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QUASI-LINEAR THEORY OF COSMIC RAY TRANSPORT AND ACCELERATION: THE ROLE OF OBLIQUE MAGNETOHYDRODYNAMIC WAVES AND TRANSIT-TIME DAMPING

REINHARD SCHLICKEISER
Max-Planck-Institut für Radioastronomie, Postfach 20 24, 53010 Bonn, Germany

AND

JAMES A. MILLER
Department of Physics, University of Alabama in Huntsville, Huntsville, AL 35899
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ABSTRACT

We calculate quasi-linear transport and acceleration parameters for cosmic ray particles interacting resonantly with undamped fast-mode waves propagating in a low-β plasma. For super-Alfvénic particles and a vanishing cross-helicity state of the fast-mode waves, we demonstrate that the rate of adiabatic deceleration vanishes, and that the momentum and spatial diffusion coefficients can be calculated from the Fokker-Planck coefficient $D_{\mu\mu}$. Adopting isotropic fast-mode turbulence with a Kolmogorov-like turbulence spectrum, we demonstrate that $D_{\mu\mu}$ is the sum of contributions from transit-time damping and gyroresonant interactions. Gyroresonance refers to $|n| \neq 0$ resonant particle-wave interactions. Transit-time damping refers to the $n = 0$ interaction of particles with the compressive magnetic field component of the fast-mode waves. We show that transit-time damping provides the dominant contribution to pitch-angle scattering in the interval $\epsilon \leq |\mu| \leq 1$, where ϵ is the ratio of Alfvén to particle speed. In the interval $|\mu| < \epsilon$, transit-time damping does not occur, and gyroresonance provides a small but finite contribution to particle scattering. As a consequence, the momentum diffusion coefficient is mainly determined by the transit-time damping contribution. On the other hand, since the spatial diffusion coefficient and the related mean free path are given by the average over μ of the *inverse* of $D_{\mu\mu}$, these spatial transport parameters are determined by the contribution from the interval $|\mu| < \epsilon$. We also calculate the cosmic ray transport parameters for plasma turbulence consisting of a mixture of isotropic fast-mode waves and slab Alfvén waves. Here, the momentum diffusion coefficient is determined by the transit-time damping of the fast-mode waves, and is a factor $\ln \epsilon^{-1}$ larger than in the case of pure slab Alfvén wave turbulence. The mean free path and the spatial diffusion coefficient are modified significantly from the pure fast-mode case, since the crucial scattering at $|\mu| < \epsilon$ is now provided by gyroresonances with slab Alfvén waves. The mean free path is a constant at nonrelativistic energies, and may account for the legendary λ_{fit}-λ_{QLT} discrepancy of solar energetic particles.

Subject headings: acceleration of particles — cosmic rays — diffusion — magnetic fields — plasmas — Sun: particle emission

1. INTRODUCTION

Most investigations of cosmic ray transport and acceleration in magnetized cosmic plasmas are restricted to the resonant interaction of energetic charged particles with low-frequency plasma waves propagating along the ordered uniform magnetic field (for a review see Schlickeiser 1994), also referred to as the slab model of plasma wave turbulence. This restriction is remarkable for two reasons: (1) the general quasi-linear theory of cosmic ray transport and acceleration in weakly turbulent plasmas for arbitrary plasma wave properties is fairly well developed (see, e.g., Jaekel & Schlickeiser 1992; Schlickeiser & Achatz 1993a, 1993b); (2) there is ample evidence, both from in-situ turbulence measurements in the interplanetary medium (Tu, Marsch, & Thieme 1989; Rickett 1990; Minter & Spangler 1997) and from growth rate calculations in the interstellar medium, especially of the fast magnetosonic wave (Tademaru 1969), that obliquely propagating waves form a sizeable fraction of the plasma turbulence.

It is the purpose of this work to calculate quasi-linear transport and acceleration parameters for cosmic ray particles resulting from the resonant interaction with oblique low-frequency plasma waves. In particular, we consider fast magnetosonic waves, hereafter referred to as the fast-mode branch. The fast-mode branch in a plasma extends from low frequencies, past the proton cyclotron frequency $\Omega_{0,p}$, and up to the electron cyclotron frequency (e.g., Thompson 1962; Swanson 1989). Fast-mode waves on the $|\omega| \ll \Omega_{0,p}$ section of this branch have a dispersion relation given by $\omega_j = jV_A k$, $j = \pm 1$, where V_A and k denote the Alfvén speed and the magnitude of the wavevector $\boldsymbol{k}$, respectively. As $|\omega_j|$ approaches $\Omega_{0,p}$, this simple dispersion relation is no longer valid, and above $\sim 10\Omega_{0,p}$ the branch enters the Whistler regime. Low-frequency fast-mode waves have a simple polarization. If a wave is parallel, then its electric field $\delta \boldsymbol{E}_w$ is transverse to the ambient magnetic field $\boldsymbol{B}_0$ and right-hand circularly polarized; if it is oblique, then the wave is linearly polarized, and $\delta \boldsymbol{E}_w$ is in the direction of $\boldsymbol{B}_0 \times \boldsymbol{k}$. According to Faraday's law, the wave magnetic field $\delta \boldsymbol{B}_w$ has both a compressive (along $\boldsymbol{B}_0$) and a linearly polarized transverse component. The importance of this compressive magnetic field component for particle acceleration has been noted by many authors (Lee & Völk 1975; Fisk 1976; Eilek 1979; Achterberg 1981; Miller, LaRosa, & Moore 1996), since it allows for the effect of transit-time damping. In general, energetic charged particles of velocity v and Larmor frequency $\Omega = \Omega_0/\gamma$

resonantly interact with undamped waves of frequency ω if the gyroresonance interaction condition $\omega - k_\parallel v_\parallel = n\Omega$, with the integer n running from $-\infty$ to $+\infty$, is fulfilled, where $v_\parallel = v\mu$ and $k_\parallel$ are the parallel velocity component and the parallel wavenumber, respectively. Unlike the case of purely parallel propagating waves in slab turbulence, the presence of the compressive magnetic field component of $\delta \boldsymbol{B}_w$ for oblique fast-mode waves allows the cosmic ray particles to resonantly interact with these waves through the $n = 0$ resonance.

One of us (Miller 1997) has recently discussed in detail the basic physical mechanism by which oblique fast-mode waves accelerate particles. One such wave has a parallel magnetic field component, resulting in a series of compressive and rarefactive magnetic perturbations moving along $\boldsymbol{B}_0$ with the parallel wave speed $\omega/k_\parallel$, where ω is the wave frequency and $k_\parallel$ is the component of the wavevector $\boldsymbol{k}$ along $\boldsymbol{B}_0$. In the frame travelling along $\boldsymbol{B}_0$ with speed $\omega/k_\parallel$ (wave frame), these perturbations are stationary and the parallel particle velocity $v'_\parallel$ will be affected by the usual mirror force $-(mv_\perp^2/2B)\nabla_\parallel B$, where m is the particle mass, $v_\perp$ is the particle velocity component normal to $\boldsymbol{B}_0$, and B is the total (ambient plus wave parallel) magnetic field. For large-amplitude waves, essentially all particles will be reflected by a compression in the wave frame. In the plasma frame, the particle then makes either a head-on or an overtaking collision with a moving magnetic mirror, leading to either a gain or a loss of energy, respectively. Since the frequency of head-on collisions is greater than the frequency of trailing collisions as a result of the larger relative velocity between particle and wave, there will be a net gain of energy. This is just second-order Fermi (1949) acceleration. As the wave amplitude decreases, a particle's pitch angle in the wave frame must approach 90° in order to insure that reflection occurs before the particle has left the compression. Equivalently, $v'_\parallel$ must approach zero, which in the plasma frame implies that the parallel component of velocity $v_\parallel$ be $\approx \omega/k_\parallel$. Thus, for small amplitude waves, an appreciable interaction between a wave and a particle arises only when the particle is moving at nearly the parallel wave phase speed. The interaction is now resonant and $\omega - k_\parallel v_\parallel \approx 0$ is just the usual $\ell = 0$ (or Landau) resonance condition. However, given that resonance does occur, the subsequent physics is close to that of Fermi acceleration: particles with $v_\parallel$ slightly greater than $\omega/k_\parallel$ suffer a trailing collision with a wave compression and slow down; particles with $v_\parallel$ slightly less than $\omega/k_\parallel$ are struck by a compression and speed up. This process could be called resonant Fermi acceleration, but the usual term for it is transit-time acceleration or transit-time damping (Fisk 1976; Achterberg 1981; Stix 1992). The name arises because the resonance condition can be rewritten as $\lambda_\parallel/v_\parallel \approx T$, where T is the wave period and $\lambda_\parallel = 2\pi/k\eta$ is the parallel wavelength. In other words, a wave and a particle will interact strongly when the particle transit time across the wave compression is approximately equal to the period.

As we shall demonstrate in the following discussion, transit-time damping ($n = 0$) of fast-mode waves provides a dominating contribution to the stochastic acceleration rate of cosmic ray particles. As previous studies (Fisk 1976; Miller et al. 1996) have done, we will base our discussion on quasi-linear theory (QLT), which, of course, is a crucial approximation and needs justification. In the aforementioned work, Miller (1997) has examined the validity of QLT and the conditions for its applicability to this case by comparing it with extensive test particle simulations. In these numerical simulations it is crucial to construct a discrete wave spectrum (that is, to mimic the continuous one) in such a way that the individual monochromatic waves are so placed in propagation angle and wavenumber that neighboring waves have overlapping resonances and the spectral density that results from binning the discrete waves matches the continuous one. Miller (1997) then found that QLT provides an excellent description of transit-time acceleration, even when the total energy density of the fast-mode wave turbulence is about equal to the ambient magnetic field energy density. This condition is certainly fulfilled in the two astrophysical plasmas where we want to apply our results: (1) in the interstellar medium of our Galaxy, where the measured mean of the magnetic field fluctuations due to turbulence of $\delta B \simeq 0.9$ μG is much less than the ordered magnetic field of about 4 μG (Spangler 1991; Minter & Spangler 1997), and (2) in the interplanetary medium during the occurrence of three solar particle events on 1977 Nov 22, 1977 Dec 27, and 1978 Apr 11 (Dröge et al. 1993) that provided the best determinations of the scattering mean free path of solar particles.

2. CALCULATION OF QUASI-LINEAR FOKKER-PLANCK COEFFICIENTS

2.1. *Basic Quasi-linear Equations*

Modern descriptions of the particle transport and acceleration in interplanetary and interstellar plasmas are based on two transport equations derived from the collisionless Boltzmann-Vlasov equation into which the electromagnetic fields of the interplanetary and interstellar medium enter by the Lorentz force term. The first of the equations, the Fokker-Planck equation, results from applying the quasi-linear approximation (Kennel & Engelmann 1966; Lerche 1968; Hasselmann & Wibberenz 1968) to the behavior of energetic charged particles in a uniform magnetic field, $\boldsymbol{B}_0 = B_0 \boldsymbol{e}_z$, with superposed small-amplitude plasma turbulence ($\delta \boldsymbol{E}$, $\delta \boldsymbol{B}$) in the rest frame of the plasma turbulence supporting fluid (e.g., the solar wind). The effect of the plasma turbulence on the particles is studied by calculating first-order corrections to the particles' orbit in the uniform magnetic field $\boldsymbol{B}_0$, and ensemble averaging over the statistical properties of the plasma turbulence (Jokipii 1966). In the *mixed comoving coordinate system*, in which the space coordinates are measured in the laboratory system and the particle's momentum coordinates are measured in the rest frame of the background plasma, which supports the plasma turbulence and in which the turbulence is homogeneous in space and time, the gyrophase-averaged phase space density $f(z, p, \mu, t)$ evolves according to the Fokker-Planck equation (see, e.g., Kirk, Schlickeiser, & Schneider 1988). The respective Fokker-Planck coefficients

$$D_{\mu\mu} \equiv \lim_{t \to \infty} \frac{1}{2t} \langle \Delta\mu(t) \Delta\mu^*(t+\tau) \rangle = \Re \int_0^\infty d\tau \langle \dot{\mu}(t) \dot{\mu}^*(t+\tau) \rangle \,, \tag{1a}$$

$$D_{\mu p} \equiv \lim_{t \to \infty} \frac{1}{2t} \langle \Delta\mu(t) \Delta p^*(t+\tau) \rangle = \Re \int_0^\infty d\tau \langle \dot{\mu}(t) \dot{p}^*(t+\tau) \rangle \,, \tag{1b}$$

$$D_{pp} \equiv \lim_{t\to\infty} \frac{1}{2t} \langle \Delta p(t) \Delta p^*(t+\tau)\rangle = \Re \int_0^\infty d\tau \langle \dot{p}(t)\dot{p}^*(t+\tau)\rangle \,, \tag{1c}$$

must be calculated (Hall & Sturrock 1967; Krommes 1984; Achatz, Steinacker, & Schlickeiser 1991) from the ensemble-averaged first-order particle-orbit correction to the equation of motion (eqs. [21] and [22] of Jaekel & Schlickeiser 1992):

$$\dot{\mu} = \frac{\Omega(1-\mu^2)^{1/2}}{B_0} \Re \sum_j \sum_{n=-\infty}^{\infty} \int d^3k \exp\,[\imath n(\psi - \phi_0) + \imath(k_\parallel v_\parallel + n\Omega - \omega_j)t]$$
$$\times \left[\frac{c}{v}(1-\mu^2)^{1/2} J_n\!\left(\frac{k_\perp v_\perp}{\Omega}\right) E_\parallel(\boldsymbol{k}) + \frac{\imath}{\sqrt{2}} \left\{\left[B_R(\boldsymbol{k})e^{\imath\psi} + \imath\mu \frac{c}{v} E_R(\boldsymbol{k})e^{\imath\psi}\right] J_{n+1}\!\left(\frac{k_\perp v_\perp}{\Omega}\right)\right.\right.$$
$$\left.\left. - \left[B_L(\boldsymbol{k})e^{-\imath\psi} - \imath\mu \frac{c}{v} E_L(\boldsymbol{k})e^{-\imath\psi}\right] J_{n-1}\!\left(\frac{k_\perp v_\perp}{\Omega}\right)\right\}\right], \tag{2a}$$

and

$$\dot{p} = \frac{\Omega p c}{v B_0} \sum_j \sum_{n=-\infty}^{\infty} \int d^3k \exp\,[\imath n(\psi - \phi_0) + \imath(k_\parallel v_\parallel + n\Omega - \omega_j)t]$$
$$\times \left\{\mu J_n\!\left(\frac{k_\perp v_\perp}{\Omega}\right) E_\parallel(\boldsymbol{k}) + \frac{(1-\mu^2)^{1/2}}{\sqrt{2}} \left[E_R(\boldsymbol{k})e^{\imath\psi} J_{n+1}\!\left(\frac{k_\perp v_\perp}{\Omega}\right) + E_L(\boldsymbol{k})e^{-\imath\psi} J_{n-1}\!\left(\frac{k_\perp v_\perp}{\Omega}\right)\right]\right\}, \tag{2b}$$

where we used the notation of Schlickeiser & Achatz (1993a). In equations (2a)–(2b), $\psi = \cot^{-1}(k_x/k_y)$, $B_{L,R} = (B_x \pm \imath B_y)/\sqrt{2}$, and $E_{L,R} = (E_x \pm \imath E_y)/\sqrt{2}$.

In the presence of low-frequency magnetohydrodynamic turbulence (e.g., Alfvén or fast-mode waves), in which the magnetic field component is much larger than the electric field component ($|\delta \boldsymbol{B}| = (c/V_A)|\delta \boldsymbol{E}|$, $V_A \ll c$), the particle's distribution function $f(z, p, \mu, t)$ adjusts very rapidly to quasi equilibrium through pitch-angle diffusion, which is close to the isotropic distribution. In this case, a second cosmic ray transport equation can be derived from the Fokker-Planck equation by a well-known approximation scheme (Jokipii 1966; Hasselmann & Wibberenz 1968; Schlickeiser 1989), commonly referred to as the diffusion-convection equation for the pitch-angle averaged phase space density $F(z, p, t)$, and which for nonrelativistic bulk speed $u \ll c$ reads

$$\frac{\partial F}{\partial t} - S_0 = \frac{\partial}{\partial z}\left(\kappa \frac{\partial F}{\partial z}\right) - \left(u + \frac{1}{4p^2}\frac{\partial}{\partial p} p^2 v a_1\right)\frac{\partial F}{\partial z} + \left(\frac{p}{3}\frac{\partial u}{\partial z} + \frac{v}{4}\frac{\partial a_1}{\partial z}\right)\frac{\partial F}{\partial p} + \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 a_2 \frac{\partial F}{\partial p}\right), \tag{3}$$

where the spatial diffusion coefficient κ (or the mean free path λ), the rate of adiabatic deceleration a_1, and the momentum diffusion coefficient a_2 are determined by pitch-angle averages of the three Fokker-Planck coefficients (eq. [1]) as

$$\kappa = \frac{v}{3}\lambda = \frac{v^2}{8}\int_{-1}^{1} d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}}\,, \qquad a_1 = \int_{-1}^{1} d\mu (1-\mu^2)\frac{D_{\mu p}}{D_{\mu\mu}}\,, \qquad a_2 = \frac{1}{2}\int_{-1}^{1} d\mu \left(D_{pp} - \frac{D_{\mu p}^2}{D_{\mu\mu}}\right), \tag{4}$$

where S_0 is the source term and v is the cosmic ray particle velocity. Note that equation (3) holds in the mixed comoving coordinate system, so that the rate of adiabatic deceleration by the plasma waves (a_1) occurs as separate term from the rate of adiabatic deceleration by the plasma velocity gradients $p(\partial u/\partial z)/3$. Inserting the equations of motion (2) in equation (1), making the usual assumption that the turbulence Fourier and Laplace components at different frequencies, at different wave speeds, and at different wavevectors are uncorrelated, and averaging over the initial phase ϕ_0, Jaekel & Schlickeiser (1992) have found for the Fokker-Planck coefficients

