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THE EXTENDED POWER LAW AS AN INTRINSIC SIGNATURE FOR A BLACK HOLE

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ABSTRACT

We analyze the exact general relativistic integrodifferential equation of radiative transfer describing the interaction of low-energy photons with a Maxwellian distribution of hot electrons in the gravitational field of a Schwarzschild black hole. We prove that, owing to Comptonization, an initial arbitrary spectrum of low-energy photons unavoidably results in spectra characterized by an extended power-law feature. We examine the spectral index by using both analytical and numerical methods for a variety of physical parameters as such the plasma temperature and the mass accretion rate. The presence of the event horizon as well as the behavior of the null geodesics in its vicinity largely determine the dependence of the spectral index on the flow parameters. We come to the conclusion that the bulk motion of a converging flow is more efficient in upscattering photons than thermal Comptonization, provided that the electron temperature in the flow is of order of a few kilo–electron volts or less. In this case, the spectrum observed at infinity consists of a soft component, which is produced by those input photons that escape after a few scatterings without any significant energy change, and a hard component (described by a power law), which is produced by the photons that underwent significant upscattering. The luminosity of the power-law component is relatively small compared to that of the soft component. For accretion into a black hole, the spectral energy index of the power law is always higher than 1 for plasma temperatures of order of a few kilo–electron volts. This result suggests that the bulk motion Comptonization might be responsible for the power-law spectra seen in the black hole X-ray sources.

Subject headings: accretion, accretion disks — black hole physics — radiation mechanisms: nonthermal — stars: neutron — X-rays: general

1. INTRODUCTION

Do black holes interact with an accretion flow in such a way that a distinct observational signature that is entirely different from those associated with any other compact object exists? In other words, can the existence of a black hole be solely inferred from the radiation observed at infinity?

These are the crucial questions that theoreticians and observers are confronting nowadays. Even though we have now accumulated enormous observational evidence in favor of the existence of black holes, it is still fair to say that their existence has not been established. Perhaps proving their existence would have been a much easier task if, for instance, an argument would have been advanced that would (1) single out the generic component (or components) of a black hole that is responsible for shaping the unique observed feature associated with black holes and (2) prove that, indeed, this generic component always results in the same observable feature independent of the environmental conditions in which the black hole finds itself.

The lack of such an argument may be traced in the plethora of various accretion flows: accretion in a state of free fall, optically thin or optically thick, accretion disks with or without relativistic corrections, shocked flows, etc. Of course, this diversity of accretion models is highly justified. On physical grounds, one expects accretion flows that describe a solar-mass black hole accreting interstellar medium to be distinct from those flows describing accretion onto a black hole in a close binary system or from a supermassive black hole at the center of an active galactic nucleus. Viewed from this angle, the detectability of a black hole appears to be a rather frustrating issue, since it is not clear a priori what type of the existing accretion models (if any) would describe a realistic accreting black hole.

In the present paper we shall show that may be not the case. The distinct feature of black hole spacetime, as opposed to the spacetimes due to other compact objects, is the presence of the event horizon. Near the horizon, the strong gravitational field is expected to dominate the pressure forces and thus to drive the accreting material into a free fall. In contrast, for other compact objects the pressure forces are becoming dominant as their surface is approached, and thus a free fall state is absent. We argue that this difference is rather crucial, as it results in an observational signature of a black hole. Roughly, the origin of this signature is due to the inverse Comptonization of low energy photons from fast-moving electrons. The presence of the low-energy photon component is expected to be generic because of, for instance, the disk structure near a black hole or bremstrahlung of the electron component from the corresponding proton component. The boosted photon component is characterized by a power-law spectrum and is entirely independent of the initial spectrum of the low-energy photons. The spectral index of the boosted photons is determined by the mass accretion rate and the bulk motion plasma temperature only. A key ingredient in proving our claim is the employment of the exact relativistic transfer to describe the Compton scattering of the low-energy radiation field of the Maxwellian distribution of fast-moving electrons.

We will prove that the power law is always present as a part of the black hole spectrum in a wide energy range (up

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to 500 keV; note that this energy band was probed by *CGRO* observations). It is worth noting that this energy band is a specific feature of the bulk inflow as well. Any extended power-law feature related to the relativistic electron distribution (for example in the case of jets) is not uniquely constrained to this energy band, i.e., not tied to $m_e c^2$.

We investigate the particular case of a nonrotating Schwartzschild black hole powering the accretion, leaving the case of a rotating black hole for the future analysis.

The presence of the power-law part in the upcomptonized spectra was rigorously proven by Titarchuk & Lyubarskij (1995; hereafter TL95). There it was demonstrated that for the wide class of the electron distributions, the power law is the solution of the full kinetic equation.

The importance of Compton upscattering of low-frequency photons in an optically thick, converging flow has been understood for a long time. Blandford and Payne were the first to address this problem in a series of papers (Blandford & Payne 1981; Payne & Blandford 1981). In the first paper they derived the Fokker-Planck radiative transfer equation, which took into account photon diffusion in space and energy, while in the second paper they solved the Fokker-Planck radiative transfer equation in the case of the steady state, spherically symmetric, supercritical accretion into a central black hole with the assumption of a power-law flow velocity of $v(r) \propto r^{-\beta}$ and neglecting thermal Comptonization. For the inner boundary condition, they assumed adiabatic compression of photons as $r \to 0$. Thus, their flow extended from $r = 0$ to infinity. They showed that all emergent spectra have a high-energy power-law tail with an index of $\alpha = 3/(2 - \beta)$ (for free fall, $\beta = \frac{1}{2}$ and $\alpha = 2$), which is independent of the low-frequency source distribution.

Chakrabarti & Titarchuk (1995) were the first to realize that it is possible to see the converging inflow spectral feature in the high (soft) state of black hole X-ray sources. They emphasized that in the soft state, the optically thick Compton cloud surrounding a black hole is cooled down and the spectrum of the converging inflow (the extended power law) will be observed.

The paper of Titarchuk, Mastichiadis, & Kylafis (1996, hereafter TMK96; see also the extended version in Titarchuk, Mastichiadis, & Kylafis 1997, hereafter TMK97) presents the exact numerical and approximate analytical solutions of the problem of spectral formation in a converging flow, taking into account the inner boundary condition, the dynamical effects of the free fall, and the thermal motion of the electrons. There, the inner boundary has been taken at *finite* radius with the spherical surface considered to be fully absorptive.

TMK96 have used a variant of the Fokker-Plank formalism where the inner boundary mimics a black hole horizon; no relativistic effects (special or general) are taken into account. Thus, their results are instructively useful, but they are not directly comparable with the observations. *By using the numerical and analytical techniques, they demonstrated that the extended power laws are present in the resulting spectra in addition to the blackbody-like emission at lower energies.*

Zane et al. (1996) presented a characteristic method and the code for solving the radiative transfer equation in differentially moving media in a curved spacetime. Some applications concerning hot and cold accretion onto nonrotating black holes were discussed there.

In our paper, the full relativistic treatment is worked out in terms of the relativistic Boltzmann kinetic equation without recourse to the Fokker-Planck approximation in either configuration and energy space.

