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# FLARE FREQUENCY DISTRIBUTIONS BASED ON A MASTER EQUATION

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## ABSTRACT

The Rosner & Vaiana model for flares is generalized to allow for flares that do not deplete all free energy from the system, a step that overcomes a number of objections to the original model. We obtain a probability balance equation, or master equation, describing the free energy $E$ of an active region subject to a prescribed growth rate, $\dot{E}$, and a prescribed distribution, $\alpha(E)$, of stochastic decay events. We argue that the solution appropriate to flares involves an energy-independent growth rate and a power-law form for $\alpha(E)$, which may be the result of an underlying avalanche process. The resulting model produces power-law flare frequency distributions below a high-energy rollover corresponding to the largest energy the system is likely to attain, which is set by the balance between the rate of growth and the rate of stochastic decay. There is a close correspondence between the resulting model and the avalanche model for flares.

*Subject heading:* Sun: flares

## 1. INTRODUCTION

The distribution of flare energies, estimated from X-ray observations, is a power law (Hudson 1991). This fact tells us something fundamental about energy storage and release in the solar corona, and in particular about the mechanism responsible for energy release in flares.

A number of models have been presented to account for the flare energy distribution, and to relate it to the underlying energy release process. The avalanche model (Lu & Hamilton 1991; Lu et al. 1993) maintains that the energetics of an active region is analogous to the behavior of a cellular automaton model. A vector field defined on a grid is evolved by a process of repeated stressing (the addition of a random vector) and spontaneous relaxation (when, subject to a local instability criterion being met, the field is redistributed locally). Eventually, the system reaches a steady state in which the field is close to instability everywhere, so that there is no natural length scale for relaxation events on the grid, and events of any size (limited only by the boundaries of the grid) may occur. The redistribution events are identified with flares in the system, of sizes defined by the number of grid points participating. The model successfully reproduces the power-law flare energy distribution, and also gives a power-law distribution of peak flux and a power-law distribution of flare duration over a limited range. Observations show that the peak flux distribution is a power law, and the distribution of flare durations may be a simple power law (over about a decade of durations), although this result is less certain (Lu et al. 1993; Crosby, Aschwanden, & Dennis 1993; Bromund, McTiernan, & Kane 1995; Crosby 1996). Remarkably, the avalanche model also reproduces the spectral indices of these power laws, with (apparently) no free parameters. However, the free parameters are hidden in the numerical details of the model and are hard to characterize. The functional form of the frequency distributions from the simulations is only a power law for certain redistribution rules and boundary conditions on the numerical box (Galsgaard 1996; Vlahos et al. 1995), and the power-law indices certainly depend on the dimensionality of the automaton.

A second, older model for the statistics of flares is that of Rosner & Vaiana (1978; hereafter RV78). Flares (and more generally, astrophysical transients) are assumed to be stochastic events with a constant mean rate $\lambda$, so that the distribution of intervals between flares is the Poisson interval distribution,

$$P(t) = \lambda e^{-\lambda t} . \tag{1}$$

Between flares, an active region is assumed to be subject to an exponential increase in free energy, $E$, above a ground-state energy $E_0$,

$$E(t) = E_0(e^{\beta t} - 1) ; \tag{2}$$

the occurrence of a flare is assumed to liberate all of this free energy, resetting the active region to the ground state. The frequency-energy distribution resulting from this model is

$$\mathcal{N}(E) = \mathcal{N}_0\left(1 + \frac{E}{E_0}\right)^{-\gamma} , \tag{3}$$

where

$$\mathcal{N}_0 = \frac{\lambda^2}{\beta E_0} \tag{4}$$

and

$$\gamma = 1 + \frac{\lambda}{\beta} . \tag{5}$$

When the stored energy $E$ is significantly greater than the ground state energy $E_0$, the distribution is a simple power law with index $\gamma$.

