THEORY OF AN EXPERIMENT IN AN ORBITING SPACE LABORATORY TO DETERMINE THE GRAVITATIONAL CONSTANT*

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Abstract. The paper analyzes an experiment in an orbiting laboratory to determine the gravitational constant G. A massive sphere, according to a suggestion of L. S. Wilk, is to have three tunnels drilled through it along mutually perpendicular diameters. The sphere either floats in the orbiting laboratory, with its center held fixed by means of external jets issuing from the spacecraft, or is tethered to the spacecraft. In either case it is free to rotate; in the second case this freedom would be achieved by a system of gimbals.

Each tunnel contains a small test object, which is held on the tunnel's axis by means of a suspension system, perhaps electrostatic, and held at rest relative to the sphere by slowly rotating the latter by means of inertia reaction wheels, governed by a servomechanism. Fundamentally, one balances the gravitational forces on the test objects by centrifugal force, determines the latter by measuring the components of angular velocity, and calculates G from the resulting balance. It is better to use three tunnels than one because their use minimizes the effects of the Earth's gravity–gradient.

Many other measurements and corrections are required. The latter arise from Earth gravity–gradient, aerodynamic drag (with the tethered sphere), gravitational forces produced by the spacecraft itself, and the force reductions produced by the empty space in all three tunnels. After the consideration of these effects there is a presentation and discussion of the equations required to reduce the observations to obtain G. There then follow the extra equations, not needed in the reduction, that are required for a computer simulation to investigate the possible extraction of a test object and to aid in designing the servomechanisms.

In Appendix B, I have devised another version of the experiment, in which the sphere is kept intact, but has short thin hollow 'vestigial tunnels' attached to the outside of the sphere, along perpendicular diameters. These external tunnels would contain the test objects and the suspension systems. The servomechanisms would then have to prevent collision of a test object with the sphere, as well as extraction. This second method could allow for some inhomogeneities in the sphere, would require no accurate drilling, and would make the suspension systems more accessible for construction and adjustment.

1. Introduction

Of all the fundamental constants of physics, G is the only one which is not known with an accuracy better than one part in a thousand. Recent theories of gravitation now make it desirable to know it much more accurately.

The lack of such an accurate value has never produced inaccuracy in dynamical astronomy, since G always appears as a factor of a mass M and the product $\mu \equiv GM$ can usually be determined accurately. E.g., for a planet moving around the Sun

$$G(M_1 + M_2) = (2\pi/P)^2 \, a_s^3, \tag{1}$$

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where M_1, M_2, P, and a_s are, respectively, the masses of the Sun and of the planet, the period, and the semi-major axis. But one does not know M_1 and M_2 accurately.

Suppose one could construct an isolated system consisting of an artificial satellite of mass M_2 moving around an artificial planet of mass M_1, with no perturbing forces from other sources. Measurement of the masses and of P and a_s would then lead to a value for G. Before the modern space program such an experiment could not be imagined, because of the pervasive presence of unwanted gravitational fields. With the advent of artificial satellites and space laboratories, however, the situation has changed.

2. A Drag-Free Spherical Laboratory in Orbit

Consider a spherical laboratory, having a spherically symmetric interior, traveling in an orbit around the Earth that is made drag-free by means of external jets. Imagine it kept inertially oriented, relative to the fixed stars, also by means of external jets.

A massive true sphere would then remain floating at the center of the laboratory. (It could be used to monitor the jets that keep the spacecraft drag-free.) A small test object in orbit about this sphere would then be in effect an artificial satellite of an artificial planet.

Let **R** be the position vector of the center of mass of the sphere of mass M_s, in some large-scale astronomer's inertial system, and let $\mathbf{f}(\mathbf{R})$ be the total gravitational field at **R** produced by all mass outside the spacecraft. Also let **s** be the position vector of a small test-object, of mass m, relative to the center of mass C of the sphere. Let it be in close orbit about M_s. Then

$$\ddot{\mathbf{R}} = \mathbf{f}(\mathbf{R}) + (Gm/s^3)\,\mathbf{s}, \tag{2}$$

$$\ddot{\mathbf{R}} + \ddot{\mathbf{s}} = \mathbf{f}(\mathbf{R}+\mathbf{s}) - (GM_s/s^3)\,\mathbf{s}. \tag{3}$$

Subtraction of (2) from (3) gives

$$\ddot{\mathbf{s}} = -\frac{G(M_s+m)}{s^3}\mathbf{s} + \mathbf{f}(\mathbf{R}+\mathbf{s}) - \mathbf{f}(\mathbf{R}), \tag{4.1}$$

$$= -\frac{G(M_s+m)}{s^3}\mathbf{s} + \mathbf{s}\cdot[\nabla\mathbf{f}(\mathbf{R})]_c + \cdots \tag{4.2}$$

Equation (4.2) is obtained from (4.1) by expanding $\mathbf{f}(\mathbf{R}+\mathbf{s})$ in a Taylor's series in the neighborhood of $\mathbf{s}=0$ and using only the first two terms. The expression $\mathbf{s}\cdot[\nabla\mathbf{f}(\mathbf{R})]_c$, the first-order gravity–gradient term, causes trouble. If this term were absent, we should have an example of the simplest two-body problem, with an inverse-square force. The orbit would be an ellipse, so that by measuring the masses M_s and m, the period P, and the semi-major axis a_s we could use Equation (1) to obtain G.

Instead of letting m move in an orbit about M_s, we could drill a narrow cylindrical hole along a diameter of M_s and let m vibrate back and forth.* In this case the force

* See e.g. Berman and Forward, 1968.

produced on m by M_s would be

$$\mathbf{f}(M_s, m) = -\left((4\pi/3)\, G\varrho\mathbf{s}\right) m\,, \tag{5}$$

where ϱ is the uniform density of the sphere. Then

$$\ddot{\mathbf{s}} = -\frac{4\pi}{3}\, G\varrho\left(1 + \frac{m}{M_s}\right)\mathbf{s} + \mathbf{f}(\mathbf{R} + \mathbf{s}) - \mathbf{f}(\mathbf{R})\,, \tag{6}$$

If a_0 is the radius of the sphere

$$(4\pi/3)\, G\varrho = GM_s/a_0^3\,, \tag{7}$$

and

$$\ddot{\mathbf{s}} = -\frac{G(M_s + m)}{a_0^3}\,\mathbf{s} + \mathbf{s}\cdot[\nabla\mathbf{f}(\mathbf{R})]_c + \cdots \tag{8}$$

If the gravity–gradient term were negligible, the test object would move in simple-harmonic motion, with the same period as that of a circular close orbit.

3. The Gravity-Gradient Term

In either of the above cases the gravity-gradient term causes difficulties. We may approximate a true inertial system by taking an inertially oriented system, with origin at the center of mass of the Earth. This procedure involves neglecting the lunar–solar perturbation, but the latter will not affect appreciably the values of $\mathbf{s}\cdot[\nabla\mathbf{f}(\mathbf{R})]_c$. If μ is the product of G and the mass of the Earth, then

$$\mathbf{f}(\mathbf{R}) = -\,\mu\mathbf{R}/R^3\,, \tag{9}$$

closely enough for present purposes. Then

$$\mathbf{s}\cdot\nabla\mathbf{f}(\mathbf{R}) = (\mu s/R^3)\,[-\,\mathbf{I}_s + 3\mathbf{I}_R(\mathbf{I}_R\cdot\mathbf{I}_s)]\,, \tag{10}$$

where $\mathbf{I}_R$ and $\mathbf{I}_s$ are unit vectors along $\mathbf{R}$ and $\mathbf{s}$, respectively. The maximum value of the bracket is 2, so that the maximum value of the gravity-gradient field is

$$\text{max. grav. grad.} = \frac{2\mu s}{R^3}\,. \tag{11}$$

For a fairly close orbit about the Earth, $\mathrm{R} \approx$ radius of the Earth. Also

$$\mu \approx G(4\pi/3)\,\varrho_e R_e^3\,, \tag{12}$$

where R_e and ϱ_e are, respectively, the radius and the mean density of the Earth. Thus

$$\text{max. grav. grad.} \approx (8\pi/3)\, G\varrho_e s\,. \tag{13}$$

In either of the two cases, the field produced by the sphere is

$$f\ (\text{sphere}) \approx (4\pi/3)\, G\varrho s\,, \tag{14}$$

where ϱ is the density of the sphere. The ratio of the maximum gravity-gradient field to this is $2\varrho_e/\varrho \approx 11/19$, if the sphere is made of tungsten.

The gravity-gradient effect is thus very serious and either of the above methods would require important corrections. We thus turn to another method, due to Wilk (1969).

4. The Three-Tunnel Method of Wilk

In its simplest form, Wilk's method would involve the use of a drag-free spherical laboratory, with a spherically symmetric interior, in orbit about the Earth, and inertially oriented relative to the fixed stars. The sphere would float at the center of mass of the spacecraft and have three tunnels along mutually perpendicular diameters. Since the three tunnels would produce a slight departure from spherical symmetry, both external and internal gravitational fields would produce a small torque on it, so that some means would have to be found to keep it inertially oriented. (Such trouble will actually be avoided in the later forms.)

With a small test object in each tunnel, their equations of motion would be

$$\ddot{x} = -(4\pi/3)\,G\varrho x + (\mathbf{s}\cdot\nabla\mathbf{f}')_x \qquad (x\text{-tunnel}), \tag{15.1}$$

$$\ddot{y} = -(4\pi/3)\,G\varrho y + (\mathbf{s}\cdot\nabla\mathbf{f}')_y \qquad (y\text{-tunnel}), \tag{15.2}$$

$$\ddot{z} = -(4\pi/3)\,G\varrho z + (\mathbf{s}\cdot\nabla\mathbf{f}')_z \qquad (z\text{-tunnel}). \tag{15.3}$$

In Equations (15) $\mathbf{f}'$ is the gravitational field produced by all matter outside the sphere. The equations would ultimately require corrections for the effects of the tunnels on the main field terms involving ϱ and also for the higher-order gravity terms.

