaystyle\sum_{n=0}^{\infty} \frac{(x-6)^n}{n^n}$ Ratio Test $c_{n+1} = \dfrac{(x-6)^{n+1}}{n^{n+1}}$ $(n+1)^{n+1}$

$$\lim_{n\to\infty}\left|\frac{(x+6)^{n+1}}{n^{n+1}} \div \frac{(x+6)^n}{n^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x+6)^n(x+6)}{n^n n}\cdot\frac{n^n}{(x+6)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x+6)}{n}\right| = 0 < 1$$

The limit is zero regardless of the value of x.

The series will converge for every x.

Radius of Convergence: $R = \infty$

The interval of convergence is:

$$-\infty < x < \infty$$

Section 10.7 - Power Series

Example: Find the Radius of Convergence and the Interval of Convergence for the following power series

❺ $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{(-3)^n}$ **Root Test**

$$\lim_{n\to\infty}\left|\sqrt[n]{\frac{(x^2)^n}{(-3)^n}}\right| \qquad \lim_{n\to\infty}\left|\sqrt[n]{\left(\frac{x^2}{-3}\right)^n}\right| \qquad \lim_{n\to\infty}\left|\frac{x^2}{-3}\right| \quad =\frac{|x^2|}{3}<1 \qquad |x^2|<3$$

$$|x|<\sqrt{3}$$

Radius of Convergence: $R=\sqrt{3}$ *The interval of convergence is*:

$$-\sqrt{3}<x<\sqrt{3}$$

Section 10.7 - Power Series

❺ $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{(-3)^n}$ **Test the Endpoints**

$x=-\sqrt{3}$

$$\sum_{n=0}^{\infty}\frac{\left(-\sqrt{3}\right)^{2n}}{(-3)^n} \qquad \sum_{n=0}^{\infty}\frac{\left(\left(-\sqrt{3}\right)^2\right)^n}{(-3)^n} \qquad \sum_{n=0}^{\infty}\frac{(3)^n}{(-3)^n} \qquad \sum_{n=0}^{\infty}\frac{(3)^n}{(-1)^n(3)^n}$$

$\displaystyle\sum_{n=0}^{\infty}(-1)^n$ *nth Term Test for Divergence* $\displaystyle\lim_{n\to\infty}(-1)^n = DNE \neq 0$

$\therefore$ *divergent* $\therefore$ *the* $-\sqrt{3}$ *is not in the interval of convergence*

Section 10.7 - Power Series

❺ $\sum_{n=0}^{\infty} \frac{x^{2n}}{(-3)^n}$

Test the Endpoints

$x = \sqrt{3} \qquad \sum_{n=0}^{\infty} \frac{(\sqrt{3})^{2n}}{(-3)^n} \qquad \sum_{n=0}^{\infty} \frac{\left((\sqrt{3})^2\right)^n}{(-3)^n} \qquad \sum_{n=0}^{\infty} \frac{(3)^n}{(-3)^n} \qquad \sum_{n=0}^{\infty} \frac{(3)^n}{(-1)^n(3)^n}$

$\sum_{n=0}^{\infty} (-1)^n \quad$ *nth Term Test for Divergence* $\quad \lim_{n\to\infty} (-1)^n = DNE \neq 0$

$\therefore$ *divergent* $\therefore \sqrt{3}$ *is not in the interval of convergence*

The interval of convergence is: $-\sqrt{3} < x < \sqrt{3}$

Section 10.7 - Power Series

10.7 Examples

$x=1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n n(1+3)^n}{4^n} = \frac{(-1)^n n \cancel{(4)^n}}{\cancel{4^n}}$$

$$\sum_{n=0}^{\infty} (-1)^n n$$

nth test for divergence

$\lim_{n\to\infty} n = \infty$ ∴ Diverges

Divergent @ $x=1$

1 cannot be included in the interval of convergence

$$\sum_{n=0}^{\infty} \frac{2^n(4x-8)^n}{n}$$

Root test $\lim_{n\to\infty} \sqrt[n]{\frac{2^n(4x-8)^n}{n}}$

$$\lim_{n\to\infty} \frac{\sqrt[n]{2^n}\cdot\sqrt[n]{(4x-8)^n}}{\sqrt[n]{n}} = \lim_{n\to\infty} \frac{2\cdot(4x-8)}{1}$$

$\lim = 2|4x-8|$

$2|4x-8| < 1$

$2\cdot 4|x-2| < 1$

$|x-2| < \frac{1}{8}$

$\boxed{R=\frac{1}{8}}$

$-\frac{1}{8} < x-2 < \frac{1}{8}$

$\boxed{\frac{15}{8} < x < \frac{17}{8}}$

Interval of convergence

test end points →

$x = \frac{15}{8}$ ∴ $\sum_{n=0}^{\infty} \frac{2^n(4 \cdot \frac{15}{8} - 8)^n}{n} = \frac{2^n(7.5-8)^n}{n}$

$\sum_{n=0}^{\infty} \frac{2^n(-0.5)^n}{n}$ root test $\lim_{n \to \infty} \sqrt[n]{\frac{2^n(-\frac{1}{2})^n}{n}}$

$\lim_{n \to \infty} \frac{\sqrt[n]{2^n} \cdot \sqrt[n]{(-\frac{1}{2})^n}}{\sqrt[n]{n}} = \lim_{n \to \infty} \left| 2 \cdot (-\frac{1}{2}) \right| = \lim = 1$ inconclusive

* Use another test

Ratio test $\sum_{n=0}^{\infty} \frac{2^n(-\frac{1}{2})^n}{n}$ $\quad a_{n+1} = \frac{2^{n+1}(-\frac{1}{2})^{n+1}}{n+1}$

$\lim_{n \to \infty} \left| \frac{\frac{2^{n+1}(-\frac{1}{2})^{n+1}}{n+1}}{\frac{2^n(-\frac{1}{2})^n}{n}} \right| = \lim_{n \to \infty} \frac{\cancel{2^n} \cdot 2 \cancel{(-\frac{1}{2})^n} \cdot (-\frac{1}{2}) \cdot n}{(n+1) \cancel{2^n} \cdot \cancel{(-\frac{1}{2})^n}}$

$\lim_{n \to \infty} \left| \frac{-1n}{n+1} \right| = \lim_{n \to} \frac{1n}{n+1} = = 1$

inconclusive

$X = \frac{15}{8}$

Use another test

$$\sum_{n=0}^{\infty} \frac{2^n(-\frac{1}{2})^n}{n} = \frac{2^n \cdot (\frac{-1^n}{2^n})}{n} = \frac{\cancel{2^n} \cdot (-1)^n}{n \cdot \cancel{2^n}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n} =$$ Alternating Harmonic Series & Converges

(3) $$\sum_{n=0}^{\infty} n!(2x+1)^n$$ ratio test

$a_{n+1} = (n+1)!\,(2x+1)^{n+1}$

$$\lim_{n\to\infty} \left| \frac{(n+1)!\,(2x+1)^{n+1}}{n!\,(2x+1)^n} \right| = \lim_{n\to\infty} \frac{\cancel{n!}(n+1)!\,\cancel{(2x+1)^n}(2x+1)}{\cancel{n!}\,\cancel{(2x+1)^n}}$$

$$\lim_{n\to\infty} (n+1)(2x+1) = \infty > 1$$ Series converge @ one point

limit is zero @ $X = -\frac{1}{2}$

Converging @ one point means $R = 0$

Interval of convergence is $X = -\frac{1}{2}$

④ $\sum_{n=0}^{\infty} \frac{(x-6)^n}{n^n}$ Root Test $\lim_{n\to\infty} \sqrt[n]{\frac{(x-6)^n}{n^n}}$

$$\lim_{n\to\infty} \frac{\sqrt[n]{(x-6)^n}}{\sqrt[n]{n^n}} = \lim_{n\to\infty} \frac{(x-6)}{n} = 0$$

$\rho = 0 < 1$ ∴ Series Converges @ 0

This means no matter what X you plug in you always get Zero

$\boxed{\begin{array}{c}\text{Interval of convergence} = \text{all values of } X \\ \text{Radius of convergence} = R = \infty \\ \text{interval is } -\infty < X < \infty\end{array}}$

5) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(-3)^n}$ Root test $\lim_{n\to\infty} \sqrt[n]{\frac{x^{2n}}{(-3)^n}}$

$\lim_{n\to\infty} \frac{\sqrt[n]{x^{2n}}}{\sqrt[n]{(-3)^n}} = \lim_{n\to\infty} \frac{x^2}{-3}$

$|\text{limit}| = \frac{1}{3}x^2$

$\frac{1}{3}x^2 < 1 \rightarrow x < \sqrt{3}$

Radius $= \sqrt{3}$ $\quad -\sqrt{3} < x < \sqrt{3}$ ← Test points

$x = -\sqrt{3}$ $\sum_{n=0}^{\infty} \frac{(-\sqrt{3})^{2n}}{(-3)^n}$ root test $\lim_{n\to\infty} \sqrt[n]{\frac{(-\sqrt{3})^{2n}}{(-3)^n}}$

$= \lim_{n\to\infty} \frac{(-\sqrt{3})^2}{(-3)} = \frac{3}{3} = 1$ inconclusive

$\sum_{n=0}^{\infty} \frac{(-\sqrt{3})^{2n}}{(-3)^n} = \sum_{n=0}^{\infty} \frac{((-\sqrt{3})^2)^n}{(-3)^n} = \sum_{n=0}^{\infty} \frac{(3)^n}{(-3)^n} = \sum_{n=0}^{\infty} \frac{\cancel{(3)^n}}{(-1)^n\cancel{(3)^n}}$

$\sum_{n=0}^{\infty} \frac{1}{(-1)^n} = \sum_{n=0}^{\infty} (-1)^n$ n^{th} test for div.

$\lim_{n\to\infty} (-1)^n = DNE \neq 0$ $\therefore$ divergent @ $x = -\sqrt{3}$

Section 10.8 - Taylor and Maclauren Series

Review: Power series:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots$$

where x is a variable, c_n represents the coefficients and a is a number.
Many other functions such as e^x, sin(x), cos(x), ln(x), etc. can be expressed as a power series.

Two assumptions:

1) The function does have a power series representation about $x = a$.
2) The function has derivatives of every order.

The values of the coefficients need to be derived.

Section 10.8 - Taylor and Maclauren Series

The values of the coefficients need to be derived at $x = a$.

$$f(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots$$

$$f(a) = c_0 + c_1 (a-a) + c_2 (a-a)^2 + c_3 (a-a)^3 + \cdots$$

$$f(a) = c_0$$

$$f'(x) = c_1 + 2c_2 (x-a)^1 + 3c_3 (x-a)^2 + \cdots$$

$$f'(a) = c_1 + 2c_2 (a-a)^1 + 3c_3 (a-a)^2 + \cdots$$

$$f'(a) = c_1$$

$$f''(x) = 2c_2 + 3(2)c_3 (x-a)^1 + \cdots$$

$$f''(a) = 2c_2 + 3(2)c_3 (a-a)^1 + \cdots$$

$$f''(a) = 2c_2$$

$$f'''(a) = 3 \cdot 2c_3$$

$$f^{(4)}(a) = 4 \cdot 3 \cdot 2c_4$$

$$f^{(5)}(a) = 5 \cdot 4 \cdot 3 \cdot 2c_5$$

Section 10.8 – Taylor and Maclauren Series

The values of the coefficients need to be derived at $x = a$.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

$f(a) = c_0$ $\quad f'(a) = c_1$ $\quad f''(a) = 2c_2$ $\quad f'''(a) = 3 \cdot 2c_3$

$$\frac{f''(a)}{2} = c_2 \qquad \frac{f'''(a)}{3 \cdot 2} = c_3$$

formula to

$f^{(4)}(a) = 4 \cdot 3 \cdot 2c_4$ $\quad f^{(5)}(a) = 5 \cdot 4 \cdot 3 \cdot 2c_5$

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

$$\frac{f^{(4)}(a)}{4 \cdot 3 \cdot 2} = c_4 \qquad \frac{f^{(5)}(a)}{5 \cdot 4 \cdot 3 \cdot 2} = c_5$$

find c_n

Section 10.8 – Taylor and Maclauren Series

Power series:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

*

Taylor Series:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Section 10.8 - Taylor and Maclauren Series

Power series:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

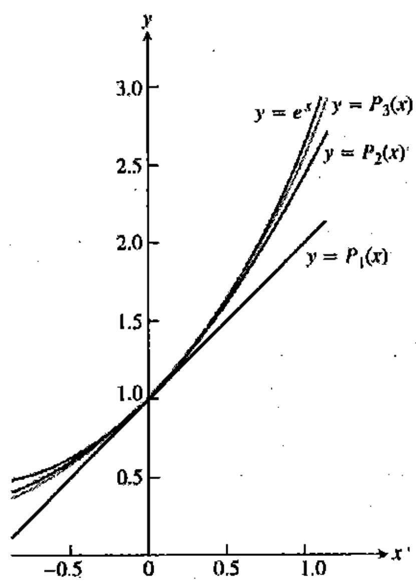
Maclauren Series: centered at $x = a = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

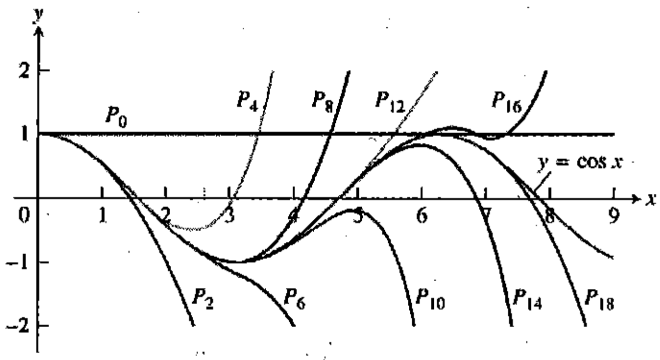
find derivative requested then plug zero for a

$$= f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \cdots$$

Section 10.8 - Taylor and Maclauren Series



Section 10.8 - Taylor and Maclauren Series



Section 10.8 - Taylor and Maclauren Series

Find the Taylor Series for $f(x) = e^x$ about $x = 0$ (Maclauren Series)

$f(x) = e^x \qquad f(0) = e^0 = 1$

$f'(x) = e^x \qquad f'(0) = e^0 = 1$

$f''(x) = e^x \qquad f''(0) = e^0 = 1$

$f'''(x) = e^x \qquad f'''(0) = e^0 = 1$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = f(0) + \underbrace{f'(0)(x)} + \underbrace{\frac{f''(0)}{2!}(x)^2} + \frac{f'''(0)}{3!}(x)^3 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Section 10.8 – Taylor and Maclauren Series

Find the Taylor Series for $f(x) = e^{-x}$ $about\ x = 0$ (Maclauren Series)

$f(x) = e^{-x}$ $\quad f(0) = e^0 = 1$

$f'(x) = -e^{-x}$ $\quad f'(0) = -e^0 = -1$

$f''(x) = e^{-x}$ $\quad f''(0) = e^0 = 1$

$f'''(x) = -e^{-x}$ $\quad f'''(0) = e^0 = -1$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!}$$

Section 10.8 – Taylor and Maclauren Series

Find the Taylor Series for $f(x) = e^{-x}$ $about\ x = -4$

$f(x) = e^{-x}$ $\quad f(-4) = e^4$ $\quad f''(x) = e^{-x}$ $\quad f''(-4) = e^4$

$f'(x) = -e^{-x}$ $\quad f'(-4) = -e^4$ $\quad f'''(x) = -e^{-x}$ $\quad f'''(-4) = -e^4$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x-a)^n = f(x) + f'(x)(x-a) + \frac{f''(x)}{2!}(x-a)^2 + \frac{f'''(x)}{3!}(x-a)^3 + \cdots$$

$$= f(-4) + f'(-4)(x+4) + \frac{f''(-4)}{2!}(x+4)^2 + \frac{f'''(-4)}{3!}(x+4)^3 + \cdots$$

$$= e^4 - e^4(x+4) + \frac{e^4}{2!}(x+4)^2 - \frac{e^4}{3!}(x+4)^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{e^4}{n!}(x+4)^n$$

Section 10.8 – Taylor and Maclauren Series

Find the Taylor Series for $f(x) = sinx\ about\ x = 0$ (Maclauren Series)

$f(x) = sinx \qquad f(0) = 0$

$f'(x) = cosx \qquad f'(0) = 1$

$f''(x) = -sinx \qquad f''(0) = 0$

$f'''(x) = -cosx \qquad f'''(0) = -1$

$f^{(4)}(x) = sinx \qquad f^{(4)}(0) = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

$$= f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \cdots$$

$$sinx = 0 + x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

$$sinx = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Section 10.8 – Taylor and Maclauren Series

TABLE 10.1 Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

10.8 Examples

1) $f(x) = e^{-x}$ $\qquad f(0) = e^{-0} = 1$

$f'(x) = -e^{-x}$ $\qquad f'(0) = -1$

$f''(x) = e^{-x}$ $\qquad f''(0) = 1$

$f'''(x) = -e^{-x}$ $\qquad f'''(0) = -1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$1 + (-x) + \frac{1}{2!}x^2 + \left(-\frac{1}{3!}x^3\right)$$

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$$

mean a

(3) $f(x) = e^{-x}$ about $\cancel{x} = -4$

$f'(x) = -e^{-x}$ $\qquad f(-4) = e^{4}$

$f''(x) = e^{-x}$ $\qquad f'(-4) = -e^{4}$

$f'''(x) = -e^{-x}$ $\qquad f''(-4) = e^{4}$ & $-e^{4}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$e^{4} - e^{4}(x+4) + \frac{e^{4}}{2!}(x+4)^2 - \frac{e^{4}}{3!}(x+4)^3$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{e^{4}}{n!}(x+4)^n$$

about $x=0$

(4) $f(x)=\sin x$ $\quad f(0)=0$
$f'(x)=\cos x$ $\quad f'(0)=1$ $\quad f''''(0)=1$
$f''(x)=-\sin x$ $\quad f''(0)=0$
$f'''(x)=-\cos x$ $\quad f'''(0)=-1$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(x)^n=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3$$

$$0+x+0-\frac{1}{3!}x^3+0+\frac{1}{5!}x^5$$

$$\sin x\approx x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

Section 10.9 – Convergence of Taylor Series

Operations on Series

Substitution

Find the Taylor series about $x = 0$

$$f(x) = e^{-3x^2}$$

Use a known series.