Litvinenko (1994) generalized the RV78 model for the case of an arbitrary energy resupply rate, $\dot{E}$ (in the RV78 model, $\dot{E} \propto E_0 + E$), by noting that the system may be described by the steady-state transport equation

$$\frac{d}{dE}(\dot{E}P) + \lambda P = 0 , \tag{6}$$

where $P(E)$ is the probability of the active region having energy $E$. Since a flare is assumed to liberate all free energy, $P(E)$ is also the probability of a flare of energy $E$ occurring, and $\lambda P(E)$ describes the frequency distribution of flares.

More recently (Litvinenko 1996), the model has been extended to describe the interaction of flaring elements, which are identified with reconnecting current sheets. Aschwanden et al. (1997) also recently reinterpreted the RV78 model, ascribing the exponential growth in the model to the growth of energy release in a constituent instability in the flare, and replacing the time between events with the saturation time of the instability.

Both the RV78 model and the avalanche models have deficiencies. The avalanche model fails to directly relate the cellular automaton to the physical properties of active regions, relying instead on an analogy based on broad physical arguments (Lu 1995a, 1995b). For the model to be put on a firm physical basis, the vector field of the cellular automaton must be identified in some way with the magnetic field of an active region, in which the free energy liberated in a flare is stored. Similarly, the rules for redistributing the field (a flare) must be identified with a mechanism for releasing energy from the magnetic field (for an alternate opinion, see Lu 1995b). Some progress has been made in these directions. Zirker & Cleveland (1993) identified a scalar automaton field with magnetic free energy, and associated its increase with results from simple models for the braiding of coronal loops. Other studies have provided some understanding of how redistribution rules and boundary conditions on the numerical box affect the frequency distributions (Vlahos et al. 1995; Galsgaard 1996).

The RV78 model also has problems; a detailed critique has been presented by Lu (1995c). Most of Lu's objections relate to the model assumption that each flare removes all of the excess energy from an active region, resetting the system to its ground state (with energy $E_0$). A variety of observations suggest that flares do not remove all of the available energy from active regions (e.g., Wang 1992). Furthermore, observations of hard X-ray microflares (Lin et al. 1984) show that the power-law distribution of flares extends down to at least $10^{26}$ergs. Equation (3) then implies that $E_0 < 10^{26}$ ergs, a value that appears to be too small, in particular since the energy supply rate is reset to the value of $\beta E_0$ after a flare. The assumption that a flare removes all of the free energy also leads to a simple relationship between the flare energy and the interval between the flare and the preceding flare, via equation (2). Biesecker (1994) tested this prediction of the RV78 model using a large sample of BATSE flares, and found no interval-energy relationship. This result was recently confirmed by investigators using a different set of data (Crosby 1996; Crosby et al. 1997).

In this paper, we describe a further generalization of the RV78 model to allow for flares that do not completely deplete the free energy of the active region, and we investigate a solution for an energy supply rate that does not depend on the free energy of the system. These steps alleviate the difficulties described by Lu (1995c). The modified model is presented as a complementary description to the avalanche picture.

The sections of this paper are divided as follows. In § 2 we describe the generalized model and discuss its properties, exemplified by an analytic solution (§ 2.1). We consider a general property of solutions to the model (§ 2.2), which allows insight into a particular solution that may be appropriate to flares (§ 2.3). In § 2.4, a numerical solution is given for this case. Finally, in § 3 we discuss the results of our study.

## 2. THE MODEL

The RV78 model may be generalized to include flares that do not reset the system to its ground state, as follows.

Consider a given active region (or constituent flaring element) with a probability $P(E)dE$ of having excess energy in the interval $I_E = (E, E + dE)$. We assume that there is a probability per unit time $\alpha(E, E')dE'$ of falling from energy $E$ to the interval $(E', E' + dE')$ as a result of a flare occurring. The energy released in this transition, $E - E'$, is the observed flare energy, so the flare frequency-energy distribution will be

$$\mathscr{N}(E) = \int_E^\infty P(E')\alpha(E', E' - E)dE' . \quad (7)$$

We further assume that the active region is subject to a rate of increase of energy $\dot{E}$. This represents a rate of storage of energy analogous to the RV78 model, although here we consider $\dot{E}$ to be an arbitrary function of $E$, following Litvinenko (Litvinenko 1994, 1996). It follows that $P(E)$ obeys the probability balance equation, or "master equation" (Gardiner 1983; Van Kampen 1992),

$$\frac{d}{dE}(\dot{E}P) + P\int_0^E \alpha(E, E')dE' - \int_E^\infty P(E')\alpha(E', E)dE' = 0 . \quad (8)$$

