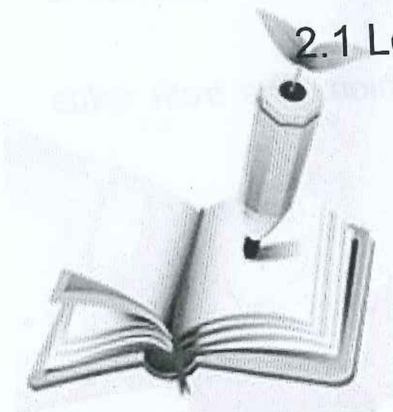
Chapter 2: The Logic of Compound Statements

2.1 Logical Form and Logical Equivalence



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Statements

- Logic: the science of reasoning
- The most basic element in logic is a statement.
- A *statement* (or *proposition*) is a declarative sentence that is *true* or *false* but not both.
- Truth Values: either true or false. Each statement must have well-defined truth value.

- Examples of statements
  - Washington, D.C., is the capital of the United States of America.
    - True
  - The declaration of independence was signed on July 4th, 1812.
    - False

Not Statements

Commands, questions, and opinions are not statements because they are neither true or false.

- Examples
  - Read this carefully. (*Command. A command is neither true nor false.*)
  - What time is it? (*Question. Not a declarative sentence. A question is neither true nor false.*)
  - Titanic is the greatest movie of all time. (*Opinion. The truth value depends on persons.*)

Using Symbols to Represent Statements

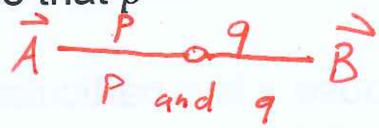
- In symbolic logic, we use lowercase letters such as p, q, r and s to represent statements.
- Standard mathematical discrete: use uppercase letters to represent sets.
- Here are two examples:

  p: January has 31 days.

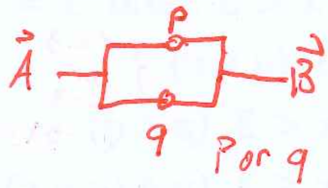
  q: February has 33 days.

Compound Statement

- One or more statements can be combined to form a single compound statement using logical operators.
- Given statements p and q,
  - * – negation of $\boxed{p: \sim p}$ is read "not p" or "It is not the case that p"
  - * – conjunction of p and q: $\boxed{p \wedge q}$ is read "p and q"



  - * – disjunction of p and q: $\boxed{p \vee q}$ is read "p or q"



p but q	means	p and q
neither p nor q	means	~p and ~q

Order of Operations

- In expressions that include the symbol ~ as well as $\wedge \, or \vee$, the order of operation is that ~ is performed first.
- The order of operations can be overridden through the use of parenthesis.

Compound Statements - Exercise

Translating from English to Symbols:

Let h = "it is hot" and s = "it is sunny"

a. It is not hot but it is sunny. ($\sim h \wedge s$)

b. It is neither hot nor sunny. ($\sim h \wedge \sim s$)

Suppose x is a particular real number. Let p, q, and r symbolize "$0 < x$", "$x < 3$", and "$x = 3$", respectively.

a. $x \leq 3$ $(q \vee r)$ $\begin{cases} x<3 \\ \text{or} \\ x=3 \end{cases}$ $p = 0<x$

b. $0 < x < 3$ $(p \wedge q)$ → x btw 0 & 3 $q = x<3$

c. $0 < x \leq 3$ $(p \wedge (q \vee r))$ $r = x=3$

x btw 0 & ($x=3$ or $x<3$)

Truth Values and Truth Tables

- Truth Value: either true or false. Each statement must have well-defined truth value.
- A truth table for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.
- Statement Form is an expression
  - made up of statement variables (such as p, q, and r) and logical connectives (such as $\sim$, $\wedge$ and $\vee$)
  - becomes a statement when actual statements are substituted for the component statement variables.

Negation

Definition

- Let p be a statement. The negation of p, denoted by $\sim p$, is the statement "It is not the case that p."
- The statement $\sim p$ is read "not p."
- The truth value of the negation of p, $\sim p$ is the opposite of the truth value of p.

Examples

- Today is Friday.
- The negation is "It is not the case that today is Friday." In simple English, "Today is not Friday." or "It is not Friday today."

p	$\sim p$
T	F
F	T

Conjunction: *And* Statements

Definition

- The compound statement formed by connecting statements with the word "*and*" is called a conjunction.
- Let p and q be simple statements. The conjunction of p and q is symbolized by $p \wedge q$. The symbol for *and* is $\wedge$.
- The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Translating from English to Symbolic Form

Let p and q represents the following two statements:

p : It is after 5 P.M.

q : They are working.

Write each compound statement below in symbolic form:

a. It is after 5 P. M. and they are working. $\boxed{p \wedge q}$

b. It is after 5 P. M. and they are not working. $\boxed{p \wedge {\sim}q}$

Common English Expressions for $p \wedge q$

- Pay attention to Words: and, but, yet, nevertheless

Symbolic Statement	English Statement	Example: p : It is after 5 P.M. q : They are working.
$p \wedge q$	p and q	It is after 5 P. M. and they are working.
$p \wedge q$	p but q	It is after 5 P. M. but they are working.
$p \wedge q$	p yet q	It is after 5 P. M. yet they are working.
$p \wedge q$	p nevertheless q	It is after 5 P. M. nevertheless, they are working.

Disjunction: *Or* Statements

Definition

- The compound statement formed by connecting statements with the word "*or*" is called a disjunction.
- Let p and q be statements. The compound statement "p *or* q", denoted by $p \vee q$, is the disjunction of p and q.
- The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.
- This definition is inclusive or: The disjunction is true when at least one of the two propositions is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Truth Tables

- Conjunction and disjunction
- Four different combinations of values for p and q

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Rows: all possible combinations of values for elementary propositions: 2^n values

Exclusive or

Definition

Let p and q be statements. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

$$\boxed{p \oplus q = (p \vee q) \wedge \sim(p \wedge q)}$$

	p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$p \oplus q$
Both	T	T	T	T	F	F
exactly one	T	F	T	F	T	T
exactly one	F	T	T	F	T	T
None	F	F	F	F	T	F

if p is true then q must be false

Constructing the Truth Table for Compound Propositions

- **Example**: Construct the truth table for

  $$(p \wedge q) \vee \sim r$$

- Simpler if we decompose the sentence to elementary and intermediate propositions

The Truth Table of $(p \wedge q) \vee \sim r$

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$

Constructing the Truth Table for Compound Propositions

- **Example**: Construct the truth table for $(p \wedge q) \vee {\sim}r$
- Simpler if we decompose the sentence to elementary and intermediate propositions.

The Truth Table of $(p \wedge q) \vee {\sim}r$		
p q r	$p \wedge q$ ${\sim}r$	$(p \wedge q) \vee {\sim}r$

Elementary propositions

Intermediate propositions

Target Compound Proposition

Constructing the Truth Table for Compound Propositions

- **Example**: Construct the truth table for $(p \wedge q) \vee {\sim}r$

The Truth Table of $(p \wedge q) \vee {\sim}r$					
p	q	r	$p \wedge q$	${\sim}r$	$(p \wedge q) \vee {\sim}r$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

Rows: all possible combinations of values for elementary propositions: 2^n values

Constructing the Truth Table for Compound Propositions

- **Example**: Construct the truth table for $(p \wedge q) \vee \sim r$

The Truth Table of $(p \wedge q) \vee \sim r$					
p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$
T	T	T	T		
T	T	F	T		
T	F	T	F		
T	F	F	F		
F	T	T	F		
F	T	F	F		
F	F	T	F		
F	F	F	F		

Constructing the Truth Table for Compound Propositions

- **Example**: Construct the truth table for $(p \wedge q) \vee \sim r$

The Truth Table of $(p \wedge q) \vee \sim r$					
p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$
T	T	T	T	F	
T	T	F	T	T	
T	F	T	F	F	
T	F	F	F	T	
F	T	T	F	F	
F	T	F	F	T	
F	F	T	F	F	
F	F	F	F	T	

Constructing the Truth Table for Compound Propositions

- **Example**: Construct the truth table for $(p \wedge q) \vee {\sim}r$

The Truth Table of $(p \wedge q) \vee {\sim}r$					
p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee {\sim}r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

Logical Equivalences

- Two statement forms P and Q are called logically equivalent, denoted as $P \equiv Q$, if and only if they have identical true values for each possible substitution of statements for their statement variables.

Example:

- Double Negative Property: $\sim(\sim p) \equiv P$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

Testing Logical Equivalences

Using truth tables to test logical equivalence of two statement forms.

1. Construct a truth table with one column for the true values of each component statement variables.
2. Check each combination of truth values of the statement variables to see whether the truth values of the two statement forms are equal.
   a. If in each row the true values of two statement forms are the same, then these two statement forms are logically equivalent.
   b. Otherwise, not logically equivalent.

What is the problem with this approach?

Exponential Growth: A truth table with n variables has 2^n rows.

Summary of Logical Equivalences

A number of logical equivalences are summarized in Theorem 2.1.1 for future reference.