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

plug in $x = -3x^2$

$$e^{-3x^2} = \sum_{n=0}^{\infty} \frac{\left(-3x^2\right)^n}{n!}$$

$$e^{-3x^2} = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{n!}$$

$$e^{-3x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{3^n x^{2n}}{n!}$$

now just plug in $n=0$, $n=1$
$n=2$, $n=3$ . . .

Section 10.9 – Convergence of Taylor Series

Operations on Series

Substitution

Find the Maclauren series and write the first four terms.

$$f(x) = \frac{1}{1+x^2}$$

Use a known series.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1+(x^2)} = \sum_{n=0}^{\infty} (-1)^n \left(x^2\right)^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$1 - x^2 + x^4 - x^6 + \cdots$$

Section 10.9 - Convergence of Taylor Series

Operations on Series

Differentiation

Find the Maclauren series.

$$f(x) = \frac{1}{x+1}$$

Differentiate

Use a known series.

$$\overbrace{ln(x+1)} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

$$\frac{d}{dx}(ln(x+1)) = 1 - 2\frac{x^1}{2} + 3\frac{x^2}{3} - \cdots \quad = 1 - x + x^2 - x^3 + \cdots$$

$$= \sum_{n=0}^{\infty}(-1)^n x^n$$

Section 10.9 - Convergence of Taylor Series

Operations on Series

Integration

Find the Maclauren series.

$$f(x) = cosx$$

Use a known series.

$$\int sinx\,dx = \int x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\ dx$$

$$-cosx = \frac{x^2}{2} - \frac{x^4}{4\cdot 3!} + \frac{x^6}{6\cdot 5!} - \frac{x^8}{8\cdot 7!} + \cdots + c$$

$$cosx = -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots - c$$

Section 10.9 - Convergence of Taylor Series

Operations on Series

Integration

Find the Maclauren series. $f(x) = cosx$

Find c

Use $x = 0$.

$$cos0 = -c \qquad 1 = -c \qquad c = -1$$

$$cosx = -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots - (-1)$$

$$cosx = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Section 10.9 - Convergence of Taylor Series

Operations on Series

Multiplication

Find the Taylor series about $x = 0$

$$f(x) = x^4 e^{-3x^2}$$

Use a known series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$x^4 e^{-3x^2} = x^4 \sum_{n=0}^{\infty} \frac{\left(-3x^2\right)^n}{n!}$$

$$x^4 e^{-3x^2} = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n} x^4}{n!}$$

$$x^4 e^{-3x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{3^n x^{2n+4}}{n!}$$

10.9 Examples

① Mc laurin Series for 4 terms

$$f(x) = \frac{1}{1+x^2}$$

Known Series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$x = x^2 \therefore$

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6$$

② Integration $f(x) = \cos x$

Use Known Series

$$\int \sin x \, dx = \int x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \, dx$$

$$-\cos x = \frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \frac{1}{8!}x^8 \dots + C$$

(3) Taylor Series for $X \cos X$ about $X=0$

$f(x) = X \cos X$

Use Known Series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$X \cos X = X \sum_{n=0}^{\infty} \frac{(-1)^n X^{2n}}{(2n)!}$$

$$X \cos X = \sum_{n=0}^{\infty} \frac{(-1)^n X^{2n+1}}{(2n)!}$$

(4) $F(x) = e^4 e^{-x^2}$ about $x = 0$

Use Known Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad x = -x^2 = e^{-x^2} \sum_{n=0}^{\infty} \frac{-x^{2n}}{n!}$$

$$e^4 e^{-x^2} = e^4 \sum_{n=0}^{\infty} \frac{-x^{2n}}{n!}$$

$$e^4 e^{-x^2} = \sum_{n=0}^{\infty} \frac{-x^{2n} e^4}{n!}$$

$$e^4 e^{-x^2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n} e^4}{n!}$$

Section 10.10 – The Binomial Series

* These are coefficients **Pascal's Triangle**

0 power 1

1 power 1 + 1

2 power 1 + 2 + 1

3 power 1 + 3 + 3 + 1

4 power 1 4 6 4 1

5 power 1 5 10 10 5 1

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Section 10.10 – The Binomial Series

Pascal's Triangle

Row 0 →	1										
Row 1 →	1	1									
Row 2 →	1	2	1								
Row 3 →	1	3	3	1							
Row 4 →	1	4	6	4	1						
Row 5 →	1	5	10	10	5	1					
Row 6 →	1	6	15	20	15	6	1				
Row 7 →	1	7	21	35	35	21	7	1			
Row 8 →	1	8	28	56	70	56	28	8	1		
Row 9 →	1	9	36	84	126	126	84	36	9	1	
Row 10 →	1	10	45	120	210	252	210	120	45	10	1

Section 10.10 - The Binomial Series

The Binomial Theorem

$$(x+a)^n = \underbrace{\binom{n}{0}}_{\text{coefficients}} x^n + \underbrace{\binom{n}{1}} ax^{n-1} + \underbrace{\binom{n}{2}} a^2x^{n-2} + \underbrace{\binom{n}{3}} a^3x^{n-3} \cdots \binom{n}{n} a^n$$

$$(x+a)^n = \sum_{i=0} \binom{n}{i} a^i x^{n-i} \quad where \quad \binom{n}{i} = \frac{n!}{i!\,(n-i)!}$$

Ex $(2x-3)^4 =$

$$\binom{4}{0}(2x)^4 + \binom{4}{1}(-3)(2x)^{4-1} + \binom{4}{2}(-3)^2(2x)^{4-2} + \binom{4}{3}(-3)^3(2x)^{4-3} + \binom{4}{4}(-3)^4(2x)^{4-4}$$

$$= 1\cdot 16x^4 + 4(-3)(2x)^3 + 6(-3)^2(2x)^2 + 4(-3)^3(2x)^1 + 1(-3)^4(2x)^0$$

$$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$$

Section 10.10 - The Binomial Series

The Binomial Theorem

$$(x+a)^n = \binom{n}{0} x^n + \binom{n}{1} ax^{n-1} + \binom{n}{2} a^2x^{n-2} + \binom{n}{3} a^3x^{n-3} \cdots \binom{n}{n} a^n$$

$$(x+a)^n = \sum_{i=0}^{n} \binom{n}{i} a^i x^{n-i} \quad where \quad \binom{n}{i} = \frac{n!}{i!\,(n-i)!}$$

$(x^3+2)^3 =$

$$\binom{3}{0}(x^3)^3 + \binom{3}{1}2(x^3)^{3-1} + \binom{3}{2}2^2(x^3)^{3-2} + \binom{3}{3}2^3(x^3)^{3-3}$$

$$= 1\cdot x^9 + 3\cdot 2(x^3)^2 + 3\cdot 4(x^3)^1 + 1\cdot 8(x^3)^0$$

$$= x^9 + 6x^6 + 12x^3 + 8$$

Section 10.10 - The Binomial Series

The Binomial Series

Formula

The Binomial Series
For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

The Binomial series is similar to the Binomial Theorem. The difference between the two is that the Binomial series is infinite.

Section 10.10 - The Binomial Series

Euler's Identity

DEFINITION

For any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$. (4)

The identity is used in working with the imaginary unit in exponential functions. The Taylor series for the exponential, sine and cosine functions are required to develop the identity

10.10 Examples

(1) $(a+b)^3 = (a^2 + ab + ab + b^2)(a+b)$

$= (a^3 + a^2b + a^2b + ab^2 + a^2b + ab^2 + ab^2 + b^3)$

$= a^3 + 3a^2b + 3ab^2 + b^3$

1

1 1

1 2 1

* 1 3 3 1 $\rightarrow 1(a)^3(b)^0 + 3(a)^2(b)^1 + 3(a)(b)^2 + 1(a)^0(b)^3$

1 4 6 4 1

(2) $(2x+3y)^4 = 1(2x)^4(3y)^0 + 4(2x)^3(3y)^1 + 6(2x)^2(3y)^2$
$+ 4(2x)^1(3y)^3 + 1(2x)^0(3y)^4$

$$\boxed{16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4}$$

On calc $(a+b)^4$

4 > Math > PRB > $\boxed{3}$ nCr > 2nd Bracket > {0, 1, 2, 3, 4}

= 1, 4, 6, 4, 1

(3) $(x^3+2)^3 = 1(x^3)^3(2)^0 + 3(x^3)^2(2)^1 + 3(x^3)^1(2)^2 + 1(x^3)^0(2)^3$
$= \boxed{x^9 + 6x^6 + 12x^3 + 8}$

(4) Find the first five terms

$\left(2 + \frac{x}{3}\right)^{12}$

$= 1(2)^{12}(x)^0 + 12(2)^{11}(x/3)^1 + 66(2)^{10}(x/3)^2 + 220(2)^9(x/3)^3 + 495(2)^8(x/3)^4$

$4096 + 24{,}576\frac{x}{3} + 67{,}584\frac{x^2}{9} + 112{,}640\frac{x^3}{27} + 126{,}720\frac{x^4}{81}$

$\boxed{4096 + 8192x + \frac{22528}{3}x^2 + \frac{112640}{27}x^3 + \frac{14080}{9}x^4}$

General Binomial Series

(5) $f(x) = \frac{1}{(1+x)^4} = (1+x)^{-4} = m = -4 \quad K = 0,1,2,3,4$

$\binom{-4}{1} = -4 \quad \binom{-4}{2} = \frac{-4(-4-1)}{2!} \quad \binom{-4}{3} = \frac{-4(-4-1)(-4-2)}{3!}$

$\binom{-4}{4} = \frac{-4(-4-1)(-4-2)(-4-3)}{4!}$

$f(x) = 1 + \binom{-4}{1}x^1 + \binom{-4}{2}x^2 + \binom{-4}{3}x^3 + \binom{-4}{4}x^4$

$1 + (-4)x + (10)x^2 + (-20)x^3 + (35)x^4 + \ldots$

$1 - 4x + 10x^2 - 20x^3 + 35x^4 \ldots$

(6) $f(x) = \left(1 - \frac{x}{3}\right)^4 \qquad m=4 \quad X = -\frac{x}{3}$

$$1 + \sum_{k=1}^{\infty} \binom{m}{k} X^k$$

$\binom{4}{1} = 4 \quad \binom{4}{2} = \frac{4(4-1)}{2!} \quad \binom{4}{3} = \frac{4(4-1)(4-2)}{3!}$

$\binom{4}{4} = \frac{4(4-1)(4-2)(4-3)}{4!}$

$$f(x) = 1 + \binom{4}{1}\left(\frac{-x}{3}\right)^1 + \binom{4}{2}\left(\frac{-x}{3}\right)^2 + \binom{4}{3}\left(\frac{-x}{3}\right)^3 + \binom{4}{4}\left(\frac{-x}{3}\right)^4$$

$$f(x) = 1 + 4\left(\frac{-x}{3}\right) + 6\left(\frac{-x^2}{9}\right) + 4\left(\frac{-x^3}{27}\right) + 1\left(\frac{-x^4}{81}\right)$$

$$\boxed{f(x) = 1 - \frac{4}{3}x + \frac{2}{3}x^2 - \frac{4}{27}x^3 + \frac{1}{81}x^4}$$

7) $f(x) = (1+x^3)^{-\frac{1}{2}}$ $\quad m = -\frac{1}{2}, \ X = x^3$

$$1 + \sum_{k=1}^{\infty} \binom{m}{k} X^k$$

$$\binom{-\frac{1}{2}}{1} = -\frac{1}{2} \quad \binom{-\frac{1}{2}}{2} = \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!} \quad \binom{-1/2}{3} \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}$$

$$\binom{-\frac{1}{2}}{4} = \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)(-\frac{1}{2}-3)}{4!}$$

$$f(x) = 1 + \binom{-\frac{1}{2}}{1}(x^3)^1 + \binom{-1/2}{2}(x^3)^2 + \binom{-1/2}{3}(x^3)^3 + \binom{-1/2}{4}(x^3)^4$$

$$f(x) = 1 + (-1/2)x^3 + \left(\frac{3}{8}\right)x^6 + \left(\frac{-5}{16}\right)x^9 + \left(\frac{35}{128}\right)x^{12}$$

$$\boxed{f(x) = 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \frac{35}{128}x^{12}}$$

8) $f(x) = (9-x)^{1/2} = 3\left(1 - \frac{x}{9}\right)^{1/2}$ $\quad m = \frac{1}{2} \ X = -\frac{x}{9}$

$$\binom{1/2}{1} = \frac{1}{2} \quad \binom{1/2}{2} = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \quad \binom{1/2}{3} \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} \quad \binom{1/2}{4} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(-}{4!}$$

$$f(x) = 3\left[1 + \binom{1/2}{1}\left(\frac{-x}{9}\right)^1 + \binom{1/2}{2}\left(\frac{-x}{9}\right)^2 + \binom{1/2}{3}\left(\frac{-x}{9}\right)^3 + \binom{1/2}{4}\left(\frac{-x}{9}\right)^4 + \dots\right]$$

$$f(x) = 3\left[1 - \frac{1}{18}x - \frac{1}{648}x^2 - \left(\frac{1}{16}\right)\left(\frac{x^3}{729}\right) - \left(\frac{5}{128}\right)\left(\frac{x^4}{6561}\right) - \dots\right]$$

$$f(x) = 3 - \frac{1}{6}x - \frac{1}{216}x^2 - \left(\frac{1}{\underset{3}{\underline{11664}}}\right)x^3 - \left(\frac{5}{\underset{3}{\underline{839808}}}\right)x^4 - \dots$$

$$\boxed{f(x) = 3 - \frac{1}{6}x - \frac{1}{216}x^2 - \frac{1}{3888}x^3 - \frac{5}{279936}x^4}$$

11.1 - Parametrizations of Plane Curves

Parametric Equations:

another process to describe a plane curve on the Cartesian coordinate system.

The coordinates (x and y) are described by the same auxiliary variable called the parameter (t).

Parametric equations are continuous on an interval.

