Calculation of G (IR 68)

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Abstract

The time-of-flight method of measuring Newton's gravitational constant, G, can begin with the following questions: 1) What is the torque experienced by the pendulum in terms of the multipole moments of the pendulum and source masses? 2) What are the multipole moments of the pendulum from it's metrology? 3) What are the multipole moments of the source masses from their metrology? 4) What is the frequency shift in terms of torque components? and 5) What is the frequency shift as measured from the data? This report addresses these questions at the initial level which is purely calculational, without recourse to measurements. Various lab conventions are explicitly expressed such as definitions of a_{lm} and K, and various approximations and corrections are identified for consideration elsewhere.

1 Torque in Terms of Multipole Moments

For discrete objects, Newton's law of gravity is just $F = \frac{GmM}{r^2}$, the potential energy is $V = -\int \mathbf{F} \bullet \mathbf{dr} = \frac{-\mathrm{GmM}}{\mathrm{r}}$, and the gravitational potential is $\Phi = \frac{V}{m} = \frac{-GM}{r}$. For continuous rigid mass distributions these potentials become $V(\mathbf{r}) = \int \rho_p(\mathbf{r})\Phi(\mathbf{r})\mathrm{d}\mathbf{r}$ and $\Phi(\mathbf{r}) = -\mathrm{G}\int \rho_\mathrm{s}(\mathbf{r}^{'})\frac{\mathrm{d}\mathbf{r}^{'}}{|\mathbf{r}-\mathbf{r}^{'}|}$ which can be expanded in terms of a spherical harmonic series where $\mathbf{r} = (r,\theta,\phi)$ and $\mathbf{r}^{'} = (r^{'},\theta^{'},\phi^{'})$ are spherical coordinates with θ or $\theta^{'}$ being the angle between the vector and the z-axis from 0 to π and ϕ or $\phi^{'}$ representing rotations about the z axis referenced to the x-axis from 0 to 2π.

$$\frac{1}{|\mathbf{r}-\mathbf{r}^{'}|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{(2l+1)}\frac{r^l}{r^{'l+1}}\, Y^*_{lm}(\theta,\phi)\, Y_{lm}(\theta^{'},\phi^{'}) \quad \text{for} \quad |\mathbf{r}^{'}| > |\mathbf{r}| \tag{1}$$

$$\Phi(\mathbf{r}) = -\mathrm{G}\int \rho_\mathrm{s}(\mathbf{r}^{'}) \left(4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{(2l+1)}\frac{r^l}{r^{'l+1}}\, Y^*_{lm}(\theta,\phi)\, Y_{lm}(\theta^{'},\phi^{'}) \right) d\mathbf{r}^{'}$$

The torque in the z-direction produced by the gravitational force when the pendulum is aligned at an angle $\phi = \psi$ is just $\tau = -\frac{dV}{d\psi} = -\frac{d}{d\psi}\left(\int \rho_p(\mathbf{r})\Phi(\mathbf{r})\mathrm{d}\mathbf{r}\right)$.

$$\tau = \frac{-d}{d\psi}\left\{ \int \rho_p(\mathbf{r}) \left[-G\int \rho_s(\mathbf{r}^{'}) \left(4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{(2l+1)}\frac{r^l}{r^{'l+1}}\, Y^*_{lm}(\theta,\phi)\, Y_{lm}(\theta^{'},\phi^{'}) \right) d\mathbf{r}^{'} \right] d\mathbf{r} \right\}$$

The integration over the pendulum mass distribution simplifies if the coordinate system is changed from lab spherical (r,θ,ϕ) to pendulum fixed spherical (r,θ,γ) where $\phi = \gamma + \psi$. Using the identity $Y^*_{lm}(\theta,\gamma+\psi) = Y^*_{lm}(\theta,\gamma)e^{-im\psi}$, and regrouping terms yields:

$$\tau = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left[\left(\frac{-d}{d\psi}e^{-im\psi}\right) \underbrace{\int \rho_p(r,\theta,\gamma)r^l Y^*_{lm}(\theta,\gamma)d\mathbf{r}}_{q_{lm}} \underbrace{\int \rho_s(r^{'},\theta^{'},\phi^{'})\frac{1}{r^{'l+1}}Y_{lm}(\theta^{'},\phi^{'})\left(\frac{-4\pi G}{2l+1}\right)d\mathbf{r}^{'}}_{a^*_{lm}(\text{ring}\#1)\ +\ a^*_{lm}(\text{ring}\#2)\ +\ a^*_{lm}(\text{other sources})} \right] \tag{2}$$