$$D_{\mu\mu} = \frac{\pi\Omega^2(1-\mu^2)}{B_0^2} \sum_j \sum_{n=-\infty}^{\infty} \int d^3k\, \delta(k_\parallel v_\parallel - \omega_j + n\Omega)\left[\frac{c^2}{v^2}(1-\mu^2) J_n^2\!\left(\frac{k_\perp v_\perp}{|\Omega|}\right) R^j_{\parallel\parallel}(\boldsymbol{k})\right.$$
$$+ \frac{1}{2} J_{n+1}^2\!\left(\frac{k_\perp v_\perp}{|\Omega|}\right)\left\{P^j_{RR}(\boldsymbol{k}) + \mu^2 \frac{c^2}{v^2} R^j_{RR}(\boldsymbol{k}) + \imath\mu \frac{c}{v}\,[T^j_{RR}(\boldsymbol{k}) - Q^j_{RR}(\boldsymbol{k})]\right\}$$
$$+ \frac{1}{2} J_{n-1}^2\!\left(\frac{k_\perp v_\perp}{|\Omega|}\right)\left\{P^j_{LL}(\boldsymbol{k}) + \mu^2 \frac{c^2}{v^2} R^j_{LL}(\boldsymbol{k}) - \imath\mu \frac{c}{v}\,[T^j_{LL}(\boldsymbol{k}) - Q^j_{LL}(\boldsymbol{k})]\right\}$$
$$- \frac{1}{2} J_{n-1}\!\left(\frac{k_\perp v_\perp}{|\Omega|}\right) J_{n+1}\!\left(\frac{k_\perp v_\perp}{|\Omega|}\right)\left(e^{2\imath\psi}\left\{P^j_{RL}(\boldsymbol{k}) - \mu^2 \frac{c^2}{v^2} R^j_{RL}(\boldsymbol{k}) + \imath\mu \frac{c}{v}\,[T^j_{RL}(\boldsymbol{k}) + Q^j_{RL}(\boldsymbol{k})]\right\}\right.$$
$$\left. + e^{-2\imath\psi}\left\{P^j_{LR}(\boldsymbol{k}) - \mu^2 \frac{c^2}{v^2} R^j_{LR}(\boldsymbol{k}) - \imath\mu \frac{c}{v}\,[T^j_{LR}(\boldsymbol{k}) + Q^j_{LR}(\boldsymbol{k})]\right\}\right)$$
$$+ \frac{\imath c\sqrt{1-\mu^2}}{\sqrt{2}v} J_n\!\left(\frac{k_\perp v_\perp}{|\Omega|}\right)\left[J_{n+1}\!\left(\frac{k_\perp v_\perp}{|\Omega|}\right)\left\{e^{\imath\psi} Q^j_{R\parallel}(\boldsymbol{k}) - e^{-\imath\psi} T^j_{\parallel R}(\boldsymbol{k}) + \imath\mu \frac{c}{v}\,[R^j_{R\parallel}(\boldsymbol{k})e^{\imath\psi} + R^j_{\parallel R}(\boldsymbol{k})e^{-\imath\psi}]\right\}\right.$$
$$\left.\left. + J_{n-1}\!\left(\frac{k_\perp v_\perp}{|\Omega|}\right)\left\{e^{-\imath\psi} Q^j_{L\parallel}(\boldsymbol{k}) + e^{\imath\psi} T^j_{\parallel L}(\boldsymbol{k}) + \imath\mu \frac{c}{v}\,[R^j_{\parallel L}(\boldsymbol{k})e^{\imath\psi} - R^j_{L\parallel}(\boldsymbol{k})e^{-\imath\psi}]\right\}\right]\right], \tag{5a}$$

$$D_{\mu p} = \frac{\pi\Omega^2(1-\mu^2)^{1/2}pc}{vB_0^2} \Re\imath \sum_j \sum_{n=-\infty}^{\infty} \int d^3k\delta(k_\| v_\| - \omega_j + n\Omega)\Bigg[-\imath \frac{c}{v} \mu(1-\mu^2)^{1/2} J_n^2\left(\frac{k_\perp v_\perp}{|\Omega|}\right) R^j_{\|\|}(\boldsymbol{k})$$
$$+ \frac{(1-\mu^2)^{1/2}}{2} \Bigg(J_{n+1}^2\left(\frac{k_\perp v_\perp}{|\Omega|}\right)\left[Q^j_{RR}(\boldsymbol{k}) + \imath\mu \frac{c}{v} R^j_{RR}(\boldsymbol{k})\right] - J_{n-1}^2\left(\frac{k_\perp v_\perp}{|\Omega|}\right)\left[Q^j_{LL}(\boldsymbol{k}) - \imath\mu \frac{c}{v} R^j_{LL}(\boldsymbol{k})\right]$$
$$+ J_{n+1}\left(\frac{k_\perp v_\perp}{|\Omega|}\right) J_{n-1}\left(\frac{k_\perp v_\perp}{|\Omega|}\right)\left\{ e^{2\imath\psi}\left[Q^j_{RL}(\boldsymbol{k}) + \imath\mu \frac{c}{v} R^j_{RL}(\boldsymbol{k})\right] - e^{-2\imath\psi}\left[Q^j_{LR}(\boldsymbol{k}) - \imath\mu \frac{c}{v} R^j_{LR}(\boldsymbol{k})\right]\right\}\Bigg)$$
$$+ \frac{1}{\sqrt{2}} J_{n-1}\left(\frac{k_\perp v_\perp}{|\Omega|}\right) J_n\left(\frac{k_\perp v_\perp}{|\Omega|}\right)\left\{ \mu e^{-\imath\psi}\left[-Q^j_{L\|}(\boldsymbol{k}) + \imath\mu \frac{c}{v} R^j_{R\|}(\boldsymbol{k})\right] - \imath(1-\mu^2)\frac{c}{v} e^{\imath\psi} R^j_{\|L}(\boldsymbol{k})\right\}$$
$$+ \frac{1}{\sqrt{2}} J_{n+1}\left(\frac{k_\perp v_\perp}{|\Omega|}\right) J_n\left(\frac{k_\perp v_\perp}{|\Omega|}\right)\left\{ \mu e^{\imath\psi}\left[Q^j_{R\|}(\boldsymbol{k}) + \imath\mu \frac{c}{v} R^j_{R\|}(\boldsymbol{k})\right] - \imath(1-\mu^2)\frac{c}{v} e^{-\imath\psi} R^j_{\|R}(\boldsymbol{k})\right\}\Bigg], \tag{5b}$$

and

$$D_{pp} = \frac{\pi\Omega^2 p^2 c^2}{B_0^2 v^2} \Re \sum_j \sum_{n=-\infty}^{\infty} \int d^3k\delta(k_\| v_\| - \omega_j + n\Omega)\Bigg(\mu^2 J_n^2\left(\frac{k_\perp v_\perp}{|\Omega|}\right) R^j_{\|\|}(\boldsymbol{k})$$
$$+ \frac{1-\mu^2}{2}\left\{ J_{n+1}^2\left(\frac{k_\perp v_\perp}{|\Omega|}\right) R^j_{RR}(\boldsymbol{k}) + J_{n-1}^2\left(\frac{k_\perp v_\perp}{|\Omega|}\right) R^j_{LL}(\boldsymbol{k}) + J_{n-1}\left(\frac{k_\perp v_\perp}{|\Omega|}\right) J_{n+1}\left(\frac{k_\perp v_\perp}{|\Omega|}\right)[R^j_{LR}(\boldsymbol{k})e^{-2\imath\psi} + R^j_{RL}(\boldsymbol{k})e^{2\imath\psi}]\right\}$$
$$+ \frac{\mu(1-\mu^2)^{1/2}}{\sqrt{2}} J_n\left(\frac{k_\perp v_\perp}{|\Omega|}\right)\left\{ J_{n-1}\left(\frac{k_\perp v_\perp}{|\Omega|}\right)[e^{\imath\psi}R^j_{\|L}(\boldsymbol{k}) + e^{-\imath\psi}R^j_{L\|}(\boldsymbol{k})] + J_{n+1}\left(\frac{k_\perp v_\perp}{|\Omega|}\right)[e^{-\imath\psi}R^j_{\|R}(\boldsymbol{k}) + e^{\imath\psi}R^j_{R\|}(\boldsymbol{k})]\right\}\Bigg), \tag{5c}$$

which involve the correlation tensors of the electric and magnetic field fluctuations

$$\langle B^j_\alpha(\boldsymbol{k}) B^{j*}_\beta(\boldsymbol{k}')\rangle = \delta(\boldsymbol{k}-\boldsymbol{k}')P^j_{\alpha,\beta}(\boldsymbol{k}), \qquad \langle E^j_\alpha(\boldsymbol{k}) B^{j*}_\beta(\boldsymbol{k}')\rangle = \delta(\boldsymbol{k}-\boldsymbol{k}')Q^j_{\alpha,\beta}(\boldsymbol{k}),$$
$$\langle B^j_\alpha(\boldsymbol{k}) E^{j*}_\beta(\boldsymbol{k}')\rangle = \delta(\boldsymbol{k}-\boldsymbol{k}')T^j_{\alpha,\beta}(\boldsymbol{k}), \qquad \langle E^j_\alpha(\boldsymbol{k}) E^{j*}_\beta(\boldsymbol{k}')\rangle = \delta(\boldsymbol{k}-\boldsymbol{k}')R^j_{\alpha,\beta}(\boldsymbol{k}). \tag{6}$$

For low-frequency MHD waves, we have from Faraday's law

$$\boldsymbol{B}^j(\boldsymbol{k}) = \frac{c}{\omega_j}\boldsymbol{k}\times\boldsymbol{E}^j(\boldsymbol{k}), \tag{7}$$

so that the electric field component is much smaller than the magnetic field component.

2.2. *The Fast-Mode Wave*

In the cold (electron temperature $T_e \to 0$) plasma limit, which is equivalent to the low beta limit, since $\beta = 2c_s^2/V_A^2 = 8\pi n m_p k_B T_e/B_0^2 \to 0$, the fast and slow magnetosonic waves in the fluid plasma (e.g., Thompson 1962) merge to the fast-mode branch with dispersion relation $\omega_j^2 = V_A^2 k^2$. Without loss of generality we may chose the particular coordinate system where the y-component of the wavevector $k_y = 0$ vanishes, implying $\psi = \cot^{-1}(k_x/k_y) = 0$ in equations (5). In this coordinate system,

$$\boldsymbol{k} = (k_\perp, 0, k_\|) = k(\sin\Theta, 0, \cos\Theta); \tag{8}$$

the y-component is the only nonvanishing electric field component of the fast-mode wave

$$\boldsymbol{e}_F \equiv \frac{\boldsymbol{E}_F}{|\boldsymbol{E}_F|} = (0, 1, 0), \tag{9}$$

so that the fast-mode wave is a transverse wave $\boldsymbol{e}_F \cdot \boldsymbol{k} = 0$. Faraday's law (eq. [7]) yields for the fast wave's normalized magnetic field component

$$\boldsymbol{b}_F \equiv \frac{\boldsymbol{B}_F}{|\boldsymbol{B}_F|} = (-\cos\Theta, 0, \sin\Theta). \tag{10}$$

Moreover, for purely transverse waves, Faraday's law (eq. [7]) can also be expressed as

$$\boldsymbol{E}^j(\boldsymbol{k}) = -\frac{\omega_j}{ck^2}\boldsymbol{k}\times\boldsymbol{B}^j(\boldsymbol{k}). \tag{11}$$

According to equations (9) and (10), we find for the respective right- and left-handed polarized and the parallel components for the fast-mode waves

$$E_L = -E_R, \qquad E_\| = 0, \tag{12}$$

$$B_L = B_R \,, \qquad B_{\|} \neq 0 \,, \tag{13}$$

implying for the components of the correlation tensors (6)

$$P_{LR} = P_{RL} = P_{LL} = P_{RR} \,, \tag{14a}$$

$$Q_{LR} = -Q_{RL} = Q_{LL} = -Q_{RR} \,, \tag{14b}$$

$$T_{LR} = -T_{RL} = -T_{LL} = T_{RR} \,, \tag{14c}$$

$$R_{LR} = R_{RL} = -R_{LL} = -R_{RR} \,, \tag{14d}$$

$$R_{\|\|} = R_{\|R} = R_{\|L} = T_{L\|} = T_{R\|} = Q_{\|R} = Q_{\|L} = 0 \,, \tag{14e}$$

$$T_{\|L} = -T_{\|R} \,, \tag{14f}$$

$$Q_{L\|} = -Q_{R\|} \,, \tag{14g}$$

$$P_{\|L} = P_{\|R} \,, \qquad P_{L\|} = P_{R\|} \,. \tag{14h}$$

Likewise, equation (11) implies for the two electric field components

$$E_L = -\frac{\imath\omega_j}{ck^2}\left(k_{\|} B_L - \frac{k_\perp}{\sqrt{2}} B_{\|}\right) , \tag{15a}$$

$$E_R = -\frac{\imath\omega_j}{ck^2}\left(-k_{\|} B_R + \frac{k_\perp}{\sqrt{2}} B_{\|}\right) , \tag{15b}$$

allowing us to express the tensors $R^j_{\alpha\beta}(\boldsymbol{k})$, $Q^j_{\alpha\beta}(\boldsymbol{k})$, and $T^j_{\alpha\beta}(\boldsymbol{k})$ in terms of the magnetic field fluctuation tensor $P^j_{\alpha\beta}(\boldsymbol{k})$. In addition to equations (14), we obtain

$$\begin{aligned} R_{RR} &= \frac{\omega_j^2}{c^2k^4}\left[k_{\|}^2 P_{RR} + \frac{k_\perp^2}{2} P_{\|\|} - \frac{k_\perp k_{\|}}{\sqrt{2}} (P_{R\|} + P_{\|R})\right] \\ &= \frac{V_{\rm A}^2}{c^2}\left[\cos^2 \Theta P_{RR} + \frac{\sin^2 \Theta}{2} P_{\|\|} - \frac{\sin \Theta \cos \Theta}{\sqrt{2}} (P_{R\|} + P_{\|R})\right] , \end{aligned} \tag{16a}$$

$$Q_{RR} = \frac{\imath\omega_j}{ck^2}\left(k_{\|} P_{RR} - \frac{k_\perp}{\sqrt{2}} P_{\|R}\right) = \frac{\imath j V_{\rm A}}{c}\left[\cos(\Theta) P_{RR} - \frac{\sin\Theta}{\sqrt{2}} P_{\|R}\right] , \tag{16b}$$

$$T_{RR} = \frac{\imath\omega_j}{ck^2}\left(-k_{\|} P_{RR} + \frac{k_\perp}{\sqrt{2}} P_{R\|}\right) = \frac{\imath j V_{\rm A}}{c}\left[-\cos(\Theta) P_{RR} + \frac{\sin\Theta}{\sqrt{2}} P_{R\|}\right] , \tag{16c}$$

$$T_{\|R} = \frac{\imath\omega_j}{ck^2}\left(-k_{\|} P_{\|R} + \frac{k_\perp}{\sqrt{2}} P_{\|\|}\right) = \frac{\imath j V_{\rm A}}{c}\left[-\cos(\Theta) P_{\|R} + \frac{\sin\Theta}{\sqrt{2}} P_{\|\|}\right] , \tag{16d}$$

$$Q_{R\|} = \frac{\imath\omega_j}{ck^2}\left(k_{\|} P_{R\|} - \frac{k_\perp}{\sqrt{2}} P_{\|\|}\right) = \frac{\imath j V_{\rm A}}{c}\left[\cos(\Theta) P_{R\|} - \frac{\sin\Theta}{\sqrt{2}} P_{\|\|}\right] . \tag{16e}$$

2.3. *Fast-Mode Wave Fokker-Planck Coefficients*

Inserting equations (14) and (16) into equations (5), we obtain for the Fokker-Planck coefficients of fast-mode waves after straightforward algebra

$$\begin{aligned} D_{pp} &= \frac{2\pi\Omega^2 p^2 c^2 (1-\mu^2)}{B_0^2 v^2} \sum_{j=\pm 1} \sum_{n=-\infty}^{\infty} \int d^3k R^j_{RR}(\boldsymbol{k}) \delta(k_{\|} v\mu - jV_{\rm A} k + n\Omega)[J_n'(Z)]^2 \\ &= \frac{2\pi\Omega^2 p^2 V_{\rm A}^2 (1-\mu^2)}{B_0^2 v^2} \sum_{j=\pm 1} \sum_{n=-\infty}^{\infty} \int d^3k \delta(k_{\|} v\mu - jV_{\rm A} k + n\Omega)[J_n'(Z)]^2 \\ &\quad \times \left[\cos^2 \Theta P^j_{RR} + \frac{\sin^2 \Theta}{2} P^j_{\|\|} - \frac{\sin \Theta \cos \Theta}{\sqrt{2}} (P^j_{R\|} + P^j_{\|R})\right] , \end{aligned} \tag{17}$$