The relativistic transport theory was developed by Lindquist (1966). He presents the appropriate radiative transfer in the curve spacetime. For completeness, in this paper we delineate some important points of that theory related with the application to the radiative transfer in the electron atmosphere.

We demonstrate that the power-law spectra are produced when low-frequency photons are scattered in the Thomson regime (i.e., when the dimensionless photon energy, $z = E'/m_e c^2$, measured in the electron rest frame satisfies $z' \ll 1$).

The eigenfunction method for the Comptonization problem employed in this paper has been offered and developed by Titarchuk & Lyubarskij (1995; hereafter TL95). Giesler & Kirk (1997) have extended the TL95 treatment by accommodating an arbitrary anisotropy of the source function. Their results for the spectral index confirm those of TL95 over a wide range of electron temperatures and optical depths; the largest difference they found is 10%, which occurs at low optical depth.

The spectral indices related with the eigenvalues of the problem are determined as functions of the optical depth of the accreting matter. Thus, for the first time, we are able to solve the full Comptonization problem in the presence of the bulk and thermal motions of electrons.

In § 2 and Appendix A, we will give the details of the derivation of the general relativistic radiative kinetic equation. Section 3 presents the method (some details are given in Appendix B). We describe the method of separation of variables and the reduction of the whole problem to the specific eigenproblem in the configuration space. We propose the numerical solution of this problem by using the iteration method (see, e.g., Sunyaev & Titarchuk 1985), integrating over characteristics (the photon trajectories in the presence of Schwarzschild black hole background). Finally, we summarize our work and draw conclusions in § 4.

2. THE MAIN EQUATION

We begin by considering background geometry, described by the following line element:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 , \tag{1}$$

where, for the Schwarzschild black hole, $f = 1 - r_s/r$, $r_s = 2GM/c^2$, and t, r, θ, and φ are the event coordinates with $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$. G is the gravitational constant, and M is the mass of a black hole.

In order to describe the photon radiation field, we shall employ the concept of the distribution function, N. The distribution function, $N(x, \boldsymbol{p})$, describes the number, dN, of photons (photon worldlines) that cross a certain spacelike volume element, dV, at $x(t, r, \theta, \varphi)$, and whose 4-momenta, $\boldsymbol{p}$, lie within a corresponding 3-surface element, dP, in momentum space. It is desirable to choose dV and dP to be coordinate invariants. Thus dN would be invariant as well, and the same would be true of $N(x, \boldsymbol{p})$.

In Appendix A we present the detailed derivation of the relativistic radiative transfer equation expressed through the distribution function, $N(x, \boldsymbol{p})$, and the interaction density function, $S(N)$ (see the definition of this function after eq. [A13]).

We will describe the electron component by a local Maxwellian distribution (see, e.g., Landau & Lifshitz 1980; Pathria 1970):

$$F(r, P_e)dP = \aleph^{-1} e^{\beta u_\mu P_e^\mu} dP , \tag{2}$$

where $\aleph$ is the normalization constant.

One has to interpret $F(r, P_e)(-P_e^a n_a)dP\,dV$, which is similar to the one implied by equation (A8), with the sole exception that considerations are restricted on the electron phase space. For our purpose, an arbitrary electron momentum state, P_e, can be represented in the form

$$P_e = \left(\frac{1}{\sqrt{1 - V^2/c^2}}, \frac{|V| \boldsymbol{n}_e}{\sqrt{1 - V^2/c^2}}\right), \tag{3}$$

where the "the thermal 3-velocity," $\boldsymbol{V}$, stands for a convenient parameterization of the electron phase space. We will take $\beta = m_e c^2/kT_e$, while u_μ stands for the hydrodynamical 4-velocity of the inflowing plasma that may be represented relative to the local orthonormal frame in the form

$$u = (u^o, u^r) = \left(\frac{1}{\sqrt{(1 - v^2)}}, -\frac{\boldsymbol{v}^r}{\sqrt{(1 - v^2)}}\right), \tag{4}$$

where the negative sign in u^r takes into account the convergent nature of the fluid flow. Note that, as a result of the hydrodynamic bulk motion, the local Maxwellian distribution exhibits a coupling of the thermal velocity, $\boldsymbol{V}$, with the hydrodynamic bulk motion, $\boldsymbol{v}$, and one gets

$$\beta u_\mu P_e^\mu = -\frac{m_e c^2}{kT_e}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}\left(1 - \frac{V^2}{c^2}\right)^{-1/2} \times \left(1 + \cos\theta \frac{Vv}{c^2}\right). \tag{5}$$

We will discuss the coupling effect in § 4, and we will consider this issue in detail in our next publication.

Within a 4-volume, dW, at the event, x, there is a decrease in the original number of worldlines due to absorption and scattering out of momentum range, dP, given by (see also the right-hand side of eq. [A13])

$$-\kappa(x, \boldsymbol{p})n(x)N(x, \boldsymbol{p})dW\,dP . \tag{6}$$

Here $n(x)$ is the proper number density of the electrons interacting with the photons, namely the number density of electrons as measured in their own local rest frame, and $\kappa(x, \boldsymbol{p})$ is the invariant absorption coefficient or invariant opacity. The κ-opacity is related to the usual scattering cross section, σ_s, via the expression (see Lindquist 1966)

$$\kappa = E \cdot \sigma_s . \tag{7}$$

On the other hand, there are increases due to pure scattering out of all other 4-momentum ranges, dP', into dP, given by

$$n(x)dW\,dP \int dP' \kappa(x, \boldsymbol{p})\zeta(x; \boldsymbol{p}' \to \boldsymbol{p})N(x, \boldsymbol{p}') . \tag{8}$$

Thus the transition probability, $\zeta(x; \boldsymbol{p}')$, can be expressed in terms of the differential cross section, $d\sigma_s/(dEd\Omega)$ (see Lindquist 1966), as

$$\kappa(x, \boldsymbol{p}')\zeta(x; \boldsymbol{p}' \to \boldsymbol{p}) = \frac{E'}{E}\frac{d\sigma_s}{dEd\Omega} . \tag{9}$$

Taking into account only Compton scattering of photons off the background electrons, one may covariantly write the transfer equation (see eq. [A13]) in the following form:

$$p^\alpha \frac{DN}{dx^\alpha} = \int N(r, P')\kappa(x, \boldsymbol{p}')\zeta(x; \boldsymbol{p}' \to \boldsymbol{p})dP' - N(r, P) \int \kappa(x, \boldsymbol{p}')\zeta(x; \boldsymbol{p} \to \boldsymbol{p}')dP' . \tag{10}$$

The first term on the right-hand side describes the increase in the photon worldlines over the infinitesimal phase-space cell centered around P, while the second term describes the processes of depletion.