The q_{lm} and a^*_{lm} are multipole moments for the pendulum and a single source mass ring or other source mass respectively. The torque, potential energy and gravitational potential can be easily expressed in these terms, where $\bar{a}^*_{lm}$ represents the average of the two rings (neglecting a^*_{lm} from all other source masses).

$$\tau(\psi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} ime^{-im\psi} q_{lm} 2\bar{a}^*_{lm}$$

$$V(\psi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{-im\psi} q_{lm} 2\bar{a}^*_{lm}$$

$$\Phi(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{-im\psi} 2\bar{a}^*_{lm} r^l Y^*_{lm}(\theta, \gamma) \tag{3}$$

Non-periodic torques such as transients, excitation and damping effects shall be discussed in Section 5 and dealt with at the data level. Fourier components of a general periodic torque can be expressed in terms of these multipole moments as follows.

$$\tau(\psi) = -\sum_{n=0}^{\infty} [\alpha_n cos(n\psi) + \beta_n sin(n\psi)] = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} ime^{-im\psi} q_{lm} 2\bar{a}^*_{lm} \tag{4}$$

By multiplying both sides by $e^{if\psi} = cos(f\psi) + isin(f\psi)$ where f is an integer, integrating over ψ from 0 to 2π, and rewriting the sum over negative m as a sum over positive m by substituting $m \rightarrow -m$, the resulting relation is:

$$-\sum_{n=0}^{\infty} \int_0^{2\pi} [cos(f\psi) + isin(f\psi)] [\alpha_n cos(n\psi) + \beta_n sin(n\psi)] \, d\psi$$

$$= \sum_{l=0}^{\infty} \sum_{m=1}^{l} \int_0^{2\pi} e^{if\psi} \left(ime^{-im\psi} q_{lm} 2\bar{a}^*_{lm} - ime^{+im\psi} q_{l-m} 2\bar{a}^*_{l-m} \right) d\psi$$

Using the normalization and orthogonality identities $\int_0^{2\pi} sin(a\psi)sin(b\psi)d\psi = \int_0^{2\pi} cos(a\psi)cos(b\psi)d\psi = \pi\delta_{ab}$ and $\int_0^{2\pi} sin(a\psi)cos(b\psi)d\psi = 0$, along with the identity $Y_{l-m}(\theta, \gamma) = (-1)^m Y^*_{lm}(\theta, \gamma)$ which leads to $q_{l-m} a^*_{l-m} = q^*_{lm} a_{lm}$, the above relation becomes:

$$-\sum_{n=0}^{\infty} (\alpha_n \pi \delta_{fn} + i\beta_n \pi \delta_{fn}) = \sum_{l=0}^{\infty} \sum_{m=1}^{l} im \left(\int_0^{2\pi} e^{i\psi(f-m)} q_{lm} 2\bar{a}^*_{lm} d\psi - \int_0^{2\pi} e^{i\psi(f+m)} q^*_{lm} 2\bar{a}_{lm} d\psi \right)$$

This integrates and simplifies to:

$$-(\alpha_f + i\beta_f)\pi = \sum_{l=0}^{\infty} \sum_{m=1}^{l} im\delta_{fm} 2\pi q_{lm} 2\bar{a}^*_{lm}$$

So as a result, the torque Fourier components α_m and β_m are expressed in terms of the multipole moments of the pendulum and average single source mass ring, q_{lm} and $\bar{a}^*_{lm}$ as:

$$\alpha_m = \sum_{l=m}^{\infty} +4mIm(q_{lm}\bar{a}^*_{lm}) = \sum_{l=m}^{\infty} +4m \left[Re(q_{lm})Im(\bar{a}^*_{lm}) + Im(q_{lm})Re(\bar{a}^*_{lm}) \right]$$

$$\beta_m = \sum_{l=m}^{\infty} -4mRe(q_{lm}\bar{a}^*_{lm}) = \sum_{l=m}^{\infty} -4m \left[Re(q_{lm})Re(\bar{a}^*_{lm}) - Im(q_{lm})Im(\bar{a}^*_{lm}) \right] \tag{5}$$

2 Pendulum Multipole Moments

As will be seen in Section 4, the pendulum moment of inertia becomes involved, and it is the ratio $\frac{q_{lm}}{I}$ of multipole moment to moment of inertia which is relevant. Both q_{lm} and I are determined from integrals over the pendulum mass distribution in the pendulum's center of mass frame. From Equation 2

$$q_{lm} \equiv \int \rho_p(r, \theta, \gamma) r^l Y^*_{lm}(\theta, \gamma) d\mathbf{r}$$