There are different parameterizations that produce the same graph or curve.

The parameter (t) usually represent time. As it increases, it gives direction to the curve.

11.1 - Parametrizations of Plane Curves

Simple curve - distinct values of the parameter yield distinct points. No overlapping occurs.

A simple curve can be drawn without lifting the pencil from the paper, and without passing through any point twice.

Closed curve - initial and final values of the parameter on an interval yield the same point.

A closed curve has the same starting and ending points, and is also drawn without lifting the pencil from the paper.

Simple; closed	Simple; not closed	Not simple; closed	Not simple; not closed

11.1 – Parametrizations of Plane Curves

$x = 2t - 1 \quad y = t + 1 \quad 0 \le t \le 2$

$x = 2t^3 + 1 \quad y = t^3 + 2 \quad -1 \le t \le 1$

$x = 3\sqrt{t-3} \quad y = 2\sqrt{4-t} \quad 3 \le t \le 4$

$x = 3\sin(t) \quad y = -\cos(t) \quad 0 \le t \le 2r$

11.1 – Parametrizations of Plane Curves

(4) $x = 3\sqrt{t-3}$ $\quad y = 2\sqrt{4-t}$ $\quad 3 \le t \le 4$

t	x	y
3	0	2
4	3	0

$x = 3\sqrt{t-3}$

$(t-3)^{1/2} = \frac{x}{3}$

$t = \left(\frac{x}{3}\right)^2 + 3$

$y = 2\sqrt{4-t} \longrightarrow y = 2\sqrt{4-\left(\frac{x}{3}\right)^2-3}$

$y = 2\sqrt{1-\frac{x^2}{9}} \longrightarrow y^2 = 4\left(1-\frac{x^2}{9}\right)$

$y^2 = 4 - \frac{4}{9}x^2 \longrightarrow \frac{y^2}{4} + \frac{x^2}{9} = 1$

$\boxed{\frac{x^2}{9} + \frac{y^2}{4} = 1}$ Same eq. as #3

(5) Find the parameterization for the line segment between (2, -1) & (6, 5)

find the slope segment

$m = \frac{5+1}{6-2} = \frac{6}{4} = \frac{3}{2}$

$y - y_1 = m(x - x_1) \longrightarrow y + 1 = \frac{3}{2}(x-2)$

Let $t = x - 2 \longrightarrow t + 2 = x$

$y + 1 = \frac{3}{2}t \longrightarrow y = \frac{3}{2}t - 1$

Domain of t:

$x=2$ $\quad$ $x=6$

$t=2-2$ $\quad$ $t=6-2$

$t=0$ $\quad$ $t=4$

$0 \leq t \leq 4$

Parametric Eqs.

$x=t+2$

$y=\frac{3}{2}t-1$

Other Method

bring it to $y=mx+b$ form

$y=\frac{3}{2}x-4$

Natural Parametrization

$\left.\begin{array}{l} \text{Let } x=t \\ y=\frac{3}{2}t-4 \end{array}\right\}$ Both eqs.

Domain of t

$x=2$ $\quad$ $x=6$

$t=2$ $\quad$ $t=6$

$2 \leq t \leq 6$

11.1 Examples

① $x = 2t - 1 \quad y = t + 1 \quad 0 \leq t \leq 2$

t	X	Y
0	-1	1
2	3	3

Solve for t

$x = 2t - 1$

$2t = x + 1$

$t = \frac{x+1}{2}$

$y = t + 1$

$y = \left(\frac{x+1}{2}\right) + 1$

$y = \frac{1}{2}x + \frac{3}{2}$

② $x = 2t^3 + 1 \quad y = t^3 + 2 \quad -1 \leq t \leq 1$

t	X	Y
-1	-1	1
0	1	2
1	3	3

$x = 2t^3 + 1 \rightarrow 2t^3 = x - 1$

$t^3 = \frac{x-1}{2} \rightarrow t = \sqrt[3]{\frac{x-1}{2}}$

$y = t^3 + 2 \rightarrow y = \frac{x-1}{2} + 2$

$\boxed{y = \frac{1}{2}x + \frac{3}{2}} \quad -1 \leq x \leq 3$

Same curve from ① but diferent parametric equation

(3) $x = 3\sin t \quad y = -2\cos t \qquad 0 \le t \le 2\pi$

t	x	y
0	0	-2
2π	0	-2

$$\cancel{t = \sin^{-1}\left(\frac{x}{3}\right)}$$

$$\cancel{y = -2\cos t}$$

$$\cancel{y = -2\cos\left(\sin^{-1}\frac{x}{3}\right)}$$

instead

$$x^2 = (3\sin t)^2 \qquad y^2 = (-2\cos t)^2$$

$$x^2 = 9\sin^2 t \qquad y^2 = 4\cos^2 t$$

$$\sin^2 t = \frac{x^2}{9} \qquad \cos^2 t = \frac{y^2}{4}$$

$$\frac{x^2}{9} + \frac{y^2}{4} = \sin^2 t + \cos^2 t$$

$$\boxed{\frac{x^2}{9} + \frac{y^2}{4} = 1}$$

11.2 - Calculus with Parametric Curves

Derivatives of Parametric Equations

First derivative:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Second Derivative:

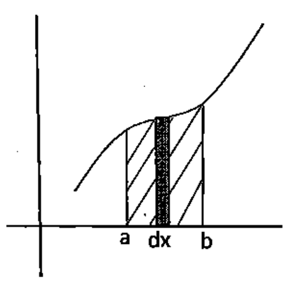
$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

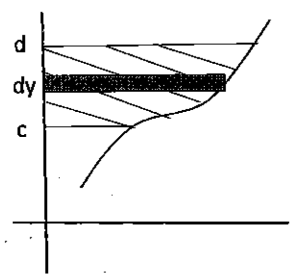
11.2 - Calculus with Parametric Curves

Integration of Parametric Equations

$$\int_a^b y\,dx$$

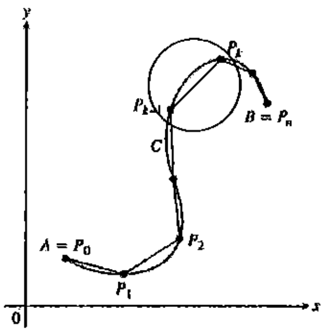
$$\int_c^d x\,dy$$

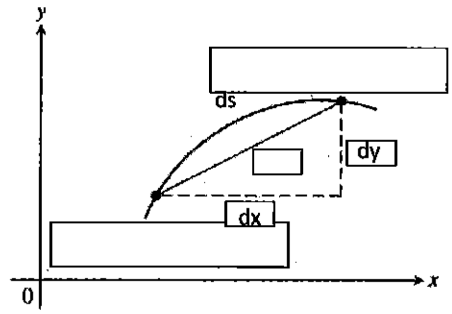




11.2 – Calculus with Parametric Curves

Arc Length





$$(dx)^2 + (dy)^2 = (ds)^2$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2$$

11.2 – Calculus with Parametric Curves

Arc Length

$$(dx)^2 + (dy)^2 = (ds)^2$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{ds}{dt}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \mathrm{ds}$$

$$\int_{LL}^{UL} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{LL}^{UL} ds$$

$$\int_{LL}^{UL} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \mathrm{s}$$

$$\boxed{arc\ length\ (s) = \int_{LL}^{UL} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Arc Length Formula

11.2 - Calculus with Parametric Curves

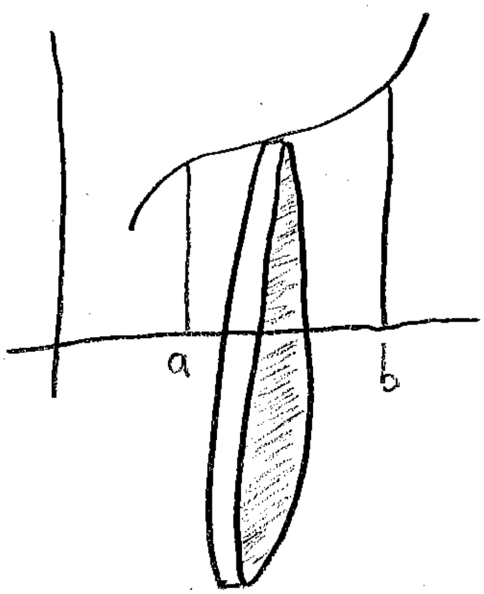
Surface Area

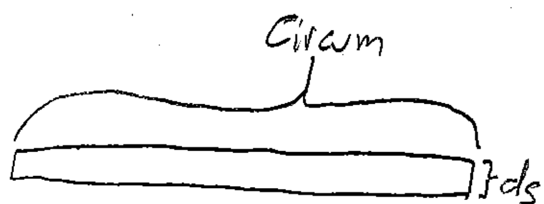
$$surface\ area\ (SA) = \int_{LL}^{UL} 2\pi r\, ds$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \mathrm{ds}$$

$$\boxed{surface\ area\ (SA) = \int_{LL}^{UL} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Surface Area formula





11.2 Examples

① $x = 6t^2 \quad , y = t^3 \qquad t \neq 0$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \frac{dy}{dt} = 3t^2, \ \frac{dx}{dt} = 12t$$

$$\frac{dy}{dx} = \frac{3t^2}{12t} = \boxed{\frac{1}{4}t} \qquad \frac{d^2y}{dx^2} = \frac{\frac{dy'}{dx}}{dt}$$

$$\frac{dy'}{dt} = \frac{d}{dt} \cdot \frac{dy}{dx} \rightarrow \frac{d}{dt} \cdot \left(\frac{1}{4}t\right) = \frac{1}{4}$$

$$\frac{d^2y}{dx^2} = \frac{1/4}{12t} = \boxed{\frac{1}{48t}}$$

② $x = t^2 \quad , y = t^3$

$$\frac{dy}{dx} \ \frac{\frac{dy}{dt}}{\frac{dx}{dt}} , \quad \frac{dy}{dt} = 3t^2 \quad , \quad \frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{3t^2}{2t} = \boxed{\frac{3}{2}t} \qquad \frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt} = \underbrace{\frac{d}{dt} \cdot \frac{dy}{dx}}$$

$$\frac{\frac{3}{2}}{2t} = \boxed{\frac{3}{4t}} \qquad \boxed{\frac{d}{dt} \cdot \frac{3}{2}t = \frac{3}{2}}$$

Eq. of a tangent line for 2 @ $t=1$

$X=1$, $y=1$ $m=\frac{dy}{dx}=\frac{3}{2}t$

$y-y_1=m(x-x_1)\rightarrow y-1=\frac{3}{2}t(x-1)$

$y-1=\frac{3}{2}(1)(x-1)\longrightarrow y=\frac{3}{2}x-\frac{3}{2}+1$

$\boxed{y=\frac{3}{2}x-\frac{1}{2}}$

3 $X=2e^{t}$, $y=\frac{1}{3}e^{-t}\rightarrow$

$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, $\frac{dy}{dt}=\frac{-1}{3}e^{-t}$, $\frac{dx}{dt}=2et$

$\frac{dy}{dx}=\frac{-\frac{1}{3}e^{-t}}{2e^{t}}=\frac{-1}{3e^{t}}\cdot\frac{1}{2e^{t}}=\boxed{\frac{-1}{6e^{2t}}}$

$\frac{d^2y}{dx^2}=\frac{d}{dt}\cdot\frac{dy}{dx}=\frac{2}{6}e^{-2t}$

$\frac{\frac{dy'}{dt}}{\frac{dx}{dt}}=\frac{\frac{1}{3e^{2t}}}{2e^{t}}=\frac{1}{3e^{2t}}\cdot\frac{1}{2e^{t}}=\boxed{\frac{1}{6e^{3t}}}$

Eq of a line @ $t=0$

$y = \frac{1}{3}e^{0} = \frac{1}{3}$ $\quad x = 2e^{0} = 2$

$y - y_1 = m(x - x_1)$, $m = \frac{-1}{6e^{2(0)}} = \frac{-1}{6}$

$y - \frac{1}{3} = -\frac{1}{6}(x-2)$

(4) $x = 2\sin t$, $y = 5\cos t$ $\quad t = \frac{\pi}{3}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, $\frac{dx}{dt} = 2\cos t$, $\frac{dy}{dt} = -5\sin t$

$\frac{dy}{dx} = \frac{-5\sin t}{2\cos t} = \frac{-5}{2}\tan t$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\cdot\frac{dy}{dx} = \frac{\frac{-5}{2}\sec^2 t}{2\cos t} = \frac{-5}{2}\sec^2 t \cdot \frac{1}{2\cos t}$

$\boxed{\frac{-5}{4}\sec^3 t}$ or $\boxed{\frac{-5}{4\cos^3 t}}$

Eq. of a line when $t = \pi/3$

$X = 2\sin\left(\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

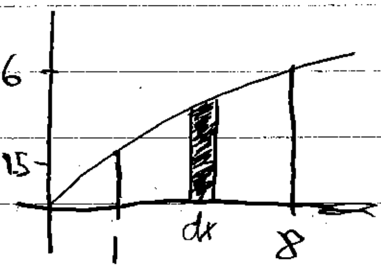
$y = 5\cos\left(\frac{\pi}{3}\right) = 5\left(\frac{1}{2}\right) = \frac{5}{2}$

$m = -\frac{5}{2}\left(\frac{\sqrt{3}}{1/2}\right) = -\frac{5\sqrt{3}}{2}$

$y - \frac{5}{2} = -\frac{5\sqrt{3}}{2}(X - \sqrt{3})$

Integrations

(5) $X = t^3$, $y = \frac{3}{2}t^2$ Area from $1 < t < 2$



$$A = \int_{t_1}^{t_2} y\,dx$$

$X = t^3$
$dx = 3t^2\,dt$ $\therefore A = \int_1^2 \frac{3}{2}t^2 \cdot 3t^2\,dt$

$$A = \int_1^2 \frac{9}{2}t^4\,dt = \frac{9}{2}\left[\frac{1}{5}t^5\right]\Big|_1^2$$

$$= \frac{9}{10}\left[2^5 - 1^5\right] = \boxed{\frac{279}{10}}$$

(6) $X = \text{Cos}\,t$ $y = \sqrt{3}\,\text{Cos}\,t$ $0 < t < \frac{\pi}{3}$

$$A = \int_0^{\pi/3} y\,dx \qquad dx = -\text{Sin}\,t\,dt$$

$$A = \int_0^{\pi/3} \sqrt{3}\,\text{Cos}\,t\,(-\text{Sin}\,t\,dt) = \int_0^{\pi/3} -\sqrt{3}\,\text{Cos}\,t\,\text{Sin}\,t\,dt \qquad U = \text{Cos}\,t,\ dU = -\text{Sin}\,t\,dt$$

$$A = \sqrt{3}\int_0^{\pi/3} U\,dU = \frac{\sqrt{3}}{2}\left[\text{Cos}^2 t\right]\Big|_0^{\pi/3} = \boxed{\frac{-3\sqrt{3}}{8}}$$

Cannot have neg area
So reverse limit boundries

(6) $x = t^{-1}$, $y = 3t^2 + 2$ $\quad 0.2 < t < 0.5$

need to reverse limits, $dx = -t^{-2}\,dt$

$$A = \int_{0.5}^{0.2} (3t^2+2)(-t^{-2})\,dt = \int_{.5}^{.2} -3t^0 - 2t^{-2}\,dt$$

$$A = \int_{.5}^{.2} -2t^{-2} - 3\,dt = \int_{.5}^{.2} -2t^{-2}\,dt - \int_{.5}^{.2} dt$$

$$A = 2\left[t^{-1}\right]\Big|_{.5}^{.2} - \left[3t\right]\Big|_{.5}^{.2}$$

$$A = 2\left[\frac{1}{.2} - \frac{1}{.5}\right] - 3\left[.2 - .5\right]$$

$$A = 6 - [-.9] = \boxed{6.9}$$

Use calculator

MAth > 9 > fint($-2t^{-2} - 3$, t, 0.5, 0.2)

= 6.9

D06164

Arc Length

7

$X = \frac{1}{3}t^3 \quad , \quad Y = \frac{1}{2}t^2 \qquad 1.5 \leq t \leq 3$

$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt \qquad \frac{dx}{dt} = t^2, \ \frac{dy}{dt} = t$