If V' is the potential corresponding to $\mathbf{f}'$, these equations reduce to

$$\frac{\ddot{x}}{x} = -\frac{4\pi}{3}\,G\varrho - \left(\frac{\partial^2 V'}{\partial x^2}\right)_0, \tag{16.1}$$

$$\frac{\ddot{y}}{y} = -\frac{4\pi}{3}\,G\varrho - \left(\frac{\partial^2 V'}{\partial y^2}\right)_0, \tag{16.2}$$

$$\frac{\ddot{z}}{z} = -\frac{4\pi}{3}\,G\varrho - \left(\frac{\partial^2 V'}{\partial z^2}\right)_0, \tag{16.3}$$

the second derivatives of V' being evaluated at the center of the sphere, since they come from a Taylor expansion in the neighborhood of $\mathbf{s}=0$. Addition then gives

$$\frac{\ddot{x}}{x} + \frac{\ddot{y}}{y} + \frac{\ddot{z}}{z} = -4\pi G\varrho - (\nabla^2 V')_0. \tag{17}$$

Since V' corresponds to $\mathbf{f}'$, the field *external* to the sphere, we then have

$$(\nabla^2 V')_0 = 0, \tag{18}$$

whether or not the tunnels go all the way through the sphere. Then

$$-4\pi G\varrho = \frac{\ddot{x}}{x} + \frac{\ddot{y}}{y} + \frac{\ddot{z}}{z}. \tag{19}$$

The method, as described, would thus eliminate the main difficulty about the gravity gradient terms. It is clear, however, that the accelerations $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ could not be measured accurately enough to improve present values of G, even if one could produce the requisite spherical symmetry and inertial orientation. Wilk has therefore turned to another procedure, although he still retains the important feature of the three tunnels.

5. Rotation of the Sphere: Wilk's Null Method

Suppose one rotates the sphere, by means of inertia wheels, about each of the three perpendicular axes defined by the tunnels. Let ω_1, ω_2, and ω_3 be the corresponding components of the angular velocity of the sphere, in that inertial system with which the tunnels momentarily coincide. In a coordinate system in which the sphere is at rest there then appear centrifugal forces $x(\omega^2-\omega_1^2)$, $y(\omega^2-\omega_2^2)$, $z(\omega^2-\omega_3^2)$ along the x, y, z tunnels respectively. Here

$$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 . \tag{20}$$

On such rotation there also appear other inertial forces, viz. Coriolis forces and $\dot{\boldsymbol{\omega}}$ forces, but they are perpendicular to the tunnels.

The equations for motion along the tunnels now become

$$\frac{\ddot{x}}{x} = -\frac{4\pi}{3} G\varrho + \omega^2 - \omega_1^2 - \left(\frac{\partial^2 V'}{\partial x^2}\right)_0 , \tag{21.1}$$

$$\frac{\ddot{y}}{y} = -\frac{4\pi}{3} G\varrho + \omega^2 - \omega_2^2 - \left(\frac{\partial^2 V'}{\partial y^2}\right)_0 , \tag{21.2}$$

$$\frac{\ddot{z}}{z} = -\frac{4\pi}{3} G\varrho + \omega^2 - \omega_3^2 - \left(\frac{\partial^2 V'}{\partial z^2}\right)_0 . \tag{21.3}$$

Addition then gives

$$4\pi G\varrho = 2\omega^2 - \frac{\ddot{x}}{x} - \frac{\ddot{y}}{y} - \frac{\ddot{z}}{z} . \tag{22}$$

To achieve the validity of (21) and (22), one has to avoid friction of the test objects with the walls of the tunnels. This requires the use of a suspension system, perhaps electrostatic, to balance out the perpendicular Coriolis and $\dot{\boldsymbol{\omega}}$ forces.

To get rid of the acceleration difficulties, i.e., the troubles produced by the terms in $\ddot{x}$, $\ddot{y}$, and $\ddot{z}$, Wilk then proposes to use servomechanisms that will keep the inertia wheels turning at such rates that the centrifugal forces just balance the gravitational forces. Equation (22) would then become

$$2\pi G\varrho = \omega^2 . \tag{23}$$

Measurements of ω_1, ω_2, and ω_3 over an appreciable time interval would then give a value for $G\varrho$ and thus for G. The servomechanisms will so govern the actions of the inertia reaction wheels as to produce known torques on the sphere. On the simplest assumption, which I shall write down as an example, for concreteness, they will have the mathematical form:

$$L_1 = k_{11}x + k_{12}\dot{x} \qquad (x\text{-tunnel}), \tag{23.1}$$

$$L_2 = k_{21}y + k_{22}\dot{y} \qquad (y\text{-tunnel}), \tag{23.2}$$

$$L_3 = k_{31}z + k_{32}\dot{z} \qquad (z\text{-tunnel}). \tag{23.3}$$

The problem of design of the servos consists in achieving suitable values for the constants $k_{\mu\nu}$.

Even if the servos are not perfect, the term $\ddot{x}/x+\ddot{y}/y+\ddot{z}/z$ will now be much smaller so that it need not be measured with as much accuracy as if it were the main term in (22).

6. The Experiment in an Orbiting Laboratory

Within the next few years it is likely that there will be a manned space station in orbit about the Earth, called a 'Skylab'. Let us assume that it will have a mass of about 10^5 lb, and that it will have an approximately circular orbit at an altitude of about 500 km. Let us also assume it to be a cylinder, of length about 40 ft and diameter about 20 ft, so oriented by the Earth's gravity–gradient that its axis will always point approximately towards the center of mass of the Earth, and let it be free of spin about its axis.

The purpose of the rest of this paper is to develop the theory of Wilk's experiment as it might be carried out aboard the Skylab. I shall consider various possibilities, as follows.

(a) The sphere may or may not be placed at the center of mass of the spacecraft.

(b) It may be held in place as regards translation, being allowed to rotate freely by means of a system of gimbals. Or, it may be allowed to drift in translation, with some method of moving it back after a certain amount of drift.

(c) The spacecraft, besides being equipped with external jets to regulate its orientation, may or may not be equipped with external jets to make it drag-free. If so equipped, some freely drifting test body, perhaps the sphere, would have to be kept in place by sensors and servomechanisms that could turn the drag-removing jets on or off, as the test body moves respectively away from or toward its position of equilibrium.

The theory will first involve a treatment of three reference systems, with the direction cosines to go from one to another. Then it will have to account for various corrections to be applied to Equations (21) for the motions of the three test objects in the tunnels.

For the reduction of the data to obtain a number for G, these corrections will arise from the gravity–gradient of the Earth, the non-spherical gravitational environment in the spacecraft packed full with equipment and astronauts, non-gravitational forces on the spacecraft, particularly drag, and the effects on a given test object of the finite diameter of its own tunnel and the effects of the other two tunnels.

For a computer simulation of the experiment one will also have to assume an orbit and a distribution of mass in the spacecraft. There will then be nine unknowns, viz. x, y, z, ω_1, ω_2, ω_3, and three Eulerian angles or independent direction cosines that describe the orientation of the sphere relative to the inertial system. The necessary nine equations will be Equations (21), as above corrected, Equations (23) with assumed values for the constants $k_{\mu\nu}$, and three more equations describing the changing orientation of the sphere. The purpose of the computer simulation is in general to test out the feasibility of the whole experiment and in particular to find whether any of the test objects is likely to be extracted from its tunnel. It should also help to decide on numerical values for the $k_{\mu\nu}$, in the servo Equations (23).

7. Reference Systems

A. THE EQUATORIAL SYSTEM (INERTIALLY-ORIENTED)

This is a system with origin E at the center of mass of the Earth, Z-axis pointing towards the Earth's north pole, and X-axis pointing towards the vernal equinox. (I leave the matter undecided whether corrections should be made for variations in the directions of EZ and EX, i.e., for the Earth's precession and nutation.) Then with $\mathbf{I}$ a unit vector along the X-axis, $\mathbf{K}$ a unit vector along the Z-axis, and $\mathbf{J}=\mathbf{K}\times\mathbf{I}$, the position vectors are as follows for

$$C = \text{C.M. of Skylab} \quad \mathbf{r}_c = \mathbf{I}X_c + \mathbf{J}Y_c + \mathbf{K}Z_c\,, \tag{24.1}$$

$$\text{O} = \text{C.M. of Sphere} \quad \mathbf{r}_0 = \mathbf{I}X_0 + \mathbf{J}Y_0 + \mathbf{K}Z_0\,, \tag{24.2}$$

$$\text{C.M. of a Test Object} \quad \mathbf{r} = \mathbf{I}X + \mathbf{J}Y + \mathbf{K}Z\,. \tag{24.3}$$

B. A SYSTEM FIXED IN THE SPACECRAFT

In order to specify the gravitational effects of the interior of the spacecraft on a test object, it will be necessary to express the spacecraft field on such an object by means of a Taylor's series. The constants in this series will then be unknowns, to be determined, along with G, in each observational run. These 'constants' will be truly constant, however, only if two conditions are satisfied. First, all the objects inside must be fixed in position relative to the spacecraft; this means that each run must be short, perhaps no more than ten minutes long, so that the astronauts can be persuaded to remain practically motionless during such a period. Second, the reference system, which we now characterize by unit vectors $\mathbf{i}_0$, $\mathbf{j}_0$, $\mathbf{k}_0$ along its axes, must be fixed relative to the spacecraft.

It will be convenient to take $\mathbf{k}_0$ pointing along the axis of the spacecraft. If the latter is kept pointing closely along the line to the center of mass of the Earth, then we have, approximately,

$$\mathbf{k}_0 \approx \frac{\mathbf{r}_c}{r_c} = \frac{1}{r_c}\left(\mathbf{I}X_c + \mathbf{J}Y_c + \mathbf{K}Z_c\right). \tag{25}$$

As in the case of the Moon, however, we can expect some libration of $\mathbf{k}_0$ about the unit vector $\mathbf{r}_c/r_c$.