The moment of inertia for arbitrary rotations of a general mass distribution about the z-axis (in lab Cartesian coordinates) is:

$$I = \int \rho_p(x,y,z)\left(x^2+y^2\right)d\mathbf{r} \tag{6}$$

As an exercise in calculating these integrals, convert to pendulum fixed cylindrical coordinates $(R^{'},\gamma^{'},z^{'})$ using $r^2 = R^{'2}+z^{'2}$, $\theta = Asin\left(R^{'}/r\right)$, and $\gamma = \gamma^{'}$ as well as $x^2+y^2 = R^{'2}$, $cos(\gamma^{'}) = \frac{x}{R^{'}}$, and $z^{'} = z$.

$$q_{lm} = \int \rho_p(R^{'},\gamma^{'},z^{'})\left(R^{'2}+z^{'2}\right)^{l/2} Y^{*}_{lm}(Asin(R^{'}/\sqrt{R^{'2}+z^{'2}}),\gamma^{'})\ \ R^{'}dR^{'}d\gamma^{'}dz^{'} \tag{7}$$
$$and\ \ I = \int \rho_p(R^{'},\gamma^{'},z^{'})R^{'2}\ \ R^{'}dR^{'}d\gamma^{'}dz^{'}$$

Consider an ideal rectangular pendulum of equal sides s and thickness d, perfectly positioned and oriented flat in the xz plane, with uniform density ρ_{p_o}. As will be seen in Section 5, the main coupling in Equation 2 is through the q_{22} term. Insofar as the pendulum has $m=2$ symmetry in the γ direction, the integral from 0 to 2π of the odd function $Im\left[Y_{22}\right] \sim sin(2\gamma)$ will be 0. Equation 5 then reduces to terms in $Re\left[q_{22}\right]$. Under these conditions, and referring to $Y_{22}(\theta,\gamma) = \sqrt{\frac{15}{32\pi}}sin^2(\theta)e^{2i\gamma}$ these integrals analytically reduce to:

$$Re[q_{22}] = \rho_{p_o}\sqrt{\frac{15}{32\pi}}\int_{-s/2}^{+s/2}dz^{'}\int_0^{2\pi}cos(2\gamma^{'})\left[\int_0^{R_{edge}}R^{'3}dR^{'}\right]d\gamma^{'} = \rho_{p_o}\frac{s}{4}\sqrt{\frac{15}{32\pi}}\int_0^{2\pi}cos(2\gamma^{'})R^4_{edge}d\gamma^{'}$$
$$I = \rho_{p_o}\int_{-s/2}^{+s/2}dz^{'}\int_0^{2\pi}d\gamma^{'}\int_0^{R_{edge}}R^{'3}dR^{'} = \rho_{p_o}\frac{s}{4}\int_0^{2\pi}R^4_{edge}d\gamma^{'} \tag{8}$$

In integrating over each of the four sides where $\epsilon = Atan(d/s)$ is the angle of the first corner in the $\psi^{'}$-direction, R_{edge} takes the following values:

$$\begin{array}{l|l}
\gamma^{'} = -\epsilon \text{ to } \epsilon & R_{edge} = \frac{s/2}{cos(\gamma^{'})} \\
\gamma^{'} = \epsilon \text{ to } \pi-\epsilon & R_{edge} = \frac{d/2}{cos(\gamma^{'}-\frac{\pi}{2})} \\
\gamma^{'} = \pi-\epsilon \text{ to } \pi+\epsilon & R_{edge} = \frac{s/2}{cos(\gamma^{'}-\pi)} \\
\gamma^{'} = \pi+\epsilon \text{ to } 2\pi-\epsilon & R_{edge} = \frac{d/2}{cos(\gamma^{'}-\frac{3\pi}{2})}
\end{array}$$

Applying the integral solutions, $\int\frac{dx}{cos^4x} = \frac{1}{3}tan\,x(2+\frac{1}{cos^2}x)$, $\int\frac{dx}{cos^2x} = tan\,x$, and $\int\frac{sin^2x}{cos^4x}dx = \frac{1}{3}tan\,x(\frac{1}{cos^2}x-1)$, the result for this special, ideal and approximate case becomes:

$$Re[q_{22}] = \sqrt{\frac{15}{32\pi}}\frac{d\ \ s^2\rho_{p_o}}{12}(s^2-d^2)$$
$$I = \frac{d\ \ s^2\rho_{p_o}}{12}(s^2+d^2)$$
$$\frac{Re\left[q_{22}\right]}{I} = \sqrt{\frac{15}{32\pi}}\left(\frac{s^2-d^2}{s^2+d^2}\right) \tag{9}$$

An imperfect pendulum geometry or orientation, non-uniform density, variations in pendulum temperature from run to run as well as between the 20C metrology and 2K data would alter the value of q_{22}.