$\left(\frac{dx}{dt}\right)^2 = t^4, \ \left(\frac{dy}{dt}\right)^2 = t^2$

$S = \int_{1.5}^{3} \sqrt{t^4 + t^2}\, dt = \int_{1.5}^{3} t\sqrt{t^2+1} \qquad U = t^2+1$

$du = 2t\, dt$

$S = \frac{1}{2}\int_{1.5}^{3} 2t\, (t^2+1)^{1/2} dt = \frac{1}{2}\int_{1.5}^{3} U^{1/2} du$

$S = \frac{1}{2}\left[\frac{U^{3/2}}{3/2}\right]\Big|_{1.5}^{3} = \frac{1}{3}\left[(t^2+1)^{3/2}\right]\Big|_{1.5}^{3}$

$\boxed{8.588 \text{ units}}$

(8) $X = 2\sin t \quad Y = 2\cos t \quad 0 \le t \le \pi$

$$S = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad X' = 2\cos t,\ (X')^2 = 4\cos^2 t$$
$$Y' = -2\sin t,\ (y')^2 = 4\sin^2 t$$

$$S = \int_0^{\pi} (4\cos^2 t + 4\sin^2 t)^{1/2} dt$$

$$S = \int_0^{\pi} 2(\cos^2 t + \sin^2 t)^{1/2} dt$$

$$S = \int_0^{\pi} 2(1)^{1/2} dt = \int_0^{\pi} 2\, dt$$

$$S = \left[2t\right]_0^{\pi} = \boxed{2\pi}$$

Surface Area

(9) $X = t \qquad Y = t^3 \qquad 0 \leq t \leq 1$

$$SA = \int_{t^1}^{t^2} 2\pi y \, ds \qquad \begin{array}{l} x' = 1, \ (x')^2 = 1 \\ y' = 3t^2, \ (y')^2 = 9t^4 \end{array}$$

$$ds = = \left((x')^2 + (y')^2\right)^{1/2} dt = ds = (1 + 9t^4)^{1/2} dt$$

$$SA = \int_0^1 2\pi (t^3)(1+9t^4)^{1/2} dt \qquad \begin{array}{l} U = 1 + 9t^4 \\ dU = 36t^3 dt \end{array}$$

$$SA \quad \frac{1}{18} \int_0^1 2(18)\pi t^3 (1+9t^4)^{1/2} dt$$

$$SA = \frac{1}{18} \int_0^1 36t^3 \pi (1+9t^4)^{1/2} dt$$

$$SA = \frac{1\pi}{18} \int_0^1 U^{1/2} dU \rightarrow \frac{1\pi}{18} \left[\frac{2}{3} U^{3/2}\right]\Big|_0^1$$

$$SA = \frac{2\pi}{54} \left[U^{3/2}\right]\Big|_0^1 = \frac{\pi}{27} \left[(1+9t^4)^{3/2}\right]\Big|_0^1$$

$$SA = \frac{\pi}{27} \left[10^{3/2}\right] = \boxed{SA = 3.563}$$

(10) $X = \frac{2}{3} t^{3/2}$ $\quad y = 2\sqrt{t} = 2t^{1/2}$

$0.5 \leq t \leq \sqrt{3}$

$\left(\frac{dx}{dt}\right)^2 = t \quad \left(\frac{dy}{dt}\right)^2 = t^{-1}$

$$SA = \int_{.5}^{\sqrt{3}} 2\pi \frac{2}{3} t^{3/2} \cdot \sqrt{t + t^{-1}}\, dt$$

$$= 2\pi \int \frac{2}{3} t^{3/2} \sqrt{\frac{t^2+1}{t}}$$

$$= 2\pi \int \frac{2}{3} t^{3/2} \frac{\sqrt{t^2+1}}{t^{1/2}}\, dt \qquad \frac{3}{2} - \frac{1}{2} = 1 = t^1$$

$$= 2\pi \int \frac{2}{3} t^1 (t^2+1)^{1/2} dt \qquad U = t^2 + 1) \quad du = 2t\, dt$$

$$\frac{2\pi}{3} \int 2t\, dt\, (t^2+1)^{1/2}$$

$$\frac{2\pi}{3} \int U^{1/2} du = \frac{2\pi}{3} \left[\frac{2}{3} U^{3/2}\right]\Big|_{.5}^{\sqrt{3}}$$

$$= \frac{4\pi}{9} \left[(t^2+1)^{3/2}\right]\Big|_{.5}^{\sqrt{3}}$$

$$= \boxed{9.219 \text{ Sq Units}}$$

11.3 – Polar Coordinates

Polar Coordinate System

Another method to plot and specify the location of points based on simple trigonometry.

Used to plot and analyze equations of conics (circles, parabolas, ellipses, and hyperbolas.

The polar coordinates consist of a length (radius) and an angle.

$$(radius, angle) = \underbrace{(r, \theta)}_{\text{in this order}}$$

11.3 – Polar Coordinates

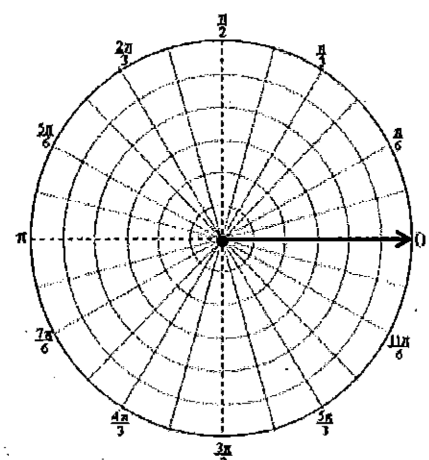
Polar Coordinate System

Polar axis – the positive x-axis

Pole – the point at which the polar axis begins and any radius begins. (0,0)

Radius – the first coordinate in an ordered pair of polar coordinates, measured from the pole and is a directed distance.

Angle – the second coordinate in an ordered pair of polar coordinates, where positive direction is measured from the polar axis in a counter-clockwise direction.



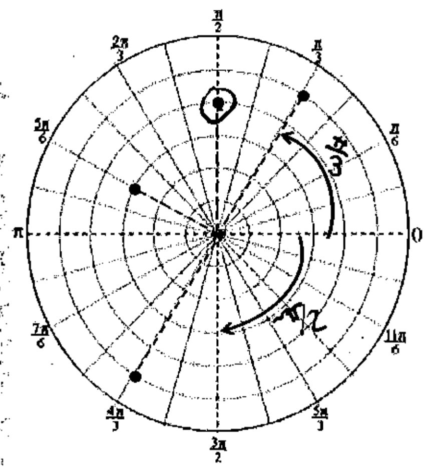
11.3 – Polar Coordinates

Polar Coordinate System

Examples

$\left(5, \frac{\pi}{3}\right)$ $\left(-5, \frac{\pi}{3}\right)$

$\left(3, \frac{5\pi}{6}\right)$ $\boxed{\left(-4, -\frac{\pi}{2}\right)}$



11.3 – Polar Coordinates

Polar Coordinate System

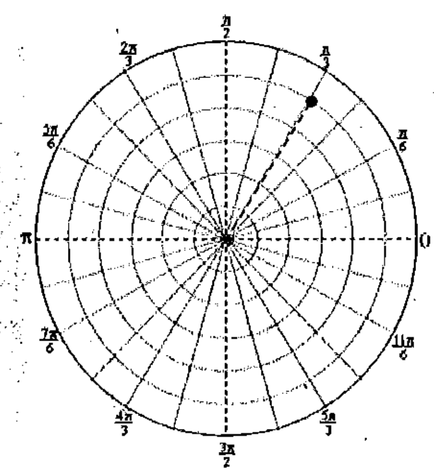
The same point in a polar coordinate system may be described using an infinite number of polar coordinates.

Examples:

$\left(5, \frac{\pi}{3}\right)$ $\left(-5, \frac{4\pi}{3}\right)$

$\left(-5, -\frac{2\pi}{3}\right)$ $\left(5, -\frac{5\pi}{3}\right)$

$\left(5, \frac{7\pi}{3}\right)$ $\left(-5, -\frac{8\pi}{3}\right)$



11.3 – Polar Coordinates

Trigonometry

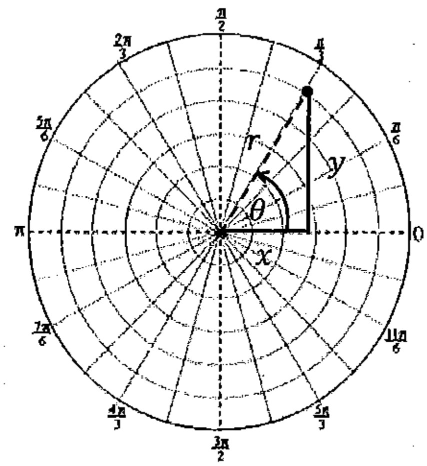
(r, θ)

$x = r\cos\theta$

$y = r\sin\theta$

$\tan\theta = \dfrac{y}{x}$

$x^2 + y^2 = r^2$



11.3 – Polar Coordinates

Calculate the polar coordinates given the Cartesian coordinates of a point.

$(8, 6)$

$x^2 + y^2 = r^2$

$8^2 + 6^2 = r^2$

$10 = r$

$\tan\theta = \dfrac{y}{x}$

$\tan\theta = \dfrac{6}{8}$

$\theta = \tan^{-1}\dfrac{6}{8}$

$\theta = 0.644 \ \ rad.$

$(10, 0.644)$

$(-3, -5) \quad 3rd\ quadrant$

$x^2 + y^2 = r^2$

$(-3)^2 + (-5)^2 = r^2$

$5.831 = r$

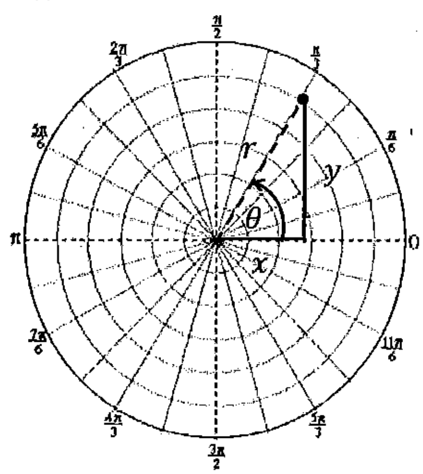
$\tan\theta = \dfrac{y}{x}$

$\tan\theta = \dfrac{-5}{-3}$

$\theta = \tan^{-1}\dfrac{-5}{-3}$

$\theta = 1.030 \ \ rad.$

$(-5.831, 1.030)$



11.3 – Polar Coordinates

Calculate the Cartesian coordinates given the polar coordinates of a point.

$\left(13, \frac{3\pi}{10}\right)$

$x = r\cos\theta$

$x = 13\cos\frac{3\pi}{10}$

$x = 7.641$

$y = r\sin\theta$

$y = 13\sin\frac{3\pi}{10}$

$y = 10.517$

$(7.641, 10.517)$

$(-3, -3.5)$ $4th\ quadrant$

$x = r\cos\theta$

$x = -3\cos(-3.5)$

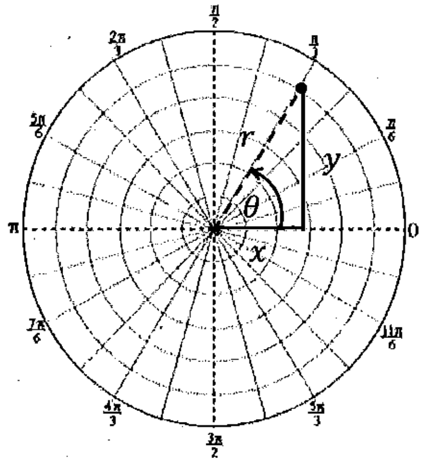
$x = 2.810$

$y = r\sin\theta$

$y = -3\sin(-3.5)$

$y = -1.052$

$(2.810, -1.052)$



11.3 – Polar Coordinates

Convert each of the given equations from Cartesian to polar or from polar to Cartesian .

$x = r\cos\theta \quad y = r\sin\theta \quad \tan\theta = \frac{y}{x} \quad x^2 + y^2 = r^2$

$x = 3$

$r\cos\theta = 3$

$r = \frac{3}{\cos\theta}$

$y = 3x + 5$

$r\sin\theta = 3r\cos\theta + 5$

$r\sin\theta - 3r\cos\theta = 5$

$r(\sin\theta - 3\cos\theta) = 5$

$r = \frac{5}{\sin\theta - 3\cos\theta}$

$2x^2 + 3y^2 = 4$

$2r^2\cos^2\theta + 3r^2\sin^2\theta = 4$

$r^2(2\cos^2\theta + 3\sin^2\theta) = 4$

$r^2 = \frac{4}{2\cos^2\theta + 3\sin^2\theta}$

11.3 – Polar Coordinates

Convert each of the given equations from Cartesian to polar or from polar to Cartesian .

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \tan\theta = \frac{y}{x} \qquad x^2 + y^2 = r^2$$

$$r\sin\theta + 6 = 0$$
$$y + 6 = 0$$
$$y = -6$$

$$\theta = \frac{7\pi}{6}$$
$$\tan\theta = \tan\frac{7\pi}{6}$$
$$\frac{y}{x} = \frac{-1/2}{-\sqrt{3}/2}$$
$$\frac{y}{x} = \frac{1}{\sqrt{3}}$$
$$y = \frac{\sqrt{3}}{3}x$$

$$r = \frac{2}{1 - \cos\theta}$$
$$r - r\cos\theta = 2$$
$$r - x = 2$$
$$r = x + 2$$
$$r^2 = (x + 2)^2$$
$$x^2 + y^2 = x^2 + 4x + 4$$
$$y^2 = 4x + 4$$
$$\frac{1}{4}y^2 - 1 = x$$

11.3 – Polar Coordinates

Convert each of the given equations from Cartesian to polar or from polar to Cartesian .

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \tan\theta = \frac{y}{x} \qquad x^2 + y^2 = r^2$$

$$r = 4\tan\theta\sec\theta$$
$$r = \frac{4\sin\theta}{\cos\theta\cos\theta}$$
$$r = \frac{4\sin\theta}{\cos^2\theta}$$
$$r\cos^2\theta = 4\sin\theta$$
$$r^2\cos^2\theta = 4r\sin\theta$$
$$x^2 = 4y$$
$$y = \frac{1}{4}x^2$$

11.3 – Polar Coordinates

Convert each of the given equations from Cartesian to polar or from polar to Cartesian .