With the cylindrical spacecraft free of spin about its axis and moving in an almost circular orbit, the orbital angular momentum $\mathbf{r}_c \times \dot{\mathbf{r}}_c$ (per unit mass) will remain almost fixed with respect to some direction in the spacecraft. Let us then choose the *fixed* $\mathbf{j}_0$ so that

$$\mathbf{j}_0 \approx p^{-1}\mathbf{r}_c \times \dot{\mathbf{r}}_c, \tag{26.1}$$

$$p \equiv |\mathbf{r}_c \times \dot{\mathbf{r}}_c|. \tag{26.2}$$

Then

$$\mathbf{i}_0 = \mathbf{j}_0 \times \mathbf{k}_0 \approx \frac{1}{r_c p}\left[r_c^2\dot{\mathbf{r}}_c - \mathbf{r}_c(\mathbf{r}_c\cdot\dot{\mathbf{r}}_c)\right]. \tag{26.3}$$

The directions of the unit vectors $\mathbf{i}_0$, $\mathbf{j}_0$, k_0 can be accurately determined at any moment only by observations of the orientation of the spacecraft relative to the fixed stars. On the other hand, if we let

$$\mathbf{k}_0' \equiv \mathbf{r}_c/r_c, \tag{27.1}$$

$$\mathbf{j}_0' \equiv p^{-1}\mathbf{r}_c \times \dot{\mathbf{r}}_c, \tag{27.2}$$

$$\mathbf{i}_0' = \mathbf{j}_0' \times \mathbf{k}_0' = \frac{1}{r_c p}\left[r_c^2\dot{\mathbf{r}}_c - \mathbf{r}_c(\mathbf{r}_c\cdot\dot{\mathbf{r}}_c)\right], \tag{27.3}$$

these primed unit vectors can be determined from the values of X_c, Y_c, Z_c, $\dot{X}_c$, $\dot{Y}_c$, $\dot{Z}_c$, given to us as functions of time by the orbit trackers.

For a sufficiently short run, with sufficiently good suppression of libration and of rotary oscillation about the axis, it may suffice to use the primed unit vectors, instead of the unprimed ones, as the spacecraft system. (Such an approximation would tend to be better in Case α of Section 8 than in Case β.) Under such circumstances one could avoid observation of the inertial orientation of the spacecraft.

C. THE TUNNEL SYSTEM, FIXED IN THE SPHERE

Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be unit vectors along the tunnels. Then, for a test object

$$\mathbf{r} - \mathbf{r}_0 \equiv \mathbf{s} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z, \tag{28.1}$$

where y and z will vanish along the axis of the x-tunnel, z and x along that of the y-tunnel, and x and y along that of the z-tunnel.

With respect to the system (a):

$$\mathbf{s} = \mathbf{I}(X - X_0) + \mathbf{J}(Y - Y_0) + \mathbf{K}(Z - Z_0), \tag{28.2}$$

and with respect to the system (b):

$$\mathbf{s} = \mathbf{i}_0\xi_1 + \mathbf{j}_0\xi_2 + \mathbf{k}_0\xi_3. \tag{28.3}$$

D. TRANSFORMATIONS AMONG THE ABOVE SYSTEMS

Let us denote the unit vectors **I**, **J**, **K** in the equatorial system by $\mathbf{I}_\mu$ ($\mu=1, 2, 3$) and the unit vectors $\mathbf{i}_0$, $\mathbf{j}_0$, $\mathbf{k}_0$ in the spacecraft system by $\mathbf{i}_{0\mu}$ ($\mu=1, 2, 3$). Also let β be the rotation matrix that carries the spacecraft system into the equatorial system. Then

$$\mathbf{I}_\nu = \sum_\mu \beta_{\nu\mu}\mathbf{i}_{0\mu}, \tag{29.1}$$

$$\mathbf{i}_{0\nu} = \sum_\mu \tilde{\beta}_{\nu\mu}\mathbf{I}_\mu = \sum_\mu \mathbf{I}_\mu \beta_{\mu\nu}, \tag{29.2}$$

where

$$\beta_{\nu\mu} = \mathbf{I}_\nu \cdot \mathbf{i}_{0\mu}, \tag{29.3}$$

$$\tilde{\beta}_{\nu\mu} = \beta_{\mu\nu} = \mathbf{i}_{0\nu} \cdot \mathbf{I}_\mu. \tag{29.4}$$

The direction cosines $\tilde{\beta}_{\nu\mu}$ are to be obtained by measurement of the inertial orientation of the spacecraft or – if it turns out to be sufficiently accurate – from Equations (27). Then for a given test object, with its position vector **s** relative to the center of mass of the sphere having components ξ_μ relative to the spacecraft system, we can dot (28.3), viz.

$$\mathbf{s} = \sum_\mu \mathbf{i}_{0\mu}\xi_\mu, \tag{28.3}$$

with $\mathbf{I}_\nu$, to obtain

$$X_\nu - X_{0\nu} = \sum_\mu \mathbf{I}_\nu \cdot \mathbf{i}_{0\mu}\xi_\mu = \sum_\mu \beta_{\nu\mu}\xi_\mu, \tag{29.5}$$

the components of **s** in the equatorial system.

Next let α be the rotation matrix that carries the tunnel system into the spacecraft system. Then

$$\mathbf{i}_{0\nu} = \sum_\mu \alpha_{\nu\mu}\mathbf{i}_\mu, \tag{30.1}$$

$$\mathbf{i}_\nu = \sum_\mu \alpha_{\mu\nu}\mathbf{i}_{0\mu}, \tag{30.2}$$

$$\alpha_{\nu\mu} = \mathbf{i}_{0\nu} \cdot \mathbf{i}_\mu, \tag{30.3}$$

$$\xi_\nu = \sum_\mu \alpha_{\nu\mu}x_\mu, \tag{30.4}$$

$$x_\nu = \sum_\mu \alpha_{\mu\nu}\xi_\mu. \tag{30.5}$$

Finally, let γ be the rotation matrix that carries the tunnel system into the equatorial system. Then

$$\mathbf{I}_\nu = \sum_\mu \gamma_{\nu\mu}\mathbf{i}_\mu, \tag{31.1}$$

$$\mathbf{i}_\nu = \sum_\mu \gamma_{\mu\nu}\mathbf{I}_\mu, \tag{31.2}$$

$$\gamma_{\mu\nu} = \mathbf{i}_\nu \cdot \mathbf{I}_\mu. \tag{31.3}$$

The components $X_\nu - X_{0\nu}$ of $\mathbf{s}$ in the equatorial system then follow from the x_μ in the tunnel system by dotting (28.1), viz.

$$\mathbf{s} = \sum_\mu \mathbf{i}_\mu x_\mu , \tag{28.1}$$

with $\mathbf{I}_\nu$ and using (31.3). We obtain

$$X_\nu - X_{0\nu} = \sum_\mu \gamma_{\nu\mu} x_\mu . \tag{31.4}$$

Now we have seen how to obtain the $\beta_{\mu\nu}$ and it is planned to measure the inertial orientation of each tunnel, so that we shall thereby obtain the $\gamma_{\mu\nu}$.

From the β's and γ's we can then obtain the α's, since the α, β, and γ matrices are connected by the equation

$$\gamma = \beta\alpha . \tag{32}$$

To prove (32), use (29.1), (30.1), and (31.3). Adopting the summation convention, we find from (29.1) and (30.1)

$$\mathbf{I}_\nu = \beta_{\nu\mu}\mathbf{i}_{0\mu} = \beta_{\nu\mu}\alpha_{\mu\sigma}\mathbf{i}_\sigma = (\beta\alpha)_{\nu\sigma}\mathbf{i}_\sigma . \tag{33}$$

On comparing (33) with (31.1), we find

$$\gamma_{\nu\mu} = (\beta\alpha)_{\nu\mu} , \tag{34}$$

so that (32) is proved. Then, from (32),

$$\alpha = \beta^{-1}\gamma = \tilde{\beta}\gamma , \tag{35}$$

since β is a rotation matrix. Then

$$\alpha_{\mu\nu} = \sum_\sigma \tilde{\beta}_{\mu\sigma}\gamma_{\sigma\nu} = \sum_\sigma \beta_{\sigma\mu}\gamma_{\sigma\nu} . \tag{36}$$

Equation (36) then permits evaluation of the α's from the β's and γ's.

8. Equations of Motion of the Spacecraft and of the Test Objects

Let $\mathbf{r}_c$ be the position vector of the center of mass C of the Skylab relative to that of the Earth, $\mathbf{f}_E(\mathbf{r}_c)$ be the Earth's gravitational field at C, and M be the total mass of the Skylab. If we consider the extreme case in which we represent it by two masses, each $M/2$ at a separation equal to its length, we find that the total Earth-gravitational force on the Skylab is given by

$$\mathbf{F}(\text{total}) = M\mathbf{f}_{\mathrm{E}}(\mathbf{r}_{\mathrm{C}}) + \mathbf{F}' , \tag{37.1}$$

where

$$|\mathbf{F}'| \approx 10^{-12} M\, |\mathbf{f}_E(\mathbf{r}_c)| . \tag{37.2}$$

We may henceforth write (37.1) without the $\mathbf{F}'$, for the total gravitational force of the Earth on the Skylab. Its equation of motion is then

$$\ddot{\mathbf{r}}_c = \mathbf{f}_E(\mathbf{r}_c) + \mathbf{D}/M + \mathbf{J}/M + \mathbf{f}_{Ls} . \tag{38}$$

Here $\mathbf{J}$ is the force on the spacecraft produced by any external jets that we may use, $\mathbf{D}$ is the total non-gravitational force, mostly aerodynamic drag, and $\mathbf{f}_{Ls}$ is the lunar-solar perturbation, which we shall ordinarily neglect.

Now let O be the center of the sphere, $\mathbf{b} \equiv \mathbf{CO}$, and $\mathbf{s}$ be the position vector of a test object relative to O. Its position vector $\mathbf{r}$, relative to the center of mass E of the Earth, is then given by

$$\mathbf{r} = \mathbf{r}_c + \mathbf{b} + \mathbf{s} = \mathbf{r}_0 + \mathbf{s}, \tag{39}$$

where

$$\mathbf{r}_0 \equiv \mathbf{r}_c + \mathbf{b}, \tag{40}$$

the position vector of O relative to E.