3 Ring Multipole Moments

The a^*_{lm} are determined from integrals over the source mass distributions. For the purposes herein, a single ring a^*_{lm} is calculated, with the understanding that this may be used to approximate the average ring $\bar{a}^*_{lm}$ and a similar calculation should be made for each mass distribution potentially influencing the pendulum. From Equation 2

$$a^*_{lm} = \int \rho_s(r^{'},\theta^{'},\phi^{'})\frac{1}{r^{'l+1}}Y_{lm}(\theta^{'},\phi^{'})\left(\frac{-4\pi G}{2l+1}\right)d\mathbf{r}^{'}$$

Now transform coordinates from lab spherical $(r^{'},\theta^{'},\phi^{'})$ to ring cylindrical with the same origin $(R^{'},\eta^{'},z^{'})$ using $r^{'2} = R^{'2}+z^{'2}$, $r^{'2}sin^2(\theta^{'}) = z^{'2}+R^{'2}cos^2(\eta^{'})$, $sin\phi^{'} = \frac{R^{'}cos\eta^{'}}{r^{'}sin\theta^{'}}$, and $cos\phi^{'} = \frac{z^{'}}{r^{'}sin\theta^{'}}$. Here $R^{'}$ and $\eta^{'}$ are the radial distance and angle around the ring as positioned on the positive x-axis (lab frame) which is called $z^{'}$ in this ring frame. This equation then becomes:

$$a^*_{lm} = \int \rho_s(R^{'}, \eta^{'}, z^{'})(z^{'2} + R^{'2})^{-(l+1)/2} Y_{lm}\left(\frac{1}{2}Asin\left(\frac{(z^{'2}+R^{'2}cos^2(\eta^{'})}{(z^{'2}+R^{'2})}\right), Asin(\frac{R^{'}cos\eta^{'}}{r^{'}sin\theta^{'}})\right)\left(\frac{-4\pi G}{2l+1}\right) d\mathbf{r}^{'} \tag{10}$$

Consider an ideal ring as a cylindrical annulus which extends from z_1 to z_2 in thickness in the $z^{'}$ direction, with ID R_1 and OD R_2 in the $R^{'}$ direction, and of course extending 2π radians around $\eta^{'}$. Presume it is perfectly positioned and oriented, with uniform density ρ_{s_o}. As is noted in Section 5, the main coupling is through the a^*_{22} term. As with the pendulum, if the ring has $m = 2$ symmetry in the $\eta^{'}$ direction, then the $Im\,[a^*_{lm}]$ term of β_{22} is zero. Under these conditions and referring to Y_{22} as above, Equation 10 becomes

$$Re\,[a^*_{22}] = \sqrt{\frac{15}{32\pi}}\left(\frac{-4\pi G}{5}\right)\int_{R_1}^{R_2} R^{'}dR^{'}\int_{z_1}^{z_2} dz^{'}\int_0^{2\pi} d\eta^{'}\frac{\rho_s}{(R^{'2}+z^{'2})^{5/2}}\left(z^{'2} - R^{'2}cos^2\eta^{'}\right) \tag{11}$$

Integrating over $\eta^{'}$ using $cos2x = 2cos^2x - 1$, and integrating over $z^{'}$ and $R^{'}$ using the identities

$$\begin{aligned}\int\frac{dx}{(x^2+a^2)^{5/2}} &= \frac{x}{3a^2(x^2+a^2)^{3/2}} + \frac{2}{3a^2}\int\frac{dx}{(x^2+a^2)^{3/2}}\\ \int\frac{dx}{(x^2+a^2)^{3/2}} &= \frac{x}{a^2(x^2+a^2)^{1/2}}\\ \int\frac{xdx}{(x^2+a^2)^{3/2}} &= \frac{-1}{(x^2+a^2)^{1/2}}\end{aligned}$$

gives the result for this special, ideal, and approximate case as:

$$Re\,[a^*_{22}] = -G\rho_{s_o}\pi\sqrt{\frac{3\pi}{10}}\left[\frac{z_2}{\sqrt{R_2^2+z_2^2}} - \frac{z_2}{\sqrt{R_1^2+z_2^2}} - \frac{z_1}{\sqrt{R_2^2+z_1^2}} + \frac{z_1}{\sqrt{R_1^2+z_1^2}}\right] \tag{12}$$

