HCC MAC 2313 Calculus and Analytic Geometry III

COURSE SYLLABUS

GENERAL INFORMATION	
Section Number:	80278
Meeting Times:	MW 6pm – 8:15pm (SSCI 106)
Credits:	5
Term:	Spring 2016

INSTRUCTOR INFORMATION	
Instructor:	Ms. Wendy Pogoda
Office:	SMPF 222
Office Hours:	TTH 11:30am – 2pm (SMPF 222), MW 3-5:30pm (online via e-mail)
Telephone:	(813) 253-7000 ext. 5728
E-mail:	wpogoda@hccfl.edu – main method

Course Description

A continuation of MAC 2312. Focuses on arc length and surface area, vectors in two and three dimensional space, planes, lines and surfaces in three-dimensional space, functions of more than one variable, partial derivatives, double and triple integrals and their applications, cylindrical and spherical coordinates, vector fields, line integrals, Green's theorem and Stoke's theorem.

Prerequisites: MAC 2312 with a minimum grade of C.

Course Objectives

1. Students will be able to define, determine, and/or solve problems involving:

   a. plane curves
   b. tangent lines and arc length

2. Students will be able to define, determine, and/or solve problems involving:

   a. vectors in the plane
   b. vectors in three dimensions
   c. the dot product
   d. the vector product
   e. lines and planes
   f. surfaces

3. Students will be able to define, determine, and/or solve problems involving:

   a. functions and space curves
   b. calculus of vector-values functions (limits, derivatives, integrals)
   c. motion
   d. curvature
   e. tangential and normal components of acceleration

4. Students will be able to define, determine, and/or solve problems involving:

   a. functions of several variables
   b. limits of functions of more than one variable
   c. continuity of functions of more than one variable
   d. partial derivatives
   e. differentiability and the total differential
   f. the chain rule
   g. directional derivatives
   h. tangent planes and normal lines
   i. extrema of functions of several variables
   j. Lagrange multipliers

5. Students will be able to define, determine, and/or solve problems involving:

   a. the double integral
   b. area and volume
   c. the double integral in polar coordinates
   d. surface area
   e. the triple integral
   f. center of mass and moments of inertia
   g. cylindrical coordinates
   h. spherical coordinates
   i. change of variables and Jacobians.

6. Students will be able to define, determine, and/or solve problems involving:

   a. vector fields
   b. line integrals
   c. line integrals independent of path
   d. Green's theorem
   e. surface integrals
   f. the divergence theorem
   g. Stoke's theorem

Course Materials

Text: Thomas Calculus Multivariable 13th Edition by Thomas

Calculator: A graphing calculator is required. (TI-83 or TI-84 recommended)

Graded Assignments

Exams: 4 exams, plus a cumulative final. Lowest of the 5 exam grades are dropped. Each of the remaining exams are worth 100 points.

Lowest dropped

Cumulative Final replaces Lowest Grade

Tentative Lesson and HW Schedule

Date	Lesson	Suggested HW
M 1/11	12.1 – 3D Coordinate System	1,5,9,15,17,19,21,25,27,29,37,43,45,47,53,57,59,61
W 1/13	12.2 – Vectors	1,3,5,9,15,17,21,25,33,35,39,41,43,47
	12.3 – The Dot Product	1,3,9,13,21,23,31,33,37,41,43
M 1/18	**College Closed - MLK**	
W 1/20	12.4 – The Cross Product	1, 5, 7, 9, 11, 15, 21, 25, 37, 39, 41, 45
	12.5 – Lines and Planes in Space	1, 3, 13, 19, 21, 23, 33, 37, 39, 53, 59
M 1/25	12.6 – Cylinders and Quadric Surfaces	1–11 odd
	13.1 – Curves in Space & Their Tangents	1, 5, 9, 15, 19, 23
W 1/27	13.2 – Integrals of Vector Functions & Projectile Motion	1, 5, 9, 11, 15, 23
	13.3 – Arc Length in Space	1, 3, 11, 13
M 2/1	13.4 – Curvature & Normal Vectors of a Curve	1, 9, 13
	13.5 – Tangential & Normal Components of Acceleration	1, 9, 13
W 2/3	13.6 – Velocity & Acceleration in Polar Coordinates	
M 2/8	Review	
W 2/10	**Test 1: Chapter 12 & 13**	***HW Chapters 12 & 13 due***
M 2/15	**College Closed – Presidents' Day**	~~1, 5, 9~~
W 2/17	14.1 – Functions of Several Variables	1, 5, 9
	14.2 – Limits and Continuity in Higher Dimensions	1–29 EOO, 33, 35, 37, 41, 45, 49 55 57, 61
M 2/22	14.3 – Partial Derivatives	1–21 EOO, 23, 27, 29, 33 37 39, 43, 53, 65, 73, 81
	14.4 – The Chain Rule	3, 7, 9, 11 meh 13, 17, 21, 23, 25, 27, 29, 33, 41, ~~49~~
W 2/24	14.5 – Directional Derivatives & Gradient Vectors	1, 3, 7, 11, 15, 19, 21, 27
	14.6 – Tangent Planes and Differentials	1, 3, 9, 11, 19, 23, 29
M 2/29	14.7 – Extreme Values & Saddle Points	1, 21, 31, 35, 41
	14.8 – Lagrange Multipliers	any 3 questions
W 3/2	14.9 – Taylor's Formula for 2 Variables*	
	14.10 – Partial Derivatives w/ Constraints*	
M 3/7	Review	
W 3/9	**Test 2: Chapter 14**	***HW Chapter 14 due***
3/14-20	**College Closed - Spring Break**	
M 3/21	15.1 – Double & Iterated Integrals over Rectangles	1, 9, 25, 15,
	15.2 – Double Integrals over General Regions	9, 13, 19, 23, 33, 41, 57, 47
W 3/23	15.3 – Area by Double Integration	7, 13, 21,
	15.4 – Double Integrals in Polar Form	1, 5, 11, 29
M 3/28	15.5 – Triple Integrals in Rectangular Coordinates	1–25 odd
	15.6 – Moments and Centers of Mass	11, 13, ~~21~~ 23
W 3/30	15.7 – Triple Integrals in Cylindrical & Spherical Coordinates*	
	15.8 – Substitutions in Multiple Integrals*	
M 4/4	Review	
W 4/6	**Test 3: Chapter 15**	***HW Chapter 15 due***
M 4/11	16.1 – Line Integrals	9, 13, 35
	16.2 – Vector Fields & Line Integrals	3, 7, 21, 23, 29
W 4/13	16.3 – Path Independence, Conservative Fields, etc.	1, 3, 13, 19
	16.4 – Green's Theorem in the Plane	5, 21, 27
M 4/18	16.5 – Surfaces and Area	3, 7, 21, 39 3 = Cyl, 7 = Sph
	16.6 – Surface Integrals	7, 13
W 4/20	16.7 – Stokes' Theorem	5,
M 4/25	16.8 – The Divergence Theorem & a Unified Theory	
W 4/27	Review	
M 5/2	**Test 4: Chapter 16**	***HW Chapter 16 due***
W 5/4	*Final Exam Week (our class will not meet this Wednesday)*	
M 5/9	**Final Exam – 6pm to 7:50pm**	

The rest of the HW will be announced in class and/or posted on Canvas.

THEOREM 5 The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots ar^{n-1} + ar^n$$

A Geometric Series will converge to $S_n = \frac{a}{1-r}$ provided that $|r| < 1$.

If $|r| \geq 1$, then the series will diverge.

THEOREM 7 If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

The Integral Test:

Let f be a continuous, positive and decreasing function and $a_n = f(x)$ for all positive integers, then $\sum_{n=1}^{\infty} a_n$ converges iff the improper integral $\int_1^{\infty} f(x)dx$ converges

If the improper integral diverges, then the series diverges.

p-Series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ p is a constant

If p is > 1, then the series converges.

If p is ≤ 1, then the series diverges.

Comparison Test (Ordinary Comparison Test)

Let $\sum_{n=1}^{\infty} U_n$ be a positive series.

If $\sum_{n=1}^{\infty} V_n$ is a positive convergent series and $0 \leq U_n \leq V_n$, then the series $\sum_{n=1}^{\infty} U_n$ converges.

If $\sum_{n=1}^{\infty} V_n$ is a positive divergent series and $0 \leq V_n \leq U_n$, then the series $\sum_{n=1}^{\infty} U_n$ diverges.

Limit Comparison Test

Let $\sum_{n=1}^{\infty} U_n$ be a positive series and U_n is a rational expression.

Choose V_n to be a positive series. If $\lim_{b\to\infty} \frac{U_n}{V_n} = L$, then

$\sum_{n=1}^{\infty} U_n$ will converge or diverge depending on $\sum_{n=1}^{\infty} V_n$ provided $0 < L < \infty$.

$\sum_{n=1}^{\infty} U_n$ will converge if $\sum_{n=1}^{\infty} V_n$ converges provided $L = 0$.

$\sum_{n=1}^{\infty} U_n$ will diverge if $\sum_{n=1}^{\infty} V_n$ diverges provided $L = \infty$.

If:

1) $a_n > 0$
2) $a_{n+1} \leq a_n$
3) $\lim_{n\to\infty} a_n = 0$,

then the alternating series is convergent.

The Ratio Test:

Let $\sum_{n=1}^{\infty} a_n$ be a positive series and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$.

If $\rho < 1$, then the series converges.

If $\rho > 1$, then the series diverges.

If $\rho = 1$, then the test is inconclusive.

The Root Test:

Let $\sum_{n=1}^{\infty} a_n$ be a positive series and $\lim_{n\to\infty} |\sqrt[n]{a_n}| = \rho$.

If $\rho < 1$, then the series converges.

If $\rho > 1$, then the series diverges.

If $\rho = 1$, then the test is inconclusive.

Taylor Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 +$$

For any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$.

Maclauren Series: centered at $x = a = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

$$= f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 +$$

The Binomial Series

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \quad \text{for } k \geq 3.$$

There are only three ways for a power series to converge

1) The series only converges at $x = a$. $(R = 0)$
2) The series converges for all x values. $(R = \infty)$
3) The series converges for some interval of x.

$|x - a| < R$

$(a - R < x < a + R)$

The end values of the interval must be tested for convergence.

Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}, \qquad -1 < x \leq 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \leq 1$$

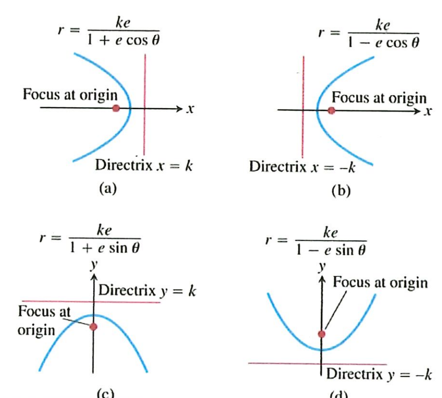
Polar Equation for a Conic with Eccentricity e

$$r = \frac{ke}{1 + e\cos\theta}$$

The vertical directrix is represented by k.

$$r = \frac{ke}{1 + e\sin\theta}$$

The horizontal directrix is represented by k.



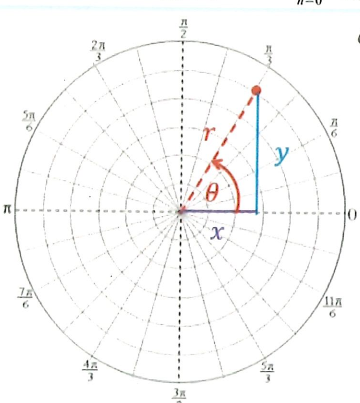
(r, θ)

$x = r\cos\theta$

$y = r\sin\theta$

$\tan\theta = \frac{y}{x}$

$x^2 + y^2 = r^2$



Lim r = a+b cos θ
a > b
Card r = a+b cos θ
a = b
Lim loop r = a+b cos θ
a < b
Circ r = b cos θ

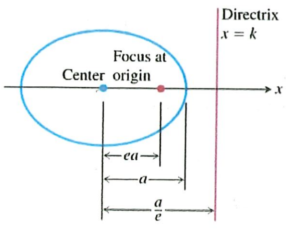
Equations of an Ellipse: Center at (h, k); Major Axis Parallel to a Coordinate Axis

Center	Major Axis	Foci	Vertices	Equation
(h, k)	Parallel to the x-axis	$(h + c, k)$ $(h - c, k)$	$(h + a, k)$ $(h - a, k)$	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$ $a > b > 0$ and $b^2 = a^2 - c^2$
(h, k)	Parallel to the y-axis	$(h, k + c)$ $(h, k - c)$	$(h, k + a)$ $(h, k - a)$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1,$ $a > b > 0$ and $b^2 = a^2 - c^2$

Hyperbola

Center	Transverse Axis	Foci	Vertices	Equation	Asymptotes
(h, k)	Parallel to the x-axis	$(h \pm c, k)$	$(h \pm a, k)$	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \quad b^2 = c^2 - a^2$	$y - k = \pm\frac{b}{a}(x - h$
(h, k)	Parallel to the y-axis	$(h, k \pm c)$	$(h, k \pm a)$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1, \quad b^2 = c^2 - a^2$	$y - k = \pm\frac{a}{b}(x - h$

Polar Equation of an Ellipse with Eccentricity e and Major Axis a



$$k = \frac{a}{e} - ea \qquad k = a\left(\frac{1}{e} - e\right)$$

$$k = a\frac{1}{e}(1 - e^2)$$

$$ke = a(1 - e^2)$$

$$r = \frac{ke}{1 + e\cos\theta}$$

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

Parabola

$y^2 = 4px$ $\qquad$ $x^2 = 4py$

$(y - k)^2 = 4p(x - h)$ $\qquad$ $(x - h)^2 = 4p(y - k)$

Length of a Polar Curve

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Area of the Region $0 \leq r_1(\theta) \leq r \leq r_2(\theta)$, $\alpha \leq \theta \leq$

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2\, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2\, d\theta = \int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

Reciprocal Identities

$\sin\theta = \frac{1}{\csc\theta}$ $\csc\theta = \frac{1}{\sin\theta}$

$\cos\theta = \frac{1}{\sec\theta}$ $\sec\theta = \frac{1}{\cos\theta}$

$\tan\theta = \frac{1}{\cot\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

Quotient Identities

$\frac{\sin\theta}{\cos\theta} = \tan\theta$

$\frac{\cos\theta}{\sin\theta} = \cot\theta$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Cofunction Identities

$\sin\theta = \cos(90^\circ - \theta)$ $\cos\theta = \sin(90^\circ - \theta)$

$\tan\theta = \cot(90^\circ - \theta)$ $\cot\theta = \tan(90^\circ - \theta)$

$\sec\theta = \csc(90^\circ - \theta)$ $\csc\theta = \sec(90^\circ - \theta)$

Opposite Angle Identities

$\sin(-A) = -\sin A$

$\cos(-A) = \cos(A)$

$\tan(-A) = -\tan(A)$

Sum and Difference Identities

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Double-Angle Identities

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

Half-Angle Identities

$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$

$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}, \cos\alpha \neq -1$

Product-Sum Identities

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

Differentiation Formulas:

1. $\frac{d}{dx}(x) = 1$
2. $\frac{d}{dx}(ax) = a$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\tan x) = \sec^2 x$
7. $\frac{d}{dx}(\cot x) = -\csc^2 x$
8. $\frac{d}{dx}(\sec x) = \sec x \tan x$
9. $\frac{d}{dx}(\csc x) = -\csc x(\cot x)$
10. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(a^x) = (\ln a)a^x$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

Integration Formulas:

1. $\int 1\,dx = x + C$
2. $\int a\,dx = ax + C$
3. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
4. $\int \sin x\,dx = -\cos x + C$
5. $\int \cos x\,dx = \sin x + C$
6. $\int \sec^2 x\,dx = \tan x + C$
7. $\int \csc^2 x\,dx = -\cot x + C$
8. $\int \sec x(\tan x)\,dx = \sec x + C$
9. $\int \csc x(\cot x)\,dx = -\csc x + C$
10. $\int \frac{1}{x}dx = \ln|x| + C$
11. $\int e^x dx = e^x + C$
12. $\int a^x dx = \frac{a^x}{\ln a} + C\ a > 0, a \neq 1$
13. $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$
14. $\int \frac{1}{1+x^2}dx = \tan^{-1} x + C$
15. $\int \frac{1}{|x|\sqrt{x^2-1}}dx = \sec^{-1} x + C$

$\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$

$\frac{d}{dx}\sinh x = \cosh x = \frac{e^x + e^{-x}}{2}$

$\frac{d}{dx}\cosh x = \sinh x = \frac{e^x - e^{-x}}{2}$

$\frac{d}{dx}\tanh x = \text{sech}^2 x$

$\frac{d}{dx}\text{sech}\, x = -\tanh x\ \text{sech}\, x$

$\frac{d}{dx}\coth x = -\text{csch}^2 x$

$\frac{d}{dx}\text{csch}\, x = -\coth x\ \text{csch}\, x$

$\frac{d}{dx}\text{arcsinh}\, x = \frac{1}{\sqrt{x^2+1}}$

$\frac{d}{dx}\text{arccosh}\, x = \frac{1}{\sqrt{x^2-1}}$

$\frac{d}{dx}\text{arctanh}\, x = \frac{1}{1-x^2}$

$\frac{d}{dx}\text{arcsech}\, x = \frac{-1}{x\sqrt{1-x^2}}$

$\frac{d}{dx}\text{arccoth}\, x = \frac{1}{1-x^2}$

$\frac{d}{dx}\text{arccsch}\, x = \frac{-1}{|x|\sqrt{1+x^2}}$

$\int \sinh x\,dx = \cosh x + c$

$\int \cosh x\,dx = \sinh x + c$

$\int \tanh x\,dx = \ln(\cosh x) + c$

$\int \coth x\,dx = \ln|\sinh x| + c$

$\int \text{sech}\, x\,dx = \tan^{-1}|\sinh x| + c$

$\int \text{csch}\, x\,dx = \ln\left|\tanh\frac{x}{2}\right| + c$

Partial Fractions:

$\int \frac{1}{(x^2-a^2)} = \frac{A}{(x+a)} + \frac{B}{(x-a)}$

$\int \frac{1}{(x+a)^2} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2}$

$\int \frac{1}{(x+a)(x^2+b)} = \frac{A}{(x+a)} + \frac{Bx+C}{(x^2+b)}$

$\int \frac{-1}{\sqrt{1-x^2}} = \cos^{-1} x$

$\int \tan u\,du = \ln|\sec u| + C$

$\int \cot u\,du = \ln|\sin u| + C$

$\int \sec u\,du = \ln|\sec u + \tan u| + C$

$\int \csc u\,du = -\ln|\csc u + \cot u| + C$

$\int \frac{du}{\sqrt{a^2+u^2}} = \text{arsinh}\left(\frac{u}{a}\right) + C$

$\int \frac{du}{\sqrt{u^2-a^2}} = \text{arcosh}\left(\frac{u}{a}\right) + C$

$\int \frac{du}{a^2-u^2} = a^{-1}\,\text{artanh}\left(\frac{u}{a}\right) + C; u^2 < a^2$

$\int \frac{du}{a^2-u^2} = a^{-1}\,\text{arcoth}\left(\frac{u}{a}\right) + C; u^2 > a^2$

$\int \frac{du}{u\sqrt{a^2-u^2}} = -a^{-1}\,\text{arsech}\left(\frac{u}{a}\right) + C$

$\int \frac{du}{u\sqrt{a^2+u^2}} = -a^{-1}\,\text{arcsch}\left|\frac{u}{a}\right| + C$

TABLE 7.4 Identities for hyperbolic functions

$\cosh^2 x - \sinh^2 x = 1$

$\sinh 2x = 2\sinh x\cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \frac{\cosh 2x + 1}{2}$

$\sinh^2 x = \frac{\cosh 2x - 1}{2}$

$\tanh^2 x = 1 - \text{sech}^2 x$

$\coth^2 x = 1 + \text{csch}^2 x$

$\text{sech}^{-1} x = \cosh^{-1}\frac{1}{x}$

$\text{csch}^{-1} x = \sinh^{-1}\frac{1}{x}$

$\coth^{-1} x = \tanh^{-1}\frac{1}{x}$

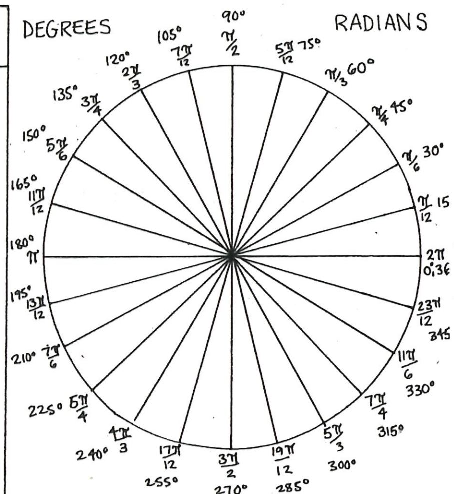
Long Division

$\frac{\text{Dividend}}{\text{Divisor}}$

Divisor | Dividend

Trigonometry

Deg.	Rad.	Sine	Cosine	Tangent
30	$\pi/6$	0.5	$\sqrt{3}/2$	$1/\sqrt{3}$
45	$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
60	$\pi/3$	$\sqrt{3}/2$	0.5	$\sqrt{3}$
90	$\pi/2$	1	0	UND.
120	$2\pi/3$	$\sqrt{3}/2$	-0.5	$-\sqrt{3}$
135	$3\pi/4$	$1/\sqrt{2}$	$-1/\sqrt{2}$	-1
150	$5\pi/6$	0.5	$-\sqrt{3}/2$	$-1/\sqrt{3}$
180	π	0	-1	0
210	$7\pi/6$	-0.5	$-\sqrt{3}/2$	$1/\sqrt{3}$
225	$5\pi/4$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1
240	$4\pi/3$	$-\sqrt{3}/2$	-0.5	$\sqrt{3}$
270	$3\pi/2$	-1	0	UND.
300	$5\pi/3$	$-\sqrt{3}/2$	0.5	$-\sqrt{3}$
315	$7\pi/4$	$-1/\sqrt{2}$	$1/\sqrt{2}$	-1
330	$11\pi/6$	-0.5	$\sqrt{3}/2$	$-1/\sqrt{3}$
0,360	2π	0	1	0



A. $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$

1) If n is odd:
   a) Factor out a sinx or a cosx.
   b) The remaining part will be raised to an even power.
   c) Use the appropriate identity as listed below.
      $\sin^2\theta = 1 - \cos^2\theta$
      $\cos^2\theta = 1 - \sin^2\theta$
   d) Replace the even power with appropriate identity, foil, and distribute the factor.
   e) Use "u du" substitution where needed.
2) If n is even:
   a) Use the half-angle identities as listed below.
      $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$
      $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$
   b) Foil and integrate. (You may need the half-angle identities again.)

B. $\int \sin^m x\cos^n x\,dx$

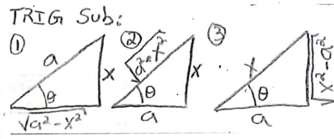
1) One of the exponents (m or n) is odd.
   a) Factor a trig functions from the one with the odd power.
   b) The remaining factor should be substituted the appropriate identity from below.
      $\sin^2\theta = 1 - \cos^2\theta$
      $\cos^2\theta = 1 - \sin^2\theta$
   c) Foil (if necessary), distribute, and integrate.