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology **t** $\nearrow$T and a contradiction **c**, $\nearrow$F the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee {\sim}p \equiv \mathbf{t}$ = T	$p \wedge {\sim}p \equiv \mathbf{c}$ = F
6. *Double negative law:*	${\sim}({\sim}p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* **t** *and* **c**:	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$

$\sim(p \wedge q)$ = Not the case both are true $\therefore$ at least one of them is false

De Morgan's Laws

- The negation of an and statement is logically equivalent to the or statement in which each component is negated.

$$\boxed{\sim(p \wedge q) \equiv \sim p \vee \sim q}$$

- Truth table proving De Morgan's laws.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

AND
$A \cdot 1 = A$
$A \cdot 0 = 0$
$A \cdot A = A$
$A \cdot A' = 0$

OR
$A + 1 = 1$
$A + 0 = A$
$A + A = A$
$A + A' = 1$

Misc
$A'' = A$
$A + (A \cdot B) = A = A \cdot (A + B)$
$A + (A'B) = A + B$
$A + (BC) = (A+B)(A+C)$

De Morgan's Laws

- The negation of an or statement is logically equivalent to the and statement in which each component is negated.

$$\boxed{\sim(p \vee q) \equiv \sim p \wedge \sim q}$$

- Truth table proving De Morgan's laws.

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

$\sim(p \vee q)$ = Not the case that at least one of them is True.
$\therefore$ Both of them must be false

Using De Morgan's Laws

Example:

Use De Morgan's laws to write the negation of $-1 < x \leq 4$.

Solution:

$= \underbrace{-1 < x}_{p}$ and $\underbrace{x \leq 4}_{q}$

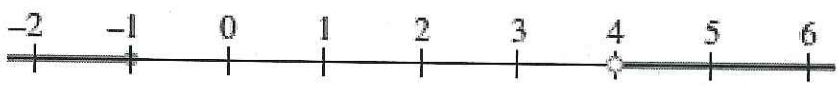
$-1 < x \leq 4 \quad \leftrightarrow (-1 < x) \wedge (x \leq 4)$

$= \sim(-1 < x \wedge x \leq 4)$

By applying the De Morgan's laws, the negation is

$= \sim(-1 < x \wedge x \leq 4)$

$$\begin{aligned} \sim((-1 < x) \wedge (x \leq 4)) &\equiv (\sim(-1 < x)) \vee (\sim(x \leq 4)) \\ &\equiv (-1 \not< x) \vee (x \not\leq 4) \\ &\equiv (-1 \geq x) \vee (x > 4) \end{aligned}$$



Proving New Logical Equivalences

- Use known logical equivalences to prove the following:

$$\sim(p \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim q$$

$\sim(p \vee (\sim p \wedge q))$

$\equiv \sim p \wedge \sim(\sim p \wedge q)$	by De Morgan's law
$\equiv \sim p \wedge (\sim(\sim p) \vee \sim q)$	by De Morgan's law
$\equiv \sim p \wedge (p \vee \sim q)$	by the double negative law
$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$	by the distributive law
$\equiv \boldsymbol{F} \vee (\sim p \wedge \sim q)$	because $\neg p \wedge p = \boldsymbol{F}$
$\equiv (\sim p \wedge \sim q) \vee \boldsymbol{F}$	by commutative laws.
$\equiv \sim p \wedge \sim q$	by the identity law.

Simplifying Statement Forms

$\sim(\sim p \wedge q) \wedge (p \vee q)$

$\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q)$	by De Morgan's law
$\equiv (\underline{p} \vee \sim q) \wedge (\underline{p} \vee q)$ $p\vee$	by the double negative law
$\equiv p \vee (\sim q \wedge q)$	by the distributive law
$\equiv p \vee (q \wedge \sim q)$	by the commutative law
$\equiv p \vee F$	because $q \wedge \sim q = F$
$\equiv p$	by the identity laws.

Logically not Equivalent

Two ways to show that statement forms *P* and *Q* are *not* logically equivalent.

1. Use a truth table to find rows for which their truth values differ.
2. The other way is to find concrete statements for each of the two forms, one of which is true and the other of which is false.

The next example illustrates both of these ways.

Example – *Showing Nonequivalence*

Show that the statement forms $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$		$\sim p \wedge \sim q$
T	T	F	F	T	F		F
T	F	F	T	F	T	$\neq$	F
F	T	T	F	F	T	$\neq$	F
F	F	T	T	F	T		T

$\sim(p \wedge q)$ and $\sim p \wedge \sim q$ have different truth values in rows 2 and 3, so they are not logically equivalent

Example – *Solution*

b. This method uses an example to show that $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent.

Let p be the statement "0 < 1" and let q be the statement "1 < 0."

Then $\sim(p \wedge q)$ is "It is not the case that both $0 < 1$ and $1 < 0$," which is true.

On the other hand,

$$\sim p \wedge \sim q \quad \text{is} \quad \text{“}0 \not< 1 \quad \text{and} \quad 1 \not< 0,\text{”}$$

which is false.

Satisfiability, Tautology, Contradiction

A statement is

- satisfiable, if its truth table contains true at least once. Example: $p \wedge q$
- unsatisable (or a contradiction) if it is not satisable; that is, all entries of its truth table are false.
- a tautology, if it is always true. Example: $p \vee \sim p$.
- a contradiction, if it always false. Example: $p \wedge \sim p$.
- A contingency, if it is neither a tautology nor a contradiction.

Example: p.

Examples of a Tautology and a Contradiction.			
p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

Chapter 2: The Logic of Compound Statements

2.2 Conditional Statements

Conditional Statements

- Conditional statements let you reason from a hypothesis to a conclusion when you make a logical interference or deduction.

If $\underbrace{\text{such and such is known}}_{\text{hypothesis}}$, then $\underbrace{\text{someting or other must be the case}}_{\text{conclusion}}$

True TRUE

Examples:

- If 4,686 is divisible by 6, then 4,686 is divisible by 3.
- If you show up for work Monday morning, then you will get the job.

Conditional Statements

> Definition
> - Let p and q be propositions. The conditional statements $p \to q$ is the proposition "if p, then q." or "p implies q"
> - The conditional statement is false when p is true and q is false, and true otherwise.
> - In the conditional statement $p \to q$, p is called the hypothesis (or *antecedent* or *premise*) and q is called the conclusion (or *consequence*).

- A conditional statement is also called an implication.

p = hypothesis

q = conclusion

if p then q = keep my word

p	q		$p \to q$	
T	T	keep	T	
T	F	break	F	→ only false
F	T	keep	T	
F	F	keep	T	

$\therefore \sim(p \to q) \equiv p \wedge \sim q$

Order of Logical Operators

- Use parentheses to specify the order in which logical operators in a compound proposition are to be applied.
- To reduce the number of parentheses, the precedence order is defined for logical operators.

Precedence of Logical Operators.	
Operator	Precedence
$\sim$	1
$\wedge$	2
$\vee$	3
$\rightarrow$ implies	4
$\leftrightarrow$	5

E.g. $\sim p \wedge q = (\sim p) \wedge q$

$p \wedge q \vee r = (p \wedge q) \vee r$

$p \vee q \wedge r = p \vee (q \wedge r)$

Conditional Statements - Examples

Construct a true table for the statement form $p \vee \sim q \rightarrow \sim p$

p	q	$\sim q$	$p \vee \sim q$	$\sim p$	$p \vee \sim q \rightarrow \sim p$
T	T	F	T	F	break F
T	F	T	T	F	break F
F	T	F	F	T	Keep T
F	F	T	T	T	Keep T

Logical Equivalence Involving →

- Show that $p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	$p \vee q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Representation of If-then as Or

- Show that $p \rightarrow q \equiv \sim p \vee q$

p	q	$\sim p$	$p \rightarrow q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

if $\sim(p \rightarrow q) \equiv p \wedge \sim q$

then $(p \rightarrow q) \equiv \sim p \vee q$

* Test question

The Negation of a Conditional Statement

The negation of "if p then q" is logically equivalent to "p and not q":

$$\boxed{\sim(p \rightarrow q) \equiv p \wedge \sim q}$$

Proof: $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q)$

$\equiv \sim(\sim p) \wedge \sim q$ by De Morgan's laws

$\equiv p \wedge \sim q$ by the double negative law

Example:
Write negations for each of the following statements:

a. If my car is in the repair shop, then I cannot get to class.
   - My car is in the repair shop and I can get to class.

b. If Sara lives in Athens, then she lives in Greece.
   - Sara lives in Athens and she does not lives in Greece.

Contrapositive, Converse and Inverse

- Other conditional statements:
  - Two operation – Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$ switch then negation
  - One operation – The Converse of $p \rightarrow q$ is $q \rightarrow p$ switch order
  - One operation – Inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$ apply negation
- <u>Example:</u>
  - If it snows, the traffic moves slowly.
  - p: it snows q: traffic moves slowly. $(p \rightarrow q)$

  The contrapositive:
  - If the traffic does not move slowly then it does not snow. $(\sim q \rightarrow \sim p)$

  The converse:
  - If the traffic moves slowly then it snows. $(q \rightarrow p)$

  The inverse:
  - If it does not snow then the traffic moves quickly. $(\sim p \rightarrow \sim q)$

- Contrapositive $\sim q \rightarrow \sim p \equiv p \rightarrow q$

Contrapositive, Converse and Inverse

- A conditional statement is logically equivalent to its contrapositive.

$$p \to q \equiv {\sim}q \to {\sim}p$$

p	q	${\sim}p$	${\sim}q$	${\sim}q \to {\sim}p$	$p \to q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

- A conditional statement and its converse are not logically equivalent.
- A conditional statement and its inverse are not logically equivalent.
- The converse and the inverse of a conditional statement are logically equivalent to each other.

$$q \to p \equiv {\sim}p \to {\sim}q$$

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Only If

Definition

If p and q are statements,

"p **only if** q" means "if not q then not p,"

Or, equivalently,

"if p then q"

Example: Converting Only If to If-Then

Join will break the world's record for the mile run only if he runs the mile in under four minutes.

