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LIKELY VALUES OF THE COSMOLOGICAL CONSTANT

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ABSTRACT

In theories in which the cosmological constant takes a variety of values in different "subuniverses," the probability distribution of its observed values is conditioned by the requirement that there be someone to measure it. This probability is proportional to the fraction of matter that is destined to condense out of the background into mass concentrations large enough to form observers. We calculate this "collapsed fraction" with a simple, pressure-free, spherically symmetric, nonlinear model for the growth of density fluctuations in a flat universe with arbitrary value of the cosmological constant, applied in a statistical way to the observed spectrum of density fluctuations at recombination. From this the probability distribution for the vacuum energy density ρ_V for Gaussian random density fluctuations is derived analytically. (The conventional quantity λ_0 is the vacuum energy density in units of the critical density at present, $\lambda_0 = \rho_V/\rho_{\rm crit,0}$, where $\rho_{\rm crit,0} = 3H_0^2/8\pi G$.) It is shown that the results depend on only one quantity, $\sigma^3\bar{\rho}$, where σ^2 and $\bar{\rho}$ are the variance and mean value of the fluctuating matter density field at recombination, respectively. To calculate σ, we adopt the flat, cold dark matter model with a nonzero cosmological constant and fix the amplitude and shape of the primordial power spectrum in accordance with data on cosmic microwave background anisotropy from the *COBE* satellite DMR experiment. A comparison of the results of this calculation of the likely values of ρ_V with present observational bounds on the cosmological constant indicates that the small, positive values of ρ_V (from 1 to 3 times greater than the present cosmic mass density) suggested recently by several lines of evidence are not very unlikely values to observe, even if there is nothing in the a priori probability distribution that favors such relatively small values.

Subject headings: cosmology: theory — galaxies: formation

1. INTRODUCTION

Though the evidence is still equivocal, there are persistent hints that the vacuum energy density[1] ρ_V is positive and up to 3 times greater than the present cosmic mass density ρ_0.[2] From the point of view of fundamental physics, such a value seems absurd. Crude estimates indicate a value of ρ_V some 120 orders of magnitude greater than ρ_0, and while it is hard enough to imagine any sort of symmetry or adjustment mechanism that could make ρ_V vanish (for a litany of failed attempts, see Weinberg 1989), it would be even more peculiar for fundamental physical theory to dictate a nonzero value for ρ_V that happens to be comparable to the cosmic mass density ρ_0 at this particular moment in the history of the universe.

As far as we know, the only way to understand a value of ρ_V comparable to ρ_0 is based upon a weak form of the anthropic principle. The application of this principle has become more plausible in recent years, because in several current theories, the cosmological constant does not have a fixed value, but takes a variety of values with varying probabilities. For instance, Hawking (1983, 1984) showed that the introduction of a three-form gauge field $A_{\mu\nu\lambda}$ yields a state vector for the universe that is a superposition of terms with different values for the cosmological constant. Coleman (1988a) subsequently showed that the effect of wormholes in quantum gravity is to make the state vector a superposition of terms, in which *any* coupling coefficient in the Lagrangian that is not fixed by symmetries takes all possible values. Also, in chaotic inflation (Linde 1986, 1987, 1988) the observed big bang is just one of an infinite number of expanding regions, in each of which the various fields that affect the vacuum energy can take different values. For brevity we will refer to parts of the "universe" in which the cosmological constant takes different values, such as terms in the state vector and local bangs, as *subuniverses*.

In any theory of this general sort, the measured value of the vacuum energy density ρ_V would be much smaller than the value expected on dimensional grounds in elementary particle physics, not because there is any physical principle that makes it small in all subuniverses, but because it is only in the subuniverses in which it is small that there would be anyone to measure it. This paper will show how to calculate the probability distribution of the values of ρ_V that would be observed under these circumstances.

An earlier paper (Weinberg 1987) pointed out that the anthropic limit on the value of ρ_V for $\rho_V > 0$ arises from the

[1] By ρ_V is meant the sum of a term $\Lambda/8\pi G$, where Λ is the cosmological constant appearing in the Einstein field equations, plus the contribution to the vacuum energy density from quantum fluctuations. The conventional quantity λ_0 is the vacuum energy density in units of the critical density at present, $\lambda_0 = \rho_V/\rho_{\rm crit,0}$, where $\rho_{\rm crit,0} = 3H_0^2/8\pi G$.

[2] For reviews and earlier references, see Ostriker & Steinhardt (1995); Krauss & Turner (1995). This conclusion has recently been challenged by preliminary results of measurements by Perlmutter et al. (1997) of redshifts and distances for distant Type Ia supernovae.

requirement that ρ_V should not be so large as to prevent the formation of galaxies. This paper suggested that this requirement implies a value of ρ_V roughly comparable to the cosmic density of nonrelativistic matter at the time that the earliest galaxies form, because, if ρ_V were much larger than this, then galaxies could not form and there would be no observers, while there did not seem to be any reason for ρ_V to be much smaller than this. Since then, galaxies have continued to be found at higher and higher redshifts, and hence at higher and higher values of the cosmic mass density, and it is becoming clear that such values of ρ_V are already ruled out. A galaxy with redshift $z \approx 4$ was formed when the cosmic mass density was more than, and perhaps considerably more than, $(1+z)^3 \approx 125$ times the present mass density, which is much greater than the observational upper limit on ρ_V.

On the other hand, it is much more likely that the value of ρ_V in our subuniverse is comparable to the average or median value measured by astronomers in all subuniverses, rather than the anthropic upper bound, so that its value should be compared with the cosmic mass density at the time of formation of typical galaxies, rather than of the earliest galaxies.[3] Here we will present a detailed analysis, which allows a calculation of the probability distribution of ρ_V from a knowledge of the spectrum of density fluctuations at recombination. The results suggest a much smaller likely value of ρ_V than the anthropic upper bound, a value that may not be in conflict with present observational bounds.

In § 2 we describe how to calculate the probability distribution for the vacuum energy density ρ_V that would be observed in various subuniverses in which there is someone to observe it. For $\rho_V > 0$, this is simply related to the fraction of matter that condenses into galaxies. We evaluate this fraction and the resulting probability distribution in § 3 under the assumption of Gaussian random density fluctuations in the cosmic mass density at the time of recombination. The results depend only on the standard deviation σ of these fluctuations at the time of recombination. In § 4 we calculate σ, adopting the cold dark matter (CDM) model for the power spectrum of these density fluctuations and assuming a flat universe with a nonzero cosmological constant (sometimes referred to as the flat "ΛCDM" model). The amplitude of the density fluctuations is fixed by the data on cosmic microwave background anisotropy from the *COBE* satellite DMR experiment. Section 5 presents a review of the range of values of the cosmological constant allowed by current observational and theoretical constraints. These values turn out to be somewhat smaller than the median for all astronomers, so in § 6 our results are used to assess how likely it is that astronomers in random subuniverses would find ρ_V to be as small as various assumed possible values. Our conclusions are presented. More details of our calculations and results may be found in the original preprint version of this paper (astro-ph/9701099).

[3] See Weinberg (1996). This is essentially the same as what was called the "principle of mediocrity" by Vilenkin (1995a, 1995b, 1996a, 1996b). Vilenkin did not undertake a detailed calculation of the probability distribution of the cosmological constant. A calculation of this sort was done by Efstathiou (1995), but it contained some errors (Vilenkin 1996b; Weinberg 1996). Efstathiou's calculation was done numerically, using linear perturbation theory and what is believed to be a realistic model of initial perturbations, while the calculation presented here is thoroughly nonlinear but concentrates on a single spherically symmetric density fluctuation, so that it is possible to understand the results analytically.

2. THE PROBABILITY DISTRIBUTION

We assume that the fundamental principles of physics prescribe some a priori probability distribution $\mathscr{P}(\rho_V)$, which gives the probability of a net vacuum energy density between ρ_V and $\rho_V + d\rho_V$ as $\mathscr{P}(\rho_V)d\rho_V$. What we want is the probability distribution $\mathscr{P}_{\rm obs}(\rho_V)d\rho_V$ that a random observer in any subuniverse will measure values of the vacuum energy density in this range. This is given by

$$\mathscr{P}_{\rm obs}(\rho_V) = \frac{\mathscr{A}(\rho_V)\mathscr{P}(\rho_V)}{\int_{-\infty}^{\infty}\mathscr{A}(\rho_V')\mathscr{P}(\rho_V')d\rho_V'}\,, \quad (1)$$

where $\mathscr{A}(\rho_V)$ is the mean number of astronomers making independent measurements of the vacuum energy density in subuniverses with vacuum energy density ρ_V.[4]

Now, either there is something in fundamental physics that makes the value $\rho_V = 0$ special, or there is not. If there is something special about the value $\rho_V = 0$, then it is to this that we should look for a solution of the cosmological constant problem, and not to anthropic considerations. For example, Coleman's (1988b) study of wormholes suggested that the probability distribution of ρ_V is sharply peaked at zero, as had previously been argued on other grounds by Hawking (1983, 1984) and Baum (1984). If there really is such a peak, then this could solve the cosmological constant problem, provided, of course, that the observed value of ρ_V is actually found to vanish. Unfortunately, arguments of Fischler et al. (1989) have cast doubt on the existence of a sharp peak at $\rho_V = 0$.

