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VISCOUS RELAXATION AND THE TRANSITION BETWEEN THE KINEMATIC AND NONLINEAR GALACTIC DYNAMOS

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ABSTRACT

This paper primarily treats the different stages in the buildup of small-scale magnetic energy in a turbulent Galactic dynamo and does not directly address the growth of a coherent Galactic-scale magnetic field. If the Galaxy is born with a very weak magnetic field, turbulence in the ISM causes the magnetic energy to grow rapidly on scales much smaller than the scale of the smallest turbulent eddy (Kulsrud & Anderson). In the early stages of growth, ambipolar diffusion (the relative motion between ions and neutrals) damps magnetic energy at the smallest scales and causes the characteristic length scale of the magnetic field to become larger as the total magnetic energy increases (Kulsrud & Anderson). When the total magnetic energy becomes as large as the kinetic energy in the smallest turbulent eddies, $\mathscr{E}_{\rm small}$, the length scale of the magnetic field becomes large enough that viscosity cannot freeze the neutrals in place in the presence of magnetic forces and ion-neutral collisions. When this happens, a new form of damping of magnetic energy termed "viscous relaxation" is shown to occur in which neutrals and ions move together to smooth out field lines. The rate of damping of magnetic energy in this regime is shown to increase over the ambipolar-diffusion damping rate. Viscous relaxation prevents the magnetic energy on scales smaller than the smallest turbulent eddy from becoming much larger than $\mathscr{E}_{\rm small}$. Numerically, $\mathscr{E}_{\rm small} \sim \mathscr{E}_{\rm kin} R^{-1/2}$, where $\mathscr{E}_{\rm kin}$ is the total turbulent kinetic energy and R is the ordinary Reynolds number of the interstellar turbulence. After the magnetic energy has saturated on scales smaller than the smallest turbulent eddy, the magnetic energy continues to grow on the scales of the turbulent eddies where viscous relaxation is ineffective. When the magnetic energy on the scale of the smallest turbulent eddy becomes comparable with $\mathscr{E}_{\rm small}$, strong MHD turbulence develops and the transition from the kinematic dynamo to the nonlinear dynamo is complete.

Subject headings: galaxies: magnetic fields — ISM: magnetic fields — MHD — plasmas — turbulence

1. INTRODUCTION

Observations show that the Galaxy is filled with a magnetic field that is a few μG in strength. This field is observed to have a component that is coherent over Galactic length scales, and also a component of similar strength that fluctuates on much smaller scales (Zweibel & Heiles 1997). The origin of the Galactic field is not understood, but a number of theories have been proposed. These theories can be loosely divided into two categories: primordial and Galactic. Primordial theories (Kulsrud 1986) assume that the Galaxy is born with a large-scale μG field. These theories include the recent protogalactic dynamo theory of Kulsrud et al. (1997a, 1997b). Galactic theories, including versions of mean-field dynamo theory (Ruzmaikin, Shukurov, & Sokoloff 1988; Beck et al. 1996), assume instead that the Galaxy is born with only a minuscule seed field and argue that the initial field is then amplified during the Galaxy's lifetime in such a way that the field becomes coherent on Galactic length scales. A crucial role is played in Galactic theories by interstellar turbulence, which is generated by supernova explosions.

This paper treats issues relevant to Galactic theories and the evolution of an initially minuscule Galactic field. Despite several unanswered questions concerning this evolution, it is clear that there are at least two stages in the field's development. The first is the "kinematic stage" in which the field is so weak that it cannot affect the turbulent velocities. Interstellar plasma is sufficiently conducting that magnetic field lines are effectively "frozen-in" to the plasma throughout the Galaxy's lifetime. This means that during the kinematic stage, field lines are passively advected by the turbulent flow in the interstellar medium (ISM). The random turbulent motions stretch and tangle field lines and cause rapid exponential growth of the magnetic energy (Batchelor 1950; Kulsrud & Anderson 1992, hereafter KA). When the field becomes sufficiently strong, the "nonlinear stage" is entered in which the field changes the nature of the turbulent velocities (Pouquet, Frisch, & Léorat 1976; Chandran 1997a). The focus of this paper is the transition between the kinematic and nonlinear stages, and in particular upon a new form of damping of magnetic energy that plays a crucial role in this transition.

It will be helpful to explain the new results presented in this paper within the context of a more detailed overall picture of the evolution of an initially minuscule Galactic field. A combination of results from KA, Chandran (1997a), and the present paper suggest that the field evolves through five stages that are described below. The evolution of the magnetic energy per unit mass per unit wavenumber $M(k)$ during these stages is depicted graphically in Figure 1. Before giving the details of the five stages, it will be helpful to describe the model that is used for the early Galactic ISM and to define in one place the many variables that play a role in the field's evolution.

1.1. *Model of the Early Galactic ISM; Definitions of Variables*

It is assumed that during the kinematic stage the interstellar velocity turbulence can be described as homoge-

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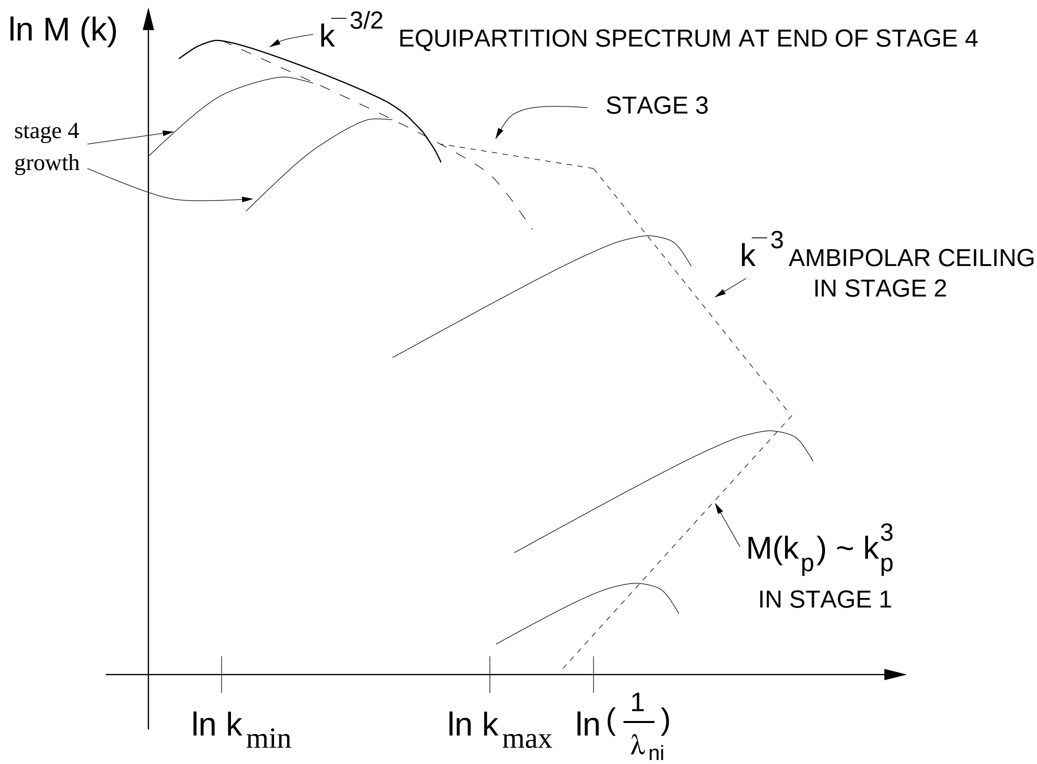

FIG. 1.—Evolution of magnetic power spectrum. Each solid line represents $M(k)$ at some instant in time. The long-dashed line represents the initial value of $E(k)$. The short dashed lines represent the curves along which the peak of the magnetic spectrum evolves during stages 1–3. The variables in this figure are defined in Tables 1 and 2.

neous hydrodynamic turbulence that is excited between some minimum wavenumber $k_{\rm min}$ (at which supernovae stir the ISM) and some maximum wavenumber $k_{\rm max}$ above which viscosity damps out turbulent motions. This means that the kinetic energy per unit mass per unit wavenumber $E(k)$ obeys a Kolmogorov $k^{-5/3}$ power law for much of the wavenumber interval ($k_{\rm min}$, $k_{\rm max}$). The value of $k_{\rm min}$ is chosen to correspond to the 100 pc spacing between randomly moving clouds in the present-day ISM. The wavenumber at the peak of the magnetic spectrum $M(k)$ will be written k_p, and the total magnetic energy per unit mass will be written $\mathscr{E}_{\rm mag}$. The magnetic energy per unit mass is the total magnetic energy in the ISM divided by the total mass of the ISM. The units of $\mathscr{E}_{\rm mag}$ are cm^2 s^{-2}. Some of these definitions and a few others are summarized in the Table 1.

