MAC 2311 Calculus w/ Analytic Geometry I

COURSE SYLLABUS

GENERAL INFORMATION

Section Number:	65987
Meeting Times:	6:00 PM to 8:15 PM – Tuesday and Thursday
Location:	SMPF 219
Credits:	5
Term:	Spring 2015

INSTRUCTOR INFORMATION

Instructor:	Mr. Thomas Carty	
Office:	STPR - E	
Office Hours:	Monday and Wednesday:	9:00 AM to 10:00 AM
		11:40 AM to 12:30 PM
	Tuesday and Thursday:	9:50 AM to 11:00 AM
		12:15 PM to 12:30 PM
		2:10 PM to 2:55 PM
		5:00 PM to 6:00 PM
Telephone:	(c) 215.534.5182	
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or

www.hccfl.edu→HCC Directory→Employee's Last Name: "carty"→Click to Search→Homepage

Course Description

Focuses on limits, continuity, derivatives, extreme value theorem, mean value theorem, differentials, integration and the fundamental theorem, with an emphasis on sine, cosine and algebraic functions. Topics include rate of change, related rates, extreme values, curve sketching, areas, arc length, work, moments and center of mass, rectilinear motion, liquid pressure and volumes.

Prerequisites: College level math skills are required with a minimum grade of C.

Possible good Calc 3 instructor recommended by Carty

MAC 2311 – 65987 – Calculus w/Analytic Geometry I

Desimone, Michael

Lesson Schedule and Homework Assignments

Instructor: Mr. Thomas Carty

Spring 2015 – 16 Weeks – TTH – 6:00 PM to 8:15 PM

Date	Topics	Homework
01/13 T L1	Introduction, Course syllabus, Homework assignments, 2.1 – Rates of Change and Tangents to Curves 2.2 – Limit of a Function and Limit Laws	~~**2.1:** #1, 3, 5, 9, 21~~ ~~41, 64~~ ~~**2.2:** #1, 3, 13, 15, 16, 19, 25, 29, 35, 43,~~ 47
01/15 TH L2	2.3 – The Precise Definition of a Limit 2.4 – One-sided Limits	~~**2.3:** #15, 23, 31, 33, 37, 39~~ ~~**2.4:** #1, 7, 11, 13, 21, 25, 29~~
01/20 T	*No class –College In-service*	
01/22 TH L3	2.5 – Continuity 2.6 – Limits Involving Infinity and Asymptotes of Graphs	~~**2.5:** #13, 15, 19, 21, 25, 29, 53, 59~~ ~~**2.6:** #13, 15, 17, 21, 23, 25, 31, 37,~~ 39, ~~43, 53, 55, 63, 65, 67, 81, 101~~
01/27 T L4	Review for Test #1 (Chapter 2)	Study for Test #1
01/29 TH	**Test #1 (Chapter 2)**	**DUE: Hmwk from 2.1 to 2.6**
02/03 T L5	3.1 – Tangents and the Derivative at a Point 3.2 – The Derivative as a Function	~~**3.1:** #5, 9, 11, 13, 19, 21, 27, 29, 31~~ ~~**3.2:** #1, 3, 7, 9, 13~~
02/05 TH L6	3.3 – Differentiation Rules 3.4 – The Derivative as a Rate of Change	~~**3.3:** #1, 5, 7, 11, 13, 17, 23, 25, 29,~~ 33, ~~45, 51, 55, 57~~ ~~**3.4:** #1, 3, 7, 9, 23, 25~~
02/10 T L7	3.5 – Derivatives of Trigonometric Functions 3.6 – The Chain Rule	~~**3.5:** #1, 3, 7, 11, 16, 19, 23, 27, 31, 35,~~ ~~37, 47, 49~~ ~~**3.6:** #9, 13, 15, 21, 23, 27, 29, 35, 43, 51,~~ ~~61, 71, 73~~
02/12 TH L8	3.7 – Implicit Differentiation 3.8 – Derivatives of Inverse Functions and Logarithms	~~**3.7:** #1, 7, 11, 15, 21, 23, 31, 33, 37~~ ~~**3.8:** #11, 17, 21, 27, 37, 41, 45, 49, 67,~~ ~~71, 83, 89, 93~~
02/17 T L9	3.9 – Inverse Trigonometric Functions 3.11 – Linearization and Differentials	~~**3.9:** #21, 23, 25, 27, 35, 39, 49~~ ~~**3.11:** #1, 3, 5, 7, 19, 25, 39, 43~~
02/19 TH L10	3.10 – Related Rates	~~**3.10:** #5, 7, 13, 17, 21, 23~~ ~~**Worksheet** – Related Rates~~
02/24 T L11	Review for Test #2 (Chapter 3)	~~Study for Test #2~~
02/26 TH	**Test #2 (Chapter 3)**	**DUE: Hmwk from 3.1 to 3.11**
03/03 T L12	4.1 – Extreme Values of Functions 4.2 – The Mean Value Theorem	~~**4.1:** #1, 3, 7, 11, 17, 19, 21, 25, 31, 39,~~ ~~45, 49, 53, 61~~ ~~**4.2:** #1, 3, 10, 13, 21, 25, 33, 37, 39, 43,~~ ~~45, 47~~
03/05 TH L13	4.3 – Monotonic Functions and the First Derivative Test 4.4 – Concavity and Curve Sketching	~~**4.3:** #3, 7, 19, 25, 35, 45, 49, 53, 57, 61~~ ~~**4.4:** #9, 11, 17, 23, 43, 53, 59, 67, 85, 89,~~ ~~97~~
03/9-13	*No Classes – Spring Break*	
03/17 T L14	4.5 – Indeterminate Forms and L'Hopital's Rule 4.6 – Applied Optimization	~~**4.5:** #7, 11, 13, 15, 27, 33, 46, 51, 63~~ **4.6:** #5, 9, 19, 23, 27, 37

nDeriv(Y, X, x)

Date	Topics	Homework
03/19 TH L15	4.7 – Newton's Method 4.8 – Antiderivatives	~~**4.7:** #1, 3, 5, 13~~ ~~**4.8:** #3, 7, 11, 15, 19, 25, 29, 35, 41, 45, 51, 55~~
03/24 T L16	Review for Test #3 (Chapter 4)	Study for Test #3
03/26 TH	**Test #3 (Chapter 4)**	**DUE: Hmwk from 4.1 to 4.8**
03/31 T L17	5.1 – Area and Estimating with Finite Sums 5.2 – Sigma Notation and Limits of Finite Sums	~~**5.1:** #1, 4, 5, 7, 13, 17~~ ~~**5.2:** #1, 3, 7, 9, 14, 15, 21, 25, 29, 33, 35, 39, 43~~
04/02 TH L18	5.3 – The Definite Integral 5.4 – The Fundamental Theorem of Calculus	~~**5.3:** #9, 41 – 55 (odd), 59~~ ~~**5.4:** #1 – 57 (every other odd)~~
04/07 T L19	5.5 – Indefinite Integrals and the Substitution Method 5.6 – Definite Integral Substitution and Area Between Curves	**5.5:** #1 – 57 (every other odd), 73, 77 no #37 **5.6:** #1, 5, 9, 11, 27, 39, 47, 51
04/09 TH L20	6.1 – Volumes Using Cross-sections	**6.1:** #1, 5, 9, 19, 21, 25, 31, 39, 41, 43, 51, 53
04/14 T L21	Review for Test #4 (Chapter 5)	Study for Test #4
04/16 TH	**Test #4 (Chapter 5)**	**DUE: Hmwk from 5.1 to 5.6**
04/21 T L22	6.2 – Volumes Using Cylindrical Shells	**6.2:** #1, 7, 9, 11, 15, 17, 25, 31, 35
04/23 TH L23	6.3 – Arc Length 6.4 – Areas of Surfaces of Revolution	**6.3:** #1, 5, 15, 17, 21 **6.4:** #1, 3, 9, 13, 15
04/28 T L24	6.5 – Work and Fluid Forces	**6.5:** #1, 3, 5, 7, 9
04/30 TH L25	Review for Test #5 (Chapter 6)	Study for Test #5
05/05 T	**Test #5 (Chapter 6) – 6:00 PM to 7:50 PM**	**DUE: Hmwk from 6.1 to 6.5** **Enjoy the Summer!**

Make-up Due Date	Test
02/06	1
03/06	2
04/03	3
04/24	4
Test 5 cannot be made up.	

Function Review

$f(x) = 2x - 7$	$f(x) = 2x - 7$	$f(x) = 2x - 7$
$f(3) = ?$	$f(a) = ?$	$f(3 + a) = ?$
$f(3) = 6 - 7$	$f(a) = 2(a) - 7$	$f(3 + a) = 2(3 + a) - 7$
$f(3) = 2(3) - 7$	$f(a) = 2a - 7$	$f(3 + a) = 6 + 2a - 7$
$f(3) = -1$	$(a, 2a - 7)$	$(3 + a, 2a - 1)$
$(3, -1)$	$(a, f(a))$	$(3 + a, f(3 + a))$

Function Review

$$f(x) = x^2 + 2x - 7$$

$$f(x + h) = ?$$

$$f(x + h) = (x + h)^2 + 2(x + h) - 7$$

$$f(x + h) = x^2 + 2xh + h^2 + 2x + 2h - 7$$

$$(x + h, x^2 + 2xh + h^2 + 2x + 2h - 7\ )$$

$$(x + h, f(x + h))$$

Function Review

$$f(x) = x^3 + 3x$$

$$f(x+h) = ?$$

$$f(x+h) = (x+h)^3 + 3(x+h)$$

Pascal's Triangle

			1					Row0
		1		1				Row1
	1		2		1			Row2
	1		3		3		1	Row3
1		4		6		4		1 Row4
1	5		10		10		5	1 Row5
1	6	15		20		15	6	1 Row6
1	7	21	35		35	21	7	1 Row7

$$f(x+h) = x^3 + +3x^2h + 3xh^2 + h^3 + 3x + 3h$$

$$(x+h, x^3 + +3x^2h + 3xh^2 + h^3 + 3x + 3h\ )$$

$$\big(x+h, f(x+h)\big)$$

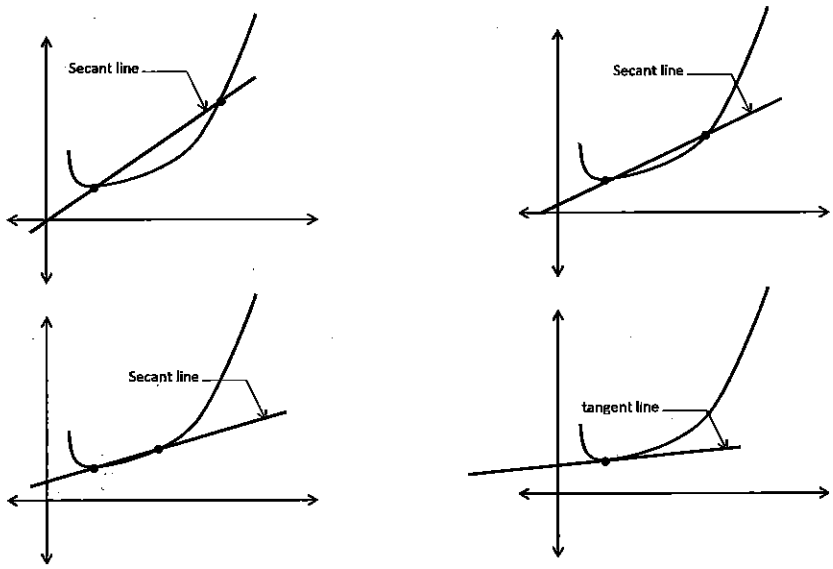
Rate of change:

describes how one quantity changes in relation to another quantity changing.

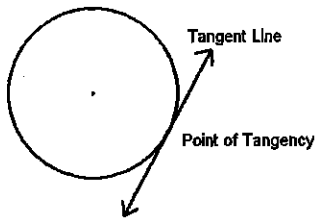
Examples:

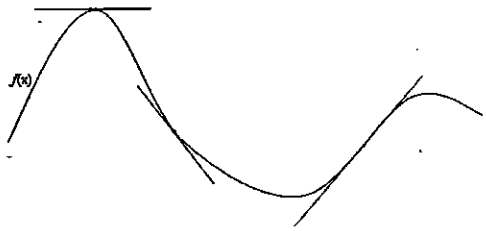
$mi/hr,\quad ft/\text{sec}$	Rate: how distance changes as time changes
$ft^2/hr,\quad cm^2/sec$	Rate: how area changes as time changes
$in^3/hr,\quad cm^3/sec$	Rate: how volume changes as time changes
$slope,\quad \dfrac{\Delta y}{\Delta x}$	Rate: how *y* changes as *x* changes

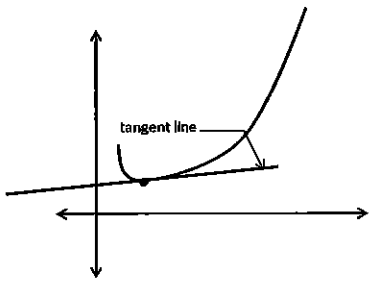
Secant Line: A line joining two points on a curve.



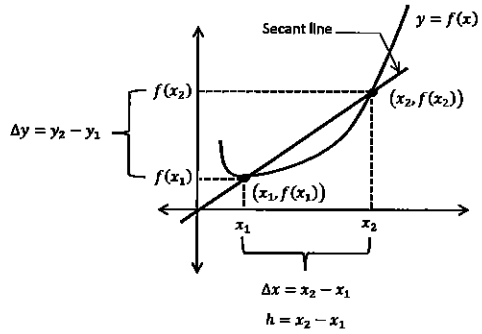
Tangent Line: A line that touches a curve a one point.







Rate of Change of a Secant Line or the Slope of a Secant Line



$$slope\ of\ a\ line: \quad m = \frac{\Delta y}{\Delta x}$$

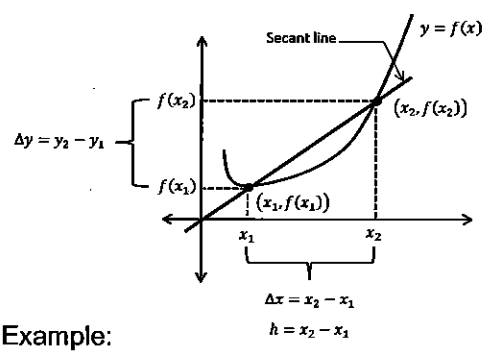
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$m = \frac{f(x_2) - f(x_1)}{\Delta x} = \frac{f(x_2) - f(x_1)}{h}$$

The slope of a secant line represents an average rate of change between two points.

$$average\ rate\ of\ change = m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Average Rate of Change



$$average\ rate\ of\ change = m: \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

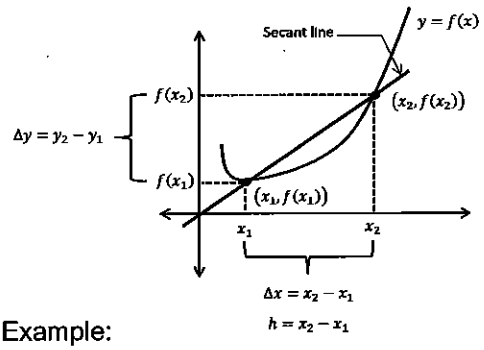
Example:

Find the average rate of change for the function and the interval.

$$f(x) = -2x^2 + 4 \qquad [1,4]$$

$$m = \frac{f(4) - f(1)}{4 - 1} = \frac{-28 - 2}{3} = \frac{-30}{3} = -10$$

Average Rate of Change



$$average\ rate\ of\ change = m: \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

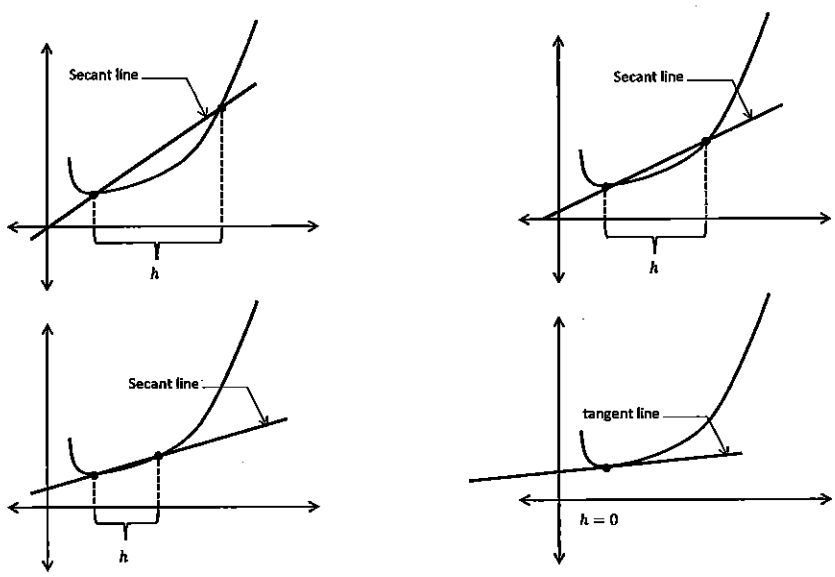
Example:

Find the average rate of change for the function and the interval.

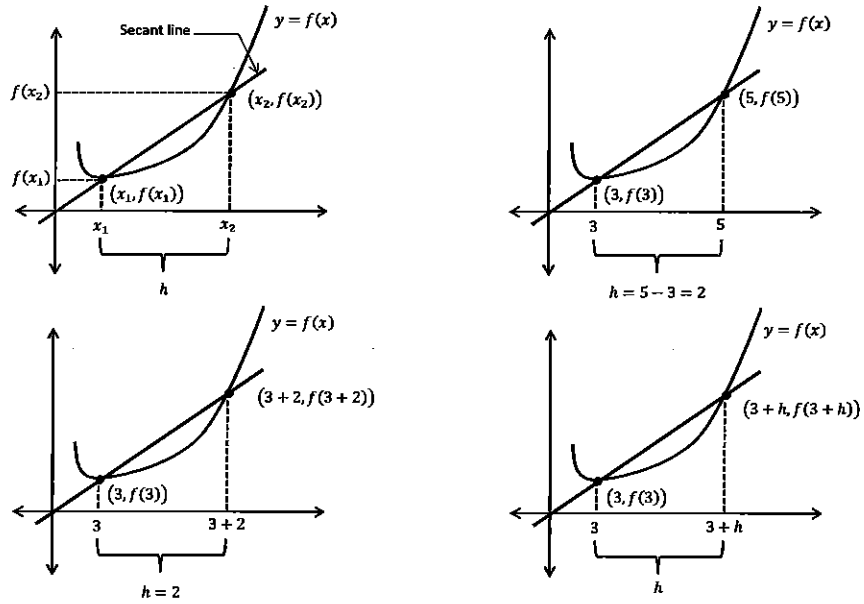
$$f(x) = \sec x \qquad \left[0, \frac{\pi}{3}\right] \qquad f(x) = \frac{1}{\cos x}$$

$$m = \frac{f\left(\frac{\pi}{3}\right) - f(0)}{\frac{\pi}{3} - 0} = \frac{\frac{1}{\cos\frac{\pi}{3}} - \frac{1}{\cos(0)}}{\frac{\pi}{3}} = \frac{\frac{1}{\frac{1}{2}} - 1}{\frac{\pi}{3}} = \frac{2-1}{\frac{\pi}{3}} = \frac{1}{\frac{\pi}{3}} = \frac{3}{\pi} = 0.9549$$

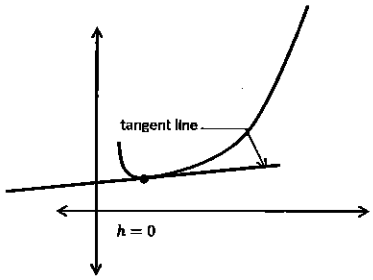
Instantaneous Rate of Change or the Slope of a Tangent Line



Instantaneous Rate of Change or the Slope of a Tangent Line



Instantaneous Rate of Change or the Slope of a Tangent Line



Slope formula requires two points.

A tangent line has one point on the curve.

Create a second point by selecting a small value of *h*.

Find the instantaneous rate of change for the function at the given point.

$f(x) = -2x^2 + 4 \qquad (1,2) \qquad h = 0.1$

$$m_{tan} = \frac{f(1+0.1) - f(1)}{(1+0.1) - 1} = \frac{1.58 - 2}{0.1} = \frac{-0.42}{0.1} = -4.2$$

$f(x) = -2x^2 + 4 \qquad (1,2) \qquad h = 0.01$

$$m_{tan} = \frac{f(1+0.01) - f(1)}{(1+0.01) - 1} = \frac{1.9598 - 2}{0.01} = \frac{-0.402}{0.01} = -4.02$$

Instantaneous Rate of Change or the Slope of a Tangent Line

$f(x) = -2x^2 + 4 \qquad (1,2) \qquad h$

$$m_{tan} = \frac{f(1+h) - f(1)}{(1+h) - 1} =$$

$$\frac{-2(1+h)^2 + 4 - (-2(1)^2 + 4)}{h} =$$

$$\frac{-2(1 + 2h + h^2) + 4 - (-2(1)^2 + 4)}{h} =$$

$$\frac{-2 - 4h - 2h^2 + 4 - 2}{h} =$$

$$\frac{-4h - 2h^2}{h} = \frac{h(-4 - 2h)}{h} =$$

$$m_{tan} = -4 - 2h$$

$$m_{tan} = -4 - 2h$$

$$h = 0.1$$

$$m_{tan} = -4 - 2(0.1)$$

$$m_{tan} = -4.2$$

$$h = 0.01$$

$$m_{tan} = -4 - 2(0.01)$$

$$m_{tan} = -4.02$$

$$h = 0.001$$

$$m_{tan} = -4 - 2(0.001)$$

$$m_{tan} = -4.002$$

m_{tan} *at* $x = 1$ *is approaching* -4 *as* h *approaches* 0

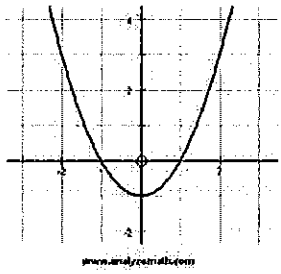
Defn: Limit

As the variable x approaches a certain value, the variable y approaches a certain value.

$$\lim_{x \to c} f(x) = L$$

Find the requested limits from the graph of the given function.

$f(x) = x^2 - 1$



$\lim_{x \to 0} f(x) = -1$

$\lim_{x \to 1} f(x) = 0$

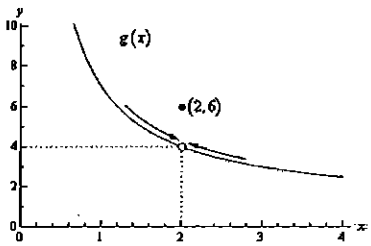
$\lim_{x \to -1} f(x) = 0$

$\lim_{x \to 2} f(x) = 3$

$\lim_{x \to -2} f(x) = 3$

$$\lim_{x \to c} f(x) = L$$

Given the following graph of a function, find the requested limit.



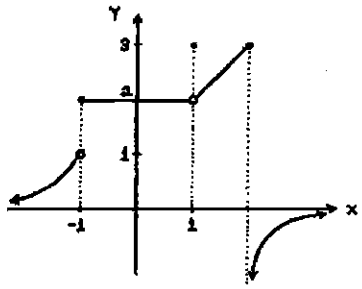
$\lim_{x \to 2} g(x) = \quad 4$

$g(2) = \quad 6$

(2,6)

$$\lim_{x \to c} f(x) = L$$

Given the following graph of a function, find the requested limits.



$\lim_{x \to 0} f(x) = \quad 2$ $\qquad f(0) = \quad 2$

(0,2)

$\lim_{x \to -1} f(x) = jump$ $\qquad f(-1) = \quad 2$

$(-1,2)$

$\lim_{x \to 1} f(x) = \quad 2$ $\qquad f(1) = \quad 3$

(1,3)

$\lim_{x \to 2} f(x) = jump$ $\qquad f(2) = \quad 3$

(2,3)

Mae 2311
Thomas Carty
1-13-15

Calculus

2.1 Rates of Change & Tangents

Review functions

$f(x) = 2x - 7$

$f(3) = ?$

$f(3) = 2(3) - 7$

$f(3) = -1 \quad (3, -1)$

$f(x) = 2x - 7$

$f(a) = 2(a) - 7$

$f(a) = 2a - 7$

$(a, 2a-7)$

ex: $f(x) = x^2 + 2x - 7$

$f(x+h) = ?$

$f(x+h) = (x+h)^2 + 2(x+h) - 7$

$f(x+h) = x^2 + 2xh + h^2 + 2x + 2h - 7$

$(x+h, x^2 + 2xh + h^2 + 2x + 2h - 7)$

ex: $f(x) = x^3 + 3x$

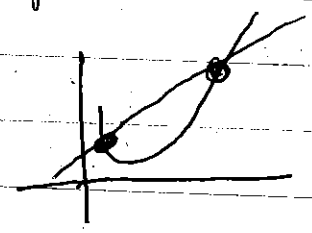
$f(x+h) = ?$

Use Pascal's Triangle to make it easier

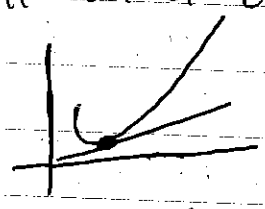
$f(x+h) = (x+h)^3 + 3(x+h)$

$f(x+h) = x^3 + 3x^2h + 3xh^2 + h^3 + 3x + 3h$

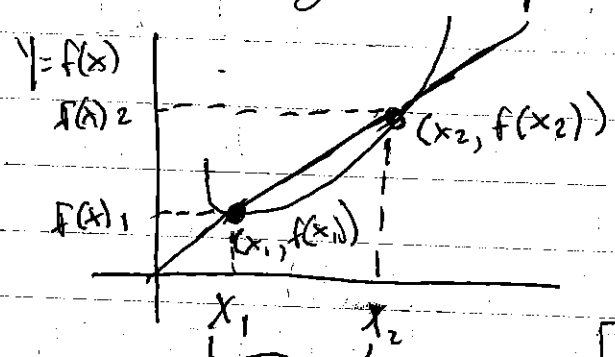
Secant line = A line that crosses two points of a curve



Tangent line = A line that touches a single point on a curve



Rate of change = Slope of a Secant line



$\Delta X = x_2 - x_1$

or $h = x_2 - x_1$

Slope $= m = \dfrac{\Delta Y}{\Delta X}$

$m = \dfrac{Y_2 - Y_1}{X_2 - X_1}$

$$\boxed{m = \frac{f(X_2) - f(X_1)}{h}}$$

Average Rate of Change

Ex: $f(x) = -2x^2 + 4$ $[1, 4]$ find Slope

$$m = \frac{f(4) - f(1)}{4 - 1} = \frac{28 - 2}{3} = \frac{-30}{3} = \boxed{-10}$$

Ex: $f(x) = \sec x$ $[0, \frac{\pi}{3}]$ find Slope

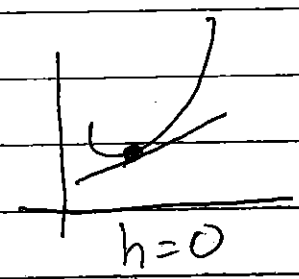
$$f(x) = \frac{1}{\cos(x)} \quad \text{because } \sec = \frac{1}{\cos}$$

$$m = \frac{f(\frac{\pi}{3}) - f(0)}{\pi/3 - 0} = \frac{\frac{1}{\cos\frac{\pi}{3}} - \frac{1}{\cos 0}}{\frac{\pi}{3}}$$

$$= \frac{\frac{1}{\frac{1}{2}} - 1}{\pi/3} = \frac{2-1}{\pi/3} = \frac{1}{\pi/3} = \frac{3}{\pi} = \boxed{0.9549}$$

Instantaneous Rate of change

Slope of tangent line



$h = 0$

$$m = \frac{f(x_1), f(x_2)}{0}$$ Cant divide by Zero

Ex: $f(x) = 2x^2 + 4$ $(1, 2)$ $h = 0.1$

$$m = \frac{f(1+0.1) - f(1)}{(1+0.1) - 1} = \frac{1.58 - 2}{0.1} = \frac{-0.42}{0.1} = \boxed{-4.2}$$

$f(x) = -2x^2 + 4 \quad (1, 2) \quad h = h$

$$m_{tan} = \frac{f(1+h) - f(1)}{(1+h) - 1}$$

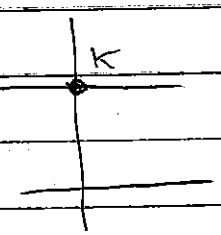
eventually = $\boxed{m_{tan} = -4 - 2h}$ for ANY slope of tangent

Ex: $m_{tan} = -4 - 2h \quad h = 0.1$

$m_{tan} = -4 - 2(0.1) = \boxed{-4.2}$

2.2 Limit Laws

$f(x) = K$ (constant) = the constant



$$\lim_{x \to c} K = K$$

- <u>Squeeze Theorem</u>

limit of 3 functions where the function in the middle is sandwiched between the upper & lower limit then the unknown middle limit is equal to the limit of the other two

$$\lim_{x \to 0} \frac{x}{x^2 - x} = \frac{0}{0} \text{ • inditerminent form}$$

$$= \lim_{x \to 0} \frac{\cancel{x}}{x(x-1)} = \frac{1}{x-1}$$

$$\lim_{x \to 0} \frac{1}{x-1} = \frac{1}{-1} = \boxed{-1}$$

$$\lim_{x \to 6} \frac{x-6}{x^2-36} = \frac{6-6}{36-36} = \frac{0}{0}$$

$$\lim_{x \to 6} \frac{x-6}{(x+6)(x-6)} = \frac{1}{x+6}$$

$$\lim_{x \to 6} \frac{1}{x+6} = \boxed{\frac{1}{12}}$$

$$\lim_{x \to -2} \frac{x+2}{x^2-3x-10} = \frac{0}{0}$$

$$\frac{x+2}{(x-5)(x+2)} = \frac{1}{x-5}$$

$$\lim_{x \to -2} \frac{1}{-2-5} = \boxed{\frac{1}{-7}}$$

- If you can't substitute then factor it out

2.2 Limits of a function

Deff: Limit

As the variable X approaches a certain value the variable y approaches a certain value

$$\lim_{x \to c} f(X) = L$$

Ex: find Limit of the given function

$f(x) = x^2 - 1$

$$\lim_{x \to 0} f(x) = (0)^2 - 1 = \boxed{-1}$$

Just Sub it in

$$\lim_{x \to 1} f(x) = (1)^2 - 1 = \boxed{0}$$

Ex: $f(x) = \dfrac{x+5}{2x-4} =$

$$\lim_{x \to 3} f(x) = \frac{3+5}{2(3)-4} = \frac{8}{2} = \boxed{4}$$

Find the requested limits for the given function.