$\frac{V \cdot U' - U \cdot V'}{V^2}$

C. $\int \tan^m x\,dx$ or $\int \sec^n x\,dx$

1) $\tan^2 x = \sec^2 x - 1$
2) $\sec^2 x = \tan^2 x + 1$

TRIG Sub:



① $x = a\sin\theta$
② $x = a\tan\theta$
③ $x = a\sec\theta$

Simpson's Rule

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1})$$

Where $\Delta x = \frac{b-a}{n}$ and n represents an even number of partitions.

$|E_S| \leq \frac{M(b-a)^5}{180n^4}$.

Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_3) + \cdots + 2(f(x_{n-1}) + f(x_n))$$

Where $\Delta x = \frac{b-a}{n}$ and n represents the number of trapezoids.

$|E_T| \leq \frac{M(b-a)^3}{12n^2}$.

First derivative:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Second Derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} \quad \frac{dy'}{dx}$$

$$\text{arc length } (s) = \int_{LL}^{UL} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$(SA) = \int_{LL}^{UL} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

x – axis symmetry

$(r, \theta) = (r, -\theta)$

y – axis symmetry

$(r, \theta) = (r, \pi - \theta)$

origin symmetry

$(r, \theta) = (r, \pi + \theta)$

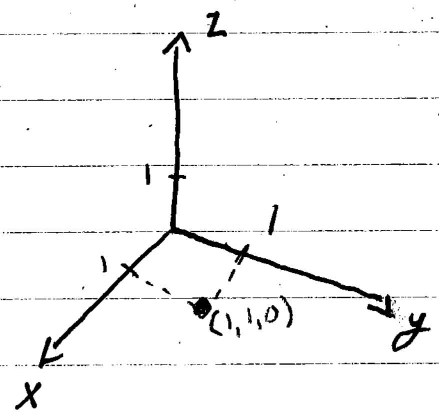
Slope of a Tangent for $r = f(\theta)$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

12.1 Three Dimensional Coordinate System

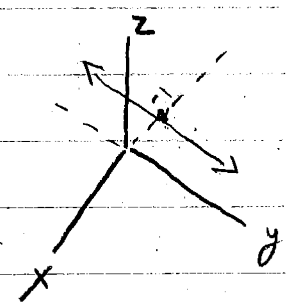
3-Dimensional Drawing



Ordered Triple (x, y, z)

Ex:

② $x = -1$, $z = 0$



A line parallel to the y-axis, going through the point $(-1, 0, 0)$

⑥ $x^2 + y^2 = 4$, $z = -2$

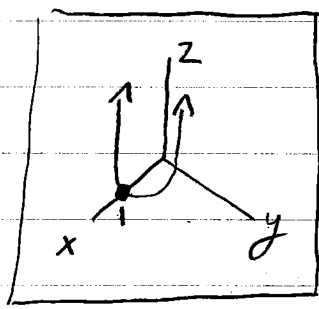
circle moved down

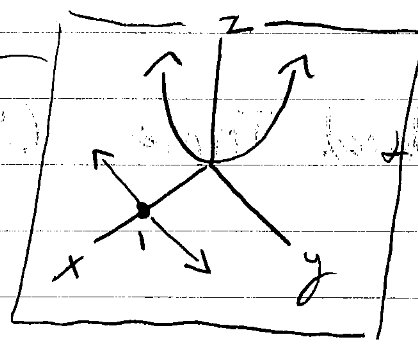
Graph is a circle with radius 2
center at $(0, 0, -2)$
on the $z = -2$ plane

⑯ $z = y^2$, $x = 1$

Parabola Shifted forward plane

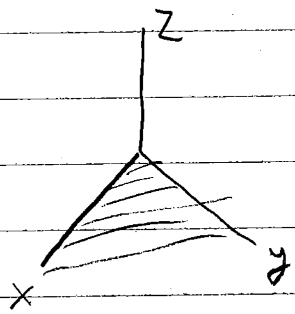
opening up





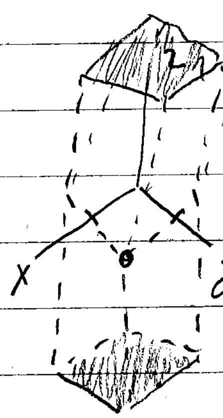
Parabola with vertex at
$(1, 0, 0)$ opening upwards
in the $x = 1$ plane

(17) a) $X \geq 0$, $Y \geq 0$, $Z = 0$



Surface area of quadrant 1 including the axes X & Y

(18) b) $0 \leq X \leq 1$, $0 \leq Y \leq 1$



Rectangular solid stretching infinitely in positive & negative Z plane bounded by the first Quadrant between 0 & 1

Distance formula
denoted by $|P_1 P_2|$ or $||P_1 P_2||$
where P_1 = point 1 & P_2 = point 2.

$$|P_1 P_2| = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}$$

Sphere Formula
All points are equal distance from the center (X_0, Y_0, Z_0)

$$r^2 = (X - X_0)^2 + (Y - Y_0)^2 + (Z - Z_0)^2$$

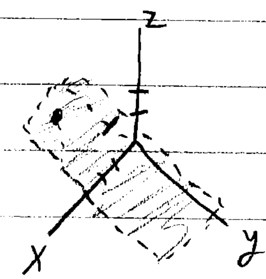
Ex:

(21) a) $1 \le x^2 + y^2 + z^2 \le 4$

ball of radius 2, centered at (0,0,0)
w/ inner ball or r=1 missing

(26) Plane through (3,-1,2) perp to...

a) x-axis



$x = 3$

(30) Circle of r=1 centered at (-3,4,1)
lying in a plane parallel to...

a) xy-Plane $(x+3)^2 + (y-4)^2 = 1^2, \ z = 1$

Cylinder centered at this point
chopped at $z = 1$

(38) Upper hemisphere of the sphere of $r=1$ centered at the origin

$x^2+y^2+z^2=1^2, \quad z>0$

(42) $P_1(-1,1,5)$ $P_2(2,5,0)$

Find distance between them

$$\sqrt{(2+1)^2+(5-1)^2+(0-5)^2} = \sqrt{(3)^2+(4)^2+(-5)^2}$$
$$= \sqrt{9+16+25} = \sqrt{50} = \boxed{5\sqrt{2}}$$

(48) Find Center & radius

$(x-1)^2+(y+1/2)^2+(z+3)^2 = 25$

ctr $= (1,-\frac{1}{2},-3)$ radius $= 5$

(52) Center of sphere $(0,-1,5)$, $r=2$

$x^2+(y+1)^2+(z-5)^2=4$

(56) $x^2+y^2+z^2-6y+8z=0$

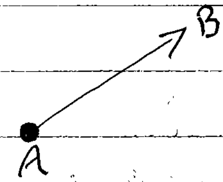
$x^2+y^2-6y+\underline{9}+z^2+8z+\underline{16} = 0+\underline{9}+\underline{16}$

$-\frac{6}{2}=-3^2=9$ $\quad$ $\frac{8}{2}=4^2=16$

$x^2+(y-3)^2+(z+4)^2=25$

ctr $=(0,3,-4)$, $r=5$

12.2 Vectors



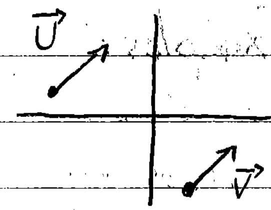
Vector $\overrightarrow{AB}$

includes both magnitude & direction.
Ex: Force, displacement, velocity

Magnitude or Length of $\overrightarrow{AB}$ is written
$|\overrightarrow{AB}|$ or $||\overrightarrow{AB}||$

Starting position & ending position of a vector do not matter. Both vectors can be the same

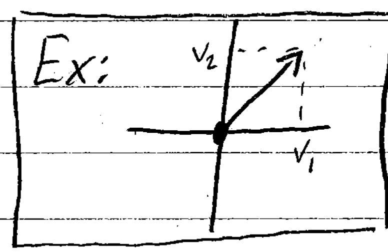
Ex:



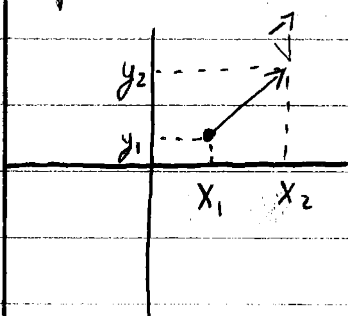
$\vec{u} = \vec{v}$

Standard position for vectors:
This can be written as $\vec{v} = \langle v_1, v_2 \rangle = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

"Components" of $\vec{v}$



If the vector is not in standard position, to get the component form:



$$\vec{V} = \langle \underbrace{X_2 - X_1}_{V_1}, \underbrace{y_2 - y_1}_{V_2} \rangle$$

Length (magnitude) of $\vec{V}$

$$= \sqrt{V_1^2 + V_2^2 + V_3^2} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Examples:

(1) $P(1,3)$ $Q(2,-1)$

Find magnitude & component form of vector $\overrightarrow{PQ}$

$$\text{mag} = \sqrt{(2-1)^2 + (-1-3)^2} = \text{mag} = \sqrt{1+16}$$

$$\text{mag} = \sqrt{17} \longrightarrow \boxed{|\overrightarrow{PQ}| = \sqrt{17}}$$

Component form : $\langle x_2 - x_1, y_2 - y_1 \rangle$

$$\langle 2-1, -1-3 \rangle$$

$$\boxed{= \langle 1, -4 \rangle}$$

Algebra: If $\vec{U} = \langle U_1, U_2, U_3 \rangle$
and $\vec{V} = \langle V_1, V_2, V_3 \rangle$ then...

a) $\vec{U} + \vec{V} = \langle U_1 + V_1, U_2 + V_2, U_3 + V_3 \rangle$

Scalar Multiplication:

If "K" is a scalar & $\vec{V}$ is a vector
then $K\vec{V} = \langle KV_1, KV_2, KV_3 \rangle$

Ex: $\vec{U} = \langle 3, 1 \rangle$, $\vec{V} \langle 1, 2 \rangle$

$\vec{U} + \vec{V} = \langle 4, 3 \rangle$, $7\vec{U} \langle 21, 7 \rangle$

Properties:

1) $\vec{U} + \vec{V} = \vec{V} + \vec{U}$ 2) $(\vec{U} + \vec{V}) + \vec{w} = \vec{U} + (\vec{V} + \vec{w})$

Ex: ② $\vec{U} \langle 3, -2 \rangle$, $\vec{V} \langle -2, -5 \rangle$

a) Component Form b) magnitude

a) $-2\vec{V} = \langle -2(-2), -2(-5) \rangle = \langle 4, 10 \rangle$

b) $\|-2\vec{V}\| = \sqrt{4^2 + 10^2} = \sqrt{116} = \boxed{2\sqrt{29}}$

⑧ $-\frac{5}{13}\vec{U} + \frac{12}{13}\vec{V} = \frac{-5}{13}\langle 3, -2 \rangle + \frac{12}{13}\langle -2, -5 \rangle$

$\langle -15/13, 10/13 \rangle + \langle -24/13, -60/13 \rangle = \langle -\frac{39}{13}, \frac{-50}{13} \rangle$

$= \boxed{\langle -3, \frac{-50}{13} \rangle}$

b) $\left\| -\frac{5}{13}\vec{u} + \frac{12}{13}\vec{v} \right\| = \sqrt{9 + \frac{2500}{169}} = \sqrt{\frac{4021}{169}}$

$= \boxed{\frac{\sqrt{4021}}{13}}$

Midpoint Formula: $\frac{x_1 + x_2}{2}$

$P(x_1, y_1, z_1) \quad Q(x_2, y_2, z_2)$

Midpt: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$

10) Vector $\vec{OP}$, O = origin

P is midpt of RS

$R(2, -1) \quad S(-4, 3)$

$P = \left(\frac{-4+2}{2}, \frac{3-1}{2} \right) = P = (-1, 1)$

$\vec{OP} = \langle -1, 1 \rangle$

Unit vector = Vector w/ magnitude 1

"Standard" unit vector

$i \langle 1, 0, 0 \rangle$

$j \langle 0, 1, 0 \rangle$

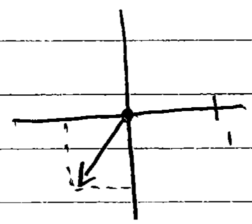
$k \langle 0, 0, 1 \rangle$

Unit vector in direction of $\vec{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$

Ex: $\vec{v}$ $\langle 3, 7, -1 \rangle$

$\|\vec{v}\| = \sqrt{59}$ So... $\left\langle \dfrac{3}{\sqrt{59}}, \dfrac{7}{\sqrt{59}}, \dfrac{-1}{\sqrt{59}} \right\rangle$

(14) Unit vector that makes an $\measuredangle$ $\theta = \dfrac{-3\pi}{4}$ with the x-axis. Find vector

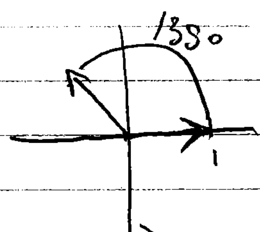


$\cos\theta = \cos -\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

$\sin\theta \quad \sin -\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

$\boxed{\left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle}$

(16) Unit vector obtained by rotating $\langle 1, 0 \rangle$ 135° counter clockwise about origin



$\left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

(18) $\overrightarrow{P_1P_2}$ if $P_1 = (1, 2, 0)$ & $P_2 = (-3, 0, 5)$

$\overrightarrow{P_1P_2} = (-3-1)\vec{i} + (0-2)\vec{j} + (5-0)\vec{k}$

$= -4\vec{i} - 2\vec{j} + 5\vec{k}$

22) $-2\vec{u} + 3\vec{v}$ if $\vec{u} = \langle -1, 0, 2\rangle$

$\vec{v} = \langle 1, 1, 1\rangle$

$\vec{u} = -i + 2k \qquad v = i + j + k$

$-2(-i + 2k) + 3(i + j + k)$

$2i - 4k + 3i + 3j + 3k = \boxed{5\vec{i} + 3\vec{j} - \vec{k}}$

26 $9\vec{i} - 2\vec{j} + 6\vec{k}$ Express as a product of its Length & direction

$\text{mag} = \sqrt{81 + 4 + 36}$

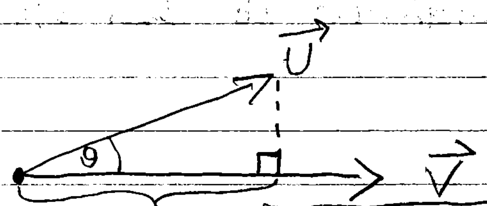
$= \sqrt{121} = \boxed{11}$

$\text{direction} = \dfrac{9\vec{i} - 2\vec{j} + 6\vec{k}}{11}$

$\text{mag} \times \text{direction} = 11\left(\dfrac{9i - 2j + 6k}{11}\right) =$

$= \boxed{9\vec{i} - 2\vec{j} + 6\vec{k}}$

12.3 The Dot Product



Dot products only between 2 vectors & gives a Scalar

$$\boxed{\text{projection of } U \text{ onto } V = \text{proj}_{\vec{V}}\vec{U}}$$

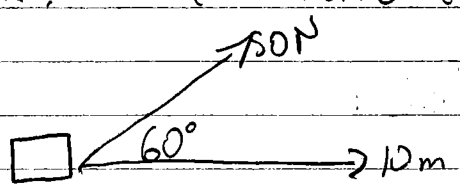
mag $\left|\text{proj}_{\vec{V}}\vec{U}\right|$ = Length of $\vec{U}$ on $\vec{V}$

$$\left|\text{proj}_{\vec{V}}\vec{U}\right| = |\vec{U}|\cos\theta$$
$$= \frac{\vec{U}\cdot\vec{V}}{|\vec{V}|}$$

dot Product:

$$\vec{U}\bullet\vec{V} = |\vec{U}||\vec{V}|\cos\theta$$

Ex: work = Force times distance



work = $\vec{U}\cdot\vec{V}$
work = $|\vec{U}|\,|\vec{V}|\cos\theta$

Work = (50N)(10m) Cos 60°
Work = 250Nm

Orthogonal = perpendicular to each other

The vectors $\vec{U}$ & $\vec{V}$ are orthogonal $\perp$ if $\vec{U} \cdot \vec{V} = 0$

Since $\cos 90° = 0$ $\quad |\vec{U}||\vec{V}|\cos 90 = 0$

Let $\vec{U} = \langle U_1, U_2, U_3 \rangle$

Let $\vec{V} = \langle V_1, V_2, V_3 \rangle$

$$\boxed{\vec{U} \cdot \vec{V} = U_1V_1, U_2V_2, U_3V_3}$$

Ex: $\vec{U} = 3\hat{i} + 4\hat{j} + 16\hat{k}$

$\vec{V} = -\hat{i} - \hat{j} + 3\hat{k}$

$\vec{U} \cdot \vec{V} = 3(-1) + 4(-1) + 16(3)$

$= \boxed{41}$

$$\text{proj}_{\vec{V}} \vec{U} = \frac{\vec{U} \cdot \vec{V}}{|\vec{V}|} \frac{\vec{V}}{|\vec{V}|} = \left(\frac{\vec{U} \cdot \vec{V}}{|\vec{V}|^2}\right)\vec{V}$$

Properties of dot product:

1) $U \cdot V = V \cdot U$
2) $(c\vec{U}) \cdot V = c(U \cdot V)$
3) $U(V + w) = U \cdot V + U \cdot w$
4) $\vec{U} \cdot \vec{U} = |\vec{U}|^2$
5) $\vec{0} \cdot \vec{U} = 0$

Examples:

2) $\vec{V} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{K}$ $\quad\quad$ $\vec{U} = 5\vec{i} + 12\vec{j}$

a) $\vec{V} \cdot \vec{U} = \left(\frac{3}{5}\right)(5) + 0 + \left(\frac{4}{5}\right)0 = \boxed{3}$

$|\vec{V}| = \sqrt{(3/5)^2 + (4/5)^2} = \sqrt{1} = 1$

b) $\cos\theta = \frac{\vec{U} \cdot \vec{V}}{|\vec{U}||\vec{V}|} = \frac{3}{13}$

c) $|\text{Proj}_{\vec{V}} \vec{U}| = |\vec{U}| \cos\theta = 13\left(\frac{3}{13}\right) = \boxed{3}$

d) $\text{proj}_{\vec{V}} \vec{U} = 3\left(\frac{\vec{V}}{|\vec{V}|}\right) = 3\left(\frac{3}{5}i + \frac{4}{5}k\right) = \boxed{\frac{9}{5}\hat{i} + \frac{12}{5}\hat{k}}$

8) $\vec{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right\rangle$, $\vec{U} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \right\rangle$

a) $\vec{V} \cdot \vec{U} = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}$

$|\vec{V}| = \sqrt{\frac{1}{2} + \frac{1}{3}} = \boxed{\sqrt{5/6}}$

$|\vec{U}| = \sqrt{\frac{1}{2} + \frac{1}{3}} = \boxed{\sqrt{5/6}}$

b) $\cos\theta = \frac{\vec{U} \cdot \vec{V}}{|\vec{U}||\vec{V}|} = \frac{1/6}{5/6} = \boxed{\frac{1}{5}}$

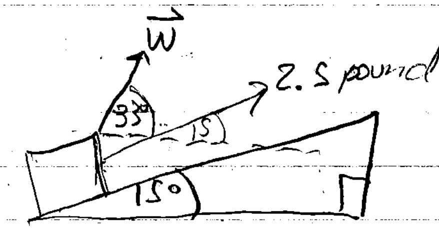
c) $|\text{Proj}_{\vec{V}} \vec{U}| = |U| \cos\theta = \left(\frac{\sqrt{5}}{\sqrt{6}}\right)\left(\frac{1}{5}\right) = \frac{\sqrt{30}}{6} \cdot \frac{1}{5}$

$\boxed{= \frac{\sqrt{30}}{30}}$

d) $\text{proj}_{\vec{v}}\vec{U} = \frac{\sqrt{30}}{30}\left(\frac{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \rangle}{\sqrt{30}/6}\right)$

$= \left\langle \frac{1}{5\sqrt{2}}, \frac{1}{5\sqrt{3}} \right\rangle$

(24)



$|\vec{w}|\cos(33° - 15°) = 2.5$ pounds

$|\vec{w}| = \frac{2.5}{\cos(18°)}$

$\vec{w} = |\vec{w}| \langle \cos 33, \sin 33 \rangle$

$\vec{w} = \frac{2.5}{\cos 18°}\langle \cos 33, \sin 33 \rangle$

$\approx \langle 2.20, 1.43 \rangle$

$\langle \cos\theta, \sin\theta \rangle = 1$ → unit vector

since pyth says $\sqrt{\cos\theta^2 + \sin^2\theta} = 1$

Ex: $2i + 7j$

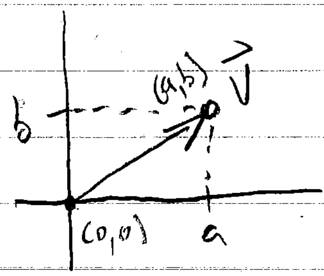
$\perp = 2x + 7y = 12$

$// = 7x - 2y = 12$

(31) Show $\vec{v} = a\vec{i} + b\vec{j}$ is $\perp$ to line $ax + by = c$

$\to by = -a x + c$

$y = \boxed{-\frac{a}{b}} x + \frac{c}{b}$ $\to$ Slopce



$\vec{v}$: $m = \frac{b-0}{a-0} = \boxed{\frac{b}{a}}$ $\to$ Slope of line

(32) Show vector $\vec{v} = a\vec{i} + b\vec{j}$ is // to $bx - ay = c$

$bx - ay = c \to -ay = -bx + c \to y = \boxed{\frac{b}{a}} x - \frac{c}{a}$

$\vec{v}$ slope $= \boxed{\frac{b}{a}}$ are equal $\therefore$ //

(34) Find eq. of line $\perp$ to $\vec{v} = -2i - j$ that goes thru $P(-1, 2)$

$a = -2$, $b - 1$ from $-2i - j$

$-2x - y = C$ plug in pts

$C = (-2(-1)) - 2 \to C = 0$

$\boxed{-2x - y = 0}$

(38) $P(0,-2)$ $\vec{v} = 2i + 3\vec{j}$ find // line

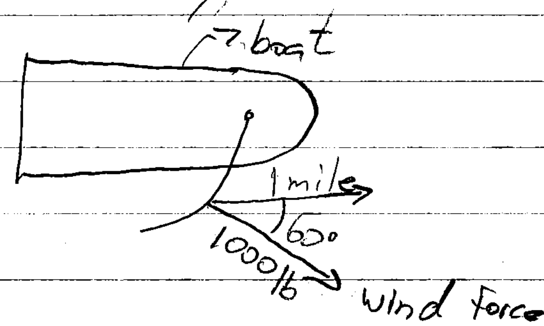
$a = 2,\ b = 3$

$bx - ay = c$

$c = 3(0) - 2(-2) \longrightarrow c = 4$

$\boxed{3x - 2y = 4}$

(44)



1 mile = 5280 feet

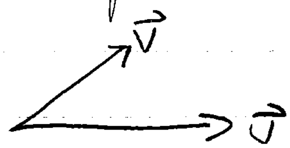
$w = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos\theta = (1{,}000)(5280) \cos 60°$