A similar calculation for an ideal ring, perfectly positioned on the positive y-axis (lab frame), is just the negative of the above result. This can be seen by selecting ring cylindrical coordinates as above, but with the $z^{''}$ axis pointing in the y-direction rather than the x-direction. The transformation differs in that $cos\phi^{'} = -\frac{R^{''}cos\eta^{''}}{r^{'}sin\theta^{'}}$, and substituting into the integrand simplifies to the negative of the integrand for the above case. A ring positioned on the negative x-axis (lab frame)would use that direction for the $z^{'}$ direction and the transformation would differ from the positive x-axis case in that $-cos(\phi^{'}) = \frac{z^{'}}{r^{'}sin\theta^{'}}$. Because the $Re\,[a^*_{22}]$ involves only $cos^2(\phi^{'})$, the result is the same as Equation 12 and the effect of the two rings add so that $Re\,[\bar{a}^*_{22}] = 2\,[a^*_{22}]$. The same is true for the rings positioned along the y-axis. When the rings are moved from one axis to the other, a volume of air is moved in the reverse direction. Simply replacing the density allows one to use the same equations to calculate the gravitational a^*_{22} due to the air.

An imperfect ring geometry orientation or placement, non-uniform density, variations in ring temperature from run to run as well as between the 20C metrology and 19C data, variations from lab standard air temperature pressure and humidity (20C, 975mBar, 0% RH), potential effects of the Ni plating on thermal expansion of the copper rings as well as the true expansion coefficient, thermal expansion of the ring-separating rods, and finally all the extras and modifications to a cylindrical ring (grooves, pin holes, pins, resister holes, resistors, vacuum grease, chamfers, sapphire windows, kevlar strings, swing mirror, resister wires safety strings and aluminum restraints, wire ties, electrical connectors, and scotch tape) would contribute to the value of $\sum a^*_{22}$. Additionally, higher order coupling to ambient field moments may be significant as discussed in Section 4. To eliminate some of the systematic uncertainty due to ring geometry and placement, data was taken with the rings in flipped and rotated configurations, and an unweighted average over these variations will reduce the systematic bias for uncorrected effects.

4 Frequency Shift in Terms of Torque Components

Given a general periodic applied torque as in Equation 4, what is the resulting motion of the pendulum – specifically it's frequency? Start by writing $\tau = I\psi^{''}$ for the torques produced by the fiber and fiber mount as well as by gravity and any other potential fields. Bantel has reviewed an extensive number of possible fiber torques

[Ph.D. Thesis, "Analastic behavior of a torsion pendulum with a CuBe fiber at low temperatures, and implications for a measurement of the gravitational constant.", UC Irvine, 1998, page 184]. Consider a "reasonable" subset here, in the torque balance equation:

$$I\psi^{''} = -k_1\psi - k_2^{''}|\psi|\psi - k_2^{'}\psi^2 - k_3\psi^3 - b\psi^{'} - k_{ss}[(A^2-\psi^2)Sign(\psi^{'}) - 2A\psi] - \sum_{n=1}^{\infty}[\alpha_n cos(n\psi) + \beta_n sin(n\psi)] \quad (13)$$

This has not been solved analytically, however an iterative substitution of an assumed solution produces a result ordered in diminishing corrections. The 0^{th} order solution is just

$$(0^{th}\ \ order\ \ solution)\ \ \psi(t) = Asin(wt), \quad w^2 = w_o^2 = \tfrac{k_1}{I} \quad (14)$$

By substituting this into Equation 13 one can solve analytically for w^2 to 1^{st} order as follows. The resulting $cos(nAsinwt)$ and $sin(nAsinwt)$ terms can be expanded as follows:

$$cos\left(nAsin(wt)\right) = \sum_{f=0}^{\infty}\left[C_{nf}cos(fwt) + D_{nf}sin(fwt)\right], \ \ sin\left(nAsin(wt)\right) = \sum_{s=0}^{\infty}\left[c_{ns}cos(swt) + d_{ns}sin(swt)\right] \quad (15)$$

$$\begin{aligned}
C_{nf} &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} cos(fwt)cos\left[nAsin(wt)\right]d(wt) \\
D_{nf} &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} sin(fwt)cos\left[nAsin(wt)\right]d(wt) = 0 \text{ from orthogonality} \\
c_{ns} &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} cos(swt)sin\left[nAsin(wt)\right]d(wt) = 0 \text{ from orthogonality} \\
d_{ns} &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} sin(swt)sin\left[nAsin(wt)\right]d(wt)
\end{aligned}$$

But this is just a combination of Bessel functions. From Bessel's first integral:

$$J_f(nA) \equiv \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} sin(f\theta)sin(nAsin\theta)d\theta + \int_{-\pi}^{\pi} cos(f\theta)cos(nAsin\theta)d\theta\right] = \frac{1}{2}(d_{nf} + C_{nf}) \quad (16)$$