When the electrostatic suspension is so adjusted that a test object in a tunnel does not touch its walls, the motion of the test object satisfies

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_c + \ddot{\mathbf{b}} + \ddot{\mathbf{s}} = \mathbf{f}_E(\mathbf{r}) + \mathbf{f}_L(\mathbf{r}) + \mathbf{f}_s(\mathbf{r}) + \mathbf{f}_\perp, \tag{41}$$

where $\mathbf{f}_L(\mathbf{r})$ is the gravitational field at the test object produced by the laboratory, $\mathbf{f}_s(\mathbf{r})$ is the gravitational field produced by the sphere, and $\mathbf{f}_\perp$ is the perpendicular force produced by the suspension and the instrumentation. It is here assumed that the suspension and the instrumentation in the Skylab produce zero longitudinal force. When we solve (41) for $\ddot{\mathbf{s}}$, we find that $-\ddot{\mathbf{b}}$ appears as an apparent force. It plays a crucial role in planning the experiment. To see why, let us estimate its value in a certain case.

Suppose the sphere is fixed, translationally, on the axis of the spacecraft, which in turn has its axis pointing approximately toward the center of mass of the Earth. Then

$$\mathbf{b} = b\mathbf{k}_0 \approx b\mathbf{r}_c/r_c. \tag{42}$$

If the orbit is approximately circular, with mean motion n, then

$$\ddot{\mathbf{b}} \approx - bn^2\mathbf{r}_c/r_c, \tag{43}$$

so that

$$|\ddot{\mathbf{b}}| \approx bn^2. \tag{44}$$

Now

$$|\mathbf{f}_s(\mathbf{r})| \approx (4\pi/3)\, G\varrho s, \tag{45}$$

where $\varrho \approx 19$ gm/cm^3 for tungsten. From the Keplerian law $\mu \approx \mathrm{n}^2 R_e^3$, we find that

$$G(4\pi/3)\,\varrho_e \approx n^2, \tag{46}$$

where ϱ_e is the Earth's mean density. Then

$$|\mathbf{f}_s(\mathbf{r})| \approx (\varrho/\varrho_e)\, n^2 s, \tag{47}$$

and

$$\left|\frac{\ddot{\mathbf{b}}}{\mathbf{f}_s(\mathbf{r})}\right| \approx \frac{\varrho_e}{\varrho}\frac{b}{s} \approx \frac{5.5}{19}\frac{b}{s}. \tag{48}$$

With $b=20$ ft $\approx$ 600 cm and $s=10$ cm, the ratio amounts to about 17. Thus $\ddot{\mathbf{b}}$ is much larger than $\mathbf{f}_s(\mathbf{r})$, which is supposed to be the main force in the experiment.

Now if the axis really pointed accurately toward E, we could evaluate $\ddot{\mathbf{b}}$ accurately from (42), with sufficiently accurate values of X_c, Y_c, Z_c, $\dot{X}_c$, $\dot{Y}_c$, and $\dot{Z}_c$, and the drag. But the most one can do in orienting the spacecraft is to point its axis, as closely as possible, along the perpendicular to the plane of the horizon. There would thus arise an error, which although small as a fraction, would undoubtedly produce too large an error in $\ddot{\mathbf{b}}$, which already swamps $\mathbf{f}_s$. It thus appears that translational restraint of the sphere would destroy the experiment, unless one placed the sphere very close to the center of mass C of the spacecraft.

I shall therefore consider only two possibilities:

(α) The sphere is not at C and is not constrained to the spacecraft by any tethers or rods, but its center actually remains at a point O fixed relative to the spacecraft by virtue of external jets. An astronaut could bring this about, watching the sphere and turning the external jets on or off to keep O fixed. That is, he could act both as sensor and servomechanism.

(β) The center O of the sphere is tied to the center of mass C of the spacecraft.

9. The Unconstrained Sphere: Case α

In this case the sphere is in free fall, so that

$$\ddot{\mathbf{r}}_0 = \mathbf{f}_E(\mathbf{r}_0) + \mathbf{f}_L(\mathbf{r}_0). \tag{49}$$

Equation (41) becomes

$$\ddot{\mathbf{r}}_0 + \ddot{\mathbf{s}} = \mathbf{f}_E(\mathbf{r}_0+\mathbf{s}) + \mathbf{f}_L(\mathbf{r}_0+\mathbf{s}) + \mathbf{f}_s(\mathbf{r}_0+\mathbf{s}) + \mathbf{f}_\perp. \tag{50}$$

Subtraction of (49) from (50) then yields

$$\ddot{\mathbf{s}} = \mathbf{f}_E(\mathbf{r}_0+\mathbf{s}) - \mathbf{f}_E(\mathbf{r}_0) + \mathbf{f}_L(\mathbf{r}_0+\mathbf{s}) - \mathbf{f}_L(\mathbf{r}_0) + \mathbf{f}_s(\mathbf{r}_0+\mathbf{s}) + \mathbf{f}_\perp. \tag{51}$$

Now, by (28.1)

$$\mathbf{s} = \sum_\mu \mathbf{i}_\mu x_\mu. \tag{52}$$

If $\boldsymbol{\omega}$ denotes the angular velocity of the sphere relative to the fixed-star inertial system, we have

$$\dot{\mathbf{i}}_\mu = \boldsymbol{\omega} \times \mathbf{i}_\mu. \tag{53}$$

Let

$$\sum_\mu \mathbf{i}_\mu \dot{x}_\mu = \mathbf{v}, \tag{54}$$

the velocity of the test object relative to the tunnel system. Two differentiations of (52) then give

$$\ddot{\mathbf{s}} = \sum_\mu \mathbf{i}_\mu \ddot{x}_\mu + \dot{\boldsymbol{\omega}} \times \mathbf{s} + 2\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{s}). \tag{55}$$

Here

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{s}) = \boldsymbol{\omega}(\boldsymbol{\omega}\cdot\mathbf{s}) - \omega^2\mathbf{s}, \tag{56}$$

so that

$$[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{s})]_\mu = \omega_\mu(\boldsymbol{\omega}\cdot\mathbf{s}) - \omega^2 x_\mu. \tag{57}$$

In a given tunnel μ, the forces $\dot{\boldsymbol{\omega}} \times \mathbf{s}$ and $2\,\boldsymbol{\omega} \times \mathbf{v}$ are perpendicular to the axis, and thus neutralized by the $\mathbf{f}_\perp$ in (50). The suspension is designed to do this. Also, in the μ-tunnel

$$s_\mu = x_\mu, \tag{58}$$

so that (57) becomes

$$[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{s})]_\mu = x_\mu(\omega_\mu^2 - \omega^2). \tag{59}$$

Equations (51), (55), and (59) thus give

$$\begin{aligned}\ddot{x}_\mu = (\omega^2 - \omega_\mu^2)\,x_\mu + \mathbf{i}_\mu\cdot[\mathbf{f}_E(\mathbf{r}_0 + \mathbf{s}) - \mathbf{f}_E(\mathbf{r}_0)]\\ + \mathbf{i}_\mu\cdot[\mathbf{f}_L(\mathbf{r}_0 + \mathbf{s}) - \mathbf{f}_L(\mathbf{r}_0)] + \mathbf{f}_{s\mu}(\mathbf{r}_0 + \mathbf{s}) \qquad (\mu = 1, 2, 3).\end{aligned} \tag{60}$$

Equation (60) is notable for its lack of any drag term. The sphere force $f_{s\mu}$ is approximately $4\pi G\varrho x_\mu/3$. The term $(\omega^2 - \omega_\mu^2)\,x_\mu$ is the centrifugal force and the other two terms are the gravity gradient terms arising respectively from the Earth and the space laboratory itself. Before we evaluate the various terms in (60) it is best to derive the corresponding equation for Case β, where the sphere is tethered to remain at the center of mass C of the spacecraft.

10. The Constrained Sphere: Case β

Here $\mathbf{CO} = 0$, by hypothesis. Equation (50) remains unchanged, but may now be written

$$\ddot{\mathbf{r}}_c + \ddot{\mathbf{s}} = \mathbf{f}_E(\mathbf{r}_c + \mathbf{s}) + \mathbf{f}_L(\mathbf{r}_c + \mathbf{s}) + \mathbf{f}_s(\mathbf{r}_c + \mathbf{s}) + \mathbf{f}_\perp. \tag{61}$$

In this case we assume no jets, so that $\mathbf{J}$ vanishes in (38). But now we do have drag $\mathbf{D}$. With neglect of $\mathbf{f}$ (lunar-solar)*, we obtain

$$\ddot{\mathbf{r}}_c = \mathbf{f}_E(\mathbf{r}_c) + \mathbf{D}/M. \tag{62}$$

Subtraction of (62) from (61) yields

$$\ddot{\mathbf{s}} = \mathbf{f}_E(\mathbf{r}_c + \mathbf{s}) - \mathbf{f}_E(\mathbf{r}_c) + \mathbf{f}_L(\mathbf{r}_c + \mathbf{s}) + \mathbf{f}_s(\mathbf{r}_c + \mathbf{s}) + \mathbf{f}_\perp - \mathbf{D}/M. \tag{63}$$

The equation for a test object in the μ'th tunnel now becomes

$$\begin{aligned}\ddot{x}_\mu = (\omega^2 - \omega_\mu^2)\,x_\mu + \mathbf{i}_\mu\cdot[\mathbf{f}_E(\mathbf{r}_c + \mathbf{s}) - \mathbf{f}_E(\mathbf{r}_c)]\\ + \mathbf{i}_\mu\cdot\mathbf{f}_L(\mathbf{r}_c + \mathbf{s}) + f_{s\mu}(\mathbf{r}_c + \mathbf{s}) - \mathbf{i}_\mu\cdot\mathbf{D}/M \qquad (\mu = 1, 2, 3).\end{aligned} \tag{64}$$

* Since the gravity gradient varies directly with the mass of the source and inversely with the cube of its distance, the lunar–solar gravity gradient is less than 10^{-7} times the Earth's gravity gradient,

Comparison of (60) and (64) shows that, in the case of tethered constraint of O at C, the gravitational force of the laboratory enters as a direct force and not just as the gravity gradient force of the unconstrained case. If the laboratory is symmetric with respect to its mid-section, however, this direct force will be zero. If it is almost symmetric, it may still not exceed the gravity gradient force in the unconstrained case. Finally, there is a drag term in the constrained case and none in the unconstrained case.

By evaluating $\mathbf{f}_E(\mathbf{r}_0+\mathbf{s})$, $\mathbf{f}_L(\mathbf{r}_0+\mathbf{s})$, and $f_{s\mu}(\mathbf{r}_0+\mathbf{s})$, we can obtain all the gravitational terms in both equations. There will then remain only the drag term in (64) to be considered.