More generally, the essence of the cosmological constant problem is that, even though the value $\rho_V \simeq 0$ has important implications for the evolution of galaxies and hence of life, there does not seem to be anything special in fundamental physics about this value. It should be kept in mind that ρ_V is not proportional to the Einstein cosmological constant Λ, but is rather the sum of a term proportional to the cosmological constant plus a term arising from zero-point fluctuations of various fields, so that *to have* $\rho_V = 0$ *requires a precise cancellation of the energy of these zero-point fluctuations by Einstein's* Λ. There may be something special about the value $\Lambda = 0$ which makes the probability distribution vanish or diverge there, but this would not imply anything about the distribution function near $\rho_V = 0$. To make an analogy, although the probability distribution of temperatures in the Antarctic ice must vanish at 0 K and 0°C, since these are the limits of the range of temperatures in which water freezes, we would hardly expect this probability distribution to vanish or diverge at 0°F.

We shall therefore assume here that there is nothing special about the behavior of the a priori probability distribution $\mathscr{P}(\rho_V)$ near $\rho_V = 0$, so that $\mathscr{P}(\rho_V)$ has neither a zero nor a singularity at $\rho_V = 0$. In particular, this rules out a probability distribution that goes as some power of ρ_V or of $\ln \rho_V$. Of course, we are not ruling out the possibility that $\mathscr{P}(\rho_V)$ may include terms that vary as powers of ρ_V, or that it may be singular at some values of ρ_V; we are only assuming that there are no singularities or zeros at $\rho_V = 0$,

[4] We have not thought through the problems associated with infinite subuniverses, where $\mathscr{A}$, if nonzero, is infinite. Presumably, in this case we should take $\mathscr{A}$ to be the ratio of the number of independent astronomers in any finite comoving volume to the entropy or baryon number in that volume.

because the essence of the cosmological constant problem is that we do not know of any reasons why there should be anything special about the behavior of $\mathscr{P}(\rho_V)$ at $\rho_V = 0$.[5]

We will also make the qualitative assumption that when ρ_V is not near some zero or singularity of $\mathscr{P}(\rho_V)$, the range of ρ_V values over which $\mathscr{P}(\rho_V)$ varies substantially is some density characteristic of elementary particle physics, such as an electron mass per cubic electron Compton wavelength, or a Planck mass per cubic Planck length. The precise value does not matter—the important point is, as we shall see, that although we assume that there is nothing special about the behavior of the a priori probability distribution $\mathscr{P}(\rho_V)$ near $\rho_V = 0$, the number $\mathscr{A}(\rho_V)$ of astronomers is very sharply peaked at $\rho_V = 0$. The range of ρ_V values for which $\mathscr{A}(\rho_V)$ is nonnegligible is so narrow, compared to the range over which $\mathscr{P}(\rho_V)$ varies substantially, that in equation (1) we may take $\mathscr{P}(\rho_V)$ as constant. The value of this constant then cancels in equation (1), which becomes

$$\mathscr{P}_{\rm obs}(\rho_V) = \frac{\mathscr{A}(\rho_V)}{\int_{-\infty}^{\infty} \mathscr{A}(\rho'_V) d\rho'_V} . \tag{2}$$

The evolution of galaxies and astronomers depends on a variety of constants of nature other than ρ_V, and the values of these other constants in the various subuniverses may be correlated with the values of ρ_V, but the range of values of ρ_V where $\mathscr{A}(\rho_V)$ is nonnegligible is so small, compared to the energy densities typical of elementary particle physics, that, in evaluating $\mathscr{A}(\rho_V)$ within this range, we can take all other fundamental constants to have fixed values, the values we observe in our subuniverse.[6]

For $\rho_V > 0$, the falloff of $\mathscr{A}(\rho_V)$ as ρ_V increases above $\rho_V = 0$ arises from the decreased chance of forming gravitationally bound systems (Weinberg 1987). For $\rho_V < 0$ (and $k = 0$), any fluctuation with positive overdensity will eventually undergo gravitational collapse; the falloff of $\mathscr{A}(\rho_V)$ as ρ_V decreases below $\rho_V = 0$ arises from the shortening of the time available for life before the recollapse of the whole subuniverse (Barrow & Tipler 1986). Roughly speaking, $\mathscr{A}(\rho_V)$ falls off at positive values of ρ_V that are larger than the cosmic density ρ_+ at the time at which typical galaxies form, because for $\rho_V > \rho_+$, galaxy formation is suppressed, and $\mathscr{A}(\rho_V)$ falls off at negative values of ρ_V that are larger in absolute value than the cosmic density ρ_- at the time at which typical astronomers live, because for $\rho_V < -\rho_-$, the appearance of astronomers is cut short by the recollapse of the universe. Since ρ_+ is bound to be much larger than ρ_- (because there are no astronomers until after galaxies form), it is plausible that $\int_{-\infty}^{0} \mathscr{A}(\rho_V) d\rho_V$ is much less than $\int_0^{\infty} \mathscr{A}(\rho_V) d\rho_V$, so that it is much more likely for astronomers to observe positive values of ρ_V than negative values. For this reason, in what follows we will estimate $\mathscr{P}_{\rm obs}(\rho_V)$ only for $\rho_V > 0$ and neglect the contribution of negative values of ρ_V in the denominator of equation (2). [If it turns out that there is an appreciable probability of observing negative values of ρ_V, then the function $\mathscr{P}_{\rm obs}(\rho_V)$ calculated below should be interpreted as giving the probability distribution of ρ_V for those astronomers who observe $\rho_V > 0$.)

Once a fluctuation in the cosmic mass distribution undergoes gravitational condensation, its subsequent evolution is essentially independent of ρ_V for $\rho_V > 0$, so the ratio of astronomers to mass in galaxies may be taken as independent of ρ_V. Also, it is reasonable to suppose that the total amount of matter in a subuniverse in theories of chaotic inflation is independent of ρ_V within the narrow range of values of ρ_V that are anthropically allowed. The number of astronomers $\mathscr{A}(\rho_V)$ who can measure ρ_V in any subuniverse should therefore be proportional to the fraction $\mathscr{F}(\rho_V)$ of matter incorporated in galaxies, so that for $\rho_V > 0$, equation (2) may be written as

$$\mathscr{P}_{\rm obs}(\rho_V) = \frac{\mathscr{F}(\rho_V)}{\int_0^{\infty} \mathscr{F}(\rho'_V) d\rho'_V} . \tag{3}$$

To calculate $\mathscr{F}(\rho_V)$ we note that the spectrum of initial density fluctuations at recombination can be regarded as independent of ρ_V, because values of ρ_V for which galaxy formation is possible are much smaller than the cosmic mass density at or before the time of recombination. Our problem is then to calculate the fraction $\mathscr{F}$ of matter that undergoes gravitational condensation into galaxies as a function of ρ_V for fixed initial conditions at recombination. Ironically, while the tininess of observationally allowed values of ρ_V creates the cosmological constant problem in the first place, it is the tininess of the range of anthropically allowed values of ρ_V that offers the possibility of a realistic calculation of $\mathscr{P}_{\rm obs}(\rho_V)$.

To see how this can work in practice, we will carry out an illustrative calculation of $\mathscr{F}(\rho_V)$ and use it to assess how well anthropic arguments succeed in accounting for a vacuum density in the observationally allowed range. Earlier work (Weinberg 1987) used a very simple model (Peebles 1967) of galaxy formation from isolated spherically symmetric pressureless fluctuations. This calculation was improved in the report of a recent conference talk (Weinberg 1996) by using the well-known model of Gunn & Gott (1972), which also assumes isolated fluctuations with spherical symmetry and zero pressure but includes the infall of matter from outside the initially overdense ball. Here we will also take into account the facts, with space filled with fluctuations, that there is a limit to the mass that can accrete onto any one fluctuation and that there must be regions of negative as well as positive overdensity, and we shall use this model to carry out detailed numerical calculations.

Consider a spherically symmetric pressureless fluctuation, consisting at recombination of a spherical core of volume V and positive average fractional overdensity δ [i.e., $\delta \equiv (\rho - \bar{\rho})/\bar{\rho}$, where ρ is the average density inside V, and $\bar{\rho}$ is the cosmic mean density at recombination], surrounded by a spherical shell of volume U of constant fractional underdensity, taken to have the value $(V/U)\delta$, so that the average overdensity within the whole fluctuation is zero. Outside this shell are other fluctuations, about which we do not need to say anything, except for the assumption that they do not seriously interfere with the spherical symmetry of the fluctuation in question. For simplicity, we will take V/U to be the same for all fluctuations.

[5] To avoid a common misapprehension about anthropic arguments, perhaps we should explain that in this work we are not using the existence of galaxies and astronomers merely as data that set constraints on the value of ρ_V in our subuniverse. Astronomical observations have already provided much stronger constraints. Our purpose here is not to decide the value of ρ_V in our subuniverse, but to find a way of understanding *why* ρ_V should take a value in the range suggested by astronomical data.