⇒ If Join does not run the mile in under four minutes, then he will not break the world's record for the mile run.

Or

⇒ If john breaks the world's record for the mile run, then he will have run the mile in under four minutes.

Exampes:

if $\underbrace{\text{n is prime}}_{p}$, $\underbrace{\text{then n is odd or n is 2}}_{q}$

neg = n is prime and n is not odd and n not 2

Inverse: if n is not prime, then n isn't odd and n not 2

Biconditional Statements

Definition

- Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q."
- The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.
- Biconditional statements are also called *bi-implications*.

- $p \leftrightarrow q$ has the same truth value as $(p \rightarrow q) \land (q \rightarrow p)$ also $\equiv \sim(p \oplus q)$
- "if and only if" can be expressed by "iff".

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional Statements

Example: Converting If and Only If to two If-Then statements:

This computer program is correct if, and only if, it produces correct answers for all possible sets of input data.

Solution:

If this computer program is correct, then it produces correct answers for all possible sets of input data; and if this program produces the correct answers for all possible sets of input data, then it is correct.

* Know

Necessary and Sufficient Conditions

Definition

If r and s are statement:

$r \to s$

r is a $\boxed{\text{sufficient}}$ condition for s means "if r then s"

r is a $\boxed{\text{necessary}}$ condition for s means "if not r then not s". also $s \to r$

but not enough

$\sim p \to \sim q$ or $q \to p$

or "if s then r"

r is a $\boxed{\text{necessary and sufficient}}$ condition for s means "r if, and only if, s"

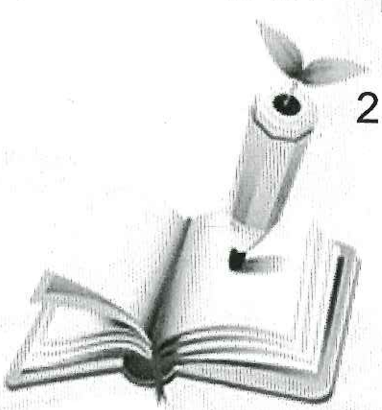
Example: converting a sufficient condition to If-Then form:

- Pia's birth on U.S soil is a sufficient condition for her to be a U.S. citizen.
  - If pia was born on U.S soil (r), then she is a U.S. citizen (s).

Example: converting a necessary condition to If-Then form:

- George's attaining age 35 is a necessary condition for his being president of the United States. $s \to r$ also $\sim r \to \sim s$
  - If George can be president of the United States, then he has attained the age of 35.
  - If George has not attained the age of 35 ($\sim r$), then he cannot be president of the United States ($\sim s$).

Chapter 2: The Logic of Compound Statements



2.3 Valid and Invalid Arguments

Arguments, Argument forms and Their Validity

Definition

- An argument is sequence of statement. All but the final statement are called premises and the final statement is called the conclusion. The symbol ∴, read "therefore" is normally placed just before the conclusion.
- An argument form is a sequence of statement forms which are compound statements involving propositional variables.
- An argument form is valid if no matter which particular statements are substituted for the statement variables in its premises, the conclusion is true if the premises are all true.
- To say that an argument is valid means that its form is valid.

In other words, an argument form with premises $p_1, p_2, \cdots, p_n$ and conclusion q is valid if and only if

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$$

is a tautology.

Ex $p \to q$ T
p T
$\therefore q$ T conclusion

Valid argument form always true conclusion = conjunction

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. A row of the truth table in which all the premises are true is called a **critical row**.
   - If there is a critical row in which the conclusion is false, then it is possible for an argument of the given form to have true premises and a false conclusion, and so the argument form is invalid.
   - If the conclusion in *every* critical row is true, then the argument form is valid.

Determining Validity or Invalidity

$$\begin{array}{lcl} p & \to & q \vee \sim r \\ q & \to & p \wedge r \\ \hline \therefore p & \to & r \end{array}$$

This statement form is invalid.

Conclusion

p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \to q \vee \sim r$	$q \to p \wedge r$	$p \to r$
T	T	T	F	T	T	T crit row	T	T
T	T	F	T	T	F	T	F does not satisfy both conditions	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T crit row	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T crit row	T	T
F	F	F	T	T	F	T crit row	T	T

Rules of Inference

- Proofs in mathematics are valid arguments that establish the truth of mathematical statements.
  - ➢ An argument is a sequence of statements that end with a conclusion.
  - ➢ The argument is valid if the conclusion (final statement) follows from the truth of the preceding statements (premises).
  - ➢ An argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false.
- Rules of inference are templates for constructing valid arguments and the basic tools for establishing the truth of statements.
- An argument form consisting of two premises and a conclusion is called a **syllogism.** The first and second premises are called the **major premise** and **minor premise**, respectively.

Rules of Inference for Propositional Logic I

- **Rules of inference,** some relatively simple valid argument forms, can be used as building blocks to construct more complicated valid argument forms.

Rules of Interference	Tautology	Name
p T $p \rightarrow q$ T $\therefore q$ T	$(p \land (p \rightarrow q)) \rightarrow q$	Modus ponens
$\sim q$ F $p \rightarrow q$ T $\therefore \sim p$ F	$(\sim q \land (p \rightarrow q)) \rightarrow \sim p$	Modus tollens
p T $\therefore p \lor q$ T	$p \rightarrow (p \lor q)$	Generalization
$p \land q$ T $\therefore p$ T	$(p \land q) \rightarrow p$	Specification
p T q T $\therefore p \land q$ T	$((p) \land (q)) \rightarrow (p \land q)$	Conjunction

Rules of Inference for Propositional Logic II

- **Rules of inference,** some relatively simple valid argument forms, can be used as building blocks to construct more complicated valid argument forms.

Rules of Interference	Tautology	Name
$p \lor q$ T $\sim p$ F $\therefore q$ T	$((p \lor q) \land \neg p) \rightarrow q$	Elimination
$p \rightarrow q$ T $q \rightarrow r$ T $\therefore p \rightarrow r$ T	$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Transitivity
$p \lor q$ T $p \rightarrow r$ T $q \rightarrow r$ T $\therefore r$ T	$((p \lor q) \land (p \rightarrow r) \land (q \rightarrow r)) \rightarrow r$	Proof by Division into Cases
$\sim p \rightarrow F$ $\therefore p$	if ~p leads to contradiction $(\sim p \rightarrow F) \rightarrow p$ then p must be true	Contradiction Rule

Rules of Inference: Modus Ponens

Rules of Interference	Tautology	Name
$p \rightarrow q$ $\underline{p}$ $\therefore q$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens

		premises		conclusion	
p	q	$p \rightarrow q$	p	q	
T	T	T	T	T	← critical row
T	F	F	T		
F	T	T	F		
F	F	T	F		

Example:
Let p be "It is snowing."
Let q be "I will study discrete math."

"It is snowing."
"If it is snowing, then I will study discrete math."

"Therefore , I will study discrete math."

55

Rules of Inference: Modus Tollens

Rules of Interference	Tautology	Name
$p \rightarrow q$ $\underline{\sim q}$ $\therefore \sim p$	$(\sim q \wedge (p \rightarrow q)) \rightarrow \sim p$	Modus tollens

Example:
Let p be "It is snowing."
Let q be "I will study discrete math."

"I will not study discrete math."
"If it is snowing, then I will study discrete math."

"Therefore , It is not snowing."

Modus Ponens and Modus Tollens

Rules of Interference	Tautology	Name
$p \rightarrow q$ $\underline{p}$ $\therefore q$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modulus ponens
$p \rightarrow q$ $\underline{\sim q}$ $\therefore \sim p$	$(\sim q \wedge (p \rightarrow q)) \rightarrow \sim p$	Modus tollens

p	q	$p \rightarrow q$	q	$\sim q$	$\sim p$
T	T	T	T	F	
T	F	F		T	
F	T	T		F	
F	F	T	crit row	T	T

Exercise
- Recognizing Modus Ponens and Modus Tollens

Use modus ponens or modus tollens to fill in the blanks of the following arguments so that they become valid inferences.

a. If $\overbrace{\text{there are more pigeons than there are pigeonholes}}^{p}$, then at least two pigeons roost in the same hole.
There are more pigeons than there are pigeonholes.

∴ At least two pigeons roost in the same hole. (Modus Ponens).

b. If $\overbrace{\text{870,232 is divisible by 6}}^{p}$, $\overbrace{\text{then it is divisible by 3}}^{q}$.
870,232 is not divisible by 3.

∴ 870,232 is not divisible by 6. (by Modus Tollens).