It can be shown from odd and even symmetry that all the C_{nf} with odd n and d_{ns} with even n vanish, so Equation 15 becomes:

$$cos\left(nAsin(wt)\right) = \sum_{f=0}^{\infty} 2J_f(nA)cos(fwt); (f\ \ even), \ \ sin\left(nAsin(wt)\right) = \sum_{s=1}^{\infty} 2J_s(nA)sin(swt); (s\ \ odd) \quad (17)$$

The resulting torque equation is

$$\begin{aligned}
-AIw^2 sin(wt) = -k_1 Asin(wt) - k_2^{'}A^2 sin^2(wt) - k_2^{''}|Asin(wt)|Asin(wt) - k_3A^3sin^3(wt) - bAwcos(wt) \\
-k_{ss}[(A^2 - A^2sin^2(wt))Sign(Awcos(wt)) - 2A^2sin(wt)] \\
-\sum_{n=1}^{\infty}\left[\alpha_n\left(\sum_{f=0,even}^{\infty} 2J_f(nA)cos(fwt)\right) + \beta_n\left(\sum_{s=1,odd}^{\infty} 2J_s(nA)sin(swt)\right)\right] \quad (18)
\end{aligned}$$

Taking a Fourier projection by multiplying by $sinwt$ and integrating over $d(wt)$ from $-\pi$ to π gives a relationship for w^2, where even and odd symmetry about 0 is used to determine that some integrals are 0. The $k_2^{'}$ and b terms are 0, and so an exact 1^{st} order solution for w^2 is then:

$$(1^{st}\ \ order\ \ solution)\ \ w^2 = w_o^2 + k_2^{''}\frac{8A}{3\pi I} + k_3\frac{3A^2}{4I} - k_{ss}\frac{2A}{I} + \sum_{n=1}^{\infty}\left[\frac{2\beta_n}{IA}J_1(nA)\right] \quad (19)$$

As noted in Section 2, and demonstrated by Equation 19 where $\beta_m \sim a_{lm}^*$ from Section 1, it is the ratio of q_{lm}/I which is relevant. The 1^{st} order solution for $\psi(t)$ can be determined by substituting into Equation 13 the general periodic solution

$$\psi(t) = Asin(wt) + \sum_{l=0}^{\infty}\left[A_l cos(lwt) + B_l sin(lwt)\right], \quad where\ \ B_1 = 0 \quad (20)$$

and solving for the A_l and B_l coefficients. Each torque other than the dominant k_1 torque is considered separately, neglecting cross terms of small torques by substituting the 0^{th} order solution into all but one small torque term at a time. Then the separate 1^{st} order solution terms are added together [$k_2^{'}$, $k_2^{''}$, k_3, and $\alpha_n + \beta_n$ terms as calculated by Newman, 1995 write up rev. 3]. The b term comes from an assumed solution of $\psi(t) = Ce^{Bt}$ to accommodate the damping effects.

$$\psi(t) = Ae^{\frac{-b_{ss}}{2mt}} sin(wt) - \frac{k_2^{''}}{k_1} \sum_{l=3,odd}^{\infty} \frac{8A^2}{\pi l(l^2-1)(l^2-4)} sin(lwt) - \frac{k_2^{'}}{k_1}\left(\frac{1}{2} + \frac{1}{6}cos(2wt)\right) A^2$$
$$-\frac{k_3}{k_1}\left(\frac{A^3}{32} sin(3wt)\right) + Ce^{-tb/2I} cos\left(t\sqrt{w_o^2 - (\frac{b}{2I})^2}\right) - \sum_{n>2,odd}^{\infty} cos(nwt)\frac{k_{ss}A^2 8Re(+i)^{m+1}}{k_1 \pi m(m^2-1)(m^2-4)} \quad (21)$$
$$+ \sum_{l=2,even}^{\infty}\left[\sum_{n=1}^{\infty} \frac{2J_l(nA)\alpha_n}{(l^2-1)k_1}\right] cos(lwt) + \sum_{l=3,odd}^{\infty}\left[\sum_{n=1}^{\infty} \frac{2J_l(nA)\beta_n}{(l^2-1)k_1}\right] sin(lwt) - \sum_{n=1}^{\infty} \frac{J_0(nA)\alpha_n}{k_1}$$

where the damping constant b_{ss} can be calculated. The above exercise gives the frequency shift to 1^{st} order in Equation 19. A 2^{nd} order solution for w^2 can be attempted by substituting the 1^{st} order solution of Equation 21 into Equation 13, but this should be explored elsewhere. The 2^{nd} order corrections of interest are between β_2 and each torque component, since it is the gravitational β_2 which is modulated by the source mass. These "cross-term" effects on the frequency have been investigated by numerical integration [Newman, 6-14-95 write up and IR38].