$\boxed{w = 2{,}640{,}000 \ \text{ft/lbs}}$

12.4 Cross Product

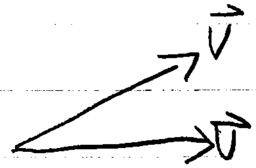
$\vec{U} \times \vec{V}$ = Cross product or vector product

$\vec{U} \times \vec{V}$



hand palm up curls towards $\vec{V}$, thumb points out towards you

$\vec{V} \times \vec{U}$



in toward book

$$\boxed{|\vec{U} \times \vec{V}| = |\vec{U}||\vec{V}| \sin\theta}$$

$$\vec{U} \times \vec{V} = (|\vec{U}||\vec{V}| \sin\theta)\,\vec{n}$$

n = unit vector $\perp$ to $\vec{U}$ + $\vec{V}$

$\vec{U} \times \vec{V} = \vec{0}$ iff $\vec{U} \parallel \vec{V}$ since $\sin\theta$ b/w them is zero, $\sin(0) = 0$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, det of $A = |A| = (1)(4) - (3)(2) = -2$

$A \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $|A| = a_{11}a_{22} - a_{21}a_{12}$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 0 \\ 8 & 12 & 6 \end{bmatrix} \qquad |A| = 2$$

$$A = 2\begin{vmatrix} 4 & 0 \\ 12 & 6 \end{vmatrix} - 1\begin{vmatrix} 5 & 0 \\ 8 & 6 \end{vmatrix} + 3\begin{vmatrix} 5 & 4 \\ 8 & 12 \end{vmatrix}$$

$$= 2(24-0) - 1(30-0) + 3(60-32)$$

$$= 48 - 30 + 84$$

$$= \boxed{102}$$

$$\vec{U} \times \vec{V} = \begin{array}{ccc} \vec{i} & \vec{j} & \vec{K} \end{array}$$

$$\begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \end{bmatrix}$$

$\vec{U} = \langle U_1, U_2, U_3 \rangle$

$\vec{V} = \langle V_1, V_2, V_3 \rangle$

$$\vec{i}\begin{vmatrix} U_2 & U_3 \\ V_2 & V_3 \end{vmatrix} - j\begin{vmatrix} U_1 & U_3 \\ V_1 & V_3 \end{vmatrix} + K\begin{vmatrix} U_1 & U_2 \\ V_1 & V_2 \end{vmatrix}$$

Examples

② $\vec{U} = 2i + 3j \quad \vec{V} = -i + j$

a) $\vec{U} \times \vec{V}$

$$\begin{vmatrix} i & j & k \\ 2 & 3 & 0 \\ -1 & 1 & 0 \end{vmatrix}$$

$$i\begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} - j\begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} + k\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix}$$

$$= i(0) - j(0) + k(5)$$

$$= \boxed{5\vec{k}}$$

b) $\vec{V} \times \vec{U} = \boxed{-5\vec{k}}$

(8) $\vec{U} = \frac{3}{2}i - \frac{1}{2}j + k$ $\vec{V} = i + j + 2k$

$$\vec{U} \times \vec{V} = \begin{vmatrix} i & j & k \\ 3/2 & -1/2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$\vec{U} \times \vec{V} = i \begin{vmatrix} -1/2 & 1 \\ 1 & 2 \end{vmatrix} - j \begin{vmatrix} 3/2 & 1 \\ 1 & 2 \end{vmatrix} + k \begin{vmatrix} 3/2 & -1/2 \\ 1 & 1 \end{vmatrix}$$

$$(-1 - 1)i - (3 - 1)j + (3/2 + 1/2)k$$

$$\vec{U} \times \vec{V} = -2\hat{i} - 2\hat{j} + 2\hat{k}$$

Length: $|\vec{U} \times \vec{V}| = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$

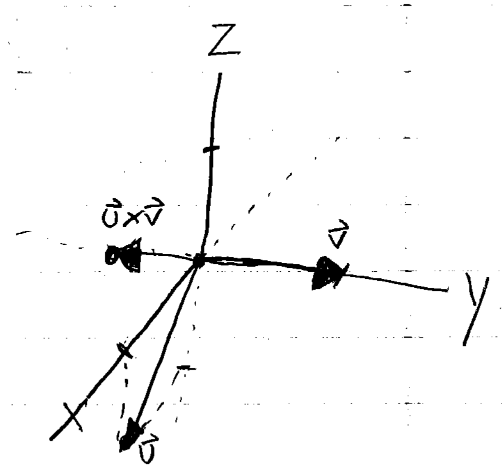
Direction: $\dfrac{\vec{U} \times \vec{V}}{|\vec{U} \times \vec{V}|} = \dfrac{-2i - 2j + 2k}{2\sqrt{3}} = \boxed{\dfrac{-\vec{i} - \vec{j} + \vec{k}}{\sqrt{3}}}$

(10) $\vec{u} = \vec{i} - \vec{k}$ $\vec{v} = \vec{j}$

$$\vec{u} \times \vec{v} \quad \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = i \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} - j \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = +1\vec{i} - 0\vec{j} + \vec{k}$$



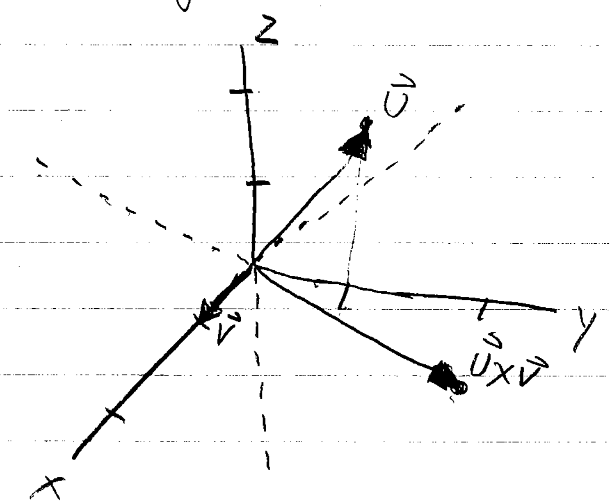
(14) $\vec{u} = \vec{j} + 2\vec{k}$

$\vec{v} = \vec{i}$

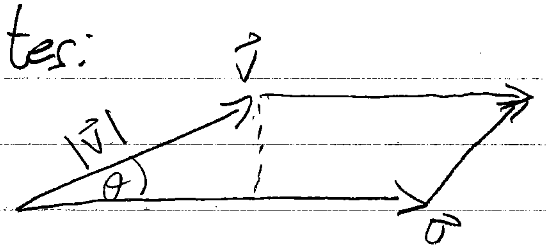
$$\vec{u}\times\vec{v} \qquad \begin{array}{ccc} i & j & k \end{array}$$
$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$\vec{u}\times\vec{v} = i\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} - j\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} + k\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$\vec{u}\times\vec{v} = 0\vec{i} + 2\vec{j} - \vec{k}$$



Notes:



$A = |\vec{u}|h$ $\quad$ Sin $= \frac{opp}{hyp}$ $\therefore$ Sin $= \frac{h}{|\vec{v}|}$

$\underbrace{|\vec{u}|}_{\text{Length of } \vec{u}} \cdot \underbrace{h}_{\text{height}}$

$h = |\vec{v}|\sin\theta \quad \therefore \quad A = |\vec{u}||\vec{v}|\sin\theta$

$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$

$A = |\vec{u} \times \vec{v}|$

Ex:

(40) $A(1,0,-1)$, $B(1,7,2)$
$C(2,4,-1)$, $D(0,3,2)$

$\vec{AB} = \langle 0,7,3 \rangle$ $\quad$ $\vec{AC}\ (1,4,0)$
$\vec{AD} = \langle -1,3,3 \rangle$

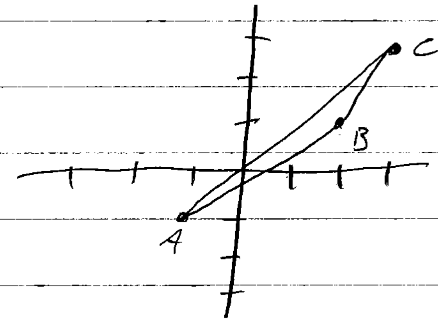
$$\vec{AD} \times \vec{AC} = \begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{vmatrix} -1, & 3, & 3 \\ 1, & 4, & 0 \end{vmatrix} \end{array}$$

$\vec{AD} \times \vec{AC} = i(-12) + 3j - 7k$

Area $= |\vec{AD} \times \vec{AC}| = \sqrt{144 + 9 + 49} = \boxed{\sqrt{202} = A}$

(42) $A(-1,-1)$, $B(3,3)$ $C(2,1)$

Area of triangle



$$A = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|$$

$\vec{AB} = (4,4)$

$\vec{AC} = (3,2)$

$$\vec{AB} \times \vec{AC} = \begin{array}{ccc} i & j & k \end{array}$$
$$\begin{vmatrix} 4 & 4 & 0 \\ 3 & 2 & 0 \end{vmatrix}$$

$$\vec{AB} \times \vec{AC} = 0\vec{i} - 0\vec{j} + 4\vec{k}$$

$$\left|\vec{AB} \times \vec{AC}\right| = \sqrt{0^2+0^2+4^2} = \sqrt{16} = \boxed{4}$$

$$A = \frac{1}{2}(4) = \boxed{A=2}$$

(46) $A(0,0,0)$ $B(-1,1,-1)$ $C(3,0,3)$

$\vec{AB} = \langle -1, 1, -1 \rangle$
$\vec{AC} = \langle 3, 0, 3 \rangle$

$$\vec{AB} \times \vec{AC} \quad \begin{vmatrix} i & j & k \\ -1 & 1 & -1 \\ 3 & 0 & 3 \end{vmatrix}$$

$\vec{AB} \times \vec{AC} = -3\vec{i} - 0\vec{j} + 3\vec{k}$

$|\vec{AB} \times \vec{AC}| = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

$\boxed{A = \frac{3\sqrt{2}}{2}}$

(26)

$|\vec{F}| = 30\ lb$

$|\vec{PQ}| = 8$ inches $\rightarrow \dfrac{8}{12\ \text{inch}} = 2/3\ ft$

$|\vec{PQ} \times \vec{F}| = |\vec{F}|\,|\vec{PQ}| \sin 135° = \boxed{10\sqrt{2}\ ft\ lb}$

Property:

$$(\vec{U} \times \vec{V}) \cdot \vec{W} = \begin{vmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{vmatrix}$$

(20) $\vec{U} = i - j + k$

$\vec{V} = 2i + j - 2k$

$\vec{W} = -i + 2j - k$

$$\begin{array}{ccc} i & j & k \\ \end{array}$$
$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix}$$

$$\text{Volume} = 1\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & -2 \\ -1 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$$

$$\text{Volume} = (-1+4) + (-2-2) + (4+1)$$
$$= 3 - 4 + 5$$
$$= \boxed{5}$$

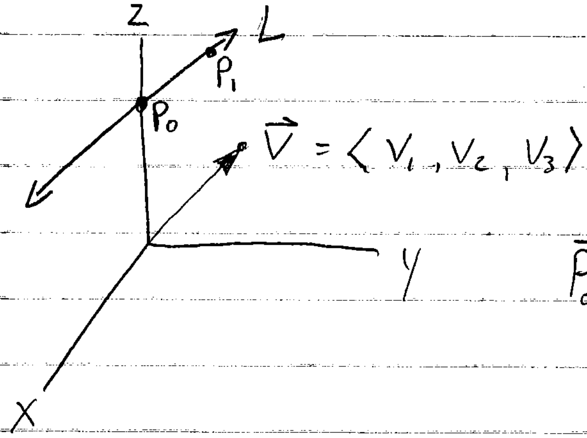
$$\vec{U} \times \vec{V} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} j + \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} k$$

$$= (2-1)i - (-2-2)j + (1+2)k$$
$$= i + 4j + 3k$$

$$(\vec{U} \times \vec{V}) \cdot \vec{W} = \langle 1, 4, 3 \rangle \cdot \langle -1, 2, -1 \rangle$$

$$= -1 + 8 - 3$$
$$\boxed{= 4}$$

12.5 Lines & Planes in Space:



$\overrightarrow{P_0P_1} = t\vec{V}$

t constant (to get the same mag)

$(X - X_0)\vec{i} + (Y - Y_0)\vec{j} + (Z - Z_0)\vec{k} = t(V_1 i, V_2 j, V_3 \vec{k})$

$Xi + yj + zk = X_0 i + y_0 i + z_0 k + t(V_1 i\ V_2 j\ V_3 k)$

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$

final position ($\vec{r}(t)$), initial position ($\vec{r}_0$)

"Standard" parametirization

$X = X_0 + tV_1$, $Y = Y_0 + tV_2$, $Z = Z_0 + tV_3$

where $-\infty < t < \infty$

Examples:

(2) Line thru $P(1,2,-1)$ & $Q(-1,0,1)$

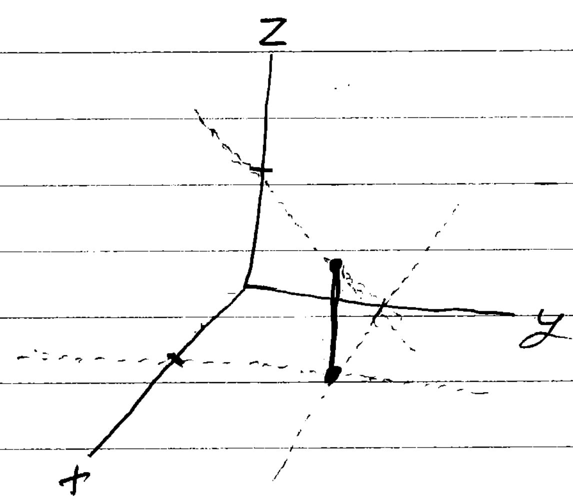
$\overrightarrow{PQ}\langle -2,-2,2\rangle$

$$\boxed{\begin{aligned} &\quad\ \ P \\ x &= 1 + t(-2) = 1-2t \\ y &= 2 + t(-2) = 2-2t \\ z &= -1 + t(2) = -1+2t \end{aligned}}$$

(16) Line Segment
$(1,1,0)$ & $(1,1,1)$

$\vec{v} = \langle 0,0,1\rangle$

$$\boxed{\begin{aligned} x &= 1 + 0t = 1 \\ y &= 1 + 0t = 1 \qquad 0 \le t \le 1 \\ z &= 0 + 1t = t \end{aligned}}$$



(18) $(0,2,0)$, $(3,0,0)$ Find parametization

Line segment connects these points

$\vec{v} = \langle 3, -2, 0 \rangle$

$$x = x_0 + tv_1$$
$$y = y_0 + tv_2$$
$$z = z_0 + tv_3$$

$x = 0 + 3t = 3t$

$y = 2 - 2t$

$z = 0 + 0t = 0$

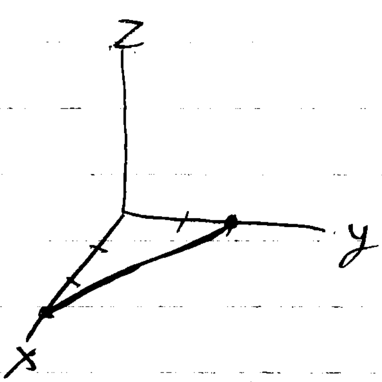
$0 \leq x \leq 3$

$0 \leq 3t \leq 3$

$0 \leq t \leq 1$

$\therefore$

$$\boxed{\begin{aligned} x &= 3t \\ y &= 2 - 2t \\ z &= 0 \\ 0 &\leq t \leq 1 \end{aligned}}$$

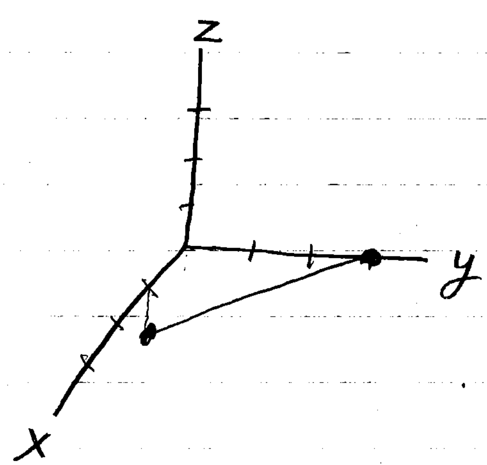


(20) $(1, 0, -1)$, $(0, 3, 0)$

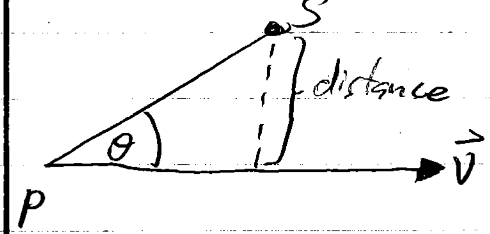
$\vec{v} = \langle -1, 3, 1 \rangle$

$$x = x_0 + tv_1$$
$$y = y_0 + tv_2$$
$$z = z_0 + tv_3$$

$$\boxed{\begin{aligned} x &= 1 - t \\ y &= 3t \\ z &= -1 + t \\ 0 &\leq t \leq 1 \end{aligned}}$$



Distance from point S to line through P, parallel to $\vec{v}$



$$d = |\vec{PS}| \sin\theta$$

$$|\vec{PS} \times \vec{v}| = |\vec{v}| \underbrace{|\vec{PS}| \sin\theta}_{\text{distance}}$$

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

(36) $S = (2, 1, -1)$, $\underset{0+2t}{x = 2t}$, $y = 1 + 2t$, $z = 2t$ $\;0+2t$

$x = x_0 + t v_1$ $\qquad P(0, 1, 0)$

$y = y_0 + t v_2$ $\qquad v_1 = 2,\ v_2 = 2,\ v_3 = 2$

$z = z_0 + t v_3$ $\qquad \vec{v} = \langle 2, 2, 2 \rangle$

$\vec{PS} = \langle 2, 0, -1 \rangle$ $\qquad \vec{PS} \times \vec{v} \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix}$

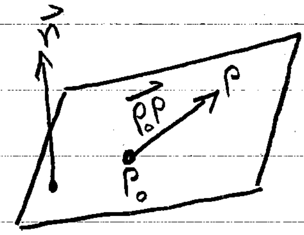
$\vec{PS} \times \vec{v} = 2\vec{i} - 6\vec{j} + 4\vec{k}$

$|\vec{PS} \times \vec{v}| = \sqrt{2^2 + 6^2 + 4^2} = \sqrt{56} = 2\sqrt{14}$

$|\vec{v}| = \sqrt{12} = 2\sqrt{3}$

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \frac{2\sqrt{14}}{2\sqrt{3}} = \boxed{\frac{\sqrt{14}}{\sqrt{3}}}$$

Equation of a Plane



① $\vec{n} \cdot \overrightarrow{P_0P} = 0$ "perpendicular"

② $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

where $\vec{n} = \langle A, B, C \rangle$

$\overrightarrow{P_0P} = \langle x-x_0, y-y_0, z-z_0 \rangle$

③ $Ax - Ax_0 + By - By_0 + Cz - Cz_0 = 0$

$Ax + By + Cz = Ax_0 + By_0 + Cz_0$

where $D = Ax_0 + By_0 + Cz_0$

Examples:

(22) Plane thru $(1, -1, 3)$ // plane $3x + y + z = 7$

A vector $\perp$ to one plane has to be $\perp$ to the other one

$\vec{n} = \langle A, B, C \rangle \therefore \vec{n} = \langle 3, 1, 1 \rangle$

$3(x-1) + 1(y+1) + 1(z-3) = 0$

$3x - 3 + y + 1 + z - 3 = 0$

$\boxed{3x + y + z = 5}$

$1st\ a \quad 2nd\ b \quad 3rd\ c$

(24) Plane $(2,4,5), (1,5,7), (-1,6,8)$

$\vec{ab} = \langle -1, 1, 2 \rangle \qquad \vec{ac} = \langle -3, 2, 3 \rangle$

$$\vec{ab} \times \vec{ac} = \begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix}$$

$\vec{ab} \times \vec{ac} = -i - 3j + \vec{k} = \vec{n} \langle A, B, C \rangle$

$\vec{n} = \langle -1, -3, 1 \rangle$

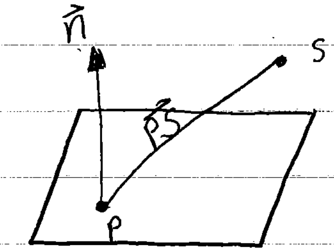
$-x - 3y + z = -1(2) - 3(4) + 1(5)$

$\boxed{-x - 3y + z = -9}$

Concept: 2 planes are // iff
their perpendicular vectors are //
(ie $\vec{n}_1 = K\vec{n}_2$) where K = Constant

Concept:

Distance from a pt to a plane



$$d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$$

$$d = \frac{|\vec{PS} \cdot \vec{v}|}{|\vec{v}|}$$

Example:

42 $S(2,2,3)$, $2x + y + 2z = 4$

$\vec{n} = \langle 2, 1, 2 \rangle$

$|\vec{n}| = \sqrt{4+1+4} = 3$

$\frac{\vec{n}}{|\vec{n}|} = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$

$P = (1, 0, 1)$

$\vec{PS} = \langle 1, 2, 2 \rangle$

$\vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} = \langle 1, 2, 2 \rangle \cdot \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$

$$\vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{2}{3} + 2\left(\frac{1}{3}\right) + \frac{4}{3} = \frac{8}{3}$$

$$\left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right| = \left| \frac{8}{3} \right| = \boxed{\frac{8}{3}}$$

Examples

(58) $3x - 6y - 2z = 3, \quad 2x + y - 2z = 2$

$n_1 = \langle 3, -6, -2 \rangle \qquad n_2 = \langle 2, 1, -2 \rangle$

$$n_1 \times n_2 \quad \begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix}$$

$n_1 \times n_2 = \underset{v_1}{14i} + \underset{v_2}{2j} + \underset{v_3}{15k}$ (vector // to line of intersection)

an arbitrary point in common $(\overset{x_0}{1}, \overset{y_0}{0}, \overset{z_0}{0})$
but you can make a variable $= 0$
& solve 2 equations
ex: $z = 0$; $\begin{cases} 3x - 6y = 3 \\ 2x + y = 2 \end{cases}$

$$x = x_0 + t v_1$$
$$y = y_0 + t v_2$$
$$z = z_0 + t v_3$$

$$\boxed{\begin{aligned} x &= 1 + 14t \\ y &= 2t \\ z &= 15t \end{aligned}}$$

54) $x = 2$, $y = 3 + 2t$, $z = -2 - 2t$

plane $= 6x + 3y - 4z = -12$

Plug them in:

$$6(2) + 3(3+2t) - 4(-2-2t) = -12$$
$$12 + 9 + 6t + 8 + 8t = -12$$
$$29 + 14t = -12$$
$$t = -41/14$$

$$x = 2 \longrightarrow x = 2$$
$$y = 3 + 2(-41/14) \longrightarrow y = -20/7$$
$$z = -2 - 2(-41/14) \longrightarrow z = 27/7$$

$$\boxed{(2, -20/7, 27/7)}$$

12.6 Examples

2) $z^2 + 4y^2 - 4x^2 = 4$

$$\frac{z^2}{2^2} + \frac{y^2}{1^2} - \frac{x^2}{1^2} = 1$$

$\boxed{\text{answer} = \text{I} \quad \text{Hyperboloid of one sheet}}$

13.1 Curves in Space & their Tangents

Path of a particle moving in space over a time interval I:

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I$$

t element of I

t element of time Interval.

$$(x, y, z) = \big(f(t), g(t), h(t)\big)$$

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

Rule #5

$$\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

#6 $$\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$$

Examples:

(2) $\vec{r}(t) = \frac{t}{t+1} i + \frac{1}{t} j, \quad t = -\frac{1}{2}$

$x = \frac{t}{t+1} \qquad y = \frac{1}{t} \rightarrow t = \frac{1}{y}$

a) $x = \frac{(1/y)}{(1/y)+1} \cdot \frac{y}{y}$

$x = \frac{1}{1+y}$

$x(1+y) = 1$

$1+y = \frac{1}{x}$

$\boxed{y = \frac{1}{x} - 1}$

b) $\vec{v}(t) = \frac{1(t+1) - 1(t)}{(t+1)^2} \vec{i} - \frac{1}{t^2} \vec{j}$

$\vec{v}(t) = \frac{1}{(t+1)^2} \vec{i} - \frac{1}{t^2} \vec{j}$

$\vec{v}(-\frac{1}{2}) = \frac{1}{(\frac{1}{2})^2} \vec{i} - \frac{1}{(\frac{1}{2})^2} \vec{j}$

$\boxed{\vec{v}(-\frac{1}{2}) = 4\vec{i} - 4\vec{j}}$

$$\vec{a}(t) = -\frac{2}{(t+1)^3}\vec{i} + \frac{2}{t^3}\vec{j}$$

$$\vec{a}\left(-\tfrac{1}{2}\right) = -\frac{2}{\left(-\frac{1}{2}+1\right)^3} i + \frac{2}{\left(-\frac{1}{2}\right)^3} \vec{j}$$

$$\boxed{\vec{a}\left(-\tfrac{1}{2}\right) = -16\vec{i} - 16\vec{j}}$$

⑥ $x^2 + y^2 = 16$

$$\vec{r}(t) = \left(4\cos\tfrac{t}{2}\right) i + \left(4\sin\tfrac{t}{2}\right) j, \quad t = \pi \ \text{ \& } \ \frac{3\pi}{2}$$

$$\vec{V}(t) = \left[-4\sin\left(\tfrac{t}{2}\right)\cdot\tfrac{1}{2}\right] i + \left[4\cos\left(\tfrac{t}{2}\right)\cdot\tfrac{1}{2}\right] j$$

$$\vec{V}(t) = -2\sin\left(\tfrac{t}{2}\right)\vec{i} + 2\cos\left(\tfrac{t}{2}\right)\vec{j}$$

$$\vec{V}(\pi) = -2\sin\left(\tfrac{\pi}{2}\right)\vec{i} + 2\cos\left(\tfrac{\pi}{2}\right)\vec{j}$$

$$\vec{V}(\pi) = -2(1)\vec{i} + 0\vec{j} \longrightarrow \boxed{\vec{V}(\pi) = -2\vec{i}}$$

$$\vec{V}\left(\tfrac{3\pi}{2}\right) = -2\sin\left(\tfrac{3\pi}{4}\right) i + 2\cos\left(\tfrac{3\pi}{4}\right) j$$

$$\vec{V}\left(\tfrac{3\pi}{2}\right) = -\sqrt{2}\, i - \sqrt{2}\, j$$

$$\vec{a}(t) = -2\cos\left(\tfrac{t}{2}\right)\cdot\tfrac{1}{2} - 2\sin\left(\tfrac{t}{2}\right)\cdot\tfrac{1}{2}$$

$$\vec{a}(t) = -\cos\left(\tfrac{t}{2}\right)i - \sin\left(\tfrac{t}{2}\right)j$$

$$\vec{a}(\pi) = -\cos\left(\tfrac{\pi}{2}\right)i - \sin\left(\tfrac{\pi}{2}\right)\vec{j}$$

$$\boxed{\vec{a}(\pi) = -\vec{j}}$$

$$\vec{a}\left(\tfrac{3\pi}{2}\right) = -\cos\left(\tfrac{3\pi}{4}\right)i - \sin\left(\tfrac{3\pi}{4}\right)\vec{j}$$

$$\boxed{\vec{a}\left(\tfrac{3\pi}{2}\right) = -\tfrac{\sqrt{2}}{2}i - \tfrac{\sqrt{2}}{2}j}$$

(8) $Y = X^2 + 1$

$$\vec{r}(t) = ti + (t^2+1)j, \quad t = -1, 0, 1$$

$$\boxed{\begin{array}{l} \vec{V}(t) = i + 2tj \\ \vec{V}(-1) = i - 2j \quad @(0) = i \quad @(1) = i + 2j \end{array}}$$

$$\boxed{\vec{a}(t) = 2\vec{j}}$$

(10) $\vec{r}(t) = (1+t)i + \frac{t^2}{\sqrt{t}} j + \frac{t^3}{3} k$, $t = 1$

$$\vec{v}(t) = i + \frac{2}{\sqrt{2}} t j + t^2 k$$

$$\boxed{\vec{v}(t) = i + \sqrt{2}\, t j + t^2 k} \Rightarrow v(1) = i + \sqrt{2} + k$$

$$\boxed{\vec{a}(t) = \sqrt{2} j + 2tk}$$

$$\text{Speed} = |\vec{v}(1)| = \sqrt{1+2+1} = 2$$

$$\text{direction} = \frac{\vec{v}(1)}{|\vec{v}(1)|} = \boxed{\frac{\vec{i} + \sqrt{2}\vec{j} + \vec{k}}{2}}$$

(16) $\vec{r}(t) = \left(\frac{\sqrt{2}}{2}t\right)\vec{i} + \left(\frac{\sqrt{2}}{2}t - 16t^2\right)\vec{j}$, $t=0$

$\vec{v}(t) = \frac{\sqrt{2}}{2}i + \left(\frac{\sqrt{2}}{2} - 32t\right)j$

$\vec{a}(t) = -32\vec{j}$

$V(0) = \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}$ $\quad \vec{a}(0) = -32j$

$\theta = \cos^{-1}\left(\frac{\vec{v}(t) \cdot \vec{a}(t)}{|\vec{v}(t)||\vec{a}(t)|}\right) = \left(\frac{0 - 16\sqrt{2}}{(1)(32)}\right)$

$\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \boxed{\frac{3\pi}{4}}$

(20) $\vec{r}(t) = t^2\vec{i} + (2t-1)\vec{j} + t^3k$, $t_0 = 2$

$\vec{V}(t) = 2ti + 2j + 3t^2k$

$\vec{V}(2) = \underset{t_1}{4i} + \underset{t_2}{2j} + \underset{t_3}{12k}$

original position $r = \vec{r}(2) = \overset{x_0}{4}i + \overset{y_0}{3}j + \overset{z_0}{8}k$

$$\boxed{\begin{aligned} X &= 4 + 4t \\ Y &= 3 + 2t \\ Z &= 8 + 12t \end{aligned}}$$

(23) a) $\vec{r}(t) = (\cos t)\,i + (\sin t)\,j \quad , \quad t \geq 0$

i) Constant Speed

$\vec{v}(t) = (-\sin t)\,i + (\cos t)\,\vec{j}$

speed $= |\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t} = \boxed{1}$ Yes

ii) $\vec{a}(t) = (-\cos t)\,i + (-\sin t)\,j$

$\vec{v} \cdot \vec{a} = \sin t \cos t - \cos t \sin t = \boxed{0}$ Yes

v) begin at $(1,0)$

$\vec{r}(0) = (\cos 0)\,i + (\sin 0)$

$\boxed{\vec{r}(0) = 1i + 0j}$ Yes

iii) Clockwise or Counter-Clockwise

13.2 Integrals of vector functions

(2) find $\int_1^2$ of $(6-6t)i + 3\sqrt{t}j + \left(\frac{4}{t^2}\right)k$

$$\int_1^2 \left[(6-6t)i + 3\sqrt{t}j + \frac{4}{t^2}k\right]dt$$

$$= \left[6t-3t^2\right]_1^2 + \left[2t^{3/2}\right]_1^2 j - \left[\frac{4}{t^3}\right]_1^2 k$$

$$= \left[(12-12)-(6-3)\right]\vec{i} + \left[2\cdot 2^{3/2}-2\right]j - \left[\frac{4}{2}-4\right]k$$

$$= \boxed{-3\vec{i} + (4\sqrt{2}-2)j + 2\vec{k}}$$

Notes: Projetile Motion

$$\vec{r}(t) = (X_0 + V_0(\cos\alpha)t)i + (y_0 + V_0(\sin\alpha)t - \tfrac{1}{2}gt^2)j$$

(22) A baseball is thrown from stands 32ft above field @ angle 30° up from horizontal.

When & how far away will the ball strike the ground if initial speed is 32ft per second

$$\text{height} = y = 32\,ft + 32\,ft/sec\,(\sin 30)t - \frac{1}{2}\cdot 32\,ft/s^2\, t^2$$

$$y = 32 + 16t - 16t^2$$

ground when $y = 0$

$$0 = (32 + 16t - 16t^2)$$

$$0 = t^2 - t - 2$$

$$\boxed{t = 2}$$

where hit ground? $X = 0 + 32(\cos 30)(2)$

$$\boxed{X = 32(\sqrt{3}) \approx 55.4\,ft}$$

$\vec{v}$ = Velocity in this chapter

13.3 Arc Length

$$L = \int_a^b |\vec{v}|\, dt \qquad |\vec{v}| = \left|\frac{dr}{dt}\right|$$

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

$$L = s(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$s(t) = \int_{t_0}^{t} \sqrt{x'(\tau)^2 + y'(\tau)^2 + z(\tau)^2}\, d\tau$$

Unit Tangent Vector

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

Examples:

② $\vec{r}(t) = [6\sin 2t]\,\vec{i} + [6\cos 2t]\,\vec{j} + 5t\vec{k}\,;\quad 0 \le t \le \pi$

$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \quad \vec{v}(t) = \frac{d\vec{r}}{dt} = 12\cos 2t\,i - 12\sin 2t\,j + 5k$

$|\vec{v}(t)| = \sqrt{(12\cos 2t)^2 + (-12\sin 2t)^2 + 5^2}$

$|\vec{v}(t)| = \sqrt{144\cos^2(2t) + 144\sin^2(2t) + 25}$

$|\vec{v}(t)| = \sqrt{144(\cos^2(2t) + \sin^2(2t)) + 25}$

$|\vec{v}(t)| = \sqrt{144(1) + 25} = \sqrt{169} = 13$

$$\boxed{\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{12\cos 2t\,\vec{i} - 12\sin 2t\,\vec{j} + 5\vec{k}}{13}}$$

$S(t) \int_0^{\pi} |\vec{v}(t)|\,dt = \int_0^{\pi} 13\,dt = \Big[13t\Big]_0^{\pi} = \boxed{13\pi}$

$0 \le t \le \pi$

(12) $\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j}, \quad \frac{\pi}{2} \leq t \leq \pi$

(product rule)

$$\vec{V} = \frac{d\vec{r}}{dt} = (\cancel{-\sin t} + \cancel{\sin t} + t\cos t)\vec{i} + (\cancel{\cos t} - \cancel{\cos t} + t\sin t)\vec{j}$$

$$\vec{V} = t\cos t\,\vec{i} + t\sin t\,\vec{j}$$

$$|\vec{V}| = \sqrt{(t^2\cos^2 t) + t^2\sin^2 t} = \sqrt{t^2(1)} = |t| \quad \text{assuming } t > 0$$

$$S(t) = \int_{\pi/2}^{\pi} t\,dt \quad \text{or} \quad S(\tau) = \int_0^t \tau\,d\tau = \left[\frac{\tau^2}{2}\right]_0^t = \frac{t^2}{2}$$

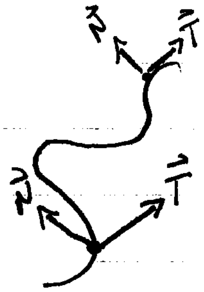
$$S(\pi) = \frac{\pi^2}{2} \quad \& \quad S\left(\frac{\pi}{2}\right) = \frac{\pi^2}{8}$$

$$S = \frac{\pi^2}{2} - \frac{\pi^2}{8} = \boxed{S = \frac{3\pi^2}{8}}$$

13.4 Curvature & Normal Vectors of a Curve

$$\underset{\text{curvature}}{K} = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{dt}\cdot\frac{dt}{ds}\right| = \frac{1}{|ds/dt|}\cdot\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$K = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| \quad , \quad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$



Principal unit vector

$$\vec{N} = \frac{1}{K}\frac{d\vec{T}}{ds} \quad , \quad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\text{Radius of curvature} = \overset{\substack{\text{rho} \\ \downarrow}}{\rho} = \frac{1}{K}$$

Examples:

Find $\vec{T}$, $\vec{N}$ & K for the following

(2) $\vec{r}(t) = (\ln(\sec t))\,\vec{i} + t\,\vec{j}, \quad -\pi/2 < t < \pi/2$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} \quad \vec{v} = \frac{d\vec{r}}{dt} = \left[\frac{1}{\cancel{\sec t}} \cdot \cancel{\sec t}\,\tan t\right]\vec{i} + \vec{j}$$

$$= \tan t\,\vec{i} + \vec{j}$$

$$|\vec{v}| = \sqrt{\tan^2 t + 1^2} = \sqrt{\sec^2 t} = \sec t$$

$$\vec{T} = \frac{\tan t\,\vec{i} + \vec{j}}{\sec t} = \boxed{\sin t\,\vec{i} + \cos t\,\vec{j}}$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} : \quad d\vec{T}/dt = \cos t\,\vec{i} - \sin t\,\vec{j}$$

$$|d\vec{T}/dt| = \sqrt{\cos^2 + (-\sin t)^2} = 1$$

$$\boxed{\vec{N} = \cos t\,\vec{i} - \sin t\,\vec{j}}$$

$$K = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{\sec t}(1) = \cos t\,(1) = \boxed{K = \cos t}$$

(10) $\vec{r}(t) = (\cos t + t\sin t)i + (\sin t - t\cos t)j + 3k$

$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$, $\vec{v} = (-\cancel{\sin t} + \cancel{\sin t} + t\cos t)i + (\cancel{\cos t} - \cancel{\cos t} + t\sin t)j$

$\vec{v} = t\cos t + t\sin t$

$|\vec{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = |t|$

$\vec{T} = \frac{t\cos t i + t\sin t j}{t} = \boxed{\cos t\, i + \sin t\, j}$

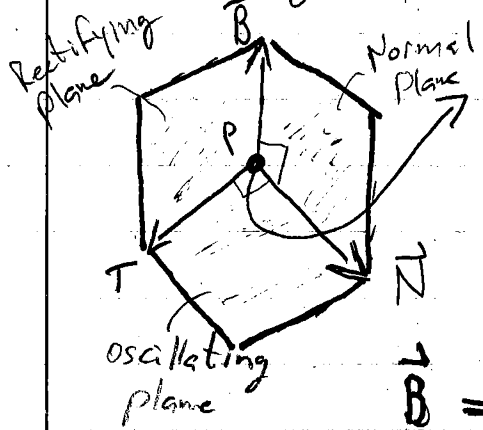
$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$; $\boxed{\frac{d\vec{T}}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j}}$

$|d\vec{T}/dt| = \sqrt{\sin^2 t + \cos^2 t} = 1$

$\boxed{\vec{N} = -\sin t\,\vec{i} + \cos t\,\vec{j}}$

$K = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{t}(1)$ $\boxed{K = \frac{1}{t}}$

13.5 Tangential & Normal components of Acceleration



$\vec{B}$ = binormal unit vector perpendicular to both $\vec{T}$ & $\vec{N}$

$$\vec{B} = \vec{T} \times \vec{N}$$

Used to calculate "torsion" (twist)

torsion

$$\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2}$$

(x, y, z) of $\vec{r}(t)$

Torsion = Rate @ which the oscillating plane turns about the $\vec{T}$ a P moves along the curve

Ex:

13.4 (12) Find $\vec{T}$, $\vec{N}$ and K

$$\vec{r}(t) = (6\sin 2t)\vec{i} + (6\cos 2t)\vec{j} + 5t\vec{k}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} : \quad \vec{v} = 12\cos 2t\, i - 12\sin 2t\, j + 5k$$

$$|\vec{v}| = \sqrt{144(\cos^2 2t + \sin^2 2t) + 25}$$

$$|\vec{v}| = \sqrt{169} = 13$$

$$\vec{T} = \frac{(12\cos 2t)i - (12\sin 2t)j + 5k}{13}$$

$$\vec{N} = \frac{(dT/dt)}{|dT/dt|} = \quad \frac{d\vec{T}}{dt} = -\frac{24}{13}\sin 2t\, i - \frac{24}{13}\cos 2t\, j$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\left(\frac{24}{13}\right)^2(\sin^2 2t + \cos^2 2t)} = \sqrt{\left(\frac{24}{13}\right)^2} = \frac{24}{13}$$

$$\vec{N} = \frac{13}{24}\left(\frac{24}{13}\left[-\sin 2t\, i - \cos 2t\, j\right]\right)$$

$$\boxed{\vec{N} = -\sin 2t\, i - \cos 2t\, j}$$

$$K = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{13}\cdot\frac{24}{13} = \boxed{\frac{24}{169}}$$

13.5 (12) Find $\vec{B}$ and τ for space curve

$$\vec{r}(t) = (6\sin 2t)i + (6\cos 2t)\vec{j} + 5t\vec{k}$$

$$\vec{B} = \vec{T} \times \vec{N} \qquad \begin{array}{ccc} i & j & k \\ \end{array}$$
$$\begin{vmatrix} 12/13\cos 2t & -\frac{12}{13}\sin 2t & 5/13 \\ -\sin 2t & -\cos 2t & 0 \end{vmatrix}$$

$$\vec{B} = \left(\tfrac{5}{13}\cos 2t\right)i - \left(\tfrac{5}{13}\sin 2t\right)j + \left(-\tfrac{12}{13}\cos^2 2t - \tfrac{12}{13}\sin^2 2t\right)K$$

$$\boxed{\vec{B} = \left(\tfrac{5}{13}\cos 2t\right)\vec{i} - \left(\tfrac{5}{13}\sin 2t\right)\vec{j} - \left(\tfrac{12}{13}(1)\right)\vec{K}}$$

$$\tau = ? \qquad \vec{v}(t) = (12\cos 2t)\vec{i} - (12\sin 2t)\vec{j} + 5\vec{k}$$

$$\vec{a}(t) = (-24\sin 2t)i - (24\cos 2t)\vec{j}$$

$$\vec{a}'(t) = (-48\cos 2t)\vec{i} + (48\sin 2t)\vec{j}$$

$$\begin{array}{ccc} i & j & k \end{array}$$
$$|\vec{v} \times \vec{a}| = \begin{vmatrix} 12\cos 2t & -12\sin 2t & 5 \\ -24\sin 2t & -24\cos 2t & 0 \end{vmatrix}$$

$$|\vec{v}\times\vec{a}| = (120\cos \cancel{8}2t)i - (120\sin 2t)j + (-288\cos^2 2t - 288\sin^2 2t)$$
$$= (120\cos 2t)i - (120\sin 2t)j - 288$$

$$|\vec{v}\times\vec{a}| = \sqrt{14{,}400\cos^2 2t + 14{,}400\sin^2 2t + 82{,}944}$$
$$|\vec{v}\times\vec{a}| = \sqrt{14{,}400 + 82{,}944} = \sqrt{97{,}344}$$

$$\tau = \frac{\begin{array}{ccc} i & j & k \\ \left| \begin{array}{ccc} 12\cos 2t & -12\sin 2t & 5 \\ -24\sin 2t & -24\cos 2t & 0 \\ -48\cos 2t & 48\sin 2t & 0 \end{array} \right| \end{array}}{97{,}344}$$

(51) 6.8

$$5\left[(-24\sin 2t * 48\sin 2t) - (-48\cos 2t * -12\sin 2t)\right]$$
$$5(-1152\sin^2 2t - 1152\cos^2 2t)$$

$$\tau = 5(-1152(1))$$

$$\tau = \frac{-5760}{97{,}344} \div \frac{576}{576} = \boxed{\frac{-10}{169}}$$

$\vec{a} = a_T \vec{T} + a_N \vec{N}$

a_T = mag of $\vec{T}$

$\boxed{a_T = \frac{d}{dt}|\vec{v}|}$ & $a_N = K|\vec{v}|^2$

a_T = Change in Length of Velocity Vector

a_N = Change in Length of velocity direction

$\boxed{a_N = \sqrt{|\vec{a}|^2 - a_T^2}}$

Ex: 6 Write accel vector in form $\vec{a} = a_T \vec{T} + a_N \vec{N}$ at a given value of t w/o finding $\vec{T}$ & $\vec{N}$

$\vec{r}(t) = (e^t \cos t)\vec{i} + (e^t \sin t)j + \sqrt{2}e^t \vec{k}$; $t = 0$

$\vec{v}(t) = e^t(\cos t - \sin t)i + e^t(\sin t + \cos t)j + \sqrt{2}e^t k$

$|\vec{v}(t)| = \sqrt{e^{2t}(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 2}$

$|\vec{v}(t)| = e^t\sqrt{2(\cos^2 t + \sin^2 t) + 2} = \boxed{2e^t}$

$a_T = \frac{d}{dt}|\vec{v}| = 2e^t$

$\vec{a} = (\cancel{e^t \cos t} - e^t \sin t - e^t \sin t - \cancel{e^t \cos t})i +$
$(\cancel{e^t \sin t} + e^t \cos t + e^t \cos t - \cancel{e^t \sin t})j + \sqrt{2}e^t K$

$\vec{a} = (-2e^t \sin t)i + (2e^t \cos t)j + (\sqrt{2}\, e^t)K$

$a_T = 2e^t$ @ $t = 0$ $= \boxed{2}$

$\vec{a} = -2e^0 \sin(0)\, i + 2e^0 \cos(0)\, j + \sqrt{2}e^0 K$

$\vec{a} = 0i + 2j + \sqrt{2}K$

$|\vec{a}(0)| = \sqrt{6}$

$a_N = \sqrt{(\sqrt{6})^2 - 2^2} = \sqrt{6-4} = \boxed{\sqrt{2}}$

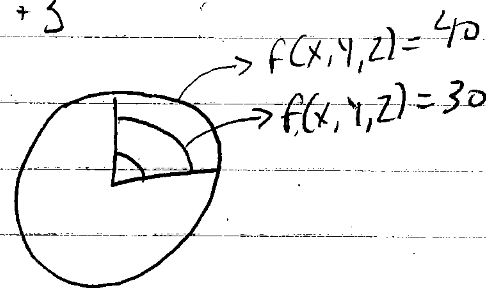
$\boxed{\vec{a} = 2\vec{T} + \sqrt{2}\,\vec{N}}$

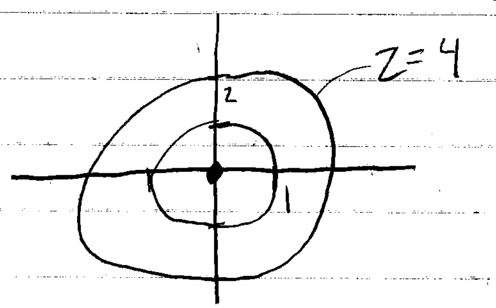
14.1 Functions of Several Variables

$f(x,y,z) = x^2 + y^2 + z^2$ where the triple cord is the radius

$f(1,2,3) = 1^2 + 2^2 + 3^2$

radius $= 14$



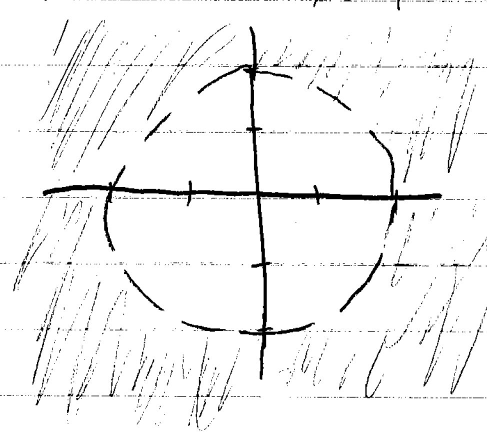


for $f(x,y) = x^2 + y^2$

Ex: 6 Find domain & sketch

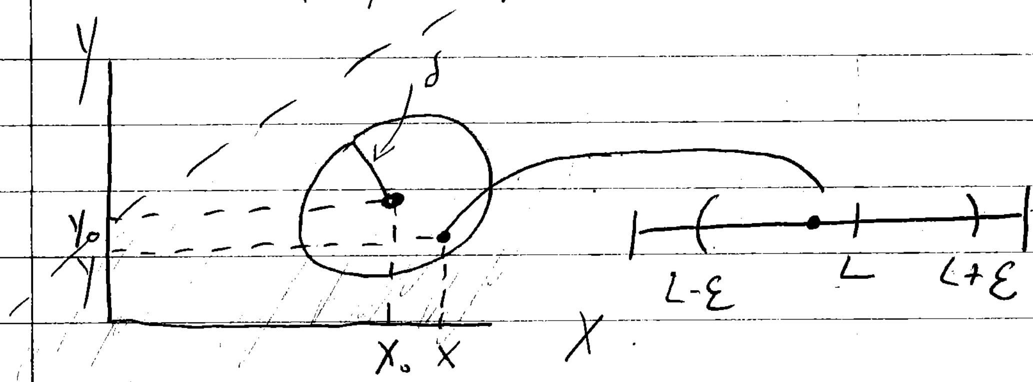
$f(x,y) = \ln(x^2 + y^2 - 4)$

restriction: $x^2 + y^2 - 4 > 0 \Rightarrow x^2 + y^2 > 4$



14.2 Limits & Continuity in Higher Dimensions

Definition: $f(x,y)$ approaches Limit L as (x,y) approaches (x_0, y_0) if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for all (x,y) in domain,
$|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$



Continuity: A function $f(x,y)$ is cont. at (x_0, y_0) if our function is defined at that point.