Rules of Inference: Generalization

Rules of Interference	Tautology	Name
$\dfrac{p}{\therefore\ p \lor q}$	$p \to (p \lor q)$	Generalization

Example:
Let *p* be "I will study discrete math."
Let *q* be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Rules of Inference: Specification

Rules of Interference	Tautology	Name
$\dfrac{p \land q}{\therefore\ p}$	$(p \land q) \to p$	Specification

Example:
Let *p* be "I will study discrete math."
Let *q* be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Rules of Inference: Elimination

Rules of Interference	Tautology	Name
$p \vee q$ $\sim p$ $\therefore q$	$((p \vee q) \wedge \sim p) \rightarrow q$	Elimination

These argument forms say that when you have only two possibilities and you can rule one out, the other must be the case.

Example:
Let *p* be "I will study discrete math."
Let *q* be "I will study English literature."

"I will study discrete math or I will study English literature."
"I will not study discrete math."

"Therefore , I will study English literature."

Rules of Inference: Transitivity

Rules of Interference	Tautology	Name
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Transitivity

- Many arguments in mathematics contain chains of if-then statements.
- From the fact that one statement implies a second and the second implies a third, you can conclude that the first statement implies the third.

Example:
Let *p* be "It is snowing."
Let *q* be "I will study discrete math."
Let *r* be "I will get an A."

"If it is snowing, then I will study discrete math." $p \rightarrow q$
"If I study discrete math, I will get an A." $q \rightarrow r$

"Therefore , If it snows, I will get an A." $p \rightarrow r$

Proof by Division into Cases

Rules of Interference	Tautology	Name
$p \lor q$ $p \to r$ $q \to r$ $\therefore r$	$((p \lor q) \land (p \to r) \land (q \to r)) \to r$	Proof by Division into Cases

It often happens that you know one thing or another is true. If you can show that in either case a certain conclusion follows, then this conclusion must also be true.

Example:
Let p be "x is positive ."
Let q be "x is negative."
Let r be "$x^2 > 0$."

"$\overbrace{x \text{ is positive}}^{p}$ or $\overbrace{x \text{ is negative}}^{q}$."
"If x is positive, then $x^2 > 0$."
"If x is negative, then $x^2 > 0$."

"Therefore , $x^2 > 0$."

Table with names provided @ exam

Which rule of inference is used in each argument below?

- "It is below freezing now. Therefore, it is either below freezing or raining now."
  - ➢ The generalization rule.
- "It is below freezing and raining now. Therefore, it is below freezing now."
  - ➢ The specification rule
- "If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow."
  - ➢ The transitivity rule

Using Rules of Inference to Build Arguments (ex1)

Show that the hypotheses:

- "It is not sunny this afternoon and it is colder than yesterday," $\sim p \wedge q$
- "We will go swimming only if it is sunny," $r \to p$
- "If we do not go swimming, then we will take a canoe trip," $\sim r \to s$
- "If we take a canoe trip, then we will be home by sunset" $s \to t$

lead to the conclusion

- "We will be home by sunset." $\therefore t$

Main steps:

1. Translate the statements into propositional logic.

   Let p: "It is sunny this afternoon," q: "It is colder than yesterday," r: "We will go swimming," s : "We will take a canoe trip," and t :"We will be home by sunset."
   The premises becomes: $\sim p \wedge q,\ r \to p,\ \sim r \to s,\ s \to t$. The conclusion is simply t .

2. Write a formal proof, a sequence of steps that state hypotheses or apply inference rules to previous steps.

Now build

Using Rules of Inference to Build Arguments (ex1) (cont')

Main steps:

1. Translate the statements into propositional logic.

   The premises becomes: $\sim p \wedge q,\ r \to p,\ \sim r \to s,\ s \to t$. The conclusion is simply t .

2. Write a formal proof, a sequence of steps that state hypotheses or apply inference rules to previous steps.

Step	**Reason**
1. $\sim p \wedge q$	Premise
2. $\sim p$	Specification using (1)
3. $r \to p$	Premise
4. $\sim r$	Modus tollens using (2) and (3)
5. $\sim r \to s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \to t$	Premise
8. t	Modus ponens using (6) and (7)

Using Rules of Inference to Build Arguments (ex2)

Show that the hypotheses:

- "If you send me an e-mail message, then I will finish writing the program,"
- "If you do not send me an e-mail message, then I will go to sleep early,"
- "If I go to sleep early, then I will wake up feeling refreshed" I

lead to the conclusion

- "If I do not finish writing the program, then I will wake up feeling refreshed."

Main steps:

1. Translate the statements into propositional logic.

   Let p: "You send me an e-mail message," q: "I will finish writing the program," r: "I will go to sleep early," and s : "I will wake up feeling refreshed."

   The premises becomes: $p \to q$, $\sim p \to r$, $r \to s$. The conclusion is simply $\sim q \to s$.

2. Write a formal proof, a sequence of steps that state hypotheses or apply inference rules to previous steps.

Using Rules of Inference to Build Arguments (ex2) (cont')

Main steps:

1. Translate the statements into propositional logic.

   The premises becomes: $p \to q$, $\sim p \to r$, $r \to s$. The conclusion is simply $\sim q \to s$.

2. Write a formal proof, a sequence of steps that state hypotheses or apply inference rules to previous steps.

Step	Reason
1. $p \to q$	Premise
2. $\sim q \to \sim p$	Contrapositive of (1)
3. $\sim p \to r$	Premise
4. $\sim q \to r$	The rule of transitivity using (2) and (3)
5. $r \to s$	Premise
6. $\sim q \to s$	The rule of transitivity using (4) and (5)

Ex 1 build

(1) $\sim p \wedge q$

(2) $r \to p$

(3) $\sim r \to s$

(4) $s \to t$

(5) $\therefore t$

1) $\sim p \wedge q$ (1)

$\therefore \sim p$ Specification

2) $\sim p$

$r \to p$ (3)

$\therefore \sim r$ tollens

3) $\sim r \to s$

$\sim r$ (2)

$\therefore s$ Pollens

4) $s \to t$

s

$\therefore t$ Ponen

Ex 2 Build

(a) $RK \to GK$

(b) $GK \to SB$

(c) $\sim SB$

(d) $RL \vee RK$

(e) $RL \to GC$

$RK \to SB$

$\sim RK$

RL

$\underset{T}{RL} \to \underset{T}{GC}$

Using Rules of Inference to Build Arguments (ex3)

You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:

a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

b. If my glasses are on the kitchen table, then I saw them at breakfast.

c. I did not see my glasses at breakfast.

d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses?

Using Rules of Inference to Build Arguments (ex3)

Solution:

Let

- *RK* = I was reading the newspaper in the kitchen.
- *GK* = My glasses are on the kitchen table.
- *SB* = I saw my glasses at breakfast.
- *RL* = I was reading the newspaper in the living room.
- *GC* = My glasses are on the coffee table.

Using Rules of Inference to Build Arguments (ex3) cont'd

Here is a sequence of steps you might use to reach the answer, together with the rules of inference that allow you to draw the conclusion of each step:

1. $RK \to GK$ by (a)

$GK \to SB$ by (b)

- $RK \to SB$ by transitivity

2. $RK \to SB$ by the conclusion of (1)

$\sim SB$ by (c)

- $\sim RK$ by modus tollens

Using Rules of Inference to Build Arguments (ex3)

3. $RL \vee RK$ by (d)

$\sim RK$ by the conclusion of (2)

- RL by elimination

4. $RL \to GC$ by (e)

RL by the conclusion of (3)

- GC by modus ponens

Thus the glasses are on the coffee table.

Fallacies – Converse Error

- Converse error or Fallacy of affirming the conclusion

Based on

$$\begin{array}{l} p \rightarrow q \\ \underline{q \qquad\quad} \\ \therefore p \end{array}$$

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

which is NOT A TAUTOLOGY.

Example:

If you do every problem in this book, then you will learn discrete mathematics.

You learned discrete mathematics.

Therefore, you did every problem in this book.

It is possible for you to learn discrete mathematics in someway other than by doing every problem in this book.

Fallacies – Inverse error

- Inverse error or Fallacy of denying the hypothesis

Based on

$$\begin{array}{l} p \rightarrow q \\ \underline{\sim p \qquad\quad} \\ \therefore \sim q \end{array}$$

which is NOT A TAUTOLOGY.

Example:

If you do every problem in this book, then you will learn discrete mathematics. And you didn't do every problem in this book. Therefore, you did not learn discrete mathematics.

It is possible that you learned discrete mathematics even if you did not do every problem in this book.

Contradiction Rule

> **Contradiction Rule**
>
> If you can show that the supposition that statement p is false leads logically to a contradiction, then you can conclude that p is true.

Show that the following argument form is valid:

$\sim p \rightarrow c$, where c is a contradiction

$\therefore p$

$\sim p \rightarrow \sim p_2$

P_1

$P_2 \wedge \sim P_2 \equiv C$

$\therefore P$

Solution:

Construct a truth table for the premise and the conclusion of this argument.

			premises	conclusion
p	$\sim p$	c	$\sim p \rightarrow c$	p
T	F	F	T	T
F	T	F	F	

There is only one critical row in which the premise is true, and in this row the conclusion is also true. Hence this form of argument is valid.

If an assumption leads to a contradiction, then that assumption must be false.