Each term in equation (8) describes a rate of change of $P(E)dE$, the probability that the active region is in the energy interval $I_E$. The first term describes the movement of the active region out of the interval $I_E$ as the result of a gradual energy increase; the second term describes the active region falling out of $I_E$ as the result of a flare, and the third term describes the active region falling into $I_E$ from above as the result of a flare. The system is assumed to be in a steady state, so the probability rates given by the three terms sum to zero.

It is easy to see how this generalized model relates to the Litvinenko model (1994, 1996); equation (8) becomes equation (6) with the choice

$$\alpha(E, E') = \lambda\delta(E') , \quad (9)$$

i.e., by requiring that only the transition to the ground state ($E' = 0$) is permitted, with a rate $\lambda$ independent of energy.

Equation (8) is made somewhat simpler if we assume that the flare transition rates depend only on the released energy, and not on the initial and final energies:

$$\alpha(E, E') = \alpha(E - E') . \quad (10)$$

Then the master equation becomes

$$\frac{d}{dE}(\dot{E}P) + P\int_0^E \alpha(E')dE' - \int_E^\infty P(E')\alpha(E' - E)dE' = 0 , \quad (11)$$

and the flare frequency-energy distribution is

$$\mathscr{N}(E) = \alpha(E)\int_E^\infty P(E')dE' . \quad (12)$$

In the following sections of this paper, we will consider solutions to equation (11). The use of equation (11) rather than equation (8) is a choice made in the interests of simplicity, and involves an assumption, viz., equation (10). It is difficult to assess whether this is a major assumption. We will return to this point in § 3.

### 2.1. *An Analytic Solution to the Model*

Equation (11) is an integro-differential equation that is difficult to solve for most choices of the transition rate $\alpha(E)$. We have been unable to obtain the general solution, i.e., the solution $P(E)$ for arbitrary $\alpha(E)$ and $\dot{E}(E)$. As a result, we will proceed by considering a simple particular solution. The case $\dot{E} = \beta_0$ (a constant), $\alpha(E) = \alpha_0$ (also constant) has the analytic solution

$$P(E) = aEe^{-(a/2)E^2} , \tag{13}$$

where $a = \alpha_0/\beta_0$. The frequency-energy distribution in this case is a simple Gaussian,

$$\mathscr{N}(E) = \alpha_0 \, e^{-(a/2)E^2} . \tag{14}$$

Clearly, this case is inappropriate for flares; there are too few small flares, and too few large flares, in comparison to a power law. However, as we shall see below, the behavior of equation (13) is typical of solutions with an energy-independent growth rate, $\dot{E} = \beta_0$, but with other forms for the transition rate $\alpha(E)$. The probability density $P(E)$ increases from zero at $E = 0$ and reaches a maximum at $E = a^{-1/2}$ before decaying essentially exponentially as $E \to \infty$. The physical interpretation is that when the growth rate $\beta_0$ is much greater than the decay rate $\alpha_0$ ($a \ll 1$), the system is more likely to reach higher energies, whereas if the growth rate is much smaller than the decay rate ($a \gg 1$), the system is more likely to be found at lower energies.

### 2.2. *A General Property of Solutions with Constant Growth Rate*

In the absence of a general solution to equation (11), it is important to determine characteristic properties of solutions. An important general property of solutions is the maximum value that the energy $E$ is likely to attain for a given growth rate $\dot{E} = \beta(E)$ and decay rate $\alpha(E)$. We will consider this question for the case of $\dot{E} = \beta_0$, a constant.