$$
\begin{aligned}
D_{\mu p} = {} & \frac{\pi\Omega^2(1-\mu^2)^{1/2}pc}{vB_0^2}\,\Re\imath \sum_{j=\pm 1}\sum_{n=-\infty}^{\infty}\int d^3k\,\delta(k_\parallel v\mu - jV_A k + n\Omega) \\
& \times\left(2(1-\mu^2)^{1/2}\left\{Q^j_{RR}(\boldsymbol{k})\,\frac{n^2J_n^2(Z)}{Z^2} + \imath\mu\,\frac{c}{v}\,R^j_{RR}(\boldsymbol{k})[J_n(Z)']^2\right\} + \sqrt{2}\,\mu Q^j_{R\parallel}(\boldsymbol{k})\,\frac{nJ_n^2(Z)}{Z}\right) \\
= {} & -\frac{\pi\Omega^2(1-\mu^2)^{1/2}pV_A}{vB_0^2}\sum_{j=\pm 1}\sum_{n=-\infty}^{\infty}\int d^3k\,\delta(k_\parallel v_\parallel - jV_A k + n\Omega) \\
& \times\left(2(1-\mu^2)^{1/2}\left\{\frac{jn^2J_n^2(Z)}{Z^2}\left[\cos(\Theta)P^j_{RR}(\boldsymbol{k}) - \frac{\sin\Theta}{\sqrt{2}}\,P^j_{\parallel R}(\boldsymbol{k})\right]\right.\right. \\
& \left. + \mu\,\frac{V_A}{v}\,[J_n(Z)']^2\left[\cos^2\Theta P_{RR} + \frac{\sin^2\Theta}{2}\,P_{\parallel\parallel} - \frac{\sin\Theta\cos\Theta}{\sqrt{2}}\,(P_{R\parallel} + P_{\parallel R})\right]\right\} \\
& \left. + \sqrt{2}\mu\,\frac{jnJ_n^2(Z)}{Z}\left[\cos(\Theta)P^j_{R\parallel}(\boldsymbol{k}) - \frac{\sin\Theta}{\sqrt{2}}\,P^j_{\parallel\parallel}(\boldsymbol{k})\right]\right),
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
D_{\mu\mu} = {} & \frac{\pi\Omega^2(1-\mu^2)}{2B_0^2}\sum_{j=\pm 1}\sum_{n=-\infty}^{\infty}\int d^3k\,\delta(k_\parallel v_\parallel - jV_A k + n\Omega) \\
& \times\left([J^2_{n+1}(Z) + J^2_{n-1}(Z)]\left\{P^j_{RR}(\boldsymbol{k}) + \mu^2\,\frac{c^2}{v^2}\,R^j_{RR}(\boldsymbol{k}) + \imath\mu\,\frac{c}{v}\,[T^j_{RR}(\boldsymbol{k}) - Q^j_{RR}(\boldsymbol{k})]\right\}\right. \\
& - 2J_{n-1}(Z)J_{n+1}(Z)\left\{P^j_{RR}(\boldsymbol{k}) + \mu^2\,\frac{c^2}{v^2}\,R^j_{RR}(\boldsymbol{k}) - \imath\mu\,\frac{c}{v}\,[T^j_{RR}(\boldsymbol{k}) - Q^j_{RR}(\boldsymbol{k})]\right\} \\
& \left. + \frac{\imath c\sqrt{2}\sqrt{1-\mu^2}}{v}\,J_n(Z)\{J_{n+1}(Z)[Q^j_{R\parallel}(\boldsymbol{k}) - T^j_{\parallel R}(\boldsymbol{k})] - J_{n-1}(Z)[Q^j_{R\parallel}(\boldsymbol{k}) + T^j_{\parallel R}(\boldsymbol{k})]\}\right) \\
= {} & \frac{2\pi\Omega^2(1-\mu^2)}{B_0^2}\sum_{j=\pm 1}\sum_{n=-\infty}^{\infty}\int d^3k\,\delta(k_\parallel v_\parallel - jV_A k + n\Omega) \\
& \times\left\{[J_n'(Z)]^2\left[P^j_{RR}(\boldsymbol{k}) + \mu^2\,\frac{c^2}{v^2}\,R^j_{RR}(\boldsymbol{k})\right] + \imath\mu\,\frac{c}{v}\,\frac{n^2J_n^2(Z)}{Z^2}\,[T^j_{RR}(\boldsymbol{k}) - Q^j_{RR}(\boldsymbol{k})]\right. \\
& \left. - \frac{\imath c\sqrt{1-\mu^2}}{\sqrt{2}\,v}\left[Q^j_{R\parallel}(\boldsymbol{k})J_n(Z)J_n'(Z) + T^j_{\parallel R}(\boldsymbol{k})\,\frac{nJ_n^2(Z)}{Z}\right]\right\} \\
= {} & \frac{2\pi\Omega^2(1-\mu^2)}{B_0^2}\sum_{j=\pm 1}\sum_{n=-\infty}^{\infty}\int d^3k\,\delta(k_\parallel v_\parallel - jV_A k + n\Omega)\left[[J_n'(Z)]^2\left(\left(1 + \frac{\mu^2V_A^2\cos^2\Theta}{v^2}\right)P^j_{RR}(\boldsymbol{k})\right.\right. \\
& \left. + \frac{\mu^2V_A^2\sin\Theta}{v^2}\left\{\frac{\sin\Theta}{2}\,P^j_{\parallel\parallel}(\boldsymbol{k}) - \frac{\cos\Theta}{\sqrt{2}}\,[P^j_{R\parallel}(\boldsymbol{k}) + P^j_{\parallel R}(\boldsymbol{k})]\right\}\right) \\
& + \frac{j\mu V_A}{v}\,\frac{n^2J_n^2(Z)}{Z^2}\left\{2\cos\Theta P^j_{RR}(\boldsymbol{k}) - \frac{\sin\Theta}{\sqrt{2}}\,[P^j_{R\parallel}(\boldsymbol{k}) + P^j_{\parallel R}(\boldsymbol{k})]\right\} \\
& \left. + \frac{jV_A\sqrt{1-\mu^2}}{\sqrt{2}\,v}\left\{\cos\Theta\left[P^j_{R\parallel}(\boldsymbol{k})J_n(Z)J_n'(Z) - P^j_{\parallel R}(\boldsymbol{k})\,\frac{nJ_n^2(Z)}{Z}\right] + \frac{\sin\Theta}{\sqrt{2}}\,P^j_{\parallel\parallel}(\boldsymbol{k})\left[\frac{nJ_n^2(Z)}{Z} - J_n(Z)J_n'(Z)\right]\right\}\right],
\end{aligned}
\tag{19}
$$

where

$$
Z \equiv \frac{k_\perp v_\perp}{|\Omega|}\,, \qquad J_n(Z)' \equiv \frac{dJ_n(Z)}{dZ}\,.
\tag{20}
$$

Equations (17)–(19) represent the exact Fokker-Planck coefficients for fast-mode waves, and they have been expressed in terms of the (measurable) components of the magnetic fluctuation correlation tensor.

For very fast particles $v \gg V_A$, we may expand the general expressions for $D_{\mu\mu}$ (eq. [19]) and $D_{\mu p}$ (eq. [18]) to first order in the small parameter $V_A/v \ll 1$, to obtain approximately

$$
\begin{aligned}
D_{\mu p}(v \gg V_A) \simeq {} & -\frac{2\pi\Omega^2(1-\mu^2)pV_A}{vB_0^2}\sum_{j=\pm 1}\sum_{n=-\infty}^{\infty}\int d^3k\,\delta(k_\parallel v\mu - jV_A k + n\Omega) \\
& \times\left\{\frac{jn^2J_n^2(Z)}{Z^2}\left[\cos(\Theta)P^j_{RR}(\boldsymbol{k}) - \frac{\sin\Theta}{\sqrt{2}}\,P^j_{\parallel R}(\boldsymbol{k})\right]\right\}
\end{aligned}
$$

$$+\frac{\mu}{\sqrt{2(1-\mu^2)}}\frac{jnJ_n^2(Z)}{Z}\left[\cos(\Theta)P^j_{R\parallel}(\boldsymbol{k})-\frac{\sin\Theta}{\sqrt{2}}P^j_{\parallel\parallel}(\boldsymbol{k})\right], \tag{21}$$

and

$$D_{\mu\mu}(v \gg V_{\rm A}) \simeq \frac{2\pi\Omega^2(1-\mu^2)}{B_0^2}\sum_{j=\pm 1}\sum_{n=-\infty}^{\infty}\int d^3k\,\delta(k_\parallel v\mu - jV_{\rm A}k + n\Omega)[J_n'(Z)]^2 P^j_{RR}(\boldsymbol{k}). \tag{22}$$

2.4. *Illustrative Example: Isotropic Magnetic Turbulence*

For the quantitative analysis of the Fokker-Planck coefficients, we must specify the magnetic turbulence tensor $P^j_{\rm mr}(\boldsymbol{k})$. Although this model is not in accord with the known polarization and damping properties of fast-mode waves at oblique angles (Tademaru 1969; Lee & Völk 1975; Jaekel & Schlickeiser 1992), mainly for illustrative reasons we adopt throughout this work the isotropic magnetic turbulence tensor (Batchelor 1953)

$$P^j_{\rm mr}(\boldsymbol{k}) = \frac{g^j(k)}{8\pi k^2}\left(\delta_{\rm mr} - \frac{k_m k_r}{k^2}\right) = \frac{g^j(k)}{8\pi k^2}\begin{pmatrix}\cos^2\Theta & 0 & -\sin\Theta\cos\Theta \\ 0 & 1 & 0 \\ -\sin\Theta\cos\Theta & 0 & \sin^2\Theta\end{pmatrix} \tag{23}$$

where we used equation (8). The magnetic energy density in wave component j is given by

$$(\delta B_j)^2 = \int d^3k[P^j_{11}(\boldsymbol{k}) + P^j_{22}(\boldsymbol{k}) + P^j_{33}(\boldsymbol{k})] = \int d^3k\,\frac{g^j(k)}{4\pi k^2} = \int_0^\infty dk g^j(k). \tag{24}$$

We also adopt a Kolmogorov-like power-law dependence (index $q > 1$) of g^j above some minimum wavenumber $k_{\rm min}$, i.e.

$$g^j(k) = g_0^j k^{-q} \quad \text{for } k > k_{\rm min}. \tag{25}$$

The normalization (eq. [24]) then implies

$$g_0^j = (q-1)(\delta B_j)^2 k_{\rm min}^{q-1}. \tag{26}$$

3. TRANSIT-TIME DAMPING AND GYRORESONANCE INTERACTION OF RELATIVISTIC PARTICLES

For ease of exposition, we will discuss the role of transit-time damping and gyroresonance interaction for cosmic ray transport and acceleration in the case of very fast particles with velocities much larger than the Alfvén velocity, i.e. $v \gg V_{\rm A}$, so that in addition to equation (17), the approximated Fokker-Planck coefficients of equations (21) and (22) apply. With equations (23) for isotropic magnetic turbulence, we obtain with $\eta = \cos\Theta$

$$D_{\mu\mu} = \frac{\pi\Omega^2(1-\mu^2)}{4B_0^2}\sum_{j=\pm1}\sum_{n=-\infty}^{\infty}\int_{-1}^{1}d\eta\int_0^\infty dk g^j(k)(1+\eta^2)\delta(kv\mu\eta - jV_{\rm A}k + n\Omega)\left[J_n'\left(\frac{kv_\perp\sqrt{1-\eta^2}}{|\Omega|}\right)\right]^2, \tag{27}$$

$$D_{\mu p} = \frac{\pi\Omega^2(1-\mu^2)pV_{\rm A}}{2vB_0^2}\sum_{j=\pm1}j\sum_{n=-\infty}^{\infty}\int_{-1}^{1}d\eta\int_0^\infty dk g^j(k)\delta(kv\mu\eta - jV_{\rm A}k + n\Omega)\,\frac{n|\Omega|}{kv_\perp\sqrt{1-\eta^2}}\,J_n^2\left(\frac{kv_\perp\sqrt{1-\eta^2}}{|\Omega|}\right)$$
$$\times\left(\frac{\mu\sqrt{1-\eta^2}}{2\sqrt{1-\mu^2}} - \frac{n|\Omega|\eta}{kv_\perp\sqrt{1-\eta^2}}\right), \tag{28}$$

and

$$D_{pp} = \frac{\pi\Omega^2p^2V_{\rm A}^2(1-\mu^2)}{4B_0^2v^2}\sum_{j=\pm1}\sum_{n=-\infty}^{\infty}\int_{-1}^{1}d\eta\int_0^\infty dk g^j(k)(1+\eta^2)\delta(kv\mu\eta - jV_{\rm A}k + n\Omega)\left[J_n'\left(\frac{kv_\perp\sqrt{1-\eta^2}}{|\Omega|}\right)\right]^2 = \frac{p^2V_{\rm A}^2}{v^2}D_{\mu\mu}. \tag{29}$$

Obviously, we can express equations (27) and (29) as the sum of two contributions

$$D_{\mu\mu} = \frac{\pi\Omega^2(1-\mu^2)}{4B_0^2}\sum_{j=\pm1}(D_T^j + D_G^j), \tag{30a}$$

$$D_{pp} = \frac{\pi\Omega^2p^2V_{\rm A}^2(1-\mu^2)}{4v^2B_0^2}\sum_{j=\pm1}(D_T^j + D_G^j) = \frac{p^2V_{\rm A}^2}{v^2}D_{\mu\mu}. \tag{30b}$$

The first term,

$$D_T^j \equiv \int_{-1}^{1}d\eta\int_0^\infty dk g^j(k)(1+\eta^2)\delta[k(v\mu\eta - jV_{\rm A})]\left[J_0'\left(\frac{kv_\perp\sqrt{1-\eta^2}}{|\Omega|}\right)\right]^2$$
$$= \frac{1}{|v\mu|}\int_0^\infty dk g^j(k)k^{-1}\int_{-1}^{1}d\eta(1+\eta^2)\delta\left(\eta - \frac{jV_{\rm A}}{v\mu}\right)J_1^2\left(\frac{kv_\perp\sqrt{1-\eta^2}}{|\Omega|}\right)$$
$$= \frac{H(|\mu| - V_{\rm A}/v)[1 + (V_{\rm A}^2/v^2\mu^2)]}{|v\mu|}\int_0^\infty dk g^j(k)k^{-1}J_1^2\left(\frac{kv_\perp\sqrt{1 - V_{\rm A}^2/v^2\mu^2}}{|\Omega|}\right), \tag{31}$$

represents the $n = 0$ contribution from transit-time acceleration, and

$$\begin{aligned} D_G^j &\equiv \sum_{n=-\infty, n\neq 0}^{\infty} \int_{-1}^{1} d\eta \int_0^\infty dk g^j(k)(1+\eta^2)\delta[k(v\mu\eta - jV_A) + n\Omega]\left[J_n'\left(\frac{kv_\perp\sqrt{1-\eta^2}}{|\Omega|}\right)\right]^2 \\ &= \sum_{n=1}^{\infty} \int_{-1}^{1} d\eta \, \frac{1+\eta^2}{|v\mu\eta - jV_A|} \int_0^\infty dk g^j(k)\left[J_n'\left(\frac{kv_\perp\sqrt{1-\eta^2}}{|\Omega|}\right)\right]^2\left[\delta\left(k + \frac{n\Omega}{v\mu\eta - jV_A}\right) + \delta\left(k - \frac{n\Omega}{v\mu\eta - jV_A}\right)\right] \\ &= \frac{1}{|v\mu|}\sum_{n=1}^{\infty} \int_{-1}^{1} d\eta \, \frac{1+\eta^2}{|\eta - (jV_A/v\mu)|}\left\{J_n'\left[\frac{n\sqrt{(1-\mu^2)(1-\eta^2)}}{|\mu\eta - (jV_A/v)|}\right]\right\}^2 \\ &\quad \times \left\{H\left[\frac{sqn(\Omega)}{\eta - (jV_A/v\mu)}\right]g^j\left(k = \frac{n\Omega}{v\mu\eta - jV_A}\right) + H\left[\frac{sqn(\Omega)}{(jV_A/v\mu) - \eta}\right]g^j\left(k = \frac{n\Omega}{jV_A - v\mu\eta}\right)\right\} \end{aligned} \tag{32}$$

represents the contribution from gyroresonant interactions with $n \geq 1$. The function $H(x) = 1(=0)$ for $x \geq 0(<0)$ in equations (31) and (32) denotes Heaviside's step function. While equation (30b) is exact, equation (30a) strictly is valid only for energetic particles with speeds much in excess of the Alfvén speed, $v \gg V_A$. In contrast to $D_{\mu\mu}$ and D_{pp}, the Fokker-Planck coefficient $D_{\mu p}$, according to equation (28), has no $n = 0$ transit-time damping contribution and after obvious resumming reduces to

$$D_{\mu p} = -\frac{\pi\Omega^2 p V_A}{2v^2 B_0^2} \sum_{j=\pm 1} j \int_{-1}^{1} d\eta\left(\frac{\mu}{2} + \frac{\eta}{1-\eta^2}\left|\mu\eta - j\frac{V_A}{v}\right|\right) \sum_{n=1}^{\infty} J_n^2\left[\frac{n\sqrt{(1-\mu^2)(1-\eta^2)}}{|\mu\eta - j(V_A/v)|}\right] g^j\left(\left|\frac{n\Omega}{v\mu\eta - jV_A}\right|\right) . \tag{33}$$

3.1. *Vanishing Cross Helicity*

We show in Appendix A that in the important case of a vanishing cross helicity of the plasma waves, i.e. equal intensity of forward and backward waves,

$$g^+(k) = g^-(k) = \tfrac{1}{2} g_{\text{tot}}(k) = \tfrac{1}{2} g_0 \, k^{-q} H(k - k_{\min}) , \tag{34}$$

the rate of adiabatic deceleration (eq. [4b]),

$$a_1 \simeq 0 , \tag{35}$$

is vanishing small, and the momentum diffusion coefficient (eq. [4c]) reduces to

$$a_2 = \tfrac{1}{2} \int_{-1}^{1} d\mu \, D_{pp}(\mu) , \tag{36}$$

which simplifies the analysis enormously. Moreover, in the vanishing cross helicity case of equation (34), both Fokker-Planck coefficients $D_{\mu\mu}$ and D_{pp} are symmetric functions in μ,

$$D_{\mu\mu}(-\mu) = D_{\mu\mu}(\mu) , \qquad D_{pp}(-\mu) = D_{pp}(\mu) . \tag{37}$$

Adopting the Kolmogorov-like turbulence spectrum (eq. [25]), we then obtain from equation (31) for the transit-time damping contribution

$$\begin{aligned} \sum_j D_T^j(\mu) &= g_0 \frac{H(|\mu| - \epsilon)[1 + (\epsilon^2/\mu^2)]}{|v\mu|} \int_{k_{\min}}^\infty dk k^{-1-q} J_1^2\left[kR_L\sqrt{(1-\mu^2)\left(1 - \frac{\epsilon^2}{\mu^2}\right)}\right] \\ &= (q-1)(\delta B)^2 \frac{R_L}{v}(R_L k_{\min})^{q-1} f_T(\mu) , \end{aligned} \tag{38}$$

where we used the normalization of equation (26) and introduced the transit-time damping function

$$f_T(\mu) \equiv H(|\mu| - \epsilon)\frac{1 + (\epsilon^2/\mu^2)}{|\mu|}\left[(1-\mu^2)\left(1 - \frac{\epsilon^2}{\mu^2}\right)\right]^{q/2} \int_U^\infty ds s^{-(1+q)} J_1^2(s) , \tag{39}$$

where the lower integration boundary is

$$U = k_{\min} R_L \sqrt{(1-\mu^2)(1-\epsilon^2\mu^{-2})} , \tag{40}$$

and where we introduced

$$R_L = \frac{v}{|\Omega|} , \qquad \epsilon = \frac{V_A}{v} \ll 1 . \tag{41}$$

Moreover, we obtain from equation (32) for the gyroresonance contributions

$$\sum_j D_G^j = \frac{1}{v}\sum_{n=1}^{\infty} \int_0^1 d\eta(1+\eta^2)[G_1(|\mu\eta - \epsilon|) + G_1(|\mu\eta + \epsilon|)] , \tag{42}$$

with

$$G_1(x) \equiv x^{-1} \sum_j g^j\left(\frac{n}{R_L x}\right)\left\{J'_n\left[\frac{n\sqrt{(1-\mu^2)(1-\eta^2)}}{x}\right]\right\}^2 H(1 - R_L x k_{\min})$$
$$= g_0 n^{-q} R_L^q x^{q-1}\left\{J'_n\left[\frac{n\sqrt{(1-\mu^2)(1-\eta^2)}}{x}\right]\right\}^2 H(1 - R_L x k_{\min}) . \quad (43)$$

Throughout this analysis we will neglect cutoff effects, i.e., $k_{\min} R_L \ll 1$. For equation (42) we then obtain

$$\sum_j D_G^j = \frac{g_0}{v} R_L^q f_G(\mu) = (q-1)(\delta B)^2 \frac{R_L}{v} (R_L k_{\min})^{q-1} f_G(\mu) , \quad (44)$$

with the gyroresonance function

$$f_G(\mu) \equiv \sum_{n=1}^{\infty} n^{-q} \int_0^1 d\eta (1+\eta^2)\left(|\mu\eta - \epsilon|^{q-1}\left\{J'_n\left[\frac{n\sqrt{(1-\mu^2)(1-\eta^2)}}{|\mu\eta - \epsilon|}\right]\right\}^2 + |\mu\eta + \epsilon|^{q-1}\left\{J'_n\left[\frac{n\sqrt{(1-\mu^2)(1-\eta^2)}}{|\mu\eta + \epsilon|}\right]\right\}^2\right) . \quad (45)$$