AND

$\cdot$

$1 \cdot 1 = 1$

$1 \cdot 0 = 0$

0 1 0

0 0 0

OR

$+$

$1 + 1 = 1$

$1 + 0 = 1$

$0 + 1 = 1$

$0 + 0 = 0$

$1 = T$

$0 = F$

Sets

Like a bag & can contain another bag

- A set is an **unordered** collection of objects. The objects in a set are called the elements, or members of the set.
- A set is said to contain its elements.

  $A = \{a_1, a_2, \ldots, a_n\}$ "A contains $a_1, a_2, \ldots, a_n$"
- $a \in A$ "a is an element of A"

  ↳ Belongs to or → "a is a member of A"
- $a \notin A$ "a is not an element of A"

  ↳ Does NOT belong

Examples:

Sets:
- **Vowels in the English alphabet**

  V = { a, e, i, o, u }
- **First seven prime numbers**.

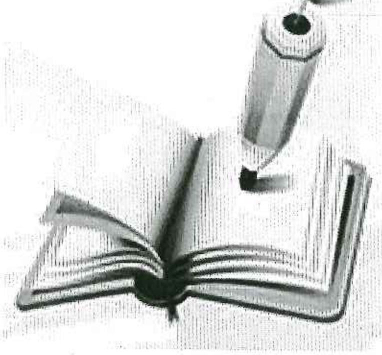
  X = { 2, 3, 5, 7, 11, 13, 17 }

Important Sets in Discrete Math

- $N = \{0, 1, 2, 3, \ldots\}$, the set of natural numbers
- $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of all integers
- $Z^+ = \{1, 2, 3, \ldots\}$, the set of positive integers → No Zero
- $Q = \{p/q \mid p \in Z, q \in Z, and\ q \neq 0\}$, the set of all rational numbers
- R, the set of all real numbers
- R^+, the set of positive real numbers
- C, the set of complex numbers.

Chapter 3: The Logic of Quantified Statements

3.1 Predicates and Quantified Statements I



Predicate vs. Propositional Logic

Predicate logic extend propositional logic by the three new features:

- Variables: $x, y, z, \cdots$ (subject of the statement)
- Predicates (i.e., propositional functions): $P(x), Q(x), R(y), M(x, y), \cdots$.
  (a property that the subject of the statement can have)
- Quantifiers: $\forall, \exists$. (make statements about groups of objects)

↳ Universal : all properties of set are true

Existential: there exists one element true

- The symbolic analysis of predicates and quantified statements is called the predicate logic.
- The symbolic analysis of ordinary compound statements is called the statement logic or propositional logic.

Predicates

- A predicate is a declarative sentence whose true/false value depends on one or more variables and becomes a statement when specific values are substituted for the variables.
- The domain of a predicate variable is the set of all values that may be substituted in place of the variable.
- A predicate $P(x)$ assigns a value **true or false** to each x in domain D depending on whether the property holds or not for x.
- The true set of $P(x)$, denoted as $\{x \in D | P(x)\}$, is the set of all elements of D that make $P(x)$ true when they are substituted for x.
  Set within a set where everything is true
- The statement $P(x)$ is also said to be the value of the **propositional function** P at x.

Predicates: Exercise

- Denote the statement "x is greater than 3" by $P(x)$, where P is the predicate "is greater than 3" and x is the variable.
- The statement "x is greater than 3" has two parts:
  - The subject: x is the subject of the statement
  - The predicate: "is greater than 3" (a property that the subject can have)

Predicates: Exercise

Given each propositional function determine its true/false value when variables are set as below.

- $\boxed{P(x) = x > 3}$ Predicate
  $P(4)$ is true. → Statement true
  $P(2)$ is false → Statement false
- $Prime(x)$ = "x is a prime number."
  Prime(2) is true, since the only numbers that divide 2 are 1 and itself.
  Prime(9) is false, since 3 divides 9.
- $Q(x, y)$ = "$x = y + 3$."
  $Q(1,2)$ is false $1 = 2+3 \rightarrow 1 = 5$ false
  $Q(3,0)$ is true

Predicates vs. Propositions

Assume a predicate $P(x)$ that represents the statement:

- x is a prime number

What are the truth values of:

- $P(2)$ T
- $P(3)$ T
- $P(4)$ F
- $P(5)$ T
- $P(6)$ F
- $P(7)$ T

All statements P(2), P(3), P(4), P(5), P(6), P(7) are propositions.

Predicates vs. Propositions

Assume a predicate P(x) that represents the statement:

- x is a prime number

What are the truth values of:

- $P(2)$ T
- $P(3)$ T
- $P(4)$ F
- $P(5)$ T
- $P(6)$ F
- $P(7)$ T

Is P(x) is a statement? No. Many possible substitutions are possible.

Predicate Truth Set: Exercise

Finding the True Set of a Predicate $Q(n)$: "n is a factor of 8."

means no remainder after division

a. The domain of n is the set of Z^+ of all positive integers.

b. The domain of n is the set of Z of all integers.

Solution:

a. The truth set is {1, 2, 4, 8} because these are exactly the positive integers that divide 8 evenly.

b. The truth set is {-8, -4, -2, -1, 1, 2, 4, 8} because these are exactly the positive integers that divide 8 evenly, without leaving a remainder.

Quantifiers

- Quantification: express the extent to which a predicate is true over a range of elements. In English, the words all, some, many, none, and few are used in quantifications.
- The two most important quantifiers are:
  - **Universal** quantifiers (∀): a predicate is true for every element under consideration Notation: $\forall x \in D$
    Example: " all CS USF students have to pass COT3100"
  - **Existential** quantifiers (∃): a predicate is true for one or more element under consideration Notation: $\exists x \in D$
    Example: "some of CS USF students graduate with honor."

Universal Statement

Definition

- Let $Q(x)$ be a predicate and D the domain of x.
- A universal statement is a statement of the form "$\forall x \in D, Q(x)$", where ∀ is called the Universal Quantifier.
- It is defined to be true if, and only if, $Q(x)$ is true for every x in D.
- It is defined to be false if, and only if, $Q(x)$ is false for at least one x in D, which is called a counterexample to the universal statement.

need only 1 counterexample to disprove

Let $P(x)$ be "$x^2 > 10$". Find the truth value of $\forall x\, P(x)$ for the following domains:

- the set of real numbers: ℝ ?
- False. 3 is a counterexample.
- the set of positive integers not exceeding 4: {1, 2, 3, 4} ?
- False. 3 is a counterexample.
- the set of real numbers in the interval [10, 39.5] ?
- True. It takes a bit longer to verify than in false statements. Let $x \in [10, 39.5]$. Then $x \geq 10$ which implies $x^2 \geq 10^2 = 100 > 10$, and so $x^2 > 10$.

Example 3 – Truth and Falsity of Universal Statements

a. Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x. \text{ ?}$$

Show that this statement is true.

$1^2 \geq 1, \; 2^2 \geq 2, \; 3^2 \geq 3, \; 4^2 \geq 4, \; 5^2 \geq 5$ ✓ T

b. Consider the statement

$$\forall x \in \mathbf{R}, x^2 \geq x. \text{ ?}$$

Find a counterexample to show that this statement is false.

Counter example

$$\left(\tfrac{1}{2}\right)^2 \geq \tfrac{1}{2} \longrightarrow \tfrac{1}{4} \geq \tfrac{1}{2} \quad \times \; F$$

Example 3 – Solution

a. Check that "$x^2 \geq x$" is true for each individual *x* in *D*.

$$1^2 \geq 1, \qquad 2^2 \geq 2, \qquad 3^2 \geq 3, \qquad 4^2 \geq 4, \qquad 5^2 \geq 5.$$

Hence "$\forall x \in D, x^2 \geq x$" is true.

- This technique used to show the truth of the universal statement is called the **method of exhaustion.**
- It consists of showing the truth of the predicate separately for each individual element of the domain.
- This method can, in theory, be used whenever the domain of the predicate variable is finite.

Example 3 – Solution

b. *Counterexample:* Take $x = \frac{1}{2}$. Then x is in $\boldsymbol{R}$ (since $\frac{1}{2}$ is a real number) and

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \ngeq \frac{1}{2}.$$

Hence "$\forall x \in \boldsymbol{R}, x^2 \geq x$" is false.

Existential Statement

Definition

- Let $Q(x)$ be a predicate and D the domain of x.
- An existential statement is a statement of the form "$\exists x \in D$ such that $Q(x)$", where $\exists$ is called the Existential Quantifier.
- It is defined to be true if, and only if, $Q(x)$ is true for at least one x in D, which is called a witness to the existential statement..
- It is defined to be false if, and only if, $Q(x)$ is false for all x in D. → all need F

Let $P(x)$ be "$x^2 > 10$". Find the truth value of $\exists x\, P(x)$ for the following domains:

- the set of real numbers: $\mathbb{R}$
- True. 10 is a witness.
- the set of positive integers not exceeding 4: {1, 2, 3, 4}
- True. 4 is a witness.
- the set of real numbers in the interval $[0, \sqrt{9.8}]$
- False. Let $x \in [0, \sqrt{9.8}]$. Then $0 \leq x \leq \sqrt{9.8}$ which implies $x^2 \leq 9.8 < 10$, and so $x^2 < 10$.

Example 4 – Truth and Falsity of Existential Statements

a. Consider the statement

$$\exists m \in Z^+ \text{ such that } m^2 = m. \quad ?$$

Show that this statement is true.

$1^2 = 1$ ✓ T

Solution:

- Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for at least one integer m. Hence "$\exists m \in Z^+$ such that $m^2 = m$" is true.