$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

& the limit is equal to the function

Ex: 3 $\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1} = \sqrt{3^2+4^2-1} = \sqrt{24} = \boxed{2\sqrt{6}}$

6 $\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^2+y^3}{x+y+1}\right) = \cos(0) = \boxed{1}$

14 $\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(1,1)} \frac{(x+y)\cancel{(x-y)}}{\cancel{x-y}} = \lim_{(x,y)\to(1,1)} 1+1 = \boxed{2}$

$x \neq y$

18 $\lim_{(x,y)\to(2,2)} \frac{x+y-4}{\sqrt{x+y}-2} = \lim_{(x,y)\to(2,2)} \frac{(\sqrt{x+y}+2)(\sqrt{x+y}-2)}{\sqrt{x+y}-2}$

$x+y \neq 4$

$\lim_{(x,y)\to(2,2)} \sqrt{4}+2 = \boxed{4}$

(26) $\lim\limits_{P \to (1,-1,-1)} \dfrac{2xy + yz}{x^2 + z^2}$ plug in

$$\frac{2(1)(-1) + (-1)(-1)}{1^2 + (-1)^2} = \frac{-2+1}{2} = \boxed{\frac{-1}{2}}$$

(32) a) $f(x,y) = \dfrac{x+y}{x-y}$

$x \neq y$ all (x,y) where $x \neq y$

(35) b) $f(x,y,z) = \sqrt{x^2+y^2-1}$

$x^2+y^2-1 \geq 0$

$x^2+y^2 \geq 1$ all (x,y,z) outside & including $x^2+y^2=1$

(36) a) $f(x,y,z) = \ln(xyz)$

$x,y,z > 0$

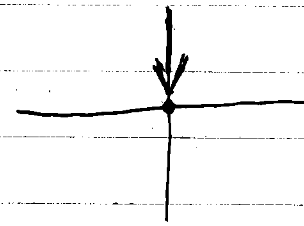
$\{(x,y,z)$ such that $xyz > 0\}$

$\{(x,y,z) \mid xyz > 0$

Limit does not exist

(42) $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$

try $y=0$ axis



$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$

$\lim_{x\to 0} \frac{x^4}{x^4} = \boxed{1}$

2 approaches

limit DNE

2 approaches

$\lim_{(x,y)\to(0,0)}$ along $y=x^2$ $\longrightarrow \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{x^4}{2x^4} = \boxed{\frac{1}{2}}$

(44) $f(x,y) = \frac{xy}{|xy|}$

$\lim_{(x,y)\to(0,0)} \frac{xy}{|xy|}$ if $y=x$ $\longrightarrow \lim_{x\to 0} \frac{x^2}{x^2} = \boxed{1}$

limit DNE

$\lim_{(x,y)\to(0,0)}$ where $y=-x$ $\frac{x(-x)}{|x(-x)|} = \boxed{-1}$

(48) $h(x,y) = \dfrac{x^2 y}{x^4+y^2}$ $\qquad \lim_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4+y^2}$

$\lim_{(x,y)\to(0,0)}$ along $y = kx^2$ $\quad \dfrac{x^2(kx^2)}{x^4+(kx^2)^2} \longrightarrow \dfrac{x^2 k x^2}{x^4+k^2x^4} \longrightarrow \dfrac{\cancel{x^4} k}{(k^2+1)\cancel{x^4}}$

$= \dfrac{k}{k^2+1}$ $\quad \boxed{\text{Limit DNE}}$

(58) Does knowing $\left|\cos\left(\frac{1}{y}\right)\right| \le 1$ tell anything about $\lim_{(x,y)\to(0,0)} x\cos\frac{1}{y}$?

$-1 \le \cos\frac{1}{y} \le 1$

$x(-1) \le x(\cos 1/y) \le x(1)$

$-x \le x\cos 1/y \le x$

if $x > 0$
$-x \le x\cos 1/y \le x$

if $x < 0$
$-x \ge x\cos 1/y \ge x$

Since $-x \to 0$ & $x \to 0$ as $(x,y)\to(0,0)$ then $x\cos 1/y \to 0$, So, $\lim_{(x,y)\to(0,0)} x\cos 1/y = 0$

polar

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$

(62) $f(x,y) = \cos\left(\frac{x^3-y^3}{x^2+y^2}\right)$

$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3-y^3}{x^2+y^2}\right) \to \cos\left(\frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2}\right)$$

$$\lim_{r\to 0} \cos\left(r\cos^3\theta - r\sin^3\theta\right) = \cos 0 = \boxed{1}$$

Show exist $\delta > 0$ for all (x,y,z)

$\sqrt{x^2+y^2+z^2} < \delta$ implies

$|f(x,y,z) - f(0,0,0)| < \varepsilon$

(72) $f(x,y) = \frac{x+y}{2+\cos x}$

$\sqrt{x^2+y^2} < \delta$

since $|x| = \sqrt{x^2} < \sqrt{x^2+y^2} < \delta$

& $|y| = \sqrt{y^2} < \sqrt{x^2+y^2} < \delta$

$$\left| f(x,y) = \frac{x+y}{2+\cos x} - \frac{0+0}{2+1} \right| = \left| \frac{x+y}{2+\cos x} \right| \leq \left| \frac{x+y}{2+(-1)} \right| \leq |x|+|y|$$

$$\left| \frac{x+y}{1} \right| < 2\delta \quad \text{if } 2\delta = \varepsilon \to \delta = \frac{\varepsilon}{2}$$

14.3 Partial Derivatives

Partial deriv with respect to X at point (X_0, Y_0):

$$\left.\frac{\partial f}{\partial X}\right|_{(X_0, Y_0)} = \lim_{h \to 0} \frac{f(X_0 + h, Y_0) - f(X_0)}{h}$$

$$= \left.\frac{d}{dX} f(X, Y_0)\right|_{X = X_0}$$

partial Derivative wrt X, is the total deriv wrt X holding all other variables constant

$$\frac{\partial f}{\partial X}(X_0, Y_0) = f_x(X_0, Y_0)$$

Second derivative

f_{xx} or $\frac{\partial^2 f}{\partial X^2}$

Examples   14.3 Partial Derivatives

(2) $f(x,y) = x^2 - xy + y^2$

$$\frac{\partial f}{\partial x} = 2x - y \qquad \frac{\partial f}{\partial y} = 2y - x$$

(10) $f(x,y) = \dfrac{x}{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\boxed{= \frac{y^2-x^2}{(x^2+y^2)^2}}$$

$$\frac{\partial f}{\partial y} = \frac{0 - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

(12) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$$f_x = \frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = \boxed{\frac{-y}{x^2+y^2}}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) = \frac{1}{x+\frac{y^2}{x}} \cdot \frac{x}{x} = \boxed{\frac{x}{x^2+y^2}}$$

(16) $f(x,y) = e^{xy} \ln y$

$$\frac{\partial f}{\partial x} = \boxed{y e^{xy} \ln y}$$

$$\frac{\partial f}{\partial y} = e^{xy} \cdot x \cdot \ln y + \frac{1}{y} \cdot e^{xy} = \boxed{x e^{xy} \ln y + \frac{e^{xy}}{y}}$$

22) $f(x,y) = \sum_{n=0}^{\infty} (xy)^n$; $(|xy| < 1)$

Recall Geometric Series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, |r| < 1$

$\therefore = \frac{1}{1-xy} = (1-xy)^{-1}$

$f_x = -(1-xy)^{-2} \cdot (-y) = \boxed{\frac{y}{(1-xy)^2}}$

$\boxed{f_y = \frac{x}{(1-xy)^2}}$

(21) $f(x,y) = \int_x^y g(t)\,dt = G(y) - G(x)$

$\frac{\partial f}{\partial x} = -\frac{\partial}{\partial x} G(x) = \boxed{-g(x)}$

$\frac{\partial f}{\partial y} = -\frac{\partial}{\partial y} = \boxed{g(y)}$

(30) $f(x,y,z) = yz\ln(xy)$

$f_x = \frac{yz}{xy} \cdot (y) = \boxed{\frac{yz}{x}}$

$f_y = (yz)(\ln(xy)) = z\ln(xy) + \frac{\cancel{y}z}{\cancel{xy}}(\cancel{x})$

$= \boxed{z\ln(xy) + z}$

$\boxed{f_z = y\ln(xy)}$

(36) $g(u,v) = v^2 e^{2u/v}$

$g_u = v^2 e^{2u/v}\left(\frac{2}{v}\right) = \boxed{2ve^{2u/v}}$

$g_v = 2ve^{2u/v} + \cancel{v^2} e^{2u/v}\left(-\frac{2u}{\cancel{v^2}}\right) = \boxed{2ve^{2u/v} - 2ue^{2u/v}}$

(42) $f(x,y) = \sin xy$

f_{xx} f_{xy} f_{yy}

derivative of $\frac{d}{dx}$ with respect y

$f_x = y\cos(xy)$

$\boxed{f_{xx} = -y^2\sin(xy)}$

$1\cdot\cos(xy) + y(-\sin(xy)(x))$

$\boxed{f_{xy} = \cos(xy) - xy\sin(xy)}$

$f_y = \cos(xy)\cdot x$

$\boxed{f_{yy} = -x^2\sin(xy)}$

Implicit Diff

(66) find $\frac{\partial x}{\partial z}$ @ $(1,-1,-3)$

$xz + y\ln x - x^2 + 4 = 0$

y is a const

$$\frac{\partial}{\partial z}(xz) + \frac{\partial}{\partial z}(y\ln x) - \frac{\partial}{\partial z}(x^2) + \frac{\partial}{\partial z}(4) = 0$$

$$\frac{\partial x}{\partial z}\cdot z + \frac{\partial z}{\partial z}\cdot x + y\cdot\frac{1}{x}\cdot\frac{\partial x}{\partial z} - 2x\frac{\partial x}{\partial z} + 0 = 0$$

$$\frac{\partial x}{\partial z}\left(z + \frac{y}{x} - 2x\right) + x = 0$$

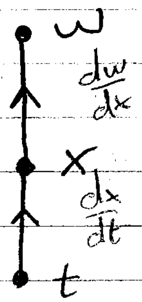
$$\frac{\partial x}{\partial z}\left(z + \frac{y}{x} - 2x\right. = -x$$

$$\frac{\partial x}{\partial z} = \frac{-x}{z + y/x - 2x}$$

$$\left.\frac{\partial x}{\partial z}\right|_{(1,-1,-3)} = -\frac{1}{-3+\frac{-1}{1}-2(1)} = \boxed{\frac{1}{6}}$$

New Section

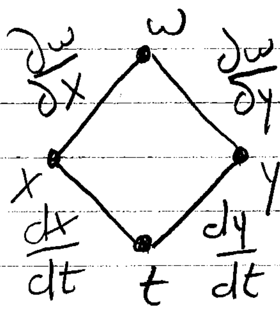
14.4 Chain Rule



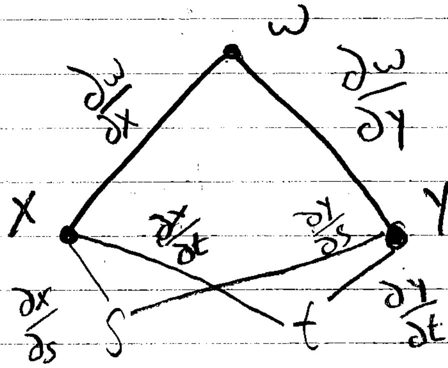
$w(x(t))$

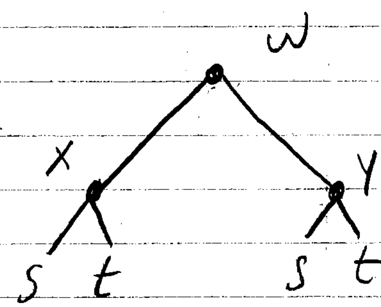
$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

"x" is an intermediate variable
"t" is an independant variable



$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

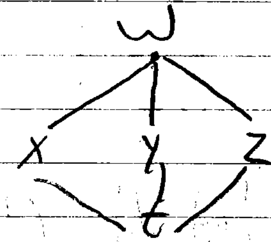




$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Examples:

(4) $w = \ln(x^2 + y^2 + z^2)$



$x = \cos t$
$y = \sin t$
$z = 4\sqrt{t}$
$t = 3$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= \frac{1}{x^2+y^2+z^2}(2x)(-\sin t) + \frac{1}{x^2+y^2+z^2}(2y)(\cos t) + \frac{1}{x^2+y^2+z^2}(2z)(2t^{-1/2})$$

$\underbrace{\sin^2 t + \cos^2 t}_{(1)} + 16t$

$$= \cancel{\frac{-2(\cos t)(\sin t)}{1+16t}} + \cancel{\frac{2(\sin t)(\cos t)}{1+16t}} + \frac{4\cdot 4\cancel{\sqrt{t}}}{(1+16t)\cancel{\sqrt{t}}}$$

$$\boxed{\frac{dw}{dt} = \frac{16}{1+16t}}$$

OR plug in

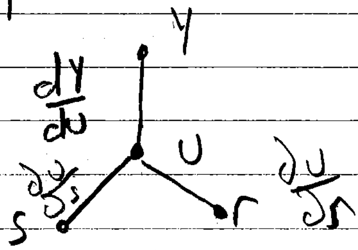
$w = \ln(\sin^2 t + \cos^2 t + (4\sqrt{t})^2)$

$w = \ln(1 + 16t)$

$$\frac{dw}{dt} = \frac{1}{1+16t}\cdot 16 = \boxed{\frac{dw}{dt} = \frac{16}{1+16t}}$$

$$\left.\frac{dw}{dt}\right|_{t=3} = \frac{16}{1+16(3)} = \boxed{\frac{16}{49}}$$

(20) $\frac{\partial y}{\partial r}$ for $y = f(u)$, $u = g(r,s)$



$$\frac{\partial y}{\partial r} = \frac{dy}{du} \cdot \frac{\partial u}{\partial r}$$

Notes: if $F(x, y) = 0$ then
(we want $\frac{dy}{dx}$):

$$F_x \cdot \frac{dx}{dx}^{(1)} + F_y \frac{dy}{dx} = 0$$

$$F_y \frac{dy}{dx} = -F_x$$

$$\boxed{\frac{dy}{dx} = \frac{-F_x}{F_y}}$$

goal: $\frac{dy}{dx}$

(26) $xy + y^2 - 3x - 3 = 0 \qquad (-1, 1)$

method 1:

$$1 \cdot y + x\frac{dy}{dx} + 2y\frac{dy}{dx} - 3(1) = 0$$

$$\frac{dy}{dx}(x + 2y) = 3 - y$$

$$\boxed{\frac{dy}{dx} = \frac{3-y}{x+2y}}$$

Method 2

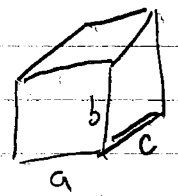
y is const $F_x = y - 3 = 0$

$$F_y = x + 2y = 0$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \boxed{\frac{-(y-3)}{x+2}}$$

$$@(-1, 1) \quad \frac{3-1}{(-1)+2(1)} = \frac{2}{1} = \boxed{2}$$

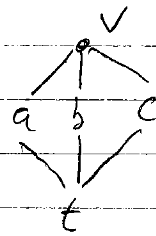
(42)



$$\frac{da}{dt} = \frac{db}{dt} = 1\,m/s$$

$$\frac{dc}{dt} = -3\,m/s$$

$a = 1m$
$b = 2m$
$c = 3m$



rate at which Vol. is changing

$$Vol = abc = \frac{dV}{dt} = \frac{\partial V}{\partial a}\cdot\frac{da}{dt} + \frac{\partial V}{\partial b}\cdot\frac{db}{dt} + \frac{\partial V}{\partial c}\cdot\frac{dc}{dt}$$

$$= (bc)\cdot\frac{da}{dt} + (ac)\cdot\frac{db}{dt} + (ab)\cdot\frac{dc}{dt}$$

$$= \left.\frac{dV}{dt}\right|_{\text{given values}} = (2\cdot3)(1) + (1\cdot3)(1) + (1\cdot2)(-3)$$

$$\frac{dV}{dt} = 6 + 3 - 6 \longrightarrow \boxed{\frac{dV}{dt} = 3\,m^3/s}$$

14.5 Directional Derivatives & Gradient Vectors

formal Definition

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

where u = components of this vector

OR $D_{\vec{u}} F = \nabla f \cdot \vec{u} = |\nabla f| \cos\theta$

derivative in direction of u — gradient

gradient: $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$

Examples:

2) $f(x,y) = \ln(x^2+y^2)$; $(1,1)$

$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$

$\frac{\partial f}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x \rightarrow @(1,1) = \frac{2}{2} = \boxed{1}$

$\frac{\partial f}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y \rightarrow @(1,1) = \frac{2}{2} = \boxed{1}$

$$\boxed{\nabla f = 1i + 1j}$$

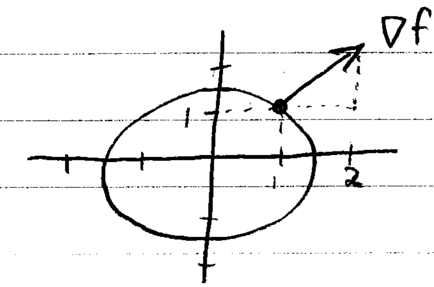
Level Curves:

$$f(1,1) = \ln(1^2+1^2) = \ln 2 = \text{height or } z$$

$$\ln(x^2+y^2) = \ln 2$$

$$e^{\ln(x^2+y^2)} = e^{\ln 2}$$

$$x^2+y^2 = 2 \qquad r = \sqrt{2}$$



(4) $g(x,y) = \dfrac{x^2}{2} - \dfrac{y^2}{2}$, $(\sqrt{2}, 1)$

$g_x = x \rightarrow g_x(\sqrt{2},1) = \sqrt{2}$

$g_y = -y \rightarrow g_y(\sqrt{2},1) = -1$

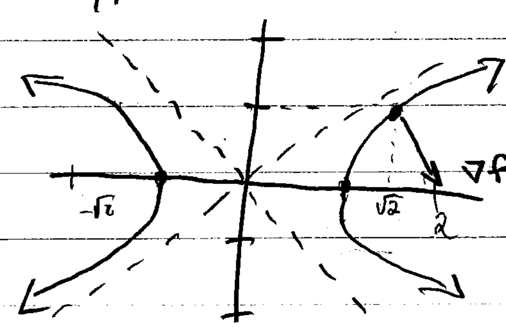
$\nabla g = \sqrt{2}\,\vec{i} - \vec{j}$

Level curve: $g(\sqrt{2},1) = \dfrac{(\sqrt{2})^2}{2} - \dfrac{1^2}{2}$

$= 1 - \dfrac{1}{2} = \boxed{\dfrac{1}{2}}$

So $\dfrac{x^2}{2} - \dfrac{y^2}{2} = \dfrac{1}{2}$ is hyperbola

$x^2 - y^2 = 1$



find $D_{\vec{u}} f$

(12) $f(x, y) = 2x^2 + y^2, \quad P(-1, 1)$

$\vec{u} = 3i - 4j$ needs to be a unit vector

$|\vec{u}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\dfrac{u}{|u|} = \dfrac{3i - 4j}{5}$ = unit vector

$\nabla f - f_x = 4x \rightarrow f_x(-1, 1) = -4$

$f_y = 2y \rightarrow f_y(-1, 1) = 2$

$\nabla f = -4i + 2j$

$D_{\vec{u}} f = \nabla F \cdot \dfrac{u}{|u|} = (-4)\left(\frac{3}{5}\right) + (2)\left(\frac{4}{5}\right)$

$D_{\vec{u}} f = -12/5 - 8/5$

$\boxed{D_{\vec{u}} f = -4} = 2$

(18) $h(x,y,z) = \cos(xy) + e^{yz} + \ln(zx)$, $P(1,0,1/2)$, $\vec{u} = i + 2j + 2k$

$|u| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$

$$\frac{\vec{u}}{|\vec{u}|} = \frac{i + 2j + 2k}{3}$$

$\nabla f = f_x = -y\sin(xy) + \frac{1}{zx} \cdot z = -y\sin(xy) + \frac{1}{x}$

$f_x(1,0,1/2) = \boxed{1}$

$f_y = -x\sin xy + ze^{yz}$

$f_y(1,0,1/2) = -0 + \frac{1}{2}(1) = \boxed{\frac{1}{2}}$

$f_z = ye^{yz} + \frac{1}{z} \longrightarrow f_z(1,0,1/2) = 0 + \frac{1}{1/2} = \boxed{2}$

$\nabla f = i + \frac{1}{2}j + 2k$

$D_{\vec{u}} f = \nabla h \cdot \frac{\vec{u}}{|\vec{u}|} = \langle 1, 1/2, 2\rangle \cdot \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle$