The upper limit to $E$ is set by the balance between growth and stochastic decay. Consider a time interval $T$. In this time, the system grows in energy by an amount $\Delta E_+ = \beta_0 T$, and most likely decays by an amount

$$\Delta E_- = T \int_0^E \alpha(E')E' \, dE' . \tag{15}$$

A characteristic upper limit to $E$, which we will denote $E_m$, is provided by setting $\Delta E_- = \Delta E_+$, i.e., by the implicit solution to

$$\beta_0 = \int_0^{E_m} \alpha(E')E' \, dE' . \tag{16}$$

For $E \gg E_m$, the likely decrease in energy, $\Delta E_-$, will be much greater than the increase, $\Delta E_+$, and so the system is unlikely to persist at energy $E$ for very long. Hence, $P(E \gg E_m)$ is small.

Applying equation (16) to the constant-$\alpha$ solution ( eq. [13]), we find $E_m^2 = 2/a$, a good estimate of the value of $E$ above which the probability density is small.

### 2.3. *A Solution Appropriate to Flares*

We will consider solutions to equation (11) for the case in which the energy supply rate does not depend on the free energy of the system, i.e., $\dot{E} = \beta_0$ (a constant). Although it is not known how energy is supplied to active regions, one possibility is that nonpotential flux elements are continuously emerging (Leka et al. 1996). In that case, the energy growth rate may be proportional to the rate of emergence of new flux, and an energy-independent value of $\dot{E}$ is an appropriate choice. More generally, if active regions represent externally driven systems, then the occurrence of a flare will not affect the rate of energy supply.

The form of equation (12) suggests that we seek a solution with $\alpha(E)$ as a power law. To make the second term in equation (11) finite, a low energy cutoff, $E_c$, is needed in the power law, so we choose

$$\alpha(E) = \alpha_0 \left(\frac{E}{E_c}\right)^{-\gamma} \theta(E - E_c) , \tag{17}$$

where $\theta(x)$ is the step function. Provided that a solution, $P(E)$, exists for this choice of $\alpha(E)$, the form of equation (12) implies that the resulting flare distribution will be a power law (with index $\gamma$) above $E_c$ and below a rollover at large energy due to the decline of $P(E)$ at large $E$. The location of the rollover may be estimated from our equation (16) for the energy above which $P(E)$ becomes small. Substituting equation (17) into (16) gives

$$\epsilon_m = \left(1 + \frac{2-\gamma}{A}\right)^{1/(2-\gamma)} , \tag{18}$$

where $\epsilon_m = E_m/E_c$ and

$$A = \frac{\alpha_0 E_c^2}{\beta_0} . \tag{19}$$

The quantity $A$ is a nondimensional measure of the ratio of the decay rate to the growth rate. By a suitably small choice of $A$, $\epsilon_m$ may be made arbitrarily large, so it is possible to have any number of decades of power-law behavior between the low-energy cutoff at $E_c$ and the high-energy rollover near $E_m$.

A physical justification for our choice of $\alpha(E)$ is the possibility that the underlying energy release process in flares is describable by an avalanche model. The distribution of flare energies that we observe is the power-law distribution that results from the avalanche process, modified only by the probability that the active region has sufficient energy for a given avalanche event. In this interpretation, our model describes the global energetics of the avalanche system.

### 2.4. *Numerical Solution*

To confirm our general predictions about the behavior of the model for the power-law choice of $\alpha(E)$, we have numerically solved equation (11). A numerical method for this solution is arrived at by considering the equation as the average over an ensemble of identical active regions. If the ensemble is started from some initial distribution of energies, averaging at time $t$ defines a probability distribution, $P(E, t)$, that evolves according to the time-dependent extension of equation (11):

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial E}(\dot{E}P) - P \int_0^E \alpha(E')dE' + \int_E^\infty P(E', t)\alpha(E' - E)dE' . \tag{20}$$