It is assumed that the ISM is homogeneous with parameters listed in Table 2. A high degree of ionization seems likely in the early Galaxy because of supernovae and shocks associated with the collapse of the disk. For this reason, equal ion and neutral densities are assumed. The value of the neutral viscosity given in Table 2 requires some explanation, and this is provided in Appendix A.

1.2. *Stage 1: Kinematic Dynamo*

The first of the five stages of the Galactic dynamo begins just after Galaxy formation when the magnetic field is, by assumption, 10^{-17} G. The equation for the evolution of the field is Ohm's law,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v}_i \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} , \tag{1}$$

where $\boldsymbol{v}_i$ is the ion velocity. The $\nabla \times (\boldsymbol{v}_i \times \boldsymbol{B})$ term represents convection of field lines by the plasma, and the $\eta\nabla^2\boldsymbol{B}$ term represents diffusion of field lines across the plasma. As mentioned above, this diffusion term is extremely small in the interstellar medium, so the field lines are effectively frozen-in to the plasma. Because the field is too weak to affect the

TABLE 1

DEFINITIONS

Variable or Phrase	Definition
$\mathscr{E}_{\rm mag}$	Total magnetic energy per unit mass
$\mathscr{E}_0$	Value of $\mathscr{E}_{\rm mag}$ when the Galaxy is born
$\mathscr{E}_{\rm kin}$	Total kinetic energy per unit mass
$\mathscr{E}_{\rm small}$	Kinetic energy per unit mass in the smallest eddies
$\mathscr{E}_{\rm thermal}$	Thermal energy per unit mass
$M(k)$	Magnetic energy per unit mass per unit k
$E(k)$	Kinetic energy per unit mass per unit k
k_p	Wavenumber at peak of magnetic spectrum
"Inertial range"	The wavenumber interval ($k_{\rm min}$, $k_{\rm max}$)

TABLE 2

ISM PARAMETERS

Parameter	Physical Significance	Value
$k_{\rm min}$ (cm^{-1})	Wavenumber at which supernovae stir the ISM	2×10^{-20}
$k_{\rm max}$ (cm^{-1})	Initial viscous cutoff	10^{-16}
$v_{k_{\rm min}}$ (cm s^{-1})	rms turbulent velocity at $k_{\rm min}$	10^6
$v_{k_{\rm max}}$ (cm s^{-1})	rms turbulent velocity at $k_{\rm max}$	6×10^4
$v_{\rm thermal}$ (cm s^{-1})	Neutral thermal speed	10^6
T (K)	Temperature	10^4
B_0 (G)	Initial magnetic field in Galaxy	10^{-17}
$\mathscr{E}_0$ (cm^2 s^{-2})	Initial magnetic energy per unit mass	3×10^{-11}
$\mathscr{E}_{\rm thermal}$ (cm^2 s^{-2})	Thermal energy per unit mass	10^{12}
n_0 (cm^{-3})	Neutral number density	1
n_i (cm^{-3})	Ion number density	1
λ_{nn} (cm)	Distance a neutral travels between collisions with other neutrals	10^{16}
λ_{in} (cm)	Distance an ion travels between collisions with neutrals	5×10^{14}
λ_{ni} (cm)	Distance a neutral travels between collisions with ions	5×10^{14}
ω_{in} (s^{-1})	Ion-neutral collision rate	2×10^{-9}
ν (cm^2 s^{-1})	Kinematic neutral viscosity	5×10^{20}
η (cm^2 s^{-1})	Resistivity	10^7
R	Reynolds number ($v_{k_{\rm min}}/k_{\rm min}\nu$)	10^5

velocity in this stage, equation (1) is linear in $\boldsymbol{B}$. By making approximations for the statistics of the hydrodynamic velocity turbulence, one can obtain an equation for $M(k)$. KA, for example, used a short correlation time approximation, and Kraichnan & Nagarajan (1967) used the Lagrangian history direct interaction approximation. In both approximations, the equation for $M(k)$ is linear in M in the limit that the magnetic field is very weak. KA's solution for $M(k)$ during this stage can be approximated by the formula

$$M(k) \sim M(k_{\rm max}, t = 0)e^{3\gamma t/4}(k/k_{\rm max})^{3/2} \quad \text{for } k \in (k_{\rm max}, k_p) , \tag{2}$$

where γ is the inverse turnover time of the smallest turbulent eddies (which is roughly 5×10^3 yr in the model ISM of § 1.1), and

$$k_p \sim k_{\rm max}\, e^{\gamma t/2} . \tag{3}$$

At wavenumbers above k_p, the magnetic spectrum decreases exponentially with increasing k. In equations (2) and (3), it is assumed that at $t = 0$ (the moment of Galaxy formation) the magnetic spectrum is peaked at $k_{\rm max}$. This assumption has no basis and is only made so that $M(k)$ can be described by a single formula during stage 1. The actual initial condition for $M(k)$ if the Galaxy is born with a minuscule field is not known. A different initial condition, however, would not dramatically affect the claims made in this paper. For example, $M(k)$ approaches the same functional form almost independently of initial conditions (KA).

Equation (2) differs from KA's formula for $M(k)$ during stage 1 (eq. [3.19] in KA) near the peak of the spectrum, but the two formulas share three important features: (1) they both scale like $k^{3/2}$ throughout most of the spectrum, (2) they both fall off exponentially at high enough k, and (3) the total magnetic energy grows like $e^{2\gamma t}$ in both expressions. Because k_p becomes much larger than $k_{\rm max}$, and because the magnetic energy is dominated by the part of the spectrum near k_p, KA's calculation shows that the magnetic field becomes concentrated on scales much smaller than the scales of the velocity turbulence. The above results also show that as k_p evolves in time, the value of M at the peak of the magnetic spectrum obeys the scaling

$$M(k_p) \sim k_p^3 . \tag{4}$$

This is depicted in Figure 1.

One source of uncertainty in KA's calculation is their assumption that the correlation time of the turbulent velocities is much shorter than the time required for the field to change. As KA were aware, this assumption is not correct. Nevertheless, numerical calculations without the short correlation time approximation (Chandran 1997b) have shown that the total magnetic energy grows roughly half as fast as KA predicted, suggesting that KA's spectral analysis is also qualitatively correct. (To avoid problems of limited spatial resolution in Eulerian codes, these numerical calculations used a Lagrangian model for the turbulent velocities to evolve the field at a single point in a frame moving with the plasma and thus did not provide direct information on the magnetic spectrum.)

1.3. *Stage 2: Ambipolar Saturation of Small-Scale Field*

The value of k_p cannot increase indefinitely because of mechanisms that damp magnetic energy at very small scales. For example, a sinusoidal perturbation in the magnetic field with wavenumber k is damped by resistivity at the rate $k^2\eta$. Because this rate of damping increases with k, while the rate of amplification of small-scale magnetic perturbations by the velocity turbulence is fixed at the rate γ, the magnetic field cannot be amplified effectively at scales smaller than the resistive scale k_R^{-1}, where k_R is defined by the relation $k_R^2\eta = \gamma$. In other words, k_R is an upper bound on k_p.

In the partially ionized interstellar medium there is another damping mechanism, ambipolar diffusion, that can be more effective than resistivity. Ambipolar diffusion is the process in which magnetic forces cause the ions to slip by the neutrals in such a way that the fine-scale structures in the magnetic field lines are smoothed out. This smoothing out of field lines corresponds to damping of small-scale magnetic energy. Ambipolar diffusion occurs even before the magnetic field is able to influence the turbulent velocities at wavenumbers lying within the inertial range of the turbulence ($k_{\rm min}$, $k_{\rm max}$). KA estimated the rate of ambipolar damping at the peak of the magnetic spectrum as

$$\xi_2 \sim k_p^2 \mathscr{E}_{\rm mag}/\omega_{in} . \quad (5)$$

There is an extra factor in KA's expression for ξ_2 that is of order unity for equal ion and neutral densities. Equation (5) reflects the fact that ambipolar damping is more effective when the magnetic structures have a finer scale and when the magnetic field is stronger.