$= \frac{1}{3} + \frac{1}{3} + \frac{4}{3} = \boxed{2}$

(20) $f(x,y) = x^2y + e^{xy}\sin y$, $P_0(1,0)$

$f_x = 2xy + ye^{xy}\sin y$ @$(1,0)$ = 0

$f_y = x^2 + xe^{xy}\sin y + \cos y\, e^{xy}$

$f_y(1,0) = 1 + 0 + 1 = \boxed{2}$

$\nabla f(1,0) = 0i + 2j$ $\qquad \dfrac{\nabla f}{|\nabla f|} = \dfrac{2j}{2} = \vec{j}$

$|\nabla f| = \sqrt{2^2} = 2$

increase most rapidly in $\vec{j}$ direction
decreases most rapidly in $-\vec{j}$ direction

$(D_{\hat{j}} f)_{P_0} = \nabla f \cdot \hat{j} = 2j \cdot j = \boxed{2} = Z$

$(D_{-\hat{j}} f)_{P_0} = \nabla f \cdot -\hat{j} = 2j \cdot (-j) = \boxed{-2} = Z$

(22) $g(x,y,z) = xe^y + z^2$, $P_0(1, \ln 2, 1/2)$

$g_x = e^y \rightarrow g_x(1, \ln 2, 1/2) = e^{\ln 2} = \boxed{2}$

$g_y = xe^y \rightarrow g_y(1, \ln 2, 1/2) = 1 \cdot e^{\ln 2} = \boxed{2}$

$g_z = 2z \rightarrow g_z(1, \ln 2, 1/2) = 2(1/2) = \boxed{1}$

$\nabla g \quad 2i + 2j + k$ $\qquad \dfrac{\nabla g}{|\nabla g|} = \dfrac{2}{3}i + \dfrac{2}{3}j + \dfrac{1}{3}k$

$|\nabla g| = \sqrt{9} = 3$

$(D_{\vec{u}} g)_{P_0} = \boxed{3}$ $\qquad$ Because $|\vec{u}||\vec{v}|\cos\theta$ where $\theta = 0$

$(D_{-\vec{u}} g)_{P_0} = \boxed{-3}$ $\qquad$ & one of these is a unit vector

$\therefore |\vec{u}|$

Notes:

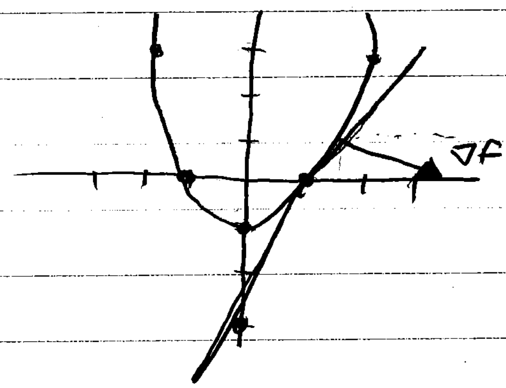
* Tangent line to a level curve

$$f_x\big|_{(x_0,y_0)}(x-x_0) + f_y\big|_{(x_0,y_0)}(y-y_0) = 0$$

(26) $x^2 - y = 1$, $(\sqrt{2}, 1)$ → level curve

$f(x,y) = x^2 - y - 1$ → original fxn

$y = x^2 - 1$



$f_x = 2x$ @ $(\sqrt{2},1) = 2\sqrt{2}$

$f_y = -1$

$\nabla f = 2\sqrt{2}\, i - j$

Tangent Line:

$f_x\big|_{(x_0,y_0)} = 2\sqrt{2}(x - x_0) \rightarrow 2\sqrt{2}(x-2)$

$f_y\big|_{(x_0,y_0)} = -1(y - y_0) \longrightarrow -1(y-1)$

$2\sqrt{2}x - 4 - y + 1 = 0$

$y = 2\sqrt{2}x - 3$

14.6 Tangent Planes & Differentials

Normal Line thru a point & perpendicular to the gradient at that point $\nabla F|_{P_0}$

Normal line: $X = X_0 + F_x|_{P_0}(t)$
$Y = Y_0 + f_y|_{P_0}(t)$
$Z = Z_0 + F_z|_{P_0}(t)$

Tangent Plane to $f(x,y,z) = C$, at point $P_0(X_0, Y_0, Z_0)$

$$F_x|_{P_0}(X - X_0) + F_y|_{P_0}(Y - Y_0) + F_z|_{P_0}(Z - Z_0) = 0$$

Examples:

2) $X^2 + Y^2 - Z^2 = 18$, $P_0(3, 5, -4)$

$f_x = 2x \; @(3,5,-4) = \boxed{6}$
$f_y = 2y \longrightarrow \boxed{10}$
$f_z = -2z \longrightarrow \boxed{8}$

a) $\overset{3}{\cancel{6}}(X-3) + \overset{5}{\cancel{10}}(Y-5) + \overset{4}{\cancel{8}}(Z+4) = 0$
$3x - 9 + 5y - 25 + 4z + 16 = 0$

$$\boxed{3x + 5y + 4z = 18}$$

b) $\boxed{\begin{aligned} X &= 3 + 3t \\ Y &= 5 + 5t \\ Z &= -4 + 4t \end{aligned}}$

(4) $x^2 + 2xy - y^2 + z^2 = 7 \quad P_0(1, -1, 3)$

$f_x = 2x + 2y \rightarrow 0$

$f_y = 2x - 2y \longrightarrow 4$

$f_z = 2z \longrightarrow 6$

$\cancel{0(x - x_0) + 4(y - y_0) + 6(z - z_0)}$

$= 4(y+1) + 6(z-3) = 0$

$= 4y + 4 + 6z - 18 = 0$

$4y + 6z - 14 = 0$

$4y + 6z = 14$

$\boxed{2y + 3z = 7}$

$\perp$ line:
$$\boxed{\begin{aligned} x &= 1 \\ y &= -1 + 4t \\ z &= 3 + 6t \end{aligned}}$$

Notes:

* plane tangent to a surface $z = f(x, y)$

$$f_x\Big|_{(x_0,y_0)}(x-x_0) + f_y\Big|_{(x_0,y_0)}(y-y_0) - (z-z_0) = 0$$

Examples:

(10) $z = e^{-(x^2+y^2)}$, $(0,0,1)$

$f_x = -2xe^{-(x^2+y^2)} \longrightarrow 0$ plugged in

$f_y = -2ye^{-(x^2+y^2)} \longrightarrow 0$

$$0(x-0) + 0(y-0) - (z-1) = 0$$

$$z = 1$$

(12) $z = 4x^2 + y^2$, $(1,1,5)$

$f_x = 8x \longrightarrow 8$

$f_y = 2y \longrightarrow 2$

$$8(x-1) + 2(y-1) - (z-5) = 0$$

$$8x - 8 + 2y - 2 - z + 5 = 0$$

$$\boxed{8x + 2y - z = 5}$$

Notes:

* Estimating change in a function in direction of $\vec{u}$ (unit vector)

$$df = \underbrace{(\nabla f|_{P_0} \cdot \vec{u})}_{\text{directional derivative}} \underbrace{ds}_{\text{incremental movement along curve}}$$

$$df = f_x|_{(x_0,y_0)}\,dx + f_y|_{(x_0,y_0)}\,dy$$

Total differential of a fxn

Examples:

20) $F(x,y,z) = e^x \cos yz$

$2i + 2j - 2k$, $ds = 0.1$

$f_x = e^x \cos yz \rightarrow f_x|_{P_0} = e^0 \cos 0 = \boxed{1}$

$f_y = -z e^x \sin yz \rightarrow f_y|_{P_0} = 0e^0 \sin 0 = \boxed{0}$

$f_z = -y e^x \sin yz \rightarrow f_z|_{P_0} = 0e^0 \sin 0 = \boxed{0}$

$\nabla f = \langle 1, 0, 0 \rangle = i$

$|\langle 2,2,2\rangle| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$

$\vec{u} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle$

$df = (\langle 1,0,0\rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle)(.1)$

$df = (\frac{1}{\sqrt{3}})(.1) \approx \boxed{0.058}$

Notes:

* Linearization

$$L(x,y) = f(x_0,y_0) + f_x\big|_{(x_0,y_0)}(x-x_0) + f_y\big|_{(x_0,y_0)}(y-y_0)$$

28 $f(x,y) = x^3y^4$ at $(1,1)$

$f(x_0,y_0) = 1$

$f_x = 3x^2y^4 \rightarrow f_x|_{p_0} = 3$
$f_y = 4x^3y^3 \rightarrow f_y|_{p_0} = 4$

$$f(x,y) \approx L(x,y) = 1 + 3(x-1) + 4(y-1)$$
$$= 3x + 4y - 6$$

[illegible]

14.7 Extreme Values & Saddle Points

Def:

1) $f(a,b)$ is a local max value of the fxn if $f(a,b) \geq f(x,y)$ for all (x,y) in open disk centered at the point (a,b)

$f_x(a,b) = 0$ and $f_y(a,b) = 0$

2nd Deriv Test:

i) fxn has a local max at point (a,b) if the $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$ at point (a,b)

ii) fxn has local min at point (a,b) if $f_{xx} >$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$

iii) Therer a saddle point at (a,b) if $f_{xx}f_{yy} - (f_{xy})^2 < 0$ at (a,b)

iv) $f_{xx}f_{yy} - (f_{xy})^2 = 0$ test is inconclusive

Examples:

(2) $f(x,y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$

$f_x = 2y - 10x + 4 \longrightarrow y - 5x + 2 = 0 \rightarrow \boxed{y = 5x - 2}$

$f_y = 2x - 4y + 4 \longrightarrow 2x - 4(5x-2) + 4$

$X = 2/3 \qquad Y = 4/3$

$\boxed{f_{xx} = -10} < 0$

$f_{xy} = 2$

$f_{yy} = -4$

$f_{xx} f_{yy} - (f_{xy})^2 = 36 > 0$

Max of $(\frac{2}{3}, 4/3)$ at original fxn

(4) $f(x,y) = 5xy - 7x^2 + 3x - 6y + 2$

$f_x = 5y - 14x + 3 = 0 \longrightarrow$

$f_y = 5x - 6 = 0 \longrightarrow \boxed{X = \frac{6}{5}}$

$5y - 14(\frac{6}{5}) + 3 = 0$

$\boxed{Y = \frac{69}{25}}$

$\boxed{f_{xx} = -14} < 0$

$f_{xy} = 5$

$f_{yy} = 0$

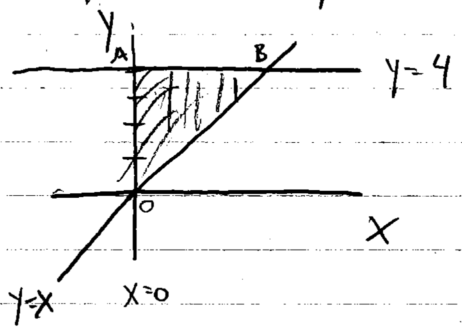
$f_{xx} f_{yy} - (5)^2 = \boxed{-25 < 0}$ Saddle Point

(32) Find abs max/min

$D(x,y) = x^2 - xy + y^2 + 1$

on closed triangular plate on first Quadrant band by $x=0$

$y=4$ & $y=x$



$f_x = 2x - y \rightarrow 2x - y = 0 \rightarrow y = 2x \rightarrow y = 0$

$f_y = -x + 2y \rightarrow -x + 2y = 0 \rightarrow -x + 4x = 0$

$x = 0$

$f(0,0) = 0^2 - (0)(0) + 0^2 + 1$

$\boxed{f(0,0) = 1}$

on line Origin to A:

Since $x = 0$, then $f(0,y)$

$f(0,y) = y^2 + 1$

$D_y = f'(0,y) = 2y = 0 \qquad y = 0$ already tested $(0,0)$

$A = (0,4) \qquad D(0,y) = (4)^2 + 1 = \boxed{17}$

On line AB: $\quad y = 4$

$D(x,4) = x^2 - 4x + 17$

$D_x = 2x - 4 = 0 \rightarrow x = 2 \quad (2,4)$ critical pt

$D(2,4) = \boxed{13}$

Line end point B : $D(4,4) = \boxed{17}$

On line Origin to B : $y = x$

$\therefore D(x,x) = x^2+1$

$D_x = 2x = 0 \rightarrow x=0$ tested $(0,0)$

Absolute max is 17 on $(0,4)$ and $(4,4)$

Absolute min is 1 on $(0,0)$

(34) $T(x,y) = x^2 + xy + y^2 - 6x \qquad 0 \le x \le 5, \quad -3 \le y \le 3$

* interior of rect:

$T_x = 2x + y - 6 = 0 \rightarrow 2(-2y) + y - 6 = 0$

$T_y = x + 2y = 0 \rightarrow x = -2y$

$-4y + y - 6 = 0$

$-3y - 6 = 0$

$y = -2$

$x = 4$

$T(4,-2) = 16 - 8 + 4 - 24$

$T(4,-2) = -12$

* Along AB = $T(0,y) = y^2$ on $-3 \le y \le 4$

$T_y = 2y = 0$

$y = 0 \qquad T(0,0) = \boxed{0}$

@ $T(0,-3) = (-3)^2 = 9$

@ $T(0,3) = (3)^2 = 9$

* Along BC : $T(x,3) = x^2 + 3x + 9 - 6x$

$= x^2 - 3x + 9$

$T_x = 2x - 3 = 0$

$x = 3/2$

$T(3/2, 3) = (3/2)^2 - 3(3/2) + 9 = \frac{27}{4}$

$T(5,3) = 5^2 - 3(5) + 9 = 19$

$$\text{* Along } \overset{x=5}{CD}: \ T(5,y) = 5^2 + 5y + y^2 - 6(5)$$
$$y^2 + 5y - 5$$

$$T_y = 2y + 5 = 0$$
$$y = -5/2 \rightarrow T(5, -5/2) = (-5/2)^2 + 5(-\tfrac{5}{2}) - 5$$
$$= -45/4$$

@ point D: $T(5,-3) = (-3)^2 + 5(-3) - 5 = -11$

$$\text{*Along } DA: \ T(x,-3) = x^2 - 3x + 9 - 6x =$$
$$= x^2 - 9x + 9$$
$$T_x = 2x - 9 = 0$$
$$x = 9/2$$
$$T(9/2, -3) = (9/2)^2 - 9(9/2) + 9 = -\frac{45}{4}$$

$$\boxed{\begin{array}{l} \text{max of } 19 \text{ @ } (5,3) \\ \text{min of } -12 \text{ @ } (4,-2) \end{array}}$$

14.8 Lagrange Multipliers

Examples:

(2) Find Extreme value of fxn $(X,Y) = XY$
Subject to the constraint that $g(x,y) = X^2 + Y^2 - 10 = 0$

$\nabla f = \lambda \nabla g$ $\qquad$ λ = Lagrange Multiplier

$\nabla f = Yi + Xj$ $\qquad$ $Yi + Xj = \lambda(2xi + 2yj)$

$yi + xj = 2x\lambda i + 2y\lambda j$

$\nabla g = 2xi + 2yj$

System of eq.

$$\begin{cases} Y = 2x\lambda \\ X = 2y\lambda \\ X^2 + Y^2 - 10 = 0 \end{cases} \qquad \frac{Y}{X} = \frac{2x\cancel{\lambda}}{2y\cancel{\lambda}}$$

$$\frac{Y}{X} = \frac{X}{Y} \rightarrow X^2 = Y^2$$

$X^2 + X^2 - 10 = 0$

$2x^2 - 10 = 0$

$X = \pm\sqrt{5}$ $\qquad$ $X^2 = Y^2 \therefore$ $\quad y = \pm\sqrt{5}$

$f(x,y) = (\sqrt{5})(\sqrt{5}) = 5$
$f(x,y) = (-\sqrt{5})(\sqrt{5}) = -5$
$f(x,y) = (-\sqrt{5})(-\sqrt{5}) = 5$
$f(x,y) = (\sqrt{5})(-\sqrt{5}) = -5$

*Another Method

$$\mathcal{L} = f(x,y) - \lambda\left(\overbrace{g(x,y) - c}^{0}\right)$$

$$f(x,y) = xy$$

$$\mathcal{L} = xy - \lambda(x^2 + y^2 - 10)$$

$$\frac{\partial \mathcal{L}}{\partial x} = y - 2\lambda x = 0, \quad \frac{\partial \mathcal{L}}{\partial y} = x - 2\lambda y = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x^2 + y^2 - 10) = 0$$

$$\begin{matrix} y = 2\lambda x \\ x = 2\lambda y \end{matrix} \quad = \quad \frac{y}{x} = \frac{\cancel{2\lambda} x}{\cancel{2\lambda} y} \longrightarrow x^2 = y^2$$

(4) $f(x,y) = x^2 y$ on line $g(x,y) = x+y=3$

$g(x,y) = x+y=0$

$\mathcal{L} = x^2y + \lambda(x+y-3)$

$\mathcal{L}_x = 2xy + \lambda = 0 \rightarrow 2xy = -\lambda$

$\mathcal{L}_y = x^2 + \lambda = 0 \rightarrow x^2 = -\lambda$

$\frac{2xy}{x^2} = \frac{\lambda}{\lambda}$

$\downarrow$

$2y = x$

$\mathcal{L}_\lambda = x + y - 3 = 0 \longleftarrow 2y = x$

$2y + y - 3 = 0$

$y = 1$

$x = 2(1)$

$x = 2$

$f(2,1) = (2)^2(1) = f(2,1) = 4$

(6) find pt on curve $x^2y = 2$ nearest to the origin

$f(x,y) = x^2 + y^2 \rightarrow$ Distance w/o squared

$g(x,y) = x^2y - 2 = 0$

$\mathcal{L} = x^2 + y^2 + \lambda(x^2y - 2)$

$\mathcal{L}_\lambda = x^2y - 2 = 0 \rightarrow 2y^2y - 2 = 0 \rightarrow y = 1,\ x = \pm\sqrt{2}$

$\mathcal{L}_x = 2x + 2xy\lambda = 0$

$\mathcal{L}_y = 2y + x^2\lambda = 0$

$\frac{2x}{2y} = \frac{2xy\lambda}{x^2\lambda} \rightarrow x^2 = 2y^2$

$f(\pm\sqrt{2}, 1) = (\sqrt{2})^2 + (1)^2 = 3 \therefore \boxed{\text{distance is } \sqrt{3}}$

(16) Tank holds $8000 m^3$
$h = ?$ $r = ?$

$$\text{Area} = 2\pi rh + 4\pi r^2 = A(r,h) = 2\pi rh + 4\pi r^2$$

$$V_s = \tfrac{4}{3}\pi r^3$$

$$V(r,h) = \tfrac{4}{3}\pi r^3 + \pi r^2 h = 8000$$

$$V_c = \pi r^2 \cdot h =$$

$$\mathcal{L} = 2\pi rh + 4\pi r^2 + \lambda(\tfrac{4}{3}\pi r^3 + \pi r^2 h - 8000)$$

$$\mathcal{L}_\lambda = \tfrac{4}{3}\pi r^3 + \pi r^2 h - 8000 = 0$$

$$\mathcal{L}_r = 2\pi h + 8\pi r + 4\pi\lambda r^2 + 2\pi rh\lambda =$$

$$\mathcal{L}_h = 2\pi r + \pi r^2 \lambda = 0$$

$$\frac{2\pi h + 8\pi r}{2\pi r} = \frac{4\pi r^2 \lambda - 2\pi rh\lambda}{\pi r^2 \lambda}$$

$$\frac{h + 4r}{r} = \frac{r - 2h}{r}$$

$$h + 4r = 4r - 2h$$

$$3h = 0$$

$$h = 0$$ no height cylinder so just a sphere

$$V = V_s = \tfrac{4}{3}r^3 = 8000$$

$$r^3 = \frac{6000}{\pi} \longrightarrow r = \sqrt[3]{\frac{6000}{\pi}} = \boxed{r = 10\sqrt[3]{\frac{6}{\pi}}}$$

$$A\left(10\sqrt[3]{\tfrac{6}{\pi}}, 0\right) = 2\pi\left(10\sqrt[3]{\tfrac{6}{\pi}}\right)(0) + 4\pi\left(10\sqrt[3]{\tfrac{6}{\pi}}\right) =$$

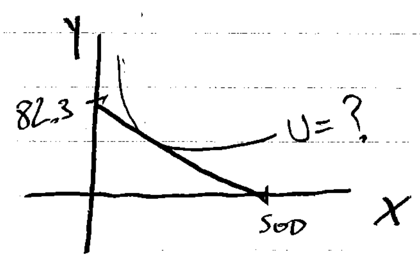
$X = \#$ songs   price song $= \$1$

$Y = \#$ expo markers   price marker $= \$6$

with $\$500$ year

$$U = X^{1/3} Y^{2/3}$$

max U such that $1 \cdot X + 6Y = 500$



$$\mathcal{L} = X^{1/3} Y^{2/3} + \lambda (X + 6Y - 500)$$

$$\mathcal{L}_\lambda = X + 6Y - 500 = 0$$

$$\mathcal{L}_X = \frac{1}{3} \frac{Y^{2/3}}{X^{2/3}} + \lambda = 0$$

$$\mathcal{L}_Y = \frac{2}{3} \frac{X^{1/3}}{Y^{1/3}} + 6\lambda = 0$$

$$\frac{\frac{1}{3} \frac{Y^{2/3}}{X^{2/3}} = +\lambda}{\frac{2}{3} \frac{X^{1/3}}{Y^{1/3}} = +6\lambda} \rightarrow \frac{Y}{2X} = \frac{1}{6}$$

$$\downarrow$$

$$6Y = 2Y$$

$$3Y = X$$

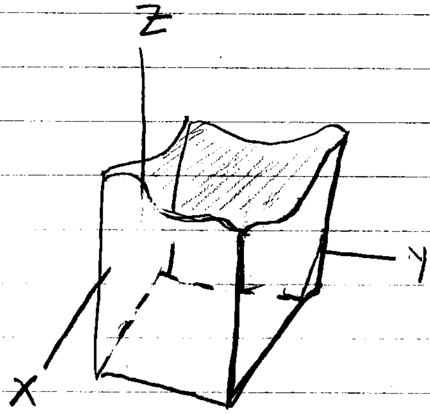
$$3Y + 6Y - 500 = 0$$

$$9Y = 500$$

$$Y \approx 55.5$$

$$X \approx 166.7$$

15.1 Double Integrals over Rectangles



Double Integrals give you the volume of a solid shape

$$V = \iint_R f(x, y)\, dA$$

R = Domain of the sum

Fubini's Theorem:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Examples:

(2) Evaluate the iterated (double Integral) Integral

$$\int_0^2 \left[\int_{-1}^{1} (x-y)\,dy\right] dx$$ inside then outside

X is const. $\int_{-1}^{1} x - y\,dy = \int_0^2 \left[xy - \frac{y^2}{2}\right]_{-1}^{1} dx$

$$\int_0^2 \left[x - \frac{1}{2} - \left(-x - \frac{1}{2}\right)\right] dx \rightarrow \int_0^2 2x\,dx$$

$$\int_0^2 2x\,dx \rightarrow \left[x^2\right]_0^2 \rightarrow 4-0 = \boxed{4}$$

(10) $$\int_0^1\int_1^2 xye^x\,dy\,dx \rightarrow \int_0^1 \left[xe^x \frac{y^2}{2}\right]_{y=1}^{y=2} dx$$

$$\int_0^1 xe^x\left[\frac{4}{2}-\frac{1}{2}\right]dx \rightarrow \frac{3}{2}\int_0^1 xe^x\,dx \qquad \begin{matrix} u=x & dv=e^x dx \\ du=dx & v=e^x \end{matrix}$$

$$\frac{3}{2}\left[xe^x\Big|_0^1 - \int_0^1 e^x\,du\right] = \frac{3}{2}\left[(1)e^1-0-e^x\Big|_0^1\right]$$

$$= \frac{3}{2}\left[e-(e^1-e^0)\right] = \frac{3}{2}\left[e^1-e^1\right] = \boxed{\frac{3}{2}}$$