For the case of $\dot{E} = \beta_0$, equation (20) may be rewritten in the nondimensional form

$$\frac{\partial p}{\partial \tau} = -\frac{\partial p}{\partial \epsilon} - Ap \int_0^\epsilon r(\epsilon')d\epsilon' + A \int_\epsilon^\infty p(\epsilon', \tau) r(\epsilon' - \epsilon) d\epsilon' , \tag{21}$$

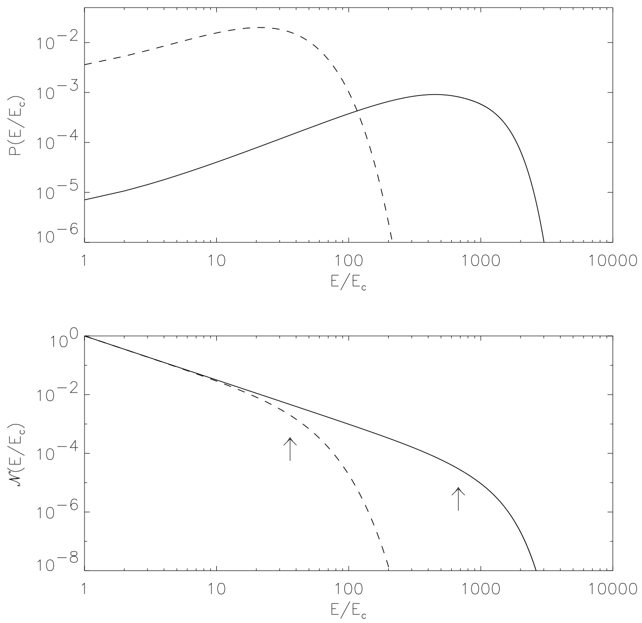


FIG. 1.—Upper panel shows the model probability distributions for $A = 0.02$ (*solid line*) and $A = 0.1$ (*dashed line*). Lower panel shows the corresponding flare frequency distributions, with the predictions for departure from power-law behavior (*arrows*).

where $\epsilon = E/E_c$, $\tau = t/(E_c/\beta_0)$, $p(\epsilon, \tau) = P(E_c \epsilon, E_c \tau/\beta_0)E_c$, and $r(\epsilon) = \alpha(E_c \epsilon)/\alpha_0$. Equation (21) was discretized and hence numerically solved for the case $r(\epsilon) = \theta(\epsilon - \epsilon_c)\epsilon^{-\gamma}$ by forward time-stepping to a steady state. The analytic solution, equation (13), provided a means of testing the numerical procedure.

Figure 1 shows the results of the calculation. The upper panel shows the probability density obtained for two cases, $A = 0.02$ (*solid line*) and $A = 0.1$ (*dashed line*). The lower panel shows the corresponding flare frequency distributions. Both cases were calculated for the choice of the power-law index $\gamma = 1.5$ (which matches the power-law index for the flare energy distribution inferred from observations); the results have been displayed as a function of $E/E_c$. The upper panel shows that the solutions have the same general behavior as the analytic (constant-$\alpha$) case; when $A$ (the ratio of decay rate to growth rate) is large, the system is more likely to be found at a low energy, and when $A$ is small, the system is more likely to be found at a high energy. The lower panel confirms the reproduction of power-law behavior above the low-energy cutoff and below the rollover at large energy. The arrows on the lower panel show the predictions of equation (18) for the departure from power-law behavior. The simple estimate $E_m$ gives a good indication of the rollover location in both cases. The normalization in the lower panel in Figure 1 assumes that the energy growth rate $\beta_0$ varies, while the transition rate parameter $\alpha_0$ is fixed. The power-law part of the curve scales linearly with the parameter $\alpha_0$, according to equation (12).

## 3. DISCUSSION

The calculation presented above demonstrates that the master equation leads to power-law flare frequency distributions for an assumed constant energy growth rate $\beta_0$ and an assumed power-law transition rate per unit energy $\alpha(E) = \alpha_0(E/E_c)^{-\gamma}\theta(E - E_c)$. An arbitrary number of decades of power-law behavior may be obtained between the low-energy cutoff, $E_c$, and a high-energy rollover at $E_m$. The location of the rollover is determined mostly by the ratio $\alpha_0 E_c^2/\beta_0$, i.e., by the ratio of decay to growth rates. Hence, the model is able to reproduce the flare frequency-energy distribution inferred from X-ray observations, which has a simple power-law form from the smallest to the largest flares (Hudson 1991).