[6] If various constants of nature and initial conditions vary from one subuniverse to another independently of the values of ρ_V, then $\mathscr{P}_{\rm obs}(\rho_V) d\rho_V$ is the probability that, if the other constants and initial conditions take the values we observe, the vacuum energy density will be observed to be between ρ_V and $\rho_V + d\rho_V$.

As shown in earlier work (Weinberg 1987), for $k = 0$, the core will undergo gravitational collapse if[7]

$$\delta \geq \left(\frac{729\rho_V}{500\bar{\rho}}\right)^{1/3} . \tag{4}$$

In addition, that portion U' of the outer shell will fall into the core, for which the average fractional overdensity within the volume $V + U'$ saturates this inequality:

$$\frac{V\delta - U'(V/U)\delta}{V + U'} = \left(\frac{729\rho_V}{500\bar{\rho}}\right)^{1/3} . \tag{5}$$

The fraction of the total mass $\bar{\rho}(U + V)$ that suffers gravitational contraction will be

$$\mathscr{F}(\delta, \rho_V) = \frac{(1 + \delta)V + [1 - (V/U)\delta]U'}{U + V} . \tag{6}$$

Solving equation (5) for U' and using the result in equation (6), we find for the fraction of mass that undergoes gravitational contraction:

$$\mathscr{F}(\delta, \rho_V) = (V/U)\delta\left[\frac{1 + (729\rho_V/500\bar{\rho})^{1/3}}{(729\rho_V/500\bar{\rho})^{1/3} + (V/U)\delta}\right] . \tag{7}$$

We require that the total density be everywhere nonnegative, so that

$$\delta \leq U/V . \tag{8}$$

For δ satisfying this inequality, equation (7) gives $\mathscr{F} \leq 1$, so that no fluctuation can get more than its fair share of mass.

In what follows we will assume that the fluctuation number density $\mathscr{N}(\delta)$ is negligible for initial fluctuations that are not everywhere weak, so that we will be integrating only over fluctuations for which $\delta \ll 1$ *and* $\delta \ll U/V$. Also, for any anthropically allowed cosmological constant, ρ_V is much less than the mass density $\bar{\rho}$ at recombination, so we will drop the term $(729\rho_V/500\bar{\rho})^{1/3}$ in the numerator (but not in the denominator) of the fraction in square brackets in equation (7). The fraction of mass winding up in galaxies is then

$$\mathscr{F}(\rho_V) = \int_{(729\rho_V/500\bar{\rho})^{1/3}}^{\infty} d\delta \mathscr{N}(\delta)\mathscr{F}(\delta, \rho_V)$$

$$= \int_{(729\rho_V/500\bar{\rho})^{1/3}}^{\infty} d\delta \frac{(V/U)\delta\mathscr{N}(\delta)}{(729\rho_V/500\bar{\rho})^{1/3} + (V/U)\delta} , \tag{9}$$

where $\mathscr{N}(\delta)d\delta$ is the fraction of all positive fluctuations that have average core fractional overdensity between δ and $\delta + d\delta$, normalized so that

$$\int_0^{\infty} \mathscr{N}(\delta)d\delta = 1 . \tag{10}$$

The normalization integral in equation (3) can be calculated by interchanging the order of integration over δ and ρ_V and expressing ρ_V in terms of a dimensionless variable x, defined by

$$\rho_V = \frac{500x^3\bar{\rho}\delta^3}{729} . \tag{11}$$

[7] It was originally assumed (Weinberg 1987) that the overdensity $\delta\rho$ was uniform, but these results actually hold for arbitrary spherically symmetric fluctuations, with δ interpreted as the *average* fractional overdensity.

Equation (9) then gives

$$\int_0^{\infty} \mathscr{F}(\rho_V)d\rho_V = \frac{500}{243} \bar{\rho}\langle\delta^3\rangle(V/U)I_0(V/U) , \tag{12}$$

where $I_0(s)$ is the function

$$I_0(s) \equiv \int_0^1 \frac{x^2dx}{x + s} = \frac{1}{2} - s - s^2 \ln\left(\frac{s}{1 + s}\right) , \tag{13}$$

and the brackets denote an average over all positive fluctuations

$$\langle f(\delta)\rangle \equiv \int_0^{\infty} \mathscr{N}(\delta)f(\delta)d\delta . \tag{14}$$

The normalized probability distribution (3) for the observed vacuum energy density is then

$$\mathscr{P}_{\rm obs}(\rho_V) = \frac{243}{500} \frac{1}{\langle\delta^3\rangle I_0(V/U)} \times \int_{(729\rho_V/500\bar{\rho})^{1/3}}^{\infty} d\delta \frac{\delta\mathscr{N}(\delta)}{(729\rho_V/500\bar{\rho})^{1/3} + (V/U)\delta} , \tag{15}$$

with all quantities on the right-hand side referring to the time of recombination. In using equation (15), we will need to make some assumption about the shape parameter $s \equiv V/U$. The value $s = 0$ corresponds to the limit in which each positive fluctuation is isolated, surrounded by an infinite volume of compensating underdensity (at a total density infinitesimally below its mean value $\bar{\rho}$), the case considered by Weinberg (1996). Values of s much greater than unity correspond to the limit in which the additional mass associated with the compensating underdense volume U is insignificant compared with that contained within the positive fluctuation in volume V, the case considered in Weinberg (1987). The value $s = 1$ corresponds to the case in which every positive fluctuation is surrounded by an *equal* volume of compensating negative fluctuation. This latter value is the one most relevant to a Gaussian random distribution of linear density fluctuations, since the volumes occupied by positive and negative density fluctuations of equal amplitude are exactly equal in that case. Thus we will concentrate on the value $s = 1$ when we apply our analysis to the observed universe in what follows.

Fortunately most of our results turn out to be almost independent of the value chosen for s. For instance, the mean value of any power of ρ_V calculated from equation (15) is

$$\langle\rho_V^n\rangle = \left(\frac{500}{729}\right)^n \frac{\bar{\rho}^n\langle\delta^{3n+3}\rangle}{\langle\delta^3\rangle} \frac{I_n(V/U)}{I_0(V/U)} , \tag{16}$$

where $I_n(s)$ is the function

$$I_n(s) \equiv \int_0^1 \frac{x^{3n+2}dx}{x + s} . \tag{17}$$

The average $\langle\rho_V^n\rangle$ in equation (16) is taken over all subuniverses, in contrast with the average over fluctuations in $\langle\delta^n\rangle$, which is calculated using the weight function $\mathscr{N}(\delta)$, as in equation (14). In particular, the mean value observed for ρ_V is given by

$$\langle\rho_V\rangle = \frac{500}{729} \frac{\langle\delta^6\rangle\bar{\rho}}{\langle\delta^3\rangle} \frac{I_1(V/U)}{I_0(V/U)} , \tag{18}$$

with

$$I_1(s) = \frac{1}{5} - \frac{s}{4} + \frac{s^2}{3} - \frac{s^3}{2} + s^4 + s^5 \ln\left(\frac{s}{1+s}\right) . \quad (19)$$

The ratio $I_1(s)/I_0(s)$ in equation (18) turns out to be nearly constant; it drops from a value of 0.5 when $s \equiv V/U \gg 1$, corresponding to no infall, to a value of 0.4 when $s \ll 1$, corresponding to well-separated fluctuations. In the case $s = 1$ of greatest interest, the ratio I_1/I_0 takes the value

$$I_1(1)/I_0(1) = \frac{47/60 - \ln 2}{-1/2 + \ln 2} = 0.467 . \quad (20)$$

The insensitivity of our results to the value of s suggests that they also may not be much affected by the crudeness of our treatment of the effect of one fluctuation on another.

3. GAUSSIAN DENSITY FLUCTUATIONS

To go further, we must make some assumption about the form of the fluctuation probability distribution $\mathscr{N}(\delta)$ at an early epoch, such as that of recombination. Current data on the anisotropy of the cosmic microwave background on large angular scales and on the large-scale clustering properties of galaxies, as well as theoretical predictions of the origin of density fluctuations by quantum processes in the early universe in inflationary cosmology models, are consistent with the assumption that the primordial fluctuations were isotropic Gaussian random noise of very small amplitude. In this case the fluctuation distribution has the form

$$\mathscr{N}(\delta) = \frac{1}{\sigma}\left(\frac{2}{\pi}\right)^{1/2} \exp\left(-\frac{\delta^2}{2\sigma^2}\right) . \quad (21)$$

The mean values of powers of δ are given in terms of the variance σ^2 by

$$\langle \delta^N \rangle = \frac{2^{N/2}\sigma^N}{\pi^{1/2}} \Gamma\left(\frac{N+1}{2}\right) . \quad (22)$$

Equation (16) then gives

$$\langle \rho_V^n \rangle = \Gamma\left(\frac{3n+4}{2}\right)\left[\frac{1000(2^{1/2}\sigma^3\bar{\rho})}{729}\right]^n \frac{I_n(s)}{I_0(s)} , \quad (23)$$

and, in particular, the mean vacuum energy density is

$$\langle \rho_V \rangle = \left[\frac{625(2\pi)^{1/2}\sigma^3\bar{\rho}}{243}\right]\frac{I_1(s)}{I_0(s)} , \quad (24)$$

where, as before, $s \equiv V/U$. This gives the numerical values

$$\langle \rho_V \rangle = \sigma^3\bar{\rho} \times \begin{cases} 2.579 & s = 0 , \\ 3.010 & s = 1 , \\ 3.224 & s = \infty . \end{cases} \quad (25)$$