Using equations (38) and (44), we obtain for the Fokker-Planck coefficients of equation (30)

$$D_{\mu\mu} = \frac{\pi|\Omega|(q-1)(1-\mu^2)}{4} \frac{(\delta B)^2}{B_0^2} (k_{\min} R_L)^{q-1}[f_T(\mu) + f_G(\mu)] , \quad (46a)$$

$$D_{pp} = p^2\epsilon^2 D_{\mu\mu} . \quad (46b)$$

With the symmetry relations (eq. [37]) and equations (4) and (36), we obtain from equations (46) for the mean free path and the momentum diffusion coefficient

$$\lambda = \frac{3}{\pi(q-1)} R_L (R_L k_{\min})^{1-q}\left(\frac{B_0}{\delta B}\right)^2 I_\lambda , \quad (47)$$

and

$$a_2 = \int_0^1 d\mu\, D_{pp}(\mu) = \frac{\pi(q-1)}{4} |\Omega| (R_L k_{\min})^{q-1}\left(\frac{\delta B}{B_0}\right)^2 \frac{V_A^2 p^2}{v^2} I_2 , \quad (48)$$

involving the two integrals

$$I_\lambda = \int_0^1 d\mu \frac{1-\mu^2}{f_T(\mu) + f_G(\mu)} \quad (49)$$

and

$$I_2 = \int_0^1 d\mu (1-\mu^2)[f_T(\mu) + f_G(\mu)] . \quad (50)$$

With equations (4a), (47), and (48) we obtain for the product of the spatial and momentum diffusion coefficients, related to the product of the related diffusive escape and acceleration timescales (see Schlickeiser 1989),

$$\kappa a_2 = \left(\frac{V_A}{2}\right)^2 p^2 I_\lambda I_2 . \quad (51)$$

Since the transit-time damping function f_T vanishes for $\mu \le \epsilon$ (see eq. [39]), the two integrals (49) and (50) obviously can be separated as

$$I_\lambda = \int_0^\epsilon d\mu \frac{1-\mu^2}{f_G(\mu)} + \int_\epsilon^1 d\mu \frac{1-\mu^2}{f_T(\mu) + f_G(\mu)} \quad (52)$$

and

$$I_2 = \int_0^1 d\mu (1-\mu^2) f_G(\mu) + \int_\epsilon^1 d\mu (1-\mu^2) f_T(\mu) . \quad (53)$$

In Appendices B and C, we investigate in detail the variation of the functions $f_T(\mu)$ and $f_G(\mu)$ in the two pitch-angle intervals $0 \le \mu \le \epsilon$ and $\epsilon < \mu \le 1$. The transit-time damping function f_T is nonzero only in the interval $\mu > \epsilon$. According to equation (B7), it can be approximated as

$$f_T(1 \le x \le \epsilon^{-2}, q \le 2) = \frac{c_1(q)}{\epsilon} \frac{1+x}{x^{(3+q)/2}} [(1+\epsilon^2)x - 1 - \epsilon^2 x^2]^{q/2} \quad (54)$$

for flat turbulence spectra, where $x = (\mu/\epsilon)^2$, $c_1(2) = \frac{3}{4}$, and

$$c_1(q < 2) = 2^{1-q} \frac{q}{4 - q^2} \frac{\Gamma(q)\Gamma(2 - q/2)}{\Gamma^3(1 + q/2)} . \tag{55}$$

According to equation (B25), for steeper turbulence spectra we find

$$f_T(q > 2, k_{\min} R_L \ll 1) = c_4(q)(R_L k_{\min})^{2-q} H(|\mu| - \epsilon) \frac{(1 - \mu^2)[1 - (\epsilon^4/\mu^4)]}{|\mu|} , \tag{56}$$

where

$$c_4(q) = \frac{2q^2 - 3q + 4}{4q(2q - 3)} . \tag{57}$$

For the gyroresonance function $f_G(\mu)$, we obtain according to equations (C9) and (C13) for all turbulence spectra the approximation

$$f_G(\mu) \simeq \begin{cases} \dfrac{3\zeta(q+1)\epsilon^q}{2\sqrt{1-\mu^2}} & \text{for } 0 \le \mu \le \sqrt{2}\,\epsilon \\ \dfrac{3\zeta(q+1)\epsilon^q}{2\sqrt{1-\mu^2}} \left[1 + \dfrac{4(3+2q)}{3(1+q)} \left(\dfrac{\mu}{\epsilon}\right)^q\right] & \text{for } \sqrt{2}\,\epsilon < \mu \le 2^{-1/2} , \\ \dfrac{\zeta(q)\mu^{q-1}}{2q} & \text{for } 2^{-1/2} < \mu \le 1 \end{cases} \tag{58}$$

where $\zeta(q)$ denotes Riemann's zeta function. In Figures 1–4, we numerically calculate the transit-time damping function f_T (*dashed curves*) and the gyroresonance function f_G (*solid curves*) for $\epsilon = 0.01$, $q = 1.5$, and $q = 5/3$ as a function of μ using the approximations of equations (54) and (58). Whereas Figures 1 and 3 show the variation over the whole $0 \le \mu \le 1$ interval, Figures 2 and 4 highlight the variation for small $\mu \le 5\epsilon$. First, we note from Figures 1 and 3 that the maximum value of f_G is much less than the maximum value of f_T, in agreement with equations (B14) and (C14). Second, Figures 2 and 4 indicate that in the interval $|\mu| < \epsilon$, where no transit-time damping occurs, the gyroresonant interactions f_G provide a small (of order ϵ^q; see eq. [58]) but finite contribution to the scattering rate of particles. These properties have profound consequences for the behavior of the two integrals (eqs. [52] and [53]) that we discuss next.

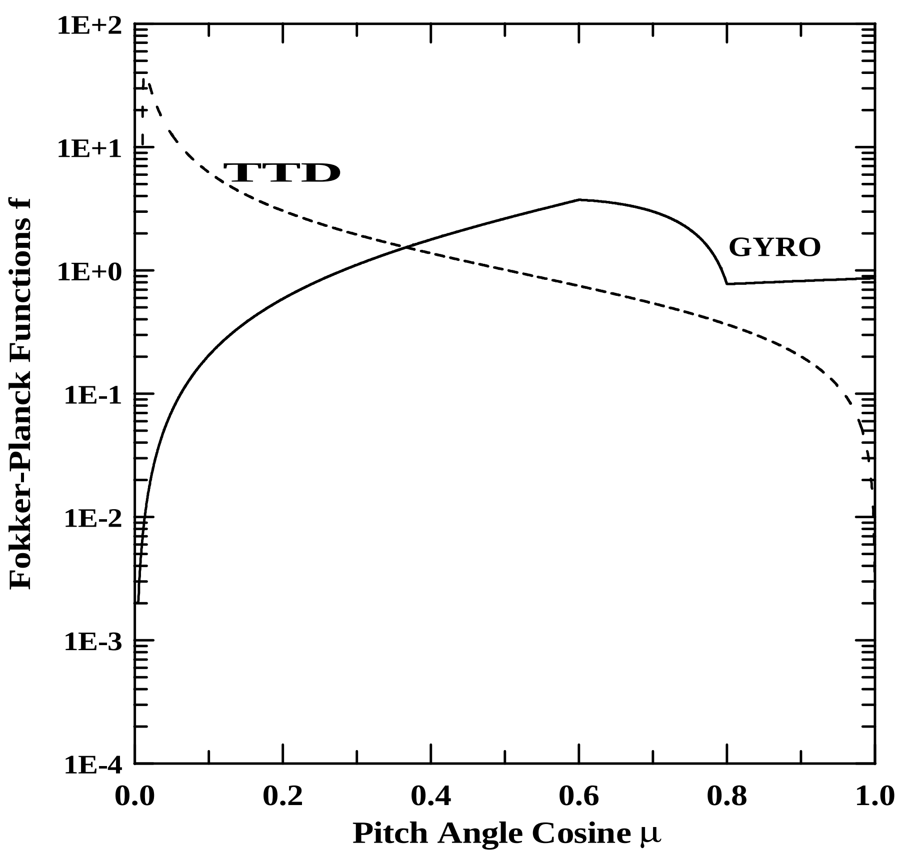

FIG. 1.—The gyroresonance function f_G (*solid curve*) and the transit-time damping function f_T (*dashed curve*) as a function of the pitch-angle cosine for $\epsilon = V_A/v = 0.01$ and $q = 1.5$.

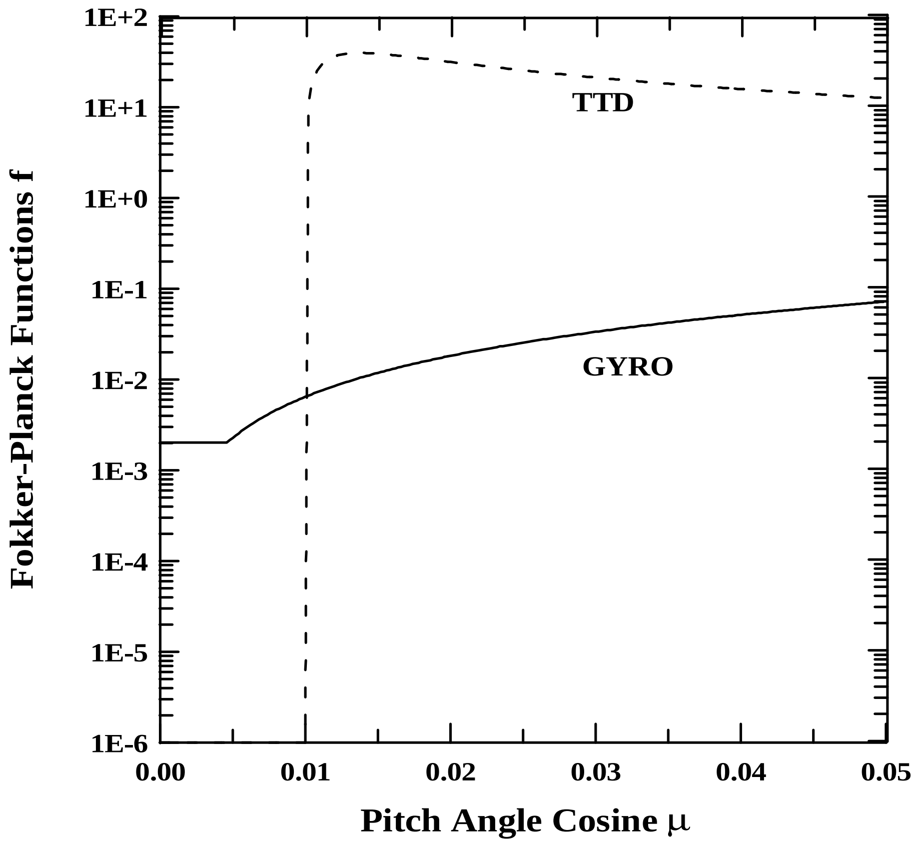

FIG. 2.—Same as Fig. 1, but for the small interval $0 \leq \mu \leq 0.05$

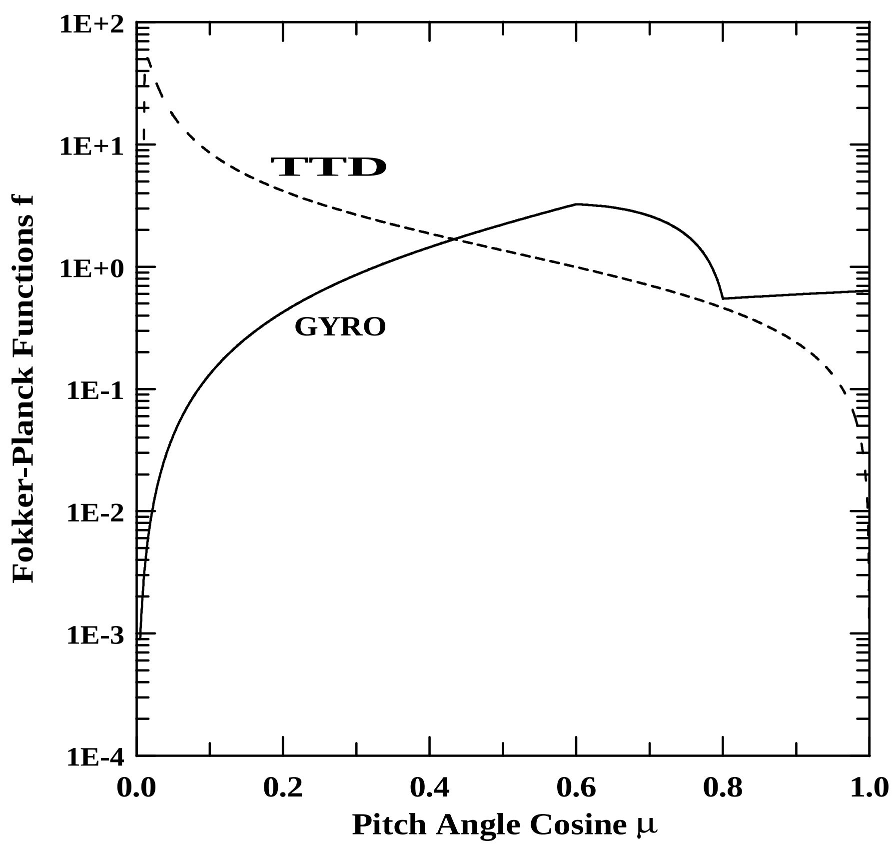

FIG. 3.—The gyroresonance function f_G (*solid curve*) and the transit-time damping function f_T (*dashed curve*) as a function of the pitch-angle cosine for $\epsilon = V_A/v = 0.01$ and $q = 5/3$.

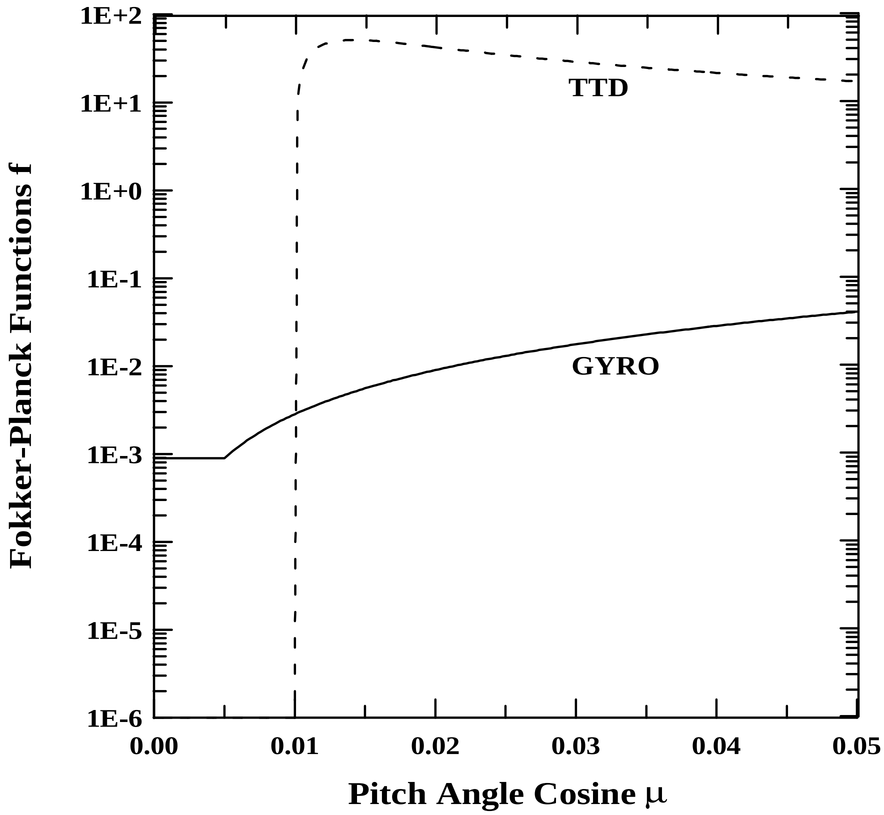

FIG. 4.—Same as Fig. 3, but for the small interval $0 \le \mu \le 0.05$

3.2. *Calculation of the Integrals I_λ and I_2*

We are now in a position to calculate the integrals given in equations (52) and (53) under the assumption that no other than the fast-mode waves interact with the cosmic ray particles.

3.2.1. *The Integral I_λ for Flat Turbulence Spectra $q \le 2$*

Using equation (58) we obtain for the first integral in equation (52)

$$\int_0^\epsilon d\mu \, \frac{1-\mu^2}{f_G(\mu)} \simeq \frac{2}{3\zeta(q+1)} \, \epsilon^{1-q} \, , \tag{59}$$

which is much larger than unity since $\epsilon \ll 1$ and $q > 1$. The second integral in equation (52) can be approximated as

$$\int_\epsilon^1 d\mu \, \frac{1-\mu^2}{f_T(\mu) + f_G(\mu)} \simeq \int_\epsilon^{\mu_T} d\mu \, \frac{1-\mu^2}{f_T(\mu)} + \int_{\mu_T}^1 d\mu \, \frac{1-\mu^2}{f_G(\mu)} \, , \tag{60}$$

where μ_T is that pitch-angle cosine for which

$$f_T(\mu_T) = f_G(\mu_T) \, . \tag{61}$$

As the inspection of Figures 1 and 3 shows, μ_T is much larger than ϵ, in particular much larger than μ_M where f_T has its maximum. Using the approximation (eq. [54]) for f_T for values of $q \le 2$ and $s \gg s_M$, that is,

$$f_T(s_M \ll s) \simeq \frac{c_1(q)}{\epsilon} \, s^{-1/2}(1 - \epsilon^2 s)^{q/2} \, , \tag{62}$$

and expressing the approximations (eq. [58]) beyond μ_M in terms of the variable $s = x - 1 = (\mu/\epsilon)^2 - 1$, that is,

$$f_G(s) \simeq \frac{2(3+2q)\zeta(q+1)\epsilon^q}{1+q} \, \frac{s^{q/2}}{\sqrt{1-\epsilon^2 s}} \, , \tag{63}$$

equation (61) readily yields (for $\mu_T < 2^{-1/2}$)

$$\mu_T(q) = \sqrt{\frac{c_5(q)}{1 + c_5(q)}} \, , \tag{64}$$

where

$$c_5(q) = \left[\frac{c_1(q)(1+q)}{2(3+2q)\zeta(q+1)}\right]^{2/(q+1)} . \tag{65}$$

The value of μ_T is solely determined by the value of q and independent of the value of ϵ. For the first integral in equation (60) we derive

$$I_T \equiv \int_\epsilon^{\mu_T} d\mu\, \frac{1-\mu^2}{f_T(\mu)} = \frac{1}{c_1(q)} \int_\epsilon^{\mu_T} d\mu\, \frac{\mu(1-\mu^2)^{(2-q)/2}}{(1+\epsilon^2\mu^{-2})(1-\epsilon^2\mu^{-2})^{q/2}} = \frac{2\epsilon^2}{c_1(q)} \int_1^{(\mu_T/\epsilon)^2} dx\, \frac{x^{(2+q)/2}(1-\epsilon^2 x)^{(2-q)/2}}{(x+1)(x-1)^{q/2}} , \tag{66}$$

after obvious substitution. An upper bound to the integral (eq. [66]) is given by

$$\begin{aligned} I_T < I_0 &= \frac{2\epsilon^2}{c_1(q)} \int_1^{(\mu_T/\epsilon)^2} dx\, \frac{x^{(2+q)/2}}{(x+1)(x-1)^{q/2}} \\ &\simeq \frac{2\epsilon^2}{c_1(q)} \left[\left(\frac{\mu_T}{\epsilon}\right)^2 - 4\ln\left(\frac{\mu_T}{\epsilon}\right) + \ln 9 + \frac{2q-3}{2-q}\right] \simeq \frac{2\mu_T^2}{c_1(q)} = \frac{2c_5(q)}{c_1(q)[1+c_5(q)]} , \end{aligned} \tag{67}$$

which is of order unity. For the second integral in equation (60) we derive with approximations (eq. [58])

$$\begin{aligned} I_G \equiv \int_{\mu_T}^1 d\mu\, \frac{1-\mu^2}{f_G(\mu)} &\simeq \frac{1+q}{2(3+2q)\zeta(q+1)} \int_{\mu_t}^{2^{-1/2}} d\mu \mu^{-q}(1-\mu^2)^{3/2} + \frac{2q}{\zeta(q)} \int_{2^{-1/2}}^1 d\mu \mu^{1-q}(1-\mu^2) \\ &\simeq \frac{1+q}{2(q-1)(3+2q)\zeta(q+1)} [\mu_T^{1-q} - 2^{(q-1)/2}] + \frac{4q}{(2-q)(4-q)\zeta(q)} [1-(6-q)2^{(6+q)/2}] , \end{aligned} \tag{68}$$

which also is of order unity, since it is only determined by the value of q.