Recall that the scattering cross section of a photon from an electron in the electron's rest frame is described by the Klein-Nishina formula,

$$\sigma(\nu \to \nu', \xi) = \frac{3}{16\pi} n_e \sigma_\mathrm{T} \frac{1 + \xi^2}{[1 + z(1 - \xi)]^2} \times \left\{1 + \frac{z^2(1 - \xi)^2}{(1 + \xi^2)[1 + z(1 - \xi)]}\right\}\delta\left[\nu' - \frac{\nu}{1 + z(1 - \xi)}\right], \tag{11}$$

where $z = h\nu/m_e c^2$ is a dimensionless photon energy, ξ is the cosine of scattering angle, σ_T is the Thomson cross section, and recall also that if all quantities on the right-hand side of the above formula are computed in the rest frame of the electron, one may explicitly write the transfer equation on the black hole background.

By rewriting equation (10) for the orthonormal frame of equation (1) (see eq. [A25]), we get the following equation:

$$\mu\sqrt{f}\frac{\partial N}{\partial r} - \nu\mu\frac{\partial\sqrt{f}}{\partial r}\frac{\partial N}{\partial \nu} - (1 - \mu^2)\left(\frac{\partial\sqrt{f}}{\partial r} - \frac{\sqrt{f}}{r}\right)\cdot\frac{\partial N}{\partial \mu} = \int_0^\infty d\nu_1 \int_{4\pi} d\Omega_1\left[\left(\frac{\nu_1}{\nu}\right)^2 \sigma_s(\nu_1 \to \nu, \xi)N(\nu_1, \mu_1, r) - \sigma_s(\nu \to \nu_1, \xi)N(\nu, \mu, r)\right]. \tag{12}$$

The scattering kernel can be calculated by performing a Lorentz boost of σ_s, multiplying it by $F(r, P_e)$ (see eq. [2]), and integrating over P_e. Then, the scattering kernel is given by

$$\sigma_s(\nu \to \nu_1, \xi, \beta) = \frac{3}{16\pi}\frac{n_e \sigma_\mathrm{T}}{\nu z}\int_0^\pi \sin\theta\, d\theta \int d^3\boldsymbol{v}\,\frac{F(r, P_e)}{\gamma} \times \left[1 + \left(1 - \frac{1 - \xi}{\gamma^2 DD'}\right)^2 + \frac{zz'(1 - \xi)^2}{\gamma^2 DD'}\right] \times \delta\left(\xi - 1 + \frac{\gamma D'}{z} - \frac{\gamma D}{z'}\right), \tag{13}$$

where $D = 1 - \mu V$, $D_1 = 1 - \mu' V$, $\gamma = (1 - V^2)^{-1/2}$, and $\xi = \boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}$ is the cosine of scattering angle. In deriving the above equation we have chosen

$$\aleph(\beta) = m_e c \int_0^\pi \int_0^c \exp(\beta u_\mu P_e^\mu)\gamma^5 \frac{V^2}{c^2} \sin\theta\, dv^2\, d\theta , \tag{14}$$

so that the distribution of electrons is normalized by a fixed electron density, n, as measured in the orthonormal frame associated with equation (1).

3. THE METHOD OF SOLUTION

3.1. *Separation Variables*

As long as the ejected low-energy photons satisfy $z_0 = h\nu_0/m_e c^2\gamma \ll 1$, the integration over incoming frequencies, ν_0, is trivially implemented provided that the explicit function of $N(r, \nu_0, \nu, \mathbf{\Omega})$ is known. Thus, we need to describe the main properties of Green's function, $N(r, \nu_0, \nu, \mathbf{\Omega})$, in a situation where the low-energy photons are injected into the atmosphere with the bulk motion.

The power-law part of the spectrum (Sunyaev & Titarchuk 1980; TL95) occurs at frequencies lower than that of the Wien cutoff ($E < E_e$, where E_e is the average electron energy). In this regime, the energy change due to the recoil effect of the electron can be neglected in comparison to the Doppler shift of the photon. Hence we can drop the third term in parenthesis and the term $\xi - 1$ of the δ-function argument in the scattering kernel (eq. [13]), transforming that into the classical Thomson scattering kernel (see also TL95; Gieseler & Kirk 1997).

Now we seek the solution of the Boltzmann equation (eq. [12]) with the aforementioned simplifications in the form

$$N(r, \nu, \mathbf{\Omega}) = \nu^{-(3+\alpha)}J(r, \mu) . \tag{15}$$

Then we can formally get from equation (12) that

$$\mu\sqrt{f}\frac{\partial J}{\partial r} + (\alpha + 3)\mu\frac{\partial\sqrt{f}}{\partial r}J - (1 - \mu^2)\left(\frac{\partial\sqrt{f}}{\partial r} - \frac{\sqrt{f}}{r}\right)\frac{\partial J}{\partial \mu} = n_e\sigma_T\left[-J + \frac{1}{4\pi}\int_{-1}^{1}d\mu_1\int_0^{2\pi}d\varphi R(\xi)J(\mu_1, \tau)\right] . \tag{16}$$

Here the phase function, $R(\xi)$, is as follows:

$$R(\xi) = \frac{3}{4}\int_0^{\pi}\sin\theta\, d\theta\int d^3\mathbf{v}\,\frac{F(r, P_e)}{\gamma^2}\left(\frac{D_1}{D}\right)^{\alpha+2}\frac{1}{D_1} \times [1 + (\xi')^2] , \tag{17}$$

where ξ' is the cosine of the scattering angle between photon incoming and outgoing directions in the electron rest frame. The reduced integrodifferential equation is two dimensional, and it can be treated and solved much more easily than the original equation (eq. [12]). The whole problem is reduced to the eigenvalue problem for equation (16). We can not claim that the kinetic equation allows a power-law solution (eq. [15]) unless first α is found and $J(r, \mu)$ is specified.

In order to derive an equation for the determination of a spectral index, we expand the phase function, $R(\xi)$, in a series of Legendre polynomials (see also Sobolev 1975; TL95):

$$R(\xi) = p^0(\mu, \mu') + 2\sum_{m=1}^{n}p^m(\mu, \mu')\cos m(\varphi - \varphi') , \tag{18}$$

$$p^m(\mu, \mu') = \sum_{i=m}^{n}c_i^m P_i^m(\mu)P_i^m(\mu') , \tag{19}$$

and

$$c_i^m = C_i\frac{(i - m)!}{(i + m)!} , \tag{20}$$

for $m = 0, 1, 2, \ldots n$. Since the phase function, $R(\xi)$, is given by the series (eq. [18]) in cos $m\varphi$, the source function (the second term in brackets on the right-hand side of eq. [16]) and $J(r, \mathbf{\Omega})$ can be expanded over cos $m\phi$, too. Under the assumption of spherical symmetry for the source, we are interested in the zero-term of the expansion that satisfies the following equations:

$$\ell J^0(r, \mu) = -\left[n_e\sigma_T + (\alpha + 3)\mu\frac{\partial\sqrt{f}}{\partial r}\right]J^0(r, \mu) + (n_e\sigma_T)B^0(r, \mu) , \tag{21}$$

where

$$\ell J^0(r, \mu) = \mu\sqrt{f}\frac{\partial J^0}{\partial r} + (1 - \mu^2)\left(\frac{\partial\sqrt{f}}{\partial r} - \frac{\sqrt{f}}{r}\right)\frac{\partial J^0}{\partial \mu} , \tag{22}$$

and the source function is

$$B^0(r, \mu) = \frac{1}{2}\int_{-1}^{1}p^0(\mu, \mu')J^0(r, \mu')d\mu' . \tag{23}$$