Also, using the Gaussian distribution (21) in equation (15) and writing

$$\delta = \left(\frac{729\rho_V}{500\bar{\rho}}\right)^{1/3}\left(\frac{x}{\beta}\right)^{1/2} , \quad (26)$$

we find the differential probability distribution

$$\mathscr{P}_{\rm obs}(\rho_V)d\rho_V = \frac{\beta^{1/2}d\beta}{2I_0(s)}\int_\beta^\infty \frac{e^{-x}dx}{sx^{1/2} + \beta^{1/2}} , \quad (27)$$

where

$$\beta \equiv \frac{1}{2\sigma^2}\left(\frac{729\rho_V}{500\bar{\rho}}\right)^{2/3} . \quad (28)$$

The probability of a vacuum energy density less than ρ_V is then

$$\mathscr{P}(\le \rho_V) = 1 - (1+\beta)e^{-\beta} + \frac{1}{2I_0(s)}\int_\beta^\infty e^{-x} \times \left\{-2s(\beta x)^{1/2} + \beta + 2s^2 x \ln\left[\frac{1}{s}\left(\frac{\beta}{x}\right)^{1/2} + 1\right]\right\}dx . \quad (29)$$

By combining equations (24) and (28), we see that the parameter β in equations (27) and (29) may be expressed in terms of $s \equiv V/U$ and the ratio $\rho_V/\langle\rho_V\rangle$ of the vacuum energy density to its mean value

$$\beta = \frac{\pi^{1/3}}{4}\left[15\frac{\rho_V}{\langle\rho_V\rangle}\frac{I_1(s)}{I_0(s)}\right]^{2/3} . \quad (30)$$

Thus, the probability of observing a vacuum energy density in a certain range depends only on the values of $\rho_V/\langle\rho_V\rangle$ and s. The analysis of initial fluctuations will enter here only as a means of calculating the parameter $\bar{\rho}\sigma^3$, and hence $\langle\rho_V\rangle$. Figure 1 shows the logarithmic differential probability distribution $\rho_V\, d\mathscr{P}(\le\rho_V)/d\rho_V$ plotted against $\rho_V/\langle\rho_V\rangle$ for various values of s.

For $s = 1$ the values of ρ_V for which $\mathscr{P}(\le\rho_V)$ is 0.05, 0.5, and 0.95 are $0.0590\sigma^3\bar{\rho}$, $1.49\sigma^3\bar{\rho}$ $(=0.49\langle\rho_V\rangle)$, and $11.1\sigma^3\bar{\rho}$, respectively. We see that the distribution of ρ_V values is quite broad; it would not be very unlikely for a subuniverse to have a value of ρ_V that is 50 times smaller or 4 times larger than the average.

On the other hand, it would be extremely unlikely to observe a value of ρ_V that differs from the mean by more than a few orders of magnitude. Not only are large values of ρ_V unlikely, as we might previously have guessed, based upon the fact that large ρ_V suppresses galaxy formation, but values of ρ_V that are extremely close to zero are also unlikely; there are simply too many other subuniverses to observe that have larger values of ρ_V that are not large enough to prevent galaxy formation.

The analysis of initial fluctuations will enter here only as a means of calculating the parameter $\bar{\rho}\sigma^3$ that enters in equation (28) and the quoted numerical results. It is impor-

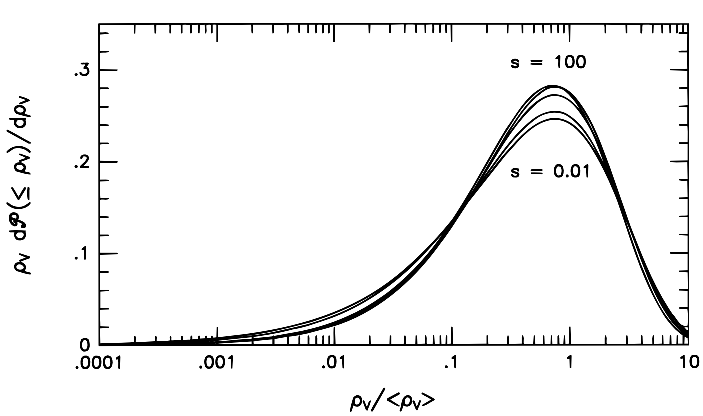

FIG. 1.—Logarithmic differential probability $\rho_V\, d\mathscr{P}(\le\rho_V)/d\rho_V$ vs. $\rho_V/\langle\rho_V\rangle$.

tant to note that during the era of recombination, when ρ_V is negligible and fluctuations are small, σ grows as $t^{2/3}$ and $\bar{\rho}$ falls as t^{-2}, so the combination $\sigma^3\bar{\rho}$ is time independent. Therefore equation (30) shows that our results (23)–(27) and (29) do not depend on what we take as the precise moment of recombination at which σ and $\bar{\rho}$ are evaluated.

4. EVALUATION OF σ

4.1. *Filtered Density Fluctuation Spectrum*

Now we must consider how to calculate σ. From equation (22) we have $\sigma^2 = \langle\delta^2\rangle$. But the variance σ^2 that is appropriate for our purpose here is that which reflects the range of wavenumbers that might possibly contribute to the formation of gravitational condensations that are large enough to lead to "astronomer formation." Only wavenumbers corresponding to density fluctuations encompassing such sufficiently large masses should be allowed to contribute. This implies that the appropriate σ for our purpose here is one calculated by filtering the underlying density field to eliminate the contribution from small wavelengths. This is accomplished by smoothing the density field before calculating the variance σ^2 according to

$$\sigma^2 = \langle\tilde{\delta}^2(\boldsymbol{r})\rangle \,, \tag{31}$$

where

$$\tilde{\delta}(\boldsymbol{r}) \equiv \int \delta(\boldsymbol{x})W(\boldsymbol{x}-\boldsymbol{r})d^3x \,. \tag{32}$$

Here W is a smoothing "window function," and $\boldsymbol{x}$ and $\boldsymbol{r}$ are comoving coordinates, which, following convention, we shall normalize to give the proper distance at present. This yields the following familiar expression for the variance σ^2:

$$\sigma^2 = \frac{1}{2\pi^2}\int_0^\infty P(k)\hat{W}^2(kR)k^2\,dk \,, \tag{33}$$

where $P(k)$ is the power spectrum (assuming statistical translation and rotation invariance),

$$P(|\boldsymbol{k}|) \equiv \int d^3x\langle\delta(\boldsymbol{x}+\boldsymbol{r})\delta(\boldsymbol{r})\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \,, \tag{34}$$

and $\hat{W}(kR)$ is the Fourier transform of the window function

$$\hat{W}(kR) \equiv \int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}W(x) \,, \tag{35}$$

with R being a length parameter to be specified below, introduced to make the argument of $\hat{W}$ dimensionless. (The bracket in eq. [34] implies an average over space; the assumptions of isotropy and homogeneity ensure that P depends only on $k = |\boldsymbol{k}|$.) The window functions in which we are interested here are those that filter out modes of wavelength smaller than some characteristic value R. There are two conventional choices for the window function, the Gaussian window function,

$$\hat{W}_G(u) = e^{-u^2/2} \,, \tag{36}$$

and the top-hat window function,

$$\hat{W}_{TH}(u) = \frac{3}{u^3}(\sin u - u\cos u) \,, \tag{37}$$

(Peebles 1980). The radii for which both window functions enclose the same mass are related by

$$\frac{R_G}{R_{TH}} = \frac{(4\pi/3)^{1/3}}{(2\pi)^{1/2}} = 0.6431 \,. \tag{38}$$

We will express our results throughout in terms of various assumed values of R_G.

The particular value of R_G appropriate for use in calculating the mass fraction that collapses out of the background is uncertain. Our understanding of the detailed conditions necessary for the formation of planets and intelligent life has not yet advanced to the point of determining what the minimum mass condensation is which is capable of forming astronomers. Roughly speaking, we should filter out condensations that are too small to retain metals produced in the first generation of stars. The minimum mass condensation that is capable of this is currently unknown, however. It is not yet established, for example, whether globular clusters of mass 10^5–$10^6\ M_\odot$ are capable of self-enrichment, whereby a first generation of stars generates and releases heavy elements without expelling them from the cluster, so that they can subsequently be incorporated in a second generation of stars. Dwarf galaxies of even greater mass, in fact, are often postulated to undergo an initial burst of massive star formation that leads to supernova-driven expulsion of their interstellar gas (containing heavy elements). Even the typical galaxy in a rich cluster of galaxies is widely believed to have released most of its heavy elements into the intracluster medium, in order to account for the nearly solar metallicity of that gas, which dominates the baryonic mass of the cluster. In short, we do not currently know what the minimum mass scale (or associated wavelength of density fluctuations) is that satisfies the necessary condition that the metals produced by the first generation of stars be retained. Nor do we know if this is a *sufficient* condition for the formation of astronomers.

