tSVD

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Outline

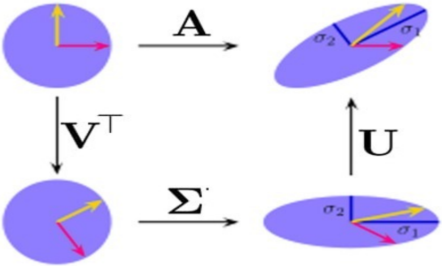
Introduction

SVD : Singular Value Decomposition

SVD is a fundamental matrix factorization technique in linear algebra, applicable to any $m \times n$ matrix. It decomposes a matrix $A_{m \times n}$ into three matrices: $A = U\Sigma V^T$, where:

- U is an $m \times m$ orthogonal matrix.
- Σ is $m \times n$ diagonal matrix whose diagonal elements are non-negative singular values(decreasing order) of A
- V^T (the conjugate transpose of V) is an $n \times n$ orthogonal matrix.

Geometric Interpretation



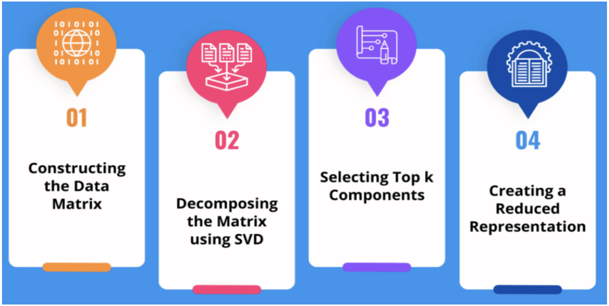
$$\mathbf{A} = \mathbf{U\Sigma V}^\top$$

tSVD : Truncated Singular Value Decomposition

tSVD is a variant of SVD where the decomposition is truncated to retain only the first k largest singular values and their corresponding singular vectors. For a matrix A, tSVD approximates A as $A \approx U_k \Sigma_k V_k^T$, where:

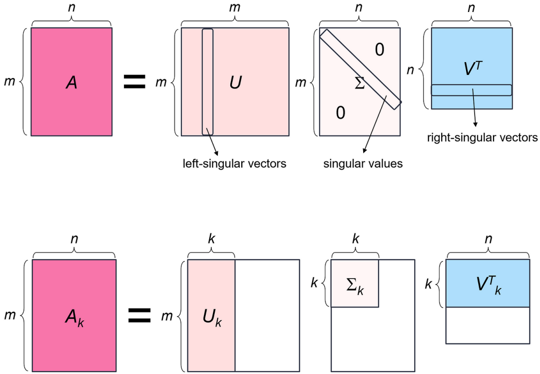
- U_k and V_k^T are truncated versions of U and V^T containing only the first k columns.
- Σ_k is a diagonal matrix containing only the first k largest singular values from Σ.

How tSVD works?



- 2 basic steps
- Data Matrix $\Rightarrow$ Decomposition $\Rightarrow$ Truncation

SVD vs tSVD



- Second figure (tSVD) : $k < m$ and $k < n$ [1]

[1] reference : https://en.wikipedia.org/wiki/SingularValueDecomposition

Example

Under determined example

$A = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}$. Lets decompose A using SVD.

SVD example

$A = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}$. Lets decompose A using SVD.

- $A = U\Sigma V^T$

SVD example

$A = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}$. Lets decompose A using SVD. $A = U\Sigma V^T$

- $U = \begin{pmatrix} 0.50453315 & 0.76077568 & -0.81649658 \\ 0.5745157 & 0.05714052 & -0.82 \\ 0.64449826 & -0.64649464 & 0.40824829 \end{pmatrix}$
- $\Sigma = \begin{pmatrix} 25.4624074 & 0 & 0 & 0 \\ 0 & 1.29066168 & 0 & 0 \\ 0 & 0 & 2.32149215 \times 10^{-15} & 0 \end{pmatrix}$
- $V^T = \begin{pmatrix} 0.14087668 & 0.34394629 & 0.54701591 & 0.75008553 \\ -0.82471435 & -0.42626394 & -0.02781353 & 0.37063688 \\ -0.54725121 & 0.71273523 & 0.21628317 & -0.38176719 \\ -0.02271807 & 0.43818773 & -0.80822125 & 0.39275159 \end{pmatrix}$

tSVD example

$A = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}$. Lets approximate A using tSVD.

tSVD example

$A = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}$. Lets decompose A using tSVD (up to rank 2).