Example 4 – Truth and Falsity of Existential Statements

b. Let $E = \{5, 6, 7, 8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m. \quad ?$$

Show that this statement is false.

Solution:

- Note that $m^2 = m$ is not true for any integers m from 5 through 8:

$$5^2 = 25 \neq 5, \qquad 6^2 = 36 \neq 6, \qquad 7^2 = 49 \neq 7, \qquad 8^2 = 64 \neq 8.$$

Thus "$\exists m \in E$ such that $m^2 = m$" is false.

F

Summary of Quantified Statements

- If the domain is empty,
  - $\forall x\, P(x)$ is **true** for any propositional function $P(x)$, since there are no counterexamples in the domain.
  - $\exists x\, Q(x)$ is false because there can is no element in the domain for which $Q(x)$ is true.
- When $\forall x\, P(x)$ and $\exists x\, P(x)$ are true and false?

Summary

	Statement	When True?	When False?
Universal	$\forall x\, P(x)$	$P(x)$ is true for every x	There is an x for which $P(x)$ is false.
Existential	$\exists x\, P(x)$	There is some x for which $P(x)$ is true	$P(x)$ is false for every x

- Suppose the elements in the universe of discourse can be enumerated as $x_1, x_2, \ldots, x_N$ then:
  - **$\forall x\, P(x)$ is true whenever $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_N)$ is true** Conjunction
  - **$\exists x\, P(x)$ is true whenever $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_N)$ is true.** disjunction

Example 5 – Translating from Formal to Informal Language

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol $\forall$ or $\exists$.

a. $\forall x \in \mathbf{R}, x^2 \geq 0.$

Solution:

All real numbers have nonnegative squares.

Or: Every real number has a nonnegative square.

Or: Any real number has a nonnegative square.

Or: The square of each real number is nonnegative.

Example 5 – Translating from Formal to Informal Language

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol ∀ or ∃.

b. $\forall x \in \mathbf{R}, x^2 \neq -1.$

Solution:

All real numbers have squares that are not equal to −1.

Or: No real numbers have squares equal to −1.

(The words *none are* or *no . . . are* are equivalent to the words *all are not*.)

Example 5 – Translating from Formal to Informal Language

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol ∀ or ∃.

c. $\exists m \in \mathbf{Z}^+$ such that $m^2 = m$.

Solution:

There is a positive integer whose square is equal to itself.

Or: We can find at least one positive integer equal to its own square.

Or: Some positive integer equals its own square.

Or: Some positive integers equal their own squares.

Universal Conditional Statements

- The universal conditional statement:

$$\forall x, \text{if } P(x) \text{ then } Q(x)$$

Example: Writing universal conditional statements formally.

- If a real number is an integer, then it is a rational number.
  - $\forall x \in R, \text{if } x \in Z \text{ then } x \in Q$
- All bytes have eight bits.
  - $\forall x$ if x is a byte then x has eight bits.
- No fire trucks are green.
  - $\forall x$ if x is a fire truck then x is not green.

Example 8 – Writing Universal Conditional Statements Informally

Rewrite the following statement informally, without quantifiers or variables.

$$\forall x \in \boldsymbol{R}, \text{if } x > 2 \text{ then } x^2 > 4.$$

Solution:

If a real number is greater than 2 then its square is greater than 4.

Or: Whenever a real number is greater than 2, its square is greater than 4.

Or: The square of any real number greater than 2 is greater than 4.

Or: The squares of all real numbers greater than 2 are greater than 4.

Equivalent Forms of Universal Statements

- A statement of the form
$$\forall x \in U, \text{if } P(x) \text{ then } Q(x)$$
can always be rewritten in the form
$$\forall x \in D, Q(x)$$
by narrowing U to be the domain D consisting of all values of the variable x that make $P(x)$ true.

- Conversely, a statement of form
$$\forall x \in D, Q(x)$$
can be rewritten as
$$\forall x, \text{if } x \text{ in } D \text{ then } Q(x)$$

Example 10 – Equivalent Forms for Universal Statements

Rewrite the following statement in the two forms "∀x, if ______ then ______" and "∀ ______x, ______":
All squares are rectangles.

Solution:

∀x, if x is a square then x is a rectangle.

∀ squares x, x is a rectangle.

Equivalent Forms of Existential Statements

Similarly, a statement of the form

"$\exists x$ such that $p(x)$ and $Q(x)$"

can be rewritten as

"$\exists x \in D$ such that $Q(x)$,"

where D is the set of all x for which $P(x)$ is true.

Example 11 – Equivalent Forms for Existential Statements

A **prime number** is an integer greater than 1 whose only positive integer factors are itself and 1. Consider the statement "There is an integer that is both prime and even."

Let $Prime(n)$ be "n is prime" and $Even(n)$ be "n is even." Use the notation $Prime(n)$ and $Even(n)$ to rewrite this statement in the following two forms:

a. $\exists n$ such that ______ $\wedge$ ______ .

b. $\exists$ ______ n such that ______.

Solution:

a. $\exists n$ such that $Prime(n) \wedge Even(n)$.

b. Two answers: $\exists$ a prime number n such that $Even(n)$.

$\exists$ an even number n such that $Prime(n)$.

S = students, D = domain, E(s) = Engineer
M(s) = Math, C(s) = Computer Science

a. There is an eng. student who is math
$\exists s \in D,\ E(s) \wedge M(s)$

b. Every Comp. Scie student is an Eng
$\forall s \in D,\ C(s) \rightarrow E(s)$

c. No computer Scie student is an Eng
$\forall s \in D,\ C(s) \rightarrow \sim E(s)$

d. Some Comp Sci. students are Math
$\exists s \in D,\ C(s) \wedge M(s)$

e. Some Comp Sci students are Eng & some are not
$\exists x, y \in D$, such that $C(x) \wedge E(x) \wedge$
$C(y) \wedge E(y)$

Chapter 3: The Logic of Quantified Statements

3.2 Predicates and Quantified Statements II

Negation of a Universal Statement

- The negation of a statement of the form

  $$\forall x \in D, Q(x)$$

  Is logically equivalent to a statement of the form

  $$\exists x \in D \text{ such that } {\sim}Q(x)$$

  Symbolically, ${\sim}(\forall x \in D, Q(x)) \equiv \exists x \in D \text{ such that } {\sim}Q(x)$

- The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").

Negation of an Existential Statement

- The negation of a statement of the form
  $$\exists x \in D \text{ such that } Q(x)$$ cannot find one true = neg
  Is logically equivalent to a statement of the form
  $$\forall x \in D, \sim Q(x)$$
  Symbolically, $\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x)$

- The negation of an existential statement ("some are") is logically equivalent to a universal statement ("none are" or "all are not").

Example 1 – Negating Quantified Statements

Write formal negations for the following statements:

a. $\forall$ primes *p*, *p* is odd.

b. $\exists$ a triangle *T* such that the sum of the angles of *T* equals 200°.

Solution:

a. By applying the rule for the negation of a $\forall$ statement, you can see that the answer is

$\exists$a prime *p* such that *p* is not odd.

b. By applying the rule for the negation of a $\exists$ statement, you can see that the answer is

$\forall$ triangles *T*, the sum of the angles of *T* does not equal 200°.

Negation of Quantified Statements

- The rules for negations for quantifiers are called **De Morgan's laws for quantifiers**

De Morgan's Laws for Quantifiers			
Negation	Equivalent Statement	When is Negation True?	When False?
$\sim\exists x\, P(x)$	$\forall x \sim P(x)$	$P(x)$ is false for every x.	There is an x for which $P(x)$ is true
$\sim\forall x\, P(x)$	$\exists x \sim P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Negations of Universal Conditional Statements

- Negation of a universal conditional statement

$$\sim(\forall x, \text{if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x)$$

or

$$\sim(\forall x, P(x) \to Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x))$$

from chapter 2

- Proof: only one case that is false. When P(x)T ∧ Q(x)F

By the definition of the negation of a universal statement:

$$\sim(\forall x, P(x) \to Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \to Q(x))$$

Since $\sim(P(x) \to Q(x)) \equiv P(x) \wedge \sim Q(x)$

We have $\sim(\forall x, P(x) \to Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x))$

Example 4 – Negating Universal Conditional Statements

Write a formal negation for statement (a) and an informal negation for statement (b).

a. $\forall$ people p, $\overbrace{\text{if } p \text{ is blond then}}^{p} \xrightarrow{} \overbrace{p \text{ has blue eyes.}}^{q}$

- $\exists$ a person p such that p is blond and p does not has blue eyes.

b. If a computer program has more than 100,000 lines, then it contains a bug.

- There is at least one computer program that has more 100,000 lines and does not contain a bug.

Variants of Universal Conditional Statements

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

- Its contrapositive is the statement:

$$\forall x \in D, \text{ if } {\sim}Q(x) \text{ then } {\sim}P(x)$$

- Its converse is the statement:

$$\forall x \in D, \text{ if } Q(x) \text{ then } P(x)$$

- Its inverse is the statement:

$$\forall x \in D, \text{ if } {\sim}P(x) \text{ then } {\sim}Q(x)$$

Variants of Universal Conditional Statements - Exercise

Write a formal and an informal contrapositive, converse, and inverse for the statement:

If a real number is greater tan 2, then its square is greater than 4.