$f(x) = x^2 - 1$

$\lim_{x\to 0} f(x) = (0)^2 - 1 = -1$

$\lim_{x\to 1} f(x) = (1)^2 - 1 = 0$

$\lim_{x\to -1} f(x) = (-1)^2 - 1 = 0$

$\lim_{x\to 2} f(x) = (2)^2 - 1 = 3$

$\lim_{x\to -2} f(x) = (-2)^2 - 1 = 3$

THEOREM 2—Limits of Polynomials
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

A rational function is the ratio of two polynomial functions

THEOREM 3—Limits of Rational Functions
If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

$f(x) = \dfrac{x+5}{2x-4}$

$\lim_{x\to 3} \dfrac{x+5}{2x-4} = \dfrac{3+5}{2(3)-4} = \dfrac{8}{2} = 4$

Substitution Theorem:
If f(x) is a polynomial function or a rational function, then

$\lim_{x\to c} f(x) = f(c)$ or $\lim_{x\to c} \frac{k(x)}{g(x)} = \frac{k(c)}{g(c)}$.

If f(x) is a rational function, then the denominator cannot equal zero.

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

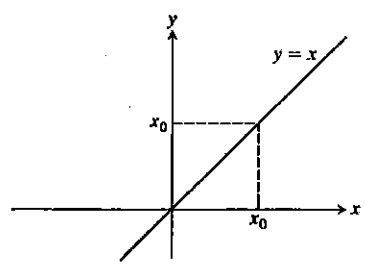
(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

Additional Limit Rules

Identity Function

$$f(x) = x$$

$$\lim_{x \to c} x = c$$

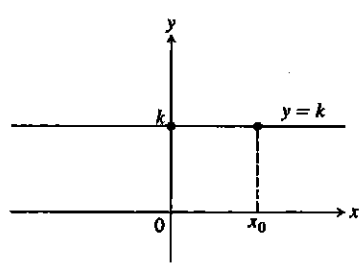


(a) Identity function

Constant Function

$$f(x) = k$$

$$\lim_{x \to c} k = k$$

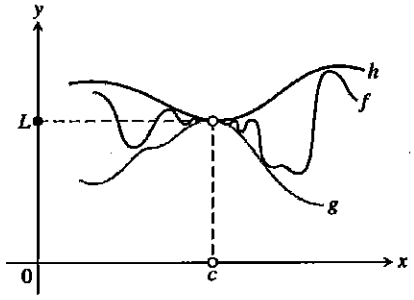


(b) Constant function

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.



Find the following limits:

$$\lim_{x \to 3} \frac{1}{4 - x^2} = -\frac{1}{5}$$

$$\lim_{x \to 5} \frac{x}{x^2 - x} = \frac{1}{4}$$

$$\lim_{x \to -4} \frac{x - 6}{x^2 - 36} = \frac{1}{2}$$

$$\lim_{x \to 6} \frac{x + 2}{x^2 + x - 10} = \frac{1}{4}$$

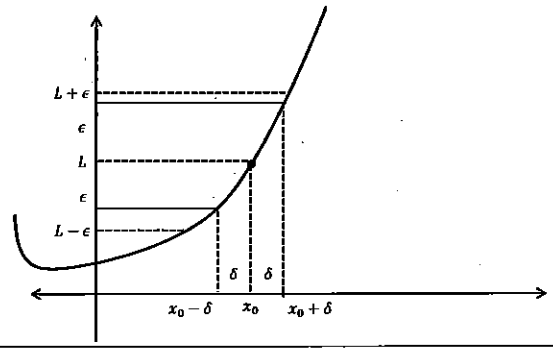
Find the following limits:

$$\lim_{x\to 0}\frac{x}{x^2-x}=\lim_{x\to 0}\frac{x}{x(x-1)}=\lim_{x\to 0}\frac{1}{(x-1)}=-1$$

$$\lim_{x\to 6}\frac{x-6}{x^2-36}=\lim_{x\to 6}\frac{x-6}{(x+6)(x-6)}=\lim_{x\to 6}\frac{1}{(x+6)}=\frac{1}{12}$$

$$\lim_{x\to -2}\frac{x+2}{x^2-3x-10}=\lim_{x\to -2}\frac{x+2}{(x+2)(x-5)}=\lim_{x\to -2}\frac{1}{(x-5)}=-\frac{1}{7}$$

Graphical Representation of a Precise Limit



DEFINITION Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Find a value of delta (δ) given epsilon (ε).

$$\lim_{x \to 2}(3x - 1) = 5, \qquad \varepsilon = 0.5$$

$$f(x) = 3x - 1 \qquad L = 5 \qquad x_0 = 2$$

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |(3x - 1) - 5| < 0.5$$

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |3x - 1 - 5| < 0.5$$

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |3x - 6| < 0.5$$

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |3(x - 2)| < 0.5$$

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |3||x - 2| < 0.5$$

$$0 < |x - 2| < \delta \quad \Rightarrow \quad 3|x - 2| < 0.5$$

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |x - 2| < \frac{0.5}{3}$$

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |x - 2| < \frac{1}{6}$$

$$\delta = \frac{1}{6}$$

Find a value of delta (δ) given epsilon (ε).

$\lim_{x\to 2}(3x-1)=5,\qquad \varepsilon=0.5\qquad f(x)=3x-1\qquad L=5\qquad x_0=2$

$$\delta=\frac{1}{6}$$

$$f(2)=5$$

$f(x_0+\delta)\qquad f\left(2+\frac{1}{6}\right)=\ 5.5\qquad L+\varepsilon\qquad 5+0.5=\ 5.5$

$f(x_0-\delta)\qquad f\left(2-\frac{1}{6}\right)=\ 4.5\qquad L-\varepsilon\qquad 5-0.5=\ 4.5$

$$If\ \delta<\frac{1}{6}, then\ \epsilon<0.5$$

$Let\ \delta=0.1$

$f(x_0+\delta)\qquad f(2+0.1)=\ 5.3$

$f(x_0-\delta)\qquad f(2-.01)=\ 4.7$

$\epsilon=0.3\ \ <0.5$

Prove the following limit.

$$\lim_{x\to -3}(7x-9)=-30$$

$f(x)=7x-9\qquad L=-30\qquad x_0=-3$

$0<|x-x_0|<\delta\quad\Rightarrow\quad |f(x)-L|<\varepsilon$

$0<|x-(-3)|<\delta\quad\Rightarrow\quad |(7x-9)-(-30)|<\varepsilon$

$0<|x+3|<\delta\quad\Rightarrow\quad |7x-9+30|<\varepsilon$

$0<|x+3|<\delta\quad\Rightarrow\quad |7x+21|<\varepsilon$

$0<|x+3|<\delta\quad\Rightarrow\quad |7(x+3)|<\varepsilon$

$0<|x+3|<\delta\quad\Rightarrow\quad |7||x+3|<\varepsilon$

$0<|x+3|<\delta\quad\Rightarrow\quad 7|x+3|<\varepsilon$

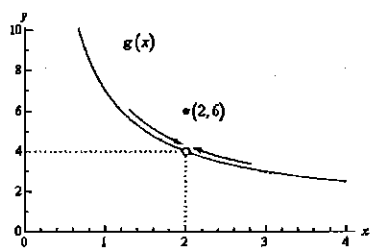
$0<|x+3|<\delta\quad\Rightarrow\quad |x+3|<\frac{\varepsilon}{7}\qquad \delta=\frac{\varepsilon}{7}$

$If\ \varepsilon>0, and\ \delta<\frac{\varepsilon}{7}, then\ this\ will\ keep\ the\ value\ of\ L\ between\ -30+\varepsilon\ and\ -30-\varepsilon.$

A limit of a function can be analyzed from the left and right sides of a particular value of x.

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$



$$\lim_{x\to 2^-} g(x) = 4 \qquad \lim_{x\to 2^+} g(x) = 4$$

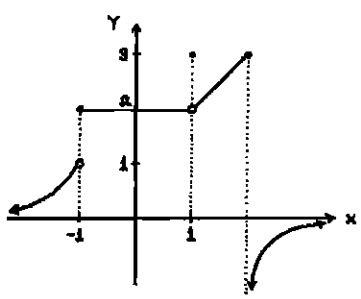
therefore

$$\lim_{x\to 2} g(x) = 4$$

A limit of a function can be analyzed from the left and right sides of a particular value of x.

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$



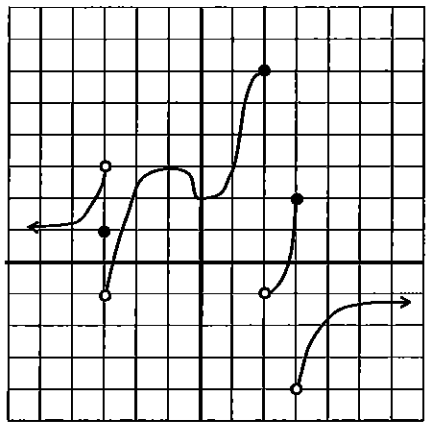
$$\lim_{x\to 1^-} f(x) = 2 \qquad \lim_{x\to 1^+} f(x) = 2$$

$$\therefore \lim_{x\to 1} f(x) = 2$$

$$\lim_{x\to -1^-} f(x) = 1 \qquad \lim_{x\to -1^+} f(x) = 2$$

$$\therefore \lim_{x\to -1} f(x) = \textit{Does not exist} \quad (DNE)$$

Given the graph of a function, find the requested limits.



$\lim_{x\to -3^-} f(x) = 3$ $\quad$ $\lim_{x\to -3^+} f(x) = -1$

$\lim_{x\to -3} f(x) = DNE$ $\quad$ $f(-3) = 1$

$\lim_{x\to 0^-} f(x) = 2$ $\quad$ $\lim_{x\to 0^+} f(x) = 2$

$\lim_{x\to 0} f(x) = 2$ $\quad$ $f(0) = 2$

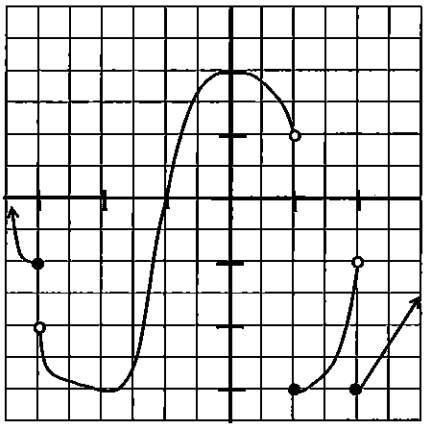
$\lim_{x\to 2^-} f(x) = 6$ $\quad$ $\lim_{x\to 2^+} f(x) = -1$

$\lim_{x\to 2} f(x) = DNE$ $\quad$ $f(2) = 6$

$\lim_{x\to 3^-} f(x) = 2$ $\quad$ $\lim_{x\to 3^+} f(x) = -4$

$\lim_{x\to 3} f(x) = DNE$ $\quad$ $f(3) = 2$

Given the graph of a function, find the requested limits.



$\lim_{x\to -3^-} f(x) = -1$ $\quad$ $\lim_{x\to -3^+} f(x) = -2$

$\lim_{x\to -3} f(x) = DNE$ $\quad$ $f(-3) = -1$

$\lim_{x\to 0^-} f(x) = 2$ $\quad$ $\lim_{x\to 0^+} f(x) = 2$

$\lim_{x\to 0} f(x) = 2$ $\quad$ $f(0) = 2$

$\lim_{x\to 1^-} f(x) = 1$ $\quad$ $\lim_{x\to 1^+} f(x) = -3$

$\lim_{x\to 1} f(x) = DNE$ $\quad$ $f(1) = -3$

$\lim_{x\to 2^-} f(x) = -1$ $\quad$ $\lim_{x\to 2^+} f(x) = -3$

$\lim_{x\to 2} f(x) = DNE$ $\quad$ $f(2) = -3$

Limits involving $\frac{sin\theta}{\theta}$

THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

$$\lim_{\theta \to 0} \frac{sin^2\theta}{\theta^2 + 3\theta}$$

$$\lim_{\theta \to 0} \frac{sin^2 0}{0^2 + 3(0)} = \frac{0}{0}$$

$$\lim_{\theta \to 0} \frac{sin\theta\ sin\theta}{\theta(\theta + 3)}$$

$$\lim_{\theta \to 0} \frac{sin\theta}{\theta} \cdot \frac{sin\theta}{(\theta + 3)}$$

$$\lim_{\theta \to 0} 1 \cdot \frac{sin0}{(0 + 3)}$$

$$1 \cdot \frac{0}{3}$$

$$0$$

$$\lim_{\theta \to 0} \frac{sin^2\theta}{\theta^2 + 3\theta} = 0$$

Limits involving $\frac{sin\theta}{\theta}$

THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

$$\lim_{x \to 0} \frac{sin6x}{5x}$$

$$\lim_{x \to 0} \frac{\sin(6 \cdot 0)}{5(0)} = \frac{0}{0}$$

$$Let\ t = 6x, \quad then\ x = \frac{t}{6}$$

$$As\ x \to 0, \text{then } t \to 0$$

$$\lim_{t \to 0} \frac{sint}{5\left(\frac{t}{6}\right)}$$

$$\lim_{t \to 0} \frac{sint}{\left(\frac{5t}{6}\right)}$$

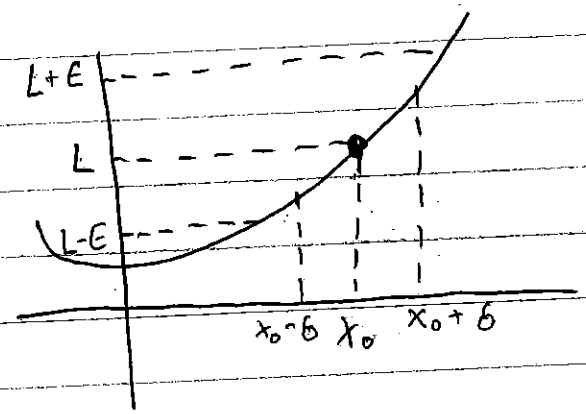
$$\lim_{t \to 0} \frac{sint}{t} \cdot \frac{6}{5}$$

$$1 \cdot \frac{6}{5}$$

$$\frac{6}{5}$$

$$\lim_{x \to 0} \frac{sin6x}{5x} = \frac{6}{5}$$

2.3 The Precise Definition of a Limit



$$\boxed{\lim_{x \to x_0} f(x) = L}$$

Ex: find the value of delta(δ) given epsilon(ε)

$$\lim_{x \to 2} (3x-1) = 5 \qquad \varepsilon = 0.5$$

$$f(x) = 3x - 1 \qquad L = 5 \qquad x_0 = 2$$

$$0 < |x - x_0| < \delta \rightarrow |f(x) - L| < \varepsilon$$

$$0 < |x-2| < \delta \rightarrow |(3x-1) - 5| < 0.5$$

$$\begin{aligned}
&\quad\ \ \text{''} \qquad |3x - 1 - 5| < .5 \\
&\quad\ \ \text{''} \qquad |3x - 6| < .5 \\
&\quad\ \ \text{''} \qquad |3(x-2)| < .5 \\
&\quad\ \ \text{''} \qquad |3| \cdot |(x-2)| < .5 \\
&\quad\ \ \text{''} \qquad 3|x-2| < .5 \\
&\quad\ \ \text{''} \qquad |x-2| < \frac{.5}{3} \\
&\quad\ \ \text{''}
\end{aligned}$$

$$\boxed{|x-2| < \delta \qquad |x-2| < 1/6}$$

Since $|x-2| < \frac{1}{6}$ then δ must be $\frac{1}{6}$

$\delta = \frac{1}{6}$

$f(2) = 5$

$f(x_0 + \delta) = f\left(2 + \frac{1}{6}\right) = 5.5$

$f(x_0 - \delta)\ f\left(2 - \frac{1}{6}\right) = 4.5$

$L + \varepsilon = 5 + .5 = 5.5$

$L - \varepsilon = 5 - .5 = 4.5$

if $\delta < \frac{1}{6}$ then $\varepsilon < 0.5$

Now let $\delta = 0.1$

$f(x_0 + \delta) = (2 + 0.1) = 5.3$

$f(x_0 - \delta) = (2 - 0.1) = 4.7$

$\varepsilon = 0.3 < 0.5$ ✓

Ex: Prove this limit $\lim_{x \to -3} (7x-9) = -30$

So $f(x) = 7x-9$ $L = -30$ $x_0 = -3$

$0 < |x - x_0| < \delta \rightarrow |f(x) - L| < \varepsilon$

$0 < |x - (-3)| < \delta \rightarrow |(7x-9) - (-30)| < \varepsilon$

$0 < |x+3| < \delta \rightarrow |7x+21| < \varepsilon$

$0 < |x+3| < \delta \rightarrow 7|x+3| < \varepsilon$

$0 < |x+3| < \delta \rightarrow |x+3| < \frac{\varepsilon}{7}$

So $\boxed{\delta = \frac{\varepsilon}{7}}$

2.4 One Sided Limits

$$\lim_{x \to c^-} f(x) = \text{Left}$$

$$\lim_{x \to c^+} f(x) = \text{Right}$$

As long as the Limit from the Left = Right then that is the Limit

$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

Ex: $\lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta^2 + 3\theta} = \frac{\sin^2 0}{0 + 3(0)} = \frac{0}{0}$

$$\frac{\sin^2 \theta}{\theta(\theta+3)} = \frac{\sin\theta(\sin\theta)}{\theta(\theta+3)}$$

$$\frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{(\theta+3)} = 1 \cdot \frac{\sin\theta}{\theta+3}$$

$$\lim_{\theta \to 0} = 1 \cdot \frac{\sin 0}{0+3} = 1 \cdot \frac{0}{3} = \boxed{0}$$

Ex: $\lim_{x \to 0} \frac{\sin 6x}{5x}$ Need Trig identity

Let "t" $= 6x$ then $x = \frac{t}{6}$

as $x \to 0$ then $t \to 0$ because they're equal

So $\lim_{t \to 0} \frac{\sin 6\left(\frac{t}{6}\right)}{5\left(\frac{t}{6}\right)} = \frac{\sin t}{5\left(\frac{t}{6}\right)} = \frac{\sin t}{\frac{5t}{6}}$

$\frac{\sin t}{t\left(\frac{5}{6}\right)} = \frac{\sin t}{t} \cdot \frac{6}{5} = 1 \cdot \frac{6}{5}$

$\lim_{x \to 0} \frac{\sin 6x}{5x} = \boxed{\frac{6}{5}}$

Just worry about manipulating the angle in the numerator to make it one variable & the rest will take care of itself

Ex: $\lim_{x \to -5} \frac{x^2+3x-10}{x+5} = \frac{0}{0}$

$= \frac{(x+5)(x-2)}{x+5} = x-2$

$\lim_{x \to -5} = -5-2 = \boxed{-7}$

Ex: $\lim_{x \to 5} \frac{x-5}{\sqrt{x+4}-3} = \frac{0}{0}$

$= \frac{x-5}{\sqrt{x+4}-3} \cdot \frac{\sqrt{x+4}+3}{\sqrt{x+4}+3}$

$= \frac{(x+5)(\sqrt{x+4}+3)}{x+4-9} = \frac{\cancel{(x-5)}(\text{stuff})}{\cancel{x-5}}$

$= \lim_{x \to 5} = \sqrt{x+4}+3 = \sqrt{9}+3 = \boxed{6}$

Get an LCD $= 3(3+x)$

Ex: $\lim_{x \to -3} \dfrac{x}{\frac{1}{3+x} - \frac{1}{3}}$

$$= \frac{x}{\frac{3}{3} \cdot \frac{1}{(3+x)} - \frac{1}{3} \cdot \frac{(3+x)}{(3+x)}}$$

$$= \frac{x}{\frac{3}{3(3+x)} - \frac{(3+x)}{3(3+x)}}$$

$$= \frac{x}{\frac{3-3-x}{3(3+x)}} = \frac{x}{\frac{-x}{3(3+x)}}$$

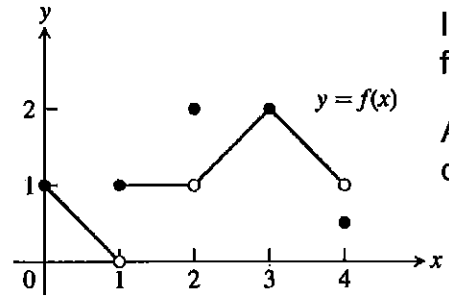
$$= \frac{\cancel{x}\left[(3)(3+x)\right]}{-\cancel{x}} = -1\left[(3)(3+x)\right]$$

$$= -3(3+x)$$

Now $\lim_{x \to -3} -3(3+x)$

$$= -3(3-3) = \boxed{0}$$

A continuous function is one that can be plotted without the plot being broken.



Is the graph of f(x) a continuous function on the interval [0, 4]? No

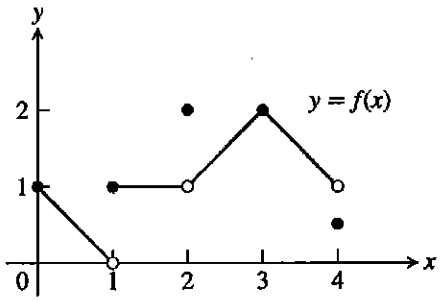
At what values of x is the function discontinuous and why?

$x = 1$ *There is a jump*

$x = 2$ *There is a hole*

$x = 4$ *There is a jump*

Is the graph of f(x) continuous at $x = 3$? Yes



$\lim_{x\to 3^-} f(x) = 2$ $\lim_{x\to 3^+} f(x) = 2$

$\lim_{x\to 3} f(x) = 2$ $f(3) = 2$

$\lim_{x\to 1^-} f(x) = 0$ $\lim_{x\to 1^+} f(x) = 1$

$\lim_{x\to 1} f(x) = DNE$ $f(1) = 1$

$\lim_{x\to 2^-} f(x) = 1$ $\lim_{x\to 2^+} f(x) = 1$

$\lim_{x\to 2} f(x) = 1$ $f(2) = 2$

$\lim_{x\to 4^-} f(x) = 1$ $\lim_{x\to 4^+} f(x) = none$

$\lim_{x\to 4} f(x) = none$ $f(4) = 0.5$

What are the rules for continuity at a point?

1) function value must exist
2) Limit value must exist
3) function value & Limit value must ~~be~~ equal

DEFINITION

Interior point: A function $y = f(x)$ is **continuous at an interior point** c of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint** a or is **continuous at a right endpoint** b of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

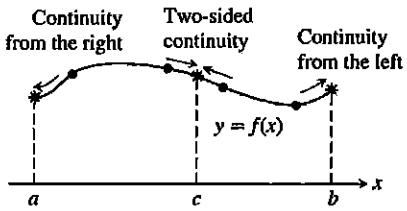


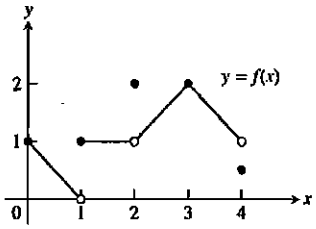
FIGURE 2.36 Continuity at points a, b, and c.

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

Rule



$x = 1$

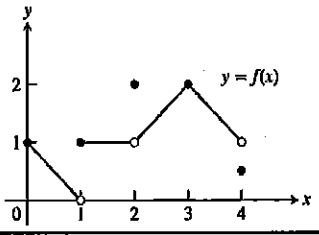
✓ $f(c) exists$ $\quad f(1) = 1$

✗ $\lim_{x \to c} f(x)\ exists$ $\quad \lim_{x \to 1} f(x) = DNE$

$\therefore \quad f(x) is\ not\ continuous\ at\ x = 1.$

Continuity Test
A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value).



$x = 2$

✓ $f(c) exists$ $\quad f(2) = 2$

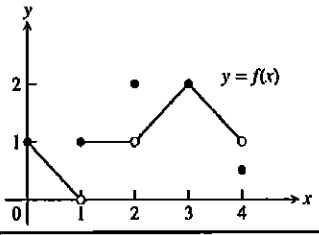
✓ $\lim_{x\to c} f(x)$ *exists* $\quad \lim_{x\to 2} f(x) = 1$

✗ $\lim_{x\to c} f(x) = f(c)$ $\quad 2 \neq 1$

$\therefore$ $f(x) is\ not\ continuous\ at\ x = 2.$

Continuity Test
A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value).



$x = 3$

✓ $f(c) exists$ $\quad f(2) = 2$

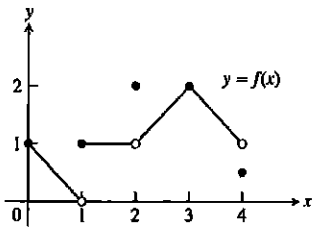
✓ $\lim_{x\to c} f(x)$ *exists* $\quad \lim_{x\to 3} f(x) = 2$

✓ $\lim_{x\to c} f(x) = f(c)$ $\quad 2 = 2$

$\therefore$ $f(x)\ is\ continuous\ at\ x = 3.$

Continuity Test
A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value).



$x = 0$ (left end point)

✓ $f(c) exists$ $\quad f(0) = 1$

✓ $\lim_{x\to c^+} f(x)$ *exists* $\quad \lim_{x\to 0^+} f(x) = 1$

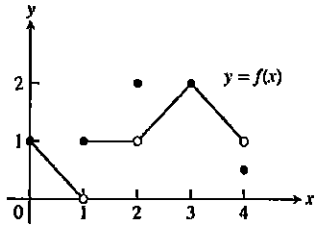
$x \to 0^+$

✓ $\lim_{x\to c^+} f(x) = f(c)$ $\quad 1 = 1$

$\therefore$ $f(x)$ *is continuous at the left endpoint,* $x = 0$.

Continuity Test
A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value).



$x = 4$ (right end point)

✓ $f(c) exists$ $\quad f(4) = 0.5$

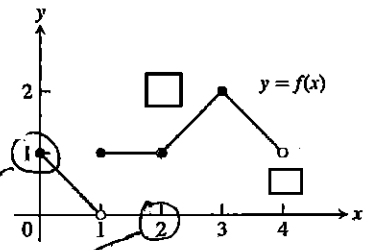
✓ $\lim_{x\to c^-} f(x)$ *exists* $\quad \lim_{x\to 4^-} f(x) = 1$

✗ $\lim_{x\to c^+} f(x) = f(c)$ $\quad 0.5 \neq 1$

$\therefore$ $f(x)$ *is not continuous at the right endpoint,* $x = 4$.

Removable Discontinuity

Removable discontinuity occurs at a point where the function has a hole but does not have a function value.



Create a similar function, $g(x)$, that is continuous at $x = 2$.

1. $\lim_{x\to 2} g(x)$ *exists* $\quad \lim_{x\to 2} g(x) = 1$
3. $\lim_{x\to 2} g(x) = g(2)$ $\quad 1 = 1$
2. $g(2)$*exists* $\quad g(2) = 1$

$$g(x) = \begin{cases} f(x) & if \quad x \neq 2 \\ 1 & if \quad x = 2 \end{cases}$$

$f(x)$ *is discontinuous at* $x = 2$

A hole exist at $x = 2$.

Removable discontinuity at $x = 2$.

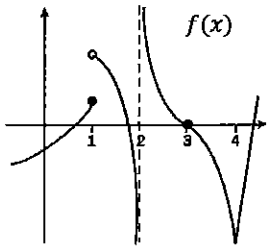
$g(x)$*is continuous at* $x = 2$.

Needs a Hole

we're essentially filling in the Hole

Removable Discontinuity

Removable discontinuity occurs at a point where the function has a hole but does not have a function value.



Create a similar function, $g(x)$, that is continuous at $x = 3$.

$\lim_{x\to 3} g(x)$ *exists* $\quad \lim_{x\to 3} g(x) = 0$

$\lim_{x\to 3} g(x) = g(3)$ $\quad 0 = 0$

$g(3)$*exists* $\quad g(3) = 0$

$$g(x) = \begin{cases} f(x) & if \quad x \neq 3 \\ 0 & if \quad x = 3 \end{cases}$$

$f(x)$*is discontinuous at* $x = 3$

A hole exist at $x = 3$.

Removable discontinuity at $x = 3$.

$g(x)$*is continuous at* $x = 3$.