In fact, all we can say with certainty is that our own Milky Way galaxy met the necessary and sufficient conditions for forming planets, life, and astronomers. The Milky Way has a luminosity that makes it roughly an L^* galaxy, the characteristic luminosity in the bright end of the galaxy luminosity function. If the minimum mass scale M_f that can be responsible for astronomer formation corresponds to that of an L^* galaxy, then the data on the galaxy luminosity function and the mass-to-light ratio of the bright inner parts of field spiral galaxies yields an estimate of the baryonic mass of the bright inner part of an L^* galaxy of $M_f \approx 10^{11}\ h^{-1}\ M_\odot$ (see, e.g., Peebles 1993, p. 122), where h is the Hubble constant H_0 in units of 100 km s^{-1} Mpc^{-1}. From equation (38), $M_f = (2\pi)^{3/2}\rho_{B0}R_G^3$, so this estimate of M_f leads to an estimate of $R_G \cong h^{-1/3}(\Omega_{B0}h^2/0.015)^{-1/3}$ Mpc in present units, assuming a cosmic mean baryon density that is consistent with the current big bang nucleosynthesis abundance constraints. (Here Ω_{B0} is the present baryon mass density in units of the present critical density $\rho_{\text{crit},0} \equiv 3H_0^2/8\pi G$.) If, on the other hand, we take M_f to be the mass of all the baryons initially within a comoving sphere whose volume equals that which, on average, typically contains just one L^* galaxy today, this gives $R_G \cong 2\ h^{-1}$ Mpc. In view of the fact that the Milky Way is actually not an isolated galaxy, but is, instead, a member of the Local Group of galaxies, which includes more than one L^* galaxy, we might even wish to consider the possibility that galaxy group membership is somehow essential to the formation of astronomers.[8] In that case, a value as large as $R_G \sim 3\ h^{-1}$ Mpc would even be reasonable.

For the case in which M_f corresponds to the bright inner part of an L^* galaxy, we shall take $R_G = 1$ Mpc in present units (where, for simplicity, we shall drop the weak dependence of R_G on h for a fixed value of $\Omega_B h^2$). We will also bracket the range of possible outcomes by considering values of R_G that are smaller and larger than this, respectively. On the low side, we take $R_G = 0.01$ Mpc, which is relevant, for instance, if M_f corresponds to the mass of a globular cluster. On the high side, we take $R_G = 2$ or 3 Mpc to illustrate the possibilities that M_f corresponds either to the total mass within the mean volume per L^* galaxy or else the mass of a small group of galaxies, respectively.

4.2. *The Cold Dark Matter Model*

In order to evaluate σ for a given value of R_G, we must adopt a model for the density fluctuation power spectrum at recombination. The cosmic microwave background anisotropy measured at large angles by the *COBE* satellite is consistent with Gaussian random noise density fluctuations with a scale-invariant primordial power spectrum $P(k) \propto k^n$, where $n = 1$, the case referred to as the Harrison–Zeldovich spectrum. The range currently allowed by a statistical analysis of the first 4 yr of data from the *COBE* DMR experiment is, in fact, $n = 1.2 \pm 0.3$ (Bennett et al. 1996). In what follows, we shall generally assume $n = 1$, which is the standard prediction of inflationary cosmology. Later, we can consider the effect of a "tilt" in the primordial spectrum away from the shape for $n = 1$. (Values of $n < 1$ can result, for example, if the primordial fluctuations include a gravitational wave contribution.)

In general, the power spectrum at recombination differs from the primordial shape k^n, except in the long wavelength limit measured directly by *COBE*. The difference reflects the linear growth of the density fluctuations prior to the recombination epoch, which is different for different wavelengths. The best-studied and most successful model for the growth of density fluctuations to date is the CDM model. This model treats the CDM density fluctuations as adiabatic fluctuations in a cold, pressure-free gas. Since we are interested in the growth of density fluctuations in the baryon-electron fluid, which must be present to form stars, planets, and people, we make the assumption that this component collapses out in lock-step with the dark matter component, at least for density fluctuations that are of large enough wavelength to behave in a pressure-free manner. As long as we restrict our attention to the epoch of recombination and later epochs, and to wavelengths larger than the baryon Jeans length in the intergalactic medium, that is, the CDM and baryon power spectra should be identical. (For a detailed discussion of the effects of Jeans-mass filtering on the linear growth of baryon density fluctuations in a flat, matter-dominated CDM model in which the Jeans mass is increased by the reheating of the intergalactic medium that accompanies its reionization, see Shapiro, Giroux, & Babul 1994.)

[8] For example, group membership might enable a galaxy that undergoes an early burst of star formation to expel its heavy elements into the surrounding intragroup environment. The latter might then act as a reservoir from which the galaxy could later accrete some of its lost metals, after the expelled gas has cooled off.

For the CDM power spectrum, we use the expression given by Liddle et al. (1996) and references therein:

$$P(k, z) = 2\pi^2 \left(\frac{c}{H_0}\right)^{3+n} (\delta_H)^2 k^n T^2(q) A^{-2}(z, 0) , \quad (39)$$

where

$$A(z, 0) = \frac{\delta_+(0)}{\delta_+(z)} , \quad (40)$$

$$T(q) = \frac{\ln(1 + 2.34q)}{2.34q} \times [1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4]^{-1/4} , \quad (41)$$

$$q = \frac{k}{h\Gamma\ \mathrm{Mpc}^{-1}} , \quad (42)$$

$$\Gamma = \Omega_0 h e^{-\Omega_{B0} - \Omega_{B0}/\Omega_0} , \quad (43)$$

and δ_H is the dimensionless amplitude at horizon crossing, which must be taken from observations of anisotropies in the microwave radiation background. Here, Ω_0 is the total matter density parameter, $\Omega_0 = \rho_0/\rho_{\mathrm{crit},0}$; δ_+ is the pure growing mode solution for the evolution of linear density fluctuations in this flat universe with nonzero cosmological constant, and $A(z, 0) = \delta_+(0)/\delta_+(z)$ is the linear growth factor between redshift z and the present. For $n = 1$, these formulae describe the case of the Harrison–Zeldovich scale-invariant power-law primordial spectrum, modified by the growth of fluctuations in a CDM model universe, for a flat universe with a nonzero cosmological constant $\lambda_0 = 1 - \Omega_0 = \rho_V/\rho_{\mathrm{crit},0}$. The fitting formula for $T(q)$ is from Bardeen et al. (1986), but with Γ given by a fit by Sugiyama (1995) in the form quoted by Liddle et al. (1996). (The numerical coefficients in this formula depend on the present microwave radiation energy density and on the quantities 100 km s^{-1} and 1 Mpc used in defining the dimensionless quantities q and h, but not on ρ_0 or H_0.) The formula for Γ given in equation (43) includes an exponential correction factor for the effect of nonzero baryon density. Since the variable Γ is often used in the literature to refer, instead, just to the product $\Omega_0 h$ (i.e., without the exponential correction factor), the so-called shape parameter for CDM models, we will also define $\Gamma_0 \equiv \Omega_0 h$, to use whenever we wish to refer only to this product. In the following calculations, we use $\Omega_{B0} = 0.015\ h^{-2}$, consistent with big bang nucleosynthesis constraints from the abundance of light elements (e.g., Copi, Schramm, & Turner 1995).

It is important to note that the explicit dependences of equations (39)–(43) on the values of Ω_0 and h does not mean that the power spectrum at recombination is different for different subuniverses with different values of the cosmological constant. All factors that depend upon Ω_0 and h reflect the fact that a knowledge of the local values of Ω_0 and h in our own subuniverse is required in order to interpret present-day observations in our own subuniverse (such as those of cosmic microwave background anisotropy) unambiguously to determine the power spectrum at recombination *assumed common to all subuniverses*. We must, therefore, distinguish clearly between the vacuum energy density in our own subuniverse, on which our inference of the universal power spectrum depends, and the vacuum energy densities in different subuniverses, on which the probabilities of galaxy formation in any subuniverse depend. To avoid any possible confusion on this point, we will, henceforth, indicate the value of the vacuum energy density in our subuniverse as ρ_V^*. We will not, however, place an asterisk on ρ_0, $\lambda_0 \equiv \rho_V^*/(\rho_V^* + \rho_0)$, H_0, or $h \equiv H_0/(100 \text{ km s}^{-1} \text{ Mpc}^{-1})$, because the subscript zero already indicates that these quantities are evaluated at the present moment in our subuniverse. Also, an asterisk is not necessary for the quantities $\bar{\rho}$, $\bar{\rho}_B$, $P(k)$, or σ, since these are assumed not to vary from one subuniverse to another.