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \tan\theta = \frac{y}{x} \qquad x^2 + y^2 = r^2$$

$$r^2 - 6r\cos\theta - 4r\sin\theta + 9 = 0$$

$$x^2 + y^2 - 6x - 4y = -9$$

$$x^2 - 6x + y^2 - 4y = -9$$

complete the square

$$\frac{-6}{2} = -3 \quad (-3)^2 = 9 \qquad \frac{-4}{2} = -2 \quad (-2)^2 = 4$$

$$x^2 - 6x + 9 + y^2 - 4y + 4 = -9 + 9 + 4$$

$$(x-3)^2 + (y-2)^2 = 4$$

$$circle: \ center\ (3,2), \qquad radius = 2$$

11.3 Examples

(1) $(-3, -5)$

$r^2 = x^2 + y^2$

$r = \sqrt{(-3)^2 + (-5)^2} \rightarrow \boxed{r = -5.831}$ when plotting

$\tan\theta = \frac{y}{x} \rightarrow \theta = \tan^{-1}\left(\frac{-5}{-3}\right) = \boxed{\theta = 1.030}$

(2) Calculate Cartesian Coord

$(\underset{r}{13}, \underset{\theta}{\tfrac{3\pi}{10}})$ 1st quad

$x = r\cos\theta \rightarrow x = 13\cos\left(\frac{3\pi}{10}\right) \rightarrow \boxed{x = 7.641}$

$y = r\sin\theta \rightarrow y = 13\sin\left(\frac{3\pi}{10}\right) \rightarrow \boxed{y = 10.517}$

(3) Cartesian Coord of $(-3, -3.5)$

$x = r\cos\theta \rightarrow -3\cos(-3.5) \rightarrow \boxed{x = 2.809}$

$y = r\sin\theta \rightarrow y = -3\sin(-3.5) \rightarrow \boxed{y = -1.052}$

(4) $y = 3x + 5$ solve for r

$r\sin\theta = 3r\cos\theta + 5$

$r\sin\theta - 3r\cos\theta = 5$

$r(\sin\theta - 3\cos\theta) = 5$

$r = \dfrac{5}{\sin\theta - 3\cos\theta}$

(5) $2x^2 + 3y^2 = 4$

$2(r\cos\theta)^2 + 3(r\sin\theta)^2 = 4$

$2r^2\cos^2\theta + 3r^2\sin^2\theta = 4$

$r^2(2\cos^2\theta + 3\sin^2\theta) = 4$

$$r^2 = \frac{4}{2\cos^2\theta + 3\sin^2\theta}$$

(6) $\theta = \frac{7\pi}{6}$

$\tan\theta = \tan\left(\frac{7\pi}{6}\right) \Rightarrow \frac{y}{x} = \frac{-1/2}{-\sqrt{3}/2}$

$\frac{y}{x} = \frac{1}{\sqrt{3}} \longrightarrow \boxed{y = \frac{\sqrt{3}}{3}x}$

(7) $r = \frac{2}{1-\cos\theta}$

$r(1-\cos\theta) = 2 \longrightarrow r(\sin\theta - r\cos\theta) = 2$

$r\sin\theta - r\cos\theta = 2 \longrightarrow$

OR $r - r\cos\theta = 2 \longrightarrow r - x = 2$

$r = x + 2 \longrightarrow r^2 = (x+2)^2$

⑧ $r = 4\tan\theta \sec\theta$

$r = 4\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right) \rightarrow r = \frac{4\sin\theta}{\cos^2\theta}$

$r\cos^2\theta = 4\sin\theta \rightarrow r^2\cos^2\theta = 4r\sin\theta$

$x^2 = 4y \rightarrow \boxed{y = \frac{1}{4}x^2}$

⑨ $r^2 - 6r\cos\theta - 4r\sin\theta + 9 = 0$

$x^2 + y^2 - 6x - 4y + 9 = 0$

$x^2 - 6x + y^2 - 4y = -9$

$x^2 - 6x + \underline{9} + y^2 - 4y + \underline{4} = -9 + \underline{9} + \underline{4}$ Complete the Square

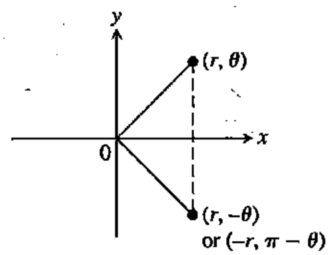
$(x-3)^2 + (y-2)^2 = 4$

Circle: center $(3,2)$ radius $= 2$

$\boxed{-6\left(\frac{1}{2}\right) = (-3)^2 = \boxed{9}}$

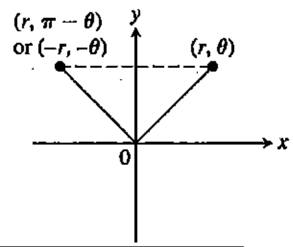
11.4 – Graphing in Polar Coordinates

Polar Symmetries



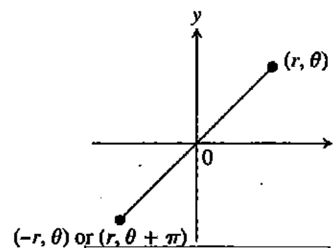
$x - axis\ symmetry$

$(r, \theta) = (r, -\theta)$



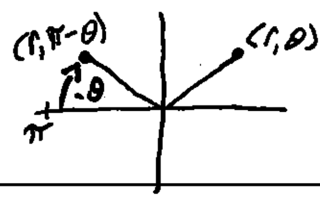
$y - axis\ symmetry$

$(r, \theta) = (r, \pi - \theta)$



$origin\ symmetry$

$(r, \theta) = (r, \pi + \theta)$



11.4 – Graphing in Polar Coordinates

Polar Symmetries

$x - axis\ symmetry$	$y - axis\ symmetry$	$origin\ symmetry$
$(r, \theta) = (r, -\theta)$	$(r, \theta) = (r, \pi - \theta)$	$(r, \theta) = (r, \pi + \theta)$

Find the symmetries $\quad r = \cos\theta - 4$

$r\left(\frac{\pi}{3}\right) = -3.5$	$r\left(\frac{\pi}{3}\right) = \boxed{-3.5}$	$r\left(\frac{\pi}{3}\right) = -3.5$
$r\left(-\frac{\pi}{3}\right) = -3.5$	$r\left(\pi - \frac{\pi}{3}\right) = \boxed{-4.5}$ not the same	$r\left(\pi + \frac{\pi}{3}\right) = -4.5$
$x - axis\ sym.$	$no\ y - axis\ sym.$	$no\ origin\ sym.$
$\cos\left(\frac{\pi}{3}\right) - 4 = -3.5$	$\cos\left(\pi - \frac{\pi}{3}\right) - 4 = -4.5$	$\cos\left(\pi + \frac{\pi}{3}\right) = -4.5$

11.4 - Graphing in Polar Coordinates

Polar Symmetries

$x-axis\ symmetry$	$y-axis\ symmetry$	$origin\ symmetry$
$(r,\theta)=(r,-\theta)$	$(r,\theta)=(r,\pi-\theta)$	$(r,\theta)=(r,\pi+\theta)$

Find the symmetries $\quad r=\dfrac{-6}{sin\theta}$

$r\left(\frac{\pi}{6}\right)=-12$	$r\left(\frac{\pi}{6}\right)=-12$	$r\left(\frac{\pi}{6}\right)=-12$
$r\left(-\frac{\pi}{6}\right)=12$	$r\left(\pi-\frac{\pi}{6}\right)=-12$	$r\left(\pi+\frac{\pi}{6}\right)=12$
$no\ x-axis\ sym.$	$y-axis\ sym.$	$no\ origin\ sym.$

11.4 - Graphing in Polar Coordinates

Slope of a Tangent for $r=f(\theta)$

$$m=\frac{dy}{dx} \qquad \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \qquad x=rcos\theta \qquad y=rsin\theta$$

$$x=f(\theta)cos\theta \qquad y=f(\theta)sin\theta$$

$$\frac{dy}{d\theta}=f'(\theta)sin\theta+f(\theta)cos\theta \qquad \frac{dx}{d\theta}=f'(\theta)cos\theta-f(\theta)sin\theta$$

$$\boxed{\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{f'(\theta)sin\theta+f(\theta)cos\theta}{f'(\theta)cos\theta-f(\theta)sin\theta}}$$

$$\boxed{\frac{dy}{dx}=\frac{f'(\theta)sin\theta+f(\theta)cos\theta}{f'(\theta)cos\theta-f(\theta)sin\theta}} \qquad \frac{dy}{dx}=\frac{r'sin\theta+rcos\theta}{r'cos\theta-rsin\theta}$$

11.4 Examples

① Find Symmetries $r = \frac{-6}{\sin\theta}$

X-axis $(r,\theta = r,-\theta)$

$r\left(\frac{\pi}{6}\right) = \frac{-6}{\sin\left(\frac{\pi}{6}\right)} = \boxed{-12}$ $r\left(\frac{\pi}{6}\right) = \frac{-6}{\sin\left(-\frac{\pi}{6}\right)} = \boxed{12}$

no X-axis symmetry

Y-axis Symetry $(r,\theta) = (r, \pi - \theta)$

$r\left(\frac{\pi}{6}\right) = \frac{-6}{\sin\left(\frac{\pi}{6}\right)} = \boxed{-12}$ $r\left(\frac{\pi}{6}\right) = \frac{-6}{\sin\left(\pi - \frac{\pi}{6}\right)} = \boxed{-12}$

there <u>is</u> y-axis Symm

②

11.4 – Graphing in Polar Coordinates

Slope of a Tangent for $r = f(\theta)$

$$\frac{dy}{dx} = \frac{f'(\theta)sin\theta + f(\theta)cos\theta}{f'(\theta)cos\theta - f(\theta)sin\theta} \qquad \frac{dy}{dx} = \frac{r'sin\theta + rcos\theta}{r'cos\theta - rsin\theta}$$

When $r = f(\theta)$ passes through the pole, $\underline{r = 0}$ and the slope of the tangent is:

$$\frac{dy}{dx} = \frac{f'(\theta)sin\theta + \cancel{f(\theta)}^{0}\cancel{cos\theta}^{0}}{f'(\theta)cos\theta - \cancel{f(\theta)}^{0}\cancel{sin\theta}^{0}} = \frac{\cancel{f'(\theta)}sin\theta}{\cancel{f'(\theta)}cos\theta} = \frac{sin\theta}{cos\theta} = tan\theta$$

11.4 – Graphing in Polar Coordinates

Slope of a Tangent for $r = f(\theta)$

$$\frac{dy}{dx} = \frac{f'(\theta)sin\theta + f(\theta)cos\theta}{f'(\theta)cos\theta - f(\theta)sin\theta} \qquad \frac{dy}{dx} = \frac{r'sin\theta + rcos\theta}{r'cos\theta - rsin\theta}$$

Given the polar equation, find the polar coordinates and the slope of the tangent at $\theta = {}^{\pi}/_{4}$.

$r = sin(2\theta)$ = eq.

$r = sin\left(2\left(\frac{\pi}{4}\right)\right)$

$r = 1$

$\left(1, \frac{\pi}{4}\right)$ = polar coord.
r θ

$r' = 2cos(2\theta)$

$r' = 2cos\left(2\left(\frac{\pi}{4}\right)\right)$ 2·0

$r' = 0$

$$\frac{dy}{dx} = \frac{(0)sin({}^{\pi}/_{4}) + (1)cos({}^{\pi}/_{4})}{(0)cos({}^{\pi}/_{4}) - (1)sin({}^{\pi}/_{4})}$$

$$\frac{dy}{dx} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1$$

At $\left(1, \frac{\pi}{4}\right)$ $m = -1$

11.4 - Graphing in Polar Coordinates

Polar Equations

Basic Equations:

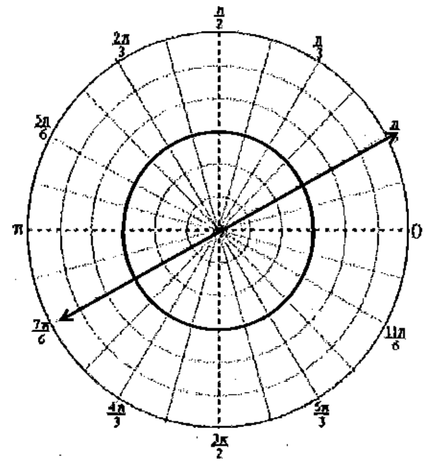
$r = a$ **A circle with its center at the pole.**

$\theta = \theta_0$ **A line through the pole at the angle θ.**

Examples:

$r = 3$

$\theta = \frac{\pi}{6}$

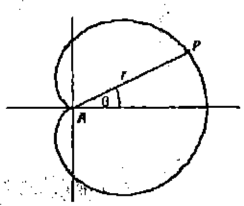


11.4 - Graphing in Polar Coordinates

Polar Equations

Plotting Polar Curves

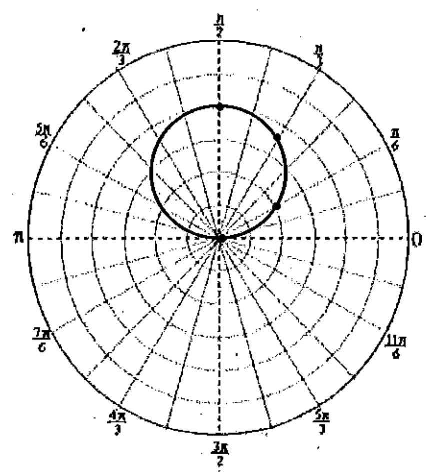
(r, θ)



Example: $r = 4sin\theta$

$\theta = 0$ $r = 0$ $\theta = \frac{\pi}{6}$ $r = 2$

$\theta = \frac{\pi}{3}$ $r = 3.46$ $\theta = \frac{\pi}{2}$ $r = 4$

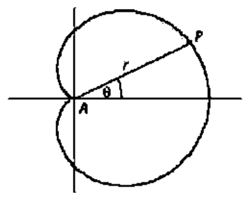


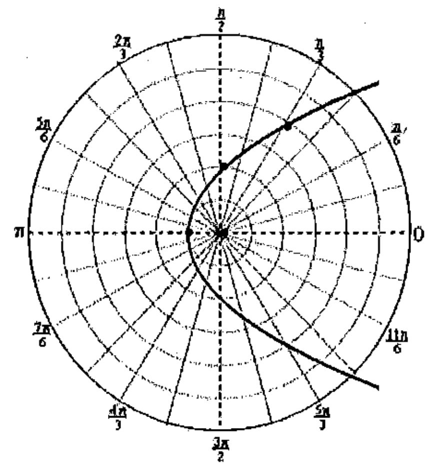
11.4 – Graphing in Polar Coordinates

Polar Equations

Plotting Polar Curves

(r, θ)





Example: $r = \dfrac{2}{1 - cos\theta}$

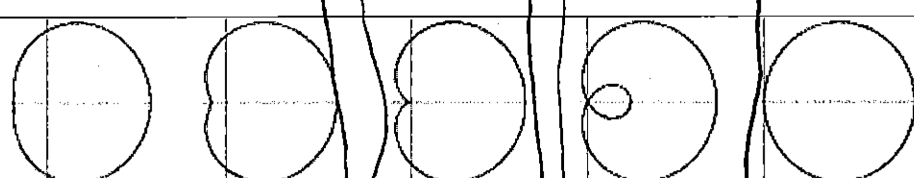
$\theta \neq 0 \quad r = und. \quad \theta = \dfrac{\pi}{6} \quad r = 14.93$

$\theta = \dfrac{\pi}{3} \quad r = 4 \quad \theta = \dfrac{\pi}{2} \quad r = 2 \quad \theta = \pi \quad r = 1$

11.4 – Graphing in Polar Coordinates

Special Polar Curves

$r = a + bcos\theta$	$r = a + bcos\theta$	$r = a + bcos\theta$	$r = bcos\theta$
$a > b$	$a = b$	$a < b$	

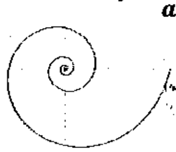


Limacon w/ a dimple — Cardioid — Limacon w/ a loop — Circle

11.4 – Graphing in Polar Coordinates

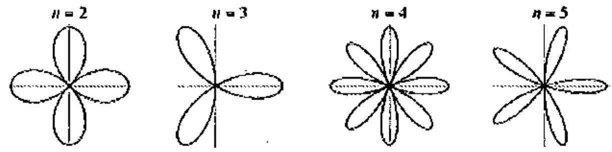
Special Polar Curves

$r = a\theta$ $\quad$ $r = \dfrac{b}{a\theta}$



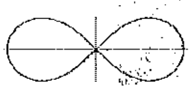
Spiral out $\quad$ Spiral in

$r = a\cos(n\theta)$



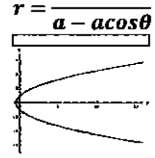
Roses

$r^2 = a\cos(2\theta)$



Lemniscate

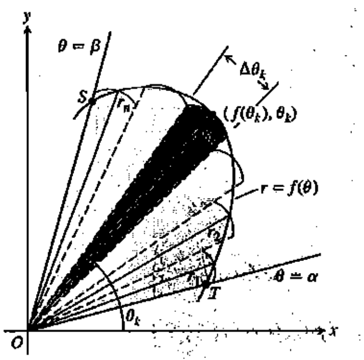
$r = \dfrac{2}{a - a\cos\theta}$

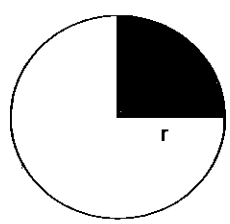


Parabola

11.5 – Area and Lengths in Polar Coordinates

Area of a Polar Curve





$circle\ area = \pi r^2$

$area\ of\ wedge = \pi r^2\left(\dfrac{\theta}{2\pi}\right)$

$area\ of\ wedge = \dfrac{1}{2}r^2\theta$

$$\boxed{area\ of\ a\ polar\ curve = \int_{\alpha}^{\beta}\frac{1}{2}r^2\,d\theta}$$