Removable Discontinuity

Example:

The given function is discontinuous. Where is it discontinuous and is it removable?

$$f(x) = \frac{x^2 - 4}{x - 2}$$

discont because of the denominator @2 if X=2 then we get 0

$$x - 2 = 0$$

$$x = 2$$

$$f(x) \text{ is discontinuous at } x = 2$$

$$f(x) = \frac{(x-2)(x+2)}{x-2}$$

$$\text{The factor } (x-2) \text{ cancels.}$$

$$\text{There is a hole at } x = 2.$$

$$\text{It is removable at } x = 2.$$

if factors of Numerator & denominator cancel each other we have a Hole @ X = +2

Removable Discontinuity

Example:

The given function is discontinuous. Where is it discontinuous and is it removable?

$$f(x) = tan\left(\frac{\pi}{2}x\right)$$

$$f(x) = \frac{sin\left(\frac{\pi}{2}x\right)}{cos\left(\frac{\pi}{2}x\right)}$$

$$cos\left(\frac{\pi}{2}x\right) = 0$$

$$let\ \theta = \frac{\pi}{2}x$$

$$cos\theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$$

$$\frac{\pi}{2}x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots = \frac{\pi}{2} \cdot \frac{2}{\pi} = \frac{\pi}{2} \cdot \frac{2}{\pi}, \frac{3\pi}{2} \cdot \frac{2}{\pi}, \frac{5\pi}{2} \cdot \frac{2}{\pi} \cdots$$

$$x = 1, 3, 5, 7, \cdots$$

$$f(x) \text{ is discontinuous at } x = 1, 3, 5, 7, \cdots$$

$$f(x) = \frac{sin\left(\frac{\pi}{2}x\right)}{cos\left(\frac{\pi}{2}x\right)}$$

$$\text{The factor } \left(cos\left(\frac{\pi}{2}x\right)\right) \text{ does not cancel.}$$

$$\text{There is no removable discontinuity for } f(x)$$

2.5 Examples

$\lim_{x \to 3} g(x) = 0$ $\quad$ $g(x) = 0$ at $x = 3$

$$\lim_{x \to 3} g(x) = g(x)$$

$$g(x) \begin{cases} f(x) & \text{if } x \neq 3 \\ 0 & \text{if } x = 3 \end{cases}$$

Ex: $f(x) = \dfrac{x^2}{x+2}$ $\quad$ $x+2=0$ $\quad$ $x=-2$

disc $= -2$

cant cancel so non-removable

Ex: $g(x) = \dfrac{x+1}{x^2-1} = \dfrac{x+1}{(x+1)(x-1)}$

disc @ $x = 1$ & $x = -1$

removable disc @ $x = -1$

$$f(x) = \frac{x^2 - 16}{x - 5} = \frac{(x-4)(x+4)}{x-5}$$

disc @ $x = 5$

Not removable factor does not cancel

$$h(x) = \frac{x^3 - 27}{x - 3} = \frac{(x-3)(x^2+3x+9)}{x-3}$$

disc @ $x = 3$ or removable @ $x = 3$

[redacted]

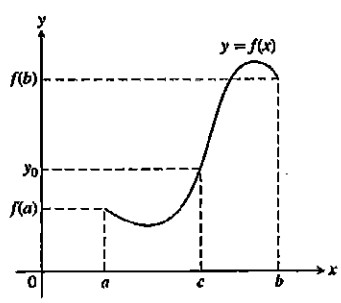
Removable Discontinuity

Examples

[redacted]

[redacted]

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



$f(x) = 2x^3 - 16x^2 + 38x - 22$ [1,5] $f(x)$ *is a polynonmial* $\therefore$ *continuous*

$f(1) = 2$ $f(5) = 18$ $f(x) = 8$

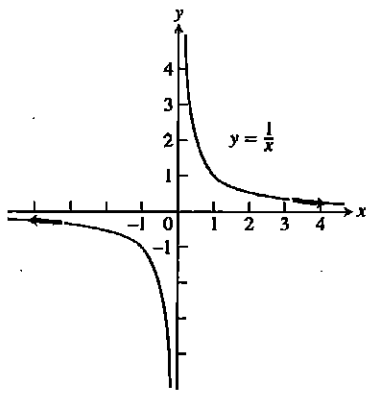
$8 = 2x^3 - 16x^2 + 38x - 22$ $x = 4.547$ use calculator here

$As\ x \to 0^+, y \to \infty$

$\lim_{x\to 0^+} f(x) = \infty$

$As\ x \to -\infty, y \to 0$

$\lim_{x\to -\infty} f(x) = 0$



$As\ x \to \infty, y \to 0$

$\lim_{x\to \infty} f(x) = 0$

$As\ x \to 0^-, y \to -\infty$

$\lim_{x\to 0^-} f(x) = -\infty$

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty.$$

$\lim_{x\to-\infty} f(x) = 5$	$\lim_{x\to\infty} f(x) = 0$	$\lim_{x\to 4^-} f(x) = -\infty$
$H.A.\ at\ y = 5$	$H.A.\ at\ y = 0$	$V.A.\ at\ x = 4$
$\lim_{x\to\infty} f(x) = -2$	$\lim_{x\to 2^+} f(x) = \infty$	$\lim_{x\to -7^+} f(x) = \infty$
$H.A.\ at\ y = -2$	$V.A.\ at\ x = 2$	$V.A.\ at\ x = -7$

write as an equation

$$\lim_{x \to \infty} \frac{x^2}{x+2} = \frac{\frac{x^2}{x^2}}{\frac{x}{x^2} + \frac{2}{x^2}} = \frac{1}{\frac{1}{x} + \frac{2}{x^2}}$$

If you enter large numbers for x like $x = 1000+$ you get small values in the denom that approach 0^+
The smaller the number in the denom the larger this limit will be

$$\frac{1}{.00001} = \text{large} \qquad \frac{1}{.00000001} = \text{larger}$$

So $\frac{1}{0^+} = \infty$

$\boxed{\text{No H.A}}$

$$\lim_{x \to \infty} \frac{3x^2+2}{5x^2-x} = \frac{\frac{3x^2}{x^2} + \frac{2}{x^2}}{\frac{5x^2}{x^2} - \frac{x}{x^2}}$$

$$= \frac{3 + \frac{2}{x^2}}{5 - \frac{1}{x}} = \frac{3 + \cancel{\frac{2}{x^2}}\,0}{5 - \cancel{\frac{1}{x}}\,0} = \frac{3}{5}$$

So H.A. @ $\boxed{y = \frac{3}{5}}$

$$\lim_{x\to\infty} \frac{x^4}{4x^6-4} = \frac{\frac{x^4}{x^6}}{\frac{4x^6}{x^6} - \frac{4}{x^6}}$$

$$= \frac{\frac{1}{x^2}}{4 - \frac{4}{x^6}} = \frac{0}{4-0} = 0$$

H.A. @ $\boxed{y = 0}$

Rule only applied when approaching infinity

(1) if largest exponent is in the numerator $\frac{x^2}{x}$ then there is no H.A.

(2) if largest exponent in numerator equals largest ~~[illegible]~~ exponent in the denominator $\frac{3x^2}{5x^2}$ you have a H.A. at the coefficients $\frac{3}{5}$

(3) if largest exponent is in the denominator then we always have H.A. @ $y=0$

(4) if largest exponent in the numerator is only ONE bigger than exponent in den then O.A.

Long Division Yay!

$$f(x) = \frac{x^2-3}{2x-4}$$

$$\begin{array}{r|l} & \frac{1}{2}x + 1 \\ \hline 2x-4 & x^2 + 0x - 3 \\ & \ominus x^2 \oplus 2x \\ \hline & 0 \quad 2x - 3 \\ & \ominus 2x \oplus 4 \\ \hline & 0 \quad 1 \end{array}$$

draw the line
change the sign

$\rightarrow$ remainder disregard

$$\boxed{y = \frac{1}{2}x + 1}$$

this is the O.A.

$$f(x) = \frac{x^2+x-6}{x-3}$$

$$\begin{array}{r|l} & x + 4 \\ \hline x-3 & x^2 + x - 6 \\ & -x^2 + 3x \\ \hline & 4x - 6 \\ & -4x + 12 \\ \hline & 0 \quad 6 \end{array}$$

O.A. on $\boxed{y = x+4}$

If you have an O.A you don't have an H.A.

THEOREM 12 All the limit laws in Theorem 1 are true when we replace $\lim_{x\to c}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

Limits Involving Infinity

Examples

[redacted]

Oblique Asymptotes

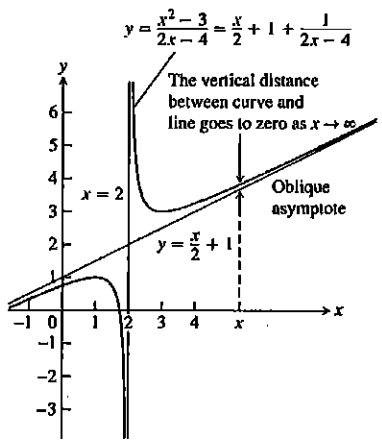


FIGURE 2.58 The graph of the function in Example 10 has an oblique asymptote.

Defn: Derivative:

- The slope of the tangent line to the graph of a function at a given point.
- The instantaneous rate of change of a function with respect to its variable.
- The limiting value of the ratio of the change in a function to the corresponding change in its independent variable.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists. $f'(x)$ *reads, "f prime of x."*

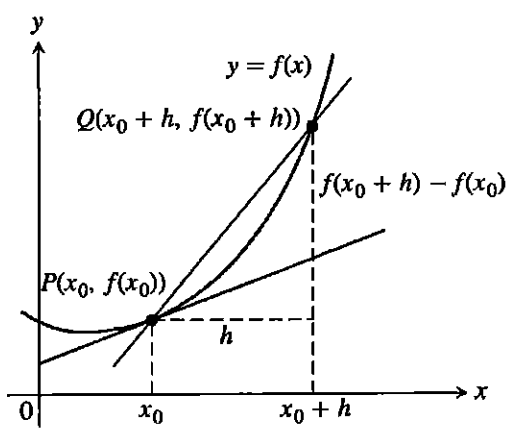


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

Instantaneous Rate of Change / the Slope of a Tangent Line at a Point

$f(x) = -2x^2 + 4 \qquad (1,2)$

$$m_{tan} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} =$$

$$m_{tan} = f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{(1+h) - 1} =$$

$$m_{tan} = f'(1) = \lim_{h \to 0} \frac{-2(1+h)^2 + 4 - (-2(1)^2 + 4)}{h}$$

$=$

$$m_{tan} = f'(1) = \lim_{h \to 0} \frac{-2(1 + 2h + h^2) + 4 - (-2(1)^2 + 4)}{h}$$

$$m_{tan} = f'(1) = \lim_{h \to 0} \frac{-2 - 4h - 2h^2 + 4 - 2}{h}$$

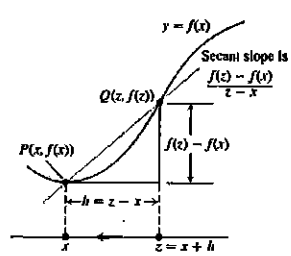
$$m_{tan} = f'(1) = \lim_{h \to 0} \frac{-4h - 2h^2}{h}$$

$$m_{tan} = f'(1) = \lim_{h \to 0} \frac{h(-4 - 2h)}{h}$$

$$m_{tan} = f'(1) = \lim_{h \to 0} -4 - 2h$$

$$m_{tan} = f'(1) = -4$$

Alternate Definition for the Derivative at a Point



Derivative of f at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

Instantaneous Rate of Change / the Slope of a Tangent Line at a Point

$f(x) = -2x^2 + 4 \qquad (1,2)$

$$m_{tan} = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$m_{tan} = f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} =$$

$$m_{tan} = f'(1) = \lim_{x \to 1} \frac{-2x^2 + 4 - (-2(1)^2 + 4)}{x - 1} =$$

$$m_{tan} = f'(1) = \lim_{x \to 1} \frac{-2x^2 + 4 - 2}{x - 1}$$

$$m_{tan} = f'(1) = \lim_{x \to 1} \frac{-2x^2 + 2}{x - 1}$$

$$m_{tan} = f'(1) = \lim_{x \to 1} \frac{-2(x^2 - 1)}{x - 1}$$

$$m_{tan} = f'(1) = \lim_{x \to 1} \frac{-2(x + 1)(x - 1)}{x - 1}$$

$$m_{tan} = f'(1) = \lim_{x \to 1} -2(x + 1)$$

$$m_{tan} = f'(1) = -4$$

You end up factoring in this method

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided the limit exists.

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

Both will give you the same result

Differentiation is the process used to develop the derivative.

Differentiating a function will create the derivative.

Example: $f(x) = x^2 + x + 1$

$$m_{tan} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$m_{tan} = f'(x) = \lim_{h \to 0} \frac{(x+h)^2 + (x + h) + 1 - (x^2 + x + 1)}{h} =$$

$$m_{tan} = f'(x) = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h + 1 - x^2 - x - 1}{h} =$$

$$m_{tan} = f'(x) = \lim_{h \to 0} \frac{2xh + h^2 + h}{h} =$$

$$m_{tan} = f'(x) = \lim_{h \to 0} \frac{h(2x + h + 1)}{h} =$$

$$m_{tan} = f'(x) = \lim_{h \to 0} 2x + h + 1$$

$$m_{tan} = f'(x) = 2x + 1$$

Example: $f(x) = x^2 + x + 1$

$$m_{tan} = f'(x) = \lim_{a \to x} \frac{f(a) - f(x)}{a - x}$$

$$m_{tan} = f'(x) = \lim_{a \to x} \frac{a^2 + a + 1 - (x^2 + x + 1)}{a - x} =$$

$$m_{tan} = f'(x) = \lim_{a \to x} \frac{a^2 + a + 1 - x^2 - x - 1}{a - x} =$$

$$m_{tan} = f'(x) = \lim_{a \to x} \frac{a^2 + a - x^2 - x}{a - x} =$$

$$m_{tan} = f'(x) = \lim_{a \to x} \frac{a^2 - x^2 + a - x}{a - x} =$$

$$m_{tan} = f'(x) = \lim_{a \to x} \frac{(a + x)(a - x) + (a - x)}{a - x} =$$

$$m_{tan} = f'(x) = \lim_{a \to x} \frac{(a - x)(a + x + 1)}{a - x}$$

$$m_{tan} = f'(x) = \lim_{a \to x} (a + x + 1)$$

$$m_{tan} = f'(x) = x + x + 1$$

$$m_{tan} = f'(x) = 2x + 1$$

3.2 Notes

Quick Example

$$f(x) = \frac{2x^2 + 5x - 6}{2x + 3}$$

V.A. $= 2x + 3 = 0$

$\boxed{x = -\frac{3}{2}}$

$$\lim_{x \to \infty} \frac{2x^{\cancel{2}} + 5x - 6}{2x^{\cancel{2}} + 3}$$

$$\frac{1}{0^+} = \boxed{\infty}$$

$\boxed{\text{No H.A.}}$

O.A. =

$$\begin{array}{r} x + 1 \\ 2x+3 \overline{\big) 2x^2 + 5x - 6} \\ -2x^2 \mp 3x \\ \hline 0 \quad 2x \\ 2x + 3 \\ \hline 0 \quad -9 \end{array}$$

$\boxed{O.A = y = x + 1}$

Derivative of $-2x^2+4$

$$f'(x) = \lim_{h \to 0} \frac{(f x + h) - f(x)}{h}$$

$$= \frac{(-2(x+h)^2 + 4) - (-2x^2 + 4}{h}$$

$$= \frac{-2(x^2 + 2xh + h^2) + 4 - (-2x^2 + 4)}{h}$$

$$= \frac{\cancel{-2x^2} - 4xh - 2h^2 + \cancel{4} + \cancel{2x^2} - \cancel{4}}{h}$$

$$= \frac{-4xh - 2h^2}{h} = \frac{h(-4x - 2h)}{h}$$

$$= \boxed{-4x - 2h}$$

$f'(1) = -4x - 2h$

$f'(1) = -4(1) - 2h = \boxed{-4 - 2h}$

$\lim_{h \to 0} -4 - 2h = \lim_{h \to 0} -4 - 2(0) = \boxed{-4}$

$\boxed{f'(1) = -4}$

Example

$f(x) = x^2 + x + 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \frac{(x+h)^2 + x + h + 1 - (x^2 + x + 1)}{h}$$

$$= \frac{x^2 + 2xh + h^2 + x + h + 1 - (x^2 + x + 1)}{h}$$

$$= \frac{2xh + h^2 + h}{h} = \frac{h(2x + h + 1)}{h}$$

$$= 2x + h + 1$$

$$\lim_{h \to 0} 2x + h + 1 = \boxed{f'(x) = 2x + 1}$$

Ex: $f(x)\, x^2 + x + 1$

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\lim_{x \to a} \frac{x^2 + x + 1 - a^2 - a - 1}{x - a}$$

$$\lim_{x \to a} \frac{x^2 + x - a^2 - a}{x - a}$$

$$\cancel{\frac{x(x+1) - a(a+1)}{x - a}}$$

Cant do that

$$\cancel{\lim_{x \to a} = x + 1 + a + 1}$$

$$\cancel{= a + 1 + a + 1}$$

$$\cancel{f'(x) = 2a + 2}$$

$$y = \frac{2}{x}$$

$$y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \frac{\frac{2}{x+h} - \frac{2}{x}}{h}$$

$$= \frac{\frac{2x}{x^2+xh} - \frac{2x+2h}{x^2+xh}}{h}$$

$$= \frac{\frac{2x - 2x - 2h}{x^2+xh}}{h}$$

$$= \frac{-2h}{x^2+xh} \cdot \frac{1}{h} = \frac{-2h}{(x^2+xh)(h)}$$

$$= \frac{-2}{x(x+h)} = \lim_{h \to 0} \frac{-2}{x^2}$$

$$\boxed{f'(x) = -\frac{2}{x^2}}$$

Ex: $f(x) \frac{1}{x+2}$ $f'(x) = \frac{f(x+h) - f(x)}{h}$

$$= \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} = \frac{\frac{x+2}{(x+2)(x+h+2)} - \frac{x+h+2}{(x+2)(x+h+2)}}{h}$$

$$= \frac{-h}{(x+2)(x+h+2)} \cdot \frac{-1}{h} = \frac{-1}{(x+2)(x+h+2)}$$

$$= \lim_{h \to 0} \frac{-1}{(x+2)(x+h+2)} = \frac{-1}{x^2+4x+4}$$

$$\boxed{f'(x) = \frac{-1}{x^2+4x+4} \text{ or } \frac{-1}{(x+2)^2}}$$

Ex: $f'(x) = 3x^2 + 2x$

$$= \frac{3(x+h)^2 + 2(x+h) - (3x^2+2x)}{h}$$

$$= \frac{3x^2 + 6xh + 3h^2 + 2x + 2h - 3x^2 - 2x}{h}$$

$$= \frac{6xh + 3h^2 + 2h}{h} = 6x + 3h + 2$$

$$\lim_{h \to 0} 6x + 3h + 2 = \boxed{f'(x) = 6x + 2}$$

Ex:

$$f(x) = \sqrt{4x-5}$$

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{\sqrt{4(x+h)-5} - \sqrt{4x-5}}{h}$$

$$= \frac{\sqrt{4(x+h)-5} - \sqrt{4x-5}}{h} \cdot \frac{\sqrt{4(x+h)-5} + \sqrt{4x-5}}{\sqrt{4(x+h)-5} + \sqrt{4x-5}}$$

$$= \frac{4(x+h)-5-(4x-5)}{h(\text{stuff})}$$

$$= \frac{\cancel{4x} + 4h \cancel{-5} \cancel{-4x} \cancel{+5}}{h(\text{stuff})} = \frac{4h}{h(\text{stuff})}$$

$$\lim_{h \to 0} \frac{4}{\sqrt{4(x+h)-5} + \sqrt{4x-5}}$$

$$= \frac{4}{\sqrt{4x-5} + \sqrt{4x-5}} = \frac{4}{2\sqrt{4x-5}}$$

$$\boxed{f'(x) = \frac{2}{\sqrt{4x-5}}}$$

REVIEW: Use the Limit Definition to find the derivative of the given function.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = 3x - x^2 + 2$

$$f'(x) = \lim_{h \to 0} \frac{(3(x+h) - (x+h)^2) + 2 - (3x - x^2 + 2)}{h}$$

$$= \lim_{h \to 0} \frac{(3x + 3h - (x^2 + 2xh + h^2)) + 2 - (3x - x^2 + 2)}{h}$$

$$= \lim_{h \to 0} \frac{3x + 3h - x^2 - 2xh - h^2 + 2 - 3x + x^2 - 2}{h}$$

$$= \lim_{h \to 0} \frac{3h - 2xh - h^2}{h} \quad = \lim_{h \to 0} \frac{h(3 - 2x - h)}{h}$$

$$= \lim_{h \to 0} 3 - 2x - h \quad = 3 - 2x \qquad f'(x) = 3 - 2x$$

What are the rules?

$f(x) = -9x + 10 \quad f'(x) = -9$

$f(x) = 5x - 8 \quad f'(x) = 5$

$f(x) = 3x^2 \quad f'(x) = 6x$

$f(x) = 52 \quad f'(x) = 0$

$f(x) = 5x^5 - 6x^3 + x \quad f'(x) = 25x^4 - 18x^2 + 1$

$f(x) = 7x^6 + 9x^{-4} - 12x + 5 \quad f'(x) = 42x^5 - 36x^{-5} - 12$

$f(x) = 2x^7 - 4x^5 + x^2 \quad f'(x) = 14x^6 - 20x^4 + 2x$

$f'(x) = nx^{n-1}$

$f(x) = 4x^2 + x \quad f'(x) = 8x + 1$

$f(x) = 6x^{-3} + 8x^3 + 2x + 7 \quad f'(x) = -18x^{-4} + 24x^3 + 2$

Derivative of a Constant Function
If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

$$\frac{d}{dx}(c) = 0$$

$f(x) = 5 \quad f'(x) = 0$ $\qquad$ $f(x) = -17 \quad f'(x) = 0$

Power Rule (General Version)
If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

for all x where the powers x^n and x^{n-1} are defined.

$f(x) = x^2 \quad f'(x) = 2x$ $\qquad$ $f(x) = x^7 \quad f'(x) = 7x^6$

$f(x) = x^{-5} \quad f'(x) = -5x^{-6}$ $\qquad$ $f(x) = x^{-8} \quad f'(x) = -8x^{-9}$

$f(x) = x^{\frac{5}{7}} \quad f'(x) = \frac{5}{7}x^{-\frac{2}{7}}$

Derivative Constant Multiple Rule
If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

$f(x) = 4x^2 \quad f'(x) = 4 \cdot 2x = 8x$

$f(x) = -9x^7 \quad f'(x) = -9 \cdot 7x^6 = -63x^6$

$f(x) = 6x^{-8} \quad f'(x) = 6 \cdot (-8)x^{-9} = -48x^{-9}$

$f(x) = -3x^{-5} \quad f'(x) = -3(-5)x^{-6} = 15x^{-6}$

$f(x) = 2x^{\frac{5}{7}} \quad f'(x) = 2 \cdot \frac{5}{7}x^{-\frac{2}{7}} = \frac{10}{7}x^{-\frac{2}{7}}$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}. \quad \frac{d}{dx}(U+V) = \frac{du}{dx} + \frac{dv}{dx}$$

$f(x) = 4x^2 - 9x^7 \qquad f'(x) = 8x - 63x^6$

$f(x) = -6x^{-5} + 6x^{-8} \qquad f'(x) = 30x^{-6} - 48x^{-9}$

$f(x) = 3x^4 + 2x^{\frac{5}{7}} - 8 \qquad f'(x) = 12x^3 + \frac{10}{7}x^{-\frac{2}{7}}$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x \quad \frac{d}{dx}(e^x) = e^x$$

$f(x) = 4e^x \qquad f'(x) = 4e^x \qquad f(x) = -6e^x + 5x^7 + 2 \quad f'(x) = -6e^x + 35x^6$

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}. \quad \frac{d}{dx}(UV) = U \cdot \frac{dv}{dx} + V\frac{du}{dx}$$

$f(x) = (4x + x^4)(3x + 2x^2)$

$u = (4x + x^4) \quad v = (3x + 2x^2)$

$f'(x) = (4x + x^4)(3 + 4x) + (3x + 2x^2)(4 + 4x^3)$

$f'(x) = 12x + 16x^2 + 3x^4 + 4x^5 + 12x + 12x^4 + 8x^2 + 8x^5$

$f'(x) = 12x^5 + 15x^4 + 24x^2 + 24x$

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$f(x) = (1+x^2)\left(x^{\frac{3}{4}} - x^{-3}\right)$$

$$u = (1+x^2) \quad v = \left(x^{\frac{3}{4}} - x^{-3}\right)$$

$$f'(x) = (1+x^2)\left(\frac{3}{4}x^{-\frac{1}{4}} + 3x^{-4}\right) + \left(x^{\frac{3}{4}} - x^{-3}\right)(2x)$$

$$f'(x) = \frac{3}{4}x^{-\frac{1}{4}} + 3x^{-4} + \frac{3}{4}x^{\frac{7}{4}} + 3x^{-2} + 2x^{\frac{7}{4}} - 2x^{-2}$$

$$f'(x) = \frac{11}{4}x^{\frac{7}{4}} + \frac{3}{4}x^{-\frac{1}{4}} + x^{-2} + 3x^{-4}$$

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

den(deriv of num)

$$g(x) = \frac{x}{e^x}$$

$$u = x \quad v = e^x$$

$$\frac{d}{dx}\left(\frac{x}{e^x}\right) = \frac{e^x(1) - x\,e^x}{\boxed{(e^x)^2}}$$

$$\frac{d}{dx}\left(\frac{x}{e^x}\right) = \frac{e^x - xe^x}{e^{2x}}$$

$$\frac{d}{dx}\left(\frac{x}{e^x}\right) = \frac{e^x(1-x)}{e^{2x}}$$

$$\frac{d}{dx}\left(\frac{x}{e^x}\right) = \frac{(1-x)}{e^x}$$

3.3 Notes & Examples

$$f(x) = (1+x^2)(x^{3/4} - x^{-3})$$

$$f'(x) = (1+x^2)\left(\tfrac{3}{4}x^{-\frac{1}{4}} + 3x^{-4}\right) + (x^{3/4} - x^{-3})(2x)$$

$$f'(x) = \left(\tfrac{3}{4}x^{-\frac{1}{4}} + 3x^{-4} + \tfrac{3}{4}x^{\frac{7}{4}} + 3x^{-2}\right) + (2x^{\frac{7}{4}} - 2x^{-2})$$

$$\boxed{f'(x) = \frac{3}{4}x^{-\frac{1}{4}} + 3x^{-4} + \frac{11}{4}x^{\frac{7}{4}} + x^{-2}}$$

find slope of 2 $(x=2)$
on calculator

$Y_1 = (1+x^2)(x^{\frac{3}{4}} - x^{-3})$

$Y_2 = \text{nDeriv}(Y_1, X, X)$

now just $Y_2(2)$ & calculator will give answer

$$g(x) = \frac{3x}{3x^2+2}$$

$$g'(x) = \frac{(3x^2+2)(3) - 3x(6x)}{(3x^2+2)^2}$$

$$g'(x) = \frac{9x^2+6-18x^2}{(3x^2+2)^2}$$

$$\boxed{g'(x) = \frac{-9x^2+6}{(3x^2+2)^2}}$$

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

$$g(x) = \frac{3x}{3x^2+2}$$

$$u = 3x \qquad v = 3x^2 + 2$$

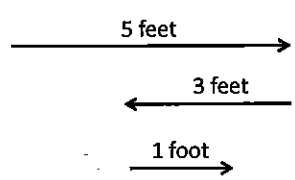
$$\frac{d}{dx}\left(\frac{3x}{3x^2+2}\right) = \frac{(3x^2+2)(3) - 3x(6x)}{(3x^2+2)^2}$$

$$\frac{d}{dx}\left(\frac{3x}{3x^2+2}\right) = \frac{9x^2+6-18x^2}{(3x^2+2)^2} \qquad \frac{d}{dx}\left(\frac{3x}{3x^2+2}\right) = \frac{-9x^2+6}{(3x^2+2)^2}$$

Position, Displacement, Velocity, Speed, and Acceleration

Position: the location of an object as determined by a position function.

Displacement: the distance an object is away from its initial position after travelling for a certain interval of time.



$$5 - 3 + 1$$

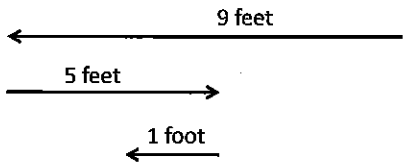
Displacement: 3 feet from the initial point

positive 3 feet → right of initial point

Position, Displacement, Velocity, Speed, and Acceleration

Position: the location of an object as determined by a position function.

Displacement: the distance an object is away from its initial position after travelling for a certain interval of time.



$$-9 + 5 - 1$$

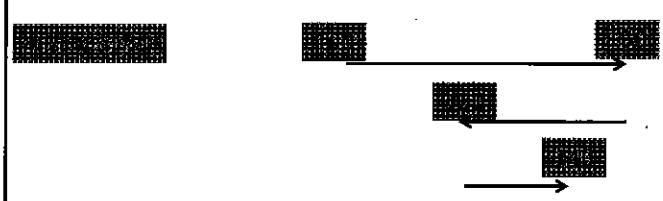
$$Displacement: -5\ feet\ from\ the\ initial\ point$$

$$negative\ 5\ feet\ \rightarrow\ left\ of\ initial\ point$$

Position, Displacement, Velocity, Speed, and Acceleration

Position: the location of an object as determined by a position function.

Displacement: the distance an object is away from its initial position after travelling for a certain interval of time.



$$150 - 100 = 50$$

$$120 - 150 = -30$$

$$140 - 120 = 20$$

$$50 - 30 + 20$$

$$Displacement: 40\ miles\ from\ the\ initial\ point$$

$$positive\ 40\ miles\ \rightarrow\ right\ of\ initial\ point$$

Position, Displacement, Velocity, Speed, and Acceleration

Velocity: the change in position with respect to a change in time. It is a rate of change with direction.

The velocity function, $v(t)$, is obtain by differentiating the position function with respect to time.

$$v(t) = s'(t) = \frac{ds}{dt}$$

$s(t) = 4t^2 + t$ $\qquad$ $s(t) = 5t^3 - 6t^2 + 6$ position function

$v(t) = s'(t) = 8t + 1$ $\qquad$ $v(t) = s'(t) = 15t^2 - 12t$ Velocity function

Speed: the absolute value of velocity. It is a rate of change without direction.

Speed is always positive

$$speed = |v(t)| = |s'(t)| = \left|\frac{ds}{dt}\right|$$

Position, Displacement, Velocity, Speed, and Acceleration

Acceleration: the change in velocity with respect to a change in time. It is a rate of change with direction.

acceleration = how velocity changes over time

The acceleration function, $a(t)$, is obtain by differentiating the velocity function with respect to time. It is also the 2nd derivative of the position function.

$$a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2s}{dt^2}$$

$s(t) = 4t^2 + t$ $\qquad$ $s(t) = 5t^3 - 6t^2 + 6$

$v(t) = s'(t) = 8t + 1$ $\qquad$ $v(t) = s'(t) = 15t^2 - 12t$

$a(t) = v'(t) = s''(t) = 8$ $\qquad$ $a(t) = v'(t) = s''(t) = 30t - 12$

Economics

Cost of production: a function of the units produced (x) that generates the cost of producing those units .

$c(x)$ cost of product (x)

Average cost of production: the cost of production function divided by the number of units produced at that cost.

$$average\ cost = \frac{c(x)}{x}$$

$\frac{\text{cost of product}}{\text{how many we produced}}$

Marginal cost of production: the rate of change of costs with respect to the level of production.

Marginal cost is an approximation of the cost to produce one more unit after producing x units

$$marginal\ cost = c'(x)$$

if I produced 50 products, how much will it cost to produce one more product?