In order to evaluate equation (39) for the CDM power spectrum for any particular value of the vacuum energy density, we must evaluate the growth factor $A(z, 0; \lambda_0)$ in equation (40) as a function of z and $\lambda_0 \equiv \rho_V/\rho_{\text{crit},\,0}$. It is convenient to express this in terms of a function $f(\lambda_0, z)$, defined as the ratio of the growth factor $(1 + z)$ in an Einstein–de Sitter universe to $A(z, 0; \lambda_0)$:

$$f(\lambda_0, z) \equiv \frac{1+z}{A(z, 0)} = (1+z)\,\frac{\delta_+(z)}{\delta_+(0)}\,, \tag{44}$$

where δ_+ is the amplitude of the linear growing mode, which is given for general $\lambda_0 \neq 0$ by

$$\delta_+(z; \lambda_0) = \left(\frac{1}{y} + 1\right)^{1/2} \int_0^y \frac{dw}{w^{1/6}(1+w)^{3/2}} \tag{45}$$

(Martel 1991), with

$$y \equiv \frac{\rho_V}{\bar{\rho}(z)} = \frac{\lambda_0}{\Omega_0}\,(1+z)^{-3} \tag{46}$$

and $\Omega_0 \equiv 1 - \lambda_0$. Using equations (44)–(46), we obtain, after some algebra,

$$\begin{aligned} f(\lambda_0, z) = {} & \Omega_0^{1/2}(1+z)^{5/2}\left[1 + \frac{\lambda_0}{\Omega_0(1+z)^3}\right]^{1/2} \\ & \times \left[\int_0^{\lambda_0/\Omega_0} \frac{dw}{w^{1/6}(1+w)^{3/2}}\right]^{-1} \\ & \times \int_0^{\lambda_0/[\Omega_0(1+z)^3]} \frac{dw}{w^{1/6}(1+w)^{3/2}}\,. \end{aligned} \tag{47}$$

For $1 + z \gg 1$, this gives the z-independent result,

$$f(\lambda_0, z) \simeq f(\lambda_0) = \frac{6\lambda_0^{5/6}}{5\Omega_0^{1/3}}\left[\int_0^{\lambda_0/\Omega_0} \frac{dw}{w^{1/6}(1+w)^{3/2}}\right]^{-1}\,. \tag{48}$$

The corrections are of order $(1 + z)^{-3}$, which for the case that interests us, where $z \approx 1000$, is entirely negligible. We have evaluated the integral in equation (48) numerically and find that it differs substantially from unity only for relatively large values of λ_0.

According to Bunn & White (1996), the first 4 yr of data on the cosmic microwave background temperature anisotropy detected by the *COBE* DMR experiment may be fitted with a dimensionless amplitude at horizon crossing given by the formula

$$\begin{aligned} \delta_H = {} & 1.94 \times 10^{-5}(\Omega_0)^{-0.785 - 0.05 \ln \Omega_0} \\ & \times \exp\,[a(n-1) + b(n-1)^2]\,. \end{aligned} \tag{49}$$

There are two sets of values of the constants a and b, which correspond to the cases of $n \neq 1$ without any gravitational wave contribution ($a = -0.95$, $b = -0.169$) and of power-law inflation with gravitational waves ($a = 1$, $b = 1.97$), respectively.

Equations (39)–(44), along with equations (48) and (49), can now be evaluated to compute the power spectrum at recombination for any flat model for any values of λ_0 (or, equivalently, Ω_0) and h.

4.3. *Results for σ and $\sigma^3\bar{\rho}$*

The variance σ^2 at recombination is given by equations (33) and (39)–(44) as

$$\begin{aligned} \sigma(z_{\text{rec}}) & = (c_{100})^{(n+3)/2}[\Gamma^{(n+3)/2}\delta_H A(z_{\text{rec}}, 0)^{-1} K_n^{1/2}(q_{\max})]_* \\ & = (c_{100})^{(n+3)/2}(1 + z_{\text{rec}})^{-1} \\ & \quad \times [\Gamma^{(n+3)/2}\delta_H f(\lambda_0) K_n^{1/2}(q_{\max})]_*\,, \end{aligned} \tag{50}$$

where the asterisk labeling the brackets (and in what follows) indicates that all quantities inside the brackets are evaluated using the values λ_0 and h in our own subuniverse; $c_{100} = 2997.9$ is the speed of light in units of 100 km s^{-1}; the second equality refers to the result to leading order in $(1 + z_{\text{rec}})^{-3}$; and

$$K_n(q_{\max}) \equiv \int_0^\infty q^{n+2}T^2(q)\hat{W}_G^2\left(2\pi\,\frac{q}{q_{\max}}\right)dq\,. \tag{51}$$

The integral in equation (51) has been evaluated numerically for $n = 1$, 0.9, and 0.8. (Note that for $n \neq 1$, we hereafter adopt constants a and b for the case of $n \neq 1$ with no gravitational waves. The case with gravitational waves yields a slightly smaller value of σ for the same value of n.)

As already mentioned, results for the probability distribution of ρ_V actually depend not on σ or on $\bar{\rho}$, but on the parameter $\sigma^3\bar{\rho}$. The total matter density $\bar{\rho}$ at recombination is related to the present matter density ρ_0 by $\bar{\rho} = \rho_0(1 + z_{\text{rec}})^3$, so, as promised, $\sigma^3\bar{\rho}$ is independent of the precise value chosen for z_{rec}:

$$\sigma^3\bar{\rho} = \rho_0(c_{100})^{(3n+9)/2}[\Gamma^{(n+3)/2}\delta_H f(\lambda_0)K_n^{1/2}(q_{\max})]_*^3\,. \tag{52}$$

It is customary to report the normalizations of the power spectra for different models in terms of the value of σ evaluated at the present for a particular filter scale, assuming that fluctuations continue to grow at the linear growth rate. For our purpose here, however, we must evaluate σ at recombination, thereby undoing the effects of the growth of fluctuations since that epoch that influence the value of σ at the present. Our goal, as mentioned earlier, is to use the observations of the cosmic microwave background anisotropy made by astronomers in our *own* subuniverse to infer the universal density fluctuation distribution common to all subuniverses at z_{rec}. Unfortunately, our ability to infer this universal density fluctuation distribution is limited by the fact that we must know the values of λ_0 and h in our own subuniverse in order to interpret the cosmic microwave background anisotropy measurements unambiguously. We illustrate the dependence of the inferred density fluctuations on the assumed values of λ_0 and h in Figure 2, where we have plotted the value of σ at recombination (taken as $z_{\text{rec}} = 1000$) as a function of λ_0 for $R_G = 0.01$, 1, 2, and 3 Mpc, for $n = 1$, 0.9, and 0.8. The effect of a "tilt" to $n < 1$ is to decrease σ, relative to its value for $n = 1$, for the same value of $q_{\max}$, or equivalently, of R_G.

With all of the ingredients necessary to evaluate the probability distribution $\mathscr{P}_{\text{obs}}(\rho_V)$ thus assembled, we can now

