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A measurement of the frequency dependence of the spring constant

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Abstract

We measured the inelasticity of a tungsten fiber by using a torsion balance. We concluded that the spring constant increases along with the angular frequency. This measurement supports the statement that the inelasticity of a torsional fiber causes a systematic error in the measurement of the Newtonian gravitational constant using the time-of-swing method.
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1. Introduction

The inelasticity of suspension fibers has been inspected in recent years concerning interferometric gravitational wave (GW) detectors by several authors [1–3]. Since the damping characteristics in the suspension and the mirror, itself, generally affect the thermal noise spectrum of suspended mirrors and of their internal vibration, it has been important to fix the damping model [4]. The experimental results published so far support an empirical inelasticity model showing that the loss coefficient is constant in frequency. If this is true, the spring constant of a fiber with such inelasticity should be dependent on the frequency. On the other hand, the present Newtonian gravitational constant was measured by a torsion balance using the time-of-swing method, which assumes a constancy of the spring constant of the torsion fiber. One of the authors pointed out a possible systematic error involving the Newtonian gravitational constant [5], and Luther showed experimentally the expected change in the torsional constant of a tungsten fiber applied to his new torsion balance [6].

We have measured the frequency dependence of the torsional spring constant of a tungsten fiber based on the systematic change in the inertial moment of a suspended balance. This measurement was complementary to a measurement of the mechanical loss.

2. Characteristics of inelasticity

The inelasticity of a spring is represented by a complex spring constant. Assume that it is represented by a

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function of angular frequency, $k_r(\omega) + \mathrm{i}k_i(\omega)$, where $k_i(\omega)/k_r(\omega) \equiv \phi(\omega)$ is experimentally a function of small magnitude. Since $k_i(\omega)$ is assumed to be constant for a wide range of frequencies, it can be expressed by

$$k_i(\omega) = \epsilon\omega^\alpha, \qquad \omega > 0, \\ = -\epsilon(-\omega)^\alpha, \qquad \omega < 0, \tag{1}$$

where α and ϵ are positive and α is small compared with unity, thus satisfying the Kramers–Kronig relation, which arose from the causality principle. Also, the real part of the spring constant should be

$$k_r(\omega) = \frac{2\epsilon}{\pi\alpha}\omega^\alpha, \tag{2}$$

where α turns out to be $2\phi/\pi$. Eq. (2) represents the frequency dependence of the spring constant.

The velocity damping model assumes that the damping term in the equation of motion is proportional to the velocity. We can thus write the equation of motion as

$$I\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + b\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} + k\theta(t) = N_{\mathrm{ext}}(t), \tag{3}$$

where I is the moment of inertia of the torsion balance, $\theta(t)$ the torsion angle of the balance, b the damping coefficient, k the spring constant, and $N_{\mathrm{ext}}(t)$ an external torque. In the case of velocity damping, the quality factor (Q) is

$$Q = \frac{\omega_0 I}{b} = \frac{k}{\omega_0 b}, \tag{4}$$

where $\omega_0 = \sqrt{k/I}$ is the eigen angular frequency of the torsion balance.

Contrary to the velocity damping model, the inelasticity model can be represented in the frequency domain of the equation of motion as

$$\{-I\omega^2 + k[1 + \mathrm{i}\phi(\omega)]\}\theta(\omega) = N_{\mathrm{ext}}(\omega). \tag{5}$$

From Eq. (5) the quality factor (Q) of the torsion balance at the eigen angular frequency is

$$Q = \frac{1}{\phi(\omega_0)}. \tag{6}$$

If $\phi(\omega)$ is independent of the torsional angular frequency, the quality factor Q is independent of the torsional angular frequency.

3. Experimental procedure

We used a torsion balance to measure the spring constant of a tungsten fiber (see Fig. 1). The fiber was 50 μm in diameter and 0.38 m in length. The balance consisted of three parts, where two masses with horizontal shafts were symmetrically connected at the center part. The masses were made of copper. The horizontal shafts were made of brass. The center part, made of aluminium, was suspended by the fiber. The moment of inertia of the balance was changed by displacing the position of the mass. This was done by replacing the horizontal shafts with ones having different lengths. The total mass of the balance was 0.122 kg.

The balance was set in a vacuum of about 10^{-4} Pa. We mounted a tiny right-angle-prism at the center part of the balance and measured the angle of the balance using an optical lever. A He–Ne laser (632.8 nm, 0.5 mW) was used. The reflected beam position was detected by a photodiode array having 46 photodiode segments. Since the distance from the balance to the photodiode array was 0.80 m and the segment pitch was 1 mm, the angular resolution of the optical lever was 0.006 rad. The sampling frequency of the torsional angle of the balance was 1 Hz. We calculated the period of the swing using the time-series data (see Fig. 2). We fitted the signal every two cycles with a sinusoidal function

$$\theta(t) = a\sin(\omega t + \varphi) + d, \tag{7}$$

where the amplitude (a), torsional angular frequency (ω), initial phase (φ), and offset (d) were fitted parameters. Based on a successive series of angular frequency measurements we made a histogram and evaluated the angular frequency by another fitting of the histogram with a Gaussian distribution. In each run, we changed the moment of inertia of the balance and measured the variation in the angular frequency of the balance (see Table 2). We measured the angular frequency twice in the case of each dumbbell (dumbbell number 1–6). First, we conducted the measurement in the increasing order of the dumbbell number from number 1 to number 6. It took almost one day for one measurement. Thus, one week was spent for this run. Second, we did the same measurement in decreasing order from number 6 to number 1, which