Economics

Cost of production: $c(x)$ $\quad average\ cost = \frac{c(x)}{x}$ $\quad mariginal\ cost = c'(x)$

A cost function is given as follows: $c(x) = 100 + 8x + 0.1x^2$

Find the average cost in producing 50 units.

$$average\ cost = \frac{c(x)}{x} = \frac{c(50)}{50} = \frac{100 + 8(50) + 0.1(50)^2}{50} = \frac{750}{50} = \$15$$

Find the marginal cost to produce the 51st unit.

$$c'(x) = 8 + 0.2x \qquad c'(50) = 8 + 0.2(50) \quad = \$18$$

Find the actual cost to produce the 51st unit.

$$c(50) = 100 + 8(50) + 0.1(50)^2 \quad = 750$$

$$c(51) = 100 + 8(51) + 0.1(51)^2 \quad = 768.1$$

$$768.1 - 750 \quad = \$18.10$$

3.4 Notes + Examples

$c(x) = 100 + 8x + 0.1x^2$

find average cost to produce 50 units

$$\text{average cost} = \frac{c(x)}{x} = \frac{c(50)}{50}$$

$$= \frac{100 + 8(50) + 0.1(50)^2}{50}$$

$$= \boxed{\$15}$$

find marginal cost to produce 51st item

$c'(x) = 8 + 0.2x$

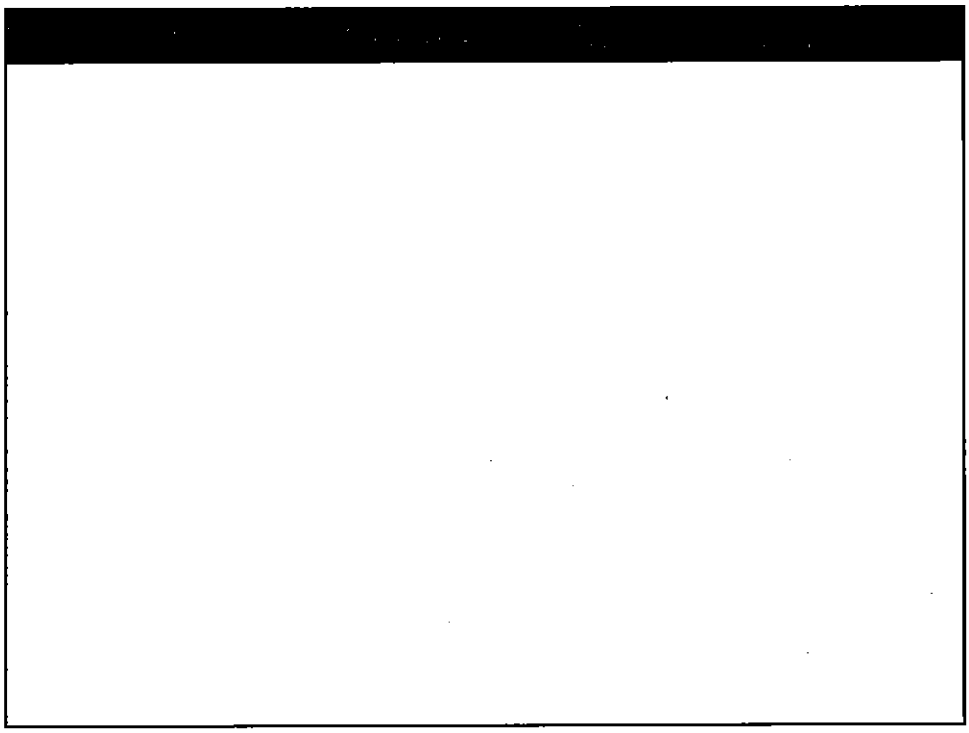
$c'(50) = 8 + 0.2(50) = \boxed{\$18}$

find the actual cost to produce the 51st

$c(50) = 750$

$c(51) = 768.10$

$768.1 - 750 = \boxed{\$18.10}$



REVIEW:

Function Notation

$$y = f(x) \qquad u = f(x) \qquad v = g(x) \qquad s = s(t)$$

Derivative Notation

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}\big(f(x)\big) = \frac{d}{dx}y = D_x f(x) = D_x y$$

$$u' = f'(x) = \frac{du}{dx} = \frac{d}{dx}\big(f(x)\big) = \frac{d}{dx}u = D_x f(x) = D_x u$$

$$v' = g'(x) = \frac{dv}{dx} = \frac{d}{dx}\big(g(x)\big) = \frac{d}{dx}v = D_x f(x) = D_x v$$

$$s' = s'(t) = \frac{ds}{dt} = \frac{d}{dt}(s(t)) = \frac{d}{dt}s = D_t s(t) = D_t s$$

Derivative of Sine

$$y = f(x) = \sin x$$

$$y' = f'(x) = \frac{dy}{dx} = \cos x$$

Derivative of Cosecant

$$y = f(x) = \csc x$$

$$y' = f'(x) = \frac{dy}{dx} = -cotx\ cscx$$

Derivative of Cosine

$$y = f(x) = \cos x$$

$$y' = f'(x) = \frac{dy}{dx} = -\sin x$$

Derivative of Secant

$$y = f(x) = \sec x$$

$$y' = f'(x) = \frac{dy}{dx} = \tan x \sec x$$

Derivative of Tangent

$$y = f(x) = \tan x$$

$$y' = f'(x) = \frac{dy}{dx} = sec^2 x$$

Derivative of Cotangent

$$y = f(x) = \cot x$$

$$y' = f'(x) = \frac{dy}{dx} = -csc^2 x$$

Derivative of tangent:

$$D_x \tan x = D_x\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} =$$

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Derivative of cosecant:

$$D_x \csc x = D_x\left(\frac{1}{\sin x}\right) = \frac{\sin x(0) - 1(\cos x)}{\sin^2 x} =$$

$$\frac{-\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = -\cot x \csc x$$

Derivative of secant:

$$D_x \sec x = D_x\left(\frac{1}{\cos x}\right) = \frac{\cos x(0) - 1(-\sin x)}{\cos^2 x} =$$

$$\frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x$$

Derivative of cotangent:

$$D_x \cot x = D_x\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} =$$

$$\frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} =$$

$$-\csc^2 x$$

3.5 Examples

$y = 2\sin x + 3\cos x$

$\boxed{y' = 2\cos x - 3\sin x}$

$y = x^3 \sin x$

$y' = (3x^2)(\sin x) + (x^3)(\cos x)$

$\boxed{y' = 3x^2 \sin x + x^3 \cos x}$

$y = e^x \cos x$

$y' = (e^x)(\cos x) + (e^x)(-\sin x)$

$\boxed{y' = e^x \cos x - e^x \sin x}$

$$f(x) = \frac{\sec x}{\sin x}$$

$$f'(x) = \frac{\sin x(\tan x \sec x) - \sec x(\cos x)}{(\sin x)^2}$$

$$f'(x) = \frac{\sin x\left(\frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\right) - \frac{1}{\cos x}(\cos x)}{(\sin x)^2}$$

$$f'(x) = \frac{\left(\frac{\sin x}{\cos x}\right)^2 - \frac{\cos x}{\cos x}}{(\sin x)^2} = \frac{\tan^2 x - 1}{(\sin x)^2}$$

$$y = \frac{\sin x + \cos x}{\cos x} \quad \text{or} \quad \frac{\sin x}{\cos x} + \frac{\cos x}{\cos x} = \tan x + 1$$

$$= \boxed{\sec^2 x}$$

$$y' = \frac{\cos x(\cos x - \sin x) - (\sin x + \cos x)(-\sin x)}{(\cos x)^2}$$

$$y' = \frac{\cos^2 x - \sin x \cos x - (-\sin^2 x - \cos x \sin x)}{(\cos x)^2}$$

$$y' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\boxed{y' = \sec^2 x}$$

$$y = \tan^2 x$$

$$y = (\tan x)(\tan x)$$

$$y' = (\tan x)(\sec^2 x) + (\tan x)(\sec^2 x)$$

$$\boxed{y' = 2\tan x \sec^2 x}$$

Practice Problems – Worksheet
Derivatives of Trigonometric Functions

Review of the Product Rule:

$y = (3x^3 + 2x^2)^2 \;\; = (3x^3 + 2x^2)(3x^3 + 2x^2)$

$y' = (3x^3 + 2x^2)(9x^2 + 4x) + (9x^2 + 4x)(3x^3 + 2x^2)$

$y' = 2(3x^3 + 2x^2)(9x^2 + 4x)$

$y = (6x^2 + x)^3 \;\; = (6x^2 + x)(6x^2 + x)(6x^2 + x)$

$y' = (6x^2 + x)(6x^2 + x)(12x + 1) + (6x^2 + x)(12x + 1)(6x^2 + x) + (12x + 1)(6x^2 + x)(6x^2 + x)$

$y' = (6x^2 + x)^2(12x + 1) + (6x^2 + x)^2(12x + 1) + (6x^2 + x)^2(12x + 1)$

$y' = 3(6x^2 + x)^2(12x + 1)$

$y = (3x^3 + 2x^2)^2 \qquad y' = 2(3x^3 + 2x^2)(9x^2 + 4x)$

$y = (6x^2 + x)^3 \qquad y' = 3(6x^2 + x)^2(12x + 1)$

Additional Problems:

$y = (x^3 + 2x)^9 \qquad y' = 9\,(x^3 + 2x)^8\,(3x^2 + 2)$

$y = (5x^2 + 1)^4 \qquad y' = 4\,(5x^2 + 1)^3\,(10x)$

$y = (2x^5 - 3x^4 + x - 3)^{13} \qquad y' = 13\,(2x^5 - 3x^4 + x - 3)^{12}\,(10x^4 - 12x^3 + 1)$

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

3.6 Notes

$y = u^9$, $u = x^3 + 2x$

$\frac{dy}{du} = 9u^8$ $\frac{du}{dx} = 3x^2 + 2$

find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$$

$$\frac{dy}{dx} = 9u^8 \cdot (3x^2 + 2)$$

$$= 9(x^3 + 2x)^8 (3x^2 + 2)$$

So... $y = (x^3 + 2x)^9$

$y' = 9(x^3 + 2x)^8 (3x^2 + 2)$

Ex: $S = e^{5t}$

$S = (e^t)^5$

$S' = 5e^{4t}(e^t)$

$S' = 5e^{5t}$

Ex: $e^{\sin x}$

$y' = e^{\sin x} \cdot \cos x$

Ex: $y = e^{(7x^2 + 2x + 5)}$

$y' = e^{(7x^2 + 2x + 5)} \cdot (14x + 2)$

3.6 Examples

$y = \sqrt{5x^2+1}$

$y' = (5x^2+1)^{\frac{1}{2}}$

$y' = \frac{1}{2}(5x^2+1)^{-\frac{1}{2}}(10x)$

$$\boxed{y' = (5x^2+1)^{-\frac{1}{2}}(5x)}$$

$y = \tan(x^3)$

$y' = \sec^2(x^3)\cdot(3x^2)$

$$\boxed{y' = 3x^2 \cdot \sec^2(x^3)}$$

$r = \sin^3\theta$

$r' = 3\sin^2\theta(\cos\theta)(1)$ because $\theta' = 1$, $x' = 1$

$r = \sin^3(2x+3)$

$r' = 3\sin^2(2x+3)(\cos(2x+3))(2)$

$$\boxed{r' = 6\sin^2(2x+3)(\cos(2x+3))}$$

$$f(x) = \left(\frac{x}{x+1}\right)^5$$

$$f'(x) = 5\left(\frac{x}{x+1}\right)^4 \cdot \left(\frac{(x+1)(1) - x(1)}{(x+1)^2}\right)$$

$$f'(x) = 5\left(\frac{x}{x+1}\right)^4 \cdot \left(\frac{x+1-x}{(x+1)^2}\right)$$

$$f'(x) = 5\left(\frac{x}{x+1}\right)^4 \left(\frac{1}{(x+1)^2}\right)$$

$$f'(x) = \frac{x^4}{(x+1)^4} \cdot \frac{5}{(x+1)^2}$$

$$\boxed{f'(x) = \frac{5x^4}{(x+1)^6}}$$

Practice Problems – Worksheet
The Chain Rule

An Implicit function is one where the variable "y" can not be easily solved for in terms of only "x".

Examples:

$$4x^2 = 2y^3 + 4y$$

$$x = \sqrt{\tan y}$$

$$3x^2y^2 = 4x^2 - 4xy$$

$$x^2 \sin x + y^2 \cos y = 1$$

Almost impossible to separate the y's here.

We need Implicit Differentiation

Differentiating an implicit function will create the derivative ($^{dy}/_{dx}$) needed to calculate the slope of any tangent on the curve or the rate of change at any value of the independent variable.

Find $^{dy}/_{dx}$

$$\frac{d\,\text{variable}}{dx} = \frac{dy}{dx}, \frac{dx}{dx}, \frac{dq}{dx}$$

$$3x^2 + y^2 = 14$$

$$6x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$$

$$6x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -6x$$

$$\frac{dy}{dx} = -\frac{6x}{2y} \qquad \frac{dy}{dx} = -\frac{3x}{y}$$

Find $^{dy}/_{dx}$

$$x^3y^5 + 3x = 8y^3 + 1$$

$$x^3 5y^4\frac{dy}{dx} + 3x^2\frac{dx}{dx}y^5 + 3\frac{dx}{dx} = 24y^2\frac{dy}{dx} + 0$$

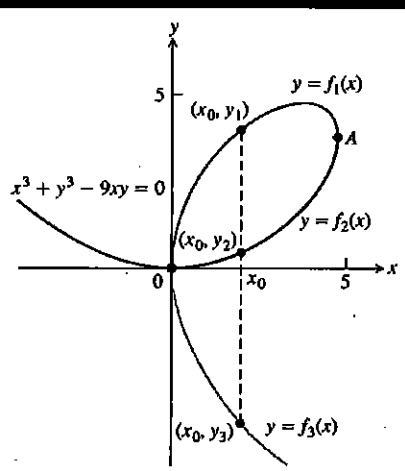
$$5x^3y^4\frac{dy}{dx} + 3x^2y^5 + 3 = 24y^2\frac{dy}{dx}$$

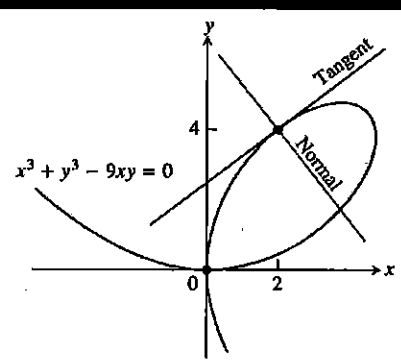
$$5x^3y^4\frac{dy}{dx} - 24y^2\frac{dy}{dx} = -3x^2y^5 - 3$$

$$\frac{dy}{dx}(5x^3y^4 - 24y^2) = -3x^2y^5 - 3$$

factored out $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{-3x^2y^5 - 3}{5x^3y^4 - 24y^2}}$$





Slope of a line thats normal to the tangent is the perpendicular line $m = \frac{-1}{m}$

3.7 Notes & Examples

find equation to the line & normal lines for the following implicit function

$3x^2 + y^2 = 14 \quad (2, \sqrt{2})$

$6x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -6x$

$\boxed{\frac{dy}{dx} = \frac{-3x}{y}}$

$\frac{dy}{dx} = \frac{-3(2)}{\sqrt{2}}$

$\frac{dy}{dx} = \frac{-6}{\sqrt{2}} = \frac{-6\sqrt{2}}{2} = \boxed{-3\sqrt{2}}$

$\boxed{y - \sqrt{2} = -3\sqrt{2}(x-2)}$

Normal line slope $= -\frac{1}{m} = +\frac{2}{6\sqrt{2}} = \frac{1}{3\sqrt{2}} = \boxed{\frac{\sqrt{2}}{6}}$

$\boxed{y - \sqrt{2} = \frac{\sqrt{2}}{6}(x-2)}$

Find the equation of the tangent and normal lines for the following implicit function at the given point.

$3x^2 + y^2 = 14 \quad (2, \sqrt{2})$

$$\frac{dy}{dx} = -\frac{3x}{y}$$

$$m_{tan} = \frac{dy}{dx} = -\frac{3(2)}{\sqrt{2}}$$

$$m_{tan} = \frac{dy}{dx} = -\frac{6}{\sqrt{2}} = -3\sqrt{2}$$

$$y - \sqrt{2} = -3\sqrt{2}(x - 2)$$

$$m_{tan} = -\frac{6}{\sqrt{2}} = -3\sqrt{2}$$

$$m_{normal} = \frac{\sqrt{2}}{6}$$

$$y - \sqrt{2} = \frac{\sqrt{2}}{6}(x - 2)$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}, \quad u > 0 \qquad (2)$$

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}. \qquad (5)$$

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}. \qquad (7)$$

Ex: $y = \ln x = \frac{1}{x} \cdot 1 = \boxed{\frac{1}{x}}$

$y = \ln(3x+2) = \frac{1}{3x+2} \cdot 3 = \boxed{\frac{3}{3x+2}}$

Find $^{dy}/_{dx}$

$y = lnx^3$

$\frac{dy}{dx} = \frac{1}{x^3}(3x^2)$

$\frac{dy}{dx} = \frac{3}{x}$

$y = lnx^3 \ = 3lnx$

$\frac{dy}{dx} = 3\frac{1}{x}$

$\frac{dy}{dx} = \frac{3}{x}$

$y = \cos\left(ln(4x^3)\right)$

$\frac{dy}{dx} = -\sin\left(ln(4x^3)\right)\ \frac{1}{4x^3}\ (12x^2)$

$\frac{dy}{dx} = -\frac{3\sin\left(ln(4x^3)\right)}{x}$

Find $^{dy}/_{dx}$

$y = 4^{4x^4}$

$\frac{dy}{dx} = 4^{4x^4}\,(ln4)(16x^3)$

$y = 3^{cosx^4}$

$\frac{dy}{dx} = 3^{cosx^4}\ (ln3)\ (-sinx^4)\ (4x^3)$

$\frac{dy}{dx} = -4x^3 3^{cosx^4} sinx^4 ln3$

3.8 Examples & Notes

$y = e^x$

is actualy

$$\frac{dy}{dx} = e^x \cdot \ln e \cdot (1) = e^x \cdot 1 \cdot 1 = \boxed{e^x}$$

find $\frac{dy}{dx}$

$y = \ln x^3$

$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{x^3} \cdot 3x^2 = \boxed{\frac{3}{x}}$$

or $y = \ln x^3 = 3 \ln x$

$$\frac{dy}{dx} = 3 \frac{1}{x} = \boxed{\frac{3}{x}}$$

$y = \cos(\ln(4x^3))$

$\frac{1}{u} \cdot \frac{du}{dx}$ you already have $\frac{du}{dx}$

$$\frac{dy}{dx} = -\sin(\ln(4x^3)) \cdot \left(\frac{1}{4x^3} \cdot 12x^2\right)$$

$$\boxed{\frac{dy}{dx} = \frac{-3\sin(\ln(4x^3))}{x}}$$

$y = 4^{4x^4}$ $\quad \frac{du}{dx} a^u = a^u \ln a \frac{du}{dx}$

$$\boxed{\frac{dy}{dx} = 4^{4x^4} \cdot \ln 4 \cdot 16x^3}$$

$y = 3^{\cos x^4}$

$$\frac{dy}{dx} = 3^{\cos x^4} \cdot \ln 3 \cdot (-\sin x^4)(3x^3)$$

$$\boxed{\frac{dy}{dx} = -4x^3 \cdot 3^{\cos x^4} \cdot \sin x^4 \ln 3}$$

$$y = \log_2(4x^5)$$

$$\frac{dy}{dx} = \frac{1}{4x^5 \ln 2} \cdot 20x^4$$

$$\frac{dy}{dx} = \frac{5x^4}{x^5 \ln 2} = \boxed{\frac{5}{x \ln 2}}$$

$$y = \log_6 3x^2$$

$$\frac{dy}{dx} = \frac{1}{3x^2} \frac{1}{\ln 6} \cdot 6x$$

$$\boxed{\frac{dy}{dx} = \frac{2}{x \ln 6}}$$

$$Y = \log_3 (3x^5+5)^5$$

$$\frac{dy}{dx} = \frac{5(3x^5+5)^4 \cdot 15x^4}{(3x^5+5)^5 \ln 3} = \boxed{\frac{75x^4}{(3x^5+5)\ln 3}}$$

Find $^{dy}/_{dx}$

$y = log_2 4x^5$

$\frac{dy}{dx} = \frac{1}{4x^5}\ \frac{1}{ln2}\ 20x^4$

$\frac{dy}{dx} = \frac{5}{xln2}$

$y = log_6 3x^2$

$\frac{dy}{dx} = \frac{1}{3x^2}\ \frac{1}{ln6}\ 6x$

$\frac{dy}{dx} = \frac{2}{xln6}$

$y = log_3(3x^5 + 5)^5$

$\frac{dy}{dx} = \frac{1}{(3x^5+5)^5}\ \frac{1}{ln3}\ 5(3x^5+5)^4\ (15x^4)$

$\frac{dy}{dx} = \frac{75x^4}{(3x^5+5)ln3}$

Variable to a variable power

Logarithmic Differentiation

This technique is useful in cases where it is easier to differentiate the logarithm of a function rather than the function itself.

$y = 2x^{2x}$ $\quad y = x^{\sin x}$ $\quad y = (1-x)^2(x+1)^4$

$y = 2x^{2x}$

$lny = \ln(2x^{2x})$

$lny = \ln 2 + \ln(x^{2x})$

$lny = \ln 2 + 2x\ln x$

$\frac{1}{y}\frac{dy}{dx} = 0 + 2x\frac{1}{x} + 2\ln x$

$\frac{1}{y}\frac{dy}{dx} = 2 + 2\ln x$

$\frac{dy}{dx} = y(2 + 2\ln x)$

$\frac{dy}{dx} = 2x^{2x}(2 + 2\ln x)$

$\boxed{\frac{dy}{dx} = 4x^{2x}(1 + \ln x)}$

Logarithmic Differentiation

$$y = x^{\sin x}$$

$$lny = \ln(x^{\sin x})$$

$$lny = \sin x \ \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x \ \frac{1}{x} + \cos x \ln x$$

$$\frac{dy}{dx} = y(\frac{1}{x}\sin x \ + \cos x \ln x)$$

$$\boxed{\frac{dy}{dx} = x^{\sin x}(\frac{1}{x}\sin x \ + \cos x \ln x)}$$

Logarithmic Differentiation

$$y = (1-x)^2(x+1)^4$$

$$lny = \ln((1-x)^2(x+1)^4)$$

$$lny = \ln(1-x)^2 + ln(x+1)^4$$

$$lny = 2\ln(1-x) + 4\ln(x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\frac{1}{1-x}(-1) + 4\frac{1}{x+1}$$

$$\frac{dy}{dx} = y\left(\frac{-2}{1-x} + \frac{4}{x+1}\right)$$

$$\frac{dy}{dx} = (1-x)^2(x+1)^4\left(\frac{-2}{1-x} + \frac{4}{x+1}\right)$$

$$y = (1-x)^2(x+1)^4$$

$$\ln y = \ln\left((1-x)^2(x+1)^4\right)$$

$$\ln y = \ln(1-x)^2 + \ln(x+1)^4$$

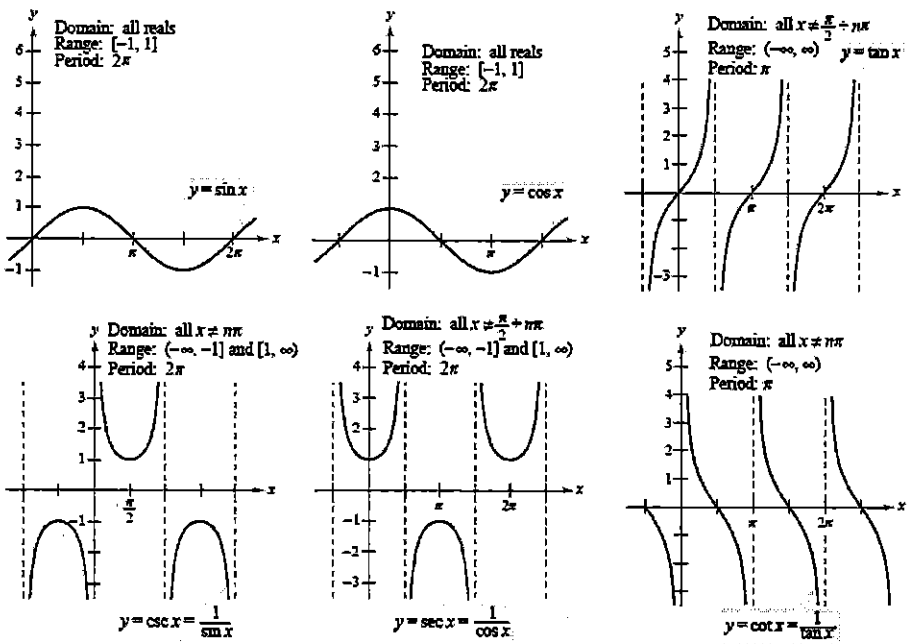
$$\ln y = 2\ln(1-x) + 4\ln(x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\cdot\left(\frac{1}{1-x}\right)(-1) + 4\cdot\left(\frac{1}{x+1}\right)(1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-2}{1-x} + \frac{4}{x+1}$$

$$\frac{dy}{dx} = y\left(\frac{-2}{1-x} + \frac{4}{x+1}\right)$$

$$\boxed{\frac{dy}{dx} = (1-x)^2(x+1)^4\left(\frac{-2}{1-x} + \frac{4}{x+1}\right)}$$



The graphs of the six trigonometric functions

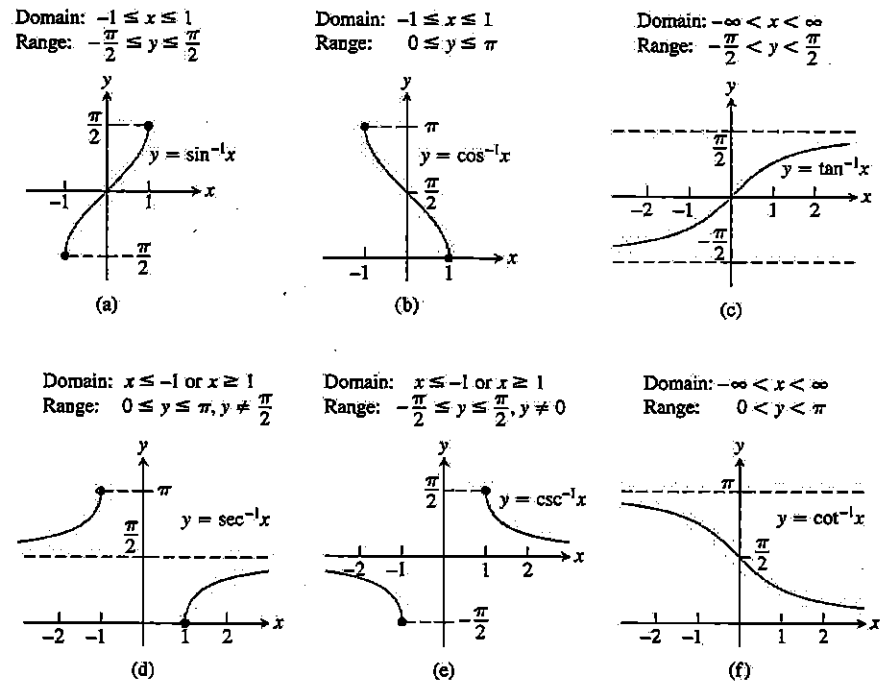


FIGURE 1.65 Graphs of the six basic inverse trigonometric functions.

$$\sin^{-1}\theta \neq \frac{1}{\sin\theta}$$

$$\sin^{-1}\theta = \arcsin\theta$$

TABLE 3.1 Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{du/dx}{1+u^2}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{du/dx}{1+u^2}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$

3.9 Notes

$y = \sin^{-1} x^2$

$D_x \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$ $\quad u = x^2$

$y' = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$

$\boxed{y' = \frac{2x}{\sqrt{1-x^4}}}$

$y = \tan^{-1}(3x)$ $\quad u = 3x$

$D_x \tan^{-1}(3x) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$

$D_x \tan^{-1}(3x) = \frac{1}{1+(3x)^2} \cdot 3$

$D_x \tan^{-1}(3x) = \frac{3}{1+9x^2}$

$y = \sec^{-1}(e^{2x})$ $\qquad$ $U = e^{2x}$

$$\frac{d}{dx} \sec^{-1} U = \frac{1}{|U|\sqrt{U^2-1}} \cdot \frac{dU}{dx}$$

$$y' = \frac{1}{e^{2x}\sqrt{e^{4x}-1}} \cdot 2e^{2x}$$

$$y' = \frac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}}$$

$$\boxed{y' = \frac{2}{\sqrt{e^{4x}-1}}}$$

$$y = \left(\sin^{-1}(5x^2)\right)^3 \qquad U = 5x^2$$

$$y' = 3\left(\sin^{-1}(5x^2)\right)^2 \cdot \frac{1}{\sqrt{1-25x^4}} \cdot 10x$$

$$y' = 3\left(\sin^{-1}(5x^2)\right)^2 \cdot \frac{10x}{\sqrt{1-25x^4}}$$

$$y' = \left(\sin^{-1}(5x^2)\right)^2 \cdot \frac{30x}{\sqrt{1-25x^4}}$$

$$\boxed{y' = \frac{30x\left(\sin^{-1}(5x^2)\right)^2}{\sqrt{1-25x^4}}}$$

$y = u \cdot v$

$$\frac{dy}{dx} = u' \cdot v + u \cdot v'$$

$y = x^2 \tan^{-1}(x^4)$ $\qquad u = x^4$

$$y' = (2x)(\tan^{-1}(x^4)) + (x^2)\left(\frac{1}{1+x^8} \cdot 4x^3\right)$$

$$\boxed{y' = 2x \tan^{-1} x^4 + \frac{4x^5}{1+x^8}}$$

Examples:

$y = sin^{-1}x^2$

$D_x sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

$u = x^2$

$y' = \frac{1}{\sqrt{1-(x^2)^2}} \quad 2x$

$y' = \frac{2x}{\sqrt{1-x^4}}$

$y = tan^{-1}(3x)$

$D_x tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$

$u = 3x$

$y' = \frac{1}{1+(3x)^2} \quad 3$

$y' = \frac{3}{1+9x^2}$

$y = sec^{-1}(e^{2x})$

$D_x sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

$u = e^{2x}$

$y' = \frac{1}{|e^{2x}|\sqrt{(e^{2x})^2-1}} \quad e^{2x} \quad 2$

$y' = \frac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}}$

$y' = \frac{2}{\sqrt{e^{4x}-1}}$

Examples:

$$y = \left(sin^{-1}(5x^2)\right)^3$$

$$y' = 3\left(sin^{-1}(5x^2)\right)^2 \frac{1}{\sqrt{1-(5x^2)^2}} \; 10x$$

$$y' = \frac{30x\left(sin^{-1}(5x^2)\right)^2}{\sqrt{1-25x^4}}$$

$$y = x^2 tan^{-1}(x^4)$$

$$y' = x^2 \frac{1}{1+(x^4)^2} \; 4x^3 \quad + \quad 2x \quad tan^{-1}(x^4)$$

$$y' = \frac{4x^5}{1+x^8} + 2x tan^{-1}(x^4)$$

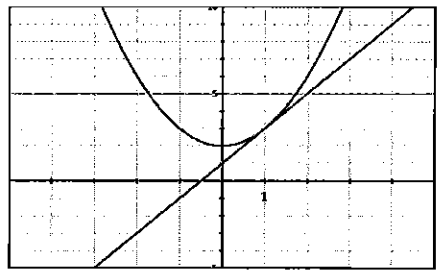
$y = x^2 + 2 \quad at\ x = 1 \quad y = 3$