11. Field at Test Object from the Earth's Gravity Gradient

The field at the test object, arising from the Earth's gravity gradient, is

$$\mathbf{f}_{EG}(s) = \mathbf{f}_E(\mathbf{r}_0 + \mathbf{s}) - \mathbf{f}_E(\mathbf{r}_0), \tag{65}$$

where $\mathbf{r}_0 = \mathbf{r}_c + \mathbf{b}$, $\mathbf{b} \neq 0$ for Case α, and $\mathbf{b} = 0$ for Case β. The Earth's gravitational potential V is given by

$$V = -\frac{\mu}{r} + \frac{\mu J_2 R_e^2}{r^3} P_2\left(\frac{Z}{r}\right) + \cdots, \tag{66}$$

where $J_2 = (1.08)\,10^{-3}$ and

$$Z = Z_0 + \mathbf{K}\cdot\mathbf{s}, \tag{67}$$

Then

$$\mathbf{f}_E(\mathbf{r}_0 + \mathbf{s}) = -\nabla_s V. \tag{68}$$

Since

$$P_2\left(\frac{Z}{r}\right) = \tfrac{3}{2}\frac{Z^2}{r^2} - \tfrac{1}{2}, \tag{69}$$

we have

$$V = -\frac{\mu}{r} - \frac{\mu J_2 R_e^2}{2r^3} + \tfrac{3}{2}\frac{\mu J_2 R_e^2 Z^2}{r^5} + \cdots. \tag{70}$$

Thus we need expressions for r^{-1}, r^{-3}, and $Z^2 r^{-5}$ as functions of the tunnel coordinates x, y, z of a test object. By (39)

$$r^2 = r_0^2 + 2\mathbf{r}_0\cdot\mathbf{s} + s^2. \tag{71}$$

Also, by (40)

$$\mathbf{r}_0 = \sum_v \mathbf{I}_v X_{cv} + \sum_v \mathbf{i}_{0v} b_v. \tag{72}$$

Here the b_v's are the components of $\mathbf{b}$ in the spacecraft system. The X_{cv} are the components of $\mathbf{r}_c$ in the equatorial system and thus known from the specification of the orbit. We need, however, to express $\mathbf{r}_0$ in the tunnel system. To do so, apply (30.1) and

(31.1) to (72). Then

$$\mathbf{r}_0 = \sum_\sigma \mathbf{i}_\sigma Q_\sigma, \tag{73.1}$$

$$Q_\sigma \equiv \sum_\nu (X_{c\nu}\gamma_{\nu\sigma} + b_\nu \alpha_{\nu\sigma}). \tag{73.2}$$

Here α is the matrix that rotates the tunnel system into the spacecraft system and γ the matrix that rotates the tunnel system into the equatorial system. Also

$$l_\sigma = Q_\sigma / r_0 \quad (\sigma = 1, 2, 3), \tag{74}$$

are the direction cosines of $\mathbf{r}_0$, the position vector of the center of the sphere, in the tunnel system.

Inserting

$$r_0^2 = \sum_\sigma Q_\sigma^2, \tag{75}$$

$$\mathbf{s} = \sum_\sigma \mathbf{i}_\sigma x_\sigma, \tag{76}$$

into (71), we find

$$\frac{r^2}{r_0^2} = 1 + \frac{2}{r_0^2} \sum_\sigma Q_\sigma x_\sigma + \frac{s^2}{r_0^2}, \tag{77}$$

so that

$$\frac{r_0}{r} = (1 - 2\lambda h + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(\lambda), \tag{78}$$

where

$$h \equiv -\frac{s}{r_0}, \tag{79.1}$$

$$\lambda \equiv \frac{1}{r_0 s} \sum_\sigma Q_\sigma x_\sigma. \tag{79.2}$$

Here $P_n(\lambda)$ is the Legendre polynomial. We thus obtain

$$\frac{r_0}{r} = 1 - \frac{\lambda s}{r_0} + \frac{s^2}{r_0^2} \left(\tfrac{3}{2}\lambda^2 - \tfrac{1}{2}\right) + \cdots, \tag{80}$$

which results in

$$\frac{\mu}{r} = \frac{\mu}{r_0} - \frac{\mu}{r_0^2} \sum_\sigma l_\sigma x_\sigma - \frac{\mu(x^2 + y^2 + z^2)}{2r_0^3} + \frac{3\mu}{2r_0^3} \left(\sum_\sigma l_\sigma x_\sigma \right)^2 + \cdots. \tag{81}$$

To find r^{-3}, we differentiate (78) with respect to λ. The result is

$$\frac{r_0^3}{r^3} = (1 - 2\lambda h + h^2)^{-3/2} = \sum_{n=1}^{\infty} h^{n-1} P_n'(\lambda). \tag{82}$$

To find r^{-5}, differentiate (82) with respect to λ, to find

$$\frac{r_0^5}{r^5} = (1 - 2\lambda h + h^2)^{-5/2} = \tfrac{1}{3} \sum_{n=2}^{\infty} h^{n-2} P_n''(\lambda). \tag{83}$$

Equations (82) and (83) become

$$\frac{r_0^3}{r^3} = 1 - \frac{3}{r_0}\sum_\sigma l_\sigma x_\sigma - \frac{3(x^2+y^2+z^2)}{2r_0^2} + \frac{15}{2r_0^2}\left(\sum_\sigma l_\sigma x_\sigma\right)^2 + \cdots. \tag{84}$$

$$\frac{r_0^5}{r^5} = 1 - \frac{5}{r_0}\sum_\sigma l_\sigma x_\sigma - \frac{5(x^2+y^2+z^2)}{2r_0^2} + \frac{35}{2r_0^2}\left(\sum_\sigma l_\sigma x_\sigma\right)^2 + \cdots. \tag{85}$$

But we need $Z^2 r^{-5}$. Here

$$Z = Z_0 + \mathbf{K}\cdot\mathbf{s}. \tag{85.1}$$

By (31.1)

$$\mathbf{K} = \sum_\mu \gamma_{3\mu}\mathbf{i}_\mu. \tag{86}$$

Thus

$$Z = Z_0 + \sum_\mu \gamma_{3\mu} x_\mu \tag{87}$$

and

$$Z^2 = Z_0^2 + 2Z_0\sum_\mu \gamma_{3\mu}x_\mu + \left(\sum_\mu \gamma_{3\mu}x_\mu\right)^2. \tag{88}$$

Multiplication of (85) by (88) then yields

$$\begin{aligned}\frac{r_0^5 Z^2}{r^5} = {} & Z_0^2 + 2Z_0\sum_\mu \gamma_{3\mu}x_\mu + \left(\sum_\mu \gamma_{3\mu}x_\mu\right)^2 \\ & - \frac{5Z_0^2}{r_0}\sum_\sigma l_\sigma x_\sigma - \frac{10Z_0}{r_0}\left(\sum_\mu \gamma_{3\mu}x_\mu\right)\left(\sum_\sigma l_\sigma x_\sigma\right) \\ & - \frac{5Z_0^2}{2r_0^2}(x^2+y^2+z^2) + \frac{35Z_0^2}{2r_0^2}\left(\sum_\sigma l_\sigma x_\sigma\right)^2 + O(s^3).\end{aligned} \tag{89}$$

On inserting (81), (84), and (89) into (70), we find

$$\begin{aligned}U \equiv -V = {} & \frac{\mu}{r_0}\left[1 - \frac{1}{r_0}\sum_\sigma l_\sigma x_\sigma - \frac{x^2+y^2+z^2}{2r_0^2} + \frac{3}{2r_0^2}\left(\sum_\sigma l_\sigma x_\sigma\right)^2\right] \\ & + \frac{\mu J_2 R_e^2}{2r_0^3}\left[1 - \frac{3}{r_0}\sum_\sigma l_\sigma x_\sigma - \frac{3}{2r_0^2}(x^2+y^2+z^2) + \frac{15}{2r_0^2}\left(\sum_\sigma l_\sigma x_\sigma\right)^2\right] \\ & - \frac{3}{2}\frac{\mu J_2 R_e^2}{r_0^5} Z_0^2\left[1 + \frac{2}{Z_0}\sum_\mu \gamma_{3\mu}x_\mu + \frac{1}{Z_0^2}\left(\sum_\mu \gamma_{3\mu}x_\mu\right)^2\right. \\ & - \frac{5}{r_0}\sum_\sigma l_\sigma x_\sigma - \frac{10}{Z_0 r_0}\left(\sum_\mu \gamma_{3\mu}x_\mu\right)\left(\sum_\sigma l_\sigma x_\sigma\right) - \frac{5}{2r_0^2}(x^2+y^2+z^2) \\ & \left. + \frac{35}{2r_0^2}\left(\sum_\sigma l_\sigma x_\sigma\right)^2\right] + O(s^3).\end{aligned} \tag{90}$$

Here

$$r_0^2 = (X_c + \sum_\mu \beta_{1\mu} b_\mu)^2 + (Y_c + \sum_\mu \beta_{2\mu} b_\mu)^2 + (Z_c + \sum_\mu \beta_{3\mu} b_\mu)^2, \tag{91.1}$$

$$Z_0 = Z_c + \sum_\mu \beta_{3\mu} b_\mu, \tag{91.2}$$

where the b_μ's are the components of $\mathbf{b}$ in the spacecraft system, $\beta_{1\mu}$, $\beta_{2\mu}$, and $\beta_{3\mu}$ are the direction cosines of $\mathbf{i}_{0\mu}$ in the equatorial system, and X_c, Y_c, Z_c are the components of $\mathbf{r}_c$ in the equatorial system. Also

$$\begin{aligned} l_\sigma \equiv \frac{Q_\sigma}{r_0} = \frac{1}{r_0} [X_c \gamma_{1\sigma} + Y_c \gamma_{2\sigma} + Z_c \gamma_{3\sigma} + b_1 \alpha_{1\sigma} \\ + b_2 \alpha_{2\sigma} + b_3 \alpha_{3\sigma}] \qquad (\sigma = 1, 2, 3) \end{aligned} \tag{91.3}$$

where the l_σ are the direction cosines of $\mathbf{r}_0$ in the tunnel system; $\gamma_{1\sigma}$, $\gamma_{2\sigma}$, $\gamma_{3\sigma}$ are the direction cosines in the equatorial system of a unit vector along the σ'th tunnel; $\alpha_{1\sigma}$, $\alpha_{2\sigma}$, $\alpha_{3\sigma}$ are the direction cosines in the spacecraft system of a unit vector along the σ'th tunnel, and the b's are the same as above.