If the Galaxy is born with a sufficiently weak magnetic field, then k_p can increase all the way to k_R before ambipolar diffusion becomes as effective as resistivity in damping the small-scale magnetic field. Given equations (2) and (3), it is shown in this paper that the resistive scale is reached only when the initial magnetic energy is less than

$$\mathscr{E}_0^R = \mathscr{E}_{\rm thermal}\left(\frac{\eta}{\nu}\right)^3 = \mathscr{E}_{\rm thermal} P_{\rm M}^{-3} , \quad (6)$$

where $P_{\rm M} = \nu/\eta$ is the magnetic Prandtl number. Given the parameters in Table 2, $P_{\rm M} \sim 5 \times 10^{13}$ and $\mathscr{E}_0^R$ corresponds to an initial magnetic field between 10^{-27} and 10^{-26} G. The initial magnetic field, however, should be no less than 10^{-18} G because of thermal battery effects in the pre-Galactic plasma (Kulsrud et al. 1997a) and compression of the pre-Galactic plasma during the formation of the Galactic disk. Thus, as argued by KA, the magnetic field in the Galaxy never evolves to the resistive scale because of ambipolar diffusion.

Once the exponential growth of k_p described by equation (3) is halted by ambipolar diffusion, stage 1 ends and stage 2 begins. It is shown in this paper that at the beginning of stage 2,

$$\mathscr{E}_{\rm mag} \sim \mathscr{E}_0^{1/3} \mathscr{E}_{\rm thermal}^{2/3} , \quad (7)$$

and

$$k_p \sim k_{\rm max}\left(\frac{\mathscr{E}_{\rm thermal}}{\mathscr{E}_0}\right)^{1/6} . \quad (8)$$

Two sources of error in these relations should be mentioned. First, they are derived using equations (2) and (3), which are approximations of KA's expression for $M(k)$. Second, KA's expression for $M(k)$ is to some degree inaccurate because of their use of the short correlation time approximation.

KA showed that the behavior of $M(k)$ during stage 2 is described by the equation

$$M(k, t) \sim M(k_{\rm max}, 0) e^{3\gamma t/4} \left(\frac{k}{k_{\rm max}}\right)^{3/2} \quad \text{for } k \in (k_{\rm max}, k_p) , \quad (9)$$

with $M(k)$ dropping off exponentially for $k > k_p$. That is, for $k \in (k_{\rm max}, k_p)$, $M(k)$ grows during stage 2 just as it grew during stage 1. However, in stage 2 k_p decreases in time:

$$k_p \sim e^{-\gamma t/6} . \quad (10)$$

Equations (9) and (10) imply that $\mathscr{E}_{\rm mag} \sim e^{3\gamma t/4} k_p^{5/2} \sim e^{\gamma t/3}$ (KA). Additional insight into the evolution of $M(k)$ can be gained by noting that k_p is always approximately that k for which the rate of ambipolar damping is equal to γ. Above this wavenumber, ambipolar damping dominates and the spectrum decreases with increasing k; below this wavenumber, ambipolar damping is ineffective and the spectrum increases with k. This reasoning helps explain why k_p must decrease during stage 2 if $\mathscr{E}$ is to increase. Furthermore, by setting $\gamma = k_p^2 \mathscr{E}_{\rm mag}/\omega_{in}$ and noting that $\mathscr{E}_{\rm mag} \sim k_p M(k_p)$ (since the magnetic energy is dominated by the wavenumber range in which $M \sim k^{3/2}$), one finds that in stage 2

$$M(k_p) \sim k_p^{-3} \quad (11)$$

as k_p decreases in time. In other words, there is a k^{-3} curve in the $M(k)$ vs. k plane that defines an "ambipolar ceiling" above which $M(k)$ cannot grow (R. Kulsrud 1996, private communication). This curve is depicted in Figure 1.

1.4. *Stage 3: Viscous Relaxation*

During stage 2, the magnetic field is concentrated at scales smaller than the neutral mean free path. At such small scales, the Navier-Stokes equations do not describe the neutrals and viscosity is not an appropriate description of the frictional damping of neutral motions. Instead, small-scale neutral motions are damped by a process known as phase mixing (Stix 1992). It is shown in this paper that phase mixing effectively freezes neutrals in place during the ambipolar damping stage of the Galactic dynamo. Because the neutrals are frozen during stage 2, and because field lines are frozen to the ions, the smoothing out of field lines in stage 2 is dominated by the relative velocity, or "drift," between the ions and neutrals. However, as k_p decreases and the neutrals become freer to move in response to magnetic forces, the neutrals and ions eventually move together and the ion-neutral drift ceases to dominate field-line smoothing. This third stage in the Galactic dynamo is reached when the field becomes concentrated at scales comparable to the neutral mean free path λ_{ni}, so that during stage 3

$$k_p \lesssim 1/\lambda_{ni} . \quad (12)$$

It is shown in this paper that at the beginning of stage 3,

$$\mathscr{E}_{\rm mag} \sim R^{-1/2} \mathscr{E}_{\rm kin} . \quad (13)$$

Given the homogeneous model of the ISM summarized in Table 2, the value of $\mathscr{E}_{\rm mag}$ given in equation (13) is much larger than the value of $\mathscr{E}_{\rm mag}$ at the beginning of stage 2 given in equation (7).

During stage 3, the neutral motions on the scale of the magnetic field are described by the Navier-Stokes equations, and the damping of such neutral motions by viscosity becomes sufficiently weak that the neutral motions on the scale of the magnetic field are greater than the ion-neutral drift velocity. The total ion velocity induced by magnetic forces in stage 3 increases relative to its value during stage 2 because the ions are no longer slowed by friction with stationary neutrals. This increase in the ion velocity increases the rate at which small-scale structures in the magnetic field are smoothed out, leading to more effective damping of small-scale magnetic energy. This faster form of magnetic-energy damping is the main result of this paper and is termed "viscous relaxation."

To clarify terminology, "viscous relaxation" is a coined phrase referring to the reduction of small-scale magnetic energy as field lines straighten when the neutral velocity is greater than the relative velocity between ions and neutrals. On the other hand, "viscosity" refers to the well-known diffusive term in the fluid or plasma momentum equation that arises because individual particles within a fluid or plasma move a finite distance between collisions. The phrase "viscous damping" refers directly to the reduction of fluid motion (not magnetic energy) by viscosity.

It is shown in this paper that the rate of damping of magnetic energy at k_p during stage 3 is

$$\xi_3 = \frac{\mathscr{E}_{\rm mag}}{\nu} . \quad (14)$$

Importantly, ξ_3 is independent of k_p. This introduces a qualitative change in the evolution of $M(k)$. During stage 2, continuous growth in the magnetic energy is made possible by a reduction in the value of k_p. This reduction in k_p during stage 2 keeps the rate of magnetic energy damping below the turbulent stretching rate γ. In contrast, continuous increase in $\mathscr{E}_{\rm mag}$ is not possible during stage 3 because changes in k_p do not affect the rate of magnetic energy damping at k_p.

The consequence of this is that viscous relaxation places a limit on the small-scale magnetic energy. This limit can be estimated by equating ξ_3 with γ and is found to be

$$\mathscr{E}_{\rm mag} = R^{-1/2}\mathscr{E}_{\rm kin} , \quad (15)$$

which is also the value of $\mathscr{E}_{\rm mag}$ at the beginning of stage 3. It is important to note, however, that this is only a limit on the magnetic energy at wavenumbers greater than $k_{\rm max}$, because viscous relaxation does not occur within the inertial range of the Kolmogorov turbulence ($k_{\rm min}$, $k_{\rm max}$), where the dynamics are controlled by turbulent pressure forces and inertia. Thus, $M(k)$ should continue to grow in the inertial range during stage 3 until the value of $M(k)$ at $k_{\rm max}$ becomes as large as $E(k_{\rm max})$, at which point strong MHD turbulence develops and stage 4 is entered. The precise evolution of $M(k)$ during stage 3 is unknown. One possibility, the scenario depicted in Figure 1, is that k_p decreases continuously from $1/\lambda_{ni}$ to $k_{\rm max}$, while the magnetic energy at wavenumbers greater than $k_{\rm max}$ stays roughly constant. Whether this occurs or not, $M(k)$ should be peaked at $k_{\rm max}$ at the end of stage 3, as opposed to a smaller wavenumber, because within the inertial range the smallest eddies are most effective at amplifying magnetic energy (KA), and these eddies are more effective at producing magnetic energy with a characteristic wavenumber $k_{\rm max}$ than magnetic energy with a smaller characteristic wavenumber.