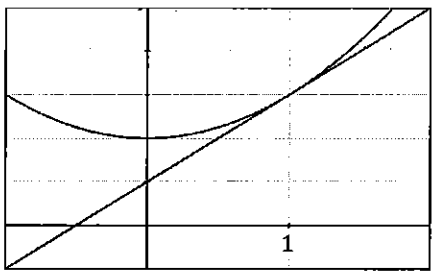
$\frac{dy}{dx} = 2x \quad at\ x = 1 \quad \frac{dy}{dx} = 2$

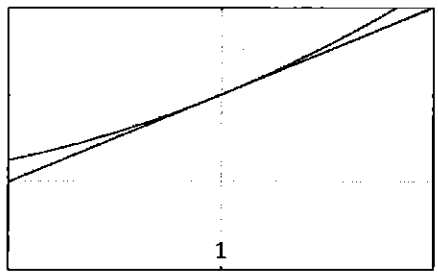
Equation of the tangent line at $x = 1$

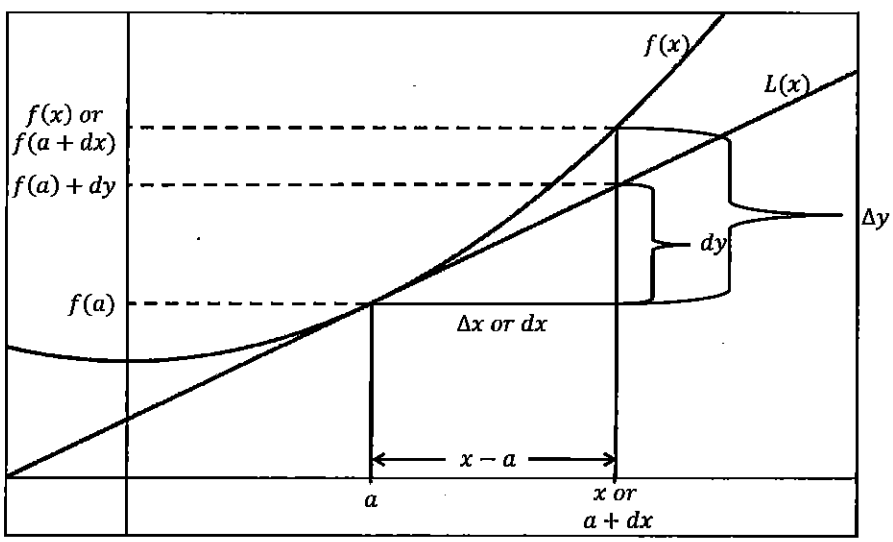
$$y - 3 = 2(x - 1)$$

$$y = 2x + 1$$









$$\frac{\Delta y}{\Delta x} = m_{sec}$$

$$\frac{dy}{dx} = f'(x) = m_{tan}$$

$$dy = f'(x)\,dx$$

Approximation function:

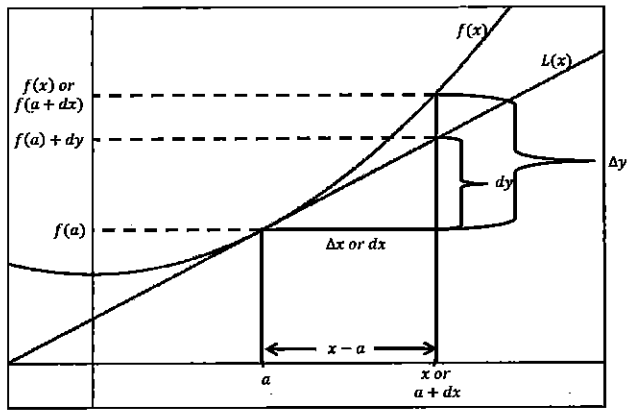
$$L(x)$$

$$L(x) = f(a) + dy$$

$$L(x) = f(a) + f'(a)\,dx$$

$$L(x) = f(a) + f'(a)\,(x - a)$$ *Linearization of* $f(x)$

$$L(x) \approx f(x)$$



DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

Example:

Find the linearization for $f(x) = \sqrt{x^2+16}$ *near* $x = -3$

$$f(x) = \sqrt{x^2+16} = (x^2+16)^{1/2}$$

$$f'(x) = \frac{1}{2}(x^2+16)^{-1/2}\ 2x$$

$$f'(x) = \frac{x}{\sqrt{x^2+16}}$$

$$L(x) = f(a) + f'(a)\,(x-a)$$

$$L(x) = f(-3) + f'(-3)\,(x-(-3))$$

$$L(x) = \sqrt{(-3)^2+16} + \frac{-3}{\sqrt{(-3)^2+16}}\ (x+3)$$

$$L(x) = 5 - \frac{3}{5}(x+3) \qquad \textit{Linearization of } f(x)$$

Find the linear approximation for $f(x) = \sqrt{x^2+16}$ *at* $x = -2.95, -3.05$ *and* -4.

$$L(x) = 5 - \frac{3}{5}(x+3)$$

$$L(-2.95) = 5 - \frac{3}{5}(-2.95+3)$$

$$L(-2.95) = 4.97$$

$$f(-2.95) = 4.97016$$

$$L(-3.05) = 5 - \frac{3}{5}(-3.05+3)$$

$$L(-3.05) = 5.03$$

$$f(-3.05) = 5.03016$$

$$L(-4) = 5 - \frac{3}{5}(-4+3)$$

$$L(-4) = 5.6$$

$$f(-4) = 5.65685$$

Example:

Find the linearization for $f(x) = \tan x$ *near* $x = \frac{\pi}{4}$

$f(x) = \tan x$

$f'(x) = sec^2 x$

$L(x) = f(a) + f'(a)\,(x - a)$

$L(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$

$L(x) = \tan\frac{\pi}{4} + sec^2\frac{\pi}{4}\left(x - \frac{\pi}{4}\right)$

$L(x) = 1 + \frac{1}{cos^2\frac{\pi}{4}}\left(x - \frac{\pi}{4}\right)$

$L(x) = 1 + 2\left(x - \frac{\pi}{4}\right)$ *Linearization of* $f(x)$

Find the linear approximation for $f(x) = \tan x$ *at* $x = \frac{2\pi}{9}, \frac{5\pi}{18}$ *and* $\frac{\pi}{3}$.

$L(x) = 1 + 2\left(x - \frac{\pi}{4}\right)$

$L\left(\frac{2\pi}{9}\right) = 1 + 2\left(\frac{2\pi}{9} - \frac{\pi}{4}\right)$ $L\left(\frac{5\pi}{18}\right) = 1 + 2\left(\frac{5\pi}{18} - \frac{\pi}{4}\right)$ $L\left(\frac{\pi}{3}\right) = 1 + 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$L\left(\frac{2\pi}{9}\right) = 0.82547$ $L\left(\frac{5\pi}{18}\right) = 1.17453$ $L\left(\frac{\pi}{3}\right) = 1.5236$

$f\left(\frac{2\pi}{9}\right) = 0.8391$ $f\left(\frac{5\pi}{18}\right) = 1.19175$ $f\left(\frac{\pi}{3}\right) = 1.73205$

Example:

Use linearization to approximate the value of $\sqrt{99.4}$

$f(x) = \sqrt{x} \ = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Choose a value of x near 99.4.

$x = 100$

$L(x) = f(a) + f'(a)\,(x - a)$

$L(x) = f(100) + f'(100)\,(99.4 - 100)$

$L(x) = \sqrt{100} \ + \frac{1}{2\sqrt{100}} \ (99.4 - 100)$

$L(x) = 10 + \frac{1}{20}(-.6)$

$L(x) = 10 - 0.03 = 9.97$ *approximate value of* $\sqrt{99.4}$

actual value of $\sqrt{99.4}$

$f(99.4) = \sqrt{99.4} = 9.96995$

Differentials $\frac{dy}{dx} = f'(x)$ $dy = f'(x)\,dx$

Examples:

$y = e^{3x}$

$dy = e^{3x} \ 3dx$

$dy = 3e^{3x}dx$

$y = \cos x^2$

$dy = -\sin x^2 \ 2xdx$

$dy = -2x\sin x^2 dx$

$xy^2 - 4x^{3/2} - y = 0$

$x2ydy - 6x^{1/2}dx \ - dy = 0$

$2xydy - dy = 6x^{1/2}dx$

$dy(2xy - 1) = 6x^{1/2}dx$

$dy = \frac{6x^{1/2}}{2xy - 1}dx$

$dy = \frac{6\sqrt{x}}{2xy - 1}dx$

3.11 Notes

Find the linearization ($L(x)$) for

$f(x) = \sqrt{x^2+16}$ near $x = -3$

$f'(x) = \frac{1}{2}(x^2+16)^{-\frac{1}{2}} \cdot 2x$

$f'(x) = \frac{x}{\sqrt{x^2+16}}$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = f(-3) + f'(-3)(x-(-3))$

$L(x) = \sqrt{(-3)^2+16} + \frac{-3}{\sqrt{(-3)^2+16}} \cdot (x+3)$

$$\boxed{L(x) = 5 - \frac{3}{5}(x+3)}$$

Linearization of $f(x) = \tan x$ near $x = \frac{\pi}{4}$

$f'(x) = \sec^2 x$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = \tan\left(\frac{\pi}{4}\right) + \frac{1}{\cos^2 \frac{\pi}{4}}\left(x - \frac{\pi}{4}\right)$

$$\boxed{L(x) = 1 + 2 \cdot \left(x - \frac{\pi}{4}\right)}$$

Find linear approximation at $x = \frac{2\pi}{9}, \frac{5\pi}{18}, \frac{\pi}{3}$

$L\left(\frac{2\pi}{9}\right) = 1 + 2\left(\frac{2\pi}{9} - \frac{\pi}{4}\right) = .82547$

$L\left(\frac{5\pi}{18}\right) = 1.1745$

$L\left(\frac{\pi}{3}\right) = 1.5236$

Linearization to approximate the value of $\sqrt{99.4}$ or $f(x) = \sqrt{x}$ near $x = 99.4$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Choose a value near 99.4 for x

$x = 100$

$$L(x) = f(100) + f'(100)(99.4 - 100)$$

$$L(x) = 10 + \frac{1}{20}(-.6)$$

$$\boxed{L(x) = 10 - 0.03 = 9.97}$$

approximation value of $\sqrt{99.4}$

Differentials

Ex: $y = \cos x^2$

$$dy = -\sin x^2 \cdot 2x\,dx$$
$$dy = -2x \sin x^2\,dx$$

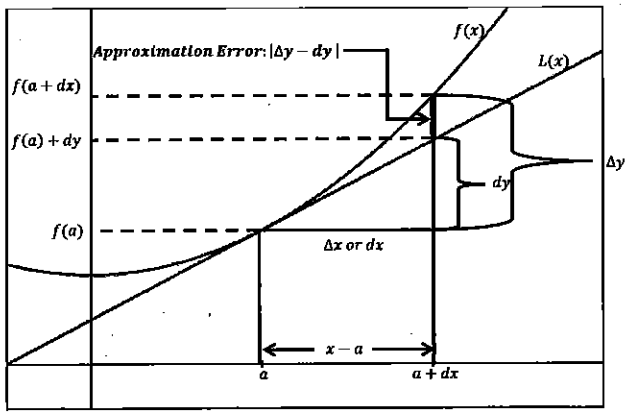
$$xy^2 - 4x^{3/2} - y = 0$$

$$x2y\,dy + dx\,y^2 - 6x^{\frac{1}{2}}dx - dy = 0$$

$$dy\,(2xy - 1) = 6x^{\frac{1}{2}}dx - y^2 dx$$

$$\boxed{dy = \frac{6x^{\frac{1}{2}} - y^2}{2x - 1}\,dx}$$

Approximation Error: $|\Delta y - dy|$



Approximation Error: $|\Delta y - dy|$

Example:

Find Δy, dy, and the approximation error.

$$f(x) = 2x^2 + 4x - 3 \ \ for\ x = 1\ and\ dx = 0.1$$

$\Delta y = f(1 + 0.1) - f(1)$

$\Delta y = f(1.1) - f(1)$

$\Delta y = 3.82 - 3$

$\Delta y = 0.82$

$f(x) = y = 2x^2 + 4x - 3$

$dy = 4xdx + 4dx$

$dy = (4x + 4)dx$

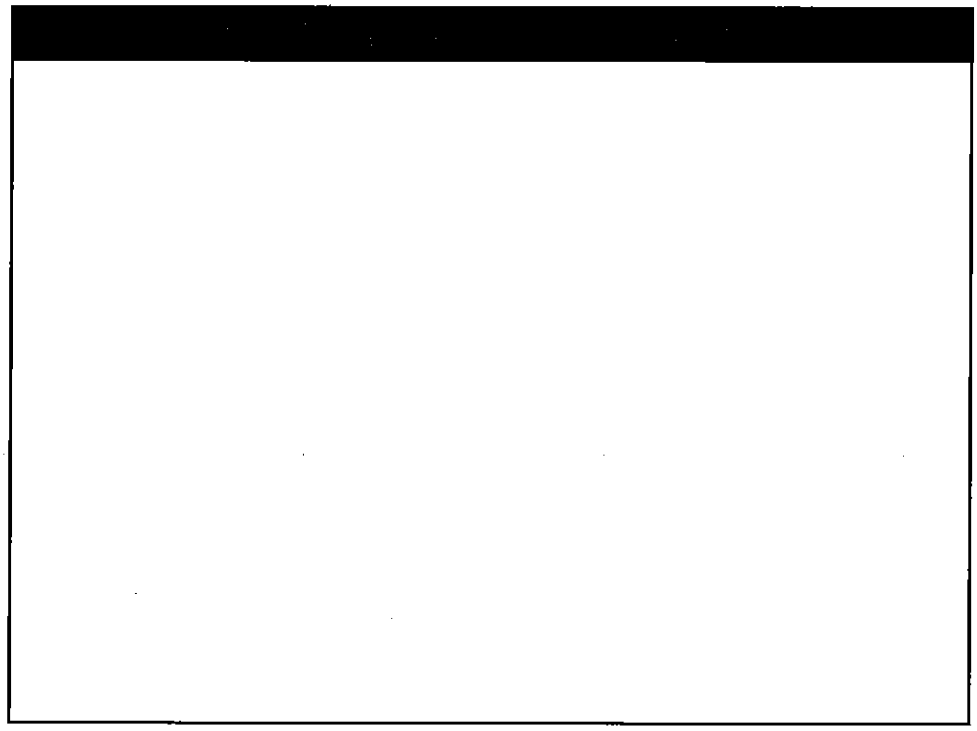
$dy = (4(1) + 4)(0.1)$

$dy = 0.8$

$|\Delta y - dy|$

$|0.82 - 0.8|$

Approximation Error: ~~0.2~~ 0.02



Related rates problems involve finding a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known.

1. Read the problem, pull out essential information and identify a formula to be used.
2. Sketch a diagram if possible.
3. Write down any known rate of change & the rate of change you are looking for.
4. Be careful with signs…if the amount is decreasing, the rate of change is negative.
5. Pay attention to whether quantities are constant or varying.
6. Set up an equation involving the appropriate quantities.
7. Differentiate with respect to t using implicit differentiation.
8. Plug in known items (you may need to find some quantities using geometry).
9. Solve for the item you are looking for, most often this will be a rate of change.
10. State your final answer with the appropriate units.

Related rates problems involve finding a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known.

Example

1) $If x = y^3 - y \text{ and } \frac{dy}{dt} = 5, \text{then what is the value of } \frac{dx}{dt} \text{ when } y = 2?$

$$x = y^3 - y$$

$$\frac{dx}{dt} = 3y^2\frac{dy}{dt} - \frac{dy}{dt}$$

$$\frac{dx}{dt} = 3(2)^2(5) - 5$$

$$\frac{dx}{dt} = 55$$

Example

2) $If\, r + s^2 + v^3 = 12, \frac{dr}{dt} = 4, and \frac{ds}{dt} = -3, then\ what\ is\ the\ value\ of \frac{dv}{dt} when\ r = 3, s = 1, and\ v = 4?$

$$r + s^2 + v^3 = 12$$

$$\frac{dr}{dt} + 2s\frac{ds}{dt} + 3v^2\frac{dv}{dt} = 0$$

$$4 + 2(1)(-3) + 3(4)^2\frac{dv}{dt} = 0$$

$$48\frac{dv}{dt} = 2$$

$$\frac{dv}{dt} = 0.042$$

Example

3) The radius r and the height h of a right circular cone are related to the cone's volume V by the equation $V = \frac{1}{3}\pi r^2 h$.

How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if r is constant?

$$V = \frac{1}{3}r^2 h$$

$$\frac{dV}{dt} = \frac{1}{3}r^2\frac{dh}{dt}$$

How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ if h is constant?

$$V = \frac{1}{3}r^2 h$$

$$\frac{dV}{dt} = \frac{1}{3}2r\frac{dr}{dt}h$$

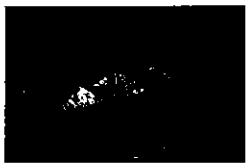
$$\frac{dV}{dt} = \frac{2}{3}hr\frac{dr}{dt}$$

How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ & $\frac{dr}{dt}$ if h and r are not constant?

$$V = \frac{1}{3}r^2 h$$

$$\frac{dV}{dt} = \frac{1}{3}r^2\frac{dh}{dt} + \frac{1}{3}2r\frac{dr}{dt}h$$

4) An oil tanker strikes an iceberg and a hole is ripped open on its side. Oil is leaking out in a near circular shape. The radius of the oil spill is changing at a rate of 1.5 miles per hour. How fast is the area of the oil spill changing when the radius is 0.6 mile?



$$A = \pi r^2$$

$$\frac{dr}{dt} = 1.5\ mi/hr$$

$$Find\ \frac{dA}{dt}\ when\ r = 0.6\ mi$$

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi 2r\frac{dr}{dt}$$

$$\frac{dA}{dt} = \pi 2(0.6)(1.5)$$

$$\frac{dA}{dt} = 5.655\ \ mi^2/hr$$

5) A balloon is being inflated at a rate of 10 cubic centimeters per second. How fast is the radius of a spherical balloon changing at the instant the radius is 5 centimeters?

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 10\ cm^3/hr$$

$$Find\ \frac{dr}{dt}\ when\ r = 5\ cm$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt}$$

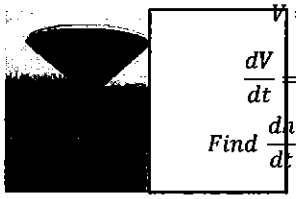
$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$10 = 4\pi(5)^2\frac{dr}{dt}$$

$$\frac{10}{4\pi(5)^2} = \frac{dr}{dt}$$

$$\frac{dr}{dt} = 0.032\quad cm/sec$$

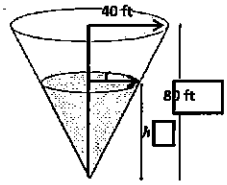
6) A water authority is filling an inverted conical water storage tower at a rate of 9 cubic feet per minute. The height of the tank is 80 feet and the radius at the top is 40 feet. How fast is the water level inside the tank changing when the water level is 60 feet deep?



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = 9\ ft^3/min$$

$$Find\ \frac{dh}{dt}\ when\ h = 60\ ft$$



$$not\ given\ \frac{dr}{dt}$$

find a relationship with r in terms of h.

use similar triangles

$$\frac{r}{h} = \frac{40}{80} \qquad r = \frac{1}{2}h$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$

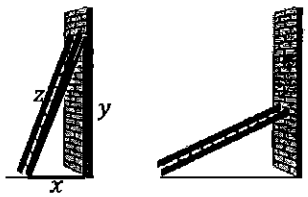
$$V = \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{12}\pi 3h^2\frac{dh}{dt}$$

$$9 = \frac{\pi}{4}(60)^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.003\ \ ft/min$$

7) A 25-foot ladder is leaning against a wall. The bottom of the ladder is being pulled away from the wall at a rate of 3 feet per second. How fast is the top of the ladder moving at the instant the bottom of the ladder is 15 feet away from the wall?



$$x^2 + y^2 = z^2$$

$$z = 25\ ft$$

$$\frac{dx}{dt} = 3\ fps$$

$$Find\ \frac{dy}{dt}\ when\ x = 15\ ft$$

$$x^2 + y^2 = z^2$$

$$x^2 + y^2 = 25^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(15)(3) + 2y\frac{dy}{dt} = 0$$

find y when x = 15 and z = 25.

$$15^2 + y^2 = 25^2$$

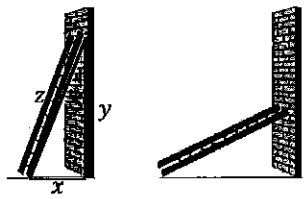
$$y = 20$$

$$2(15)(3) + 2(20)\frac{dy}{dt} = 0$$

$$40\frac{dy}{dt} = -90$$

$$\frac{dy}{dt} = -2.25\ \ fps$$

8) A 25-foot ladder is leaning against a wall. The bottom of the ladder is being pulled away from the wall at a rate of 3 feet per second. How fast is the top of the ladder moving at the instant the bottom of the ladder is 15 feet away from the wall? At the same instant, what is the rate of change of the area between the ground, the wall, and the ladder?



$$A = \frac{1}{2}bh = \frac{1}{2}xy$$

$x = 15\,ft \quad y = 20\,ft$

$\frac{dx}{dt} = 3\,fps \quad \frac{dy}{dt} = -2.25\,fps$

$Find\ \frac{dA}{dt}\ at\ that\ instant.$

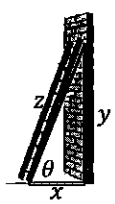
$$A = \frac{1}{2}xy$$

$$\frac{dA}{dt} = \frac{1}{2}x\frac{dy}{dt} + \frac{1}{2}\frac{dx}{dt}y$$

$$\frac{dA}{dt} = \frac{1}{2}(15)(-2.25) + \frac{1}{2}(3)(20)$$

$$\frac{dA}{dt} = 13.125\ ft^2/sec$$

9) A 25-foot ladder is leaning against a wall. The bottom of the ladder is being pulled away from the wall at a rate of 3 feet per second. How fast is the top of the ladder moving at the instant the bottom of the ladder is 15 feet away from the wall? At the same instant, what is the rate of change of the angle (θ) between the ground and the ladder?



$\tan\theta = \frac{y}{x}$

$x = 15\,ft \quad y = 20\,ft$

$\frac{dx}{dt} = 3\,fps \quad \frac{dy}{dt} = -2.25\,fps$

$Find\ \frac{d\theta}{dt}\ at\ that\ instant.$

$$\tan\theta = \frac{y}{x}$$

$$sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - \frac{dx}{dt}y}{x^2}$$

$$\frac{1}{cos^2\theta}\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - \frac{dx}{dt}y}{x^2}$$

$$\frac{1}{cos^2\theta}\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - \frac{dx}{dt}y}{x^2}$$

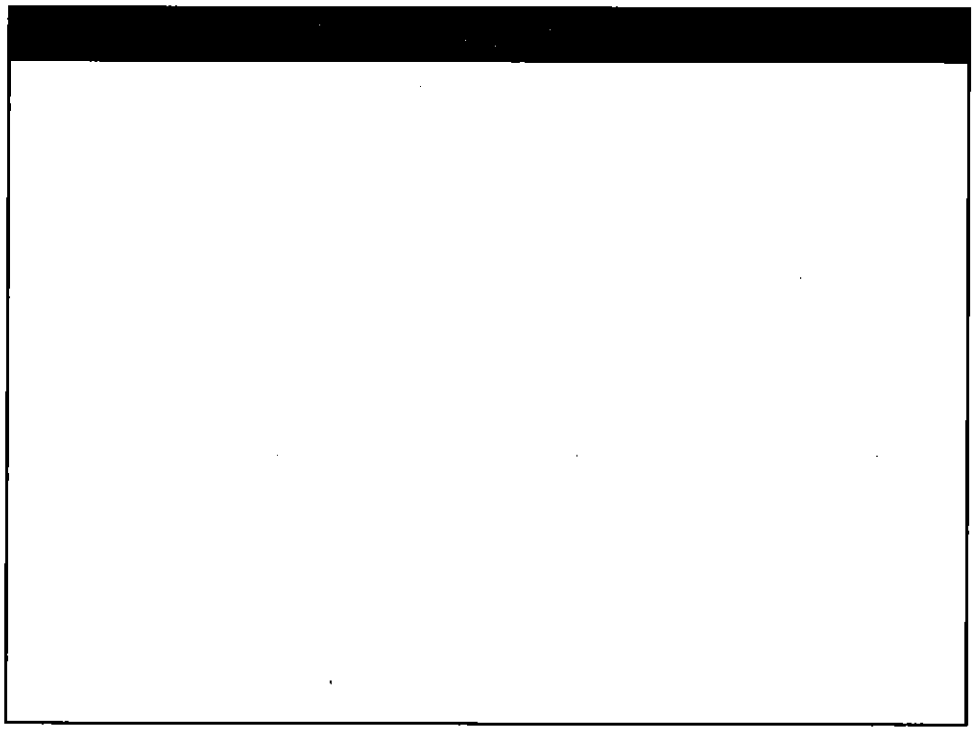
$find\ \theta\ when$
$x = 15\ and\ y = 20$

$$\tan\theta = \frac{20}{15}$$

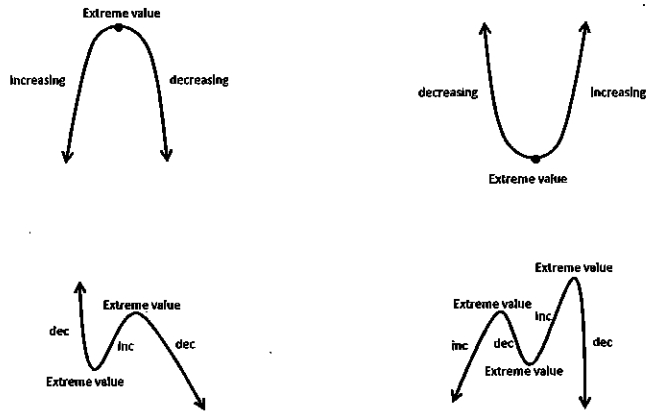
$$\theta = tan^{-1}\frac{4}{3} = 0.927\ rad.$$

$$\frac{1}{cos^2(0.927)}\frac{d\theta}{dt} = \frac{(15)(-2.25) - (3)(20)}{15^2}$$

$$\frac{d\theta}{dt} = -0.150\ \ rad/sec$$



Extreme Values of a function are created when the function changes from increasing to decreasing or from decreasing to increasing



Extreme Value or Extrema

Classifications of Extreme Values

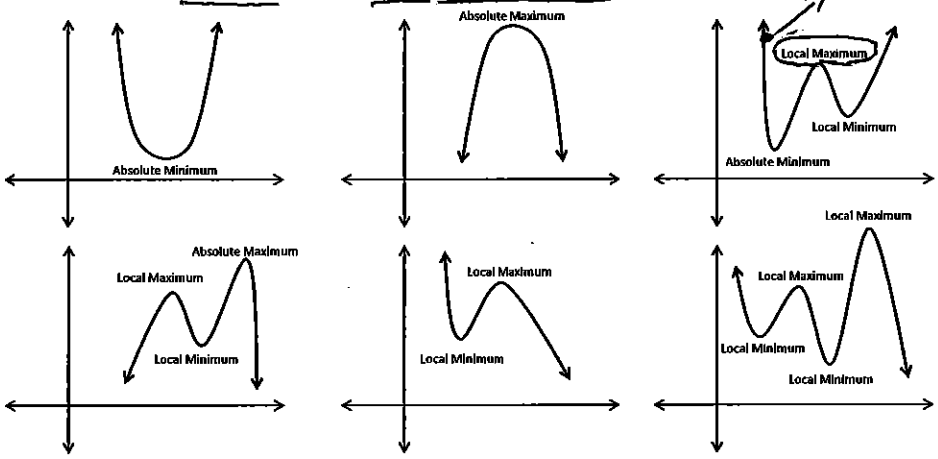
Absolute Minimum – the smallest function value in the domain

Absolute Maximum – the largest function value in the domain

Local Minimum – the smallest function value in an open interval in the domain

Local Maximum – the largest function value in an open interval in the domain

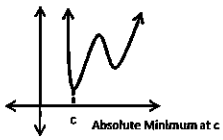
higher value here



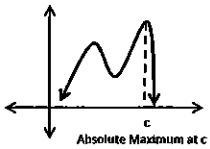
Definitions:

c = any number

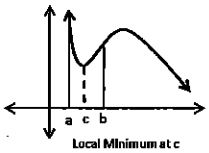
Absolute Minimum – occurs at a point c if $f(c) \leq f(x)$ for x all values in the domain.



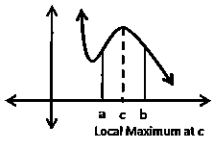
Absolute Maximum – occurs at a point c if $f(c) \geq f(x)$ for all x values in the domain.



Local Minimum – occurs at a point c in an open interval, (a, b), in the domain if $f(c) \leq f(x)$ for all x values in the open interval.



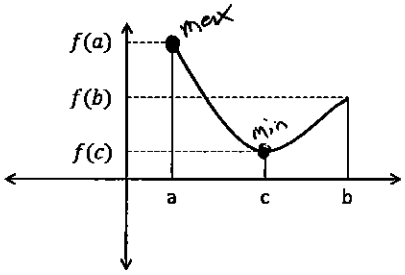
Local Maximum – occurs at a point c in an open interval, (a, b), in the domain if $f(c) \geq f(x)$ for all x values in the open interval.



The Extreme Value Theorem (Max-Min Existence Theorem)

If a function is continuous on a closed interval, [a, b], then the function will contain both an absolute maximum value and an absolute minimum value.

will have

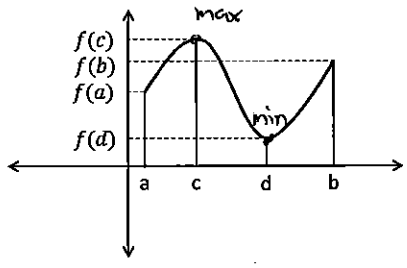


Absolute maximum value: f(a)

Absolute minimum value: f(c)

The Extreme Value Theorem (Max-Min Existence Theorem)

If a function is continuous on a closed interval, [a, b], then the function will contain both an absolute maximum value and an absolute minimum value.