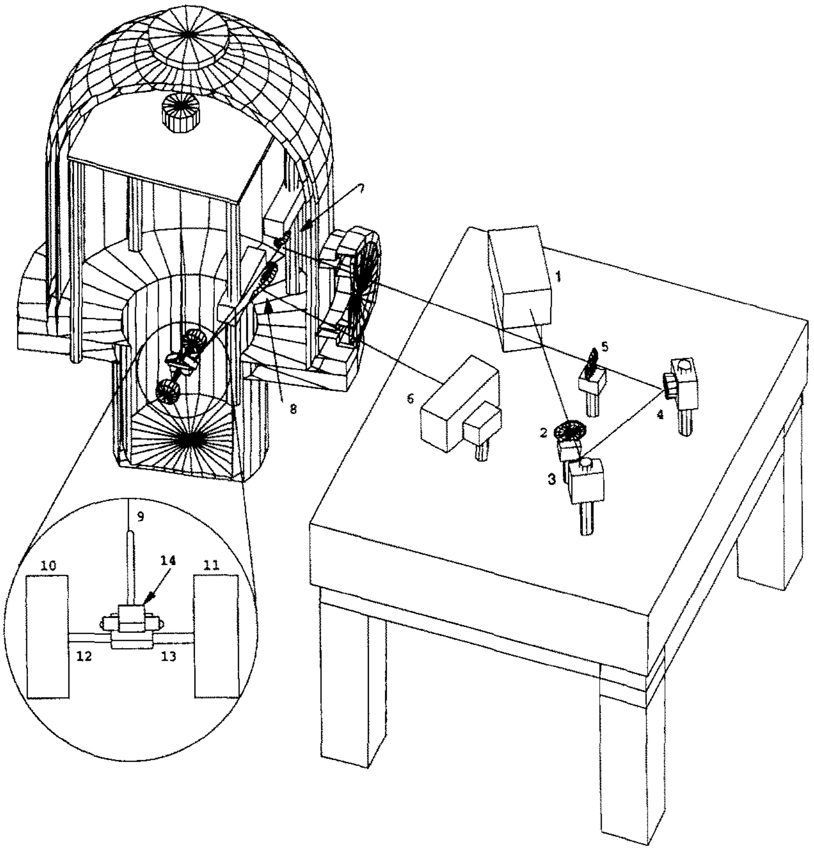

Fig. 1. Setup of the torsion balance and the optical lever where 1 is the laser source; 2 and 5 are lens; 3, 4, 7 and 8 are mirrors; 6 is a photodiode array; 9 is a fiber; 10 and 11 are the mass; 12 and 13 are horizontal shafts; 14 is a right-angle-prism.

took another week. Angular frequencies obtained in this way at the same dumbbell in different runs coincided with each other within the measurement error. Thus we concluded that the effect due to the aging of the tungsten fiber is negligible. We measured both the mass and length of the balance precisely and calculated the moment of inertia (see Table 1).

From these measurements, we obtained the spring constant of the torsion fiber during each run (see Table 2). These are plotted in Fig. 3 as the torsional spring constant of the tungsten fiber,

Table 1
Error budgets for the spring constant (k)

		ppm
Statistic:	$\Delta\omega$	110–210
Systematic:	Position of the masses	130–200
	Mass of the masses	43
	ΔI	160–220
Total:		270–440

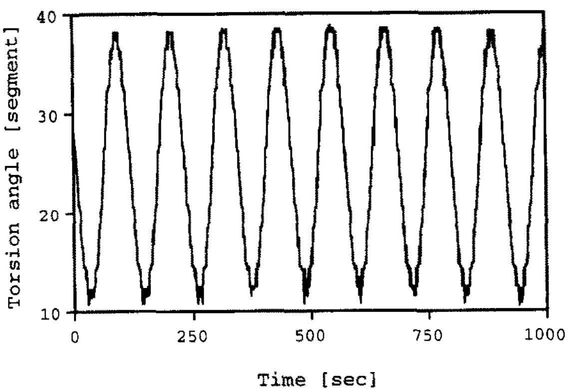

Fig. 2. Swing of the torsion balance.