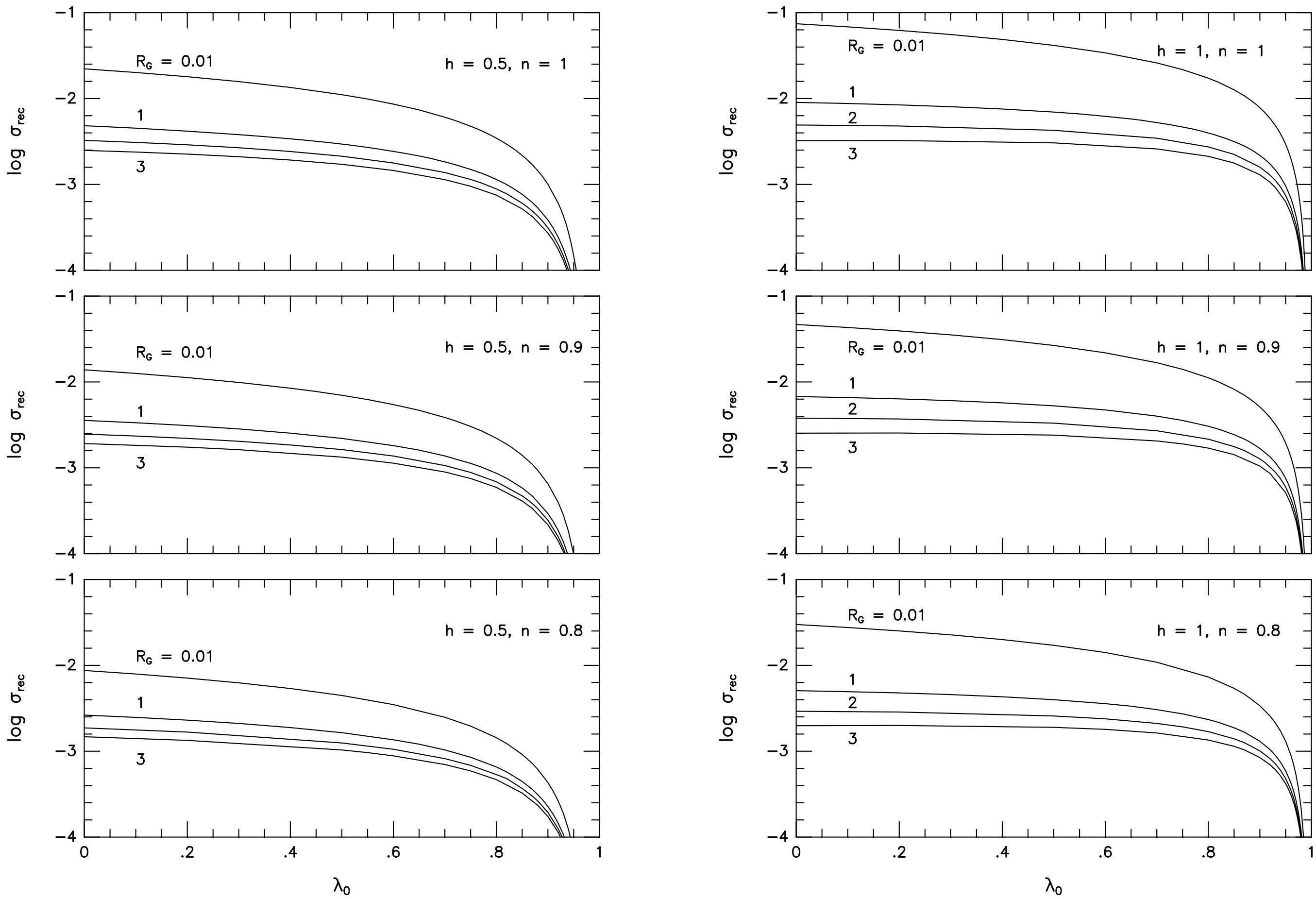

FIG. 2*a* FIG. 2*b*

FIG. 2.—(*a*) The rms density fluctuation at recombination (i.e., at $z_{\rm rec} = 1000$), $\sigma = \sigma_{\rm rec}$, in the *COBE*-normalized flat CDM model vs. λ_0, for $R_{\rm G} = 0.01$, 1, 2, and 3 Mpc, respectively, as labeled, for $h = 0.5$, for $n = 1$ (*top panel*), 0.9 (*middle panel*), and 0.8 (*bottom panel*). (*b*) As in (*a*), but for $h = 1$.

evaluate the probability of observing any particular value of ρ_V anywhere in the universe, as well as the average or median values observed, as functions of the values we adopt for ρ_V^* (or $\lambda_0 = 1 - \Omega_0$) and h in our own subuniverse. It will be useful first to consider what values of ρ_V^* and h are suggested by observation.

5. OBSERVATIONAL BOUNDS

Ostriker & Steinhardt (1995) and Krauss & Turner (1995) have argued that the apparent discrepancies that had previously been identified between observations and the predictions of the standard CDM model (i.e., CDM in a flat universe with zero cosmological constant and the scale-invariant Harrison-Zeldovich primordial power spectrum) can be reconciled if the standard CDM model is modified to admit a nonzero cosmological constant, in a range given by Ostriker & Steinhardt as roughly $\lambda_0 = 0.65 \pm 0.1$. We have reproduced the observational and theoretical constraints that led Ostriker & Steinhardt (1995) to this conclusion, plotted here in Figure 3 as a series of upper and lower bounds in the (λ_0, h)-plane. The constraint curves plotted in Figure 3 are based on Ostriker & Steinhardt (1995) and references therein, with the following additions.

The data on the large-scale clustering of galaxies from galaxy surveys constrain the flat CDM model by requiring that the spatial and angular correlation statistics of the observed galaxies in our local universe at the present epoch agree with the predictions of structure formation by gravitational instability in the CDM model. This leads to upper and lower bounds on the so-called shape parameter $\Gamma_0 = \Omega_0 h$, given by $\Gamma_0 = 0.25 \pm 0.05$ (assuming $n = 1$; see curves labeled "Γ_0" and "LSS" in Fig. 3). A similar constraint results from the requirement that the CDM model reproduce the observed space density and luminosity function of X-ray clusters in the present universe. We plot this constraint separately from that of the shape parameter Γ_0 that Ostriker & Steinhardt (1995) plotted. This X-ray cluster abundance constraint is expressed by Viana & Liddle (1996) as bounds on the rms density fluctuation $\sigma_8(z = 0)$ for a smoothing radius (in present units) given by $R_{\rm TH} \approx 8\ h^{-1}$ Mpc. Assuming $n = 1$, these bounds are given by

$$\sigma_8 = [0.6\Omega^{-(0.59 - 0.016\Omega_0 + 0.06\Omega_0{}^2)}]^{+32\phi(\Omega_0)\%}_{-24\phi(\Omega_0)\%}\ , \tag{53}$$

where

$$\phi(\Omega_0) \equiv \Omega_0^{0.26 \log_{10} \Omega_0}\ . \tag{54}$$

This amounts to a constraint on λ_0 and h that is similar to that from the correlation statistics. (The above-mentioned bounds from the statistics of large-scale structure ["LSS"] in the galaxy distribution also refer to the part of the CDM power spectrum at wavelengths $\lambda \gtrsim 8\ h^{-1}$ Mpc.)

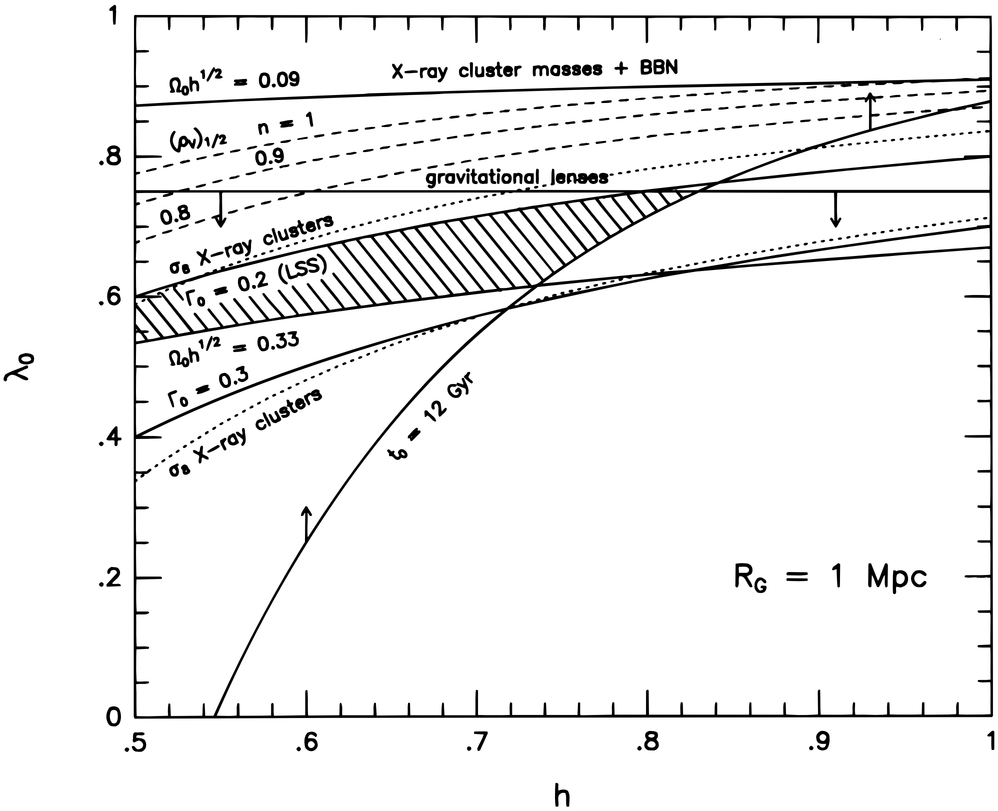

FIG. 3.—Observational constraints on $\lambda_0 \equiv \rho_V^*/(\rho_V^* + \rho_0)$ and $h \equiv H_0/(100\ \mathrm{km\ s^{-1}\ Mpc^{-1}})$. (Curves labeled "LSS" and "$\Gamma_0 = 0.2$" or "$\Gamma_0 = 0.3$" bound the region allowed by the constraint $\Gamma_0 = \Omega_0 h = 0.25 \pm 0.05$, derived by matching the spatial and angular correlation statistics from galaxy surveys with the theoretical predictions of the large-scale clustering of galaxies in a *COBE*-normalized flat CDM model with primordial power spectrum index $n = 1$. The curves labeled "σ_8 X-ray clusters" bound the values of λ_0 and h that make this CDM model satisfy the constraint on the present space density of X-ray clusters. The curve labeled "$t_0 = 12$ Gyr" indicates the lower limit that makes the age of the universe at least as large as current estimates of the minimum age of globular clusters. The curves labeled "$\Omega_0 h^{1/2}$" indicate the boundaries defined by the X-ray–measured total and baryonic masses of clusters of galaxies, together with the big bang nucleosynthesis limits on the baryon mean density and the assumption that the ratio of baryon to total mass inside each cluster equals the ratio of universal mean values. The curve labeled "gravitational lenses" indicates the upper limit imposed by counts of quasars lensed by intervening galaxies. The dashed curves labeled "$(\rho_V)_{1/2}$" are the values for which our own subuniverse has the median value of ρ_V for all subuniverses, if $R_G = 1$ Mpc and $n = 1$ (*top dashed curves*), 0.9 (*middle dashed curve*), or 0.8 (*bottom dashed curve*).