Collecting terms in the integral (eq. [52]), we find that both contributions (eqs. [67] and [68]) are negligibly small compared to the contribution from equation (59), so that the integral (eq. [52]) for flat turbulence spectra $q \le 2$ becomes

$$I_\lambda(q \le 2) \simeq \frac{2}{3\zeta(q+1)}\, \epsilon^{1-q} . \tag{69}$$

The result (eq. [69]) can also be reproduced from Figure 5, where we present the variation of the integrand of integral (eq. [52]) as a function of μ. The presence of massive scattering due to transit-time interactions of particles with $|\mu| > \epsilon$ leads to a vanishingly small value of the integrand at these pitch angles, so that the integral is solely determined by the integrand's values at small pitch angles $|\mu| \le \epsilon$, in agreement with equation (69).

We thus have shown that for the quasi-linear interaction with fast-mode waves with a flat power-law turbulence spectrum ($q \le 2$), the mean free path λ and the associated spatial diffusion coefficient κ are solely determined by the scattering rate of particles in the interval $|\mu| < \epsilon$. In this interval, transit-time damping does not contribute to the pitch-angle scattering, which instead solely relies on gyroresonant interactions with $|n| \ge 1$. The small gyroresonance scattering in this interval is crucial to have quasi-linear scattering through $\mu = 0$ (and to establish an isotropic particle distribution function), and therefore determines the quantitative value of the spatial diffusion coefficient.

3.2.2. *The Integral I_λ for Steep Turbulence Spectra* $2 < q < 6$

Repeating the analysis for steep power spectra, we must use equation (56) instead of equation (54) for the transit-time damping function in equations (61) and (66), respectively. Except near the endpoints $\mu \simeq \epsilon$ and $\mu \simeq 1$, the transit-time damping function (eq. [56]) is much larger than the gyroresonance function (eq. [58]), so we obtain for the second integral in equation (52)

$$\int_\epsilon^1 d\mu\, \frac{1-\mu^2}{f_T(\mu)+f_G(\mu)} \simeq \int_{\epsilon(1+x_1)}^{1-x_2} d\mu\, \frac{1-\mu^2}{f_T(\mu)} + \int_\epsilon^{\epsilon(1+x_1)} d\mu\, \frac{1-\mu^2}{f_G(\mu)} + \int_{1-x_2}^1 d\mu\, \frac{1-\mu^2}{f_G(\mu)} , \tag{70}$$

where

$$x_1 \simeq \frac{3}{8}\frac{\zeta(q+1)}{c_4(q)}\, \epsilon^{q+1}(R_{\rm L}\, k_{\rm min})^{q-2} \ll 1 \tag{71}$$

and

$$x_2 \simeq \frac{\zeta(q)}{4qc_4(q)}\, (R_{\rm L}\, k_{\rm min})^{q-2} \ll 1 \tag{72}$$

are determined by the conditions

$$f_T(\epsilon(1+x_1)) = f_G(\epsilon(1+x_1)) , \qquad f_T(1-x_2) = f_G(1-x_2) . \tag{73}$$

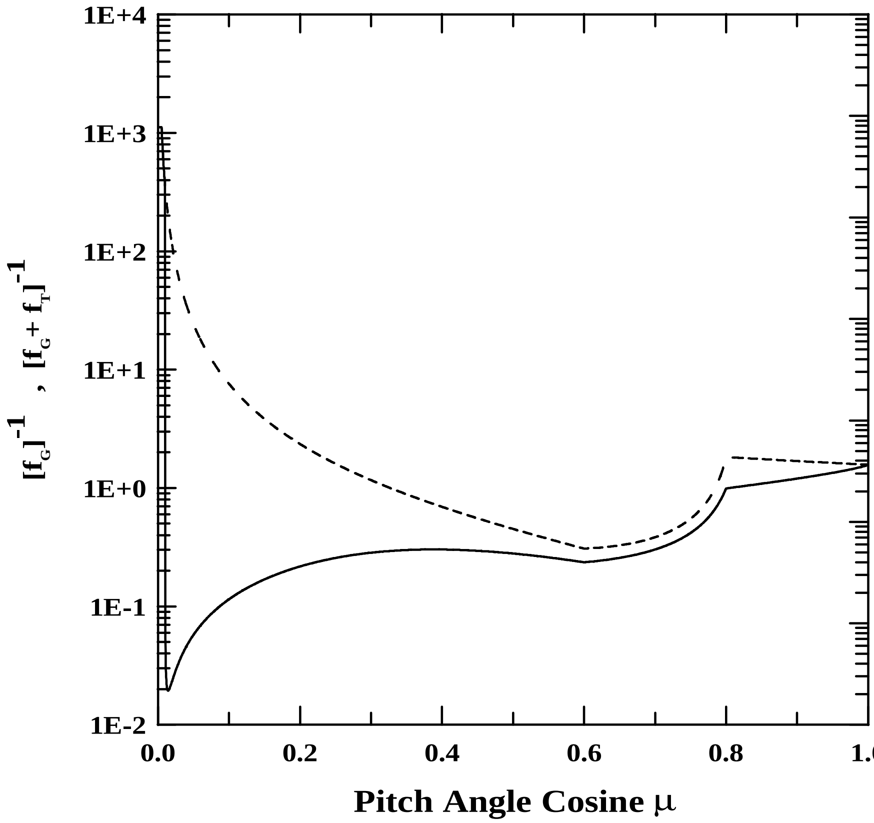

FIG. 5.—Inverse of the gyroresonance function f_G^{-1} (*dashed curve*) and the inverse of the sum $(f_T+f_G)^{-1}$ (*solid curve*) for $\epsilon = V_A/v = 0.01$ and $q = 5/3$

Performing the integrals, equation (70) becomes

$$\int_\epsilon^1 d\mu\,\frac{1-\mu^2}{f_T(\mu)+f_G(\mu)} \simeq \frac{(R_L\,k_{\min})^{q-2}}{c_4(q)}\left\{\frac{1}{2}\,[(1-x_2)^2-\epsilon^2(1+x_1)^2] + \frac{\epsilon^4}{4}\left[\ln\frac{(1-x_2)^2-\epsilon^2}{(1-x_2)^2+\epsilon^2} - \ln\frac{(1+x_1)^2-1}{(1+x_1)^2+1}\right] + \frac{\epsilon^2}{4c_4(q)} + \frac{\zeta(q)}{8qc_4(q)}\,(R_L\,k_{\min})^{q-2}\right\} \simeq \frac{(R_L\,k_{\min})^{q-2}}{2c_4(q)}\,, \tag{74}$$

where the last approximation holds to lowest order in $\epsilon \ll 1$. Compared with the contribution (eq. [59]), which also holds in the case of steep turbulence spectra, we find that equation (74) is negligibly small, and we reproduce equation (69) also for steep turbulence spectra,

$$I_\lambda(q>2) \simeq \frac{2}{3\zeta(q+1)}\,\epsilon^{1-q}\,. \tag{75}$$

3.2.3. *The Integral I_2*

The integral (eq. [53]) is easier to evaluate since the contributions simply add. With equations (58), we obtain for the gyroresonance contribution

$$H_G \equiv \int_0^1 d\mu(1-\mu^2)f_G(\mu) \simeq \frac{3(\pi+2)}{16}\,\zeta(q+1)\epsilon^q + \frac{\zeta(q)}{q^2(2+q)}\,[1-(q+4)2^{-(4+q)/2}] + \frac{2(3+2q)\zeta(q+1)}{1+q}\,H_0\,, \tag{76}$$

where

$$H_0 \equiv \int_0^{2^{-1/2}} d\mu\mu^q\sqrt{1-\mu^2} = \int_0^{\pi/4} d\theta\,\sin^q\theta(1-\sin^2\theta)\,. \tag{77}$$

An upper bound to the integral (eq. [77]) is given by

$$H_0 < \int_0^{2^{-1/2}} d\mu\mu^q = \frac{2^{-(1+q)/2}}{1+q}\,. \tag{78}$$

The first term on the right-hand side of equation (76) is much smaller than the other two, since $\epsilon \ll 1$, and with equation (78) used to estimate H_0, we obtain for the gyroresonance contribution

$$H_G \leq \frac{\zeta(q)}{q^2(2+q)}\,[1-(q+4)2^{-(4+q)/2}] + \frac{2^{(1-q)/2}(3+2q)\zeta(q+1)}{(1+q)^2}\,, \tag{79}$$

which is of order unity and solely determined by the value of q.

Expressing equation (54) in terms of $\mu = \epsilon x^{1/2}$, the transit-time damping contribution to the integral equation (53) for flat turbulence spectra is

$$H_T(q \leq 2) \equiv \int_\epsilon^1 d\mu (1-\mu^2) f_T(\mu, q \leq 2)$$
$$= c_1(q) \int_\epsilon^1 d\mu \frac{1-\mu^2}{\mu} \left(1 + \frac{\epsilon^2}{\mu^2}\right) \left[(1-\mu^2)\left(1 - \frac{\epsilon^2}{\mu^2}\right)\right]^{q/2} . \tag{80}$$

Substituting

$$y = \sqrt{(1-\mu^2)\left(1 - \frac{\epsilon^2}{\mu^2}\right)} , \tag{81}$$

the integral equation (80) can be cast into the form

$$H_T(q \leq 2) = c_1(q)(1-\epsilon^2)(1-\epsilon)^q \int_0^1 dy \frac{y^{q/2}}{[(1-y)\{[(1+\epsilon)/(1-\epsilon)]^2 - y\}]^{1/2}}$$
$$= c_1(q)(1-\epsilon)^{q+2} D(q, \alpha) , \tag{82}$$

with

$$D(q, \alpha) \equiv \int_0^1 dy y^{q/2}(1-y)^{-1/2}(1-\alpha y)^{-1/2} \tag{83}$$

and

$$\alpha = \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 . \tag{84}$$

Obviously, the integral (eq. [83]) can be expressed as the hypergeometric function

$$D(q, \alpha) = \frac{\pi^{1/2}\Gamma[(2+q)/2]}{\Gamma[(3+q)/2]} F\left(\frac{2+q}{2}, \frac{1}{2}; \frac{3+q}{2}; \alpha\right) . \tag{85}$$

Using the quadratic transformation formula 15.4.14 of Abramowitz & Stegun (1972), we can express the hypergeometric function in terms of the associated Legendre function of the second kind, $Q_{q/2}$, of zeroeth order and degree $(q/2)$, so that equation (85) reduces to

$$D(q, \alpha) = 2^{1/2} \frac{(\chi^2 - 1)^{1/4} Q_{q/2}(\chi)}{\alpha^{(1+q)/4}(1-\alpha)^{1/2}} , \tag{86}$$

where

$$\chi = \frac{1+\alpha}{2\alpha^{1/2}} = \frac{1+\epsilon^2}{1-\epsilon^2} . \tag{87}$$

Equation (86) then becomes

$$D(q, \alpha) = \left(\frac{1+\epsilon}{1-\epsilon}\right)^{(q+2)/2} Q_{q/2}\left(\frac{1+\epsilon^2}{1-\epsilon^2}\right) . \tag{88}$$

Inserting equation (88) into equation (82), we find for the transit-time contribution

$$H_T(q \leq 2) = c_1(q)(1-\epsilon^2)^{(q+2)/2} Q_{q/2}\left(\frac{1+\epsilon^2}{1-\epsilon^2}\right) . \tag{89}$$

For arguments $\chi \to 1$ from above, the Legendre function Q_ν logarithmically diverges (Magnus, Oberhettinger, & Soni 1966), with the leading term

$$\lim_{\chi \to 1^+} Q_\nu(\chi) \simeq -\gamma_{\rm E} - \psi\left[\frac{(2+q)}{2}\right] - \frac{1}{2} \ln \frac{\chi - 1}{2} , \tag{90}$$

in terms of Euler's constant $\gamma_{\rm E} = 0.5772$ and the Digamma function ψ. The limit $\chi \to 1^+$ corresponds to the limit $\epsilon \to 0$. Applying this limit to equation (88), we find with equation (90)

$$\lim_{\epsilon \to 0} H_T(q \leq 2) \to c_1(q)(1-\epsilon^2)^{(q+2)/2}\left[-\frac{1}{2} \ln \frac{\epsilon^2}{1-\epsilon^2} - \gamma_{\rm E} - \psi\left(\frac{(2+q)}{2}\right)\right] . \tag{91}$$

To lowest order in $\epsilon \ll 1$, the transit-time damping contribution in case of flat turbulence spectra is

$$H_T(\epsilon \ll 1, q \le 2) \simeq c_1(q) \ln \epsilon^{-1} , \tag{92}$$

which, at least for relativistic particles, is about one order of magnitude larger than unity. Comparing the transit-time contribution (eq. [92]) with the gyroresonance contribution (eq. [79]), we find that for relativistic particles, transit-time damping provides the dominant contribution. Collecting terms in equation (53), we find in this case for the integral

$$I_2(q \le 2) \simeq c_1(q)[\ln \epsilon^{-1} + c_6(q)] \simeq c_1(q) \ln \frac{v}{V_{\rm A}} , \tag{93}$$

where

$$c_6(q) \equiv \left\{ \frac{\zeta(q)}{q^2(2+q)} [1 - (q+4)2^{-(4+q)/2}] + \frac{2^{(1-q)/2}(3+2q)\zeta(q+1)}{(1+q)^2} \right\} \Big/ c_1(q) , \tag{94}$$

as indicated in equation (93), can be neglected for fast particles $v \gg V_{\rm A}$.

For steep power spectra we use equation (56) to obtain after simple integrations

$$\begin{aligned} H_T(q > 2) &\equiv \int_\epsilon^1 d\mu(1-\mu^2) f_T(\mu, q > 2) = c_4(q)(R_{\rm L} k_{\rm min})^{2-q} \int_\epsilon^1 d\mu \mu^{-1}(1-\mu^2)^2(1-\epsilon^4\mu^{-4}) \\ &= c_4(q)(R_{\rm L} k_{\rm min})^{2-q}(1-\epsilon^2)[(1+\epsilon^2)\ln \epsilon^{-1} - 1 + \epsilon^2] \simeq c_4(q)(R_{\rm L} k_{\rm min})^{2-q} \ln \epsilon^{-1} , \end{aligned} \tag{95}$$

which is much larger than the gyroresonance contribution (eq. [79]) since $R_{\rm L} k_{\rm min} \ll 1$. Here we obtain

$$I_2(q > 2) \simeq c_4(q)(R_{\rm L} k_{\rm min})^{2-q} \ln \frac{v}{V_{\rm A}} . \tag{96}$$

Inspecting equations (93) and (96) we find, quite to the contrary to our results for the spatial transport of cosmic ray particles, that transit-time damping provides the bulk of the contribution to the stochastic acceleration of fast cosmic ray particles, whereas the contribution from gyroresonant acceleration is negligibly small. The results (eqs. [93] and [94]) are supported by the calculations shown in Figure 6, where we present the variation of the integrand of the integral (eq. [53]) compared with the transit-time damping function f_T as a function of μ. The near equality of the two curves, especially near its maximum value, indicates that transit-time damping provides the dominant contribution to the value of the integral I_2 and therefore to the momentum diffusion coefficient.