Table 2

Dumbbell	Moment of inertia ($kg\,m^2$)	Frequency (Hz)	Spring constant (N m/rad)	Quality factor
1	3.3217×10^{-5}	1.3591×10^{-2}	2.422×10^{-7}	3.2×10^3
2	5.3947×10^{-5}	1.0664×10^{-2}	2.422×10^{-7}	3.5×10^3
3	7.8836×10^{-5}	8.8212×10^{-3}	2.422×10^{-7}	3.2×10^3
4	1.1092×10^{-4}	7.4352×10^{-3}	2.421×10^{-7}	3.3×10^3
5	2.4712×10^{-4}	4.9798×10^{-3}	2.419×10^{-7}	3.1×10^3
6	4.4743×10^{-4}	3.7013×10^{-3}	2.420×10^{-7}	3.3×10^3

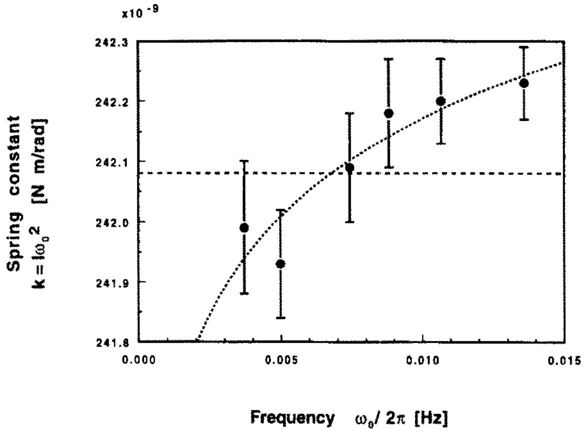

Fig. 3. Frequency dependence of the real part of the spring constant. The dotted line shows the result of fitting the spring constant with Eq. (2). The dashed line shows the result of fitting the spring constant with a constant.

$$k = I\omega_0{}^2 . \tag{8}$$

We also measured the quality factor (Q) of the torsion balance (see Table 2). The amplitude decay was fitted in each run by a function $a(t) = a_0 e^{-t/2\tau}$, where τ is the relaxation time and a_0 is the initial amplitude. The measured Q at different frequencies is plotted in Fig. 4 as

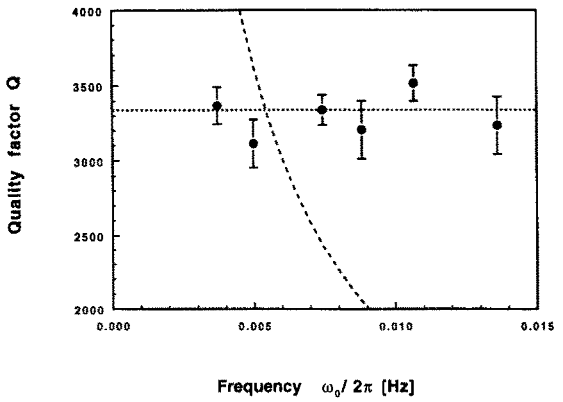

Fig. 4. Frequency dependence of Q. Some errors are within the markers. The dotted line shows the result of the fitting measured Q with a constant. The dashed line shows the result of the fitting measured Q with Eq. (4).

$$Q = \omega_0 \tau . \tag{9}$$

4. Discussion

We measured the spring constant of a tungsten fiber used in a torsion balance. If we applied the velocity damping model to the fiber, the spring constant was constant in frequency. We fitted the measured spring constant with a function $k_r(\omega)$ = constant and obtained $\chi^2/N = 11.5/5$, where N is the degree of freedom. Fitting the measurement with Eq. (2) produced

$$\phi^{-1} = 697^{+339}_{-172}, \quad \chi^2/N = 2.2/4 \tag{10}$$

(Fig. 3). From this we concluded that the spring constant increases along with the angular frequency. This value should be compared with the observed mechanical loss. Measurements showed that the quality factor of the fiber was constant from 3.7 to 13.6 mHz within the error of the measurment, that is, the result of fitting the quality factor with a function Q = constant was

$$Q = 3338 \pm 58 . \tag{11}$$

The value of Q in Eq. (11) is five times larger than that in Eq. (10). However, positive and negative errors of ϕ are asymmetric and non-linear. The significance of $\phi^{-1} = 3338$ is 3σ. Although the disagreement between the fitted value and the Q measurement is not small, the trend of change is in agreement with the theoretical model of inelasticity. This discrepancy would be smaller if the change rate becomes less. If we applied the velocity damping model, the quality factor should have been inversely proportional to the torsional frequency (see Eq. (4)), which did not satisfy the measurement. If we take the inelasticity model with constant structure damping, the Q calculated by Kramers–Kronig relation does not completely agree with that of the direct measurement. Since we have measured the frequency in only a small portion of frequency range and we have not had complete knowledge of the Q in whole frequency range, the Kramers–Kronig relation might not be applied. What we can say is that it suggests the existence of inelasticity with the accuracy presented here. We need to do further precise experiments to dtermine how the inelasticity can be applied. This measurement supports the statement that the inelasticity of a torsional fiber causes a systematic error in the measurement of the Newtonian gravitational constant using the time-of-swing method.

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