There are two boundary conditions that our solution must satisfy. The first is that there is no scattered radiation outside of the atmosphere:

$$J^0(0, \mu) = 0 \quad \text{for } \mu < 0 . \tag{24a}$$

The second boundary condition is that we have an absorptive boundary at radius r_s:

$$J^0(r_s, \mu) = 0 \quad \text{for } \mu > 0 . \tag{24b}$$

Thus the whole problem is reduced to the standard radiative transfer problem for the space part of the solution, $J(r, \mathbf{\Omega})$. Inversion of the differential operator, ℓ, on the left-hand side of equation (21) leads to the integral equation for $B^0(r, \mu)$:

$$\begin{aligned} B^0(r, \mu) = {} & \tfrac{1}{2}\int_{-1}^{0}p^0(\mu, \mu')d\mu'\int_0^{T(r_{\rm bn}, r, \mu')} \\ & \times \exp\{-T[r_{\rm bn}, r'(r, \mu'), \mu']\}B^0(r', \mu')dT \\ & + \tfrac{1}{2}\int_0^{1}p^0(\mu, \mu')d\mu'\int_0^{T(r_{\rm bn}, r, \mu')} \\ & \times \exp\{-T[r_{\rm bn}, r'(r, \mu'), \mu']\}B^0(r', \mu')dT , \end{aligned} \tag{25}$$

where $T(r_{\rm bn}, r, \mu)$ is the optical path along the characteristic curve of the differential operator, $\hat{\ell}$, determined by the initial point, r, μ' toward the boundary radius, $r_{\rm bn}$ ($r_{\rm bn} = r_s$ and ∞ for the inner and outer boundaries, respectively).

The phase function component, $p^0(\mu, \mu')$, entered in equations (21) and (23) is determined by the sum

$$p^0(\mu, \mu') = \sum_{i=0}^{n}C_i P_i(\mu)P_i(\mu') . \tag{26}$$

Thus, we can present the source function, B^0, also as a sum:

$$B^0(r, \mu) = \sum_{i=0}^{n}C_i P_i(\mu)\int_{-1}^{1}P_i(\mu')J^0(r, \mu')d\mu' . \tag{27}$$

This form of the source function is used for the solution of the boundary problem (eqs. [21]–[24b]) by the iteration method (see, e.g., Sunyaev & Titarchuk 1985). In order to proceed with the iteration method, one has to assume some initial field distribution (in terms of the intensity, J^0) and then calculate B^0 in accordance to equation (27), which is followed by the solution of the differential equation (eq. [21]).

This iteration formalism is identical to the integral-equation formalism in which

$$B^0(r, \mu) = \sum_{i=0}^{n} C_i P_i(\mu) B_i^0(r) , \tag{28}$$

where the set of $B_i^0(r)$ components of the source function, $B(r, \Omega)$, is determined by the system of the integral equations (compare with TL95):

$$B_i^0(r) = \frac{1}{2} \sum_{j=0}^{n} C_j \Bigg[\int_{-1}^{0} p_i(\mu') p_j(\mu') d\mu' \int_0^{T(r_{\rm bn}, r, \mu')} \times \exp \{ -T[r_{\rm bn}, r'(r, \mu'), \mu'] \} B_j^0(r') dT + \int_0^1 p_i(\mu') p_j(\mu') d\mu' \times \int_0^{T(r_{\rm bn}, r, \mu')} \exp \{ -T[r_{\rm bn}, r'(r, \mu'), \mu'] \} B_j^0(r') dT \Bigg] . \tag{29}$$

Thus the eigenvalue problem (eqs. [21]–[24b]) can be reduced to an eigenproblem for a system of integral equations (eq. [29]) where the optical paths, $T(r_{\rm bn}, r, \mu)$, and the expansion coefficients of the phase function, C_i, depend on the spectral index, α, as a parameter. In other words, one has to find the values of α that guarantee the existence of the nontrivial solution of equation (29). In § 3.2 and Appendix B, we shall proceed with the numerical solution of the eigenvalue problem by presenting the bulk motion phase function, $R_{\rm b}(\xi_{\rm b})$, in the degenerated form (see eq. [26]).

Now it is worth noting that in the case of the pure thermal motion in the isothermal plasma cloud, the problem is substantially simplified. The source function, $B^0(r, \mu)$, can be replaced by its zeroth moment, $B_0^0(r)$ (TL95), which guarantees the accuracy of the spectral index determination to better than 10% in the worst cases (Giesler & Kirk 1997).

For example, the equation for the zeroth moment, $B_0^0(r)$, reads

$$B_0^0(r) = \frac{C_0}{2} \Bigg[\int_{-1}^{0} d\mu' \int_0^{T(r_{\rm bn}, r, \mu')} \times \exp \{ -T[r_{\rm bn}, r'(r, \mu'), \mu'] \} B_0^0(r') dT + \int_0^1 d\mu' \times \int_0^{T(r_{\rm bn}, r, \mu')} \exp \{ -T[r_{\rm bn}, r'(r, \mu'), \mu'] \} B_0^0(r') dT \Bigg] , \tag{30}$$

where C_0 is the zero-moment of the phase function.

3.2. *Photon Trajectories and the Characteristics of the Space Operator, ℓ*

The characteristics of the differential operator, ℓ, are determined by the following differential equation:

$$\left[-\frac{1}{2x^2(1 - x^{-1})} + x^{-1} \right] dx = d[\ln(1 - \mu^2)^{-1/2}] , \tag{31}$$

where $x = r/r_s$ is a dimensionless radius. The integral curves of this equation (the characteristic curves) are given by

$$\frac{x(1 - \mu^2)^{1/2}}{(1 - x^{-1})^{1/2}} = \frac{x_0(1 - \mu_0^2)^{1/2}}{(1 - x_0^{-1})^{1/2}} = p , \tag{32}$$

where p is an impact parameter at infinity; p can also be determined at a given point in a characteristic by the cosine of an angle between the tangent to and the radius vector to the point and by the given point position, x_0.

In the flat geometry, the characteristics are just straight lines:

$$x(1 - \mu^2)^{1/2} = p , \tag{33}$$

where an impact parameter, p, is the distance from a given point to the center.

We can resolve equation (32) with respect to μ to get

$$\mu = \pm(1 - p^2/y^2) , \tag{34}$$

where $y = x^{3/2}/(x - 1)^{1/2}$. The graph of y as a function of x is presented in Figure 1, which allows us to comprehend the possible range of radii for the given impact parameter, p, through the inequality $p \le y$. For example, if $p \le (6.75)^{1/2}$, then the photon can escape from the inner boundary (the black hole horizon) toward the observer or vice versa; all photons going toward the horizon having these impact parameters are gravitationally attracted by the black hole. However, if $p > (6.75)^{1/2}$, then the finite trajectories are possible with the radius range between $1 \le x \le 1.5$ or the infinite trajectories with $p \le y(x)$ (x is always more than 1.5).