- Formal version of this statement is $\forall x \in R$, if $x > 2$ then $x^2 > 4$.
- Contrapositive: ~ & switch
  - $\forall x \in R$, if $x^2 \leq 4$ then $x \leq 2$.
  - If the square of a real number is less or equal to 4, then this real number is less or equal to 2.
- Converse:
  - $\forall x \in R$, if $x^2 > 4$ then $x > 2$. switch
  - If the square of a real number is greater than 4, then this real number is greater than 2.
- Inverse: invert sign
  - $\forall x \in R$, if $x \leq 2$ then $x^2 \leq 4$.
  - If a real number is less than or equal to 2, then the square of the number is less than or equal to 4.

*Know

Variants of Universal Conditional Statements

A universal conditional statement is logically equivalent to its contrapositive:

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \equiv \forall x \in D, \text{ if } \sim Q(x) \text{ then } \sim P(x)$$

- Any particular *x* in *D* that makes "if *P*(*x*) then Q(*x*)" true also makes "if ~Q(*x*) then ~*P*(*x*)" true (by the logical equivalence between $p \to q$ and $\sim q \to \sim p$).
- It follows that the sentence "If *P*(*x*) then Q(*x*)" is true for all *x* in *D* if, and only if, the sentence "If ~Q(*x*) then ~*P*(*x*)" is true for all *x* in *D*.

A universal conditional statement is not logically equivalent to its converse.

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \not\equiv \forall x \in D, \text{ if } Q(x) \text{ then } P(x)$$

A universal conditional statement is not logically equivalent to its inverse.

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \not\equiv \forall x \in D, \text{ if } \sim P(x) \text{ then } \sim Q(x)$$

Necessary and Sufficient Conditions, Only If

The definitions of *necessary, sufficient,* and *only if* can also be extended to apply to universal conditional statements.

Know? Possible.

• Definition

- "$\forall x, r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$."
- "$\forall x, r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$."
- "$\forall x, r(x)$ **only if** $s(x)$" means "$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$."

*

Ex #43 $\forall x \in \mathbb{Z}$, $\overset{P}{x \text{ is div by } 8}$ is not necessary cond. for x is div by 4

$\sim(\forall x \in \mathbb{Z}$, x divis by 8 is necessary for x div by 4)

$\equiv \exists x \in \mathbb{Z}$, is not div by 8 but/And is div by 4

Example 6 – Necessary & Sufficient Conditions

Rewrite the following statements as quantified conditional statements. Do not use the word necessary or sufficient.

a. Squareness is a sufficient condition for rectangularity.

$\to P(x) \to Q(x)$

Solution:

A formal version of the statement is

$\forall x$, if x is a square, then x is a rectangle.

Or, in informal language:

If a figure is a square, then it is a rectangle.

Example 6 – Necessary & Sufficient Conditions

Rewrite the following statements as quantified conditional statements. Do not use the word necessary or sufficient.

b. Being at least 35 years old is a necessary condition for being President of the United States. $\equiv q(x) \to P(x)$ or $\sim P(x) \to \sim q(x)$

Solution:

A formal version of the statement is

∀ people *x*, if *x* is younger than 35, then *x* cannot be President of the United States.

Or, by the equivalence between a statement and its contrapositive:

∀ people *x*, if *x* is President of the United States, then *x* is at least 35 years old.

Chapter 3: The Logic of Quantified Statements

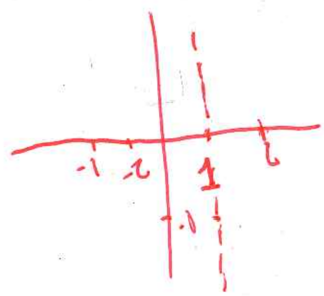
3.3 Statements with Multiple Quantifiers or Nested Quantifiers

Nested Quantifiers

- Two quantifiers are nested if one is in the scope of the other.
- Everything within the scope of a quantifier can be thought of as a propositional function.

Example

- "$\forall x\, \exists y\, (x + y = 0)$" is the same as "$\forall x\, Q(x)$", where $Q(x)$ is "$\exists y\, (x + y = 0)$".
  - for every real number x there is a real number y such that $x + y = 0$.
  - This states that every real number has an additive inverse.



$(x+y)=0$ T

Summary of Quantification of Two Variables

Statement	When True?	When False?
$\forall x \forall y\, P(x,y)$ $\forall y \forall x\, P(x,y)$	$P(x,y)$ is true for every pair x, y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y\, P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y\, P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which $P(x,y)$ is false.
$\exists x \exists y\, P(x,y)$ $\exists y \exists x\, P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	$P(x,y)$ is false for every pair x, y.

Translating Math Statements into Nested Quantifiers

Translate the following statements:

1. "The sum of two positive real numbers is always positive."
   - "For every two integers, if these integers are both positive, then the sum of these integers is positive"
     $\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$, where $x \in \mathbb{R}, y \in \mathbb{R}$
   - "For every two positive integers, the sum of these integers is positive"
     $\forall x \forall y(x + y > 0)$, where $x \in \mathbb{R}^+, y \in \mathbb{R}^+$
2. "Every real number except zero has a multiplicative inverse." (a multiplicative inverse of x is y such that $xy = 1$).
   - "For every real number x except zero, x has a multiplicative inverse"
   - "For every real number x, if $x \neq 0$, then there exists a real number y such that $xy = 1$"
     $\forall x\,((x \neq 0) \rightarrow \exists y(xy = 1))$

Negations of Nested Quantifiers

- Negations of Nested Quantifiers

$$\sim(\forall x \in D, \exists y \in E \text{ such that } P(x,y)) \equiv$$
$$\exists x \in D \text{ such that } \forall y \in E, \sim P(x,y)$$
$$\sim(\exists x \in D \text{ such that } \forall y \in E, P(x,y)) \equiv$$
$$\forall x \in D, \exists y \in E \text{ such that } \sim P(x,y)$$

Proof:

- Recall De Morgan's Laws for Quantifiers:
  - $\sim(\forall x \in D, P(x)) \equiv \exists x \in D \text{ such that } \sim P(x)$
  - $\cancel{\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x)}$

$$\sim(\forall x \in D, \exists y \in E \text{ such that } P(x,y))$$
$$\equiv \forall x \in D \text{ such that } \sim(\exists y \in E \text{ such that } P(x,y))$$
$$\equiv \forall x \in D \text{ such that } \forall y \in E, \sim P(x,y)$$

Negating Nested Quantifiers - Exercise

- Express the negation of the statement $\forall x \exists y(xy = 1)$ so that no negation precedes a quantifier.
  - ➢ The negation of $\forall x \exists y(xy = 1)$ is $\sim\forall x \exists y(xy = 1)$
  - ➢ Now use De Morgan's Laws to move the negation as far inwards as possible.

$\sim\forall x \exists y(xy = 1) \equiv \exists x \sim\exists y(xy = 1)$ by De Morgan's for $\forall$

$\equiv \exists x \forall y \sim(xy = 1)$ by De Morgan's for $\exists$

$\equiv \exists x \forall y(xy \neq 1)$

The Order of Quantifiers

Let $P(x, y)$ be the statement "$x + y = y + x$".
Consider $\forall x \forall y\, P(x, y)$ and $\forall y \forall x\, P(x, y)$:

- What is the meaning of each of these statements?
  - ➢ $\forall x \forall y\, P(x, y)$: "For all real numbers x, for all real numbers y, $x + y = y + x$."
  - ➢ $\forall y \forall x\, P(x, y)$: "For all real numbers y, for all real numbers x, $x + y = y + x$."
- What is the truth value of each of these statements?
  - ➢ Based on the commutative law for addition, both are true.
- Are they equivalent?
  - ➢ They have the same meaning and they are equivalent.

- Principle: the order of nested universal quantifiers in a statement without other quantifiers can be changed without changing the meaning of the quantified statement.

The Order of Quantifiers

Let $Q(x, y)$ be the statement "$x + y = 0$"

Consider $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$:

- What is the meaning of each of these statements?
  - ➢ $\exists y \forall x Q(x, y)$: "There is a real number y such that for every real number x, $Q(x, y)$"
  - ➢ $\forall x \exists y Q(x, y)$: "For every real number x, there is a real number y such that $Q(x, y)$"
- What is the truth value of each of these statements?
  - ➢ No matter what value of y is chosen, there is only one value of x for which $x + y = 0$. Because there is no real number y such that $x + y = 0$ for all real numbers x, the statement $\exists y \forall x Q(x, y)$ is false.
  - ➢ Given a real number x, there is a real number y such that $x + y = 0$; namely, $y = -x$. Hence, the statement $\forall x \exists y Q(x, y)$ is true.
- Are they equivalent?
  - ➢ No

- In a statement containing both $\forall$ and $\exists$, changing the order of the quantifiers usually changes the meaning of the statement.

Chapter 3: The Logic of Quantified Statements

3.4 Arguments with Quantified Statements

Universal Instantiation

- The rule of universal instantiation: if some property is true of every element in a set, then it is true of any particular element in a set.

