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COT 3100 Introduction to Discrete Mathematics Exam 3

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Exam Rules

- Use the back of the exam paper as necessary. But indicate clearly which problems that the answers on the back correspond to.
- Make sure that your writing is legible; otherwise your grades may be adversely affected.
- Close book, notes and HW.
- All electronics must be turned off.
- Show all work to get partial credits except yes/no problems.

Problem	Points	Score
1	16	16
2	10	3
3	10	10
4	12	9
5	10	6
6	12	10
Total	70	54

Problem 1 [16 points]: Prove the statement by mathematical induction.

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2, \text{for all integers } n \geq 0$$

Step 1: Let $P(n): \sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$

Show $P(0)$ to be true

LHS: $\sum_{i=1}^{1} i \cdot 2^i = 1 \cdot 2^1 = \boxed{2}$ RHS: $0 \cdot 2^{0+2} + 2 = \boxed{2}$ LHS = RHS ✓

Step 2: Induction. If $P(K)$ for all $K \geq 0$ is true, then $P(K+1)$ is true

Suppose $P(K)$: $\sum_{i=1}^{K+1} i \cdot 2^i = K \cdot 2^{K+2} + 2$

Show $P(K+1)$ is true: $\sum_{i=1}^{K+2} i \cdot 2^i = \underline{(K+1) \cdot 2^{K+3} + 2}$ RHS

$\boxed{\sum_{i=1}^{K+1} i \cdot 2^i} + (K+2 \cdot 2^{K+2}) = K \cdot 2^{K+2} + 2 + K+2 \cdot 2^{K+2}$

sub in $P(K)$

$K \cdot 2^{K+2} + (K+2) \cdot 2^{K+2} + 2 = 2^{K+2}(K + K + 2) + 2$

$2^{K+2}(2K+2) + 2 = 2^{K+2} \cdot 2(K+1) + 2 = 2^{K+3}(K+1) + 2$

$\underline{(K+1) \cdot 2^{K+3} + 2} = RHS$

Problem 2 [10 points]: Solving Recurrence Relations by Iteration

User iteration to guess an explicit formula for the sequence below defined recursively. Use the sequence formulas to simplify your answer whenever possible.

$$h_k = 4h_{k-1} + 5 \text{ , for all integers k} \geq 1$$
$$h_0 = 2$$

$h_1 = 4h_0 + 5 = 4(2) + 5 = 8 + 5 = 2^3 + 5 = 4(2^1) + 5$

$h_2 = 4h_1 + 5 = 4(4(2) + 5) + 5 = 4 \cdot 4 \cdot 2 + 5 + 5 = 4^2 \cdot 2 + 2(5)$

$h_3 = 4h_2 + 5 = 4(4(4(2)+5)+5) = 4 \cdot 4 \cdot 4 \cdot 2 + 5 + 5 + 5 = 4^3 \cdot 2 + 3(5)$

$h_n = 4^n \cdot 2 + n(5)$

$$2\sum_{n=1}^{m} 4^n \quad + 5\sum_{n=1}^{n} n = 2\left(\frac{4^{m+1}-1}{4-1}\right) + 5\left(\frac{m(m+1)}{2}\right)$$

$$2\left(\frac{4^{m+1}-1}{3}\right) + 5\left(\frac{m(m+1)}{2}\right)$$

Problem 3 [10 points]: Second-Order Linear Homogeneous Recurrence Relations with Constant Coefficients

Suppose a sequence satisfies the below given recurrence relation and initial conditions. Find an explicit formula for the sequence.

$a_k = 7a_{k-1} - 10a_{k-2}$, for all integers k ≥ 2

$a_0 = 2,\ a_1 = 2$

Roots: $t=2$, $t=5$ (r, s)

$t^2 = 7t - 10 \Rightarrow t^2 - 7t + 10 \Rightarrow (t-2)(t-5)$

two distinct roots r & S

$a_n = C(2)^n + D(5)^n \Rightarrow a_0 = C + D \Rightarrow 2 = C + D$ ①

$a_1 = C(2)^1 + D(5)^1 \Rightarrow 2 = 2C + 5D$ ②

multiply ① by -2 : $(-2)(2) = (C+D)(-2)$

① $-4 = -2C - 2D$

+ ② $2 = 2C + 5D$

$-2 = 3D \Rightarrow \underline{D = \frac{-2}{3}}$

① $2 = C + D \Rightarrow 2 = C + \left(\frac{-2}{3}\right)$

$C = 2 + \frac{2}{3} \Rightarrow \underline{C = \frac{8}{3}}$

$a_n = C2^n + D5^n$

$a_n = \frac{8}{3}(2)^n - \frac{2}{3}(5)^n$

Problem 4 [12 points]: Loop Invariant

Use the loop invariant theorem to prove the correctness of the loop with respect to the pre- and post-conditions.

[pre-condition: $largest = A[1]$ and $i = 1$]

while $(i \neq m)$

1. $i := i + 1$
2. **if** $A[i] > largest$ **then** $largest := A[i]$

end while

[post condition: $largest = maximum\ value\ of\ A[1], A[2], \ldots, A[m]$]

Loop invariant: $I(n)$ is "$largest = maximum\ value\ of\ A[1], A[2], \ldots, A[n+1]$ and $i = n + 1$."

1. Basic Property: $I(0)$: largest = max value of $A[1], A[2] \ldots, A[0+1] \wedge i = 0+1$
   $I(0)$: largest = max value of $A[1]$ and $i = 1$ ✓
   Satisfies pre-condition.