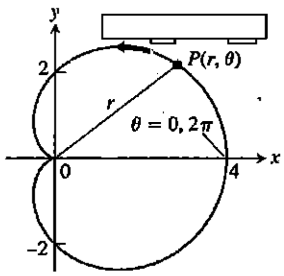
$$area\ of\ a\ polar\ curve = \int_{\alpha}^{\beta}\frac{1}{2}f(\theta)^2\,d\theta$$

11.5 – Area and Lengths in Polar Coordinates

Area of a Polar Curve

$r = 2 + 2cos\theta$

$$area\ of\ a\ polar\ curve = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$



Cardioid

$0 \leq \theta \leq 2\pi$

$$A = \int_{0}^{2\pi} \frac{1}{2}(2 + 2cos\theta)^2 d\theta$$

$$A = \int_{0}^{2\pi} \frac{1}{2}(2(1 + cos\theta))^2 d\theta$$

$$A = \int_{0}^{2\pi} 2(1 + cos\theta)^2 d\theta$$

$A = 18.850$

fint = {½(2+2cosθ)², θ, 0, 2π}

$$A = 2\int_{0}^{\pi} \frac{1}{2}(2 + 2cos\theta)^2 d\theta$$

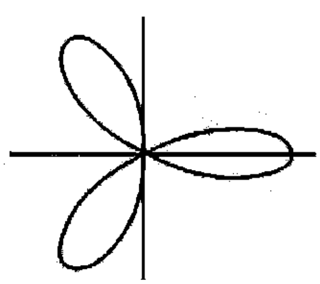
$A = 18.850$

11.5 – Area and Lengths in Polar Coordinates

Area of a Polar Curve

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

$r = 5\cos(3\theta)$



Rose w/3 petals

$0 \leq \theta \leq \pi$

Area of entire region

$$A = \int_{0}^{\pi} \frac{1}{2}(5\cos(3\theta))^2 d\theta$$

$$A = \int_{0}^{\pi} \frac{1}{2}(5cos\,(3\theta))^2 d\theta$$

$A = 19.635$

Area of a one petal

$$A = \frac{19.635}{3}$$

$A = 6.545$

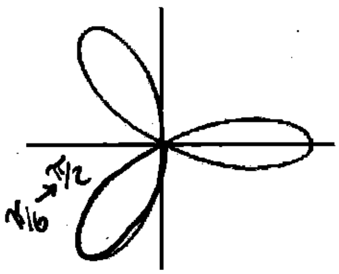
On test

11.5 – Area and Lengths in Polar Coordinates

Area of a Polar Curve

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

$r = 5\cos(3\theta)$



Rose w/3 petals

$0 \le \theta \le \pi$

Area of a one petal

$r = 5\cos(3\theta)$

$0 = 5\cos(3\theta)$

$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2},$

$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6},$

$$A = 2\int_{0}^{\pi/6} \frac{1}{2}(5\cos(3\theta))^2 d\theta \qquad A = 6.545$$

$$A = \int_{\pi/6}^{\pi/2} \frac{1}{2}(5cos\ (3\theta))^2 d\theta$$

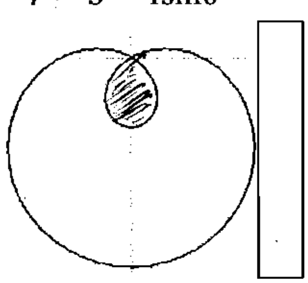
$$A = \int_{\pi/2}^{5\pi/6} \frac{1}{2}(5cos\ (3\theta))^2 d\theta$$

11.5 – Area and Lengths in Polar Coordinates

Area of a Polar Curve

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

$r = 3 - 4\sin\theta$



Limacon w/ a loop

$0 \le \theta \le 2\pi$

Area of the inner loop

$r = 3 - 4\sin\theta$

$0 = 3 - 4sin\theta$

$\theta = \sin^{-1}(3/4)$

$\theta = 0.848$

$\theta = \pi - 0.848$ = Sym about y-axis

$\theta = 2.294$

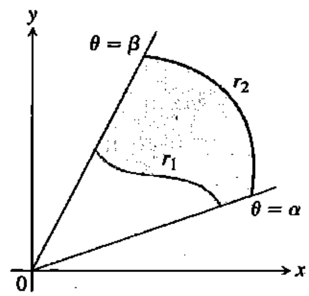
$$A = \int_{0.848}^{2.294} \frac{1}{2}(3 - 4sin\theta)^2 d\theta$$

$A = 0.381$

or

$$A = 2\int_{0.848}^{\pi/2} \frac{1}{2}(3 - 4sin\theta)^2 d\theta$$

11.5 - Area and Lengths in Polar Coordinates



Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \quad \alpha \le \theta \le \beta$

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2 \, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 \, d\theta = \boxed{\int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta} \quad (1)$$

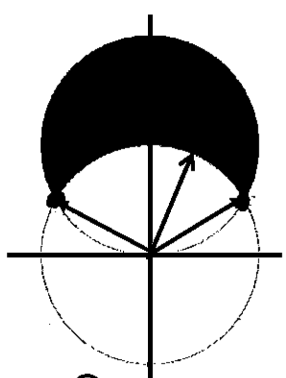
11.5 - Area and Lengths in Polar Coordinates

Area of a Polar Curve

Area of the region outside r = 2 and inside r = 4sinθ

$r = 2$

$r = 4\sin\theta$



• = POI

Need Pts. Of Intersection for limits

Set them = to each other

$4sin\theta = 2$

$sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\theta = 0.524$

$\theta = 2.618$

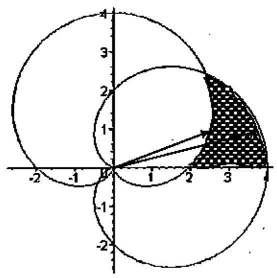
$$A = \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2) d\theta$$

$$A = \int_{0.524}^{2.618} \frac{1}{2}\left((4sin\theta)^2 - 2^2\right) d\theta$$

$A = 7.653$

11.5 – Area and Lengths in Polar Coordinates

Calculate the area of the region outside $r = 2 + 2sin\theta$, inside $r = 2 + 2cos\theta$, and in the first quadrant.



Points Of Intersection

$$2 + 2sin\theta = 2 + 2cos\theta$$

$$2sin\theta = 2cos\theta$$ or $\sin\theta = \cos\theta$

$$tan\theta = 1$$ @ $\pi/4$

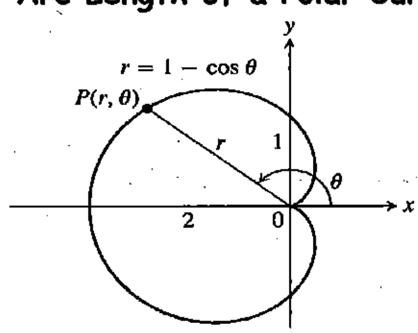
$$\theta = \frac{\pi}{4} = 0.785$$

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_2^2 - r_1^2)d\theta$$

$$A = \int_{0}^{0.785} \frac{1}{2}\left((2 + 2cos\theta)^2 - (2 + 2sin\theta)^2\right)d\theta \qquad A = 2.657$$

11.5 – Area and Lengths in Polar Coordinates

Arc Length of a Polar Curve



Length of a Polar Curve
If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

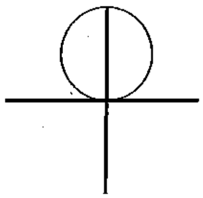
$$\boxed{L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.} \qquad (3)$$

11.5 - Area and Lengths in Polar Coordinates

Find the arc length of the polar curve between the given angle interval.

$r = 5\sin\theta$

$0 \le \theta \le \pi$



$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\frac{dr}{d\theta} = 5\cos\theta$$

$$L = \int_{0}^{\pi} \sqrt{(5sin\theta)^2 + (5cos\theta)^2} d\theta$$

$$L = \int_{0}^{\pi} \sqrt{25sin^2\theta + 25cos^2\theta}\, d\theta$$

$$L = \int_{0}^{\pi} \sqrt{25(sin^2\theta + cos^2\theta)}\, d\theta$$

$$L = \int_{0}^{\pi} 5\, d\theta$$

$$L = 5\,\theta \Big|_{0}^{\pi}$$

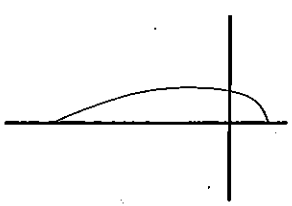
$$L = 5\pi = 15.708$$

11.5 - Area and Lengths in Polar Coordinates

Find the arc length of the polar curve between the given angle interval.

$r = e^{\theta}$

$0 \le \theta \le \pi$



$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\frac{dr}{d\theta} = e^{\theta}$$

$$L = \int_{0}^{\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2} d\theta$$

$$L = \int_{0}^{\pi} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta$$

$$L = \int_{0}^{\pi} \sqrt{2e^{2\theta}}\, d\theta$$

$$L = \sqrt{2}\int_{0}^{\pi} e^{\theta} d\theta$$

$$L = \sqrt{2}e^{\theta}\Big|_{0}^{\pi}$$

$$L = \sqrt{2}e^{\pi} - \sqrt{2}e^{0}$$

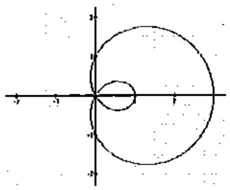
$$L = 31.312$$

Exam Qst.

11.5 – Area and Lengths in Polar Coordinates

Find the arc length of half of the inside loop

$r = 2\cos\theta + 1$



$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \frac{dr}{d\theta} = -2\sin\theta \qquad L = \int_{\frac{2\pi}{3}}^{\pi} \sqrt{(2\cos\theta + 1)^2 + (-2\sin\theta)^2}\, d\theta$$

$$L = \int_{\frac{2\pi}{3}}^{\pi} \sqrt{4\cos^2\theta + 4\cos\theta + 1 + 4\sin^2\theta}\, d\theta$$

Know unit circle & how this graphs

$\theta = 0$

$r = 2\cos 0 + 1 = 3$

$\theta = \pi$

$r = 2\cos\pi + 1 = -1$

$0 = 2\cos\theta + 1$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$L = \int_{\frac{2\pi}{3}}^{\pi} \sqrt{4\cos\theta + 4\cos^2\theta + 4\sin^2\theta + 1}\, d\theta$$

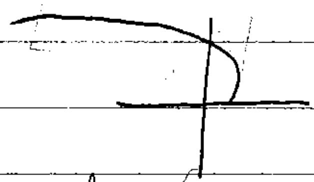
$$L = \int_{\frac{2\pi}{3}}^{\pi} \sqrt{4\cos\theta + 4 + 1}\, d\theta$$

$$L = \int_{\frac{2\pi}{3}}^{\pi} \sqrt{4\cos\theta + 5}\, d\theta \qquad L = 1.341$$

11.5 – Area and Lengths in Polar Coordinates

11.5 Examples – Arc Length

$r = e^{\theta} \qquad 0 \leq \theta \leq \pi$



$$L = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\frac{dr}{d\theta} = e^{\theta} = e^{2\theta}$$

$$r^2 = e^{2\theta}$$

$$L = \int_0^{\pi} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta$$

$$L = \int_0^{\pi} \sqrt{2e^{2\theta}}\, d\theta$$

$$L = \int_0^{\pi} \sqrt{2}\sqrt{e^{2\theta}}\, d\theta$$

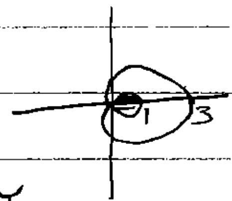
$$L = \sqrt{2} \int_0^{\pi} \left(e^{2\theta}\right)^{1/2} = L = \sqrt{2} \int_0^{\pi} e^{\theta}\, d\theta$$

$$L = \sqrt{2}\left(e^{\theta}\right)_0^{\pi} = \sqrt{2}\left(e^{\pi} - 1\right)$$

$$L = \sqrt{2}e^{\pi} - \sqrt{2}$$

$$L = 31.312$$

$r = 2\cos\theta + 1 \qquad 0 \leq \theta \leq 2\pi$



$\dfrac{dr}{d\theta} = -2\sin\theta = 4\sin^2\theta$

$r^2 = 4\cos^2\theta + 4\cos\theta + 1$

when $\theta = \pi$

$r = 2\cos\pi + 1 = -1$ $\longrightarrow$ at $\theta = \pi$ move -1 radians

when $r = 0$

$0 = 2\cos\theta + 1$

$2\cos\theta = -1 \longrightarrow \theta = \cos^{-1}(-\tfrac{1}{2}) = \tfrac{2\pi}{3}$ & $\tfrac{4\pi}{3}$

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

from π to $4\pi/3$ because $2\pi/3$ to $4\pi/3$ is the whole inner loop

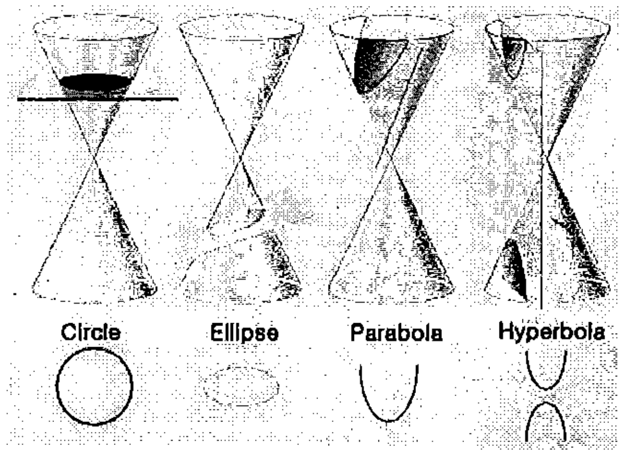
$$L = \int_{\pi}^{4\pi/3} \sqrt{(2\cos\theta + 1)^2 + 4\sin^2\theta}\, d\theta$$

$$L = \int_{\pi}^{4\pi/3} \sqrt{4\cos^2\theta + 4\cos\theta + 1 + 4\sin^2\theta}\, d\theta$$

$$L = \int_{\pi}^{4\pi/3} \sqrt{4\cos\theta + 5}\, d\theta$$

$$\boxed{L = 1.341}$$

Conics – curves that are created by the intersection of a plane and a right circular cone.



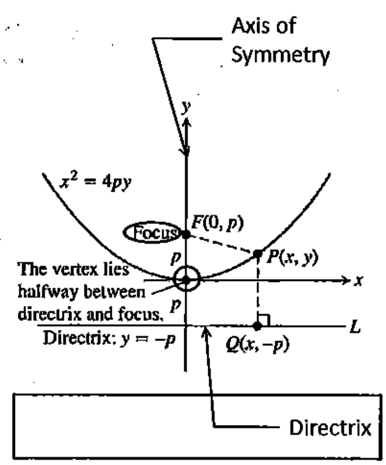
Parabola – set of points in a plane that are equidistant from a fixed point (d(F, P)) and a fixed line (d (P, Q)).

Focus - the fixed point of a parabola.

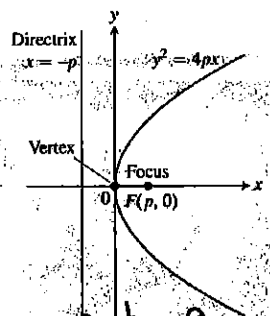
Directrix - the fixed line of a parabola.

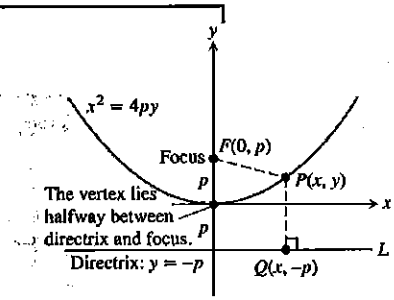
Axis of Symmetry – The line that goes through the focus and is perpendicular to the directrix.

Vertex – the point of intersection of the axis of symmetry and the parabola.