3.3. *Spectral Index Determination*

We are assuming a free fall for the background flow, where the bulk velocity of the infalling plasma is given by $v(r) = c(r_s/r)^{1/2}$. In the kinetic equations (eqs. [12] and [16]), the density, n, is measured in the local rest frame of the flow, and it is $n = \dot{m}(r_s/r)^{1/2}/(2r\sigma_{\rm T})$. Here $\dot{m} = \dot{M}/\dot{M}_E$, $\dot{M}$ is the

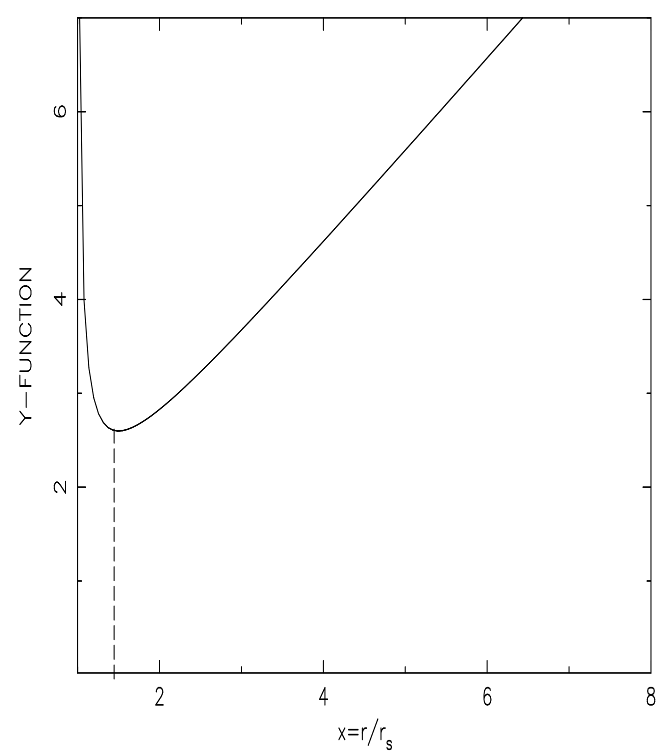

FIG. 1.—Plot of the photon trajectory phase space: the phase-space function, y, vs. the dimensionless radius, $x = r/r_s$.

mass accretion rate, and $\dot{M}_E \equiv L_E/c^2 = 4\pi G M m_p/\sigma_T c$ is the Eddington accretion rate.

For the cold converging inflow ($kT_e = 0$ keV), the electron distribution is the δ-function, $F(r, P_e) = \delta(\boldsymbol{v} - \boldsymbol{v}_b)$, defined in the velocity phase space in the way that

$$\int_0^\pi \sin\theta\, d\theta \int d^3\boldsymbol{v} F(r, P_e) = 1 .$$

In this case the phase function is

$$R_b(\xi) = \frac{3}{4}\frac{1}{\gamma_b^2}\left(\frac{D_{1b}}{D_b}\right)^{\alpha+2}\frac{1}{D_{1b}}\,[1 + (\xi_b')^2] , \qquad (35)$$

where the subscript b is related with the bulk velocity direction (the case of arbitrary temperature will be considered elsewhere). In the case of zero temperature, the directions of incoming and outcoming photons are related. Our goal is to find the nontrivial solution, $J^0(r, \mu)$, of this homogeneous problem and the appropriate spectral index, α, for which this solution exists. This problem can be solved by the iteration method, which involves the integration of the differential equation (eq. [21]) with the given boundary conditions (eqs. [24a]–[24b]) along the characteristics (eq. [32]) by using the method of Runge-Kutta.

The integration starts from the inner or the outer boundary, depending on the particular impact parameter, p (eq. [32]), which is in turn determined by the dimensionless radius, x ($x = r/r_s$), and the cosine, μ, of the angle between the photon direction and the radius vector at the given point, x.

If μ is positive at x, then the photon trajectory (the characteristics) can start at the inner boundary [if $x < 1.5$ and $p < x^{3/2}/(x-1)^{1/2}$ or if $x > 1.5$ and $p < (6.75)^{1/2}$] or at the outer boundary [if $x > 1.5$ and $p > (6.75)^{1/2}$].

All cases can be understood from Figure 1. The trajectories with the given p are related to the parallel lines to the x-axis, $y = p$. These lines start at $x = 1$ or at infinity. For example, if they start at infinity (i.e., having negative μ) and $p \geq (6.75)^{1/2}$, they must have the turning point with $\mu = 0$. Thus they must pass through the point with radius x_*, where $p = x_*^{3/2}/(x_* - 1)$. At this point, the cosine, μ, changes sign from minus to plus, and after that the trajectory enters through the point with radius x at the positive angle, $\theta = \cos^{-1}\mu$.

If the trajectory starts at $x = 1$ (i.e., having positive μ) and $p < (6.75)^{1/2}$, the parallel line, $y = p$, has no turning point.

If μ is negative at x, the trajectories starting at the internal boundary (having positive cosine, μ) have to pass through the turning point, $\mu = 0$ (changing the cosine sign), at radius x_*, where $p = x_*^{3/2}/(x_* - 1)$. However, if the trajectory that starts at the outer boundary has no turning points, the cosine, μ, is always negative along the trajectory.

This space integration is followed by integration, $J^0(r, \mu')$, over the angular variable, μ', in equation (23). As the initial distribution for $B^0(r, \mu)$ or $J^0(r, \mu)$ we can choose, for example, the uniform one. We use the Gaussian integration to calculate equation (23) (see, e.g., Abramowitz & Stegan 1970 for details of the methods). After quite a few iterations the iterative process converges, and it produces the eigenfunction source distribution, $B^0(r, \mu)$. The number of iterations, n, is related to the average number of scatterings that the soft photons undergo to transform into the hard ones (see, e.g., Titarchuk 1994). It is determined by the Thomson optical depth of the bulk motion atmosphere, $\tau_b = \tau_T(r_s)$. For the cold atmosphere ($T_e = 0$), the iteration number, n, is $\simeq 2\tau_b$. The convergence of the process can be done only with the proper choice of the value of a spectral index, α.

4. RESULTS OF CALCULATIONS AND DISCUSSION

Figure 2 presents the results of the calculations of the spectral indices as a function of mass accretion rates. It is clearly seen that the spectral index is a weak function of mass accretion rate in a wide range of $\dot{m}$, from 3–10. The asymptotic value of the spectral index for the high mass accretion rate is 1.75, which is between $\alpha = 2$, the value found by Blandford & Payne (1981) for the infinite medium, and $\alpha \approx 1.4$, the value found by TMK96 for the finite bulk motion atmosphere.

The latter two results are obtained in the nonrelativistic Fokker-Planck approximation. We see that the efficiency of the hard photon production in the cold bulk motion atmosphere is quite low. This is not the case if the plasma temperature is of order of a few kilo–electron volts or higher. The coupling effect between the bulk and local Maxwellian motion occurs when the bulk motion velocity is very close to the speed of light, i.e., when the matter is very close to the horizon. The upscattering effect increases significantly in the latter case. In the regime of the relativistic bulk motion, the electron distribution (eq. [2]) has a sharp maximum at $\theta = \pi$ and $V = c$. In the vicinity of the maximum, the distribution is characterized by the exponential shape, $F(r, P_e) \propto \exp(-\beta\gamma_b/2\gamma)$ (see also eq. [5]). More results and details regarding the relativistic coupling will be presented elsewhere.