- $A \approx U_k \Sigma_k V_k^T$

tSVD example

$$A = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}.$$ Lets decompose A using tSVD. $A \approx U_k \Sigma_k V_k^T$

- $U_k = \begin{pmatrix} 0.50453315 & 0.76077568 \\ 0.5745157 & 0.05714052 \\ 0.64449826 & -0.64649464 \end{pmatrix}$
- $\Sigma_k = \begin{pmatrix} 25.46240744 & 0 \\ 0 & 1.29066168 \end{pmatrix}$
- $V_k^T = \begin{pmatrix} 0.14087668 & 0.34394629 & 0.54701591 & 0.75008553 \\ -0.82471435 & -0.42626394 & -0.02781353 & 0.37063688 \end{pmatrix}$

Singular values comparison

- Singular values approximated to 2 decimals

SVD	tSVD
$\begin{pmatrix} 25.46 & 0 & 0 & 0 \\ 0 & 1.29 & 0 & 0 \\ 0 & 0 & 2.32 \times 10^{-15} & 0 \end{pmatrix}$	$\begin{pmatrix} 25.46 & 0 \\ 0 & 1.29 \end{pmatrix}$

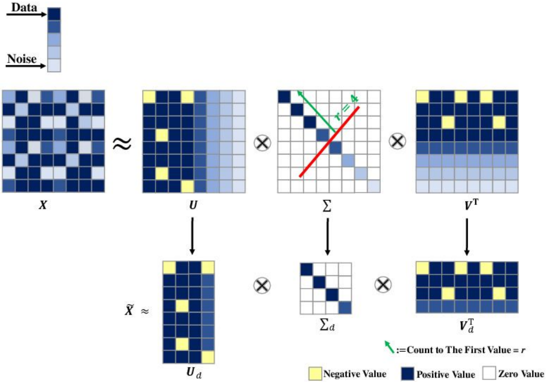
- Truncation up to Rank: $k = 2$ in SVD to get tSVD.[2]

[2] remove the minimum Singular value 2.32×10^{-15}

Applications of tSVD

Applications

1. **Dimensionality Reduction**
2. **Handling ill-conditioned matrices**
3. **Noise Reduction**



tSVD for Highly Oscillatory Integrals

Condition Number

- $Ax = b$, the Condition Number : $cond(A)$
- $$cond(A) = \frac{LargestSingularValue}{SmallestSingularValue} \tag{1}$$
- $cond(A) \gg 1 \Rightarrow$ ill-conditioned [3]
- Otherwise, well-conditioned [4]

[3] small change in input cause to a huge change in output

[4] https://en.wikipedia.org/wiki/ConditionNumber

Highly Oscillatory Integrals

- Li et al. developed a modified Levin method and show

$$I(\omega) = \int_a^b f(x) e^{i\omega g(x)} dx \tag{2}$$

  can be solved using Chebyshev spectral collocation method using tSVD to solve the resulting linear system of coefficients.

- $$\left(D + \frac{b-a}{2} i\omega\Sigma\right) p = \frac{b-a}{2} f \tag{3}$$

  where $D =$ Chebyshev differential matrix and $\Sigma = diag(g'(x))$.

How do we solve the linear system ?

- Depending on the condition number of the linear system use tSVD or linear solver[5] to find p values
- If $\left(D + \frac{b-a}{2} i\omega\Sigma\right)$'s condition number is $\gg 1$, use tSVD
- Start with SVD and then tSVD by truncating Singular values
- Improved Conditioning

[5] LU factorization method, QR *etc*

Only tSVD ?

- **Pseudo-Inverse :** Using the Moore-Penrose Pseudo-Inverse for solving linear systems, especially if the matrix is not full rank.
- **Iterative Methods :** Conjugate Gradient Method is useful for large, sparse, symmetric, and positive-definite matrices.
- **QR Decomposition :** Particularly effective if the matrix is more raw-dominant.
- **LU Decomposition :** Can be useful, especially with partial pivoting to handle ill-conditioned behavior.


Thank you