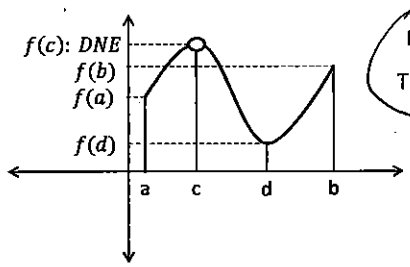


Absolute maximum value: f(c)

Absolute minimum value: f(d)

The Extreme Value Theorem (Max-Min Existence Theorem)

If a function is continuous on a closed interval, [a, b], then the function will contain both an absolute maximum value and an absolute minimum value.



F is not continuous at c.

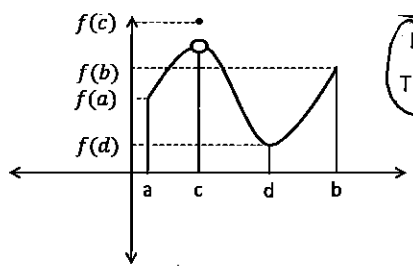
Theorem does not apply.

Absolute maximum value: none

Absolute minimum value: f(d)

The Extreme Value Theorem (Max-Min Existence Theorem)

If a function is continuous on a closed interval, [a, b], then the function will contain both an absolute maximum value and an absolute minimum value.



F is not continuous at c.

Theorem does not apply.

but...

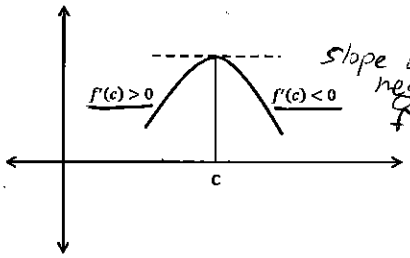
Absolute maximum value: f(c)

Absolute minimum value: f(d)

The value exists, it's just not continuous.

The First Derivative Theorem for Local Extreme Values

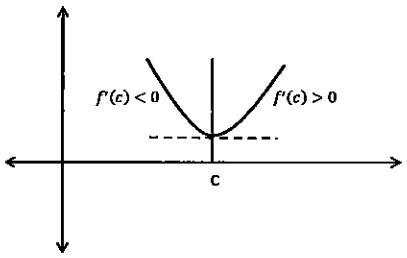
If a function has a local maximum or minimum value at a point (c) in the domain and the derivative is defined at that point, then $f'(c) = 0$.



slope of neg so function is decreasing

Slope of the tangent line at c is zero.

$f'(c) = 0$



Critical Points

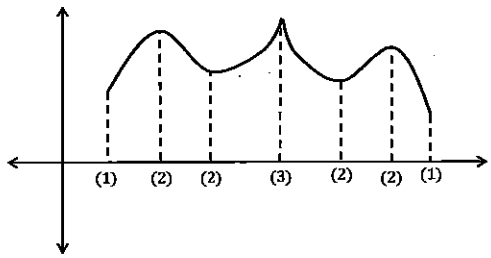
If a function has an extreme value, then the value of the domain at which it occurs is defined as a critical point.

Three Types of Critical Points

(1) *Endpoints of an interval*

(2) *Stationary Points*: $f'(c) = 0$

(3) *Singular Points*: $f'(c)$ *does not exist* = having zero in the denominator



Which table best describes the graph?

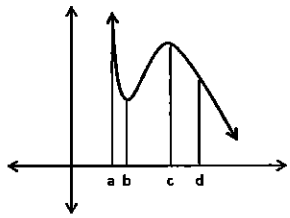


Table A

x	$f'(x)$
a	27
b	0
c	0
d	-5

Table B

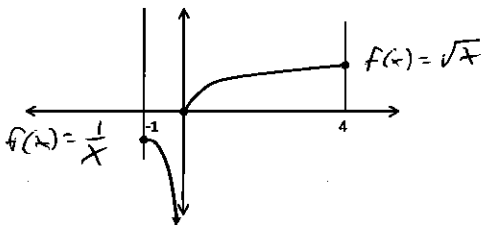
x	$f'(x)$
a	-30
b	5
c	0
d	-7

Table C

x	$f'(x)$
a	-22
b	0
c	0
d	-9

Graph the function. State the location(s) of any absolute extreme values, if applicable. Does the Extreme Value Theorem apply?

$$f(x) = \begin{cases} \frac{1}{x} & if\ -1 \le x < 0 \\ \sqrt{x} & if\ 0 \le x \le 4 \end{cases}$$



No absolute minimum

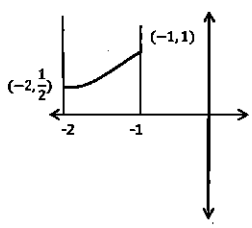
Absolute maximum at x = 4

The Extreme Value Theorem does not apply

The function is not continuous at x = 0.

Graph the function. Calculate any absolute extreme values, if applicable. Plot them on the graph and state the coordinates.

$$f(x) = -\frac{1}{x} \qquad -2 \le x \le -1$$



$$f(x) = -x^{-1}$$

$$f'(x) = x^{-2} = \frac{1}{x^2}$$

$$f'(x) \ne 0$$

$$f'(x)\ is\ undefined\ at\ x = 0$$

$$x = 0\ is\ not\ a\ critical\ point;\ \ not\ in\ [-2,-1]$$

Critical points

$$x = -2, -1$$

$f(-2) = \frac{1}{2}$ Absolute minimum

$f(-1) = 1$ Absolute maximum

nDeriv(Y1, X, X)

Calculate any absolute extreme values. State their identities and coordinates.

$$f(x) = \frac{x+1}{x^2+2x+2}$$

$$f'(x) = \frac{(x^2+2x+2)(1) - (x+1)(2x+2)}{(x^2+2x+2)^2}$$ Quotient Rule

$$f'(x) = \frac{-x^2-2x}{(x^2+2x+2)^2} = \frac{-x(x+2)}{(x^2+2x+2)^2}$$

Critical points

$f'(x) = 0$ Set numerator $= 0$

$x = -2, 0$

Is $f'(x)$ undefined?

$x^2 + 2x + 2 = 0$ Set denom $= 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)}$$

no real solutions because of negative under radical

$f(-2) = -0.5$ Absolute minimum

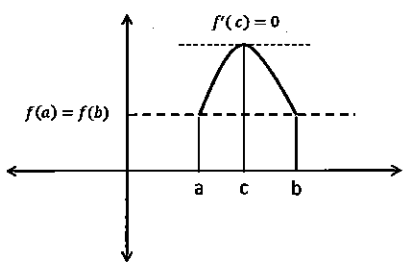
$(-2, -0.5)$

$f(0) = 0.5$ Absolute maximum

$(0, 0.5)$

Rolle's Theorem

A function is given that is continuous on every point of a closed interval,[a, b], and it is differentiable on every point of the open interval (a, b). If $f(a) = f(b)$, then there exists at least one value in the open interval,(a, b), where $f'(c) = 0$.



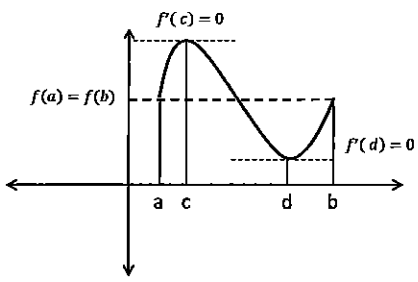
$$slope_{segment\ ab} = \frac{f(b) - f(a)}{b - a} = \frac{0}{b - a} = 0$$

$$slope_{tangent\ @\ c} = 0$$

$$f'(c) = 0$$

Rolle's Theorem

A function is given that is continuous on every point of a closed interval,[a, b], and it is differentiable on every point of the open interval (a, b). If $f(a) = f(b)$, then there exists at least one value in the open interval,(a, b), where $f'(c) = 0$.



$$slope_{segment\ ab} = \frac{f(b) - f(a)}{b - a} = \frac{0}{b - a} = 0$$

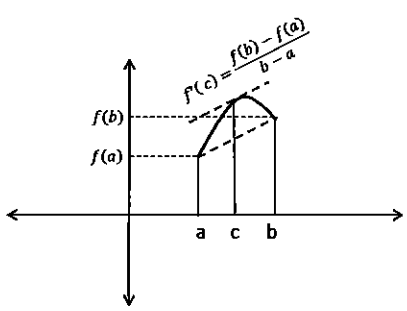
$$slope_{tangent\ @\ c} = 0 \qquad slope_{tangent\ @\ d} = 0$$

$$f'(c) = 0 \qquad f'(d) = 0$$

The Mean Value Theorem

A function is given that is continuous on every point of a closed interval,[a, b], and it is differentiable on every point of the open interval (a, b). If $f(a) = f(b)$, then there exists at least one value (c) in the open interval,(a, b), where

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$



$$slope_{segment\ ab} = \frac{f(b) - f(a)}{b - a}$$

$$slope_{tangent\ @\ c} = f'(c)$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The Mean Value Theorem

A function is given that is continuous on every point of a closed interval,[a, b], and it is differentiable on every point of the open interval (a, b). If $f(a) = f(b)$, then there exists at least one value (c) in the open interval,(a, b), where

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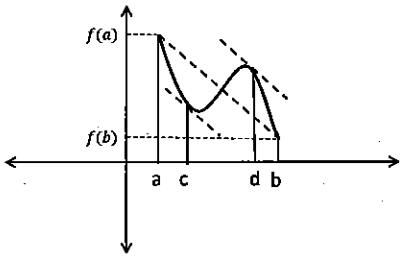


$$slope_{segment\ ab} = \frac{f(b)-f(a)}{b-a}$$

$$slope_{tangent\ @\ c} = f'(c)$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$slope_{tangent\ @\ d} = f'(d)$$

$$f'(d) = \frac{f(b)-f(a)}{b-a}$$

Find the values of *x* that satisfy the Mean Value Theorem: $f(x) = \sqrt{x-1}$ $[1, 3]$

$$\frac{f(b)-f(a)}{b-a} = f'(x)$$

$$slope\ _{secant\ line} = \frac{f(3)-f(1)}{3-1}$$

$$slope\ _{secant\ line} = \frac{\sqrt{2}}{2}$$

$$f(x) = (x-1)^{1/2}$$

$$f'(x) = \frac{1}{2}(x-1)^{-1/2}(1)$$

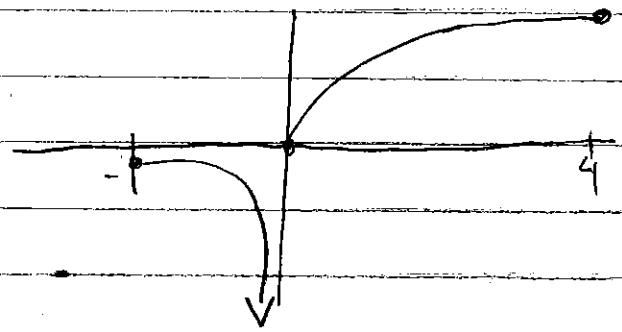
$$f'(x) = \frac{1}{2(x-1)^{1/2}} = \frac{1}{2\sqrt{x-1}}$$

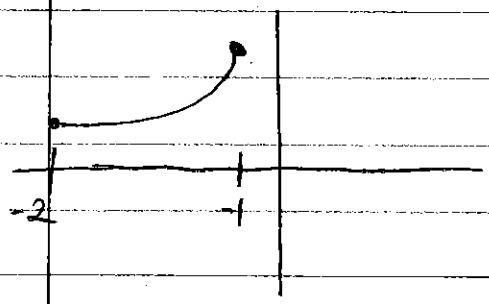
$$\frac{\sqrt{2}}{2} = \frac{1}{2\sqrt{x-1}} \rightarrow \sqrt{2}2\sqrt{x-1} = 2$$

$$\sqrt{2}\sqrt{x-1} = 1 \rightarrow \sqrt{x-1} = \frac{1}{\sqrt{2}}$$

$$x-1 = \frac{1}{2} \rightarrow x = \frac{3}{2}$$

$$f(x) \begin{cases} \frac{1}{x} \\ \sqrt{x} \end{cases}$$





$f(x) = 3x^4 - 16x^3 + 18x^2$

$f'(x) = 12x^3 - 48x^2 + 36x$

$f'(x) = 0$

$f'(x) = 12x^3 - 48x^2 + 36x = 0$

$12x(x^2 - 4x + 3)$

$x = 0 \qquad (x-3)(x-1)$

$x = 3$

$x = 1$

$f(-2) = 248$

$f(1) = 5$

$f(0) = 0$

$f(2) = -8$

$\cancel{f(3)}$ = not in domain

$f(-2) = 248$ absolute max

$f(2) = -8$ absolute min

$f(\theta) = \theta^2 + \sin\theta \qquad \left[-\frac{\pi}{3}, \frac{\pi}{2}\right]$

$f'(\theta) = 2\theta + \cos\theta$

$2\theta + \cos\theta = 0 \qquad f\left(-\frac{\pi}{3}\right) = .231$

$\theta = -.450$

$f(-.450) = -.232$ Abs Min

$f\left(\frac{\pi}{2}\right) = 3.467$ Absolute Max

$f(x) = 5 - \frac{4}{x} \qquad [1, 4]$

$\frac{f(4)-f(1)}{4-1} = \frac{4-1}{4-1} = 1$

$f'(x) = 0 + \frac{4}{x^2} = 4x^{-2}$

So... $4x^{-2} = 1 \qquad \boxed{x = \pm 2}$

$\frac{4}{x^2} = 1 \qquad x = -2$ not in domain

$f(\theta) = \sin^2(2\theta) \qquad \left[\frac{\pi}{3}, \frac{5\pi}{6}\right]$

$f'(\theta) = 2(\sin 2\theta)(\cos 2\theta) \cdot 2$

$f'(\theta) \quad 4(\sin 2\theta)(\cos 2\theta) = 0$

$\sin 2\theta = 0 \qquad \cos 2\theta = 0 \qquad$ Let $2\theta = x$

$\sin x = 0 \qquad \cos x = 0$

$x = \sin^{-1} 0 \qquad x = \cos^{-1} 0$

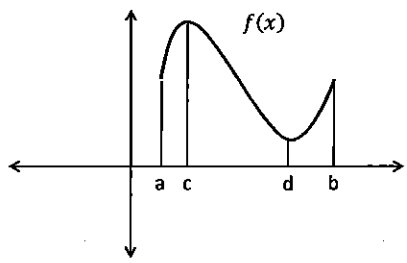
$x = 0, \pi, 2\pi, 3\pi \qquad x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$2\theta = \ldots \qquad 2\theta = \ldots$

$\theta = 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi, 2\pi \qquad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Monotonicity – defines where a function is increasing or decreasing.

A function is monotonic if it is increasing or decreasing on an interval.

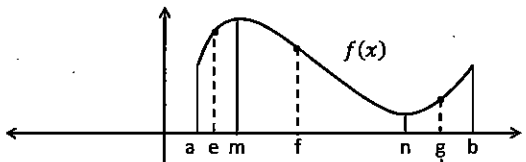


Monotonicity of $f(x)$	
Interval	Increasing/Decreasing
(a, c)	*increasing*
(c, d)	*decreasing*
(d, b)	*increasing*

The First Derivative Test

A function $f(x)$ is continuous on an open interval containing critical point(s). If $f(x)$ is differentiable on the interval, except possibly at the critical points, then $f(x)$ at the critical point(s) can be classified as follows:

1. Local Maximum if $f'(x)$ changes from positive to negative at m.
2. Local Minimum if $f'(x)$ changes from negative to positive at n.
3. If there is no sign change, then the critical point is not a local minimum or maximum.



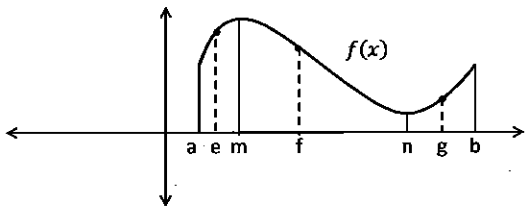
Pick point to Left M

Pick Point Right of M

Pick point right of n

+ test to check the change in signs

The First Derivative Test



Test Point	f'(test point)	f'(x)	Inc/Dec	f(critical point)	Extrema
e	$f'(e) = +$	$f'(x) > 0$	Inc.		
				$f(m)$	*local max*
f	$f'(f) = -$	$f'(x) < 0$	Dec.		
				$f(n)$	*local min*
g	$f'(g) = +$	$f'(x) > 0$	Inc.		

Example Problems

$$f(x) = x^3 - 6x - 3$$

$$f(x) = x^4 - 6x^2 + 2$$

$$f(x) = x(lnx)^2$$

$$f(x) = x^3 - 4x^2 + 3sinx$$

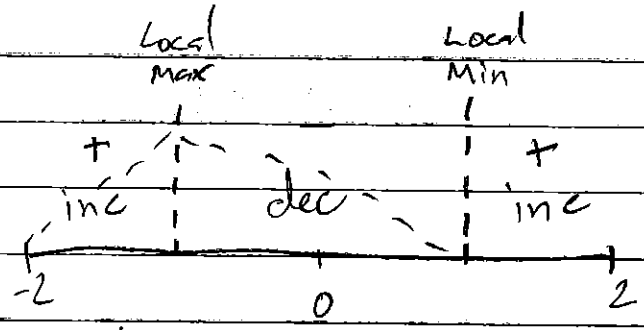
4.3 Notes + Examples

1) $f(x) = x^3 - 6x - 3$

$f'(x) = 3x^2 - 6$

$3x^2 - 6 = 0$

$x^2 = 2 \quad x = \pm\sqrt{2}$



test points -2, 0, 2 @ derivative

Now find their values by plugging the local Max + Min into the original function

$f(-\sqrt{2}) = 2.657$ Max

$f(\sqrt{2}) = -8.657$ Min

increasing : $f'(x) > 0$: $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

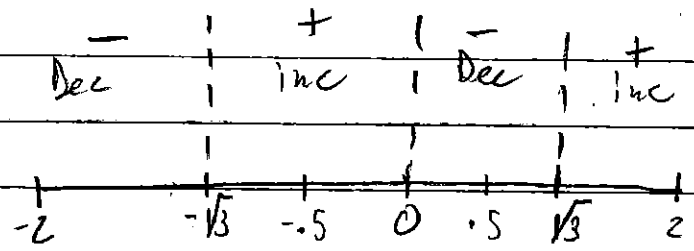
decreasing : $f'(x) < 0$: $(-\sqrt{2}, \sqrt{2})$

2) $f(x) = x^4 - 6x^2 + 2$

$f'(x) = 4x^3 - 12x$

$4x^3 - 12x = 0$

$4x(x^2 - 3) = 0$



$x = 0$ & $x = \pm\sqrt{3}$

$f(-\sqrt{3})$ -7 local Min + absolute min

$f(0) = 2$ local Max

$f(\sqrt{3}) = -7$ local Min absolute Max

increasing : $f'(x) > 0$ $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

Decreasing : $f'(x) < 0$ $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

e is inverse of $\ln$

so $\ln x$ can be cancelled out by $e^{(\ln x)}$ which is X.

3) $f(x) = x(\ln x)^2$

$f'(x) = 1(\ln x)^2 + (x)\left(2(\ln x)\left(\frac{1}{x}\right)\right)$

$f'(x) = (\ln x)^2 + 2\ln x$

$(\ln x)^2 + 2\ln x = 0$

$\ln x(\ln x + 2) = 0$

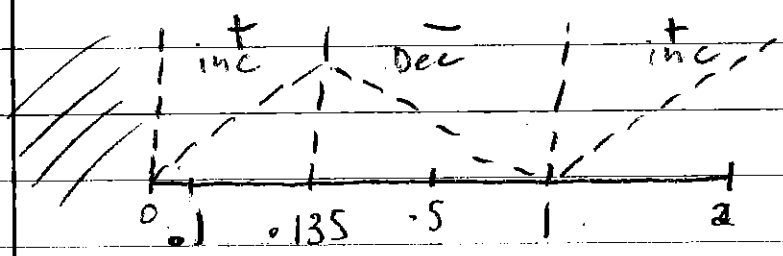
$\ln x = 0$ $\qquad$ $\ln x + 2 = 0$

$e^{\ln x} = e^0$ $\qquad$ $\ln x = -2$

$x = 1$ $\qquad$ $e^{\ln x} = e^{-2}$

$x = .135$

Can't have zero



$f'(.1) = +$

$f'(.5) = -$

$f'(2) = +$

$f(.135) = 0.541$ local Max

$f(1) = 0$ local min

increasing $= f'(x) > 0$ $(0, .135) \cup (1, \infty)$

decreasing $= f'(x) < 0$ $(.135, 1)$

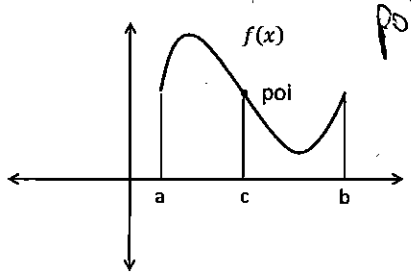
Concavity – defines the curvature of a function.

A function is concave up on an open interval if $f'(x)$ is increasing on the interval.

A function is concave down on an open interval if $f'(x)$ is decreasing on the interval.

Point of Inflection (poi) – the point on the graph where the concavity changes.

poi = where concave up changes to concave down

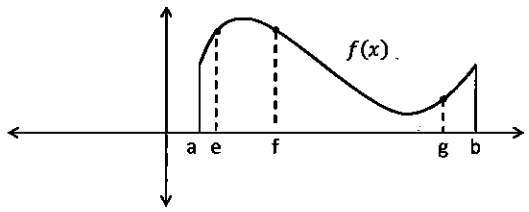


Concavity of f(x)	
Interval	Concave up/Concave down
(a, c)	*concave down*
(c, b)	*concave up*

The Second Derivative Test for Concavity

The graph of a twice-differentiable function $y = f(x)$ is:

1. Concave up on any interval where $f''(x) > 0$, and +
2. Concave down on any interval where $f''(x) < 0$. −



where $f''(x) = 0$ that is poi

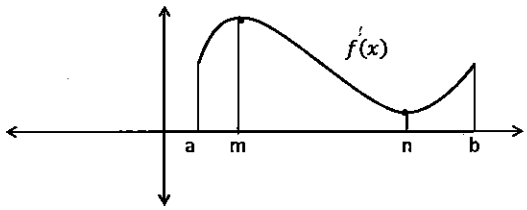
x	f''(x)	f''(x)	Concave up/Concave down
e	$f''(e) = -$	$f''(x) < 0$	Concave down
f	$f''(f) = -$	$f''(x) < 0$	Concave down
g	$f''(g) = +$	$f''(x) > 0$	Concave up

The Second Derivative Test for Local Extrema

If $f'(c) = 0$ (which makes $x = c$ a critical point) and $f''(c) < 0$, then f has a local maximum at $x = c$.

If $f'(c) = 0$ (which makes $x = c$ a critical point) and $f''(c) > 0$, then f has a local minimum at $x = c$.

NOTE: If the second derivative is equal to zero (or undefined) then the Second Derivative Test is inconclusive.



Critical Point	f''(x)	Concavity	Extrema
m	$f''(m) = -$	Concave down	Local max
n	$f''(n) = +$	Concave up	Local min

Example Problems

$$f(x) = x(lnx)^2$$

$$f(x) = x^3 - 4x^2 + 3sinx$$

$$f(x) = x^4 - 6x^2 + 2$$

$$f(x) = x^3 - 6x - 3$$

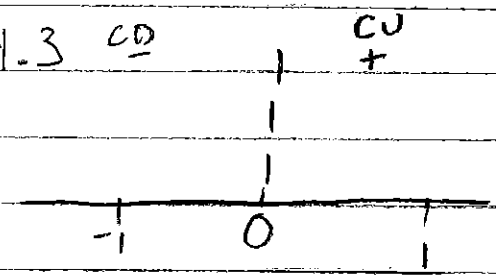
4.4 Notes & Examples

Same problems as 4.3

1) $f''(x) = 6x$

$6x = 0$

$x = 0$



test points back into second derivative

$f''(-1) = -$ CD

$f''(1) = +$ CU

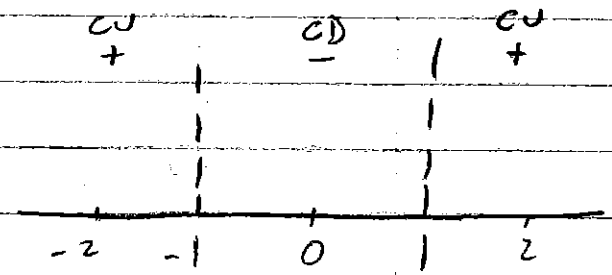
CU: $f''(x) > 0$: $(0, \infty)$

CD: $f''(x) < 0$: $(-\infty, 0)$

2) $f''(x) = 12x^2 - 12$

$f''(x) = 12(x^2 - 1)$

$12(x^2 - 1) = 0 \quad x = \pm 1$



$f''(-2) = +$

$f''(0) = -$

$f''(2) = +$

CU: $f''(x) > 0$: $(-\infty, -1) \cup (1, \infty)$

CD: $f''(x) < 0$: $(-1, 1)$

3) $f''(x) = 2\ln x + \frac{2}{x}$

$2\ln x \cdot \frac{1}{x} + \frac{2}{x} = 0$

$\frac{2\ln x + 2}{x} = 0$

$2\ln x + 2 = 0$

$2(\ln x + 1) = 0$

$\ln x + 1 = 0$

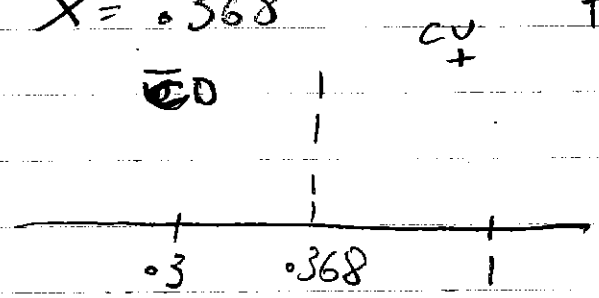
$\ln x = -1$

$e^{\ln x} = e^{-1}$

$x = .368$

$f''(.3) = -$

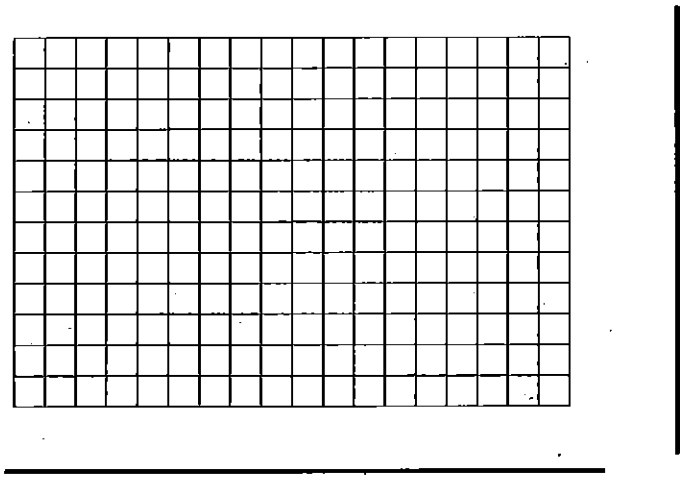
$f''(1) = +$



CU: $f''(x) > 0 = (.368, \infty)$

CD: $f''(x) < 0 = (0, .368)$

Curve Sketching



Indeterminate Forms

if you sub zero

$$\frac{0}{0} \text{ and } \frac{\infty}{\infty}$$

$$\lim_{x\to 0}\frac{2x^3-3x^2+5x}{x^3-x}=\frac{0}{0}$$

$$\lim_{x\to 0}\frac{sinx}{x}=\frac{0}{0}$$

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\frac{\infty}{\infty}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{secx}{1+tanx}=\frac{\infty}{\infty}$$

L'Hopital's Rule

Differentiate the Numerator
Differentiate the Denominator
then plug that into the function

If $\lim_{x\to u}\frac{f(x)}{g(x)}=\frac{0}{0}$ then $\lim_{x\to u}\frac{f(x)}{g(x)}=\lim_{x\to u}\frac{f'(x)}{g'(x)}$

If $\lim_{x\to u}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$ then $\lim_{x\to u}\frac{f(x)}{g(x)}=\lim_{x\to u}\frac{f'(x)}{g'(x)}$

Indeterminate Forms

$$\frac{0}{0} \text{ and } \frac{\infty}{\infty}$$

$$\lim_{x\to 0}\frac{2x^3-3x^2+5x}{x^3-x}=\frac{0}{0}$$

$$\lim_{x\to 0}\frac{6x^2-6x+5}{3x^2-1}=$$

$$\frac{5}{-1}=-5$$

$$\lim_{x\to\frac{\pi}{2}}\frac{secx}{1+tanx}=\frac{\infty}{\infty}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{secxtanx}{(secx)^2}=$$

$$\lim_{x\to\frac{\pi}{2}}\frac{tanx}{secx}=$$

$$\lim_{x\to\frac{\pi}{2}}sinx=$$

1

Indeterminate Forms

$$\frac{0}{0} \ and \ \frac{\infty}{\infty}$$

$$\lim_{x\to 0}\frac{sinx}{x} = \frac{0}{0}$$

$$\lim_{x\to 0}\frac{cosx}{1} =$$

$$\frac{1}{1} = 1$$

$$\lim_{x\to\infty}\frac{e^x}{x^2} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\frac{e^x}{2x} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\frac{e^x}{2} =$$

$$\frac{\infty}{2} = \infty$$

Other Indeterminate Forms

$$0 \cdot \infty \ and \ \infty - \infty$$

$$\lim_{x\to\frac{\pi}{2}}(tanx\ln(sinx)) = \infty \cdot 0$$

$$\lim_{x\to\frac{\pi}{2}}\frac{sinxln(sinx)}{cosx} = \frac{0}{0}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{cosxln(sinx) + sinx\ \frac{1}{sinx}\ cosx}{-sinx} =$$

$$\lim_{x\to\frac{\pi}{2}}\frac{cosxln(sinx) + \ cosx}{-sinx} = \frac{0}{-1} = 0$$

4.5 Examples

① $\lim_{x \to 0} \frac{2x^3 - 3x^2 + 5x}{x^3 - x} = \frac{0}{0}$

$\lim_{x \to 0} \frac{6x^2 - 6x + 5}{3x^2 - 1}$

$\frac{6(0)^2 - 6(0) + 5}{3(0)^2 - 1} = \frac{5}{-1} = \boxed{-5}$

$\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1 + \tan x} = \frac{\infty}{\infty}$

$\lim_{x \to \frac{\pi}{2}} \frac{\tan x \sec x}{\sec^2 x}$

$\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x} = \frac{\tan x}{\frac{1}{\cos x}} = \frac{\sin x}{\cos x} \cdot \frac{\cos x}{1}$