The importance of viscous relaxation lies in the fact that it prevents the magnetic energy from remaining on scales small compared to the velocity turbulence as the magnetic energy grows to equipartition with the kinetic energy. Such a separation of scales is predicted by the theory of KA, which includes ambipolar diffusion but not viscous relaxation. The presence or absence of such a separation of scales must be known in order to determine whether the mean-field dynamo can operate once $\mathscr{E}_{\rm mag} \sim \mathscr{E}_{\rm kin}$ and also whether the inverse cascade of Pouquet et al. (1976) might be related to the emergence of a Galactic-scale field.

1.5. *Stage 4: Nonlinear Dynamo—Approach to Equipartition*

Once the kinetic and magnetic energies become comparable at $k_{\rm max}$, the "nonlinear stage" of the dynamo is entered and the turbulence changes from strong hydrodynamic turbulence to strong MHD turbulence. It may be misleading to refer only to this stage as nonlinear, because the equation for $M(k)$ is already nonlinear during stage 2 (KA). [This nonlinearity arises because the velocity in equation (1) at wavenumber k_p depends upon $\boldsymbol{B}$.] Furthermore, even during stage 3 the field has some effect on the part of the turbulent velocity spectrum near $k_{\rm max}$. Despite the possible confusion, the term "nonlinear dynamo" will be used to describe stage 4 because it is already in use in the literature. What distinguishes this stage from previous stages is that eventually velocities throughout the inertial range of the turbulence ($k_{\rm min}$, $k_{\rm max}$) depend upon $\boldsymbol{B}$. As discussed above, it seems probable that $M(k)$ is peaked near $k_{\rm max}$ at the beginning of stage 4. The evolution of $M(k)$ and $E(k)$ when the magnetic spectrum is initially peaked near $k_{\rm max}$, when $E(k)$ is initially a $k^{-3/2}$ spectrum, and when $M(k_{\rm max})$ is initially comparable to $E(k_{\rm max})$ is treated using statistical closures in Chandran (1997a). (The fact that the initial velocity spectrum is taken as $k^{-3/2}$ instead of $k^{-5/3}$ is related to a limitation of Eulerian closures that is explained in that reference.) Figure 2 contains some numerical results from Chandran (1997a) that demonstrate the approach to equipartition between the magnetic and kinetic energies. Once equipartition is reached, both $M(k)$ and $E(k)$ scale as $k^{-3/2}$ instead of $k^{-5/3}$. This final $k^{-3/2}$ scaling is believed to be real and not merely an artifact of statistical closures (Kraichnan 1965). Because the nonlinear dynamo has been discussed in detail by Pouquet et al. (1976) and others, it will not be discussed further here.

Calculations of Pouquet et al. (1976) and Chandran (1997a) suggest that it takes tens of large-eddy turnover times for equipartition to be reached. For the parameters given in Table 2, the large-eddy turnover time in the ISM is 1.6×10^6 yr. At equipartition, the magnetic field is roughly a few μG strong, but it is coherent only on the length scale corresponding to $k_{\rm min}$, which is roughly 100 pc. Because this length is much smaller than the radius of the Galactic disk, the field at the end of stage 4 is not the field observed today.

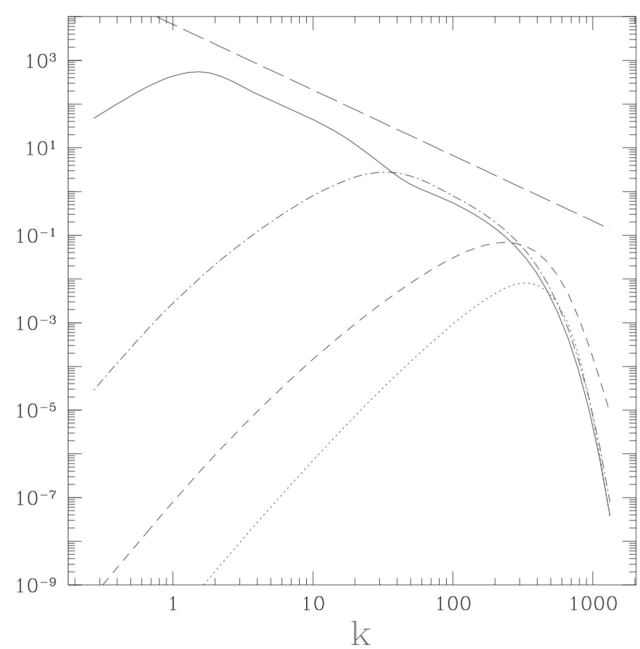

FIG. 2.—Results from statistical closure calculations described in Chandran (1997a) of the growth of the magnetic energy during stage 4. The dotted line at the bottom is $M(k)$ just after the simulation starts. The short-dashed line above it is $M(k)$ after one "large-eddy crossing time," $2\pi/k_{\rm min}\, v_{k_{\rm min}}$, and the dash-dot-dashed line above that is $M(k)$ after 4.4 large eddy crossing times. The solid line is $E(k)$ after 4.4 large-eddy crossing times. The straight long-dashed line at the top is a plot of $k^{-3/2}$. It should be noted that $M(k) = 2\pi k^2 W(k)$, where $W(k)$ is the magnetic spectrum used in Chandran (1997a).

1.6. *Stage 5: Growth of the Galactic-Scale Field*

If the initial seed field is to evolve into the present-day Galactic field, there must be a fifth stage in which the field grows at wavenumbers much smaller than $k_{\rm min}$. It is not clear, however, that such a fifth stage exists. Proponents of mean-field dynamo theory (e.g., Ruzmaikin et al. 1988) argue that the combined action of differential rotation of the Galaxy and helical turbulence lead to such Galactic-scale fields. However, mean-field dynamo theory has only been derived for the case in which the magnetic field does not influence the turbulence ($\mathscr{E}_{\rm mag} \ll \mathscr{E}_{\rm kin}$, where $\mathscr{E}_{\rm kin}$ is the total turbulent kinetic energy). Because the time required for the Galactic-scale field to exponentiate in mean-field dynamo theory is roughly 500 million yr, the results of KA, Pouquet et al. (1976), and Chandran (1997a) show that $\mathscr{E}_{\rm mag}$ becomes comparable to $\mathscr{E}_{\rm kin}$ before the Galactic-scale field can exponentiate once. It is thus not appropriate to treat the case $\mathscr{E}_{\rm mag} \ll \mathscr{E}_{\rm kin}$. Furthermore, numerical calculations by Cattaneo & Hughes (1996) suggest that the backreaction of the magnetic fields on the turbulent velocities disables the mean-field dynamo theory growth mechanism before the galactic-scale mean field can become appreciably strong.

There is another possibility for a fifth stage, one that is in fact consistent with zero mean field. According to the work of Pouquet et al. (1976), an injection of helical kinetic energy into MHD turbulence leads to an inverse cascade of magnetic energy and magnetic helicity to larger scales, even in the absence of any initial magnetic helicity. This conclusion is supported by the statistical mechanics arguments of Frisch et al. (1975). Such an injection of helical kinetic energy is provided by supernovae and the differential rotation of the Galaxy. The amount of time T required in an inverse cascade to build up strong fields (μG fields in the case of the Galaxy) coherent on a length scale L that is bigger than the largest turbulent eddy size is given by equation (6.7) in the paper by Pouquet et al. (1976):

$$T \sim L(\Omega\langle v^2\rangle l_{\rm grad}^{-1} l_{\rm inj}^2)^{-1/3} , \tag{16}$$

where Ω is the angular velocity of Galactic rotation, $\langle v^2\rangle$ is the mean-square turbulent velocity, $l_{\rm grad}$ is the scale height of the Galaxy, and $l_{\rm inj}$ is the length scale corresponding to supernova stirring. Taking $L = 10$ kpc, $\Omega = 10^{-15}$ s^{-1}, $\langle v^2\rangle = 10^{12}$ cm^2 s^{-2}, $l_{\rm grad} =$ 500 pc, and $l_{\rm inj} =$ 100 pc, one finds that $T \sim 2.4 \times 10^9$ yr, a time significantly shorter than the life of the Galaxy. Thus, *if* the isotropic three-dimensional results of Pouquet et al. (1976) carry over to the disklike geometry of the Galaxy, an inverse cascade should be capable of generating a magnetic field coherent on the scale of the Galaxy. However, in the absence of mathematical proofs, detailed explanations of field-line evolution, and high Reynolds numerical simulations to confirm the statistical closure calculations of Pouquet and her collaborators, there remains some doubt as to the prevalence of inverse cascades in MHD turbulence.