In addition to the above reasonable set of fiber effects, 19 additional torques have been considered by Bantel [c.f. Ph.D. Thesis, page 184]. In determining which torque terms are significant, Bantel has measured k_1, K_{ss}, $k_2^{''}$, and k_3 for CuBe fibers [c.f. Ph.D. Thesis, page 130].

Some more exotic possible corrections could be due to non-Lagrangian (i.e. coupling terms involving higher order derivatives than $\psi^{'}$), relativistic, or quantum effects. More likely corrections, due to non-gravitational periodic forces, would be included in the a^*_{lm} and q_{lm} formalism if they are $1/r^2$ forces. The effect of fiber temperature variations will primarily be through the temperature dependence of k_1. Tilt effects are generally geometric effects on the optoelectronic readout system, but could be electromagnetic coupling to the support frame.

5 Frequency Shift in Terms of the Data

The source mass rings are moved back and forth between two positions $\frac{\pi}{2}$ apart (assumed to be exact) and the difference in frequency Δw^2_{raw} is measured. Δw^2_{raw} is defined as the frequency difference, (ring position "+1")-(ring position "-1"), where position "+1" has the rings along the East-West axis in agreement with the datataking code. This will be taken as the x-axis herein. Recall in Section 2 the pendulum is aligned flat in the xz-plane, which operationally is also East-West. The $Re\,[\bar{a}^*_{lm}]$ for the two ideal rings of Section 3 in position "+1" is two times that of Equation 12.

Over the course of datataking, the amplitude generally decreases due to the various damping terms. Over short time intervals the amplitude may increase and decrease due to the transport of the rings between positions, non-periodic fiber torques, and transients introduced by abrupt temperature or mechanical shifts. The effect of all these on the recorded data is a drift in amplitude and a drift in period, which is near impossible to calculate exactly. In order to remove linear drifts in frequency, an 'ABA' is calculated, meaning that the above definition of Δw^2_{raw} is replaced with the following where the terms are sequential data 'sets' according to the source mass modulation at time t_1, t_2, then t_3. The $\pm$ indicates the ring state of the central value (+ if unspecified).

$$\Delta\omega^{\pm}_{raw} \equiv \left[w^2_{raw}\right]_{ABA\pm} \equiv w^2_{raw}(position \pm 1,\ \ time_2) - \frac{1}{2}\left[w^2_{raw}(position \mp 1,\ \ time_3) + w^2_{raw}(position \mp 1,\ \ time_1)\right] \quad (22)$$

Some fitting of the frequency within a set results in the set-averaged values. A similar process may be applied to the amplitude to form an A_{ABA}. The resulting measured frequency difference is then interpreted as:

$$\Delta\omega^2_{raw} \sim \left[w_o^2 + k_2^{''}\frac{8A}{3\pi I} + k_3\frac{3A^2}{4I} - k_{ss}\frac{2A}{I} + \sum_{n=1}^{\infty}\left[\frac{2\beta_n}{IA}J_1(nA)\right] + d(t)\right]_{position+1}$$
$$- \left[w_o^2 + k_2^{''}\frac{8A}{3\pi I} + k_3\frac{3A^2}{4I} - k_{ss}\frac{2A}{I} + \sum_{n=1}^{\infty}\left[\frac{2\beta_n}{IA}J_1(nA)\right] + d(t)\right]_{position-1}$$
$$= \left[\left(k_2^{''}\frac{8}{3\pi I} - k_{ss}\frac{2}{I}\right)A_{ABA} + k_3\frac{3(A^2)_{ABA}}{4I}\right] + \sum_{n=1}^{\infty}\left[\frac{2}{I}\left[\frac{\beta_n J_1(nA)}{A}\right]_{ABA}\right] + \delta\omega^2_{drift} \tag{23}$$

Where the measured $\Delta\omega^2_{raw}$ also includes $\delta\omega^2_{drift}$ which accounts for drifts $d^{\pm}(t)$ in this apparent signal due to tilt, temperature, etc. The first bracketed terms of the last line together define an amplitude correction, $\delta\omega^2_{Acorr}$ which would vanish if the amplitude varied linearly in time. The corrected frequency shift is then interpreted as calculated from the formalism discussed herein. $\delta\omega^2_{\beta J}$ is defined here; it is a small correction present when both A(time) and $\beta_n^{position+1} \neq -\beta_n^{position-1}$, where $[x]_{AVE} \equiv [x(t_2) + \frac{1}{2}(x(t_1) + x(t_3))]/2$.