$\lim_{x \to \frac{\pi}{2}} \sin x$

$\sin\left(\frac{\pi}{2}\right) = \boxed{1}$

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\cos x}{1}$$

$$\cos(0) = \boxed{1}$$

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{e^x}{2x}$$

$$\lim_{x \to \infty} \frac{e^x}{2} = \frac{e^\infty}{2} = \frac{\infty}{2} = \boxed{\infty}$$

$$\lim_{x \to \frac{\pi}{2}} (\tan x \cdot \ln(\sin x)) = \infty \cdot 0$$ Can't use L'Hopital

Break it apart

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin x \cdot \ln(\sin x)}{\cos x} = \frac{0}{0}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin x \left(\frac{1}{\sin x} \cdot \cos x\right) + \cos x \left(\ln(\sin x)\right)}{-\sin x}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin x (\cot x) + \cos x (\ln(\sin x))}{-\sin x}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x + \cos x (\ln(\sin x))}{-\sin x}$$

$$\frac{\cos \frac{\pi}{2} + \cos \frac{\pi}{2} \left(\ln\left(\sin \frac{\pi}{2}\right)\right)}{-\sin \frac{\pi}{2}}$$

$$\frac{0 + 0 (\ln(1))}{-1} = \frac{0}{-1} = \boxed{0}$$

$$\frac{1}{0^+} - \frac{1}{0^+} = \infty - \infty$$

$$\lim_{x \to 0} \frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{x - \sin x}{x \sin x}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{1(\sin x) + (x)(\cos x)}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} =$$

$$\lim_{x \to 0} \frac{0 + \sin x}{\cos x + \cos x - x \sin x}$$

$$\frac{0 + \sin(0)}{\cos(0) + \cos(0) - 0(\sin 0)} = \frac{0 + 0}{1 + 1 - 0}$$

$$\frac{0}{2} = \boxed{0}$$

Other Indeterminate Forms

$0 \cdot \infty \; and \; \infty - \infty$

$$\lim_{x\to 0} \frac{1}{sinx} - \frac{1}{x} = \frac{1}{0} - \frac{1}{0} = \infty - \infty$$

Find a common denominator

$xsinx$

$$\lim_{x\to 0} \frac{x}{xsinx} - \frac{sinx}{xsinx} = \lim_{x\to 0} \frac{x - sinx}{xsinx} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{1 - cosx}{sinx + xcosx} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{sinx}{cosx + cosx - xsinx} = \frac{0}{2} = 0$$

Other Indeterminate Forms – Indeterminate Powers

$0^0, \infty^0, 1^\infty$

Set = Y

$$\lim_{x\to 0^+} x^x = 0^0$$

$$y = \lim_{x\to 0^+} x^x$$

In both

$$lny = ln \lim_{x\to 0^+} x^x$$

$$lny = \lim_{x\to 0^+} ln\, x^x$$

$$lny = \lim_{x\to 0^+} xln\, x = \quad 0 \cdot (-\infty)$$

$$lny = \lim_{x\to 0^+} \frac{lnx}{\frac{1}{x}} = \frac{-\infty}{\infty}$$

$$lny = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} =$$

$$lny = \lim_{x\to 0^+} -x = \quad 0$$

$$lny = 0$$

$$e^{lny} = e^0$$

$$y = 1$$

$$y = \lim_{x\to 0^+} x^x = 1$$

Other Indeterminate Forms – Indeterminate Powers: $0^0, \infty^0, 1^\infty$

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = 1^\infty$$

$$y = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$$

$$lny = ln\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$$

$$lny = \lim_{x\to\infty} ln\left(1+\frac{2}{x}\right)^x$$

$$lny = \lim_{x\to\infty} xln\left(1+\frac{2}{x}\right) = \infty\cdot 0$$

Other Indeterminate Forms – Indeterminate Powers: $0^0, \infty^0, 1^\infty$

$$lny = \lim_{x\to\infty} xln\left(1+\frac{2}{x}\right) = \infty\cdot 0$$

$$lny = \lim_{x\to\infty}\frac{ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$$

$$lny = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{2}{x}}\cdot\frac{-2}{x^2}}{\frac{-1}{x^2}} =$$

$$lny = \lim_{x\to\infty}\frac{\left(1+\frac{2}{x}\right)}{\frac{1}{x}} =$$

$$lny = \lim_{x\to\infty}\frac{2}{1+\frac{2}{x}} = 2$$

$$lny = 2$$

$$e^{lny} = e^2$$

$$y = e^2$$

$$y = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^2$$

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = 1^\infty$$

$$y = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$$

$$\ln y = \lim_{x\to\infty} \ln\left(1+\frac{2}{x}\right)^x$$

$$\ln y = \lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right) = \infty\cdot 0 \text{ indeterminate}$$

$$\ln y = \lim_{x\to\infty} \frac{1}{x}\cdot \ln\left(1+\frac{2}{x}\right)$$

$$\ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \frac{0}{0} \text{ L'hopitals}$$

$$\ln y = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}}\cdot\frac{-2}{x^2}}{-\frac{1}{x^2}} = \frac{\frac{-2}{x^2(1+\frac{2}{x})}}{-\frac{1}{x^2}}$$

$$\ln y = \lim_{x\to\infty} \frac{-2}{x^2\left(1+\frac{2}{x}\right)}\cdot\frac{-x^2}{1}$$

$$\ln y = \lim_{x\to\infty} \frac{2}{1+\frac{2}{x}}$$

$$\ln y = \frac{2}{1+\frac{2}{\infty}}$$

$$\ln y = \frac{2}{1+0}$$

$$\ln y = 2$$

$$y = e^2$$

$$\boxed{y = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^2}$$

1) The position function of a projectile launched vertically upward from an elevated position is $s(t) = -16t^2 + 87t + 129$. Position (s) is measured in feet and time (t) is measured in seconds.

a) When will the projectile hit the ground?
b) What is the impact velocity?
c) When will the projectile reach its maximum height?
d) What is its maximum height?

2) A cardboard box manufacturing company has to maximize the volume of a box made from a 24-inch by 9-inch sheet of cardboard. What are the dimensions of the box?

3) A box is to be constructed where the base length is 3 times the base width. The material used to build the top and bottom cost $10 per square foot and the material used to build the sides cost $6 per square foot. If the box must have a volume of 50 cubic feet, determine the dimensions that will minimize the cost to build the box.

4) A printer need to make a poster that will have a total area of 200 in^2 and will have 1 inch margins on the sides, a 2 inch margin on the top and a 1.5 inch margin on the bottom. What dimensions will give the largest printed area?

Newton's Method to approximate the zero of $f(X)$

$$4.7 = X = X_n - \frac{f(X_n)}{f'(X_n)}$$

If the following functions represent the derivative of the original function, find the original function.

$f(x) = 3x^2$ $\longleftrightarrow$ $F(x) = x^3$

$f(x) = x^2 + 2x$ $\longleftrightarrow$ $F(x) = \frac{1}{3}x^3 + x^2$

$f(x) = 3x^2 - 1$ $\qquad$ $F(x) = x^3 - x$

$f(x) = -\frac{2}{x^3} + 4 \ = -2x^{-3} + 4$ $\qquad$ $F(x) = x^{-2} + 4x$

Antiderivative – If $F'(x) = f(x)$ on an interval, then $F(x)$ is the antiderivative of $f(x)$ for every value of x on the interval.

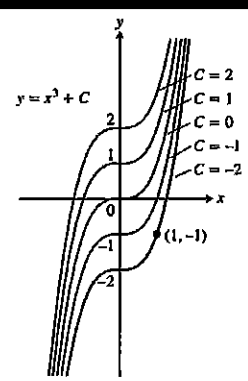
State the derivative of each function.

$f(x) = x^3$ $\qquad$ $f'(x) = 3x^2$

$f(x) = x^3 + 2$ $\qquad$ $f'(x) = 3x^2$

$f(x) = x^3 - 1$ $\qquad$ $f'(x) = 3x^2$

$f(x) = x^3 + 4$ $\qquad$ $f'(x) = 3x^2$



Theorem: – If $F(x)$ is an antiderivative of $f(x)$ on an interval I, then the general antiderivative of $f(x)$ is:

$$F(x) + C$$

$$f(x) = 3x^2 \ \rightarrow \ \underline{F(x) = x^3 + C}$$

Formulas

Antiderivative Formulas where k is a constant
(from page 281 of the textbook)

TABLE 4.2 Antiderivative formulas, k a nonzero constant

	Function	General antiderivative		Function	General antiderivative
1.	x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$	8.	e^{kx}	$\frac{1}{k}e^{kx} + C$
2.	$\sin kx$	$-\frac{1}{k}\cos kx + C$	9.	$\frac{1}{x}$	$\ln\lvert x\rvert + C, \quad x \neq 0$
3.	$\cos kx$	$\frac{1}{k}\sin kx + C$	10.	$\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1} kx + C$
4.	$\sec^2 kx$	$\frac{1}{k}\tan kx + C$	11.	$\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1} kx + C$
5.	$\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	12.	$\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1} kx + C, \quad kx > 1$
6.	$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	13.	a^{kx}	$\left(\frac{1}{k\ln a}\right)a^{kx} + C, \quad a > 0, a \neq 1$
7.	$\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$			

TABLE 4.3 Antiderivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule:*	$kf(x)$	$kF(x) + C, \quad k$ a constant
2.	*Negative Rule:*	$-f(x)$	$-F(x) + C,$
3.	*Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

TABLE 4.2 Antiderivative formulas, k a nonzero constant

	Function	General antiderivative		Function	General antiderivative
1.	x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$	8.	e^{kx}	$\frac{1}{k}e^{kx} + C$
2.	$\sin kx$	$-\frac{1}{k}\cos kx + C$	9.	$\frac{1}{x}$	$\ln\lvert x\rvert + C, \quad x \neq 0$
3.	$\cos kx$	$\frac{1}{k}\sin kx + C$	10.	$\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1} kx + C$
4.	$\sec^2 kx$	$\frac{1}{k}\tan kx + C$	11.	$\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1} kx + C$
5.	$\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	12.	$\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1} kx + C, \quad kx > 1$
6.	$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	13.	a^{kx}	$\left(\frac{1}{k\ln a}\right)a^{kx} + C, \quad a > 0, a \neq 1$
7.	$\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$			

Write the general antiderivative of each of the following functions.

1. $f(x) = x^5 \qquad F(x) = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C$

2. $f(x) = \sin(2x) \qquad F(x) = -\frac{1}{2}\cos(2x) + C$

8. $f(x) = e^{-3x}$

$F(x) = -\frac{1}{3}e^{-3x} + C$

4.8 Examples

$$\int -5\sin t \, dt = 5\cos t + c$$

$$\int (2e^{x} - 3e^{-2x})\, dx = 2e^{x} - 3\frac{e^{-2x}}{-2} = 2e^{x} + \frac{3}{2}e^{-2x} + c$$

$$\int 3x^{4}\, dx = \frac{3}{5}x^{5} + c$$

$$\int (4x^{3} - 3x^{2} + 6x - 1)\, dx = x^{4} - x^{3} + 3x^{2} - x + c$$

$$\int x^{4}(5 - x^{2})\, dx = (5x^{4} - x^{6}) = x^{5} - \frac{1}{7}x^{7} + c$$

$$\int \frac{3}{\sqrt{x}}\, dx = \int 3x^{-\frac{1}{2}}\, dx = 6x^{\frac{1}{2}} + c$$

$$\int \frac{2 + x^{2} - 8x^{3}}{x^{2}}\, dx = \int (2x^{-2} + 1 - 8x)\, dx$$

$$-2x^{-1} + x - 4x^{2} + c$$

$$\int (x^2+4x)^2 = \int (x^4 + 8x^3 + 16x^2)\,dx$$

$$\frac{1}{5}x^5 + 2x^4 + \frac{16}{3}x^3 + c$$

$$\int \frac{dy}{dx} = \int 8x^3 - 15x^2 - 2$$

$$y = 2x^4 - 5x^3 - 2x + C$$

$$11 = 2(3)^4 - 5(3)^3 - 2(3) + C$$

$$11 = 21 + C$$

$$\boxed{C = -10}$$

$$\boxed{y = 2x^4 - 5x^3 - 2x - 10}$$

$$\int \frac{d^2y}{d^2x} = \int 2 - 6x = y' = 2x - 3x^2$$

$$4 = 0 + C$$

$$\int \frac{dy}{dx} = \int 2x - 3x^2 + 4 = y = x^2 - x^3 + 4x + C$$

$$1 = (0)^2 - (0)^3 + 4(0) + C$$

$$1 = 0 - 0 + 0 + C$$

$$C = 1$$

$$f(x) = x^2 - x^3 + 4x + 1$$

$$\int \frac{x^4+2x^2-1}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\left(x^4+2x^2-1\right)dx$$

$$\int x^{3.5} + 2x^{1.5} - x^{-.5}$$

$$\int x^{\frac{7}{2}} + 2x^{\frac{3}{2}} - x^{-\frac{1}{2}}\,dx$$

$$\frac{2}{9}x^{\frac{9}{2}} + \frac{4}{5}x^{\frac{5}{2}} - 2x^{\frac{1}{2}} + C$$

Indefinite Integrals

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

$$\int (5-6x)dx = 5x - \frac{6x^{1+1}}{1+1} + C = 5x - \frac{6x^2}{2} + C = 5x - 3x^2 + C$$

$$\int -5sint\,dt = -5(-cost) + C = 5cost + C$$

$$\int (2e^x - 3e^{-2x})dx = 2e^x - \frac{3e^{-2x}}{-2} + C = 2e^x + \frac{3}{2}e^{-2x} + C$$

Initial Value Problems

Solve for the original equation if $\frac{d^2y}{dx^2} = 2 - 6x$ given $y'(0) = 4$ and $y(0) = 1$.

$$\int \frac{d^2y}{dx^2} = \int 2 - 6x$$

$$\frac{dy}{dx} = 2x - \frac{6x^2}{2} + C$$

$$\frac{dy}{dx} = 2x - 3x^2 + C$$

$$4 = 2(0) - 3(0)^2 + C$$

$$4 = C$$

$$\frac{dy}{dx} = 2x - 3x^2 + 4$$

$$\int \frac{dy}{dx} = \int 2x - 3x^2 + 4$$

$$y = \frac{2x^2}{2} - \frac{3x^3}{3} + 4x + C$$

$$y = x^2 - x^3 + 4x + C$$

$$1 = (0)^2 - (0)^3 + 4(0) + C$$

$$1 = C$$

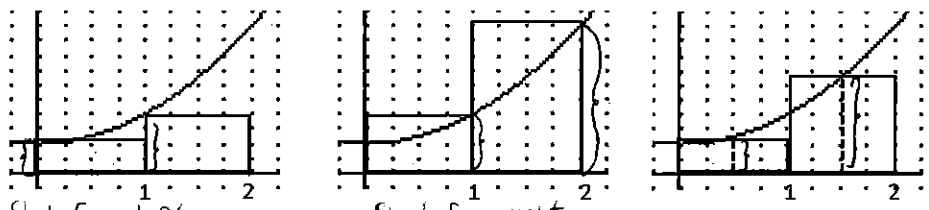
$$y = x^2 - x^3 + 4x + 1$$

Estimating Area Under a Curve

Approximate the area under the curve $f(x) = x^2 + 1$ from $x = 0$ to $x = 2$ using 2 rectangles.

$\Delta x = \frac{2-0}{2} = 1.$ $\qquad$ $Area = height \cdot length = f(x) \cdot \Delta x$

Left-hand endpoints | Right-hand endpoints | Midpoints closest approximation



Start from Left $\qquad$ Start from right

Estimated Area Under the Curve = the sum of the area of all rectangles

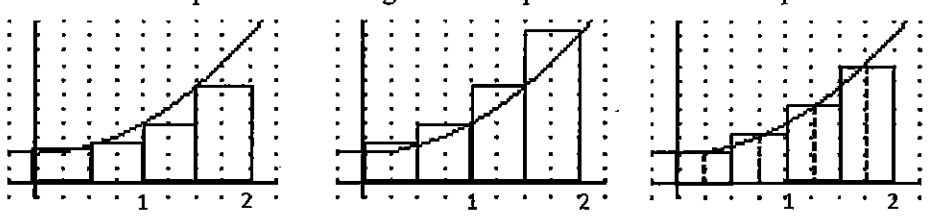
$A = f(0) \cdot 1 + f(1) \cdot 1$	$A = f(1) \cdot 1 + f(2) \cdot 1$	$A = f(.5) \cdot 1 + f(1.5) \cdot 1$
$A = 1 \cdot 1 + 2 \cdot 1$	$A = 2 \cdot 1 + 5 \cdot 1$	$A = 1.25 \cdot 1 + 3.25 \cdot 1$
$\boxed{A = 3}$	$A = 7$	$A = 4.5$
under		

Estimating Area Under a Curve

Approximate the area under the curve $f(x) = x^2 + 1$ from $x = 0$ to $x = 2$ using 4 rectangles.

$\Delta x = \frac{2-0}{4} = \frac{1}{2}.$ $\qquad$ $Area = height \cdot length = f(x) \cdot \Delta x$

Left-hand endpoints | Right-hand endpoints | Midpoints



Estimated Area Under the Curve = the sum of the area of all rectangles

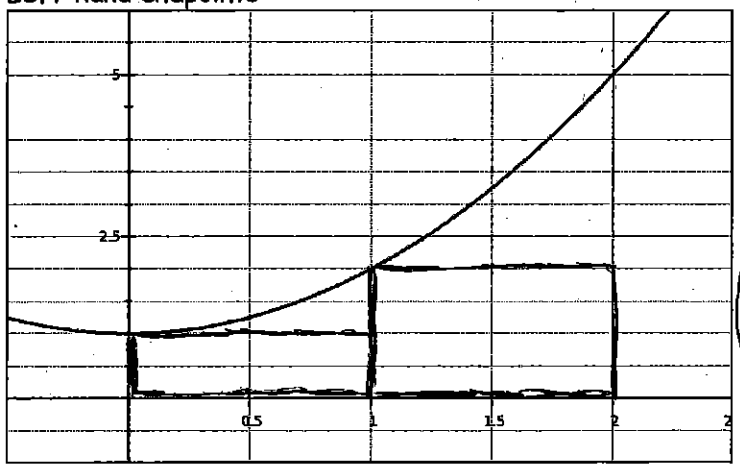
LH	$A = f(0) \cdot .5 + f(.5).5 + f(1).5 + f(1.5).5$	$= 3.75$
RH	$A = f(.5) \cdot .5 + f(1).5 + f(1.5).5 + f(2).5$	$= 5.75$
Mid	$A = f(.25) \cdot .5 + f(.75).5 + f(1.25).5 + f(1.75).5$	$= 4.625$

Area Under a Curve

Lesson Problems

1) Approximate the area under the curve $f(x) = x^2 + 1$ from $x = 0$ to $x = 2$ using 2 rectangles.

Left-hand endpoints



Area of Rectangle

$A = L \cdot W$

$W = \Delta X$

$\Delta X = \frac{2-0}{2} = \boxed{1}$

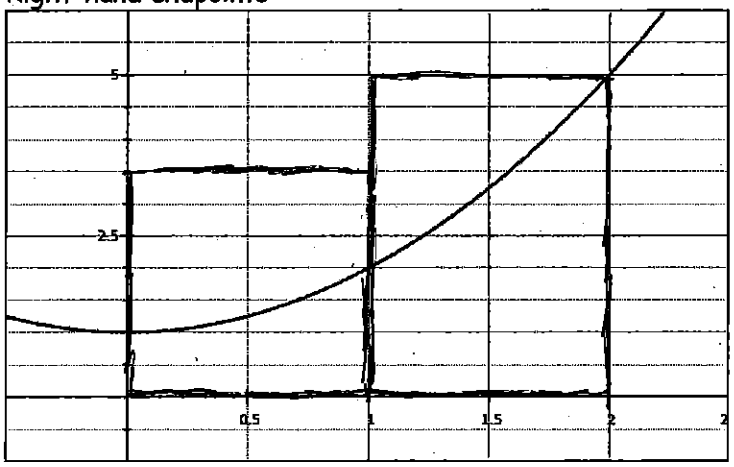
$L = f(X)$

$A = f(0) \cdot 1 + f(1) \cdot 1$

$A = 1 + 2$

$\boxed{A = 3}$

Right-hand endpoints

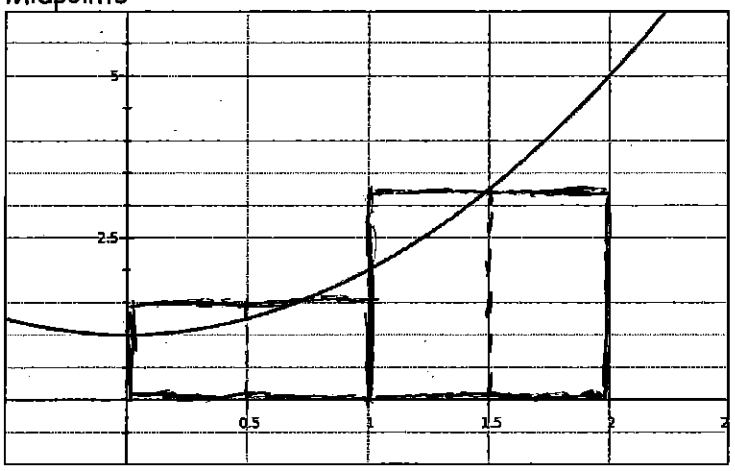


$A = f(1) \cdot 1 + f(2) \cdot 1$

$A = 2 + 5$

$\boxed{A = 7}$

Midpoints



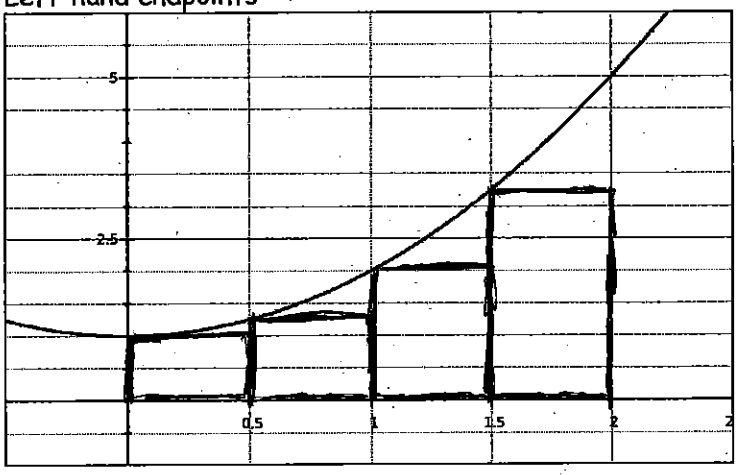
$A = f(.5) \cdot 1 + f(1.5) \cdot 1$

$A = 1.25 + 3.25$

$\boxed{A = 4.5}$

2) Approximate the area under the curve $f(x) = x^2 + 1$ from $x = 0$ to $x = 2$ using 4 rectangles.

Left-hand endpoints



$\Delta x = \frac{2-0}{4} = \frac{1}{2}$

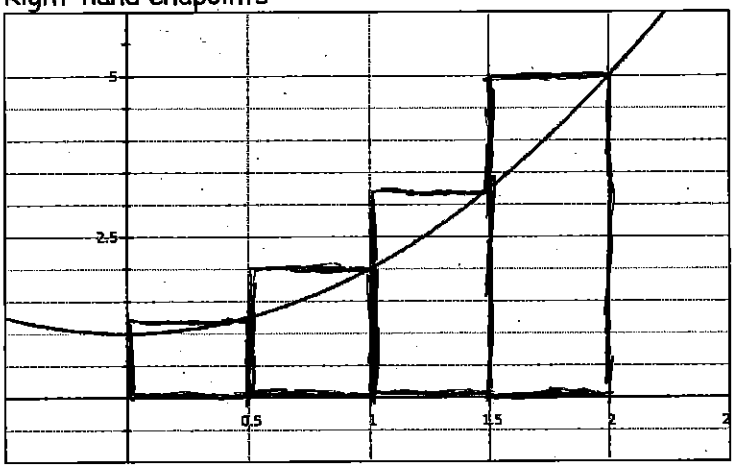
$A = f(0)\cdot\frac{1}{2} + f(.5)\cdot\frac{1}{2} + f(1)\cdot\frac{1}{2} + f(1.5)\cdot\frac{1}{2} +$

$A = .5 + \frac{5}{8} + 1 + \frac{9}{8} +$

$A = \frac{1}{2} + \frac{5}{8} + 1 + \frac{9}{8} +$

$\boxed{A = 3.75}$

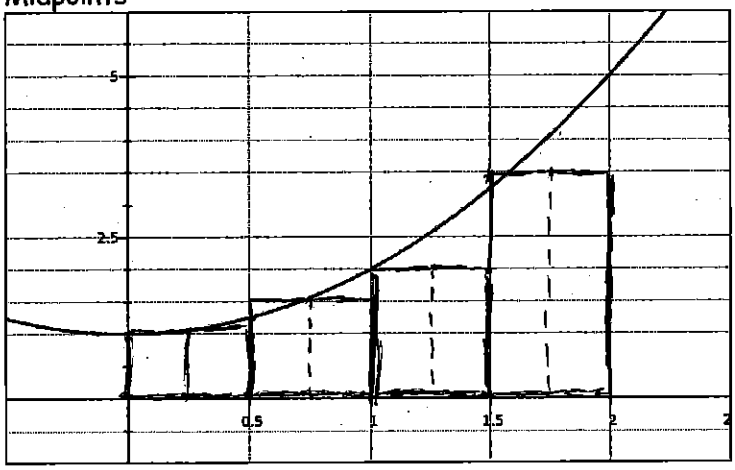
Right-hand endpoints



$f(.5)\cdot\frac{1}{2} + f(1)\cdot\frac{1}{2} + f(1.5)\cdot\frac{1}{2} + f(2)\cdot\frac{1}{2}$

$\boxed{A = 5.75}$

Midpoints



$A = f(.25)\cdot\frac{1}{2} + f(.75)\cdot\frac{1}{2} + f(1.25)\cdot\frac{1}{2} + f(.75)$

$\boxed{A = 4.625}$

$Y_1(\{.25, .75, 1.25, 1.75\})$

$\boxed{\text{List} > \text{Math} >}$ Sum(Ans) * .5 = $\boxed{4.625}$

Average Value of an Integral

Average Value: Given a closed interval for a continuous function, the average value is the function value that when multiplied by the length of the interval produces the same area as that under the curve.

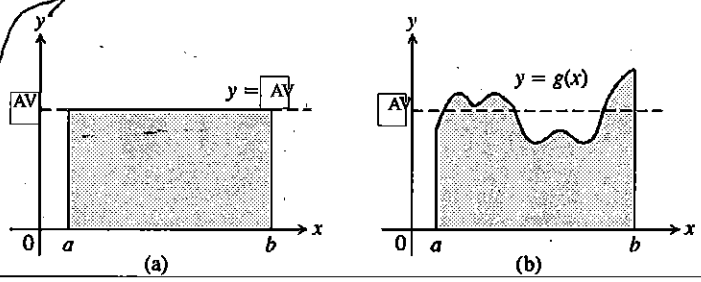
$A = L \cdot W$

$A = h \cdot W$

$h = \frac{A}{W}$

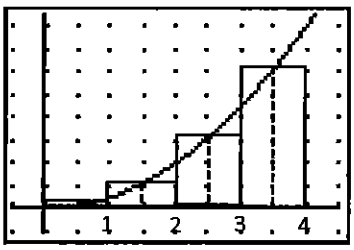
$\boxed{h = \frac{A}{\Delta x}}$

$$Average\ Value\ (AV) = \frac{area\ under\ the\ curve}{length\ of\ the\ interval}$$



Average Value of an Integral

Estimate the average value for the function $f(x) = x^2$ on the interval [0,4] using four midpoint subintervals (rectangles) on equal width.



Estimated Area Under the Curve

$$width\ of\ subintervals = \Delta x = \frac{4-0}{4} = 1.$$

$$A = f(.5) \cdot 1 + f(1.5) \cdot 1 + f(2.5) \cdot 1 + f(3.5) \cdot 1$$

$$\boxed{A = 21}$$

$$\boldsymbol{Average\ Value\ (AV) = \frac{area\ under\ the\ curve}{length\ of\ the\ interval}}$$

$$Average\ Value\ (AV) = \frac{21}{4-0}$$

$$AV = \frac{21}{4} = \boxed{5.25}$$

Sigma Notation – A mathematical notation that represents the sum of many terms using a formula.

Sequence – a function whose domain is positive integers.

Function Notation	**Sigma Notation**
$f(x), g(x), h(x)$	a_n, b_i, c_k
$f(x) = 2x + 1$	$a_n = 2n + 1$
$g(x) = x^2 - 3x + 7$	$b_i = i^2 - 3i + 7$
$h(x) = \dfrac{x+6}{x^2+2x+3}$	$c_k = \dfrac{k+6}{k^2+2k+3}$

The index k ends at $k = n$.

The summation symbol (Greek letter sigma)

$$\sum_{k=1}^{n} a_k$$

a_k is a formula for the kth term.

The index k starts at $k = 1$.

Ex: $\sum_{K=1}^{3} a_K = a_1 + a_2 + a_3$

Sigma Notation

Examples

$$\sum_{n=1}^{4} 2n$$

$2(1) + 2(2) + 2(3) + 2(4)$

$2 + 4 + 6 + 8$

20

$$\sum_{k=1}^{6} 2$$

$2 + 2 + 2 + 2 + 2 + 2$

12

Ex: $f(x) = 9$

when $x = 2$ $f(2) = 9$

$\therefore (1,9)(2,9)(3,9)(4,9) \ldots$

$$\sum_{i=1}^{\boxed{3}} (i^2 - 3i + 7)$$

$(1^2 - 3 \cdot 1 + 7) + (2^2 - 3 \cdot 2 + 7) + (3^2 - 3 \cdot 3 + 7)$

$5 + 5 + 7$

17

Calculator: List > OPS > Seq$(X^2 - 3X + 7, X, 1, 3) = \{5, 5, 7\}$

Sum(ANS) = $\boxed{17}$

5.2 Examples

$$\sum_{i=1}^{3} (i^2 - 3i + 7) = (1^2 - 3(1) + 7) + (2^2 - 3(2) + 7) + (3^2 - 3(3) + 7)$$
$$= 5 + 5 + 7$$
$$= \boxed{17}$$

$1 + 2 + 3 + 4 + \dots + 98$

$$\sum_{n=1}^{98} n$$

$1 - 2 + 3 - 4 + \dots - 98$

$$\sum_{n=1}^{98} n \cdot (-1)^{n+1}$$

$$\overset{1}{1} + \overset{2}{\tfrac{1}{2}} + \overset{3}{\tfrac{1}{3}} + \overset{4}{\tfrac{1}{4}} + \dots + \overset{70}{\tfrac{1}{70}}$$

$$\sum_{n=1}^{70}$$

$$\underset{1}{1} - \underset{2}{\tfrac{1}{4}} + \underset{3}{\tfrac{1}{9}} - \underset{4}{\tfrac{1}{16}} + \dots - \underset{49}{\tfrac{1}{49}}$$

$$\sum_{n=1}^{7} (-1)^{n+1} \cdot \frac{1}{n^2}$$

Sigma Notation

Express the sums in sigma notation.