As an example, in Figure 3 we demonstrate the zeroth moment of the source function distribution (the hard photon production). It is seen there that the distribution has a strong peak around $2r_s$. This means that the vicinity of the black hole is a place where the hard photons are produced

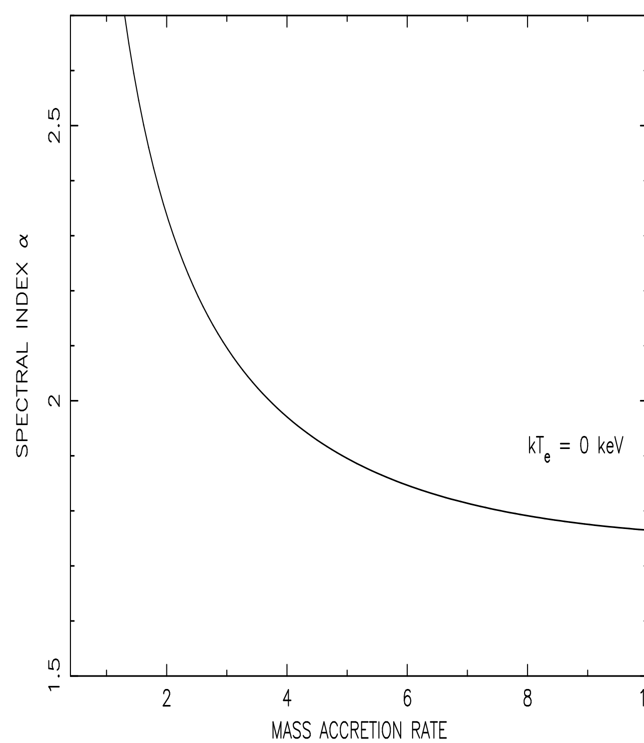

FIG. 2.—Plot of the energy spectral index (photo index − 1) vs. the total mass accretion rate.

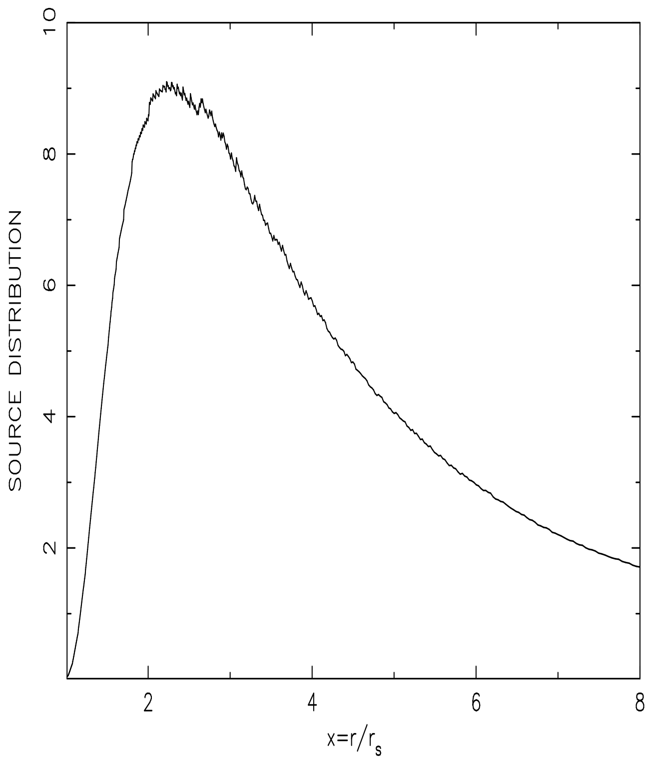

FIG. 3.—Plot of the source function distribution, eq. (B1), in arbitrary units vs. the dimensionless radius, $x = r/r_s$, for the dimensionless mass accretion rate, $\dot{m} = 4$, and $\mu = 0$.

by upscattering of the soft photons off the converging electrons.

Our calculations were made under the assumption of the free fall velocity profile. Since the energy gain due to the bulk motion Comptonization is not bigger than a factor 3 (if the spectral indices are higher than 1.5; TMK97), it follows that we can safely neglect the effects of the radiation force in our calculations if the injected photon flux in the converging inflow is of order of a few percent of the Eddington luminosity.

The assumption of Thomson scattering accepted in our solution restricts the relevant energy range to $E < m_e c^2$. Our approach cannot determine accurately the exact position of the high-energy cutoff that is formed owing to the downscattering of the very energetic photons in the bulk motion electron atmosphere. Additional efforts are required to confirm the qualitative estimates of the high-energy cutoff position as being of order $m_e c^2$ (TMK97). Laurent & Titarchuk (1998), by using Monte Carlo calculations, checked and confirmed these results for the spectral indices and the TMK97 estimates of the high-energy cutoff position.

As a conclusion, we would like to point out the definitive (according to our model) difference between black holes and neutron stars, as can be ascertained in their spectral properties while in their soft states, when their luminosity is dominated by the quasi-thermal, soft, component: in the black hole case, there should always be an additional steep power-law high-energy tail extending to energies $\sim m_e c^2$. This component should be absent in neutron star systems because the effect of the bulk motion is suppressed by the radiation pressure in this case.

We presented the full relativistic formalism and solved semianalytically the kinetic equation by using the TL95 eigenfunction method (see also Giesler & Kirk 1997) in the case of plasma infalling radially into a compact object with a soft source of input photons. We found that *the converging flow has crucial effects on the emergent spectrum for moderately super-Eddington mass accretion rates.*

Our power-law spectra can be applicable for the explanation of the observational situations in black hole candidate sources.

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APPENDIX A

GENERAL RELATIVISTIC RADIATIVE KINETIC FORMALISM

Let $\boldsymbol{u}$ be an arbitrary timelike unit vector ($\boldsymbol{u} \cdot \boldsymbol{u} \equiv u^\alpha u_\alpha = -1$, $u^0 > 0$) at some given spacetime point, x, and let $\boldsymbol{d}_1 x$, $\boldsymbol{d}_2 x$, $\boldsymbol{d}_3 x$ be the three arbitrary displacement vectors (with components $d_1 x^\alpha$, etc.) that span an element of hypersurface orthogonal to $\boldsymbol{u}$. By using orthogonality of two vectors, u_λ and

$$d\Sigma_\lambda = \sqrt{(-g)}\epsilon_{\alpha\beta\gamma\lambda} d_1 x^\alpha d_2 x^\beta d_3 x^\gamma , \tag{A1}$$

to the element of the hypersurface, one can get the invariant volume element orthogonal to the unit vector, $\boldsymbol{u}$, as follows:

$$dV = \sqrt{(-g)}\epsilon_{\lambda\alpha\beta\gamma} u^\lambda d_1 x^\alpha d_2 x^\beta d_3 x^\gamma , \tag{A2}$$