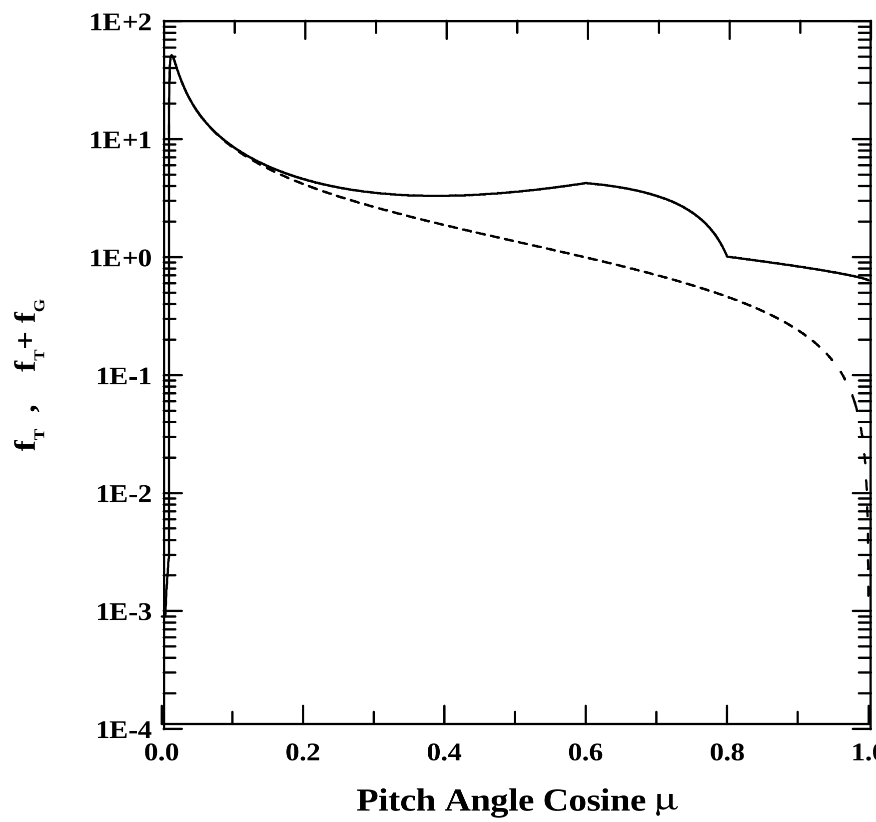

FIG. 6.—The transit-time damping function f_T (*dashed curve*) and the sum $f_T + f_G$ (*solid curve*) for $\epsilon = V_{\rm A}/v = 0.01$ and $q = 5/3$

3.3. *Cosmic Ray Mean Free Path from Fast-Mode Waves*

With the results (eqs. [69] and [75]) for the integral I_λ, we find that the cosmic ray mean free path (eq. [47]) resulting from interactions with fast-mode waves of total intensity $(\delta B_F)^2$, spectral index q_F, and minimum wavenumber $k_{\mathrm{min},F}$ is given by

$$\lambda_F(R_L k_{\mathrm{min},F} \ll 1) = \frac{2}{\pi(q_F-1)\zeta(q_F+1)}\left(\frac{B_0}{\delta B_F}\right)^2 R_L(k_{\mathrm{min},F} R_L \epsilon)^{1-q_F}$$
$$= \frac{2}{\pi(q_F-1)\zeta(q_F+1)}\left(\frac{B_0}{\delta B_F}\right)^2 \left(\left|\frac{\Omega_{p,0}}{\Omega_0}\right| \frac{ck_{\mathrm{min},F}}{\omega_{p,i}}\right)^{2-q_F} \gamma^{2-q_F} \frac{v}{V_A k_{\mathrm{min},F}} = \lambda_{F0}\beta\left(\left|\frac{\Omega_{p,0}}{\Omega_0}\right|\gamma\right)^{2-q_F}, \tag{97}$$

where $\beta = v/c$ and the constant

$$\lambda_{F0} = \frac{2c}{\pi(q_F-1)\zeta(q_F+1)V_A}\frac{B_0^2}{(\delta B_F)^2} k_{\mathrm{min},F}^{-1}\left(\frac{k_{\mathrm{min},F}\, c}{\omega_{p,i}}\right)^{2-q_F}. \tag{98}$$

The term $\omega_{p,i}$ denotes the proton plasma frequency. Apart from numerical factors (of order unity), the physical dependence is

$$\lambda_F \simeq R_L\left(\frac{B_0}{\delta B_F}\right)^2 (k_{\mathrm{min},F} R_L \epsilon)^{1-q_F} \tag{99}$$

for all values of $q_F > 1$, provided that the Larmor radius of the cosmic ray particles is smaller than the outer scale of the fast-mode turbulence, $R_L \ll k_{\mathrm{min},F}^{-1}$. In Figure 7 we calculate the mean free path from (eq. [97]) for different cosmic ray particles and the value $q_F = 5/3$. At nonrelativistic energies, the mean free path resulting from the interaction with fast-mode waves varies linearly with the particle velocity independent of the value of q_F, while at relativistic energies it is proportional to γ^{2-q_F}.

For the associated spatial diffusion coefficient we obtain

$$\kappa_F(R_L k_{\mathrm{min},F} \ll 1) = \frac{v}{3}\lambda_F = \frac{2}{3\pi(q_F-1)\zeta(q_F+1)}\left(\frac{B_0}{\delta B_F}\right)^2 \left(\left|\frac{\Omega_{p,0}}{\Omega_0}\right| \frac{ck_{\mathrm{min},F}}{\omega_{p,i}}\right)^{2-q_F} \gamma^{2-q_F}\frac{v^2}{V_A k_{\mathrm{min},F}}, \tag{100}$$

which is proportional to v^2 or E_{kin} at nonrelativistic energies, and proportional to γ^{2-q_F} at relativistic energies. The timescale for particles to diffuse a distance L by pitch-angle scattering along the ordered magnetic field can be calculated from the spatial diffusion coefficient as

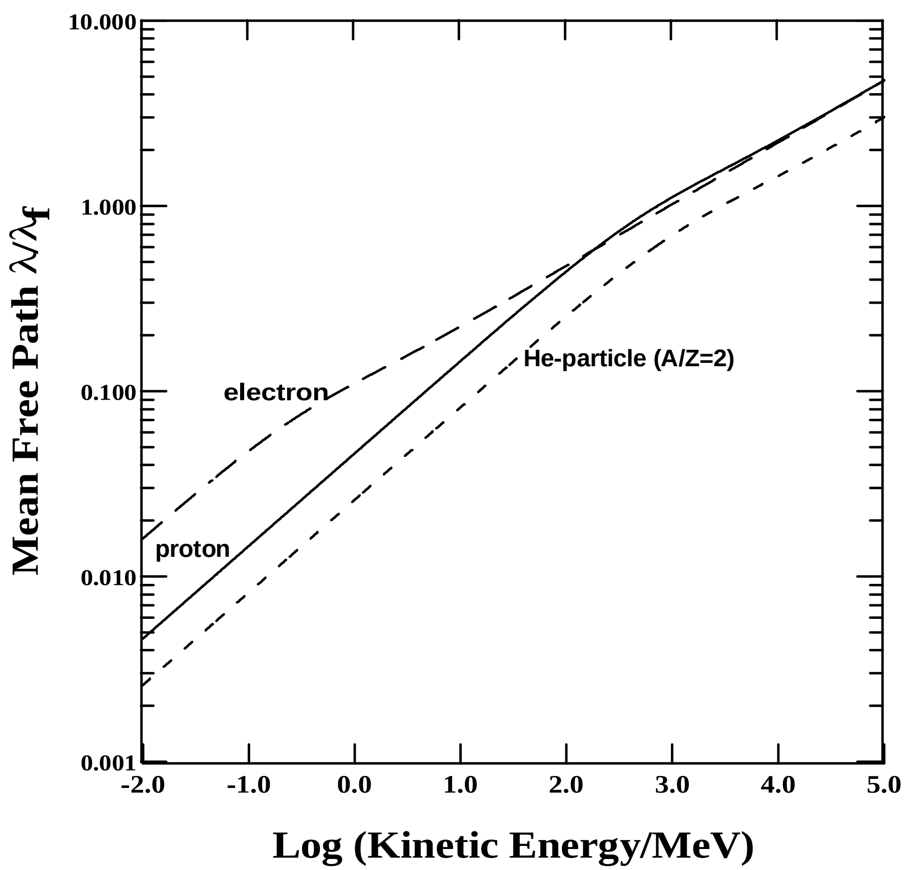

FIG. 7.—The mean free path from fast-mode waves as a function of kinetic energy for three different cosmic ray particle species, for a power spectral index $q_F = 5/3$ and neglecting any cutoff effects.

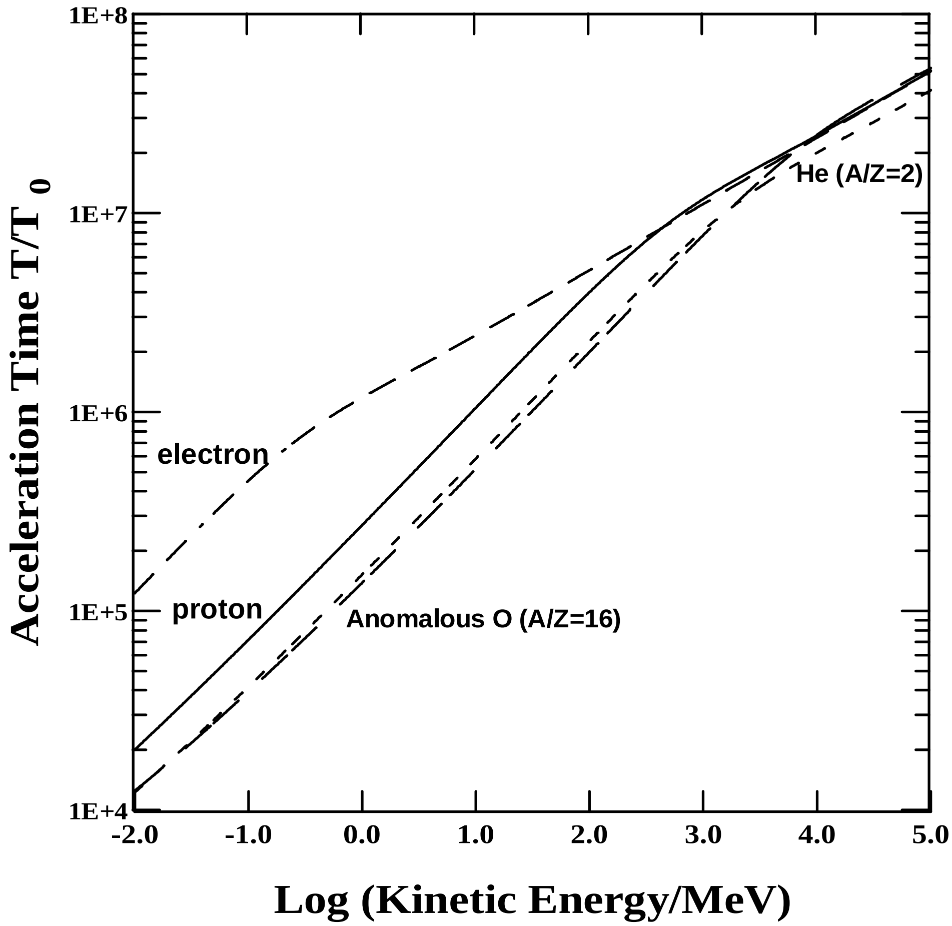

FIG. 8.—The acceleration timescale resulting from transit-time damping of fast-mode waves as a function of kinetic energy for four different cosmic ray particle species, for a power spectral index $q_F = 5/3$, Alfvén speed $V_A = 30$ km s^{-1}, and neglecting any cutoff effects.

$$T_D = \frac{L^2}{\kappa_F} = \frac{3\pi(q_F - 1)\zeta(q_F + 1)}{2}\left(\frac{\delta B_F}{B_0}\right)^2 \frac{L^2(R_L\, k_{\min,F}\,\epsilon)^{q_F - 1}}{v R_L} . \tag{101}$$

3.4. *Cosmic Ray Momentum Diffusion Coefficient from Fast-Mode Waves*

In the case of the momentum diffusion coefficient (eq. [48]) we have to consider the two cases of flat ($1 < q_F \le 2$) and steep ($q_F > 2$) turbulence power spectra. Using equations (93) and (96), respectively, we obtain

$$a_{2,F}(1 < q_F \le 2) \simeq \frac{\pi(q_F - 1)c_1(q_F)}{4}\left(\frac{\delta B_F}{B_0}\right)^2 |\Omega| (R_L\, k_{\min,F})^{q_F - 1}\, \frac{V_A^2 p^2}{v^2} \ln \frac{v}{V_A} \tag{102}$$

and

$$\begin{aligned} a_{2,F}(2 < q_F < 6) &\simeq \frac{\pi(q_F - 1)c_4(q_F)}{4}\left(\frac{\delta B_F}{B_0}\right)^2 |\Omega|\, R_L\, k_{\min,F}\, \frac{V_A^2 p^2}{v^2} \ln \frac{v}{V_A} \\ &= \frac{\pi(q_F - 1)c_4(q_F)}{4}\left(\frac{\delta B_F}{B_0}\right)^2 \frac{V_A^2\, k_{\min,F}\, p^2}{v} \ln \frac{v}{V_A} , \end{aligned} \tag{103}$$

respectively. For the associated acceleration timescale we derive

$$\begin{aligned} T_A(1 < q_F \le 2) \equiv \frac{p^2}{a_{2,F}(1 < q_F \le 2)} &= \frac{4}{\pi(q_F - 1)c_1(q_F)}\left(\frac{B_0}{\delta B_F}\right)^2 \frac{v R_L}{V_A^2 \ln(v/V_A)}\, (R_L\, k_{\min,F})^{1 - q_F} \\ &= T_0\, \frac{vc}{V_A^2 \ln(v/V_A)}\left(\left|\frac{\Omega_{p,0}}{\Omega_0}\right| \beta\gamma\right)^{2 - q_F} , \end{aligned} \tag{104}$$

with the constant

$$T_0 = \frac{4}{\pi(q_F - 1)c_1(q_F)}\left(\frac{B_0}{\delta B_F}\right)^2 (k_{\min}\, c)^{-1}\left(\frac{k_{\min}\, c}{\Omega_{p,0}}\right)^{2 - q_F} \tag{105}$$

and

$$T_A(2 < q_F < 6) = \frac{4}{\pi(q_F - 1)c_1(q_F)} \left(\frac{B_0}{\delta B_F}\right)^2 \frac{v k_{\mathrm{min},F}^{-1}}{V_A^2 \ln(v/V_A)} . \tag{106}$$

In Figure 8 we calculate the acceleration timescale (eq. [104]) as a function of kinetic energy, assuming a value of $q_F = 5/3$ for several cosmic ray particle species. At nonrelativistic energies, the acceleration timescales increases proportional to $T_A(\beta \ll 1) \propto \beta^{3-q_F}/\ln(\beta/\beta_A) \simeq E_{\mathrm{kin}}^{(3-q_F)/2}$, while at relativistic energies it increases as $T_A(\gamma \gg 1) \propto \gamma^{2-q_F} \propto E_{\mathrm{kin}}^{2-q_F}$.

3.5. *Timescale Relation*

Inspecting equations (101) and (104)–(106), we find that the product of the acceleration and diffusion timescales due to fast-mode wave interactions is

$$[T_A(p)T_D(p)]_F(1 < q_F \le 2) = \frac{6\zeta(q+1)}{c_1(q)} \left(\frac{L}{V_A}\right)^2 \left[\left(\frac{v}{V_A}\right)^{q-1} \ln \frac{v}{V_A}\right]^{-1} \tag{107}$$

for flat turbulence spectra. This product is a constant for relativistic particles, and the value of the constant is independent of microscopic quantities (e.g., the turbulence level $(\delta B)^2$) and determined just by the macroscopic properties (typical size L, Alfvén speed V_A) of the considered physical system.

In the case of a steep power spectra, we obtain

$$[T_A(p) \cdot T_D(p)]_F(2 < q_F < 6) = \frac{6\zeta(q+1)}{c_4(q)} \left(\frac{L}{V_A}\right)^2 [R_L\, k_{\mathrm{min},F}]^{q_F-2} \left[\left(\frac{v}{V_A}\right)^{q-1} \ln \frac{v}{V_A}\right]^{-1} , \tag{108}$$

which is about a factor $(R_L\, k_{\mathrm{min},F})^{q_F-2} \ll 1$ smaller than equation (107).

4. COSMIC TRANSPORT PARAMETERS FROM FAST-MODE WAVES AND SLAB ALFVÉN WAVES

In the literature, most investigations of cosmic ray transport and acceleration in magnetized cosmic plasmas are restricted to the resonant interaction of energetic charged particles with low-frequency plasma waves propagating along the ordered uniform magnetic field.

4.1. *Cosmic Ray Transport Parameters from Slab Alfvén Waves*

Schlickeiser (1989) has calculated the Fokker-Planck coefficients as well as the spatial and momentum diffusion coefficients for particle-wave interaction with these waves. Assuming vanishing magnetic and cross helicity of the Alfvén waves and adopting the same power-law turbulence spectrum given in equations (34) and (25)–(26), with total Alfvén wave intensity $(\delta B_A)^2$, spectral index q_A, and outer scale $k_{\mathrm{min},A}$, we use equations (58) of Schlickeiser (1989) for the two Fokker-Planck coefficients $D_{\mu\mu}$ and D_{pp} from Alfvén waves. Adopting the same notation from equation (46) as before, the slab Alfvén waves give rise to the additional cyclotron-resonance function

$$f_A = \tfrac{1}{2}[(1 - \mu\epsilon)^2 |\mu - \epsilon|^{q_A - 1} + (1 + \mu\epsilon)^2 |\mu + \epsilon|^{q_A - 1}] \simeq \tfrac{1}{2}(|\mu - \epsilon|^{q_A - 1} + |\mu + \epsilon|^{q_A - 1}) , \tag{109}$$

which additionally contributes to the two integrals, equations (49) and (50). Now these integrals become

$$I_\lambda = \int_0^1 d\mu \frac{1 - \mu^2}{f_A(\mu) + f_T(\mu) + f_G(\mu)} \tag{110}$$

and

$$I_2 = \int_0^1 d\mu (1 - \mu^2)[f_A(\mu) + f_T(\mu) + f_G(\mu)] . \tag{111}$$

In the interval $\epsilon < \mu \le 1$, the function

$$f_A(\epsilon < \mu \le 1) \simeq \mu^{q_A - 1} \tag{112}$$

is comparable to the gyroresonance function given in equation (58) from fast-mode waves, but clearly smaller than the transit time damping function (see eqs. [54] and [56]). In this interval we may neglect the contribution of the Alfvén wave function f_A to the integrals (110) and (111). However, in the interval $0 \le \mu < \epsilon$, the function

$$f_A(0 \le \mu \le \epsilon) \simeq \epsilon^{q_A - 1} \tag{113}$$

is larger than the gyroresonance contribution (eq. [58]), provided that the power spectrum spectral indices of Alfvén (q_A) and fast-mode (q_F) waves fulfill

$$q_A < q_F + 1 , \tag{114}$$

where we assume that the total wave intensities $(\delta B_A)^2$ and $(\delta B_F)^2$ are of comparable orders of magnitude. As a consequence, the integral (eq. [111]) remains unchanged (i.e., eqs. [93] and [96] still hold), so that the momentum diffusion will still be dominated by transit-time damping of the fast-mode waves.

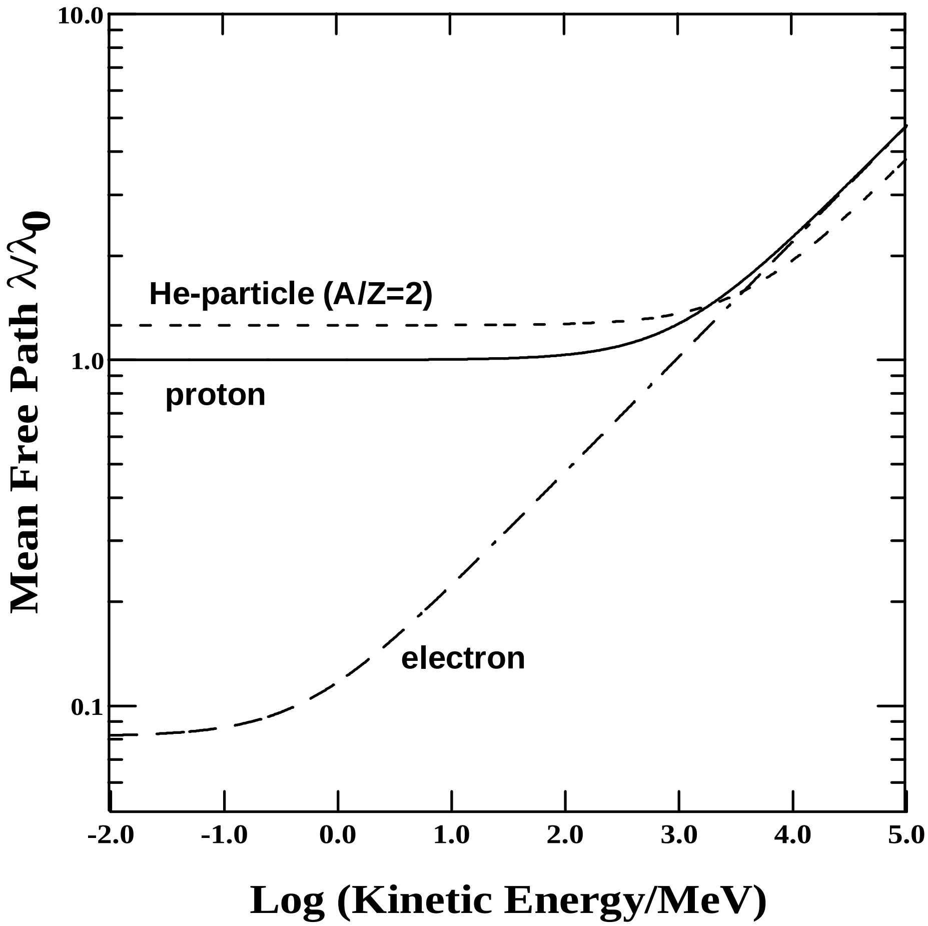

FIG. 9.—The mean free path as a function of kinetic energy for three different cosmic ray particle species in the case of an admixture of slab Alfvén waves and isotropic fast-mode waves, assuming equal spectral densities and neglecting any cutoff effects. A power-law spectral density of index $q = 5/3$ is assumed. The neglect of cutoff effects is problematic for electrons with energies below 10^3 MeV, and probably leads to a severe underestimation of the electron mean free path.