- Example:

  All men are mortal. If $\forall$ is T $\equiv \forall x \in D, P(x)$

  Socrates is a man. and S belongs to $\forall \equiv x_0 \in D$

  $\therefore$ Socrates is mortal. $\therefore$ Soc is $P(x) \equiv P(x_0)$

Universal Modus Ponens

Formal Version	Informal Version
$\forall x$, if $P(x)$ then $Q(x)$ $P(a)$ for a particular a. $\therefore\ Q(a)$	If x makes $P(x)$ true, then x makes $Q(x)$ true a makes $P(x)$ true. $\therefore\ a$ makes $Q(x)$ true.

Example: Recognizing the Form of Universal Modus Ponens

If an integer is even, then its square is even.

k is a particular integer that is even.

$\therefore\ k^2$ is even.

$\forall x, P \to Q$

$P(a)$

$\therefore Q(a)$ Ponens

Let $P(x)$ be "x is an even integer", let $Q(x)$ be "x^2 is even," and let k stand for a particular integer that is even. Then the statement has the form:

$\forall x$, if $P(x)$ then $Q(x)$

$P(k)$ for a particular k.

$\therefore\ Q(k)$

This argument has the form of universal modus ponens and is therefore valid.

Universal Modus Tollens

Formal Version	Informal Version
$\forall x$, if $P(x)$ then $Q(x)$ $\sim Q(a)$ for a particular a. $\therefore \sim P(a)$	If x makes $P(x)$ true, then x makes $Q(x)$ true a does not make $Q(x)$ true. $\therefore$ a does not make $P(x)$ true.

$\forall x, P \to Q$
$\sim Q(a)$
$\therefore \sim P(a)$ (Tollens)

Example: Recognizing the Form of Universal Modus Tollens

All human beings are mortal.

Zeus is not mortal.

$\therefore$ Zeus is not human.

Let $H(x)$ be "x is human", let $M(x)$ be "x is mortal," and let Z stand for Zeus. Then the statement has the form:

$$\forall x, \text{if } H(x) \text{ then } M(x)$$
$$\sim M(Z)$$
$$\therefore \sim H(Z)$$

This argument has the form of universal modus tollens and is therefore valid.

Universal Transitivity

Formal Version	Informal Version
$\forall x, P(x) \to Q(x)$ $\forall x, Q(x) \to R(x)$ $\therefore \forall x, P(x) \to R(x)$	Any x that makes $P(x)$ true makes $Q(x)$ true Any x that makes $Q(x)$ true makes $R(x)$ true $\therefore$ Any x that makes $P(x)$ true makes $R(x)$ true

Example: Recognizing the Form of Universal Transitivity

1. All the triangles are blue.
2. If an object is to the right of all the squares, then it is above all the circles.
3. If an object is not to the right of all the squares, then it is not blue.

$\therefore$ All the triangles are above all the circles.

Solution: Rewrite the premises and the conclusion in if-then form.

1. $\forall x$, if x is a triangle, then x is blue.
2. $\forall x$, if x is to the right of all the squares, then x is above all the circles.
3. $\forall x$, if x is not to the right of all the squares, then x is not blue. → contrapositive here

$\therefore$ $\forall x$, if x is a triangle, then x is above all the circles.

Universal Transitivity – Exercise (cont')

Solution:

I. Rewrite the premises and the conclusion in if-then form.

1. $\forall x$, if x is a triangle, then x is blue.
2. $\forall x$, if x is to the right of all the squares, then x is above all the circles.
3. $\forall x$, if x is not to the right of all the squares, then x is not blue.

$\therefore$ $\forall x$, if x is a triangle, then x is above all the circles.

II. Rewrite the premises and conclusion of the argument.

1. $\forall x$, if x is a triangle, then x is blue.
3. $\forall x$, if x is blue, then x is to the right of all the squares.
2. $\forall x$, if x is to the right of all the squares, then x is above all the circles.

$\therefore$ $\forall x$, if x is a triangle, then x is above all the circles.

Proving Validity of Arguments with Quantified Statements

Definition

To say that an argument form with quantified statements is valid means the following:

No matter what particular predicates are substituted for the predicate variables in its premises, if the resulting premise statements are all true, then the conclusion is also true.

An argument is called valid, and only if, its form is valid.

- An argument is valid if, and only if, the truth of its conclusion ~~follows necessarily from the truth of its premises~~.

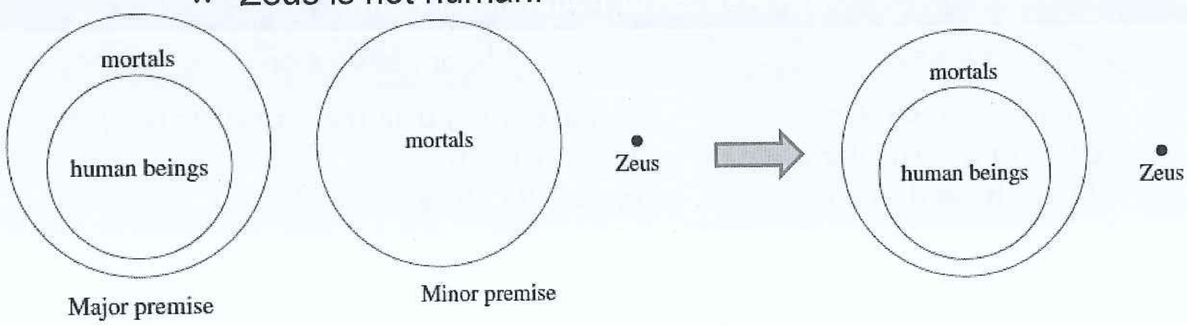
Using Diagrams to Test for Validity

To test the validity of an argument diagrammatically,

- Represent the truth of both premises with diagrams.
- Check whether they necessarily represent the truth of the conclusion as well.

Example: Using a Diagram to Show Validity

All human beings are mortal.
Zeus is not mortal.
∴ Zeus is not human.



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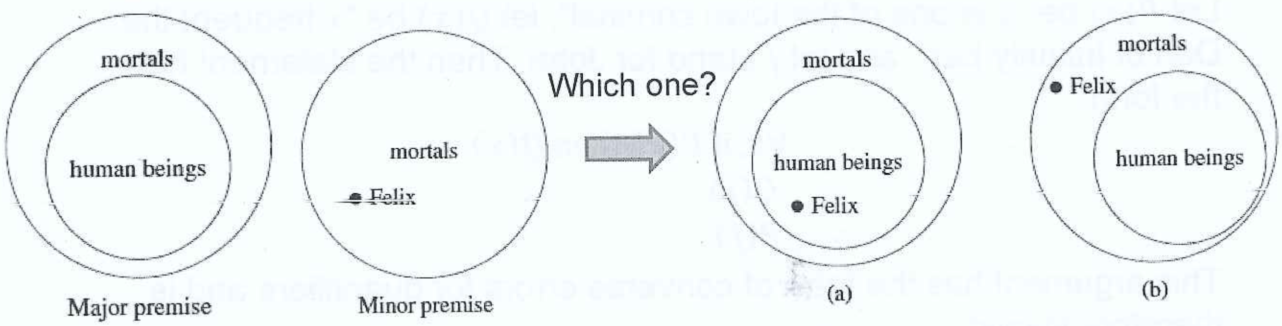
Using Diagrams to Show Invalidity

Example:

All human beings are mortal.
Felix is mortal.
∴ Felix is a human being.



Converse Errors in Quantified Form

Universal Modus Ponens	
Formal Version	**Informal Version**
$\forall x$, if $P(x)$ then $Q(x)$	If x makes $P(x)$ true, then x makes $Q(x)$ true
$P(a)$ for a particular a.	a makes $P(x)$ true.
$\therefore$ $Q(a)$	$\therefore$ a makes $Q(x)$ true.

Converse Error (Quantified Form)	
Formal Version	**Informal Version**
$\forall x$, if $P(x)$ then $Q(x)$	If x makes $P(x)$ true, then x makes $Q(x)$ true
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
$\therefore$ $P(a)$ (Invalid con.)	$\therefore$ a makes $P(x)$ true. (Invalid con.)

Converse Errors in Quantified Form

Example: Recognizing the Form of Converse Error

All the town criminals frequent the Den of Iniquity bar.
John frequents the Den of Iniquity bar.
$\therefore$ John is one of the town criminals.

Let $P(x)$ be "x is one of the town criminal", let $Q(x)$ be "x frequent the Den of Iniquity bar," and let J stand for John. Then the statement has the form:

$$\begin{array}{ll} & \forall x, \text{ if } P(x) \text{ then } Q(x) \\ & Q(J) \\ \therefore & P(J) \end{array}$$

This argument has the form of converse errors for quantifiers and is therefore invalid.

Inverse Errors in Quantified Form

Universal Modus Tollens			
	Formal Version		Informal Version
	$\forall x$, if $P(x)$ then $Q(x)$		If x makes $P(x)$ true, then x makes $Q(x)$ true
	$\sim Q(a)$ for a particular a.		a does not make $Q(x)$ true.
$\therefore$	$\sim P(a)$	$\therefore$	a does not make $P(x)$ true.

Inverse Error (Quantified Form)			
	Formal Version		Informal Version
	$\forall x$, if $P(x)$ then $Q(x)$		If x makes $P(x)$ true, then x makes $Q(x)$ true
	$\sim P(a)$ for a particular a.		a does not make $P(x)$ true.
$\therefore$	$\sim Q(a)$ (Invalid con.)	$\therefore$	a does not make $Q(x)$ true. (Invalid con.)