The remainder of this paper is dedicated to justifying and elaborating the claims made above concerning viscous relaxation and ambipolar diffusion. In § 2 the ideas underlying Kolmogorov's analysis of turbulence are briefly reviewed, because these ideas play an important role in the subsequent analysis. In § 3 a simple model of the magnetic field is used to estimate the rate of damping of magnetic energy associated with viscous relaxation. In § 4 a smaller scale version of the same magnetic field model is used to estimate the rate of ambipolar damping of magnetic energy. In § 5, approximate numerical values are given for k_p, $\boldsymbol{B}$, and the t (the time elapsed since Galaxy formation) at the beginning of each of the five stages. The main results are summarized in § 6.

2. APPROXIMATE RELATIONS IN KOLMOGOROV TURBULENCE

In Kolmogorov's picture (Kolmogorov 1941), the fluid is stirred on some large scale $k_{\rm min}^{-1}$ (in the ISM the stirring is provided by supernovae), and energy flows from this largest scale down through smaller and smaller scales until eventually viscosity damps turbulent eddies faster than the energy cascade replenishes them. A central element of Kolmogorov's analysis is that modes exchange energy only with other modes of similar scale. For example, large eddies do not effectively amplify or distort much smaller eddies; they merely transport them bodily from one location to another. Because of this, the rate of energy flow from one logarithmic band in k space to the next becomes independent of k, at least near the middle of the power spectrum.

If the energy per unit mass of the turbulence is written

$$\mathscr{E}_{\rm kin} = \int dkE(k) = \int d(\ln k)kE(k) , \tag{17}$$

then the mean square velocity at wavenumber k satisfies

$$\frac{v_k^2}{2} \sim kE(k) . \tag{18}$$

Since interactions between eddies are localized in k space, the typical rate at which an eddy of wavenumber k passes its energy on to the next smallest eddy is $kv_k \sim k^{3/2}E^{1/2}$. The time $(kv_k)^{-1}$ is referred to as the eddy turnover time. If the energy in a logarithmic band flows to the next logarithmic band in roughly one eddy turnover time, the rate of energy flow per unit mass ϵ can be written

$$\epsilon \sim [kE(k)] \cdot [k^{3/2}E(k)^{1/2}] = k^{5/2}E(k)^{3/2} . \tag{19}$$

Taking ϵ to be independent of k,

$$E(k) \sim k^{-5/3} , \tag{20}$$

and

$$v_k \sim k^{-1/3} . \tag{21}$$

This scaling only applies to the range of wavenumbers for which viscosity and the large-scale stirring of the system can be ignored. This region of k space is called the inertial range, and as described below, this range is broad when the Reynolds number $(v_{k_{\rm min}}/\nu k_{\rm min})$ is large.

At small enough scales, viscosity will cause the turbulent power spectrum to fall rapidly to zero with increasing k. The wavenumber corresponding to the transition from the inertial range to the dissipation range can be approximated by assuming that the inertial range scaling of $k^{-5/3}$ extends out to infinite and then calculating the rate at which energy is dissipated by viscosity at each scale. At some scale, this rate of dissipation will equal the rate, ϵ, at which energy cascades down the turbulent spectrum. The wavenumber corresponding to this scale, $k_{\rm max}$, will mark the transition to the dissipation range. The rate at which viscosity drains energy per unit mass from a logarithmic band at wavenumber k is $\nu k^2[kE(k)]$. The transition wavenumber, $k_{\max}$, therefore satisfies

$$k_{\max}^2 \nu[k_{\max} E(k_{\max})] \sim k_{\min} v_{k_{\min}}[k_{\min} E(k_{\min})] . \quad (22)$$

Because $\epsilon \sim k v_k[kE(k)]$ is independent of wavenumber, equation (22) can be rewritten as

$$k_{\max}^2 \nu \sim k_{\max} v_{k_{\max}} = \gamma , \quad (23)$$

where $\gamma = k_{\max} v_{k_{\max}}$ is the rate of stretching of field lines by the smallest turbulent eddies. Equation (22) can also be rewritten as

$$\frac{k_{\max}}{k_{\min}} \sim \left(\frac{v_{k_{\min}}}{k_{\min} \nu}\right)^{3/4} = R^{3/4} . \quad (24)$$

Since $v_k \sim k^{-1/3}$, equation (24) implies that

$$v_{k_{\max}} \sim v_{k_{\min}} R^{-1/4} . \quad (25)$$

If one assumes that $E(k)$ is negligible outside the inertial range, then equation (17) implies that

$$\mathscr{E}_{\rm kin} \sim v_{k_{\min}}^2 . \quad (26)$$

The energy per unit mass in the smallest turbulent eddies $\mathscr{E}_{\rm small} \sim v_{k_{\max}}^2$ can thus be written

$$\mathscr{E}_{\rm small} \sim v_{k_{\max}}^2 \sim \mathscr{E}_{\rm kin} R^{-1/2} . \quad (27)$$

Another quantity of interest is $k^* = 1/\lambda_{ni}$. As discussed in Appendix A, the viscosity is roughly $v_{\rm thermal} \lambda_{ni}$. One can thus infer that

$$\frac{k^*}{k_{\min}} \sim \frac{v_{\rm thermal}}{k_{\min} \nu} = R \frac{v_{\rm thermal}}{v_{k_{\min}}} = \frac{R}{N_{\rm M}} , \quad (28)$$

where $N_{\rm M}$ is the turbulent Mach number. Equations (24) and (28) imply that $k^* \sim R^{1/4} k_{\max} N_{\rm M}^{-1}$. In the interstellar medium, $N_{\rm M}$ should be roughly 1 since interstellar turbulence is driven by shocks from supernovae explosions, and the thermal and kinetic energies are comparable in post-shock regions.

3. VISCOUS RELAXATION: $k_{\max} \ll k_p \ll 1/\lambda_{ni}$

In this section, viscous relaxation is explained with the use of a simplistic but concrete model, and a rate of damping of magnetic energy is obtained for $k_{\max} \ll k_p \ll 1/\lambda_{ni}$. Instead of a turbulent magnetic field with a broad spectrum in k space, let us assume that at some particular instant in time the field lines are all sine waves with wavenumber k_p. In particular, let us take $B_z = 0$ and assume that the equation for any field line can be written

$$y(x) = \frac{1}{k_p} \sin k_p x + C , \quad (29)$$

where C is just a constant that labels a field line according to its placement along the y-axis. The amplitude of each sine wave (one-half the displacement in y between the wave's crest and trough) is $1/k_p$, which is comparable to the wavelength of the wave $2\pi/k_p$. It would only be justified to choose the amplitude to be much less than the wavelength if the small-scale magnetic field (at wavenumber k_p) were a fluctuation about some much stronger large-scale magnetic field. Equation (29) implies that

$$\frac{B_y}{B_x} = \frac{dy}{dx} = \cos k_p x . \quad (30)$$

If we assume that $B_x^2 + B_y^2 = B_0^2$, where B_0 is a constant, then equation (30) implies that

$$B_x = B_0 \frac{1}{(1 + \cos^2 k_p x)^{1/2}} , \quad (31)$$

$$B_y = B_0 \frac{\cos k_p x}{(1 + \cos^2 k_p x)^{1/2}} . \quad (32)$$

The Lorentz force $\boldsymbol{J} \times \boldsymbol{B}$ can be written $\nabla(-B^2/8\pi) + (1/4\pi)\boldsymbol{B} \cdot \nabla \boldsymbol{B}$, where the $\nabla(-B^2/8\pi)$ term is the force due to magnetic pressure, and the $(1/4\pi)\boldsymbol{B} \cdot \nabla \boldsymbol{B}$ term is the force due to the tension in magnetic field lines. In the case of the constant strength magnetic field given above, it is only the tension in the magnetic field lines that contributes to the force. This tension force can be written

$$\boldsymbol{B} \cdot \nabla \boldsymbol{B} = \frac{k_p B_0^2}{(1 + \cos^2 kx)^2} \left[\left(\frac{1}{2}\right) \sin 2kx \hat{x} - \sin kx \hat{y}\right] . \quad (33)$$

The y component of this force acts to straighten out the field line.