The choice of the power-law form for $\alpha(E)$ needs to be justified in terms of the underlying flare mechanism; we have argued that an avalanche process may be at work. Avalanche processes naturally produce power-law event rate distributions, owing to the lack of a characteristic scale for events in self-organized critical systems. With this interpretation, the probability $P(E)$ is the likelihood that a total free energy $E$ is available in the region in a self-organized critical state, given the energy input rate $\beta_0$ and the intrinsic power-law distribution of energy loss rates $\alpha(E)$ due to the avalanche mechanism. There is a close correspondence between our model and the cellular automaton avalanche models (Lu & Hamilton 1991; Lu et al. 1993; Zirker & Cleveland 1993; Vlahos et al. 1995; Galsgaard 1996), which also represent a statistical steady state between a (roughly) constant rate of energy input and a power-law distribution of stochastic decay rates. Our model provides a complementary description of an avalanche system.

It is also necessary to justify our choice of a constant energy input rate, $\beta_0$. We have argued that this choice may describe energy input from the emergence of magnetic flux with free energy, in the form of field-aligned currents. The rate of flux emergence determines the rate of energy input to the active region, which may vary with time, but it is not dependent on the free energy in the active region. Hence, a constant value of $\dot{E}$ is a good approximation. Lu (1995c) has argued that the energy supply rate in the RV78 model cannot depend on the energy of the system, raising theoretical objections to the energy supply rate being reset by the occurrence of a flare. The choice of $\dot{E} = \beta_0$ (a constant) avoids these criticisms.

Our model predicts a rollover in the flare distribution at large energy, because of the low probability that the system has energy above $E_m$, as described in §§ 2.3 and 2.4. In the model, the value of $E_m$ is set by the balance between a given growth rate and the distribution of stochastic decay events. In practice, the growth rate and the decay process will be determined by physical properties of the active region. If we assume that the supply of energy comes from the emergence of nonpotential flux, then we expect that $\dot{E} \sim \dot{\Phi}$, where $\Phi$ is the total flux of the active region. In general, $\dot{\Phi}$ will be smaller for small active regions, so the rollover will occur at lower energies than for large active regions. The cellular automaton models of flares predict a simple relationship, $E_r \propto L^{\delta}$ ($\delta \approx 3.9$) between the rollover energy, $E_r$, and the size of the box, $L$ (Lu & Hamilton 1991; Lu et al. 1993). The box dimensions may be identified with the size of the active region (Lu et al. 1993; Wheatland & Sturrock 1996), so this model also predicts a rollover at a lower energy in the flare distribution from a small active region. Recently, observational evidence for such a rollover in the flare frequency distribution for small active regions has been obtained (Kucera et al. 1997).

By invoking an underlying avalanche process, our model is able to account for not just the power-law distribution of flare energies, but also the observed power-law distribution of flare peak flux. Avalanche models also produce a power-law distribution of flare durations over a limited range of durations, which may be consistent with observations. (The determination of the distribution of flare durations is made difficult by selection effects, but the distribution is roughly a power law over about a decade of durations; see Lee, Petrosian, & McTiernan [1993].) The ability to reproduce all three frequency durations represents an advantage of the avalanche models over the RV78 model, which must invoke additional (ad hoc) relationships to account for the duration and peak flux distributions.