Parabolas





finding p	finding p
$\boxed{y^2 = 4px}$	$\boxed{x^2 = 4py}$
$(y-k)^2 = 4p(x-h)$	$(x-h)^2 = 4p(y-k)$

K = vertical movement
h = horizontal movement

Find the vertex, focus and the directrix

$x^2 = 16y$

parabola, opens up

vertex: (0,0) → no transf-ormation

find p

$16 = 4p$

$p = 4$

focus

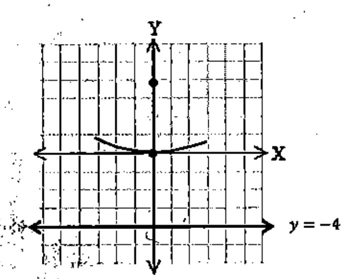
$(0, 0 + p)$

$(0, 4)$

directrix

$y = 0 - p$

$y = -4$



Find the vertex and the focus given: $y^2 + 10y + x + 20 = 0$

$y^2 + 10y + x + 20 = 0$

$y^2 + 10y = -x - 20$

complete the square

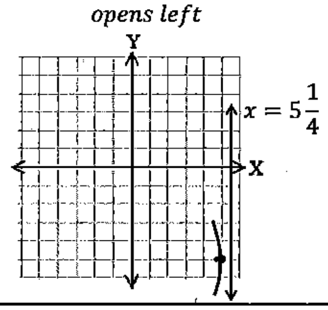
$\frac{10}{2} = 5 \qquad 5^2 = 25$

$y^2 + 10y + 25 = -x - 20 + 25$

$(y+5)^2 = -x + 5$

$(y+5)^2 = -(x-5)$

opens left



vertex

$(5, -5)$

find p

$1 = 4p$

$p = \frac{1}{4}$

focus

$(5 - \frac{1}{4}, -5)$

$(4\frac{3}{4}, -5)$

directrix

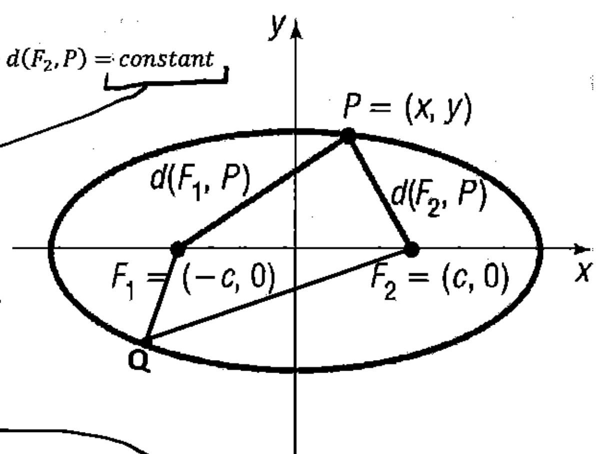
$x = 5 + \frac{1}{4}$

$x = 5\frac{1}{4}$

Ellipse – a set of points in a plane whose sum of the distances from two fixed points is a constant.

$d(F_1, P) + d(F_2, P) = \text{constant}$

Same constant



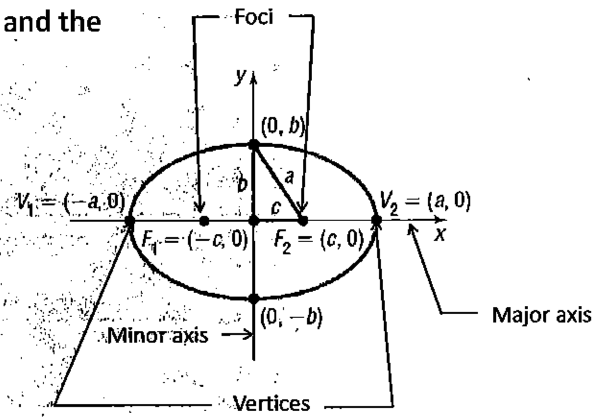
$d(F_1, Q) + d(F_2, Q) = \text{constant} = d(F_1, P) + d(F_2, P)$

Foci – the two fixed points, F_1 *and* F_2, whose distances from a single point on the ellipse is a constant.

Major axis – the line that contains the foci and goes through the center of the ellipse.

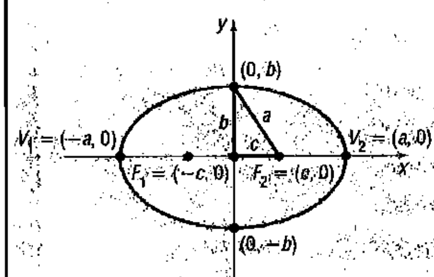
Vertices – the two points of intersection of the ellipse and the major axis, V_1 *and* V_2 .

Minor axis – the line that is perpendicular to the major axis and goes through the center of the ellipse.



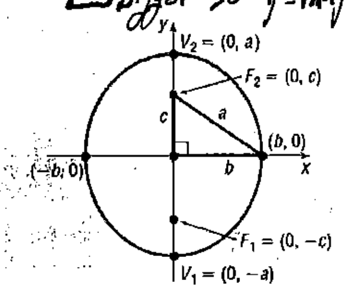
Equation of an Ellipse Centered at the Origin

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad where\ a>b$$



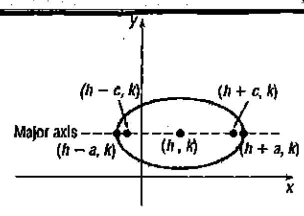
$$\frac{x^2}{b^2}+\frac{y^2}{\boxed{a^2}}=1 \quad where\ a>b$$

bigger so y = major

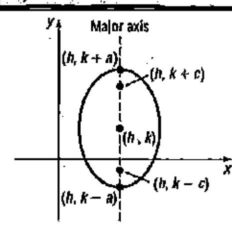


Equation of an Ellipse Centered at a Point

Equations of an Ellipse: Center at (h, k); Major Axis Parallel to a Coordinate Axis				
Center	**Major Axis**	**Foci**	**Vertices**	**Equation**
(h, k)	Parallel to the x-axis	$(h + c, k)$	$(h + a, k)$	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$
		$(h - c, k)$	$(h - a, k)$	$a > b > 0$ and $b^2 = a^2 - c^2$
(h, k)	Parallel to the y-axis	$(h, k + c)$	$(h, k + a)$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1,$
		$(h, k - c)$	$(h, k - a)$	$a > b > 0$ and $b^2 = a^2 - c^2$



(a) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$



(b) $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

Find the vertices for the major and minor axes, and the foci using the following equation of an ellipse.

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

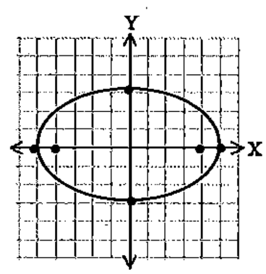
Major axis is along the x-axis

Vertices of major axis: add/Sub to X-axis

$a^2 = 25 \quad a = \pm 5 \quad (-5,0)\ and\ (5,0)$

Vertices of the minor axis add/Sub to Y-axis

$b^2 = 9 \quad b = \pm 3 \quad (0,3)\ and\ (0,-3)$



Foci

not pyth → $\boxed{c^2 = a^2 - b^2} \quad c^2 = 25 - 9$

$c^2 = 16 \quad c = \pm 4$ add/Sub from major axis

$(-4,0)\ and\ (4,0)$

Find the vertices for the major and minor axes, and the foci using the following equation of an ellipse.

$$4x^2 + 9y^2 = 36$$

$$\frac{4x^2}{36} + \frac{9y^2}{36} = 1 \qquad \frac{x^2}{9} + \frac{y^2}{4} = 1$$

Major axis is along the x-axis

Vertices of major axis:

$a^2 = 9 \quad a = \pm 3 \quad (-3,0)\ and\ (3,0)$

Vertices of the minor axis

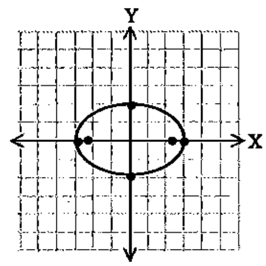
$b^2 = 4 \quad b = \pm 2 \quad (0,2)\ and\ (0,-2)$

Foci

$c^2 = a^2 - b^2 \quad c^2 = 9 - 4$

$c^2 = 5 \quad c = \pm\sqrt{5}$

$(-\sqrt{5},0)\ and\ (\sqrt{5},0)$



Find the center, the vertices of the major and minor axes, and the foci using the following equation of an ellipse.

$$16x^2 + 4y^2 + 96x - 8y + 84 = 0$$

$$16x^2 + 96x + 4y^2 - 8y = -84$$

$$16(x^2 + 6x) + 4(y^2 - 2y) = -84$$

$$\frac{6}{2} = 3 \quad 3^2 = 9 \qquad \frac{-2}{2} = -1 \quad (-1)^2 = 1$$

$$16(x^2 + 6x + 9) + 4(y^2 - 2y + 1) = -84 + 144 + 4$$

$$16(x+3)^2 + 4(y-1)^2 = 64$$

$$\frac{16(x+3)^2}{64} + \frac{4(y-1)^2}{64} = 1$$

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{16} = 1$$

11.6 Examples

Find the vertex & focus

$y^2 + 10y + x + 20 = 0$

Complete Square

$y^2 + 10y + 25 = -x + 5$

$(y+5)^2 = -x + 5$

$(y+5)^2 = -(x-5)$ opens left

vertex $= (5, -5)$

Find p

$-(y+5)^2 = \boxed{}(x-5)$

$1 = 4p \rightarrow p = \frac{1}{4}$

focus $(5 - \frac{1}{4}, -5)$

$(4\frac{3}{4}, -5)$

directrix $x = 5 + \frac{1}{4}$

$x = 5\frac{1}{4}$

$a^2 = b^2 + c^2$ find focci of $\frac{x^2}{25} + \frac{y^2}{9} = 1$

$25 = 9 + c^2$

$c^2 = 25 - 9$

$c = \sqrt{16} = \boxed{\pm 4}$ add/sub from major axis

Find Center, Vertices & Focci

$16x^2 + 4y^2 + 96x - 8y + 84 = 0$

$16x^2 + 96x + 4y^2 - 8y = -84$

$16(x^2 + 6x + 9) + 4(y^2 - 2y + 1) = -84 + 144 + 4$

$16(x^2 + 6x + 9) + 4(y^2 - 2y + 1) = 64$

$\frac{16(x+3)^2}{64} + \frac{4(y-1)^2}{64} = 1$

$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{16} = 1$

Center $(-3, 1)$, major axis = y-axis

Vertices = $a^2 = 16$, $a = \pm 4$

$(-3, 1+4)$, $(-3, 1-4)$

* Vertices $(-3, 5)$, $(-3, -3)$

* Vertices of minor $(-5, 1)$, $(-1, 1)$

Focci $a^2 = b^2 + c^2$

$c^2 = a^2 - b^2 \rightarrow c^2 = 16 - 4 \rightarrow c = \pm 2\sqrt{3}$

now add/sub to major axis

$(-3, 1 + 2\sqrt{3})$, $(-3, 1 - 2\sqrt{3})$

$$\frac{(x+3)^2}{4}+\frac{(y-1)^2}{16}=1$$

Center:

$(-3,1)$

Major axis: $x=-3\ (vertical)$

Vertices: $a^2=16 \quad a=\pm 4$

$(-3,1-4)\ and\ (-3,1+4)$

$(-3,-3)\ and\ (-3,5)$

Minor axis: $y=1\ (horizontal)$

Vertices of the minor axis

$b^2=4 \quad b=\pm 2$

$(-3-2,1)\ and\ (-3+2,1)$

$(-5,1)\ and\ (-1,1)$

Foci

$c^2=a^2-b^2$

$c^2=16-4$

$c^2=12$

$c=\pm 2\sqrt{3}$

$(-3,1-2\sqrt{3})\ and\ (-3,1+2\sqrt{3})$

$(-3,-2.464)\ and\ (-3,4.464)$

$$\frac{(x+3)^2}{4}+\frac{(y-1)^2}{16}=1$$

Center:

$(-3,1)$

Major axis vertices:

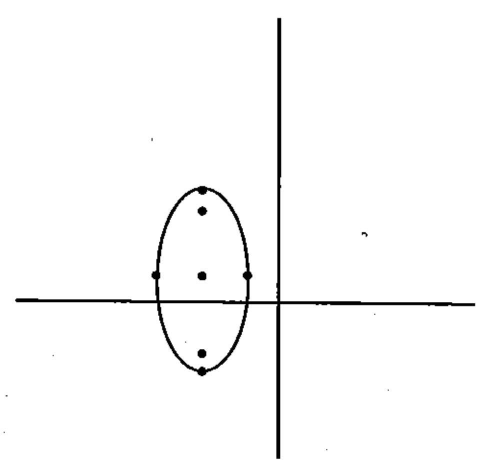
$(-3,-3)\ and\ (-3,5)$

Minor axis vertices:

$(-5,1)\ and\ (-1,1)$

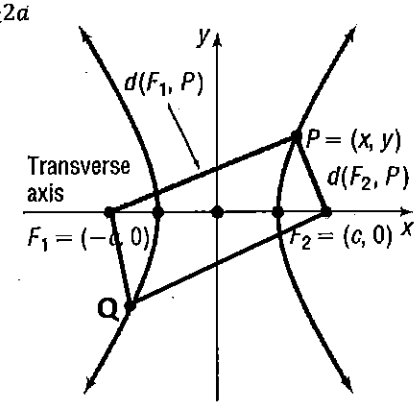
Foci

$(-3,-2.464)\ and\ (-3,4.464)$



Hyperbola – a set of points in a plane whose <u>difference</u> of the distances from two fixed points is a constant.

$d(F_1,P) - d(F_2,P) = constant = \pm 2a$



$d(F_1,Q) - d(F_2,Q) = \quad constant \quad = \pm 2a$

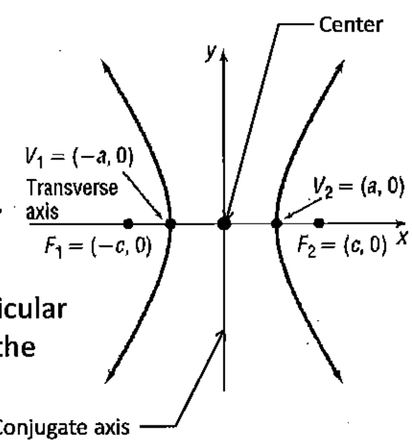
Foci – the two fixed points, F_1 *and* F_2, whose difference of the distances from a single point on the hyperbola is a constant.

Transverse axis – the line that contains the foci and goes through the center of the hyperbola.

Center – the midpoint of the line segment between the two foci.

Vertices – the two points of intersection of the hyperbola and the transverse axis, V_1 *and* V_2 .

Conjugate axis – the line that is perpendicular to the transverse axis and goes through the center of the hyperbola.



Equation of an Ellipse Centered at the Origin

Equation of a Hyperbola Center at (0, 0) Transverse Axis along the x-Axis

An equation of the hyperbola with center at (0, 0), foci at $(-c, 0)$ and $(c, 0)$, and vertices at $(-a, 0)$ and $(a, 0)$ is

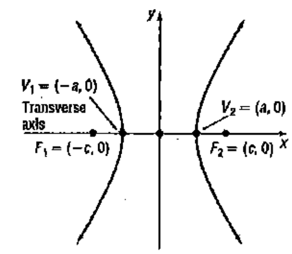
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \text{where } b^2 = c^2 - a^2$$

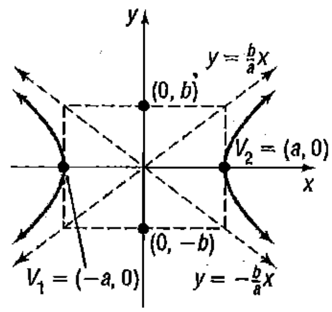
The transverse axis is the x-axis.