When the length scale is larger than the neutral mean free path ($k_p < 1/\lambda_{ni}$), the equations describing the ions and neutrals are

$$\rho_n\left(\frac{\partial \boldsymbol{v}_n}{\partial t} + \boldsymbol{v}_n \cdot \nabla \boldsymbol{v}_n\right) = -\nabla p_n + \rho_n \nu \nabla^2 \boldsymbol{v}_n - \rho_n \omega_{ni}(\boldsymbol{v}_n - \boldsymbol{v}_i) , \quad (34)$$

$$\rho_i\left(\frac{\partial \boldsymbol{v}_i}{\partial t} + \boldsymbol{v}_i \cdot \nabla \boldsymbol{v}_i\right) = -\nabla\left(p_i + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi} \boldsymbol{B} \cdot \nabla \boldsymbol{B} - \rho_i \omega_{in}(\boldsymbol{v}_i - \boldsymbol{v}_n) . \quad (35)$$

Here, $\boldsymbol{v}_n$ is the neutral velocity, $\boldsymbol{v}_i$ is the ion velocity, ρ_n is the neutral mass density, ρ_i is the ion mass density, ν is the neutral viscosity, ω_{in} is the ion-neutral collision rate (the rate at which a single ion collides with neutrals), ω_{ni} is the neutral-ion collision rate, p_n is the neutral pressure, and p_i is the ion pressure. These equations are approximate versions of transport equations derived by Braginskii (1965). For the sake of simplicity, it is assumed that the shear in the large-scale turbulent velocities is negligible, and the reference frame is chosen so that the large-scale velocity is zero. Ion viscosity is negligible for the parameters given in Table 2.

If the wavelength k_p were sufficiently small and the magnetic field sufficiently strong, viscous forces would be very weak and the evolution of the plasma and neutrals in the presence of the initial magnetic field given in equations (31) and (32) would be very complicated, even turbulent. However, because $k_p \gg k_{\max}$, and because we are considering stage 3 of the Galactic dynamo during which $\mathscr{E}_{\rm mag} \sim \mathscr{E}_{\rm small}$, damping will be very strong and magnetic forces fairly weak, and as a result the qualitative behavior of the ions and neutrals at wavenumber k_p is to some degree simplified. Physically, the motions are determined by a balance between the magnetic force, on the one hand, and the viscosity and ion-neutral collision terms on the other. In this sense, the motion of the field line is analogous to the motion of an overdamped harmonic oscillator in which the velocity is determined by a balance between the spring's force and the frictional force (see Appendix C). To obtain an approximate solution of equations (34) and (35) we can drop the inertia terms (the left-hand sides of eqs. [34] and [35]), solve for the ion and neutral velocities, and then verify that the solution is consistent with the neglect of the inertia terms.

To further simplify the analysis, let us focus upon the motion of a trough of one of the field lines (a point on this field line for which y is a minimum). The ion and neutral motions at the trough will be in the $\hat{y}$ direction by symmetry, so $\boldsymbol{v}_n = v_n \hat{\boldsymbol{y}}$ and $\boldsymbol{v}_i = v_i \hat{\boldsymbol{y}}$. Assuming that p_n and p_i are initially uniform, symmetry implies that the pressure gradients are zero at the trough. The equations are simplified by the fact that ρ_n and ρ_i are constant in space and time and equal to each other, which implies that $\omega_{ni} = \omega_{in}$. If we drop the inertia terms, approximate the viscous force term as $-\rho_n \nu k_p^2 v_n \hat{\boldsymbol{y}}$, and use $B_0^2/8\pi = (\rho_n + \rho_i)\mathscr{E}_{\rm mag}$ to write $(1/4\pi)\boldsymbol{B} \cdot \nabla \boldsymbol{B} = \hat{\boldsymbol{y}} 4\rho_i k_p \mathscr{E}_{\rm mag} f(t)$, then equations (34) and (35) reduce to

$$0 = -k_p^2 \nu v_n - \omega_{in}(v_n - v_i) , \tag{36}$$

$$0 = k_p \mathscr{E}_{\rm mag} f(t) - \omega_{in}(v_i - v_n) , \tag{37}$$

where the 4 in front of $\mathscr{E}_{\rm mag}$ has been dropped. Here, $f(t)$ is a dimensionless function of order 1 that gives the time evolution of the initial magnetic force given in equation (33).

By adding equations (36) and (37) and then rewriting equation (37), one obtains two equations:

$$v_n = \frac{\mathscr{E}_{\rm mag} f(t)}{k_p \nu} , \tag{38}$$

$$v_i - v_n = \frac{k_p \mathscr{E}_{\rm mag} f(t)}{\omega_{in}} . \tag{39}$$

As discussed in Appendix A, $\nu \sim v_{\rm thermal} \lambda_{ni}$, and $\omega_{in} \sim v_{\rm thermal}/\lambda_{in}$. Since $\lambda_{in} = \lambda_{ni}$ in our model for the ISM, the condition $k_p \ll 1/\lambda_{ni}$ implies that $\omega_{in} \gg k_p^2 \nu$. Therefore, in equations (38) and (39),

$$v_i - v_n \ll v_n , \tag{40}$$

implying that the motion of field lines is dominated by the motion of the combined plasma-neutral fluid and not by ambipolar diffusion (the relative velocity between the ions neutrals). To check the consistency of the assumption that the inertia terms are negligible, one can use equation (38) to write $\partial v_n/\partial t = v_n(1/f)df/dt$. The time required for $f(t)$ to change by an appreciable fraction is on the order of the time required for the field line to straighten. This is the amount of time required for the trough of the field line to move a distance $1/k_p$ at a velocity v_i. Thus, $\partial v_n/\partial t \sim k_p v_i v_n$, and we can write

$$\frac{|\partial \boldsymbol{v}_n/\partial t|}{|\nu \nabla^2 \boldsymbol{v}_n|} \sim \frac{k_p v_i v_n}{k_p^2 \nu v_n} \sim \frac{\mathscr{E}_{\rm mag}}{(k_p \nu)^2} \sim \frac{\mathscr{E}_{\rm mag}}{\mathscr{E}_{\rm small}} \left(\frac{k_{\rm max}}{k_p}\right)^2 , \tag{41}$$

where the last relation makes use of the rough equality $\nu \sim v_{k_{\rm max}}/k_{\rm max}$ (eq. [23]). Since $\mathscr{E}_{\rm mag} \sim \mathscr{E}_{\rm small}$ in stage 3 of the Galactic dynamo, $|\partial \boldsymbol{v}_n/\partial t|$ is indeed negligible in equation (36). The $\boldsymbol{v}_n \cdot \nabla \boldsymbol{v}_n$ term is zero at the trough, but elsewhere its magnitude is $k_p v_n^2$, which is negligible because $k_p v_n^2 \sim k_p v_i v_n \sim |\partial \boldsymbol{v}_n/\partial t|$. Similarly, the ion inertia term is much less than the magnetic force term:

$$\frac{|\partial \boldsymbol{v}_i/\partial t|}{k_p \mathscr{E}_{\rm mag}} \sim \frac{v_i^2}{\mathscr{E}_{\rm mag}} \sim \frac{\mathscr{E}_{\rm mag}}{\mathscr{E}_{\rm small}} \left(\frac{k_{\rm max}}{k_p}\right)^2 \ll 1 \tag{42}$$

In our simple model for the magnetic field, the magnetic energy is reduced appreciably when the field lines straighten. If plasma initially located at $\boldsymbol{r}_0$ is displaced to the position $\boldsymbol{r} = \boldsymbol{r}_0 - \hat{\boldsymbol{y}}(1/k_p) \sin k_p x$, thereby straightening all field lines, then Lundquist's identity[2] implies that the final magnetic field is given by

$$B_x = B_0 \frac{1}{(1 + \cos^2 k_p x)^{1/2}} , \tag{43}$$

$$B_y = 0 . \tag{44}$$

The same result can be recovered by considering flux conservation through a planar surface at constant x that is convected with the plasma. This implies that the magnetic energy is reduced by an appreciable fraction in the time $1/k_p v_i$ required for the field lines to straighten. The rate of damping of magnetic energy by viscous relaxation during stage 3, ξ_3, can thus be estimated as $k_p v_i$, or

$$\xi_3 \sim \frac{\mathscr{E}_{\rm mag}}{\nu} . \tag{45}$$

Notwithstanding the requirement that $k_{\rm max} \ll k_p \ll 1/\lambda_{nn}$, this rate of damping is independent of the value of k_p.