$$\sum_{n=1}^{\infty}\frac{2}{I}\left[\frac{\beta_n J_1(nA)}{A}\right]_{ABA} \equiv \delta\omega^2_{\beta J} + \sum_{n=1}^{\infty}\frac{2(\beta_n^+ - \beta_n^-)}{I}\left[\frac{J_1(nA)}{A}\right]_{AVE}$$
$$\Delta w^2 \equiv \Delta w^2_{raw} - \delta w^2_{Acorr} - \delta w^2_{drift} - \delta\omega^2_{\beta J} = \sum_{n=1}^{\infty}\frac{2(\beta_n^+ - \beta_n^-)}{I}\left[\frac{J_1(nA)}{A}\right]_{AVE} \tag{24}$$

In Equation 23 the w_o^2 terms cancel provided k_1 is frequency independent. In fact Kuroda [Phys.Rev.Lett. 75, 2796 (1995)] pointed out this is not the case, but Newman [Meas.Sci.Technol. 10, 445 (1999)] has placed limits on this correction to Δw^2 based on the Q of the pendulum oscillations. Now consider the various n terms. Equation 1 is a convergent series, and it follows that the q_{lm} a^*_{lm} product decreases in magnitude with m as well. In fact, their product $\sim$ $\frac{1}{r}\left(r^{'}/r\right)^l$ where $\left(r^{'}/r\right) \sim 0.1$. Equation 5 shows that β_m only involves terms with $l \geq m$, therefore as m increases, β_m decreases by $\sim m$ orders of magnitude. So by the time $m = 10$ or so, β_m is negligible. Because

$$Re(q_{lm}) \sim Re(Y^*_{lm}) \sim \int cos(m\gamma^{'})d\gamma^{'}, \quad Im(q_{lm}) \sim Im(Y^*_{lm}) \sim \int sin(m\gamma^{'})d\gamma^{'}$$

the zero terms for the ideal pendulum of Section 2 can easily be identified as all $sin(m\gamma^{'})$ terms and all odd m $cos(m\gamma^{'})$ terms. So the remaining β_m terms are β_2, β_4, β_6, and β_8. In calculating these moments above $m = 2$, two tricks are used. The first is to note that misplacement of the rings in the $\phi^{'}$ direction leads to a β_2 which can be calculated from $\beta_m = \beta^{'}_m cos(m\phi^{'}) - \alpha'_m sin(m\phi^{'})$ where the β'_m and α'_m are calculated without the misplacement. The second trick is to calculate the $Re\left[a^*_{lm}\right]$ with the ring positioned on the z-axis (lab frame) and rotate the result to the x-axis (lab frame). One rule that applies here is that the rotation mixes the various m moments, but not the l moment groups, so if the $Re\left[a^*_{lm}\right]$ for a given l are all 0 before rotating, they will remain 0 after the rotation. The ring geometry was selected with this in mind. The β_4 and a_{42} terms vanish by choosing a specific ring position in $z^{'}$ and so β_2 is significantly dominant as mentioned in Sections 2 and 3. Magnetic coupling effects would appear as a nonzero β_1. So in general, one can define a 2-position K_n as:

$$K_n \equiv \frac{\beta_n^{position+1} - \beta_n^{position-1}}{IG}, \quad G = \frac{\Delta w^2}{\sum_{n=1}^{\infty} 2K_n\left[\frac{J_1(nA)}{A}\right]_{AVE}} \tag{25}$$

Neglecting all but the $lm = 22$ term, this becomes (recall $\beta_2^{position+1} = -\beta_2^{position-1}$):

$$K_2 = \frac{2\beta_2^{position+1}}{IG}, \quad G = \frac{\Delta w^2}{2K_2\left[\frac{J_1(2A)}{A}\right]_{AVE}} \tag{26}$$

For pendulum dimensions, $s \sim 4cm$, $d \sim 0.3cm$, and $\rho_{p_o} \sim 2.2g/cc$, Equation 9 gives $Re\left(\frac{q_{22}}{I}\right) = \frac{5.76gcm^2}{15.07gcm^2}$. For ring dimensions $z_1 \sim 32.56cm$, $z_2 \sim 37.39cm$, $R_1 \sim 15.62cm$, $R_2 \sim 26.04cm$, and $\rho_{s_o} \sim 8.96g/cc$, Equation 12 gives $Re\left(\bar{a}^*_{22}\right)^{position+1} \sim -G * 0.506g/cc$. Using Equation 5, $\beta_2^{position+1} \sim +G * 23.32g^2/cm$ and from Equation 26, $K_2 \sim 3.09g/cc$. Inputing amplitudes and frequencies into Equation 26 should result in $G \sim 6.67 \times 10^{-8}cm^3/gsec^2$.