(16) $$\iint_R \frac{\sqrt{x}}{y^2}\,dA \qquad R: 0\le x\le 4,\ 1\le y\le 2$$

$$\int_1^2\int_0^4 \frac{\sqrt{x}}{y^2}\,dx\,dy \rightarrow \int_1^2\left[\frac{1}{y^2}\cdot\frac{2}{3}x^{3/2}\right]_{x=0}^{x=4} dy$$

$$\int_1^2 \frac{1}{y^2}\frac{2}{3}\left[4^{3/2}-0\right]dy \rightarrow \int_1^2 \frac{16}{3y^2}\,dy = \frac{-16}{3}\cdot\frac{1}{y}\Big|_1^2$$

$$= -\frac{16}{3}\left(\frac{1}{2}-1\right) \rightarrow \boxed{\frac{8}{3}}$$

(26) Find Volume of Region bounded above by elliptical $z = 16 - x^2 - y^2$ and bounded below by square
$R: 0 \leq x \leq 2,\ 0 \leq y \leq 2$

$$V = \iint_R (16 - x^2 - y^2)\, dA \longrightarrow \int_0^2\int_0^2 (16 - x^2 - y^2)\, dx\, dy$$

$$= \int_0^2 \left[16x - \frac{1}{3}x^3 - y^2 x\right]_0^2 dy \longrightarrow \int_0^2 \left(32 - \frac{8}{3} - 2y^2\right) dy$$

$$= \left[32y - \frac{8}{3}y - \frac{2}{3}y^3\right]_0^2 \longrightarrow 64 - \frac{16}{3} - \frac{16}{3} = \boxed{\frac{160}{3}}$$

15.2 Double Integrals over General Regions

Theorem: Fubini's Theorem

1. if $R: a \le x \le b,\ g_1(x) \le y \le g_2(x)$

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

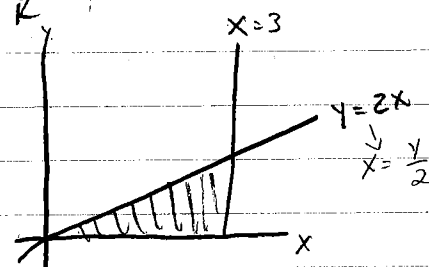
2. if $R: c \le y \le d,\ h_1(y) \le x \le h_2(y)$

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Examples:

Write the iterated Integral for $\iint_R dA$ over R — vertical & horizontal cross sections

(10)



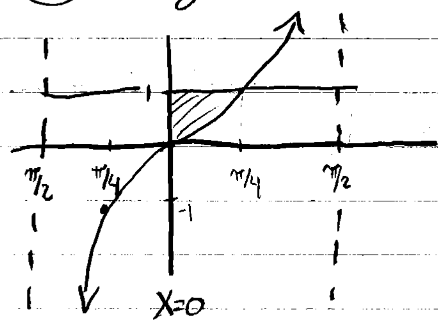
a) $0 \le y \le 2x$
$0 \le x \le 3$
start with variables
end with constants

$$\boxed{\int_0^3 \int_0^{2x} dy\,dx}$$

$y = 2(3) = 6$

$$\boxed{b)\ \int_0^6 \int_{y/2}^{3} dx\,dy}$$

(14) Region bounded by $y = \tan x$, $x = 0$, $y = 1$

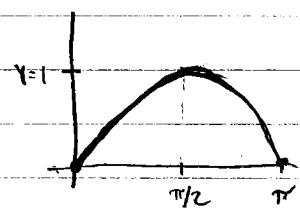


a) $\displaystyle\int_0^{\pi/4}\int_{\tan x}^{1} dy\,dx$

b) $\displaystyle\int_0^{1}\int_0^{\tan^{-1} y} dx\,dy$

(20) $\displaystyle\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx$ $\qquad 0 \le y \le \sin x$

$0 \le x \le \pi$



$$\int_0^{\pi}\left[\frac{1}{2}y^2\right]_0^{\sin x} dx \longrightarrow \int_0^{\pi}\frac{1}{2}\sin^2 x\,dx \longrightarrow \frac{1}{2}\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)dx$$

$$\frac{1}{4}\int_0^{\pi}(1-\cos 2x)\,dx \longrightarrow \frac{1}{4}\int_0^{\pi}dx - \frac{1}{4}\int_0^{\pi}\cos 2x$$

$$\frac{1}{4}\left[x\right]_0^{\pi} - \frac{1}{8}\left[\sin 2x\right]_0^{\pi} \longrightarrow \frac{1}{4}\pi - \frac{1}{8}\sin 2\pi$$

$$\frac{\pi}{4} - 0 \longrightarrow \boxed{\frac{\pi}{4}}$$

26) $f(x,y) = x^2 + y^2$ over Triangular region with vertices $(0,0)$, $(1,0)$ & $(0,1)$

$$\int_0^1 \int_0^{1-x} (x^2 + y^2)\,dy\,dx \longrightarrow \int_0^1 \left[x^2 y + \frac{1}{3} y^3\right]_0^{1-x} dx$$

$$\int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx$$

1 — 0
1 1 — 1
1 2 1 — 2
1 3 3 1 — 3 power

$$\int_0^1 \left[x^2 - x^3 + \frac{1 - 3x + 3x^2 - x^3}{3}\right] dx$$

$$\int_0^1 \left[x^2 - x^3 + \frac{1}{3} - x + x^2 - \frac{x^3}{3}\right] dx$$

$$\int_0^1 \left[-\frac{4}{3}x^3 + 2x^2 - x + \frac{1}{3}\right] dx$$

$$\left[\frac{-4}{3} \cdot \frac{1}{4}x^4 + 2 \cdot \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x\right]_0^1$$

$$-\frac{1}{3} + \frac{2}{3} - \frac{1}{2} + \frac{1}{3} \longrightarrow \frac{2}{3} - \frac{1}{2} = \boxed{\frac{1}{6}}$$

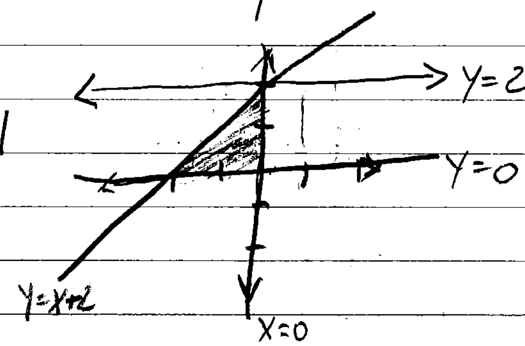
$$y = x + 2$$

$$x = y - 2 \qquad x = 0$$

(34) $\displaystyle\int_0^2\int_{y-2}^0 dx\,dy$

$$y-2 \le x \le 0$$
$$0 \le y \le 2$$

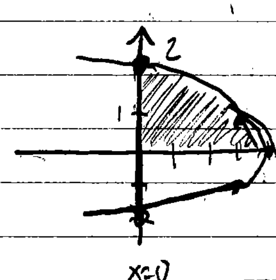
now reverse for vertical



$$\boxed{= \int_{-2}^{0}\int_{0}^{x+2} dy\,dx}$$

$$0 \le y \le x+2$$
$$-2 \le x \le 0$$

(40) $\displaystyle\int_0^2\int_0^{4-y^2} y\,dx\,dy$



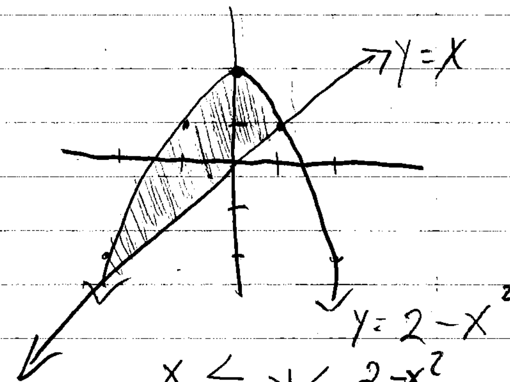
$$0 \le x \le 4 - y^2$$
$$0 \le y \le 2$$

Vert cross section

$$0 \le y \le \sqrt{4-x}$$
$$0 \le x \le 4$$

$$\boxed{= \int_0^4\int_0^{\sqrt{4-x}} dy\,dx}$$

(58) $V = \iint_R x^2 \, dA$



$x \le y \le 2-x^2$

$-2 \le x \le 1$

$$V = \int_{-2}^{1}\int_{x}^{2-x^2} x^2 \, dy\, dx$$

$$V = \int_{-2}^{1}\left[x^2 y\right]_{x}^{2-x^2} dx \longrightarrow \int_{-2}^{1} x^2(2-x^2-x)\,dx$$

$$\int_{-2}^{1}(-x^4 - x^3 + 2x^2)\,dx$$

$$= \left[-\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3\right]_{-2}^{1}$$

$$V = \frac{189}{60}$$

$$\boxed{V = \frac{63}{20}}$$

15.3 Area by Double Integration

* If you make $z=1$ your volume will equal the area times the height of 1. Anything times 1 is itself so you can find Area by making the height $= 1$

* $A = \iint_R 1\, dA$

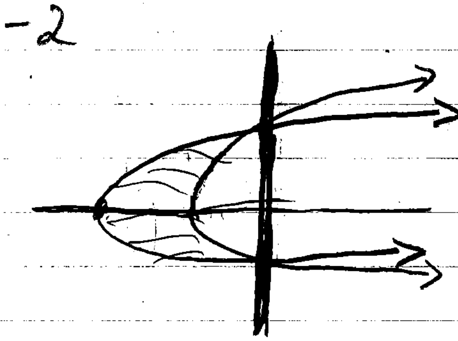
* Average value of fxn over R

$$\frac{1}{\text{area of } R} \iint_R f\, dA$$

Examples:

(8) $X = y^2 - 1$ & $X = 2y^2 - 2$

$$\text{Area} = \iint_R dx\,dy \longrightarrow \int_{-1}^{1}\int_{2y^2-2}^{y^2-1} dx\,dy$$



$$= \int_{-1}^{1} \Big[X\Big]_{2y^2-2}^{y^2-1} dy \longrightarrow \int_{-1}^{1} [y^2-1-2y^2+2]\,dy$$

$2y^2-2 \leq X \leq y^2-1$

$-1 \leq y \leq 1$

$$= \int_{-1}^{1} (-y^2+1)\,dy \longrightarrow \left[-\frac{1}{3}y^3 + y\right]_{-1}^{1}$$

$$\boxed{A = \frac{4}{3}}$$

(20) Average over $F(x,y) = xy$ over

(1) □ $0 \le x \le 1$, $0 \le y \le 1$

(2) D $x^2 + y^2 \le 1$ in Q1

Which larger?

(1) $A = \int_0^1 \int_0^1 dx\,dy = 1$

Average Value $= \frac{1}{1} \int_0^1 \int_0^1 xy\,dx\,dy$

$= \int_0^1 \left[\frac{1}{2}x^2 y\right]_0^1 \rightarrow \int_0^1 \frac{1}{2} y\,dy = \frac{y^2}{4} = \boxed{\frac{1}{4}}$ larger

(2) $A = \int_0^1 \int_0^{\sqrt{1-y^2}} dx\,dy \longrightarrow \boxed{\frac{\pi}{4}}$

Average $= \frac{1}{\pi/4} \int_0^1 \int_0^{\sqrt{1-y^2}} xy\,dx \longrightarrow \frac{4}{\pi} \int_0^1 \left[y \frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$

$\frac{2}{\pi} \int_0^1 \left[y - y^3\right]_0^1 = \frac{2}{\pi}\left[\frac{1}{2} - \frac{1}{4} - 0\right]$

$= \boxed{\frac{1}{2\pi}}$

15.4 Double Integrals in Polar Form

$x^2 + y^2 = 1 \longrightarrow r = 1$

$x^2 + y^2 = r^2$

rect $\rightarrow$ polar

$x = r\cos\theta$

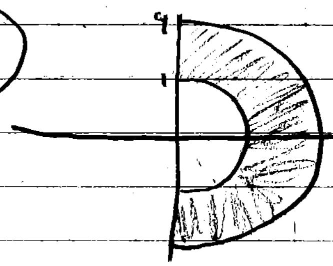
$y = r\sin\theta$

Area $= \iint_R dA$ in 15.3

New Area for Polars

$* \quad A = \iint_R r\,dr\,d\theta$

Ex: 2



$x^2 + y^2 = 4^2 \longrightarrow r = 4$

&

$x^2 + y^2 = 1^2 \longrightarrow r = 1$

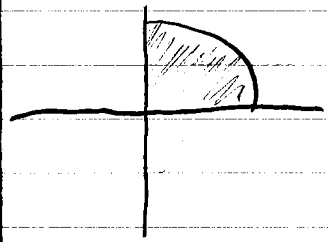
$$\boxed{\begin{array}{c} 1 \le r \le 4 \\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \end{array}}$$

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{4} r\,dr\,d\theta = \frac{15}{2}\pi$$

(10) $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$

$0 \leq x \leq \sqrt{1-y^2}$
$0 \leq y \leq 1$

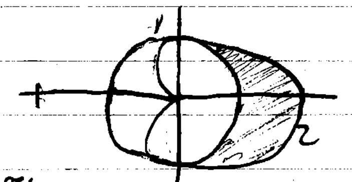
$x^2 = 1-y^2$
$x^2+y^2=1$



$$\int_0^{\pi/2}\int_0^1 r^2 \cdot r\,dr\,d\theta = \int_0^{\pi/2} \left[\frac{1}{4}r^4\right]_0^1 d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4}\,d\theta \longrightarrow \left[\frac{1}{4}\theta\right]_0^{\pi/2} \longrightarrow \boxed{\frac{\pi}{8}}$$

(28) $r = 1+\cos\theta$ & $r=1$



$1 \leq r \leq 1+\cos\theta$
$-\pi/2 \leq \theta \leq \pi/2$

$$\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta \longrightarrow \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2}r^2\right]_1^{1+\cos\theta} d\theta$$

$$\int_{-\pi/2}^{\pi/2} \left[\frac{(1+\cos\theta)^2}{2} - \frac{1}{2}\right] d\theta \longrightarrow \int_{-\pi/2}^{\pi/2} \left(\cos\theta + \frac{\cos^2\theta}{2}\right) d\theta$$

Table Int $= \left[\sin\theta + \frac{\theta}{4} + \frac{\sin 2\theta}{8}\right]_{-\pi/2}^{\pi/2} = \boxed{\frac{8+\pi}{4}}$

15.5 Triple Integrals

$$V = \iint_R z(x,y)\,dx\,dy$$

$$V = \iint \int_0^{z(x,y)} dz\,dx\,dy$$

z bound does not have to start at zero.

$$V = \iint \int_{-3}^{z(x,y)} dz\,dx\,dy$$

Average Value:

$$\frac{1}{\text{Volume}} \iiint_{\text{cube}} xyz\,dV$$

15.6 Moments & Centers of Mass

Mass: if δ (delta = density) $= \delta(x,y,z)$ is the density of an object ocupying a region D in space
Then the mass is

$$\boxed{m = \iiint_D \delta \, dV}$$

The first moment of a solid region D about a coordinate plane is the triple integral over D of the distance of a pt (x,y,z) in D to the plane multiplied by the density of the solid at that point

1st moments:

$$m_{yz} = \iiint_D x\delta \, dV$$

$$m_{xz} = \iiint_D y\,\delta \, dV$$

$$m_{xy} = \iiint_D z\,\delta \, dV$$

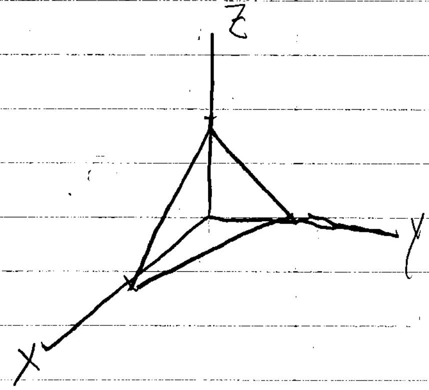
$(\bar{x}, \bar{y}, \bar{z}) \rightarrow$ Coordinates of CM

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

Example:

Find the center of mass of a tetrahedron D bounded by the planes $x+y+z=1$, $x=0$, $y=0$, $z=0$
density, $\delta = y$



$0 \le z \le 1-x-y$,
$0 \le y \le 1-x$
$0 \le x \le 1$

$$M = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \Big[zy\Big]_0^{1-x-y} \rightarrow \int_0^1 \int_0^{1-x} y - yx - y^2 \, dy \, dx$$

$$= \int_0^1 \left[\frac{1}{2}y^2 - \frac{1}{2}y^2 x - \frac{1}{3}y^3\right]_0^{1-x}$$

$$= \int_0^1 \left[\frac{1}{2}(1-x)^2 - \frac{1}{2}x(1-x)^2 - \frac{1}{3}(1-x)^3\right] dx$$

$$= \int_0^1 \left[\frac{1}{6} - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right] dx$$

$$= \left[\frac{1}{6}x - \frac{1}{4}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4\right]_0^1 = \boxed{\frac{1}{24}}$$

$$M_{yz} = \iiint_D x\,\delta\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} [xy]\,dz\,dy\,dx = \boxed{\frac{1}{120}}$$

$$M_{xz} = \iiint_D y\,\delta\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} [y^2]\,dz\,dy\,dx = \boxed{\frac{1}{60}}$$

$$M_{xy} = \iiint_D z\,\delta\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} [zy]\,dz\,dy\,dx = \boxed{\frac{1}{120}}$$

$$\bar{x} = \frac{\frac{1}{120}}{\frac{1}{24}} = \boxed{\frac{1}{5}}, \quad \bar{y} = \frac{\frac{1}{60}}{\frac{1}{24}} = \boxed{\frac{2}{5}}, \quad \bar{z} = \frac{\frac{1}{120}}{\frac{1}{24}} = \boxed{\frac{1}{5}}$$

$$\boxed{\left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)}$$

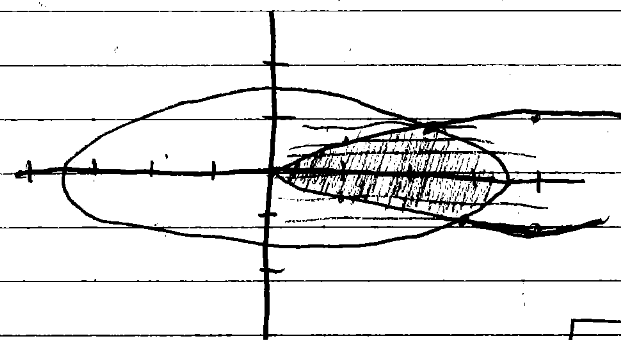
(12) Find Mass $x^2+4y^2=12$, $x=4y^2$

$\delta = 5x$

ellipse: $\frac{x^2}{12}+\frac{y^2}{3}=1$

$\sqrt{12}\approx 3.5$

$\sqrt{3}\approx 1.7$



Find point of intersections

$4y^2 \le x \le +\sqrt{12-4y^2}$

$-\sqrt{3}/2 \le y \le \sqrt{3}/2$

$x^2+4y^2=12$ $\quad x=4y^2$

$3=4y^2$

$y=\pm\sqrt{3/2}$

$x^2+x-12=0$

$(x+4)(x-3)$

$x=-4$ + $\boxed{x=3}$

$$\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\int_{4y^2}^{\sqrt{12-4y^2}} 5x\,dx\,dy \longrightarrow \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{5}{2}x^2\Big]_{4y^2}^{\sqrt{12-4y^2}}$$

$$\frac{5}{2}\left[12y-\frac{4}{3}y^3-\frac{16}{5}y^5\right]_{-\sqrt{3}/2}^{\sqrt{3}/2}$$

$$M=\frac{5}{2}\cdot 2\left[\left(12\left(\frac{\sqrt{3}}{2}\right)-\frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)^3-\frac{16}{5}\left(\frac{\sqrt{3}}{2}\right)^5\right]\right.$$

$$\boxed{M=23\sqrt{3}}$$

<u>Notes</u>: 2nd moments (moment of inertia) defines Torque needed for a desired angle of rotation

moment of inertia about x-axis

$$I_x = \iiint_D (y^2 + z^2)\,\delta\,dV$$

$$I_y = \iiint_D (x^2 + z^2)\,\delta\,dV$$

$$I_z = \iiint_D (x^2 + y^2)\,\delta\,dV$$

rotation about line L:

$$I_L = \iiint_D r^2\,\delta\,dV$$

where $r = (x, y, z)$ is a distance from a point (x, y, z) in D to L

Ex: Find moment of inertia of region D with $\delta = 1$ bounded by planes $z = x$, $z = 0$, $x = 1$ & parabolic cylinder $x = y^2$

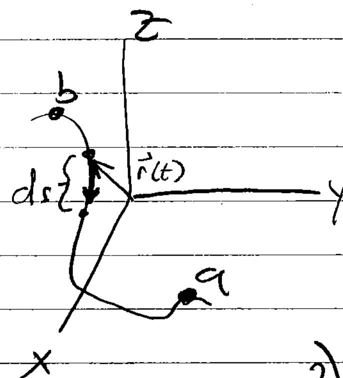
$0 \leq z \leq x$, $y^2 \leq x \leq 1$, $-1 \leq y \leq 1$

$$I_x = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} (y^2 + z^2) \cdot 1 \, dz \, dx \, dy = \frac{64}{189}$$

$$I_y = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} (x^2 + z^2) \, dz \, dx \, dy = \frac{16}{27}$$

$$I_z = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} (x^2 + y^2) \, dz \, dx \, dy = \frac{40}{63}$$

16.1 Line Integrals



To integrate $f(x,y,z)$ over curve C

1) Find smooth paramet. of C

$\vec{r}(t) = x(t)i + y(t)j + z(t)k, \quad a \leq t \leq b$

2) Evaluate as $\int_C f(x,y,z)\,ds$

$$= \int_a^b f(g(t), h(t), k(t))\,|\vec{v}(t)|\,dt$$

Ex:

(10) $\int_C (x - y + z - 2)\,ds$ where C is line segment

$x = t, \quad y = 1 - t, \quad z = 1$

from $(0,1,1)$ to $(1,0,1)$

$0 \leq t \leq 1, \quad r(t) = ti + (1-t)j + 1k$

$v(t) = \frac{dr}{dt} = i - j \rightarrow |\vec{v}| = \sqrt{2}$

$$\int_0^1 [t - (1-t) + 1 - 2]\sqrt{2}\,dt \rightarrow \int_0^1 [2t - 2]\sqrt{2}\,dt$$

$$\sqrt{2}\left[t^2 - 2t\right]_0^1 \rightarrow \sqrt{2}\,[1 - 2 - 0] = \boxed{-\sqrt{2}}$$

(14) Find Line Integral of $\frac{\sqrt{3}}{x^2+y^2+z^2}$
over curve
$r(t) = ti + tj + tk \qquad 1 \le t < \infty$