2 Inductive property: for all integers $k \geq 0$, if guard G & loop invariant $I(k)$ are both true before loop iteration, then $I(k+1)$ is true after loop iteration

Before loop $I(k)$: largest = $A[k+1]$ and $i = k+1$

after loop iteration

1. $i_{new} = k+1+1 \Rightarrow i_{new} = k+2$ ✓

2. if $A[i_{new}] >$ largest then largest = $A[i_{new}]$

$A[k+2] > A[k+1] \therefore largest_{new} = A[k+2]$

$I(k+1)$ is true after loop iteration

3. Eventual Guard Falsity: After finite loop iterations N for all $k \geq 0$ eventually $i = m$ & guard breaks $I(N)$: largest $A[N+1] \wedge i = N+1$ which means we have $m-1$ iterations ✓

2

3

2

L>

4. Correctness of Post Cond; If N is the least # of iterations after guard breaks and $I(N)$ true

then $i = N+1$ & $largest = A[N+1]$

guard breaks therefore $m = N+1$

2

$largest =$ max value of $A[1], A[2], \ldots A[m]$

which satisfies post condition

Problem 5 [10 points]: Let n and r be positive integers and suppose $r \leq n$. Then prove

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

LHS: $\dfrac{(n+1)!}{r!\,(n+1-r)!} = \dfrac{(n+1)\,n!}{r!\,(n+1-r)(n+1-r-1)!}$

$\dfrac{(\cancel{n}+1)\,n!}{r!\,(\cancel{n+1})-r\,(n-r)!} = \dfrac{n!}{r!\,(-r)\,(n-r)!} = \dfrac{n!}{r(r-1)!\,(-r)\,(n-r)!}$

$\dfrac{n!}{(r-1)!\,(n-r)!\,(r)(-r)}$

RHS: $\dfrac{n!}{(r-1)!\,(n-r+1)!} + \dfrac{n!}{r!\,(n-r)!}$

Problem 6 [12 points] : Assume that n is a positive integer. For each of the following algorithm segments, how many times will the innermost loop be iterated when the algorithm segment is implemented and run?

1) for m := 1 to n
    for k := 1 to m
        for j := 1 to k
            for i := 1 to j
                [Statements in the body of the inner loop, none containing branching statements that lead outside the loop]
            next i
        next j
    next k
next m

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\sum_{j=1}^{k}\sum_{i=1}^{j} 1 \qquad 1 \le i \le j \le k \le m \le n$$

$$\binom{n-1+4}{4} = \binom{n+3}{4}$$

3

$$\frac{(n+3)!}{4!\,(n+3-4)!} = \frac{(n+3)!}{4!\,(n-1)!} = \frac{(n+3)(n+2)(n+1)(n)\cancel{(n-1)!}}{4!\,\cancel{(n-1)!}} = \frac{n(n+3)(n+2)(n+1)}{4!} = \boxed{\frac{n(n+1)(n+2)(n+3)}{24}}$$

2) for i := 0 to n-1
    for j := i+1 to n-1
        [Statements in the body of the inner loop, none containing branching statements that lead outside the loop]
    next j
next i

Back

3) for i:=1 to n
    for j:=1 to i-1
        for k:=1 to j-1
            [Statements in the body of the inner loop, none containing branching statements that lead outside the loop]
        next k
    next j
next i

Back

4) for i:=1 to n
    for j:=1 to i
        for k:=1 to j
            [Statements in the body of the inner loop, none containing branching statements that lead outside the loop]
        next k
    next j
next i

$$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 \qquad 1 \le k \le j \le i \le n$$

$$\binom{n-1+3}{3} = \binom{n+2}{3}$$

$$\binom{n+2}{3} = \frac{(n+2)!}{3!\,(n+2-3)!} = \frac{(n+2)!}{3!\,(n-1)!}$$

$$\frac{(n+2)(n+1)(n)\cancel{(n-1)!}}{3!\,\cancel{(n-1)!}} = \boxed{\frac{n(n+1)(n+2)}{6}}$$

3

Problem 6

2) $$\sum_{i=0}^{n-1}\sum_{J=i+1}^{n-1} 1 = \sum_{i=0}^{n-1}\left(\sum_{J=1}^{n} 1\right) = \sum_{i=0}^{n-1}(n) = \sum_{i=1}^{n-2} n$$

1

$J = 0+1$ when $i = n-1$

$J = 1$ $J-1 = n-1$

$J = n$

$$\sum_{i=1}^{n-2} n = \frac{(n-2)(n-2+1)}{2} = \boxed{\frac{(n-1)(n-2)}{2}}$$

3) $$\sum_{i=1}^{n}\sum_{J=1}^{i-1}\sum_{k=1}^{J-1} 1 = \sum_{i=1}^{n}\sum_{J=1}^{i-1}(J-1) = \sum_{i=1}^{n}\left(\sum_{J=1}^{i-1} J - \sum_{J=1}^{i-1} 1\right)$$

$$\sum_{i=1}^{n}\left(\frac{(i-1)(i)}{2} - i-1\right) = \sum_{i=1}^{n}\left(\frac{i(i-1)-2(i-1)}{2}\right) = \sum_{i=1}^{n}\left(\frac{(i-1)(i-2)}{2}\right)$$

$$\sum_{i=1}^{n} \frac{i^2-3i+2}{2} = \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} 3i + \sum_{i=1}^{n} 2$$

$$\frac{n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2} + n = \frac{n(n+1)(2n+1) - 9n(n+1) + 6n}{6}$$

$$\frac{n(n+1)[2n+1-9]+6n}{6} = \boxed{\frac{n(n+1)[2n-8]+6n}{6}}$$

$\downarrow$

$$\frac{2n(n+1)(n-4)+6n}{6}$$

$$\frac{n(n+1)(n-4)}{3} + n$$

$$\boxed{\frac{n(n+1)(n-4)+3n}{3}}$$