If ξ_3 becomes larger than γ, then the magnetic energy will decrease. Since ξ_3 in stage 3 is proportional to $\mathscr{E}_{\rm mag}$, viscous relaxation leads to an upper bound on the magnetic energy when $k_p \gg k_{\rm max}$. This upper bound can be evaluated by setting $\xi_3 = \gamma$ and using equation (23) to write $\nu = v_{k_{\rm max}}/k_{\rm max}$, and it can be written as

$$\mathscr{E}_{\rm mag} \sim v_{k_{\rm max}}^2 \sim \mathscr{E}_{\rm kin} R^{-1/2} , \tag{46}$$

where the last relation follows from equation (25). It should be emphasized that this is only a limit on the magnetic energy at wavenumbers greater than $k_{\rm max}$, because viscous relaxation has been derived under the assumption that the magnetic energy is peaked at such wavenumbers, and it only describes damping of magnetic energy near the wavenumber k_p. In other words, viscous relaxation does not prevent $M(k)$ from growing at wavenumbers less than or equal to $k_{\rm max}$.

4. AMBIPOLAR DAMPING: $k_p \gg 1/\lambda_{ni}$

When $k_p \gg 1/\lambda_{ni}$, the analysis is more complicated. The neutrals are no longer described by the fluid equation (34), and the damping of neutral motions at k_p is no longer described by the viscous term $\nu \nabla^2 \boldsymbol{v}_n$. This can be seen within the context of a simple example in which magnetic forces start to induce wave motion in the neutrals of the form

$$\boldsymbol{v}_n = -\hat{\boldsymbol{y}} v_0 \sin k_p x . \tag{47}$$

The viscous term $\nu \nabla^2 \boldsymbol{v}_n$ is a diffusive term and applies only when neutral particles make many collisions during the time in which they move between regions of appreciably different $\boldsymbol{v}_n$. In other words, the diffusive viscosity term is only applicable when $k_p < 1/\lambda_{ni}$. For $k_p > 1/\lambda_{ni}$, a substantial fraction of the neutral particles that start at $x = \pi/2k_p$ will stream freely (without collisions) to $x = 3\pi/2k_p$ in a time that is on the order of $1/k_p v_{\rm thermal}$. During this time, they will carry the negative y momentum of the wave in equation (47) at $x = \pi/2k_p$ to the location $x = 3\pi/2k_p$, where that negative y momentum will cancel a substantial part of

[2] $\boldsymbol{B} = (\rho/\rho_0)\boldsymbol{B}_0 \cdot \nabla_0 \boldsymbol{r}$, where ρ_0 is the initial density, ρ is the final density, $\boldsymbol{B}_0$ is the initial magnetic field, $\boldsymbol{B}$ is the final magnetic field, and $\nabla_0 \boldsymbol{r}$ is the Jacobian matrix of the transformation $\boldsymbol{r}_0 \to \boldsymbol{r}$.

the wave's initial momentum. This process is called phase mixing (Stix 1992), and it results in a damping rate of neutral motions at wavenumber k_p on the order of $k_p v_{\rm thermal}$.

We can obtain an approximate solution for v_i and v_n by rewriting the $k_p^2 \nu$ term in the equations (36) and (37) as $k_p v_{\rm thermal}$:

$$0 = -(k_p v_{\rm thermal}) v_n - \omega_{in}(v_n - v_i) , \tag{48}$$

$$0 = k_p \mathscr{E}_{\rm mag} f(t) - \omega_{in}(v_i - v_n) . \tag{49}$$

As in the last section, the neglect of inertia will be consistent with the solutions for v_n and v_i. These solutions can be written

$$v_n = \frac{\mathscr{E}_{\rm mag} f(t)}{v_{\rm thermal}} , \tag{50}$$

$$v_i - v_n = \frac{k_p \mathscr{E}_{\rm mag} f(t)}{\omega_{in}} . \tag{51}$$

Because $\omega_{in} \sim v_{\rm thermal}/\lambda_{in}$ and $k_p \gg 1/\lambda_{ni}$, it follows that $\omega_{in} \ll k_p v_{\rm thermal}$. Thus,

$$v_i - v_n \gg v_n , \tag{52}$$

and so

$$v_i \sim \frac{k_p \mathscr{E}_{\rm mag} f(t)}{\omega_{in}} . \tag{53}$$

Because the rate of damping of neutral motions $k_p v_{\rm thermal}$ is much larger than the ion-neutral collision rate ω_{in}, the neutrals are essentially frozen in place, and the motion of ions is dominated during stage 2 by the relative drift between ions and neutrals. The rate at which magnetic energy is damped by ambipolar diffusion, $\xi_2 = k_p v_i$, satisfies the approximate relation

$$\xi_2 \sim \frac{k_p^2 \mathscr{E}_{\rm mag} f(t)}{\omega_{in}} , \tag{54}$$

as argued by KA.

One can derive equations relating $\mathscr{E}_{\rm mag}$ to k_p during both stage 1 and stage 2. By solving these equations simultaneously, one can find the values of $\mathscr{E}_{\rm mag}$ and k_p at the transition between stages 1 and 2. During stage 2, k_p is always approximately that k for which the rate of ambipolar damping is equal to γ. Above this wavenumber, ambipolar damping dominates and the spectrum decreases with increasing k; below this wavenumber, ambipolar damping is ineffective and the spectrum increases with k. By setting the damping rate given in equation (54) equal to $k_{\rm max} v_{k_{\rm max}}$ and using the relation $\omega_{in} \sim v_{\rm thermal}/\lambda_{in}$, one obtains the following equation, which is valid during stage 2:

$$k_p^2 \mathscr{E}_{\rm mag} \lambda_{in} \sim k_{\rm max} v_{k_{\rm max}} v_{\rm thermal} . \tag{55}$$

For stage 1, as mentioned in the introduction, KA find that $\mathscr{E}_{\rm mag} = \mathscr{E}_0 e^{2\gamma t}$. This relation in conjunction with equation (3) yields the following equation, which is valid during stage 1:

$$k_p^2 \sim k_{\rm max}^2 (\mathscr{E}_{\rm mag}/\mathscr{E}_0)^{1/2} . \tag{56}$$

In addition, $k_{\rm max} \lambda_{in} = k_{\rm max} \lambda_{ni} \sim R^{-1/4} N_{\rm M}$, where the last relation was discussed following equation (28). After using these relations to eliminate λ_{in}, one can solve equations (55) and (56) simultaneously. Labeling the solution for $\mathscr{E}_{\rm mag}$ at the beginning of stage 2 "$\overline{\mathscr{E}}$," one finds that

$$\overline{\mathscr{E}}^{3/2} \mathscr{E}_0^{-1/2} R^{-1/4} N_{\rm M} \sim v_{\rm thermal} v_{k_{\rm max}} . \tag{57}$$

Using equation (25) to write $v_{k_{\rm max}} \sim R^{-1/4} N_{\rm M} v_{\rm thermal}$, and defining $\mathscr{E}_{\rm thermal} = v_{\rm thermal}^2$ as the thermal energy per unit mass, equation (57) can be rewritten

$$\overline{\mathscr{E}} \sim \mathscr{E}_0^{1/3} \mathscr{E}_{\rm thermal}^{2/3} . \tag{58}$$

Labeling the solution for k_p at the beginning of stage 2 "$\overline{k_p}$," equation (56) can be used to write

$$\overline{k_p} \sim k_{\rm max} \left(\frac{\mathscr{E}_{\rm thermal}}{\mathscr{E}_0} \right)^{1/6} . \tag{59}$$

Equation (59) implies that for a sufficiently weak initial magnetic field, the peak of the magnetic spectrum would evolve to the resistive scale k_R^{-1} before the end of stage 1, where k_R is determined by the condition $k_R^2 \eta \equiv \gamma$ and η is the resistivity. Let $\mathscr{E}_0^R$ be the value of $\mathscr{E}_0$ below which the resistive scale is reached. An estimate of $\mathscr{E}_0^R$ can be obtained by setting $\overline{k_p} = k_R$. The relation $\gamma = k_{\rm max} v_{k_{\rm max}} \sim k_{\rm max}^2 \nu$ then implies that

$$\mathscr{E}_0^R = \mathscr{E}_{\rm thermal} \left(\frac{\eta}{\nu} \right)^3 = \mathscr{E}_{\rm thermal} P_{\rm M}^{-3} \tag{60}$$

where $P_{\rm M} = \nu/\eta$ is the magnetic Prandtl number.