Recent work suggests that the energies of coronal mass ejections (CMEs) are exponentially distributed (Jackson 1997). It is possible to modify the model we have presented to describe the statistics of both flares and CMEs, and in particular to account for the different distributions of these events. Consider that in addition to the flare transitions described by $\alpha_F(E)$, CMEs occur with a transition rate $\alpha_C(E)$ independently of flares (this may or may not be a good assumption—the relationship between flares and CMEs is still being debated). Then, the steady state of the system is described by equation (11) with the replacement $\alpha(E) \rightarrow \alpha_C(E) + \alpha_F(E)$, and the flare and CME frequency-energy distributions are given by

$$\mathscr{N}_i(E) = \alpha_i(E) \int_E^\infty P(E')dE' , \tag{22}$$

where $i = F, C$. If flares and CMEs have different underlying transition rates ($\alpha_F$ versus $\alpha_C$), then different frequency energy distributions will result. The observations may be accounted for by a mechanism that produces a power-law transition rate for flares and an exponential transition rate for CMEs.

Although we have argued for a power-law form for the transition rate $\alpha(E)$ to produce a power-law distribution $\mathscr{N}(E)$, there is also a simple solution to equation (11) that directly produces a power-law flare frequency distribution. If the flare transition rate does not depend on the energy of the flare, $\alpha = \alpha_0$, and the growth rate of the system is proportional to the square of the free energy, $\dot{E} = \beta_0 E^2$, then

$$P(E) = P_0 E^{-\delta} , \tag{23}$$

where

$$\delta = 1 + \frac{\alpha_0}{\beta_0} . \tag{24}$$

For the integral of the probability density to be finite, we require a low-energy cutoff, $E_c$, in the probability distribution; then the appropriate normalization is

$$P_0 = (\gamma - 1)E_c^{\delta - 1} . \tag{25}$$

The corresponding flare frequency distribution, via equation (12), is

$$\mathscr{N}(E) = \alpha_0 \left(\frac{E}{E_c}\right)^{-\alpha_0/\beta_0} . \tag{26}$$

This result is similar to the RV78 model in that the power-law index depends only on the ratio of the flaring rate to the growth rate of the system. Lu (1995c) has argued against this result of the RV78 model. It is also difficult to justify the particular choice of $\beta(E)$ needed to produce a power-law distribution of $\mathscr{N}(E)$. We can think of no a priori physical justification for this choice. In this paper we have argued for an energy-independent growth rate, which is appropriate if active regions are externally driven systems, and which meets the criticisms of Lu (1995c). For these reasons we consider equations (23)–(26) to be a mathematically interesting but not physically significant solution to the master equation.

We have presented a mathematical formalism to describe the energy of an active region—the master equation—but have not derived a general solution to the model. In the absence of a general solution, we have proceeded by reasoning, based on simple physical arguments, the mathematical form of the equations and a general property of solutions (§ 2.2). We have argued that a particular solution is relevant to flares, the solution for an energy-independent growth rate (a physical argument justifies this choice) and a power-law transition rate $\alpha(E)$ (strongly suggested by the mathematical form of eq. [12] together with the property of constant-$\dot{E}$ solutions derived in § 2.2). With these choices, the model produces a power-law flare frequency-energy distribution, and corresponds closely to the avalanche model. It is, however, possible that other solutions to the master equation exist that are of relevance to flares. For example, we have chosen to consider systems with a transition rate $\alpha(E)$ that depends only on the released energy, via equation (10). This choice was motivated by simplicity, and it is not clear whether this is a major assumption. Alternative assumptions for equation (10) and other solutions to the master equation will be considered in future work. It is also possible to generalize the existing model. For example, the evolution of a system driven by a time-varying energy input rate, $\dot{E} = \beta(t)$, is described by equation (11). We will consider further generalizations in later work.

We have argued that for our model to fit the observations, we require a power law transition rate per unit energy, $\alpha(E)$, which may be attributable to an avalanche process. This step presents our model as a complementary description to the (cellular automaton based) avalanche models for flares. While these models successfully reproduce the observed statistics, we have argued that their physical basis is somewhat unclear. In part, the deficiencies of all models for the statistics of flares are attributable to our lack of physical understanding of the details of energy release in flares. In the absence of first-principle models of flare energy storage and release, it is important to develop phenomenological models like the one outlined in this paper. The success of one such model over others may allow the reverse engineering of the flare mechanism.

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