However, equation (110) instead of equations (69) and (75) now becomes

$$I_\lambda \simeq \int_0^\epsilon d\mu \, \frac{1-\mu^2}{f_{\rm A}(\mu)} \simeq \epsilon^{2-q_{\rm A}} \, . \tag{115}$$

In the absence of fast-mode waves ($f_T = f_G = 0$), equation (110) reduces to (Schlickeiser 1989) $I_{\rm A} \simeq (2-q_{\rm A})^{-1}$, so that

$$I_\lambda \simeq \epsilon^{2-q_{\rm A}} I_{\rm A} \, , \tag{116}$$

which is less than $I_{\rm A}$ for $q_{\rm A} < 2$. Evidently, the proportionality factor $\epsilon^{2-q_{\rm A}}$ also applies to the corresponding mean free path.

4.2. *Transport Parameters in the Case of Admixture of Slab Alfvén Waves to Fast-Mode Waves*

Generalizing the result of equation (97) in the presence of slab Alfvén waves and fast-mode waves of arbitrary outer scale ($k_{{\rm min},F}$, $k_{{\rm min},A}$), total intensities (δB_F^2, $\delta B_{\rm A}^2$), and power-law shape (q_F, $q_{\rm A}$) the cosmic ray mean free path of equation (47) becomes

$$\begin{aligned}\lambda &= \frac{3}{\pi(q_{\rm A}-1)} \frac{B_0^2}{(\delta B_{\rm A})^2} k_{{\rm min},A}^{-1} (k_{{\rm min},A} R_{\rm L} \epsilon)^{2-q_{\rm A}} \left[1 + \frac{3\zeta(q_F+1)(q_F-1)}{2(q_{\rm A}-1)} \frac{(\delta B_F)^2}{(\delta B_{\rm A})^2} \epsilon k_{{\rm min},A}^{1-q_{\rm A}} k_{{\rm min},F}^{q_F-1} (R_{\rm L}\epsilon)^{q_F-q_{\rm A}}\right]^{-1} \\ &= \frac{3}{\pi(q_{\rm A}-1)} \frac{B_0^2}{(\delta B_{\rm A})^2} k_{{\rm min},A}^{-1} \left(k_{{\rm min},A} \frac{V_{\rm A}}{|\Omega_0|}\right)^{2-q_{\rm A}} \gamma^{2-q} \left[1 + \frac{3\zeta(q_F+1)(q_F-1)}{2(q_{\rm A}-1)} \frac{(\delta B_F)^2}{(\delta B_{\rm A})^2} \epsilon k_{{\rm min},A}^{1-q_{\rm A}} k_{{\rm min},F}^{q_F-1} \left(\frac{V_{\rm A}}{|\Omega_0|}\right)^{q_F-q_{\rm A}} \gamma^{q_F-q_{\rm A}}\right]^{-1} ,\end{aligned} \tag{117}$$

whereas the momentum diffusion coefficients (eqs. [102] and [103]) remain unchanged. Obviously, the detailed variation of the mean free path (eq. [117]) depends sensitively on assumptions concerning the intensities (δB_F^2, $\delta B_{\rm A}^2$), shape (q_F, $q_{\rm A}$), and outer scale ($k_{{\rm min},F}$, $k_{{\rm min},A}$) of the two wave fields.

4.2.1. *Equal Intensity, Shape and Outer Scale*

We will consider here only the most simple possibility that Alfvén and fast-mode waves have equal intensities, identical shapes and outer scales; that is,

$$\delta B_F^2 = \delta B_A^2 = \frac{\delta B_t^2}{2}\,, \qquad q_F = q_A = q\,, \qquad k_{\min,F} = k_{\min,A} = k_{\min}. \tag{118}$$

The examination of other possibilities is left as an exercise to the interested reader. The mean free path for the conditions specified in equation (118) reduces to

$$\lambda = \frac{6}{\pi(q-1)} \frac{B_0^2}{(\delta B_t)^2} k_{\min}^{-1} (k_{\min} R_L \epsilon)^{2-q} \left[1 + \frac{3\zeta(q+1)}{2} \epsilon\right]$$
$$\simeq \frac{6}{\pi(q-1)} \frac{B_0^2}{(\delta B_t)^2} k_{\min}^{-1} \left(k_{\min} \frac{V_A}{|\Omega_0|}\right)^{2-q} \gamma^{2-q} = \lambda_0 \left(\left|\frac{\Omega_{p,0}}{\Omega_0}\right| \gamma\right)^{2-q}, \tag{119}$$

where

$$\lambda_0 = \frac{6}{\pi(q-1)} \frac{B_0^2}{(\delta B_t)^2} k_{\min}^{-1} \left(\frac{k_{\min} c}{\omega_{p,i}}\right)^{2-q}. \tag{120}$$

In Figure 9 we show the mean free path (eq. [119]) for different cosmic ray species. As can be seen, the mean free path is proportional to $(\gamma/|\Omega_0|)^{2-q} \propto (E_{\rm tot}/Z)^{2-q}$, a constant determined by $|\Omega_0|^{q-2} = |m/Ze|^{2-q}$ at nonrelativistic energies. We note that this behavior may account for the legendary $\lambda_{\rm fit} - \lambda_{\rm QLT}$ discrepancy of solar energetic particles (e.g., Palmer 1982; Schlickeiser 1989; Bieber et al. 1994).

The apparent discrepancy between the quasi-linear scattering length, calculated in a simple slab model from observed power spectra of low-frequency magnetohydrodynamical fluctuations, and the phenomenologically inferred scattering length $\lambda_{\rm fit}$ obtained from fitting time-dependent diffusion models to the intensity-time variation of solar particle events (Palmer 1982; Wanner & Wibberenz 1993) has been a long-standing problem of cosmic ray physics. As has been pointed out by Palmer (1982), the discrepancy consists of two parts: (1) the fitted mean free paths are typically an order of magnitude larger than the theoretical mean free paths (referred to as the "magnitude problem" by Bieber et al. 1994); and (2) the observations are broadly consistent with a rigidity-independent mean free path from 0.5 to 5000 MV, while according to many theories, based on Alfvén wave interactions only, the mean free path should increase with increasing rigidity (referred to as the "flatness problem"). Here we have shown that in the case of turbulence consisting of a mixture of slab Alfvén waves and isotropic fast-mode waves, both the magnitude and flatness problems can be resolved within quasi-linear theory. We also avoid the classical argument of Fisk (1979) that additional scattering (by quasi-linear slab Alfvén wave interactions) would reduce the value of the scattering length λ_A further and thus worsen the situation. His argument does not apply, since the dominant scattering contribution from transit-time damping of fast-mode waves does not contribute at all pitch angles of cosmic ray particles; it thus heavily reduces the contribution of the pitch angle range $\epsilon \le |\mu| \le 1$ to the value of the integral I_λ, which now is predominantly fixed by the contribution from the small interval $0 \le |\mu| < \epsilon$. This effective reduction in the relevant integration range introduces an additional factor $\epsilon = V_A/v$ instead of unity, which resolves both the flatness and magnitude problems.

5. DISCUSSION AND SUMMARY

We have calculated the quasi-linear transport and acceleration parameters for cosmic ray particles interacting resonantly with undamped fast-mode waves propagating in a low-β plasma. In the cold plasma limit ($\beta \to 0$), the usual fast and slow magnetosonic waves in the fluid plasma merge to the fast-mode waves with the dispersion relation $\omega_j^2 = V_A^2 k^2$ propagating at all angles θ to the ordered background magnetic field. For simplicity, we have considered fast ($v \gg V_A$) cosmic ray particles and a vanishing cross helicity state of the fast-mode waves. In this case, the rate of adiabatic deceleration a_1 vanishes, and the momentum diffusion coefficient a_2 and the spatial diffusion coefficient κ can be calculated as pitch-angle averages of functions determined by the two nonvanishing Fokker-Planck coefficients $D_{\mu\mu}(\mu)$ and $D_{pp} = \epsilon^2 p^2 D_{\mu\mu}(\mu)$, where $\epsilon = V_A/v$. We consider isotropic fast-mode turbulence with a Kolmogorov-like turbulence spectrum $g_{\rm tot} \propto k^{-q_F}$ above the minimum wavenumber $k_{\min}$, with values of $1 < q_F < 6$. Our principal results are:

1. The Fokker-Planck coefficients $D_{\mu\mu}$ and D_{pp} are the sum of contributions from transit-time damping and gyroresonance interactions. Transit-time damping interaction refers to the $n = 0$ resonant interactions of particles with the compressive magnetic field component of the fast-mode waves that occur provided that the parallel particle velocity equals the fast-mode phase velocity $\pm V_A$ divided by $\cos\theta$; that is, $v_\parallel = \pm V_A/\cos\theta$. Since $\cos\theta \le 1$, only super-Alfvénic particles with $|\mu| \ge \epsilon$ are subject to transit-time damping. Gyroresonance refers to all $|n| \ne 0$ resonant particle-wave interactions, and leads to the well-known strong coupling of cosmic ray particles with plasma waves for which the wavelength is an integer multiple of the particle Larmor radius (slightly modified by finite wave propagation effects).
2. Transit-time damping provides the dominant contribution to the pitch-angle scattering of particles in the interval $\epsilon \le |\mu| \le 1$. In the interval $|\mu| < \epsilon$, where no transit-time damping occurs, the gyroresonant interactions provide a small but finite contribution to the scattering rate of particles.
3. Since the momentum diffusion coefficient is determined by the μ-average of the Fokker-Planck coefficient D_{pp}, transit-

time damping provides the bulk of the contribution to the stochastic acceleration of fast cosmic ray particles, while the contribution from gyroresonant acceleration is negligibly small.

4. Since the spatial diffusion coefficient κ and the related mean free path λ are determined by the μ average of the *inverse* of $D_{\mu\mu}$, the major contribution to these spatial transport parameters comes from the interval $|\mu| < \epsilon$. In this interval, transit-time damping does not occur and pitch-angle scattering from gyroresonance is the only contributor. This small gyroresonance scattering is crucial to have quasi-linear scattering through $\mu = 0$ (and thus establish an isotropic particle distribution function), and therefore determines the quantitative value of the spatial diffusion coefficient. The presence of massive scattering due to transit-time interactions of particles with $|\mu| > \epsilon$ leads to a vanishingly small contribution to the integral from these pitch angles.

5. The cosmic ray mean free path from isotropic fast-mode wave interactions alone increases proportionally with the particle Larmor radius R_L at nonrelativistic particle energies, while at relativistic energies it is proportional to $R_L^{2-q_F}$. The acceleration time for momentum diffusion is $T_A = p^2/a_2$. For values of $q_F \le 2$, T_A increases as $v^{3-q_F}/\ln(v/V_A)$ at nonrelativistic energies, and as $R_L^{2-q_F}$ at relativistic energies.

6. We also calculate the cosmic ray transport parameters for plasma turbulence consisting of a mixture of isotropic fast-mode waves and slab Alfvén waves, using the results of Schlickeiser (1989) for the latter. In this case, the momentum diffusion coefficient is determined by the transit-time damping of the fast-mode waves and is about a factor of $\ln(v/V_A)$ larger than in the case of pure slab Alfvén wave turbulence. The mean free path and the spatial diffusion coefficient are modified significantly from the pure fast-mode case, since the crucial scattering at small pitch angles $|\mu| < \epsilon$ is now provided by gyroresonances with the slab Alfvén waves instead of by gyroresonances with the fast-mode waves. The mean free path is proportional to $(\gamma/|\Omega_0|)^{2-q} \propto (E_{tot}/Z)^{2-q}$, which is a constant determined by $|\Omega_0|^{q-2} = |m/Ze|^{2-q}$ at nonrelativistic energies. This behavior may account for the legendary λ_{fit}–λ_{QLT} discrepancy for solar energetic particles.

In summary, we find that the extension of the theory of cosmic ray transport and acceleration, in particular by including the interactions with fast-mode waves, is very fruitful and promising. These waves provide the dominant contribution to the stochastic acceleration of cosmic ray particles, and affect the spatial transport in such a way that the particles' mean free path becomes constant at nonrelativistic energies, in agreement with observations of solar energetic particles and galactic cosmic rays. Another attractive feature of the fast-mode waves is that all super-Alfvénic particles can undergo acceleration by transit-time damping, which may help solve the injection problem of cosmic rays. However, near injection energies, possible modifications resulting from relatively large values of V_A/v (≈ 1) and deviations from the adopted nondispersive fast-mode dispersion relation at large wavenumbers must be investigated before making definite conclusions.

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APPENDIX A

VANISHING FAST-MODE WAVE CROSS HELICITY

In the case of equal intensity of forward and backward waves (eq. 34), we can express the coefficient (eq. [33]) as the sum of two terms,

$$D_{\mu p} = \frac{\pi\Omega^2 p V_A}{4v^2 B_0^2}\left[\mu D_1(\mu) + D_2(\mu)\right] , \tag{A1}$$

with

$$D_1(\mu) = \frac{1}{2}\int_{-1}^{1} d\eta \left[F_1\left(\left|\mu\eta + \frac{V_A}{v}\right|\right) - F_1\left(\left|\mu\eta - \frac{V_A}{v}\right|\right)\right] , \tag{A2}$$

$$D_2(\mu) = \int_{-1}^{1} d\eta \, \frac{\eta}{1-\eta^2}\left[F_2\left(\left|\mu\eta + \frac{V_A}{v}\right|\right) - F_2\left(\left|\mu\eta - \frac{V_A}{v}\right|\right)\right] , \tag{A3}$$

where

$$F_1(x) \equiv J_n^2\left(\frac{n\sqrt{(1-\mu^2)(1-\eta^2)}}{x}\right) g_{tot}\left(\frac{n|\Omega/v|}{x}\right) , \tag{A4}$$

$$F_2(x) \equiv x F_1(x) . \tag{A5}$$

After close inspection we find that equation (A2) is identically zero,

$$\begin{aligned} D_1(\mu) &= \frac{1}{2}\left\{\int_0^1 d\eta\left[F_1\left(\left|\mu\eta + \frac{V_A}{v}\right|\right) - F_1\left(\left|\mu\eta - \frac{V_A}{v}\right|\right)\right] + \int_0^1 d\eta\left[F_1\left(\left|-\mu\eta + \frac{V_A}{v}\right|\right) - F_1\left(\left|-\mu\eta - \frac{V_A}{v}\right|\right)\right]\right\} \\ &= \frac{1}{2}\int_0^1 d\eta\left[F_1\left(\left|\mu\eta + \frac{V_A}{v}\right|\right) - F_1\left(\left|\mu\eta - \frac{V_A}{v}\right|\right) + F_1\left(\left|\mu\eta - \frac{V_A}{v}\right|\right) - F_1\left(\left|\mu\eta + \frac{V_A}{v}\right|\right)\right] = 0 , \end{aligned} \tag{A6}$$

so that the coefficient (eq. [A1]) becomes

$$D_{\mu p}(\mu) = \frac{\pi \Omega^2 p V_{\rm A}}{2v^2 B_0^2} \sum_{n=1}^{\infty} \int_0^1 d\eta \, \frac{\eta}{1-\eta^2} \Bigg[\left| \mu\eta + \frac{V_{\rm A}}{v} \right| J_n^2\left(\frac{n\sqrt{(1-\mu^2)(1-\eta^2)}}{|\mu\eta + V_{\rm A}/v|} \right) g_{\rm tot}\left(\frac{n|\Omega/v|}{|\mu\eta + V_{\rm A}/v|} \right)$$
$$- \left| \mu\eta - \frac{V_{\rm A}}{v} \right| J_n^2\left(\frac{n\sqrt{(1-\mu^2)(1-\eta^2)}}{|\mu\eta - V_{\rm A}/v|} \right) g_{\rm tot}\left(\frac{n|\Omega/v|}{|\mu\eta - V_{\rm A}/v|} \right) \Bigg] . \quad \text{(A7)}$$

Equation (A7) is antisymmetric in μ, i.e.,

$$D_{\mu p}(-\mu) = -D_{\mu p}(\mu) , \quad \text{(A8)}$$

and its value is very close to zero, since in both intervals of integration $\eta \gg V_{\rm A}/v|\mu|$ and $\eta \ll V_{\rm A}/v|\mu|$ the integrand vanishes to lowest order in the respective small quantities $(V_{\rm A}/v|\mu|\eta) \ll 1$ and $(v|\mu|\eta)/V_{\rm A} \ll 1$. As a consequence, the case of equation (34) yields a vanishing rate of adiabatic deceleration (eq. [4b])

$$a_1 \simeq 0 , \quad \text{(A9)}$$

and the momentum diffusion coefficient (eq. [4c]) reduces to

$$a_2 = \tfrac{1}{2} \int_{-1}^{1} d\mu D_{pp}(\mu) . \quad \text{(A10)}$$

APPENDIX B

PROPERTIES OF THE TRANSIT-TIME DAMPING FUNCTION f_T

The transit-time damping function $f_T(\mu)$ given in equation (39) is different from zero only for particles with cosine of pitch angle $|\mu| \geq \epsilon$. This simply reflects the requirement that in order to undergo transit-time damping, the parallel particle speed $|v_{\|}| = v|\mu|$ must be at least as large as the fast-mode wave phase velocity $V_{\rm A}$. Substituting in equation (39) $\mu = \epsilon x^{1/2}$, we find with

$$T(U, q) \equiv \int_U^{\infty} ds s^{-(1+q)} J_1^2(s) \quad \text{(B1)}$$

that f_T can be expressed as

$$f_T(1 \leq x \leq \epsilon^{-2}) = \frac{T(U, q)}{\epsilon} \frac{1+x}{x^{(3+q)/2}} [(1+\epsilon^2)x - 1 - \epsilon^2 x^2]^{q/2} . \quad \text{(B2)}$$