Asymptotes of a Hyperbola

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has the two oblique asymptotes

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$





Equation of a Hyperbola Centered at the Origin

Equation of a Hyperbola; Center at (0, 0); Transverse Axis along the y-Axis

An equation of the hyperbola with center at (0, 0), foci at $(0, -c)$ and $(0, c)$, and vertices at $(0, -a)$ and $(0, a)$ is

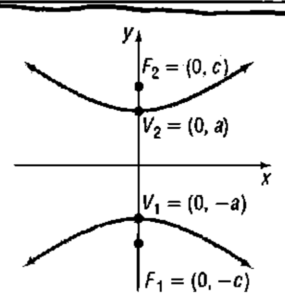
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \qquad \text{where } b^2 = c^2 - a^2$$

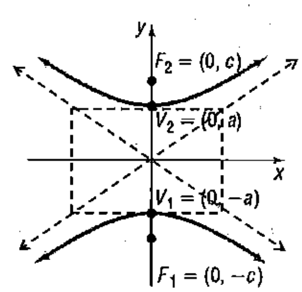
The transverse axis is the y-axis.

Asymptotes of a Hyperbola

The hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ has the two oblique asymptotes

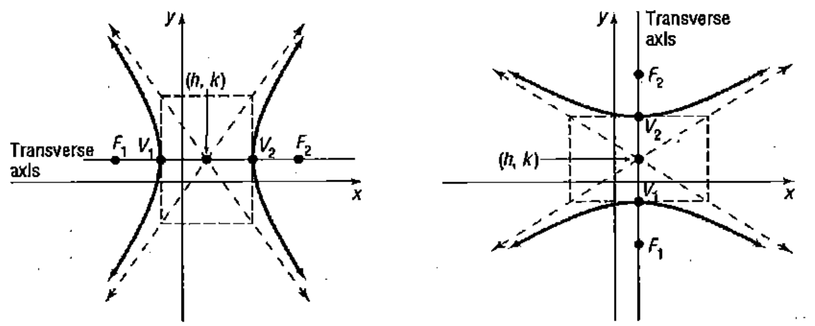
$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x$$





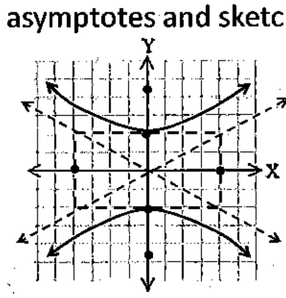
Equation of a Hyperbola Centered at a Point

Center	Transverse Axis	Foci	Vertices	Equation	Asymptotes
(h, k)	Parallel to the x-axis	$(h \pm c, k)$	$(h \pm a, k)$	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,\ b^2 = c^2 - a^2$	$y - k = \pm\frac{b}{a}(x - h)$
(h, k)	Parallel to the y-axis	$(h, k \pm c)$	$(h, k \pm a)$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1,\ b^2 = c^2 - a^2$	$y - k = \pm\frac{a}{b}(x - h)$



for hyperbolas, the lead positive term is the

Identify the direction of opening, the coordinates of the center, the vertices, and the foci. Find the equations of the asymptotes and sketch the graph.



$\frac{y^2}{4} - \frac{x^2}{16} = 1$ Center: (0,0)

Vertices of transverse axis:

$a^2 = 4$ $a = \pm 2$ $(0,-2)$ *and* $(0,2)$

Foci

For graphing the box

$b^2 = 16$ $b = \pm 4$ $(-4,0)$ *and* $(4,0)$

$b^2 = c^2 - a^2$ $16 = c^2 - 4$

$c^2 = 20$ $c = \pm 2\sqrt{5}$ of transverse

$(0,-2\sqrt{5})$ *and* $(0,2\sqrt{5})$

Equations of the Asymptotes

$$y - y_1 = \pm\frac{a}{b}(x - x_1)$$

$$y - 0 = \pm\frac{2}{4}(x - 0)$$

$$y = \pm\frac{1}{2}x$$

one that the transverse axis goes through

Find the center, the vertices of the transverse axis, the foci and the equations of the asymptotes using the following equation of a hyperbola.

$$y^2 - 4x^2 - 72x + 10y - 399 = 0$$

$$y^2 + 10y - 4x^2 - 72x = 399$$

$$(y^2 + 10y) - 4(x^2 + 18x) = 399$$

$$\frac{10}{2} = 5 \quad 5^2 = 25 \qquad \frac{18}{2} = 9 \quad 9^2 = 81$$

$$(y^2 + 10y + 25) - 4(x^2 + 18x + 81) = 399 + 25 - 324$$

$$(y+5)^2 - 4(x+9)^2 = 100$$

$$\frac{(y+5)^2}{100} - \frac{4(x+9)^2}{100} = 1$$

$$\frac{(y+5)^2}{100} - \frac{(x+9)^2}{25} = 1$$ Opening up/down

Find the center, the vertices of the transverse axis, the foci and the equations of the asymptotes using the following equation of a hyperbola.

$$y^2 - 4x^2 - 72x + 10y - 399 = 0$$

$$\frac{(y+5)^2}{100} - \frac{(x+9)^2}{25} = 1$$

Center: $(-9, -5)$

Vertices:

$a^2 = 100 \quad a = 10$

$(-9, -5 - 10)$ *and* $(-9, -5 + 10)$

$(-9, -15)$ *and* $(-9, 5)$

Foci:

$b^2 = c^2 - a^2 \quad 25 = c^2 - 100$

$c^2 = 125 \quad c = \sqrt{125} = 5\sqrt{5}$

$(-9, -5 - 5\sqrt{5})$ *and* $(-9, -5 + 5\sqrt{5})$

$(-9, -16.18)$ *and* $(-9, 6.18)$

Find the center, the vertices of the transverse axis, the foci and the equations of the asymptotes using the following equation of a hyperbola.

$$y^2 - 4x^2 - 72x + 10y - 399 = 0$$

$$\frac{(y+5)^2}{100} - \frac{(x+9)^2}{25} = 1$$

Equations of the Asymptotes

Center: $(-9, -5)$

$a = 10 \qquad b = 5$

$$y - y_1 = \pm\frac{a}{b}(x - x_1)$$

$$y - (-5) = \pm\frac{10}{5}(x - (-9))$$

$$y + 5 = \pm 2\,(x + 9)$$

Find the center

$y^2 + 10y - 4x^2 - 72x - 399 = 0$

$y^2 + 10y + 25 - 4(x^2 + 18x + 81) = 399 + 25 - 324$

$(y+5)^2 - 4(x-9)^2 = 100$

$$\frac{(y+5)^2}{100} - \frac{4(x+9)^2}{100} = 1$$

$$\frac{(y+5)^2}{100} - \frac{(x+9)^2}{25} = 1$$

$(-9, -5)$ center

opening up/down

$a^2 = 100 \rightarrow a = \pm 10$

$b^2 = 25 \rightarrow b = \pm 5$

$c^2 = a^2 + b^2 \rightarrow c^2 = 125 \rightarrow c = \pm\sqrt{125}$

Vertices $\rightarrow (-9, -5+10)$ & $(-9, -5-10)$

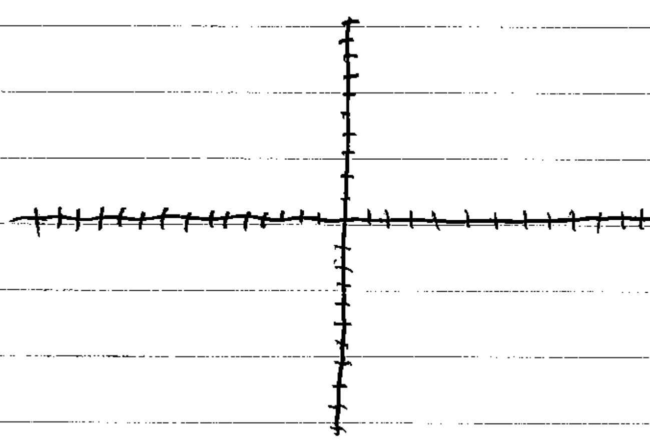
$(-9, 5)$ & $(-9, -15)$

Foci $\rightarrow (-9, -5+\sqrt{125})$ & $(-9, -5-\sqrt{125})$

$(-9, 6.180)$ & $(-9, -16.180)$

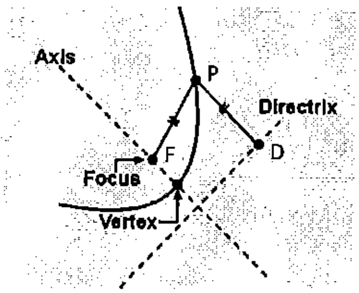
Assymptotes: $y - y_1 = \pm \frac{a}{b}(x - x_1)$

$y + 5 = \pm 2(x+9)$



Eccentricity

The ratio of the distance from a fixed point (focus) to a point on the conic to the distance from the point to the directrix is the eccentricity of a conic. It is a constant ratio and is denoted by **e**.



always in/out

$$\underline{e} = \boxed{\frac{PF_{oc}}{PD_{ir}}} \qquad e \cdot PD = PF$$

If e < 1, the conic is an ellipse.
If e = 1, the conic is a parabola.
If e > 1, the conic is a hyperbola.

Polar Equation for a Conic with Eccentricity e

$$r = \frac{ke}{1 + e\,cos\theta}$$

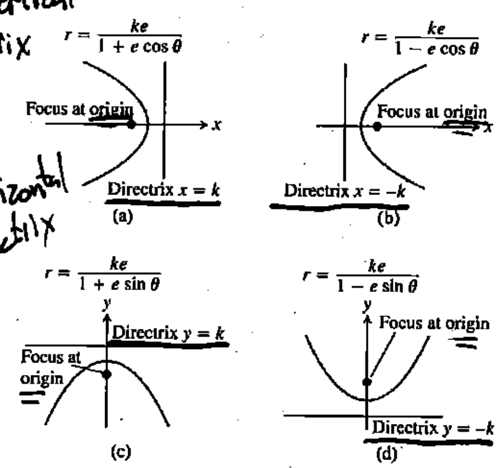
cosθ = vertical Directrix

The vertical directrix is represented by k.

$$r = \frac{ke}{1 + e\,sin\theta}$$

sinθ = horizontal Directrix

The horizontal directrix is represented by k.

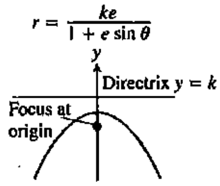


To use these polar equations, a focus is located at the origin.

K is always positive in these eq.

Given the eccentricity and the directrix corresponding to the focus at the origin, find the polar equation.

$e = \frac{1}{3}$ and $y = 6$ Directrix



$$r = \frac{ke}{1 + e\ sin\theta}$$

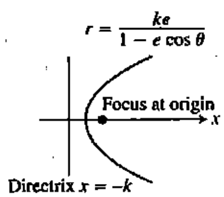
$$r = \frac{(6)\left(\frac{1}{3}\right)}{1 + \frac{1}{3}\ sin\theta}$$

$$r = \frac{(6)\left(\frac{1}{3}\right)}{\frac{1}{3}(3 +\ sin\theta)}$$

$$r = \frac{6}{3 +\ sin\theta}$$

Given the eccentricity and the directrix corresponding to the focus at the origin, find the polar equation.

$e = \frac{1}{4}$ and $x = -2$



$$r = \frac{ke}{1 - e\ cos\theta}$$

$$r = \frac{(2)\left(\frac{1}{4}\right)}{1 - \frac{1}{4}\ cos\theta}$$

$$r = \frac{(2)\left(\frac{1}{4}\right)}{\frac{1}{4}(4 -\ cos\theta)}$$

$$r = \frac{2}{4 -\ cos\theta}$$

① $e = \frac{1}{4}$ & $X = -2$ $K = +2$

$$r = \frac{Ke}{1 - e\cos\theta}$$

$$r = \frac{2(\frac{1}{4})}{1 - (\frac{1}{4})\cos\theta} \longrightarrow r = \frac{2(\frac{1}{4})}{\frac{1}{4}(4 - \cos\theta)}$$

$$\boxed{r = \frac{2}{4 - \cos\theta}}$$

② $r = \frac{6}{2 + \cos\theta} \rightarrow \frac{\cancel{6}}{\cancel{2}(1 + \frac{1}{2}\cos\theta)} \therefore e = \frac{1}{2}$

$r = \frac{3}{1 + \frac{1}{2}\cos\theta} \rightarrow$ $Ke = 3$ & $e = \frac{1}{2}$

$K(\frac{1}{2}) = 3 \rightarrow K = 6$

directrix $= X = 6$ $Ke = a(1 - e^2)$

$3 = a(1 - (\frac{1}{2})^2) \rightarrow 3 = a(\frac{3}{4})$

$\underline{\underline{a = 4}}$

$e = \frac{1}{2}$, $K = x = 6$ $a = 4$

$\frac{a}{e} = \frac{4}{\frac{1}{2}} = 8$

Center $= K - \frac{a}{e}$ (

$6 - 8 = -2$

$(-2, 0)$ or $(2, \pi)$

Vertices: from center

$-2 - 4 = -6$

Vertices:

$-2 + 4 = 2$

radius is from the pole not the center

$$r = \frac{6}{2 + \cos\theta} \longrightarrow 6 = \frac{6}{2 + \cos\theta}$$

$2 + \cos\theta = 1$

$\cos\theta = -1$

$\theta = \pi$ $(6, \pi)$

from the other vertice:

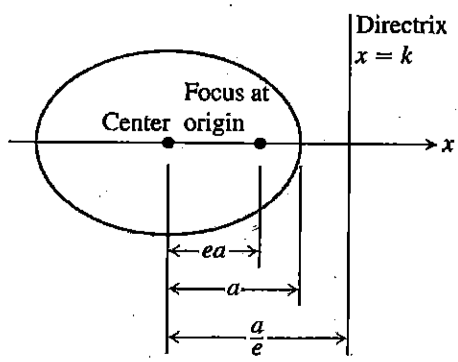
$$r = \frac{6}{2 + \cos\theta} \longrightarrow 2 = \frac{6}{2 + \cos\theta}$$

$2 + \cos\theta = 3 \longrightarrow \cos\theta = 1 \longrightarrow \theta = 0$

$(2, 0)$

This diagram is helpful

Polar Equation of an Ellipse with Eccentricity *e* and Major Axis *a*



$$k = \frac{a}{e} - ea \qquad k = a\left(\frac{1}{e} - e\right)$$

$$k = a\frac{1}{e}(1 - e^2)$$

$$ke = a(1 - e^2)$$

$$r = \frac{ke}{1 + e\,cos\theta}$$

$$r = \frac{a(1 - e^2)}{1 + e\,cos\theta}$$

Given the polar equation, find the directrix that corresponds to the focus at the origin, the polar coordinates of the vertices and the center

$$r = \frac{6}{2 + cos\theta}$$

$$r = \frac{6}{2\left(1 + \frac{1}{2}\,cos\theta\right)}$$

$$r = \frac{3}{1 + \frac{1}{2}\,cos\theta}$$

$$e = \frac{1}{2}$$

$$ke = 3$$

$$k\frac{1}{2} = 3$$

$$k = 6$$

$directrix$:
$x = 6$

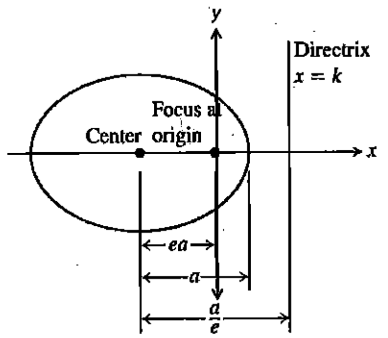
$$ke = a(1 - e^2)$$

$$3 = a\left(1 - \left(\frac{1}{2}\right)^2\right)$$

$$3 = a\left(\frac{3}{4}\right)$$

$$4 = a$$

Section [illegible] in Polar Coordinates



$e = \frac{1}{2}$ $directrix$: $x = 6$

$a = 4$

$\frac{a}{e} \rightarrow \frac{4}{\frac{1}{2}} \rightarrow 8$

$center: 6 - 8$

-2

$(-2, 0)$ or $(2, \pi)$

$vertices$:	$vertices$:
$-2 - 4$	$-2 + 4$
$-6, 0$	$2, 0$
$r = \frac{6}{2 + \cos\theta}$	$r = \frac{6}{2 + \cos\theta}$
$6 = \frac{6}{2 + \cos\theta}$	$2 = \frac{6}{2 + \cos\theta}$
$2 + \cos\theta = 1$	$2 + \cos\theta = 3$
$\cos\theta = -1$	$\cos\theta = 1$
$\theta = \pi$	$\theta = 0$
$(6, \pi)$	$(2, 0)$