where $g = \det g_{\alpha\beta}$, $g_{\alpha\beta}$ is the metric tensor (see eq. [1]) and $\epsilon_{\lambda\alpha\beta\gamma}$ is the Levi-Civita alternating symbol, $\epsilon_{0123} = +1$. Since N is independent of $\boldsymbol{u}$, we may orient dV to make $u^0 = 1$, $u^i = 0$. Thus, in a local Minkowski frame it would be

$$dV = d^3 x . \tag{A3}$$

Similarly, with any arbitrary future pointing null vector $\boldsymbol{p}(\boldsymbol{p} \cdot p = 0)$ at the point x, one associates the 3-surface element of the zero-mass shell spanned by three displacement vectors, $\boldsymbol{d}_1 p$, $\boldsymbol{d}_2 p$, $\boldsymbol{d}_3 p$. The vectorial element satisfies

$$(dP)p_\lambda = \sqrt{(-g)}\epsilon_{\alpha\beta\gamma\lambda}\, d_1 p^\alpha d_2 p^\beta d_3 p^\gamma \ . \tag{A4}$$

Thus the invariant volume, dP, is

$$dP = \sqrt{(-g)}\epsilon_{ijk} \frac{d_1 p^i d_2 p^j d_3 p^k}{-p_0} \tag{A5}$$

because all products of the right-hand side of equation (A4) with p^λ for $\lambda \neq 0$ are equal to zero [this follows from the equality $\boldsymbol{p} \cdot \boldsymbol{p} = 0$, i.e., $dp^0 = (g_{ij} p^j dp^i)/g_{0\alpha} p^{\alpha 0}$ and the fact that a number of the even and odd transpositions of the combination $\alpha\beta\gamma$ are equal].

When local Minkowskian coordinates are used, with $\boldsymbol{d}_1 p$, $\boldsymbol{d}_2 p$, $\boldsymbol{d}_3 p$ tangent to the respective coordinate lines, this reduces to the familiar Lorentz invariant 3-surface element in the momentum space, d^3p/E (see, e.g., Landau & Lifshitz 1971). If we introduce the zero-shell spherical coordinates, (E, θ, φ), defined (in a local Minkowski frame) through the relationships

$$p^1 = E \cos\theta \ , \quad p^2 = E \sin\theta \cos\varphi \ , \quad p^3 = E \sin\theta \sin\varphi \ , \tag{A6}$$

where E is a photon energy, we then obtain

$$dP = p\, dE\, d\Omega \ . \tag{A7}$$

In any coordinate system, one can introduce the invariant infinitesimal volume element over the photon space, erected above the spacetime point under consideration. In other words, one can introduce an orthonormal tetrad in each point of spacetime and get the same result as equation (A7).

The number of photon worldlines crossing an infinitesimal "3-area," $d\Sigma_\lambda$ (or dV), at x, with 4-momenta in the range dP is

$$dN = N(x, \boldsymbol{p}) = N(x, \boldsymbol{p})(-\boldsymbol{p} \cdot \boldsymbol{u}) dP\, dV \ , \tag{A8}$$

where the inclusion of the projector, $(-\boldsymbol{p} \cdot \boldsymbol{u})$, takes care of perpendicular counting of photon states along $d\Sigma_\lambda$ or dV.

The above somewhat abstract but fully covariant definition of the photon distribution function, N, incorporates, in fact, the familiar density interpretation of the classical Boltzmann distribution function defined over the six-dimensional classical phase space of the photon gas. One simply has to take dV along an arbitrary, $t = \text{const}$ (see eq. [A3]), spacelike hypersurface equipped with orthogonal coordinates to recover from (A8) and (A7) that

$$dN = N(x, \boldsymbol{p}) d^3x\, d^3P \ , \tag{A9}$$

i.e., the familiar phase-space density interpretation of the distribution function. Comparing (A9) with the classical definition,

$$dN = f(x, E, \Omega) dE\, d\Omega\, d^3x \tag{A10}$$

(Ω is the unit 3-vector in the direction of the beam), one can obtain that

$$N(x, \boldsymbol{p}) = E^{-2} f(x, E, \Omega) \tag{A11}$$

in the local proper frame. Since N is a scalar, this gives its value in any other frame as well; it can be used to infer the transformation properties of f and the specific intensity, $I_\nu = Ef = E^3 N$.

The general relativistic transfer equation essentially describes the rate of change of the number of worldlines defined above, dN, as it is pushed along an arbitrary null spacetime geodesic. In the presence of external fields, the number of photon worldlines centered around $\boldsymbol{p}$ is altered. The alteration depends on the specific interaction the photon field is subjected to. In accordance with the relativistic form of Liouville's theorem (see, e.g., Lindquist 1966), $(-\boldsymbol{p} \cdot \boldsymbol{u}) dV\, dP$ remains invariant along the given set of trajectories. Hence the change in the number of worldlines within $(-\boldsymbol{p} \cdot \boldsymbol{u}) dV\, dP\, d\tau = dW\, dP$, where τ is some parameter changing along geodesics (Landau & Lifshitz 1971), is

$$\frac{dx^\alpha}{d\tau} = p^\alpha \ , \qquad \frac{dp^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \ , \tag{A12}$$

which is simply proportional to the change in N. Thus the transfer equation takes the following form:

$$p^\alpha \frac{DN}{dx^\alpha} = p^\alpha \frac{\partial N}{\partial x^\alpha} - \frac{\partial N}{\partial p^\beta} \Gamma^\beta_{\alpha\gamma} p^\alpha p^\gamma = S(N) \ , \tag{A13}$$

where a product, $S(N) dW\, dP$, denotes the change, dN, in all possible interactions taking place between the radiation field and external sources within dW. The Christoffel symbols are $\Gamma^\alpha_{\beta\gamma}$.

In the present paper we shall be interested exclusively in the Compton scattering of the radiation field of a locally Maxwellian distribution of electrons representing the plasma inflowing into black hole. To deal with the general covariant of the theory, we shall perform all the calculations relative to the local orthonormal frame or tetrad, $\{e_a(x)\}(a = 0, 1, 2, 3)$, tied to the metric (eq. [1]) (see, e.g., Landau & Lifshitz 1971). Relative to such a local orthonormal frame, an arbitrary photon momentum, $\boldsymbol{p}$, may be represented as

$$\boldsymbol{p} = p^a \boldsymbol{e}_a. \tag{A14}$$

If p^α is the usual contravariant components of $\boldsymbol{p}$ in some coordinate system, $\{x^\alpha\}$, then

$$\boldsymbol{p} = p^\alpha \boldsymbol{e}_\alpha \,, \tag{A15}$$

where $\{\boldsymbol{e}_\alpha\}$ is the induced coordinate basis. If we define

$$\boldsymbol{e}_a = e_a^\alpha \boldsymbol{e}_\alpha \quad \text{and } \boldsymbol{e}_\alpha = e_\alpha^a \boldsymbol{e}_a \,, \tag{A16}$$

then it is evident that

$$p^a = e_\alpha^a p^\alpha \quad \text{and } p^\alpha = e_a^\alpha p^a \tag{A17}$$

correspondingly. One can introduce the transformation that leaves the coordinate system unaltered but express vectors in terms of their tetrad components, namely

$$x'^\alpha = x^\alpha \,, \qquad p'^a = e_\alpha^a p^\alpha \,. \tag{A18}$$

By using this transformation, the radiative transfer equation can be rewritten in the form, which is very similar to equation (A13),

$$p^a e_a^\alpha \frac{\partial N}{\partial x^\alpha} - \frac{\partial N}{\partial p^b} \gamma_{ac}^b p^c p^a = S(N) \,, \tag{A19}$$

with

$$\gamma_{ac}^b = e_a^\alpha e_\gamma^b e_{a;\alpha}^\gamma \tag{A20}$$

being the Ricci rotation coefficients (see, e.g., Landau & Lifshitz 1971).