The component along the σ'th tunnel of $\mathbf{f}_E$ in Equations (68) is then given by computing $\partial U/\partial x_\sigma$ and putting $x_\mu = 0$, $\mu \neq \sigma$:

$$\begin{aligned} f_{E\sigma}(\mathbf{r}_0 + \mathbf{s}) = & - \frac{\mu}{r_0^2} l_\sigma - \frac{\mu}{r_0^3} x_\sigma + \frac{3\mu}{r_0^3} l_\sigma^2 x_\sigma \\ & + \frac{\mu J_2 R_e^2}{2 r_0^3} \left[- \frac{3 l_\sigma}{r_0} - \frac{3 x_\sigma}{r_0^2} + \frac{15}{r_0^2} l_\sigma^2 x_\sigma \right] \\ & - \frac{3}{2} \frac{\mu J_2 R_e^2 Z_0^2}{r_0^5} \left[\frac{2\gamma_{3\sigma}}{Z_0} + \frac{2\gamma_{3\sigma}^2}{Z_0^2} x_\sigma - \frac{5 l_\sigma}{r_0} \right. \\ & \left. - \frac{20 l_\sigma \gamma_{3\sigma} x_\sigma}{Z_0 r_0} - \frac{5 x_\sigma}{r_0^2} + \frac{35 l_\sigma^2 x_\sigma}{r_0^2} \right] + \cdots . \end{aligned} \tag{92}$$

Let $(f_\sigma)_{g.g.}$ be the component along the σ'th tunnel of the Earth gravity gradient force, viz.

$$(f_\sigma)_{g.g.} = f_{E\sigma}(\mathbf{r}_0 + \mathbf{s}) - f_{E\sigma}(\mathbf{r}_0). \tag{93}$$

It is obtained from (92) by dropping all the terms independent of x_σ:

$$\begin{aligned} (f_\sigma)_{g.g.} = & - \frac{\mu x_\sigma}{r_0^3} + \frac{3\mu l_\sigma^2}{r_0^3} x_\sigma + \frac{\mu J_2 R_e^2}{2 r_0^3} \left(- \frac{3 x_\sigma}{r_0^2} + \frac{15 l_\sigma^2 x_\sigma}{r_0^2} \right) \\ & - \frac{3}{2} \frac{\mu J_2 R_e^2}{r_0^5} Z_0^2 \left(\frac{2\gamma_{3\sigma}^2 x_\sigma}{Z_0^2} - \frac{20 \gamma_{3\sigma} l_\sigma x_\sigma}{Z_0 r_0} - \frac{5 x_\sigma}{r_0^2} + \frac{35 l_\sigma^2 x_\sigma}{r_0^2} \right) + \cdots . \end{aligned} \tag{94}$$

This corresponds to a Taylor expansion of the Earth's potential V, through terms in the second degree, in the neighborhood of $\mathbf{s} = 0$, so that

$$\sum_\sigma \frac{(f_\sigma)_{g.g.}}{x_\sigma} = - (\nabla^2 V)_0, \tag{95}$$

which must vanish, since the *earth* potential satisfies Laplace's equation (and not Poisson's) anywhere *outside* the Earth. It is a simple matter to verify the vanishing of $\sum_\sigma (f_\sigma)_{g.g.}/x_\sigma$. We find immediately

$$\sum_\sigma \frac{(f_\sigma)_{g.g.}}{x_\sigma} = \frac{30\mu J_2 R_e^2 Z_0}{r_0^7}\left(r_0 \sum_\sigma \gamma_{3\sigma} l_\sigma - Z_0\right). \tag{96}$$

By (91.3), we then compute

$$\begin{aligned} r_0 \sum_\sigma l_\sigma \gamma_{3\sigma} = X_c \sum_\sigma \gamma_{3\sigma}\gamma_{1\sigma} + Y_c \sum_\sigma \gamma_{3\sigma}\gamma_{2\sigma} + Z_c \sum_\sigma \gamma_{3\sigma}^2 \\ + b_1 \sum_\sigma \alpha_{1\sigma}\gamma_{3\sigma} + b_2 \sum_\sigma \alpha_{2\sigma}\gamma_{3\sigma} + b_3 \sum_\sigma \alpha_{3\sigma}\gamma_{3\sigma}. \end{aligned} \tag{97}$$

Here the coefficients of X_c and Y_c vanish and that of Z_c is unity. To evaluate the coefficients of the b's, note that, since the matrices α, β, and γ satisfy $\gamma = \beta\alpha$, we have

$$\alpha\gamma^{-1} = \beta^{-1}, \tag{98}$$

from which

$$\sum_\sigma \alpha_{\mu\sigma}(\gamma^{-1})_{\sigma\nu} = \sum_\sigma \alpha_{\mu\sigma}\gamma_{\nu\sigma} = (\beta^{-1})_{\mu\nu} = \beta_{\nu\mu}. \tag{99}$$

Thus the coefficient of b_μ is $\beta_{3\mu}$, so that

$$r_0 \sum_\sigma l_\sigma \gamma_{3\sigma} = Z_c + \sum_\mu \beta_{3\mu} b_\mu = Z_0, \tag{100}$$

by (91.2). Insertion of this result into (96) shows that

$$\sum_\sigma \frac{(f_\sigma)_{g.g.}}{x_\sigma} = 0, \tag{101}$$

to this order, as expected.

12. Higher Harmonic Terms in the Gravity Gradient Force

To obtain an estimate of the effect of the J_2 term in (94), let us consider the case where the σ'th tunnel is parallel to $\mathbf{r}_0$, so that $l_\sigma = 1$. The contribution of the J_2 term is then

$$(f_\sigma)_{g.g.}(J_2) = N_\sigma x_\sigma, \tag{102}$$

where

$$N_\sigma = \frac{3\mu J_2 R_e^2}{2 r_0^5}\left(4 - 2\gamma_{3\sigma}^2 + \frac{20\gamma_{3\sigma} Z_0}{r_0} - \frac{30 Z_0^2}{r_0^2}\right). \tag{103}$$

If the tunnel and $\mathbf{r}_0$ are also parallel to the Earth's polar axis, then $X_0 = Y_0 = 0$, $Z_0 = r_0$, and $\gamma_{3\sigma} = 1$, so that

$$N_\sigma = \frac{3\mu J_2 R_e^2}{2 r_0^5}(-8) = -\frac{12\mu J_2 R_e^2}{r_0^5} \approx -\frac{12\mu J_2}{R_e^3}. \tag{104}$$

Let us compare this with the main field on the test object, viz.

$$f_\sigma(\text{sphere}) \approx \frac{4\pi}{3} G\varrho x_\sigma, \tag{105}$$

when $\varrho \approx 19$ gm cm^{-3} for tungsten. Then

$$\left|\frac{N_\sigma x_\sigma}{f_\sigma(\text{sphere})}\right| \approx \frac{12J_2}{R_e^3}\frac{4\pi}{3} G\varrho_e R_e^3 \left[\frac{4\pi}{3} G\varrho\right]^{-1}$$

$$= 12J_2 \frac{\varrho_e}{\varrho}, \tag{106}$$

where $\varrho_e \approx 5.5$ gm cm^{-3}, the mean density of the Earth. Then

$$\text{ratio} \approx (12) \times 10^{-3} \frac{5.5}{19} = (3.5) \times 10^{-3}. \tag{107}$$

Thus the J_2 gravity gradient field may amount to $\frac{1}{300}$ of the main field. The gravity gradient fields from the higher harmonics of the Earth's potential may be expected to be about 10^{-3} of this or about 3×10^{-6} of the main field.

In the formal reduction of data, however, they will not enter since (101) will still hold, when we take a sum over the three tunnels. In the computer simulation where we are concerned with each tunnel separately – to make sure that no test object is ejected – it is only the J_2 among the higher harmonics that may count and its effect is marginal.

13. Quadratic Terms in the Gravity Gradient Force

For the estimate of quadratic terms it will suffice to consider only the monopole part of the Earth's potential, viz.

$$U = \frac{\mu}{r} = \frac{\mu}{r_0}\left[1 + h\lambda + h^2\left(\tfrac{3}{2}\lambda^2 - \tfrac{1}{2}\right) + h^3\left(\tfrac{5}{2}\lambda^3 - \tfrac{3}{2}\lambda\right) + \cdots\right], \tag{108}$$

by (78). Its cubic part is

$$U_3 = \frac{\mu}{r_0}\left[\tfrac{5}{2}(\lambda h)^3 - \tfrac{3}{2}h^2(h\lambda)\right]. \tag{109}$$

With use of (79.1), (79.2), and (74), this becomes

$$U_3 = \frac{\mu}{2r_0^4}\left[-5\Big(\sum_\sigma l_\sigma x_\sigma\Big)^3 + 3(x^2+y^2+z^2)\sum_\sigma l_\sigma x_\sigma\right]. \tag{110}$$

Then

$$\frac{\partial U_3}{\partial x_\sigma} = \frac{\mu}{2r_0^4}\left[-15 l_\sigma\Big(\sum_\sigma l_\sigma x_\sigma\Big)^2 + 6x_\sigma \sum_\sigma l_\sigma x_\sigma + 3l_\sigma(x^2+y^2+z^2)\right]. \tag{111.1}$$

Thus the quadratic gravity gradient term in the σ'th tunnel is given by

$$\delta_3 f_{g.g.\sigma} = \frac{3\mu}{2r_0^4} l_\sigma (3 - 5l_\sigma^2)\, x_\sigma^2. \tag{111.2}$$

The quantity $|3l_\sigma - 5l_\sigma^3|$ has a maximum value 2, occurring for $l_\sigma = \pm 1$, so that

$$|\delta_3 f_{g.g.\sigma}|_{\max} = \frac{3\mu x_\sigma^2}{r_0^4} \approx 3G \frac{4\pi\varrho_e}{3} \frac{x_\sigma^2}{R_e}.$$

Comparison with the main field

$$f_\sigma(\text{sphere}) \approx \frac{4\pi}{3} G\varrho x_\sigma,$$

gives

$$|\text{ratio}| \approx 3\frac{\varrho_e}{\varrho}\frac{x_\sigma}{R_e},$$

where $\varrho_e \approx 5.5$ gm cm^{-3}, $\varrho \approx 19$ gm cm^{-3}, $R_e \approx (6.4)\times 10^8$ cm, and $|x| < 10$ cm. Thus the quadratic gravity gradient term is less than $(1.4)\times 10^{-8}$ of the main force on the test object.