B1. FLAT TURBULENCE SPECTRUM $q < 2$

We first investigate the integral (B1). From equation (40), we find that in the interval $\epsilon \leq \mu \leq 1$,

$$U = k_{\rm min} R_{\rm L} \sqrt{(1-\mu^2)(1-\epsilon^2\mu^{-2})} \leq k_{\rm min} R_{\rm L}(1-\epsilon) \ll 1 \quad \text{(B3)}$$

is much smaller unity if $k_{\rm min} R_{\rm L} \ll 1$ holds. For turbulence spectral indices $q < 2$, we can then calculate the integral equation (B1) by using expression 6.574.2 of Gradstheyn & Ryzhik (1965):

$$T(U \ll 1, q < 2) = \int_0^{\infty} ds s^{-(1+q)} J_1^2(s) - \int_0^{U} ds s^{-(1+q)} J_1^2(s) \simeq c_1(q) - \frac{U^{2-q}}{4(2-q)} \simeq c_1(q) , \quad \text{(B4)}$$

with the constant

$$c_1(q) \equiv \frac{\Gamma(1+q)\Gamma(1-(q/2))}{2^{1+q}\Gamma(2+(q/2))\Gamma^2(1+(q/2))} = 2^{1-q} \frac{q}{4-q^2} \frac{\Gamma(q)\Gamma(2-(q/2))}{\Gamma^3(1+(q/2))} . \quad \text{(B5)}$$

In deriving the result (eq. [B4]), we use

$$J_\nu(z \leq 1) \simeq (z/2)^\nu/\Gamma(\nu+1) < (z/2)^\nu \quad \text{(B6)}$$

as an approximation for small arguments. With equation (B4), the transit-time damping function (eq. [B2]) reduces in the case where $q < 2$ to

$$f_T(1 \leq x \leq \epsilon^{-2}, q < 2) = \frac{c_1(q)}{\epsilon} \frac{1+x}{x^{(3+q)/2}} [(1+\epsilon^2)x - 1 - \epsilon^2 x^2]^{q/2} , \quad \text{(B7)}$$

which vanishes for $x_N = 1, \epsilon^{-2}$ corresponding to $\mu_N = \epsilon$, 1, respectively, and attains a maximum in between. Substituting further $x = 1 + s$ in equation (B7), we derive

$$f_T(0 \le s \le \epsilon^{-2} - 1) = \frac{c_1(q)(1-\epsilon^2)^{q/2}}{\epsilon} \frac{(2+s)s^{q/2}}{(1+s)^{(3+q)/2}} \left[1 - \frac{\epsilon^2 s}{1-\epsilon^2}\right]^{q/2} . \tag{B8}$$

For values of $s \ll \epsilon^{-2}$, we can approximate equation (B8) by

$$f_T(0 \le s \ll \epsilon^{-2} - 1) \simeq \frac{c_1(q)(1-\epsilon^2)^{q/2}}{\epsilon} \frac{(2+s)s^{q/2}}{(1+s)^{(3+q)/2}} . \tag{B9}$$

From the first derivative of equation (B9) we find that the maximum,

$$f_{T,\max}(\epsilon, q) = \frac{(1-\epsilon^2)^{q/2}}{\epsilon} c_1(q)c_2(q) , \tag{B10}$$

with

$$c_2(q) \equiv \sqrt{2}\, \frac{(q+\sqrt{q^2+16})(q-4+\sqrt{q^2+16})^{q/2}}{(q+\sqrt{q^2+16})^{(3+q)/2}} , \tag{B11}$$

occurs at

$$s_M(q) = \frac{\sqrt{q^2+16}+q-4}{2} , \tag{B12}$$

yielding $s_M(q=1.5) = (\sqrt{73}-5)/4 = 0.886$ and $s_M(q=5/3) = 1.0$, corresponding to

$$\mu_M = \epsilon\left(\frac{\sqrt{q^2+16}+q-2}{2}\right)^{1/2} , \tag{B13}$$

yielding $\mu_M(q=1.5) = 1.373\epsilon$ and $\mu_M(q=5/3) = \epsilon\sqrt{2} = 1.414\epsilon$.

First, we note that s_M is indeed much less than $\epsilon^{-2} \simeq o(10^6)$, so that the approximation given in equation (B9) is well justified. Equation (B10) indicates that to lowest order in $\epsilon \ll 1$, the maximum value varies as

$$f_{T,\max} \simeq c_3(q)\epsilon^{-1} , \tag{B14}$$

with $c_3(q) = c_1(q)c_2(q)$. For $q = 1.5$ and $q = 5/3$, we obtain $c_3(1.5) = 0.632$ and $c_3(5/3) = 0.595$, respectively. In Figures 1–4 we have calculated numerically the transit-time damping function f_T (*dashed curve*) for the value $\epsilon = 0.01$ for the two values of $q = 1.5$ (Figs. 1 and 2) and $q = 5/3$ (Figs. 3 and 4). While Figures 1 and 3 show the variation over the whole $0 \le \mu \le 1$ interval, Figures 2 and 4 show the variation for small $\mu \le 5\epsilon$. It will be noticed that f_T vanishes for small $\mu \le \epsilon$, then sharply rises to its maximum value (eq. [B10]) close to $\mu_M \simeq \epsilon$, in full agreement with equation (B13).

B2. STEEP TURBULENCE SPECTRUM $2 \le q < 6$

For steeper turbulence spectra, with $q \ge 2$, we use Bessel function relations and Bessel's equation to derive

$$J_1(s) = -sJ_0'(s) = s^2[J_0(s) + J_0''(s)] . \tag{B15}$$

The integral (B1) then becomes, after partial integration,

$$\begin{aligned} T &= -\int_U^\infty ds s^{2-q} J_0'(J_0 + J_0'') = -\tfrac{1}{2}\int_U^\infty ds s^{2-q}[(J_0^2)' + (J_1^2/s^2)'] \\ &= \tfrac{1}{2}\left[U^{2-q}J_0^2(U) + U^{-q}J_1^2(U) + (2-q)T + (2-q)\int_U^\infty ds s^{2-q}J_0^2(s)\right] . \end{aligned} \tag{B16}$$

Solving for T, we obtain

$$qT = U^{2-q}J_0^2(U) + U^{-q}J_1^2(U) + (2-q)H_1 , \tag{B17}$$

with

$$H_1 \equiv \int_U^\infty ds s^{2-q} J_0^2(s) = \int_U^\infty ds s^{1-q}\left[J_2^2 + \frac{2}{s}(J_1^2)'\right] = 2qT - 2U^{-q}J_1^2(U) + \int_U^\infty ds s^{1-q} J_2^2(s) . \tag{B18}$$

Combining equations (B17) and (B18) yields

$$T = \frac{J_1^2(U)}{qU^q} + \frac{J_0^2(U)}{q(2q-3)U^{q-2}} + H_2(U) , \tag{B19}$$

with the integral

$$H_2(U) \equiv \frac{2-q}{q(2q-3)} \int_U^\infty ds s^{1-q} J_2^2(s) . \tag{B20}$$

For values of $1 < q < 6$, we can use expression 6.574.2 of Gradstheyn & Ryzhik (1965) to obtain in particular

$$H_2(U \ll 1, 2 \le q < 6) \simeq \frac{2-q}{q(2q-3)} \int_0^\infty ds s^{1-q} J_2^2(s) = \frac{(2-q)2^{3-q}}{q^2(2q-3)(2+q)} \frac{\Gamma(q-1)\Gamma(3-q/2)}{\Gamma^3(q/2)} . \tag{B21}$$

Since $U \ll 1$, we use again approximation (eq. [B6]) for the Bessel functions to obtain for equation (B19)

$$T(U \ll 1, q = 2) = \tfrac{3}{4} , \tag{B22}$$

$$T(U \ll 1, 2 < q < 6) \simeq \frac{2q^2 - 3q + 4}{4q(2q-3)} U^{2-q} = \frac{2q^2 - 3q + 4}{4q(2q-3)} (k_{\min} R_{\mathrm{L}})^{2-q}[(1-\mu^2)(1-\epsilon^2\mu^{-2})]^{1-(q/2)} . \tag{B23}$$

While for $q = 2$ the transit-time damping function (eq. [39]) is identical to equation (B7) with $c_1(2) = \frac{3}{4}$, for steeper turbulence spectra it becomes

$$f_T(q > 2, k_{\min} R_{\mathrm{L}} \ll 1) = c_4(q)(R_{\mathrm{L}} k_{\min})^{2-q} H(|\mu| - \epsilon) \frac{(1-\mu^2)[1-(\epsilon^4/\mu^4)]}{|\mu|} , \tag{B24}$$

where

$$c_4(q) \equiv \frac{2q^2 - 3q + 4}{4q(2q-3)} . \tag{B25}$$

APPENDIX C

PROPERTIES OF THE GYRORESONANCE FUNCTION f_G

C1. PITCH ANGLE INTERVAL $0 \le \mu \le \epsilon$

In this interval, the product $\mu\eta \le \epsilon\eta \le \epsilon$ is always smaller than ϵ, so that we may approximate the function given by equation (45) as

$$f_G(\mu \le \epsilon) \simeq 2\epsilon^{q-1} \sum_{n=1}^{\infty} n^{-q} \int_0^1 d\eta (1+\eta^2) \left[J_n'\left(\frac{n\sqrt{1-\eta^2}}{\epsilon} \right) \right]^2 . \tag{C1}$$

To evaluate this integral, we use the approximation of Bessel functions for small and large arguments (Abramowitz & Stegun 1972),

$$J_n(nz, z \le 1) \simeq \frac{n^n z^n}{2^n \Gamma(n+1)} \tag{C2a}$$

and

$$J_n(nz, z > 1) \simeq \sqrt{\frac{2}{\pi n z}} \cos\left[nz - \frac{(2n+1)\pi}{4} \right] , \tag{C2b}$$

implying

$$[J_n'(nz, z \le 1)]^2 \simeq \left[\frac{n^n z^{n-1}}{2^n \Gamma(n+1)} \right]^2 \tag{C3a}$$

and

$$\begin{aligned}[J_n'(nz, z > 1)]^2 &\simeq \frac{2}{\pi n z} \sin^2\left[nz - \frac{(2n+1)\pi}{4} \right] = \frac{1}{\pi n z} \left\{ 1 - \cos\left[2nz - \frac{(2n+1)\pi}{2} \right] \right\} \\ &= \frac{1}{\pi n z} \left\{ 1 - \sin(2nz) \sin\left[\frac{(2n+1)\pi}{2} \right] \right\} = \frac{1}{\pi n z} [1 - (-1)^n \sin(2nz)] .\end{aligned} \tag{C3b}$$

The argument $z = (1-\eta^2)^{1/2}/\epsilon$ of the Bessel function in equation (C1) is small or large compared to unity for $\eta > \eta_c$ and

$\eta \le \eta_c$, respectively, with $\eta_c = (1-\epsilon^2)^{1/2}$, so that we obtain

$$\int_0^1 d\eta(1+\eta^2)\left[J'_n\left(\frac{n\sqrt{1-\eta^2}}{\epsilon}\right)\right]^2 \simeq i_1 + \frac{\epsilon}{\pi n}(i_2+i_3)\,, \tag{C4a}$$

with

$$\begin{aligned} i_1 &= \frac{n^{2n}\epsilon^{2-2n}}{4^n\Gamma^2(n+1)}\int_{\eta_c}^1 d\eta(1+\eta^2)(1-\eta^2)^{n-1} \\ &= \frac{2n^{2n}\epsilon^{2-2n}}{4^n\Gamma^2(n+1)}\int_0^\epsilon ds s^{2n-1}\left(1-\frac{s^2}{2}\right)(1-s^2)^{-1/2} \simeq \frac{n^{2n-1}}{4^n\Gamma^2(n+1)}\epsilon^2 \end{aligned} \tag{C4b}$$

to lowest order in the small quantity ϵ,

$$\begin{aligned} i_2 &= \int_0^{\eta_c} d\eta\,\frac{1+\eta^2}{\sqrt{1-\eta^2}} = \frac{3}{2}\arcsin\left(\sqrt{1-\epsilon^2}\right) - \frac{\epsilon}{2}\sqrt{1-\epsilon^2} \\ &= \frac{3}{2}\left(\frac{\pi}{2}-\arcsin\epsilon\right) - \frac{\epsilon}{2}\sqrt{1-\epsilon^2} \simeq \frac{3\pi}{4} - 2\epsilon\,, \end{aligned} \tag{C4c}$$

again to lowest order in ϵ, and

$$\begin{aligned} i_3 &= (-1)^{n+1}\int_0^{\eta_c} d\eta\,\frac{1+\eta^2}{\sqrt{1-\eta^2}}\sin\left(2n\frac{\sqrt{1-\eta^2}}{\epsilon}\right) \\ &= (-1)^{n+1}\int_{\arcsin\epsilon}^{\pi/2} d\theta(1+\cos^2\theta)\sin\left(\frac{2n}{\epsilon}\sin\theta\right)\,, \end{aligned} \tag{C4d}$$

where we substituted $\eta = \cos\theta$ in the last integral. Using the series

$$\sin\left(\frac{2n}{\epsilon}\sin\theta\right) = 2\sum_{k=0}^{\infty} J_{2k+1}\left(\frac{2n}{\epsilon}\right)\sin[(2k+1)\theta]\,, \tag{C5}$$

we obtain for equation (C4d)

$$\begin{aligned} i_3 &= (-1)^{n+1}\sum_{k=0}^{\infty} J_{2k+1}\left(\frac{2n}{\epsilon}\right)\int_{\arcsin\epsilon}^{\pi/2} d\theta(3+\cos 2\theta)\sin[(2k+1)\theta] \\ &= \frac{1}{2}(-1)^{n+1}\sum_{k=0}^{\infty} J_{2k+1}\left(\frac{2n}{\epsilon}\right)\left\{\frac{\cos[(2k-1)\arcsin\epsilon]}{2k-1}\right. \\ &\quad \left. + 6\,\frac{\cos[(2k+1)\arcsin\epsilon]}{2k+1} + \frac{\cos[(2k+3)\arcsin\epsilon]}{2k+3}\right\}. \end{aligned} \tag{C6}$$

Since $2n/\epsilon \gg 1$ is very large, we may use approximation (C2b) again; that is,

$$J_{2k+1}\left(\frac{2n}{\epsilon}\right) \simeq \sqrt{\frac{\epsilon}{\pi n}}\cos\left[\frac{2n}{\epsilon} - \frac{\pi}{4} - \frac{(2k+1)\pi}{2}\right] = (-1)^k\sqrt{\frac{\epsilon}{\pi n}}\sin\left(\frac{2n}{\epsilon}-\frac{\pi}{4}\right)$$

to obtain after obvious resumming

$$i_3 = (-1)^{n+1}2\sqrt{\frac{\epsilon}{\pi n}}\sin\left(\frac{2n}{\epsilon}-\frac{\pi}{4}\right)\sum_{k=0}^{\infty}(-1)^k\frac{\cos[(2k+1)\arcsin\epsilon]}{2k+1} = \frac{1}{2}(-1)^{n+1}\sqrt{\frac{\epsilon\pi}{n}}\sin\left(\frac{2n}{\epsilon}-\frac{\pi}{4}\right)\,, \tag{C7}$$

where we used (from Magnus et al. 1966)

$$\sum_{k=0}^{\infty}\frac{\sin[(2k+1)\pi/2]\cos[(2k+1)\arcsin\epsilon]}{2k+1} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{\sin[(2k+1)(\pi/2-\arcsin\epsilon)] + \sin[(2k+1)(\pi/2+\arcsin\epsilon)]}{2k+1} = \frac{\pi}{4}\,. \tag{C8}$$

Collecting terms in equation (C4a), we obtain for equation (C1) to lowest order in ϵ

$$f_G(\mu\le\epsilon) \simeq 2\epsilon^{q-1}\sum_{n=1}^{\infty} n^{-(q-1)}\left[\frac{n^{2n}\epsilon^2}{4^n\Gamma^2(n+1)} + \frac{\epsilon}{\pi}\left(\frac{3\pi}{4}-2\epsilon\right) + \frac{(-1)^{n-1}}{2}\left(\frac{\epsilon}{n}\right)^{3/2} n\sin\left(\frac{2n}{\epsilon}-\frac{\pi}{4}\right)\right] \simeq \frac{3}{2}\zeta(q+1)\epsilon^q\,, \tag{C9}$$

in terms of Riemann's zeta function.

C2. PITCH ANGLE INTERVAL $\epsilon < \mu \le 1$

In this interval, we can approximate $|\mu\eta \pm \epsilon| \simeq \epsilon$ for $\eta \le \epsilon/\mu$ and $|\mu\eta \pm \epsilon| \simeq \mu\eta$ for $\eta > \epsilon/\mu$. As a consequence, the function (eq. [45]) becomes

$$f_G(\mu > \epsilon) \simeq 2 \sum_{n=1}^{\infty} n^{-q}[\epsilon^{q-1}K_1(\mu) + \mu^{q-1}K_2(\mu)] , \tag{C10}$$

with the two functions

$$K_1(\mu) \equiv \int_0^{\epsilon/\mu} d\eta(1+\eta^2)\left[J'_n\left(\frac{n\sqrt{(1-\mu^2)(1-\eta^2)}}{\epsilon}\right)\right]^2 \tag{C11}$$

and

$$K_2(\mu) \equiv \int_{\epsilon/\mu}^1 d\eta(1+\eta^2)\eta^{q-1}\left[J'_n\left(\frac{n\sqrt{(1-\mu^2)(1-\eta^2)}}{\mu\eta}\right)\right]^2 . \tag{C12}$$

As in the preceding section, these two functions are calculated using the approximations given in equation (C3). Leaving the details as an exercise to the interested reader, we find

$$f_G(\mu > \epsilon) \simeq \begin{cases} \dfrac{3\zeta(q+1)\epsilon^q}{2\sqrt{1-\mu^2}} & \text{for } \epsilon \le \mu \le \sqrt{2}\epsilon \\ \dfrac{3\zeta(q+1)\epsilon^q}{2\sqrt{1-\mu^2}}\left[1+\dfrac{4(3+2q)}{3(1+q)}\left(\dfrac{\mu}{\epsilon}\right)^q\right] & \text{for } \sqrt{2}\epsilon < \mu \le 2^{-1/2} \\ \dfrac{\zeta(q)\mu^{q-1}}{2q} & \text{for } 2^{-1/2} < \mu \le 1 \end{cases} \tag{C13}$$

According to equations (C13), the gyroresonance function attains its maximum

$$f_{G,\max}(q) = 2^{(3-q)/2}\zeta(q+1)\,\frac{3+2q}{1+q} \tag{C14}$$

at $\mu_G = 2^{-1/2}$. This maximum is orders of magnitude smaller than the maximum (eq. [B14]) of the transit-time damping function.

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