We introduce in the tangent space, at each event (t, r, θ, φ), the orthormal basis

$$\boldsymbol{e}_0 = e^{\Lambda/2} \boldsymbol{e}_t \,, \qquad \boldsymbol{e}_1 = e^{-\Lambda/2} \boldsymbol{e}_r \,, \qquad \boldsymbol{e}_2 = r^{-1} \boldsymbol{e}_\theta \,, \qquad \boldsymbol{e}_3 = (r \sin \theta)^{-1} \boldsymbol{e}_\varphi \,, \tag{A21}$$

where $e^{\Lambda/2} = f$ (see eq. [1]). Relative to this local orthormal frame, an arbitrary photon momentum can be represented as

$$\boldsymbol{p} = [p^{(0)}, p^{(1)}, p^{(2)} p^{(3)}] = (E, E \cos \theta, E \sin \theta \cos \varphi, E \sin \theta \sin \varphi) \,, \tag{A22}$$

where E is the physical photon energy as measured by an orthonormal observer, and θ, φ are local polarlike phase-space coordinates. The angle, θ, is measured relative to the outward radial direction at the point under consideration. For our problem we shall assume the steady state conditions for the accretion and radiation field, respectively, which implies that N is time independent and spherically symmetric. Furthermore, N should be axially symmetric around the radial direction, which results in $N = N(r, E, \mu)$, where $\mu = \cos \theta$.

Thus the coefficients e_a^α and e_α^a are given by

$$e_1^\alpha = \delta_1^\alpha e^{\Lambda/2} \,, \qquad e_2^\alpha = \delta_2^\alpha e^{-\Lambda/2} \,, \qquad e_3^\alpha = \delta_3^\alpha / r \,, \qquad e_4^\alpha = \delta_2^\alpha / (r \sin \theta) \tag{A23}$$

and

$$e_1^a = \delta_1^a e^{-\Lambda/2} \,, \qquad e_2^a = \delta_2^a e^{\Lambda/2} \,, \qquad e_3^a = \delta_3^a r \,, \qquad e_4^a = \delta_4^a r \sin \theta \,. \tag{A24}$$

Here δ_i^j is the Kronecker delta (which is equal to 0 if $i \neq j$ and equal to 1 if $i = j$).

By using these formulas, one can calculate the rotational coefficients, γ_{ac}^b (see eq. [A20]), and get the relativistic radiative transfer equation in the form

$$e^{-\Lambda/2} \left[\mu \frac{\partial N}{\partial r} + \frac{(1 - \mu^2)}{r} \frac{\partial N}{\partial \mu} \right] + e^{-\Lambda/2} \frac{\Lambda'}{2} \left[\mu E \frac{\partial N}{\partial E} + (1 - \mu^2) \frac{\partial N}{\partial \mu} \right] = \frac{S(N)}{E} \,. \tag{A25}$$

The terms in the first pair of brackets are obtained by using formulas (A24) for $e_{(1a)}^{(1\alpha)}$ and $e_{(2a)}^{(2\alpha)}$ and the fact that the derivative is

$$\frac{d\mu}{d\theta} = - \sin \theta e^{-\Lambda/2} \,.$$

The first term in the second set of brackets in equation (A25) is related with the Ricci coefficient, γ_{00}^1, and the second term there appears as a result of calculating the product $\gamma_{00}^1 (p^0)^2 (\partial N / \partial p^1)$. We also use the fact that the partial derivative is

$$\frac{\partial N}{\partial p^1} = \frac{1 - \mu^2}{E} \frac{\partial N}{\partial \mu} + \mu \frac{\partial N}{\partial E} \,. \tag{A26}$$

APPENDIX B

NUMERICAL CALCULATION OF THE SOURCE FUNCTION

We use the iteration method to solve the radiative transfer problem (eqs. [21]–[24b]), which involves the calculation of the source function, B^0. In order to calculate the source function, we transform it into the degenerate form by using the phase function formula (eq. [35]). We have

$$B^0(r, \mu) = \tfrac{1}{2} \int_{-1}^{1} p^0(\mu, \mu')J^0(r, \mu')d\mu' = (I_1 + I_2 + I_4 + I_7) + (I_3 + I_5)\mu + (I_6 - I_7)\mu^2 . \tag{B1}$$

The following integrals, I_i, are computed by using the Gaussian integration formula (see Abramowitz & Stegan 1970 for details of the Gaussian integration method):

$$I_1 = \frac{3}{4} \frac{1}{\gamma_b^2} \frac{1}{D_b^{\alpha+2}} \int_{-1}^{1} D_{1b}^{\alpha+1} J^0(r, \mu')d\mu' , \tag{B2}$$

$$I_2 = -\frac{3}{4} \frac{1}{\gamma_b^4} \frac{1}{D_b^{\alpha+3}} \int_{-1}^{1} D_{1b}^{\alpha} J^0(r, \mu')d\mu' , \tag{B3}$$

$$I_3 = \frac{3}{4} \frac{1}{\gamma_b^4} \frac{1}{D_b^{\alpha+3}} \int_{-1}^{1} \mu' D_{1b}^{\alpha} J^0(r, \mu')d\mu' , \tag{B4}$$

$$I_4 = \frac{3}{8} \frac{1}{\gamma_b^6} \frac{1}{D_b^{\alpha+4}} \int_{-1}^{1} D_{1b}^{\alpha-1} J^0(r, \mu')d\mu' , \tag{B5}$$

$$I_5 = -\frac{3}{4} \frac{1}{\gamma_b^6} \frac{1}{D_b^{\alpha 4}} \int_{-1}^{1} \mu' D_{1b}^{\alpha-1} J^0(r, \mu')d\mu' , \tag{B6}$$

$$I_6 = \frac{3}{8} \frac{1}{\gamma_b^6} \frac{1}{D_b^{\alpha+4}} \int_{-1}^{1} (\mu')^2 D_{1b}^{\alpha-1} J^0(r, \mu')d\mu' , \tag{B7}$$

$$I_7 = \frac{3}{16} \frac{1}{\gamma_b^6} \frac{1}{D_b^{\alpha+4}} \int_{-1}^{1} [1 - (\mu')^2] D_{1b}^{\alpha-1} J^0(r, \mu')d\mu' . \tag{B8}$$

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