14. Summary of Earth Gravity Gradient Effects on a Test Object

A. SPHERE UNCONSTRAINED, BUT MAINTAINED BY MEANS OF EXTERNAL JETS AT A POINT FIXED IN THE SPACECRAFT

In this case the gravity gradient field $f_{\sigma g.g.}$ at the test object is given by

$$\frac{f_{\sigma g.g.}}{x_\sigma} = -\frac{\mu}{r_0^3}(1 - 3l_\sigma^2) + \frac{3\mu J_2 R_e^2}{2r_0^5}\Big[-1 + 5l_\sigma^2 - 2\gamma_{3\sigma}^2 + \frac{20\gamma_{3\sigma} l_\sigma Z_0}{r_0} + \frac{5Z_0^2}{r_0^2}(1 - 7l_\sigma^2)\Big] + O(x_\sigma). \tag{112}$$

Here the position vector of the center of the sphere is $\mathbf{r}_0 = \mathbf{r}_c + \mathbf{b}$, where $\mathbf{r}_c$ is the position vector of the sphere's center of mass relative to that of the Earth.

Here

$$r_0^2 = (X_c + \sum_\mu \beta_{1\mu} b_\mu)^2 + (Y_c + \sum_\mu \beta_{2\mu} b_\mu)^2 + (Z_c + \sum_\mu \beta_{3\mu} b_\mu)^2, \tag{112.1}$$

$$Z_0 = Z_c + \sum_\mu \beta_{3\mu} b_\mu, \tag{112.2}$$

$$r_0 l_\sigma = X_c \gamma_{1\sigma} + Y_c \gamma_{2\sigma} + Z_c \gamma_{3\sigma} + b_1 \alpha_{1\sigma} + b_2 \alpha_{2\sigma} + b_3 \alpha_{3\sigma}. \tag{112.3}$$

Also X_c, Y_c, Z_c are the coordinates of the center of mass of the spacecraft in the equatorial system, $\gamma_{1\sigma}, \gamma_{2\sigma}, \gamma_{3\sigma}$ are the direction cosines of the σ'th tunnel in the equatorial system, $\alpha_{1\sigma}, \alpha_{2\sigma}, \alpha_{3\sigma}$ are the direction cosines of the σ'th tunnel in the spacecraft system $\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$, and $\beta_{3\mu}$ $(\mu = 1, 2, 3)$ are the direction cosines of the Earth's polar axis in the system $\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$. Also l_σ $(\sigma = 1, 2, 3)$ are the direction cosines of $\mathbf{r}_0$ in the tunnel system.

In (112) the terms involving X_c, Y_c, Z_c are larger than those involving the b's, by a factor of about 10^6. Error in the α's and β's, arising from possible imprecise determination of the spacecraft axes, defined by $\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$, is thus unimportant.

B. CENTER OF SPHERE CONSTRAINED TO REMAIN AT THE CENTER OF MASS OF THE SPACECRAFT

In this case the sphere is constrained relative to translation, but mounted in a system of gimbals so that it can rotate freely. Also $\mathbf{b}=0$.

The above formulas for $f_{\sigma g.g.}$ then get simplified by the omission of the terms in b_1, b_2, b_3. We may then put $\mathbf{r}_0=\mathbf{r}_c$ and $Z_0=Z_c$. The direction cosines l_σ are then those of $\mathbf{r}_c$ in the tunnel system.

15. Gravitational Effects of the Spacecraft Itself

Since the Skylab is to be a cylinder, filled with all sorts of equipment and carrying several astronauts, it will produce a gravitational field on a test object. Since the environment will be complex and indeed unpredictable, it will be impossible to calculate this field. Instead, I shall assume that everything is tied down during a run, say for 10 min. This means that the astronauts would have to be persuaded to remain as stationary as possible during this time.

To treat the gravitational potential V_L produced by the Skylab itself, let us expand it in a Taylor's series in the neighborhood of the center of the sphere. On carrying terms through the cube of $\mathbf{s}$, the position vector of the test object relative to the center of the sphere, we can then specify V_L by means of 18 coefficients. These coefficients will be constant if and only if everything is tied down during the run. If the sphere is constrained, all 18 coefficients will indeed be required to specify the field $f_{L\sigma}$ produced by the space laboratory on the test object in the σ'th tunnel. If the sphere is unconstrained, only the gravity gradient part, $f'_{L\sigma}$, of this field remains and $f'_{L\sigma}$ requires only 10 constants for its specification.

The idea behind this kind of treatment of the laboratory field is to treat the above constants, as well as G, as unknowns to be determined from the highly redundant data that will be obtained during a run. A statistical reduction of these data will then give both G and the 'constants' and will test their constancy and thus the reliability of the experiment.

Let $U_L=-V_L$. Then the gravitational field produced by the spacecraft itself on the test object in the σ'th tunnel will be given by

$$f_{L\sigma}=\frac{\partial U_L}{\partial x_\sigma}, \tag{113}$$

where x_σ is the distance from the center of the sphere to the test object in the σ'th tunnel.

Now let $\xi_\mu\,(\mu=1, 2, 3)$ be the components of $\mathbf{s}$ in a system fixed in the spacecraft. The system $\mathbf{i}_0$, $\mathbf{j}_0$, $\mathbf{k}_0$, as mentioned above, is such a system. If we now expand U_L as a Taylor's series in the ξ_μ's, the coefficients will be constants, if and only if everything is tied down during the run.

Through cubic terms, this Taylor's series takes the form (See Hildebrand, 1964 for the method)

$$U_L(\mathbf{r}_0+\mathbf{s})-U_L(\mathbf{r}_0)=\sum_\mu A_\mu\xi_\mu+\tfrac{1}{2}\Big[\sum_\mu B_\mu\xi_\mu^2+\sum_{\mu<\nu}\sum C_{\mu\nu}\xi_\mu\xi_\nu\Big]$$
$$+\tfrac{1}{3}\Big[\sum_\mu D_\mu\xi_\mu^3+\sum_\nu\xi_\mu\sum_{\nu\neq\mu}E_{\mu\nu}\xi_\nu^2+F\xi_1\xi_2\xi_3\Big]. \tag{114}$$

There are three A's, three B's, three C's, three D's, six E's and one F, or 19 coefficients in all. But

$$B_1+B_2+B_3=(\nabla^2U_L)_0=0, \tag{115}$$

since we are here not including the sphere itself (which does not have to be hollow at its center) as part of the spacecraft. Thus there are 18 independent coefficients.

If we let α be the matrix of the rotation that takes one from the tunnel system $\mathbf{i}_\mu$ to the spacecraft system $\mathbf{i}_{0\mu}$, then

$$\mathbf{i}_{0\nu}=\sum_\mu\alpha_{\nu\mu}\mathbf{i}_\mu, \tag{30.1}$$

$$\xi_\nu=\sum_\mu\alpha_{\nu\mu}x_\mu. \tag{30.4}$$

In the σ'th tunnel, however,

$$\xi_\mu=\alpha_{\mu\sigma}x_\sigma, \tag{116}$$

not summed. Insertion of (116) into (114) then yields

$$U_L-U_L(0)=x_\sigma\sum_\mu A_\mu\alpha_{\mu\sigma}+\tfrac{1}{2}x_\sigma^2\Big[\sum_\mu B_\mu\alpha_{\mu\sigma}^2+\sum_{\mu<\nu}\sum C_{\mu\nu}\alpha_{\mu\sigma}\alpha_{\nu\sigma}\Big]$$
$$+\tfrac{1}{3}x_\sigma^3\Big[\sum_\mu D_\mu\alpha_{\mu\sigma}^3+\sum_\mu\alpha_{\mu\sigma}\sum_{\nu\neq\mu}E_{\mu\nu}\alpha_{\nu\sigma}^2+F\alpha_{1\sigma}\alpha_{2\sigma}\alpha_{3\sigma}\Big]. \tag{117}$$

Then

$$f_{L\sigma}=\frac{\partial U_L}{\partial x_\sigma}=\sum_\mu A_\mu\alpha_{\mu\sigma}+x_\sigma\Big[\sum_\mu B_\mu\alpha_{\mu\sigma}^2+\sum_{\mu<\nu}\sum C_{\mu\nu}\alpha_{\mu\sigma}\alpha_{\nu\sigma}\Big]$$
$$+x_\sigma^2\Big[\sum_\mu D_\mu\alpha_{\mu\sigma}^3+\sum_{\mu\neq\nu}\sum E_{\mu\nu}\alpha_{\mu\sigma}\alpha_{\nu\sigma}^2+F\alpha_{1\sigma}\alpha_{2\sigma}\alpha_{3\sigma}\Big]. \tag{118}$$

If the sphere is constrained (and we have seen that constraining it will completely ruin the experiment unless its center is then fixed at the center of mass of the spacecraft), the net gravitational field of the spacecraft itself on the test object is given by (118), with 18 independent coefficients. If, however, the spacecraft is approximately symmetric with respect to its mid-section, the terms involving the A_μ's will be 'small'. That is, they will certainly be much smaller than they would be if the center of the sphere were 20 feet from the center of mass of the spacecraft.