$1+2+3+4+\cdots+98$

$$\sum_{i=1}^{98} i$$

Alternating Series

$1-2+3-4+\cdots-98$

$$\sum_{i=1}^{98} \boxed{(-1)^{i+1}}\, i$$

Alternator

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{70}$

$$\sum_{k=1}^{70} \frac{1}{k}$$

$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots-\frac{1}{49}$

$$\sum_{i=1}^{7} (-1)^{i+1}\frac{1}{i^2}$$

Sigma Notation

Linearity of Sigma

c = constant

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

constant $\cdot \sum_{i=1}^{n} a_i$

$$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

Example

$$\sum_{k=1}^{n}(3k^2+2k-7)$$

$$\sum_{k=1}^{n} 3k^2 + \sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 7 \quad \rightarrow \quad 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 7$$

Summation Rules

$$\sum_{k=1}^{n} c = n \cdot c$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Summation Rules Examples

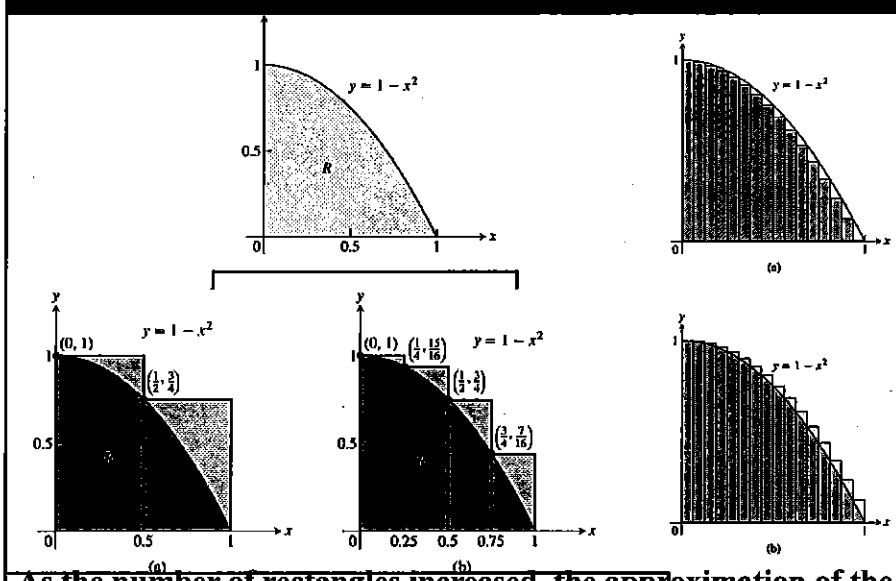
$$\sum_{k=1}^{51} 4 = 51 \cdot 4 = 204 \longrightarrow sum(seq(4, x, 1, 51, \boxed{1}))$$

increment to raise by

$$\sum_{k=1}^{32} k = \frac{32(32+1)}{2} = 528 \longrightarrow sum(seq(x, x, 1, 32, 1))$$

$$\sum_{k=1}^{15} k^2 = \frac{15(15+1)(2 \cdot 15+1)}{6} = 1240 \longrightarrow sum(seq(x^2, x, 1, 15, 1))$$

$$\sum_{k=1}^{9} k^3 = \left(\frac{9(9+1)}{2}\right)^2 = 2025 \longrightarrow sum(seq(x^3, x, 1, 9, 1))$$



As the number of rectangles increased, the approximation of the area under the curve approaches a value.

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Definition

The *definite integral* from a to b, $\int_a^b f(x)\,dx$ is the number to which all Riemann sums tend as the number of rectangles approaches infinity and as the width of all rectangles tend to zero:

where n = # of rectangles

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

where Δx_i is the width of the ith rectangle and c_i is the x-coordinate of the point where the ith rectangle touches $f(x)$.

Note: The function f(x) must be continuous on the interval [a, b].

Parts of the Definite Integral

$$\int_a^b f(x)\,dx$$

Upper limit of integration

Integral sign

Lower limit of integration

The function is the integrand.

x is the variable of integration.

Integral of f from a to b

When you find the value of the integral, you have evaluated the integral.

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Properties of the Definite Integral

TABLE 5.4 Rules satisfied by definite integrals

1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	A Definition when $f(a)$ exists
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any constant k → take out constants
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6.	*Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then $\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$	
7.	*Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
		$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)	

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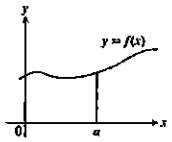
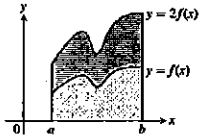
Geometric Interpretations of the Properties of the Definite Integral



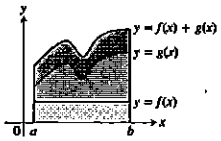
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



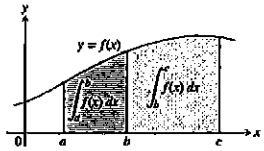
(b) *Constant Multiple:* ($k = 2$)

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



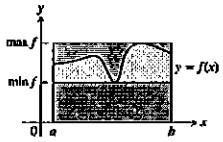
(c) *Sum:* (*areas add*)

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



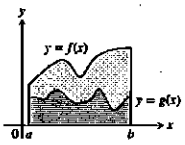
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$



(f) *Domination:*

$$f(x) \ge g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

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we'll be using these most

Using the Properties of the Definite Integral

Given: $\int_1^3 f(x)dx = 6$ $\quad \int_3^7 f(x)dx = 9$ $\quad \int_1^3 g(x)dx = -4$

$$\int_1^3 3f(x)dx = 3\boxed{\int_1^3 f(x)dx} = 3(6) = 18$$

$$\int_1^3 (2f(x) - 4g(x))dx = 2\int_1^3 f(x)dx - 4\int_1^3 g(x)dx = 2(6) - 4(-4) = 28$$

$$\int_1^7 f(x)dx = \int_1^3 f(x)dx + \int_3^7 f(x)dx = 6 + 9 = 15$$

flip

$$\int_3^1 f(x)dx = -\int_1^3 f(x)dx = \boxed{-6}$$

Rules of the Definite Integral

$$\int_a^b c\,dx = c(b-a) \qquad \int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2} \qquad \int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$$

Examples

$$\int_2^6 4\,dx = \quad 4(6-2) = \quad 16 \qquad \int_4^8 x\,dx = \frac{8^2}{2} - \frac{4^2}{2} = \quad 32 - 8 = \quad 24$$

$$\int_3^5 x^2\,dx = \frac{5^3}{3} - \frac{3^3}{3} = \frac{125}{3} - \frac{27}{3} = \frac{98}{3} = 32.67$$

$$\int_3^4 x^2 + 3x - 2\,dx = \int_3^4 x^2\,dx + 3\int_3^4 x\,dx - \int_3^4 2\,dx = \frac{4^3}{3} - \frac{3^3}{3} + 3\left(\frac{4^2}{2} - \frac{3^2}{2}\right) - \ 2(4-3) =$$

$$\frac{64}{3} - \frac{27}{3} + 3\left(\frac{16}{2} - \frac{9}{2}\right) - \ 2(1) = \ 20.83$$

Calc: Math > 9 > fnInt(X^2+3X-2, X, 3, 4) = 20.83

The Fundamental Theorem of Calculus

If f is continuous at every point in $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Examples

$$\int_1^5 5x\,dx = \left.\frac{5x^2}{2}\right|_1^5 = \frac{5(5)^2}{2} - \frac{5(1)^2}{2} = \frac{125}{2} - \frac{5}{2} = \frac{120}{2} = 60$$

$$\int_{\pi/6}^{2\pi/3} \sin x\,dx = -\cos x\Big|_{\pi/6}^{2\pi/3} = -\cos\left(\frac{2\pi}{3}\right) - \left(-\cos\left(\frac{\pi}{6}\right)\right) = -\left(-\frac{1}{2}\right) - \left(-\frac{\sqrt{3}}{2}\right) = 0.866$$

$$\frac{1+\sqrt{3}}{2}$$

The Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a).$$

Examples

$$\int_3^4 x^2 + 3x - 2\,dx = \left.\frac{x^3}{3} + \frac{3x^2}{2} - 2x\right|_3^4 = \frac{4^3}{3} + \frac{3(4)^2}{2} - 2(4) - \left(\frac{3^3}{3} + \frac{3(3)^2}{2} - 2(3)\right)$$

$$37.33 - 16.5 = 20.83$$

$$\int_1^{32} \frac{1}{x^{6/5}}\,dx = \int_1^{32} x^{-6/5}\,dx = \left.\frac{x^{-1/5}}{-\frac{1}{5}}\right|_1^{32} = \left.\frac{-5}{x^{1/5}}\right|_1^{32} = -\frac{5}{2} - \left(-\frac{5}{1}\right) = 2.5$$

Differentiating a Definite Integral

If f is continuous on $[a,b]$ and x is any point on $[a,b]$, then

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\frac{d}{dx}\int_1^x t^2 dt = \frac{d}{dx}\left(\frac{t^3}{3}\right)\Big|_1^x = \frac{d}{dx}\left(\frac{x^3}{3} - \frac{1^3}{3}\right) = \frac{d}{dx}\left(\frac{x^3}{3} - \frac{1}{3}\right) = x^2$$

$$\frac{d}{dx}\int_1^{4x} t^2\,dt = \frac{d}{dx}\left(\frac{t^3}{3}\right)\Big|_1^{4x} = \frac{d}{dx}\left(\frac{(4x)^3}{3} - \frac{1^3}{3}\right) = \frac{d}{dx}\left(\frac{64x^3}{3} - \frac{1}{3}\right) = 64x^2$$

$$\frac{d}{dx}\int_1^{x^2} t^2\,dt = \frac{d}{dx}\left(\frac{t^3}{3}\right)\Big|_1^{x^2} = \frac{d}{dx}\left(\frac{(x^2)^3}{3} - \frac{1^3}{3}\right) = \frac{d}{dx}\left(\frac{x^6}{3} - \frac{1}{3}\right) = \frac{6x^5}{3} = 2x^5$$

Differentiating a Definite Integral

If f is continuous on $[a,b]$ and x is any point on $[a,b]$, then

replace with top limit then take derivative of it

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\frac{d}{dx}\int_1^x t^2\,dt = (x)^2(1) = x^2$$

$$\frac{d}{dx}\int_1^{x^2} \sin t\;dt = (2x)sin(x^2)$$

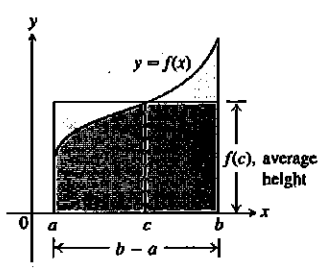
$$\frac{d}{dx}\int_1^{4x} t^2\,dt = (4x)^2(4) = 64x^2$$

$$\frac{d}{dx}\int_1^{x^2+x} \tan t\;dt = (2x+1)tan(x^2+2)$$

$$\frac{d}{dx}\int_1^{x^2} t^2\,dt = (x^2)^2(2x) = 2x^5$$

$$\frac{d}{dx}\int_{tanx}^{sinx} t^2\,dt = (sin^2x)(cosx) - (tan^2x)(sec^2x)$$

Mean Value Theorem for Integrals



THEOREM 3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$\boxed{f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.}$$

$$f(c)\cdot(b-a) = \int_a^b f(x)\,dx$$

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$$A = \text{height}\cdot\text{Length} = \int_a^b f(x)\,dx$$

Mean Value for Definite Integral

Find the mean (average) value of the function $f(x) = x^2 + 2$ over the interval $[1, 3]$

$$AV = \frac{1}{3-1}\int_1^3 (x^2 + 2)\, dx$$

$$AV = \frac{1}{2}\left(\frac{x^3}{3} + 2x\right)\Bigg|_1^3$$

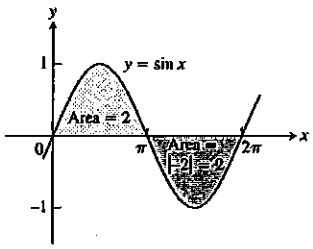
$$AV = \frac{1}{2}\left[\left(\frac{27}{3} + 6\right) - \left(\frac{1}{3} + 2\right)\right]$$

$$AV = \frac{1}{2}\left[15 - \frac{7}{3}\right]$$

$$AV = \frac{19}{3} = 6.33$$

Difference between the Value of a Definite Integral and Total Area

Find the mean (average) value of the function $f(x) = sinx$ over the interval $[0, 2\pi]$



Total Area

$$\int_0^{2\pi} sinx\, dx =$$

$$\int_0^{\pi} sinx\, dx - \int_{\pi}^{2\pi} sinx\, dx =$$

$$-cosx\Big|_0^{\pi} - \quad -cosx\Big|_{\pi}^{2\pi}$$

$$-cos(\pi) - (-cos(0)) - \left[\left(-cos(2\pi) - (-cos(\pi))\right)\right]$$

$$-(-1) - (-1) - \quad (-(1) - (1)) =$$

$$4$$

Value of the Definite Integral

$$\int_0^{2\pi} sinx\, dx = -cosx\Big|_0^{2\pi} =$$

$$-cos(2\pi) - (-cos(0)) =$$

$$-(1) - (-1) = \quad 0$$

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5.3 Examples

$$\int_3^4 (x^2+3x-2)\,dx = \frac{x^3}{3}+\frac{3}{2}x^2-2x\Big|_3^4$$

$$\left(\frac{4^3}{3}+\frac{3}{2}\cdot 4^2-2(4)\right)-\left(\frac{3^3}{3}+\frac{3}{2}\cdot 3^2-2(3)\right)$$

$$\boxed{=20.83}$$

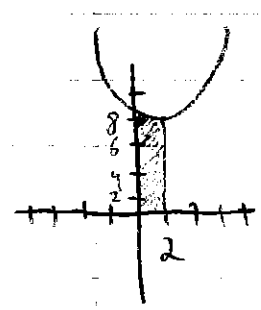
$$\int_1^{32}\frac{1}{x^{6/5}} = \int_1^{32} x^{-6/5} = \frac{x^{-6/5+1}}{-\frac{6}{5}+1} = \frac{x^{-1/5}}{-\frac{1}{5}} =$$

$$= \frac{-5}{x^{1/5}}\Big|_1^{32} = \frac{-5}{(32)^{1/5}} - \left(\frac{-5}{(1)^{1/5}}\right) = \frac{-5}{2}+5 = \boxed{2.5}$$

$$\int_1^2 (t^2-2t+8)\,dt = \frac{t^3}{3}-t^2+8t\Big|_1^2$$

$$= \frac{(2)^3}{3}-(2)^2+8(2)-\left[\frac{1}{3}-1+8(1)\right]$$

$$= \frac{8}{3}-4+16-\left[\frac{1}{3}-1+8\right] = 7.333$$



5.4 Notes

$$AV = \frac{1}{3-1}\int_1^3 (x^2+2)\,dx = \frac{1}{2}\int_1^3 (x^2+2)\,dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3} + 2x\right)\Big|_1^3$$

$$= \frac{1}{2}\left[\frac{3^3}{3} + 2(3) - \left(\frac{1}{3} + 2(1)\right)\right] = \frac{1}{2}\left(9 + 6 - \frac{7}{3}\right)$$

$$= \boxed{6.33}$$

$$AV = \frac{1}{2\pi - 0}\int_0^{2\pi} \sin x\,dx =$$

$$\frac{1}{2\pi}(-\cos x)\Big|_0^{2\pi} = \frac{1}{2\pi}\left[-\cos 2\pi - (-\cos 0)\right]$$

$$= \frac{1}{2\pi}(-1+1) = \frac{1}{2\pi}(0) = 0$$ Cant have zero Area

Backup $= \int_0^{2\pi} \sin x\,dx = \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx$

$$= -\cos x\Big|_0^{\pi} - \left(-\cos x\Big|_{\pi}^{2\pi}\right) = -(-1) - (-1) - [-(1) - (1)]$$

$$= \frac{1}{2\pi}(4) = \boxed{\frac{2}{\pi}}$$

Differentiation Review:

$$y = (5x+6)^4$$
$$dy = 4(5x+6)^3(5)dx$$

Integration: $\int dy = \int 20(5x+6)^3 dx$

THEOREM 6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

u du Substitution: $u = g(x)$

$du = g'(x)\,dx$

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Differentiation Review:

$$y = (5x+6)^4$$
$$dy = 4(5x+6)^3(5)dx$$

Integration: $\int dy = \int 20(5x+6)^3 dx$

u du Substitution: $u = \underline{5x+6}$

$du = \underline{5\,dx}$

$\int dy = \int 4(5x+6)^3\, 5dx$ $\qquad$ $\int dy = \int 4(u)^3\, du$

$y + c = \frac{4u^4}{4} + c$ $\qquad$ $y = \underbrace{u}^4 + C$ $\qquad$ $\boxed{y = (5x+6)^4 + C}$

you're allowed to move it over

replace back

Answer

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don't worry about sign

Differentiation Review:

$$y = \sin(x^4)$$

$$dy = \cos(x^4) \cdot 4x^3 dx$$

Integration: $\int dy = \int \cos(x^4) 4x^3 dx$

***u du Substitution*:** $u = x^4$

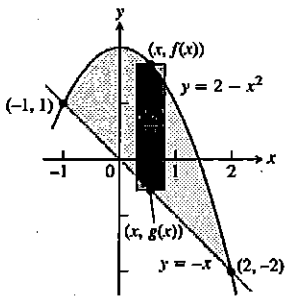
$du = 4x^3\, dx$

$\int dy = \int \cos(x^4) 4x^3 dx$ $\qquad$ $\int dy = \int \cos u\, du$

$$y = \sin u + C$$

$$y = \sin(x^4) + C$$

Area Between Curves



width of rectangle = dx

length of rectangle = f(x) − g(x)

upper function − lower function

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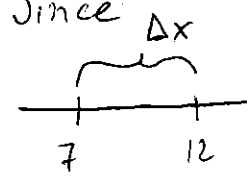
Since



$= 12 - 7 = \boxed{5}$

$\Delta y \left\{ \begin{array}{l} 5 \\ -5 \end{array} \right.$

$\Delta y = 5 - (-5) = \boxed{10}$

then

$\underline{f(x) - g(x)}$

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

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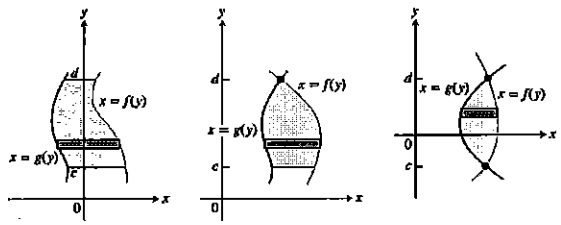
Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these



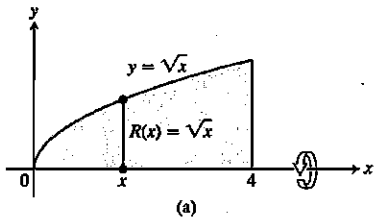
use the formula

$$A = \int_c^d [f(y) - g(y)]\, dy.$$

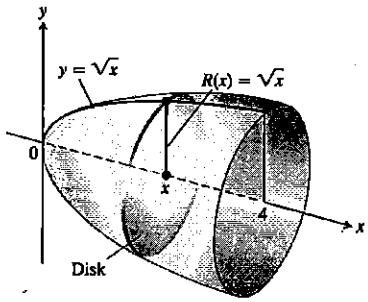
In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

width of rectangle:
$$f(y) - g(y)$$

right function − left function



(a)



(b)

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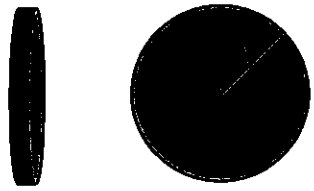
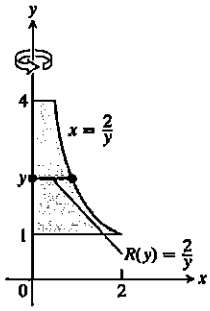
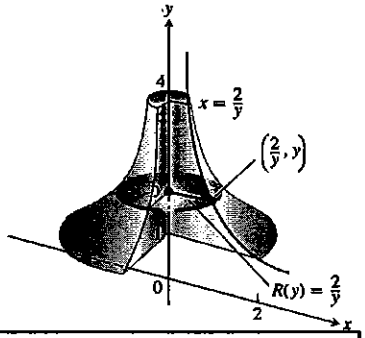


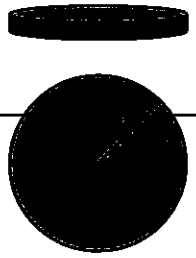
Volume of the disk

$$V = \pi r^2 \cdot thickness$$

$$V = \pi r^2 \, dx$$







Volume of the disk

$$V = \pi r^2 \cdot thickness$$

$$V = \pi r^2 \, dy$$

Solids of Revolution: The Disk Method

The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid like the one shown in Figure 6.8, we need only observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

So the definition of volume in this case gives

Volume by Disks for Rotation About the x-axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

Volume by Disks for Rotation About the y-axis

$$V = \int_c^d A(y)\,dy = \int_c^d \pi[R(y)]^2\,dy.$$

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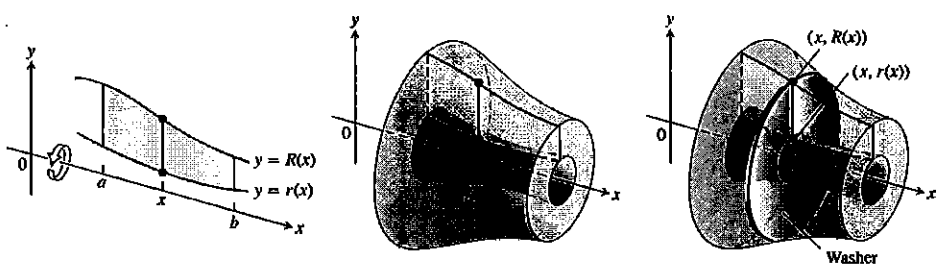
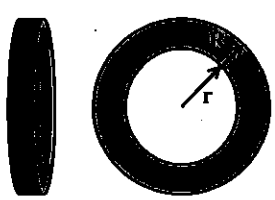


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.



Volume of the washer

$$V = \pi R^2 \cdot thickness - \pi r^2 \cdot thickness$$

$$V = \pi R^2\,dx - \pi r^2\,dx$$

$$V = \pi(R^2 - r^2)\,dx$$

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Solids of Revolution: The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are *washers* (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

Outer radius: $R(x)$

Inner radius: $r(x)$

The washer's area is

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$

Consequently, the definition of volume in this case gives

Volume by Washers for Rotation About the x-axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.$$

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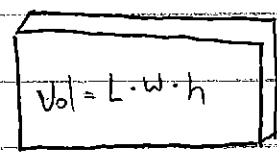
6.2 Notes

cut in half



length = Circumference

height or width = $f(x) - G(x)$ top - bottom

thickness = dx

$$\therefore \text{Volume} = \int 2\pi r \cdot \left(f(x) - g(x)\right) \cdot dx$$

Rotate about Horizontal Line

$$\text{Volume} = \int 2\pi r \cdot \left(f(y) - g(y)\right) \cdot dy$$

6.5 Work Notes

Work done by Constant force

$W = Fd$ Work = Force * distance

$$W = \int_a^b F(x)\,dx$$

Hooke's Law for Springs

$F = Kx$ x = Spring length under force minus original length

1 m

24N = 1.8 m

$F = Kx$

$24N = K(0.8m)$

$K = \frac{24N}{.8m} = 30 \frac{N}{m}$

$W = \int_0^2 30x\,dx = 15x^2 \Big|_0^2 = \boxed{60 \frac{N}{m}}$

$F = 0 \Rightarrow 45N = 30x \Rightarrow \frac{45}{30} \Rightarrow x = 1.5$

$\frac{30 lbs}{1 N} = K(1)$

$\int_3^6 30x\,dx = 15x^2 \Big|_3^6$

Bucket = 1000 lbs

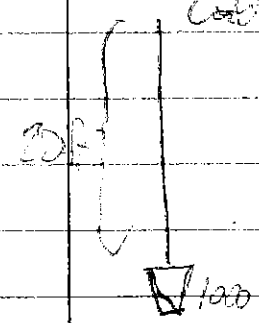
Cable = 10 lbs/ft

$W_t = ?$

$K = 10 lbs/ft$

$W_b = F \cdot d = 1000 lbs \cdot 20 ft$

$W_b = 20{,}000 lbs$

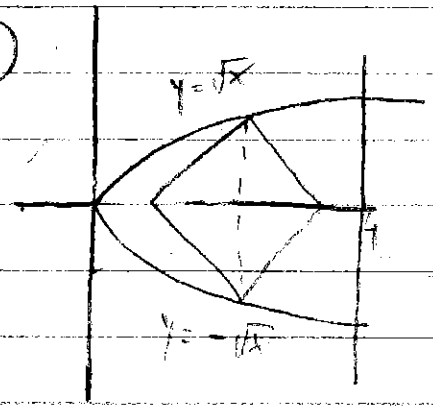


#1, 5, 9, 19, 21, 25, 31, 39, 41, 43, 51, 53

6.1 Homework

$X=0,\ X=4$

1)



$y=-\sqrt{x} \rightarrow \sqrt{x}$

$$\int_0^4 \frac{1}{2}(2\sqrt{x})^2\,dx =$$

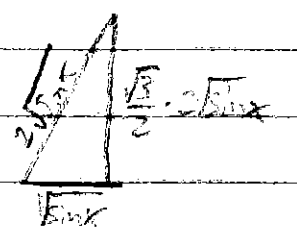
$\text{Area} = \frac{1}{2} d \cdot d_2$ $\quad d = \sqrt{x} - (-\sqrt{x})$

$\text{Area} = \frac{1}{2} d^2$ $\quad d = 2\sqrt{x}$

$\text{Area} = \frac{1}{2}(2\sqrt{x})^2$

5)

$A = \frac{1}{2} b h$

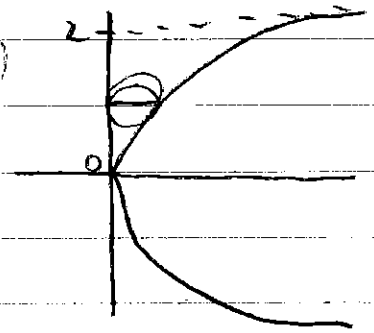


$$A = \frac{1}{2} \cdot \left(\sqrt{\sin x}\right) \cdot \left(\frac{\sqrt{3}}{2} \cdot 2\sqrt{\sin x}\right)$$

$$A = \sqrt{3}\sin x$$

$$\int_0^{\pi} \sqrt{3}\sin x\,dx$$

(9)



$x = \sqrt{5}\, y^2$ = diameter

$y = \pm\sqrt{\frac{x}{\sqrt{5}}}$

Area $= \pi r^2$

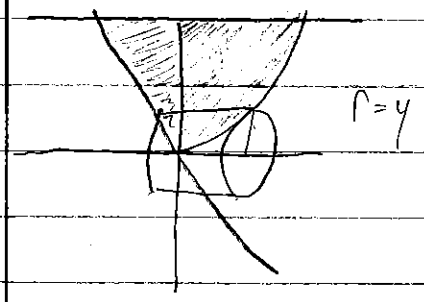
radius $= \frac{1}{2} \cdot \sqrt{5}\, y^2$

$$\int_0^2 \pi r^2\, dy = \int_0^2 \pi \left(\frac{\sqrt{5}}{2} y^2\right) dy$$

$$\int_0^2 \pi \frac{5}{4} y^4\, dy = \frac{5\pi}{4} \cdot \frac{y^5}{5} \Big|_0^2$$

6.3

(15) $x = \sqrt{y}$ $\quad x = -y$ $\quad y = 2$



$\text{Poi} = \sqrt{y} = -y$

$y = 0$

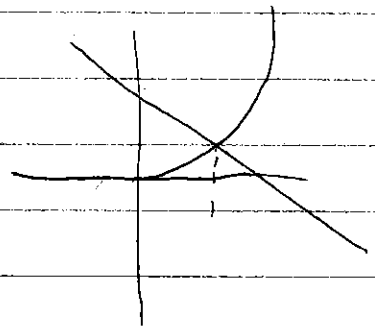
$$\int_0^2 2\pi y\left(\sqrt{y} - (-y)\right) dy$$

$$\int_0^2 2\pi y\left(y^{\frac{1}{2}} + y\right) dy$$

$$\int_0^2 2\pi\left(y^{3/2} + y^2\right) dy = 2\pi\left(\frac{2}{5}y^{\frac{5}{2}} + \frac{y^3}{3}\right)\Big|_0^2$$

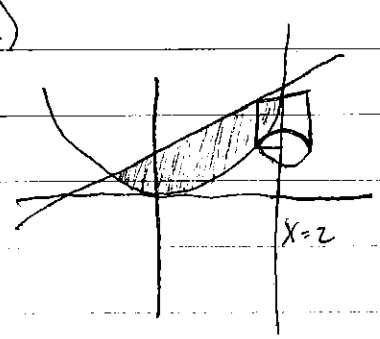
$$\boxed{30.972}$$

(9)



a)

25



P_{oi}

$x+2=x^2$

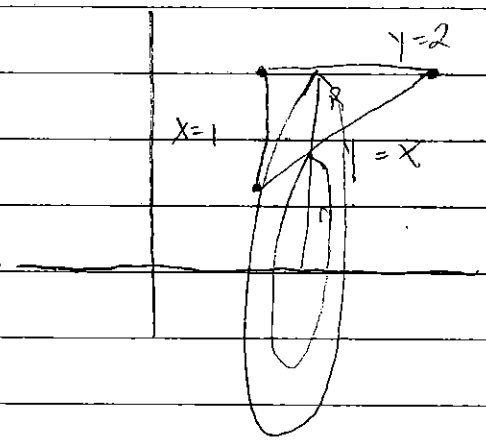
$x=-1, 2$

$r=2-x$

$$\int_{-1}^{2} 2\pi(2-x)(x+2-x^2)\,dx$$

b, c = Shells
a, d = washers

(31) (1, 1) (1, 2) (2, 2)



$R^2 - r^2$

$R = 2 \quad r = x$