$\vec{V} = i + j + k \rightarrow |\vec{v}| = \sqrt{3}$
$x = t \ , \ y = t \ , \ z = t$

$$\int_1^\infty \frac{\sqrt{3}}{3t^2}(\sqrt{3})\,dt \rightarrow \int_1^\infty \frac{3}{3t^2}\,dt \rightarrow \int_1^\infty \frac{1}{t^2}\,dt$$

$$\int_1^\infty t^{-2}\,dt \rightarrow \left[-\frac{1}{t}\right]_1^\infty = -\left[\frac{1}{\infty} - \frac{1}{1}\right] = -[0-1] = \boxed{1}$$

(18) $f(x,y,z) = -\sqrt{x^2+z^2}$
over $r(t) = a\cos t\, j + a\sin t\, k, \quad 0 \le t \le 2\pi$
$x = 0, \quad y = a\cos t, \quad z = a\sin t$

$v(t) = -a\sin t\, j + a\cos t\, k$
$|v(t)| = \sqrt{a^2(\sin^2 t + \cos^2 t)} = |a|$

$$f(x,y,z) = -\sqrt{0^2 + (a\sin t)^2} = -\sqrt{a^2\sin^2 t}$$

$$= -|a||\sin t| \begin{cases} -|a|\sin t, & 0 \le t < \pi \\ |a|\sin t, & \pi \le t \le 2\pi \end{cases}$$

$$\int_0^\pi -|a|\sin t\,|a|\,dt + \int_\pi^{2\pi} |a|\sin t\,|a|\,dt$$

$$\int_0^\pi -a^2\sin t\,dt + \int_\pi^{2\pi} a^2\sin t\,dt$$

$$= \Big[a^2 \cos t\Big]_0^{\pi} - \Big[a^2 \cos t\Big]_{\pi}^{2\pi}$$

$$a^2\Big[\overset{-1}{\cos\pi} - \overset{1}{\cos 0}\Big] - a^2\Big[\overset{1}{\cos 2\pi} - \overset{-1}{\cos\pi}\Big]$$

$$a^2(-2) - a^2(2) = \boxed{-4a^2}$$

(33) Mass of wire along a curve

$r(t) = (t^2-1)j + 2tk, \quad 0 \le t \le 1$ if $\delta = \frac{3}{2}t$

$$M = \int_C \delta \, ds \qquad \begin{aligned} v(t) &= 2tj + 2K \\ |v| &= \sqrt{4t^2+4} = 2\sqrt{t^2+1} \end{aligned}$$

$$M = \int_0^1 \frac{3}{2}t\left(2\sqrt{t^2+1}\right)dt = \int_0^1 3t\sqrt{t^2+1}\,dt$$

$u = t^2+1$

$$\frac{du}{2} = \frac{2t\,dt}{2} = \frac{du}{2} = t\,dt \rightarrow \int \frac{3}{2}\sqrt{u}\,du$$

$$= \frac{3}{2}\cdot\frac{2}{3}u^{3/2} = (t^2+1)^{3/2}\Big]_0^1$$

$$= \left[2^{3/2} - 1^{3/2}\right] = \boxed{2\sqrt{2} - 1}$$

16.2 Vector fields & Line Integrals for Work Circulation & flux

Vector fields & Gradient fields

a vector field: function that assigns a vector to each point in the domain

$\mathbb{R}^3$: $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$

$\mathbb{R}^2$: $\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$

The Gradient field of a differentiable fxn $f(x,y,z)$ is the field of gradient vectors

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial x}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Ex: 12 find Gradient field

$f(x,y,z) = \ln\sqrt{x^2+y^2+z^2}$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot 2x = \frac{x}{x^2+y^2+z^2}$$

$$\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2+z^2} \qquad \frac{\partial f}{\partial z} = \frac{z}{x^2+y^2+z^2}$$

$$\boxed{\nabla f = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}}$$

Notes: Lines integral of a vector field
$\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$
along the curve C given by
$\vec{r}(t) = g(t)i + h(t)\hat{j} + k(t)\hat{k}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

Ex: 9 b) Find line integral of $\vec{F} = \sqrt{z}\,i - 2xj + \sqrt{y}K$
from $(0,0,0)$ to $(1,1,1)$ over path
$r(t) = ti + t^2j + t^4K$, $0 \le t \le 1$

$X = t$, $Y = t^2$, $Z = t^4$

$\sqrt{t^4} = t^2 i$ $\quad -2tj$
$\sqrt{t^2} = t\,K$

$\vec{F} = t^2 i - 2t\hat{j} + t\hat{K}$

$r't = i + 2tj + 4t^3K \bullet \vec{F} = t^2 i - 2tj + tK$

$$\int_0^1 \vec{F} \cdot r'(t) = t^2 - 4t^2 + 4t^4 = \int_0^1 4t^4 - 3t^2 =$$

$$\left[\frac{4}{5}t^5 - t^3\right]_0^1 = \frac{4}{5} - 1 = \boxed{\frac{-1}{5}}$$

Notes: Line integral with respect to
dx, dy, dz
$\therefore$ Rewrite everything with respect to a parameter "t"

Ex: 19 $\displaystyle\int_C \frac{x}{y}\,dy$, $C: x = t,\ y = t^2 \rightarrow dy = 2t\,dt$
$1 \leq t \leq 2$

$$\therefore \int_1^2 \frac{t}{t^2}(2t\,dt) \rightarrow \int_1^2 2\,dt = 2t\Big]_1^2 = \boxed{2}$$

Like #23

Evaluate $\displaystyle\int_C (x + yz)\,dx + 2x\,dy + xyz\,dz$
where C is a line segment from $(1,0,0)$ to $(2,3,2)$

$r(t) = (2-1)i + (3-0)j + (2-0)k$
$r(t) = 1i + 3j + 2k$

$x = 1 + 1t$, $y = 0 + 3t$, $z = 0 + 2t$
$x = 1 + t$, $y = 3t$, $z = 2t$ $\quad 0 \leq t \leq 1$

$dx = dt$
$dy = 3dt$
$dz = 2dt$

$$\int_0^1 (1 + t + 3t\,2t)\,dt + 2(1+t)3\,dt + (1+t)3t\,2t\,2\,dt$$

$$\int_0^1 1 + t + 6t^2\,dt + (6 + 6t)\,dt + (12t^2 + 12t^3)\,dt$$

$$\int_0^1 12t^3 + 18t^2 + 7t + 7\,dt = 3t^4 + 6t^3 + \frac{7}{2}t^2 + 7t\Big]_0^1$$

$$= \boxed{\frac{39}{2}}$$

Notes: Work done by a Force over a curve in Space

Work done by a vector field F in F in moving an object from pt $A = \vec{r}(a)$ to $B = \vec{r}(b)$ along C (parame by $r(t)$, $a \le t \le b$) is:

$$* \quad \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

Ex: (20) Find work done by $\vec{F}$ over curve (In increasing direction of t)

$\vec{F} = 2y\,i + 3x\,j + (x+y)\,k$

$\vec{r}(t) = \cos t\, i + \sin t\, j + t/6\, k \qquad 0 \le t \le 2\pi$

$x = \cos t, \quad y = \sin t, \quad z = t/6$

$\vec{F} = 2\sin t\, i + 3\cos t\, j + (\sin t + \cos t)k$

$r'(t) = -\sin t\, i + \cos t\, j + \frac{1}{6}k$

$\vec{F} \cdot d\vec{r} = -2\sin^2 t + 3\cos^2 t + \frac{\sin t}{6} + \frac{\cos t}{6}$

$\vec{F} \cdot d\vec{r} = -2\sin^2 t + (3 - 3\sin^2 t) + \frac{1}{6}(\cos t + \sin t)$

$\vec{F} \cdot d\vec{r} = 3 - 5\sin^2 t + \frac{1}{6}(\cos t + \sin t)$

$\vec{F} \cdot d\vec{r} = 3 - \frac{5}{2}(1 - \cos 2t) + \frac{1}{6}(\cos t + \sin t)$

$\vec{F} \cdot d\vec{r} = \frac{1}{2} + \frac{5}{2}\cos 2t + \frac{1}{6}(\cos t + \sin t)$

$$W = \int_0^{2\pi} \left[\frac{1}{2} + \frac{5}{2}\cos 2t + \frac{1}{6}(\cos t + \sin t)\right] dt$$

$$\frac{1}{2}t + \frac{5}{4}\sin 2t + \frac{1}{6}\sin t - \frac{1}{6}\cos t = \boxed{\pi}$$

Notes: Flow + Circulation

the flow along the curve from A to B is given by

$$* \quad \int_C \vec{F} \cdot d\vec{r} \qquad \left(\begin{array}{l}\text{if } A = B \Rightarrow \\ \text{flow called a circulation} \\ \text{around } C\end{array}\right)$$

→ flow over curved closed path

Ex: Find circulation of the velocity field $\vec{F} = \langle -2y, 2x \rangle$ around the closed path C, param. by $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$

$x = \cos t$, $y = \sin t$

$\vec{F} = \langle -2\sin t, 2\cos t \rangle \cdot \vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$$\int_0^{2\pi} 2\sin^2 t + 2\cos^2 t \, dt = \int_0^{2\pi} 2\,dt = 2t \Big]_0^{2\pi} = \boxed{4\pi}$$

Notes: Flux accross a simple closed plane curve

$$* \quad \int_C \vec{F} \cdot \vec{n} \, ds \quad + \quad \oint M\,dy - N\,dx$$

ex: Find flux $\vec{F} = \langle -2y, 2x \rangle$ accross C (the unit circle: $x^2+y^2=1$) in XY plane

$r(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$

$M = \overset{-2y}{-2\sin t}$, $N = \overset{2x}{2\cos t}$ $\qquad dy = \cos t\,dt$

$dx = -\sin t\,dt$

$$\text{flux} = \int_0^{2\pi} -2\sin t\cos t\,dt - 2\cos t(-\sin t\,dt)$$

$$= \int_0^{2\pi} 0\,dt = 0$$

16.3 Path Independence, Conservative Vector Fields

$f(x\,y\,z) = 2xy - 3xz$ Evaluate $\int_C \vec{F}\cdot dr$
$(1,1,0)$ to $(2,3,4)$

$\int_C F\cdot dr = f(2,3,4) - f(1,1,0) = -12 - 2 = \boxed{-14}$

② $\vec{F} = \overbrace{y\sin z}^{M}\, i + \overbrace{x\sin z}^{N}\, j + \overbrace{xy\cos z}^{P}\, k$. $\vec{F}$ conserv.?

$\frac{\partial P}{\partial y} = x\cos z$, $\frac{\partial N}{\partial z} = x\cos z$ $\quad \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$

$\frac{\partial M}{\partial z} = y\cos z$, $\frac{\partial P}{\partial x} = y\cos z$ $\quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$

$\frac{\partial N}{\partial x} = \sin z$, $\frac{\partial M}{\partial y} = \sin z$ $\quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$\boxed{\text{Conservative!}}$

(4) $\vec{F} = -yi + xj$. $\vec{F}$ conserv?

$\frac{\partial M}{\partial y} = -1$ $\quad$ $\frac{\partial N}{\partial x} = 1$ $\quad$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\boxed{\text{Not Conservative}}$

(18) Find potential fxn for conservative field & Evaluate

$$\int_{(0,2,1)}^{(1,\pi/2,2)} \underbrace{2\cos y\,dx}_{M} + \overbrace{\left(\frac{1}{y} - 2x\sin y\right)}^{N} dy + \overbrace{\frac{1}{z}}^{P} dz$$

$$F = \left\langle \underset{\downarrow\, f_x}{2\cos y},\ \underset{\downarrow\, f_y}{\frac{1}{y} - 2x\sin y},\ \underset{\downarrow\, f_z}{\frac{1}{z}} \right\rangle$$

$$f = \int 2\cos y\,dx = \underbrace{2x\cos y}_{\text{take } f_y} + g(y,z)$$

$$f_y = -2x\sin y + g_y = \frac{1}{y} - 2x\sin y \quad \therefore\ g_y = \frac{1}{y}$$

So integral of $g_y = \int \frac{1}{y} = \ln y + h(z) = g(y,z)$

So $f = 2x\cos y + \ln y + h(z) \rightarrow f_z = h'(z) = \frac{1}{z} = h(z) = \ln z + C$

$\int \frac{1}{z}$

Finally, $f = 2x\cos y + \ln y + \ln z + C$

$$f(1,\pi/2,2) - f(0,2,1) = \ln(\pi/2) + \ln(2) - \ln(2) = \boxed{\ln(\pi/2)}$$

(14) $\displaystyle\int_{(1,1,2)}^{(3,5,0)} \underbrace{yz\,dx}_{M} + \overbrace{xz}^{N}dy + \overbrace{xy}^{P}dz$

$\frac{\partial M}{\partial y} = z \quad \frac{\partial N}{\partial x} = z \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$M_z = y \quad P_x = y$

$N_z = x, \quad P_y = x \quad \boxed{\text{exact}}$

$f = xyz + C$ through $\rightarrow f_x = yz$

$f(x,y,z) = \int yz\,dx = xyz + C$

exact $\therefore \displaystyle\int_{(1,1,2)}^{(3,5,0)} df = f = (x)(y)(z)$

$f(3,5,0) - f(1,1,2) = \boxed{-2}$

16.4 Green's Theorem

Circulation $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$

Flux $= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\,dy$

Ex: 6 Circulation + flux

$\vec{F} = (x^2 + 4y)i + (x + y^2)j$ bounded $0 \le x \le 1$, $0 \le y \le 1$

$\frac{\partial N}{\partial x} = 1 \qquad \frac{\partial M}{\partial y} = 4 \qquad \int_0^1 \int_0^1 (1-4)\,dx\,dy = \iint -3\,dx\,dy$

$= \boxed{-3}$

$\frac{\partial M}{\partial x} = 2x \qquad \frac{\partial N}{\partial y} = 2y \qquad \int_0^1 \int_0^1 (2x+2y)\,dx\,dy$

$\int_0^1 (1+2y)\,dy = y + y^2 \Big]_0^1 = \boxed{2}$

(22) $\oint 3y\,dx + 2x\,dy \qquad 0 \le x \le \pi, \quad 0 \le y \le \sin x$

$$\int_0^{\pi}\int_0^{\sin x} (2-3)\,dy\,dx \qquad \frac{\partial M}{\partial y} = 3 \qquad \frac{\partial N}{\partial x} = 2$$

$$\int_0^{\pi} \Big[-y\Big]_0^{\sin x} \rightarrow \int_0^{\pi} -\sin x\,dx \rightarrow \Big[\cos x\Big]_0^{\pi} = -1-1 = \boxed{-2}$$

16.5 Surfaces & Areas

Use paramet to define a surface in space. Then use it to find Surface Area as double Integral

Cylindrical coordinates (r, θ, z)

Converting = $x = r\cos\theta$, $y = r\sin\theta$
$z = z$, $r^2 = x^2 + y^2$, $\tan\theta = y/x$

Ex: Find paramet. of $z^2 = x^2 + y^2$, $-4 \le z \le 4$
Using cylindrical coordinates
$x = r\cos\theta$, $y = r\sin\theta$, $z^2 = r^2 \to z = r$

Let $u = r$ & $v = \theta$. Then

$\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + rK$, $-4 \le r \le 4$, $0 \le \theta \le 2\pi$

Spherical coordinates (ρ, φ, θ)
rho = ρ = distance from P to origin
phi = φ = angle OP makes with positive z-axis

Converting: $r = \rho\sin\varphi$, $x = r\cos\theta = \rho\sin(\varphi)\cos\theta$
$z = \rho\cos\varphi$, $y = r\sin\theta = \rho\sin\varphi\sin\theta$
$\rho =$

Ex: find paramet of sphere $x^2+y^2+z^2=9$.

$\rho = r = 3$, $x = 3\sin\varphi\cos\theta$, $y = 3\sin\varphi\sin\theta$

$z = 3\cos\varphi$ $\quad 0 \le \varphi \le \pi$ & $0 \le \theta \le 2\pi$

Let $u = \varphi$ & $v = \theta$

$r(\varphi, \theta) = 3\sin\varphi\cos\theta\, i + 3\sin\varphi\sin\theta\, j + 3\cos\varphi\, K$

Area of smooth Surface:

$$A = \int_c^d \int_a^b |\vec{r}_u \times \vec{r}_v| \, du\, dv \qquad \vec{r}_u = \frac{\partial r}{\partial u}, \quad \vec{r}_v = \frac{\partial r}{\partial v}$$

Ex: 20. Find Area of portion of the cone $z = \frac{\sqrt{x^2+y^2}}{3}$ between $z=1$ & $z = 4/3$

Let $x = r\cos\theta$ & $y = r\sin\theta \rightarrow z = \frac{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}{3}$

$z = \frac{r}{3}$, $\quad 1 \le z \le 4/3 \rightarrow 1 \le r/3 \le 4/3 \rightarrow 3 \le r \le 4$

$0 \le \theta \le 2\pi$

$\therefore r(\theta, r) = \langle r\cos\theta, r\sin\theta, r/3 \rangle$

$\vec{r}_r \langle \cos\theta, \sin\theta, 1/3 \rangle$, $\vec{r}_\theta \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 1/3 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left\langle -\frac{1}{3} r\cos\theta, -\frac{1}{3} r\cos^2\theta, r\cos^2\theta + r\sin^2\theta \right\rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \left\langle -\frac{1}{3} r\cos\theta, -\frac{1}{3} r\sin\theta, r \right\rangle$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{\frac{1}{9}r^2 + r^2} = \sqrt{\frac{10}{9}r^2} = \frac{r\sqrt{10}}{3}$$

$$A = \int_3^4 \int_0^{2\pi} \frac{r\sqrt{10}}{3} \, d\theta \, dr = \int_3^4 \left[\frac{\theta r\sqrt{10}}{3}\right]_0^{2\pi} = \int_3^4 \frac{2\pi r\sqrt{10}}{3} \, dr$$

$$= \left[\frac{2\pi\sqrt{10}}{3} \cdot \frac{1}{2} r^2\right]_3^4 = \boxed{\frac{7\pi\sqrt{10}}{3} = A}$$

Notes

$$\text{Surface Area} \iint_R \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} \, dA$$

$\vec{p} =$ i, j or k normal to region R

Ex: 38 Area $x^2+y^2-z=0 \quad 2 \le z \le 6$

$f(x,y,z) = x^2+y^2=z$ Let R be region on XY-plane so $\vec{p}=k$

$\nabla f = 2xi + 2yj - k$

$|\nabla f| = \sqrt{4x^2+4y^2+1}$

$\nabla f \cdot \vec{p} = 2x*0 + 2y*0 - 1*1 = -1$

$|\nabla f \cdot \vec{p}| = \sqrt{(-1)^2} = 1$

$$S = \iint \frac{\sqrt{4x^2+4y^2+1}}{1}\,dx\,dy$$

Convert to polar: $2 \le z \le 6 \rightarrow \sqrt{2} \le r \le \sqrt{6}$

since $z=r^2$

$$S = \int_{\sqrt{2}}^{\sqrt{6}}\int_0^{2\pi} \sqrt{4r^2+1}\; r\,d\theta\,dr = \int_{\sqrt{2}}^{\sqrt{6}} 2\pi r\sqrt{4r^2+1}\,dr$$

$u = 4r^2+1$

$du = 8r\,dr$

$$S = \int_9^{25} \frac{\pi}{4}\sqrt{u}\,du = \frac{\pi}{4} * \frac{2}{3}\left[25^{3/2} - 9^{3/2}\right] = \frac{\pi}{6}(125-27)$$

$$= \boxed{\frac{49\pi}{3}}$$

Notes: $A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$

if $Z = f(X, Y)$, coefficient of z is $\underline{1}$

16.6 Surface Integrals

(2) $G(x,y,z) = z$ over $y^2 + z^2 = 4$,
$z \geq 0$, $1 \leq x \leq 4$

$r(x,y) = xi + yj + \sqrt{4-y^2}\,k$

$\vec{r}_x = \langle 1, 0, 0 \rangle$, $\vec{r}_y = \langle 0, 1, \frac{-y}{\sqrt{4-y^2}} \rangle$

$\vec{r}_x \times \vec{r}_y = \langle 0, \frac{y}{\sqrt{4-y^2}}, 1 \rangle$

$|\vec{r}_x \times \vec{r}_y| = \sqrt{\frac{y^2}{4-y^2} + 1} = \sqrt{\frac{y^2 + 4 - y^2}{4-y^2}} = \frac{2}{\sqrt{4-y^2}}$

$$\iint_S G\,|\vec{r}_x \times \vec{r}_y| = \int_{-2}^{2}\int_{1}^{4} \left(\sqrt{4-y^2} * \frac{2}{\cancel{\sqrt{4-y^2}}} \right) dx\,dy$$

$$\int_{-2}^{2}\int_{1}^{4} 2\,dx\,dy = \int_{-2}^{2} 6\,dy = \boxed{24}$$

Now solve implicitly...

$f(x,y,z) = y^2 + z^2 = 4$

$\vec{\nabla f}\,\langle 0, 2y, 2z \rangle \rightarrow |\vec{\nabla f}| = \sqrt{4y^2 + 4z^2} = 2\sqrt{y^2+z^2}$

$|\nabla f \cdot k| = |2z| = 2z$

$$\iint_S G\,d\sigma = \int_{-2}^{2}\int_{1}^{4}\left(z \times \frac{2\sqrt{y^2+z^2}}{2z}\right)dx\,dy$$

$$\int_{-2}^{2}\int_{1}^{4}\sqrt{y^2+z^2}\,dx\,dy \longrightarrow \iint \sqrt{y^2+(4-y^2)^2}\,dx\,dy$$

$$\iint_{1}^{4} 2\,dx\,dy = \boxed{24}$$

Again now explicitly ...

$$z = f(\cancel{x}, y) = \sqrt{4-y^2} \quad (\text{because } z=0)$$

$$f_x = 0, \quad f_y = \frac{-2y}{2\sqrt{4-y^2}} = \frac{-y}{\sqrt{4-y^2}}$$

$$\int_{-2}^{2}\int_{1}^{4}\left(\sqrt{4-y^2} * \sqrt{f_x^2+f_y^2+1}\right)dx\,dy$$

$$\int_{-2}^{2}\int_{1}^{4}\left(\sqrt{4-y^2} * \sqrt{0^2+\left(\frac{-y}{\sqrt{4-y^2}}\right)^2+1^2}\right)dx\,dy$$

$$\int_{-2}^{2}\int_{1}^{4} 2\,dx\,dy = \boxed{24}$$

16.7 Stokes Theorem

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$\vec{F} = Mi + Nj + PK$$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Ex: 2 Curl F where $F = 2yi + 3xj - z^2k$

$$\text{curl } F \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix} = \langle 0, 0, 1 \rangle$$

Stokes Theorem:

$$\oint \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \vec{n} \, d\sigma$$

(20) $\vec{F} = 2xy\,i + x\,j + (y+z)\,k$

Surface $z = 4 - x^2 - y^2$, $z \geq 0$

$r(t) = 2\cos t\, i + 2\sin t\, j \quad 0 \leq t \leq 2\pi$

$\vec{F} = \langle 8\cos t \sin t,\ 2\cos t,\ 2\sin t \rangle$

$= \langle -2\sin(t),\ 2\cos(t),\ 0 \rangle$