The value of $\mathscr{E}_{\rm mag}$ at the end of stage 2 can be found by setting $k_p = 1/\lambda_{ni}$ in equation (55), rewriting $\lambda_{ni} v_{\rm thermal}$ as ν, and using equation (23) to write $k_{\rm max} \nu \sim v_{k_{\rm max}}$:

$$\mathscr{E}_{\rm mag} \sim v_{k_{\rm max}}^2 \sim R^{-1/2} \mathscr{E}_{\rm kin} . \tag{61}$$

5. NUMERICAL EXAMPLE

Approximate numerical values for k_p, $\boldsymbol{B}$, and t (the time elapsed since Galaxy formation) at the beginning of each of the five stages are given in Table 3. These values are estimated using the parameters given in Table 2. The value of t at the beginning of stage 4 is not known since the evolution of $M(k)$ during stage 3 has not been calculated. Nevertheless, it seems probable that the amount of time spent in stage 3 is less than the amount spent in stage 2, since the magnetic energy at $k_{\rm max}$ does not have to grow very much during stage 3 in order for $M(k_{\rm max})$ to reach the value $E(k_{\rm max})$. The results of Pouquet et al. (1976) and Chandran (1997a) suggest that stage 4 lasts for tens of large-eddy turnover times. (A large-eddy turnover time is $1/k_{\rm min} v_{k_{\rm min}} \sim 1.6 \times 10^6$ yr.)

6. CONCLUSION

In this paper it is shown that viscous relaxation, a new mechanism for damping magnetic energy, leads to satura-

TABLE 3

APPROXIMATE VALUES OF k_p, $|\boldsymbol{B}|$, AND t (TIME SINCE GALAXY FORMATION) AT THE BEGINNING OF EACH STAGE

Stage Number	k_p (cm^{-1})	$\boldsymbol{B}$ (G)	t (yr)
1	10^{-16}	10^{-17}	0
2	6×10^{-13}	3×10^{-10}	8×10^4
3	2×10^{-15}	10^{-7}	3×10^5
4	10^{-16}	10^{-7}	4×10^5
5	2×10^{-20}	2×10^{-6}	10^8

tion of the magnetic energy on scales smaller than the smallest turbulent eddy in the Galactic dynamo. The saturated energy level is roughly the kinetic energy of the smallest turbulent eddies, $\mathscr{E}_{\rm small} = R^{-1/2}\mathscr{E}_{\rm kin}$, where $\mathscr{E}_{\rm kin}$ is the total turbulent kinetic energy. Viscous relaxation does not constrain the magnetic energy within the "inertial range" of the turbulence [the wavenumber interval ($k_{\rm min}$, $k_{\rm max}$) of the velocity spectrum in which Kolmogorov's $k^{-5/3}$ law is approximately valid]. As a result, once the magnetic energy saturates at wavenumbers larger than $k_{\rm max}$, it continues to grow within the inertial range until the magnetic and kinetic power spectra are comparable at $k_{\rm max}$. Once this point is reached, the nonlinear dynamo begins and the magnetic energy eventually grows to equipartition with the kinetic energy at each scale throughout most of the inertial range, with the final magnetic and kinetic spectra scaling as $k^{-3/2}$ instead of $k^{-5/3}$ (Pouquet et al. 1976; Chandran 1997a). The importance of viscous relaxation lies largely in the fact that it prevents the magnetic energy from remaining on scales small compared to the velocity turbulence as the magnetic energy grows to equipartition with the kinetic energy. Such a separation of scales is predicted by the theory of Kulsrud & Anderson (1992) and would affect the operation of both the mean-field dynamo and the inverse cascade of Pouquet et al. (1976).

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APPENDIX A

NEUTRAL VISCOSITY IN A PARTIALLY IONIZED PLASMA

In a neutral fluid, the viscosity can be estimated as (Braginskii 1965)

$$\nu \sim \lambda_{nn} v_{\rm thermal} \sim \frac{\lambda_{nn}^2}{\delta t} , \tag{A1}$$

where $\delta t = \lambda_{nn}/v_{\rm thermal}$ is the time between collisions. The reason for writing ν in the form $\lambda_{nn}^2/\delta t$ is to bring out the analogy between viscosity and diffusion. Each neutral particle can be thought of as undergoing a random walk, taking a random step of length λ_{nn} in a time δt with a resulting diffusion coefficient of $\lambda_{nn}^2/\delta t$. In a partially ionized plasma with equal ion and neutral densities at a temperature of 10^4 K, neutrals collide with ions more often than with other neutrals (Spitzer 1978), and the step length in a neutral particle's random walk is reduced from λ_{nn} to $\lambda_{ni} \sim \lambda_{nn}/20$. At the same time, δt is reduced to $\lambda_{ni}/v_{\rm thermal}$, so that the neutral viscosity becomes

$$\nu \sim \lambda_{ni} v_{\rm thermal} . \tag{A2}$$

APPENDIX B

THE CASE OF LOW FRACTIONAL IONIZATION

A low fractional ionization has two main effects on the early stages of the Galactic dynamo. First, during the ambipolar diffusion stage, a reduction of the fractional ionization χ reduces the number of ion-neutral collisions and thus decreases the friction that opposes the magnetic field's tendency to straighten itself out. As a result, a decrease in χ increases the ambipolar drift velocity and enhances ambipolar damping of magnetic energy during stage 2. This has the effect of reducing k_p for any given value of $\mathscr{E}_{\rm mag}$. For sufficiently small values of χ, stage 3 is skipped, and the ambipolar stage leads directly into the nonlinear dynamo stage. Second, neutral viscosity increases because neutral collisions with ions become less frequent. (As described in Appendix A, neutral-ion collisions make up the majority of neutral collisions for 50% ionization.) This increase in viscosity lowers the Reynolds number of the turbulence and leads to a small decrease in both $k_{\rm max}$ and $\gamma = k_{\rm max} v_{k_{\rm max}}$.

APPENDIX C

THE OVERDAMPED HARMONIC OSCILLATOR

The simple model used in this paper to describe the motion of field lines bears some similarity to the overdamped harmonic oscillator, which is reviewed in this Appendix. The displacement of a mass on a spring is described by the equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega_0^2 x = 0 , \tag{C1}$$

where b describes the effect of some source of friction and $\omega_0^2 x$ is the acceleration induced by the spring. If $x \sim e^{-i\omega t}$, then equation (C1) implies that

$$\omega = \frac{-ib}{2}\left(1 \pm \sqrt{1 - \frac{4\omega_0^2}{b^2}}\right) . \tag{C2}$$

If $b \gg \omega_0$, then ω has two solutions:

$$\omega \sim -ib \, , \tag{C3}$$

$$\omega \sim -i\omega_0^2/b \, . \tag{C4}$$

The first of these damping rates is very large (in the sense that $b \gg \omega_0$) and represents a rapidly decaying transient. The second damping rate is very small and represents the solution in which friction balances the force of the spring, with inertia playing an insignificant role and the velocity remaining roughly constant. If the mass is initially held at rest ($\dot{x} = 0$ at $t = 0$) at some nonzero displacement and then let go, then for $t > 0$, $x(t) \sim \exp(-\omega_0^2 t/b)$. That is, the motion will be described by the "very small" damping rate, the velocity will change very slowly in time, and inertia will be negligible. The solution for these initial conditions is well approximated by neglecting the second time derivative, the inertia term, from the outset.

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