Estimates of the total masses and baryonic mass fractions of individual clusters of galaxies, derived by fitting the X-ray surface brightness profiles of each cluster and assuming the cluster intergalactic medium is an isothermal sphere in hydrostatic equilibrium with a virialized cluster gravitational potential, yield another pair of bounds. If the assumption is further made that the ratio of baryonic mass to total mass of each cluster is equal to the universal mean ratio, Ω_{B0}/Ω_0, then a comparison of this X-ray–estimated ratio with the constraints on Ω_{B0} from standard big bang nucleosynthesis and the observed light-element abundances (i.e., $0.01 \lesssim \Omega_{B0} h^2 \lesssim 0.02$; Copi, Schramm, & Turner 1995) implies a constraint on the total density parameter given by $0.09 \lesssim \Omega_0\ h^{1/2} \lesssim 0.33$. The curves labeled "$\Omega_0 h^{1/2}$" and "X-ray cluster masses + big bang nucleosynthesis" in Figure 3 indicate the bounds on λ_0 and h that result from this argument. Some recent numerical gasdynamical simulations of cluster formation in the flat, matter-dominated CDM model suggest that the upper bound on $\Omega_0 h^{1/2}$ that results from the high values estimated for cluster baryonic mass fraction by the equilibrium model described above may be too low (e.g., Bartelmann & Steinmetz 1997; Martel, Shapiro, & Valinia 1998; Valinia 1996). The simulated clusters, when properly resolved, are often found to be comprised of subclusters in the act of merging, and, together with projection effects, this can cause an observer who uses the assumption of isothermal spheres in hydrostatic equilibrium to underestimate the total mass and overestimate the baryon fraction.

The estimated minimum age of globular clusters derived by comparison of theoretical models of stellar evolution with observed globular cluster H-R diagrams is about 12×10^9 yr. This leads to a lower limit to λ_0 for each h, based on the requirement that the age of our universe exceeds this estimate of the minimum age of globular cluster stars (see curve in Fig. 3 labeled "$t_0 = 12$ Gyr").

An upper bound to λ_0 results from the comparison of the statistics of quasars observed to be gravitationally lensed by intervening galaxies with the predictions of flat cosmological models with a nonzero cosmological constant. A flat cosmology with cosmological constant tends to produce more gravitationally lensed quasars than does such a cosmology with zero cosmological constant. The resulting limit was quoted by Ostriker & Steinhardt as $\lambda_0 < 0.75$. More recently, Kochanek (1996) has argued for a somewhat tighter limit. However, limited observational data and the possibility that evolution effects on the population of lensing galaxies have not been properly taken into account suggest that this limit is still uncertain. We plot the upper bound quoted by Ostriker & Steinhardt (1995) in Figure 3, but caution that this limit is still in flux and that the future change can be up *or* down.

Finally, an independent constraint was recently derived by Perlmutter et al. (1997), using observations of Type Ia supernovae, to infer the relationship between redshift and distance. Their result is consistent with the limit $\lambda_0 < 0.5$. However, this interesting approach is too preliminary to be considered reliable at this time. We have not included it in Figure 3.

Figure 3 shows that, for the observationally favored value of $H_0 = 70$ km s^{-1} Mpc^{-1}, the above interpretation of observational data suggests a value of λ_0 in the range 0.6–0.7, corresponding to a vacuum energy density ρ_V^* in the range $1.5\rho_0$–$2.3\rho_0$. The range of allowed values shrinks for smaller and larger values of the Hubble constant, with larger values of λ_0 favored for larger values of H_0.

6. RESULTS

We might try to use our results to guess the actual value of ρ_V^* by *assuming* that we live in a typical subuniverse, defined as one in which ρ_V^* is equal to the median observed values of ρ_V for all astronomers in all subuniverses. The dashed curves in Figure 3 show the results obtained in this way for three different values of the slope parameter n and for a filter scale $R_G = 1$ Mpc. As can be seen from this figure, these "predicted" values of ρ_V are typically several times larger than those suggested by observation. But such "predictions" have low confidence anyway. As remarked in § 3, the probability distribution of ρ_V values is quite broad, so it is not unlikely that ρ_V^* in our subuniverse could be significantly different from the anthropic mean or median.

Since the cosmological constant in our subuniverse seems to be somewhat smaller than might have been expected on anthropic grounds, perhaps the best use we can make of our calculations is, for various assumed values of ρ_V^* and other parameters in our subuniverse, to use the results of § 4 to calculate σ and then to use this result and the results of § 3 to calculate how likely or unlikely it is that a random astronomer in any subuniverse would find a value of the vacuum energy density as small as or smaller than the assumed value ρ_V^*.

Our results for $H_0 = 70$ km s^{-1} Mpc^{-1} and $n = s = 1$ are shown in Table 1. To assess the implications of these results, let us consider three different possibilities for what observations may eventually reveal about the value of ρ_V.

TABLE 1

PROBABILITY THAT A RANDOM ASTRONOMER WOULD OBSERVE A VACUUM ENERGY DENSITY AS SMALL AS THE VALUE ρ_V^* IN OUR SUBUNIVERSE[a] FOR VARIOUS VALUES OF ρ_V^*

		$R_G = 1$ Mpc		$R_G = 2$ Mpc	
λ_0	ρ_V^*/ρ_0	σ	$\mathscr{P}(\leq \rho_V^*)$	σ	$\mathscr{P}(\leq \rho_V^*)$
0.1……	0.11	0.0067	0.0005	0.0042	0.0019
0.2……	0.25	0.0063	0.0013	0.0040	0.0045
0.3……	0.43	0.0059	0.0025	0.0038	0.0084
0.4……	0.67	0.0054	0.0049	0.0036	0.015
0.5……	1.00	0.0048	0.0097	0.0032	0.027
0.6……	1.50	0.0041	0.021	0.0029	0.054
0.7……	2.33	0.0033	0.054	0.0024	0.12
0.8……	4.00	0.0023	0.19	0.0017	0.35
0.9……	9.00	0.0011	0.90	0.0008	0.98

[a] For $s = 1$, $H_0 = 70$ km s^{-1} Mpc^{-1} and $n = 1$.

First, suppose that observations show that ρ_V really is in the range suggested by the data summarized in Figure 3, that is, that ρ_V^* is found to lie in the range $1.5\rho_0$–$2.3\rho_0$, corresponding to λ_0 from 0.6 to 0.7. We see from Table 1 that for a filter scale $R_G = 1$ Mpc, $H_0 = 70$ km s^{-1} Mpc^{-1}, and spectral index $n = 1$, the probability that a random observer would find a value for ρ_V as small as the value ρ_V^* in our subuniverse would be in the range of about 2%–5%. Thus, for this choice of parameters, ρ_V^* is well below the median, but not so low as to make an anthropic explanation implausible.

The probabilities go up for a larger filter scale. This implies a smaller variance σ^2 in the fluctuating density field at recombination, which means that objects of mass corresponding to this filter scale will collapse out later, when the mean matter density is smaller. In that case galaxy formation can be suppressed by smaller vacuum energy densities, so that the average *observed* value of the vacuum energy density will be smaller. Table 1 shows that, for $R_G = 2$ Mpc, the probability that a random observer would find a value for ρ_V that is as small as the value ρ_V^* in our subuniverse would vary from about 5% to 12% for ρ_V^* in the same range of $1.5\rho_0$ to $2.3\rho_0$.

The probabilities of finding small values of ρ_V^* also increase with a decrease in H_0 or in the spectral index n, either of which would decrease the estimated value of σ. For instance, for $R_G = 1$ Mpc and $\rho_V^* = 1.5\rho_0$ (corresponding to $\lambda_0 = 0.6$), the probability that a random astronomer would find a value for ρ_V as small as ρ_V^* is 8% if $n = 1$ and $H_0 = 50$ km s^{-1} Mpc^{-1} and 9% if $n = 0.8$ and $H_0 = 70$ km s^{-1} Mpc^{-1}, as compared to 2% for the previously considered case, in which $n = 1$ and $H_0 = 70$ km s^{-1} Mpc^{-1}.

We conclude that, given all uncertainties in our model and input parameters, anthropic considerations do fairly well as an explanation of a cosmological constant with λ_0 in the range 0.6–0.7. Compared to all other arguments known to us, which either leave ρ_V arbitrary or give a ρ_V that is zero or vastly larger than ρ_0, anthropic considerations are doing spectacularly well.

Second, suppose that observations definitely establish a nonzero value of λ_0, but one that is substantially less than 0.6. Unless there is something seriously wrong with our detailed calculations, we would have to conclude that the observed value of ρ_V is much less than would have been expected anthropically. On the other hand, the values of ρ_V suggested by anthropic considerations would still be far closer to any value that is large enough to be detected astronomically than those suggested by any other available arguments.

Finally, suppose that observations fail to reveal any nonzero value of ρ_V^* and set an upper limit of $0.1\rho_0$. (It is hard to see how observations could yield a much more stringent upper limit in the foreseeable future.) In this case the probability that a random astronomer would find this small a value for ρ_V would be less (and in most cases much less) than 1% for any plausible values of R_G, n, and H_0: in this case anthropic arguments would be pretty well ruled out as an explanation of the smallness of the vacuum energy density. It would then be reasonable to guess that, for some mysterious reason, ρ_V actually vanishes.

Unfortunately, although observation of a value of ρ_V^* that is anthropically likely would support the idea that there is a diversity of possible ρ_V values, astronomical observation alone cannot confirm this idea, since we only observe one subuniverse. Ultimately this issue will have to be settled by advances in fundamental physics, which we hope will tell us whether, in fact, it is correct that there are many subuniverses with different values of the cosmological constant. If this is not correct, then there is no justification for the anthropic reasoning used here, while, if it is correct, an anthropic analysis along the lines of the present paper will be unavoidable.

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