If the sphere is unconstrained, then the net field on a test object, produced by the spacecraft, is given by a gravity gradient term alone. This is

$$f'_{L\sigma}=f_{L\sigma}(\mathbf{r}_0+\mathbf{s})-f_{L\sigma}(\mathbf{r}_0), \tag{119}$$

where

$$f_{L\sigma}(\mathbf{r}_0)=\sum_\mu A_\mu\alpha_{\mu\sigma}, \tag{120}$$

Thus

$$\frac{f'_{L\sigma}}{x_\sigma} = \sum_\mu B_\mu \alpha_{\mu\sigma}^2 + \sum_{\mu<\nu}\sum C_{\mu\nu}\alpha_{\mu\sigma}\alpha_{\nu\sigma} + x_\sigma\Big[\sum_\mu D_\mu \alpha_{\mu\sigma}^3 + \sum_{\mu\neq\nu}\sum E_{\mu\nu}\alpha_{\mu\sigma}\alpha_{\nu\sigma}^2 + F\alpha_{1\sigma}\alpha_{2\sigma}\alpha_{3\sigma}\Big]. \tag{121}$$

This gives the net field on the test object, when the sphere is unconstrained, but $\mathbf{b}\neq 0$. (Of course the constants in (121) will have values differing from those when $\mathbf{b}=0$.) When one sums the equations of motion over the three tunnels, there is a further simplification, as follows.

$$\begin{aligned}\sum_\sigma \Big(\sum_\mu B_\mu \alpha_{\mu\sigma}^2 + \sum_{\mu<\nu}\sum C_{\mu\nu}\alpha_{\mu\sigma}\alpha_{\nu\sigma}\Big) &= \sum_\mu B_\mu \sum_\sigma \alpha_{\mu\sigma}^2 + \sum_{\mu<\nu}\sum_\sigma C_{\mu\nu}\alpha_{\mu\sigma}\alpha_{\nu\sigma} && (122.1)\\ &= \sum_\mu B_\mu, && (122.2)\end{aligned}$$

since α is an orthogonal matrix. Also $\sum_\mu B_\mu = 0$, by (115), so that

$$\sum_\sigma \frac{f_{L\sigma'}}{x_\sigma} = \sum_\mu D_\mu \sum_\sigma x_\sigma \alpha_{\mu\sigma}^3 + \sum_{\mu\neq\nu}\sum E_{\mu\nu}\sum_\sigma x_\sigma\alpha_{\mu\sigma}\alpha_{\nu\sigma}^2 + F\sum_\sigma x_\sigma \alpha_{1\sigma}\alpha_{2\sigma}\alpha_{3\sigma}. \tag{123}$$

In this case only 10 coefficients remain to be determined, in the reduction of the data. Of course there will still be 18 in the computer simulation, where one must consider each tunnel separately, in searching for possible extraction of a test object.

16. The Effects of Drag on a Test Object

If the sphere is unconstrained, it is in free fall in the gravitational field of the Earth and the spacecraft, and the aerodynamic drag on the spacecraft has no effect on the test object.

Suppose now that the sphere is constrained, as in Case β, at the center of mass of the spacecraft. The effect of spacecraft drag on a test object in the σ'th tunnel then appears as a force per unit mass

$$f_{D\sigma} = -\frac{\mathbf{D}}{M}\cdot\mathbf{i}_\sigma. \tag{124}$$

Here M is the total mass of the spacecraft, of the order 10^5 pounds and $\mathbf{i}_\sigma$ is a unit vector along the σ'th tunnel. The vector $\mathbf{D}$ is the total non-gravitational force acting on the spacecraft. If this force is entirely aerodynamic drag, it is given by

$$\mathbf{D} = -\tfrac{1}{2}C_D A \varrho_a v_a \mathbf{v}_a. \tag{125}$$

Here ϱ_a is the atmospheric density, $\mathbf{v}_a$ is the resultant velocity of the spacecraft relative to the (rotating) atmosphere, $v_a \equiv |\mathbf{v}_a|$, A is the projected area of the spacecraft perpendicular to $\mathbf{v}_a$, and C_D is a dimensionless coefficient ≈ 2.2. Here A will be of the order $20' \times 40' = 800$ ft^2. Even if the axis of the cylinder is perpendicular to the spacecraft velocity $\dot{\mathbf{r}}_c$, this figure is not exact because $\mathbf{v}_a \neq \dot{\mathbf{r}}_c$.

Indeed

$$\mathbf{v}_\mathrm{a} = \dot{\mathbf{r}}_c - \Omega_e \mathbf{K} \times \mathbf{r}_c , \tag{126}$$

which says that the net velocity of the spacecraft, relative to the atmosphere, is its velocity in the equatorial system minus the velocity in the same system of the atmosphere, considered to be rotating like a rigid body along with the Earth. Here Ω_e is the sidereal rate of rotation of the Earth and $\mathbf{K}$ a unit vector along its polar axis.

If $\mathbf{I}_\mu$ $(\mu = 1, 2, 3)$ are the unit vectors in the equatorial system, we have

$$\dot{\mathbf{r}}_c = \sum_\mu \mathbf{I}_\mu \dot{X}_{c\mu} , \tag{127}$$

and

$$\mathbf{K} \times \mathbf{r}_c = \mathbf{J} X_c - \mathbf{I} Y_c . \tag{128}$$

Also

$$\mathbf{I}_\mu \cdot \mathbf{i}_\sigma = \gamma_{\mu\sigma} . \tag{31.3}$$

Then, from Equations (124) through (128) and (31.3), we find

$$f_{D\sigma} = \frac{1}{2} C_D \frac{A}{M} \varrho_\mathrm{a} v_\mathrm{a} \left[(\dot{X}_c + \Omega_e Y_c) \gamma_{1\sigma} + (\dot{Y}_c - \Omega_e X_c) \gamma_{2\sigma} + \dot{Z}_c \gamma_{3\sigma} \right] , \tag{129}$$

where the γ's are the direction cosines of the σ'th tunnel in the equatorial system and where

$$v_\mathrm{a}^2 = \dot{\mathbf{r}}_c^2 - 2\Omega_e (X_c \dot{Y}_c - Y_c \dot{X}_c) + \Omega_e^2 (X_c^2 + Y_c^2) , \tag{130}$$

Estimates show that the term in Ω_e^2 is about $\frac{1}{200}$ of the main term $\dot{\mathbf{r}}_c^2$, and the term in Ω_e is about $\frac{2}{15}$ of it. Since $\Omega_e > 0$ and $X_c \dot{Y}_c - Y_c \dot{X}_c > 0$ for a direct orbit, as the Skylab will have, the rotation of the atmosphere diminishes the effect of drag, as expected.

To estimate the ratio of $f_{D\sigma}$ to the main field $4\pi G \varrho x_\sigma / 3$, we consider the case where the given tunnel is parallel to $\mathbf{v}_a$. Then by (124) and (125)

$$f_{D\sigma} = \frac{1}{2} \frac{C_D A}{M} \varrho_\mathrm{a} v_a^2 . \tag{131}$$

Here $v_a^2 \approx \dot{\mathbf{r}}_c^2 \approx \mu / r_c$, for an orbit with small eccentricity. Approximating r_c by R_e and writing

$$\mu \approx G \frac{4\pi}{3} \varrho_e R_e^3 , \tag{132}$$

where ϱ_e is the Earth's mean density, we then obtain

$$f_{D\sigma} \approx \frac{C_D A}{2M} \varrho_a G \frac{4\pi}{3} \varrho_e R_e^2 . \tag{133}$$

But

$$f_{\text{main}} \approx G \frac{4\pi}{3} \varrho x_\sigma, \tag{134}$$

where ϱ is the density of tungsten. Then

$$\frac{f_{D\sigma}}{f_{\text{main}}} \approx \frac{C_D A}{2M} \frac{\varrho_a \varrho_e}{\varrho} \frac{R_e^2}{x_\sigma}. \tag{135}$$

With $C_D = 2.2$, $A = 800\ \text{ft}^2 = 743200\ \text{cm}^2$, $\varrho_e = 5.5\ \text{gm cm}^{-3}$, $\varrho = 19\ \text{gm cm}^{-3}$, $R_e =$ $= (6.4) \times 10^8$ cm, $M = 10^5\ \text{lb} = (4.54) \times 10^7$ gm, we find

$$\frac{f_{D\sigma}}{f_{\text{main}}} = (2.14) \times 10^{14} \varrho_a, \tag{136}$$

if $x_\sigma = 10$ cm and ϱ_a is in gm cm^{-3}.

We consider an orbit at an altitude of 500 km. The atmospheric density ϱ_a is a highly variable quantity. According to Jacchia (1969), at 500 km the exospheric temperature T_{ex} can vary from about 700 K at sunspot minimum to 1900 K at sunspot maximum. There is also a variation of a few hundred degrees between day and night. Reference to the U.S. Standard Atmosphere Supplements (1966) furnishes the following data (which at 500 km do not depend on which seasonal model is used)

T_{ex}	ϱ_a (in gm cm^{-3})	$f_{D\sigma}/f_{\text{main}}$ at $x = 10$ cm
800 K	1.54×10^{-16}	0.033
1900 K	5.74×10^{-15}	1.23

At $x_\sigma = 2$ cm, these values of the ratio would be five times larger. Thus at certain phases of the sunspot cycle, the effect of drag can be very serious. At 400 km, the ratio is about twice that at 500 km for $T_{ex} = 1900$ K and about eight times that at 500 km for $T_{ex} = 800$ K.

If the sphere is kept constrained, so that drag enters the picture, its effects can thus be very important and highly variable, depending strongly on the phase of the sunspot cycle, time of day, and altitude.

17. Reduction of the Observations: Unconstrained Sphere

To reduce the observations when the sphere is not constrained, we may use Equations (60), (94), and (121). In so doing, we write

$$f_{s\sigma}(\mathbf{r}_0 + \mathbf{s}) = -\frac{4\pi}{3} G \varrho x_\sigma - f_{G\sigma}^T, \tag{137}$$

where $f_{G\sigma}^T$ is the attractive correction to the gravitational field of the sphere, produced by the finite diameter of the σ'th tunnel and the presence of the two other tunnels.

I defer the evaluation of $f^T_{G\sigma}$ to Appendix A. Then

$$
\begin{aligned}
\ddot{x}_\sigma = & -\frac{4\pi}{3} G\varrho x_\sigma - f^T_{G\sigma} + (\omega^2 - \omega_\sigma^2)\, x_\sigma \\
& + x_\sigma \Bigg[-\frac{\mu}{r_0^3}(1 - 3l_\sigma^2) + \frac{3\mu J_2 R_e^2}{2r_0^5} \Bigg\{ -1 + 5l_\sigma^2 - 2\gamma_{3\sigma}^2 \\
& + \frac{20\gamma_{3\sigma} l_\sigma Z_0}{r_0} + 5\frac{Z_0^2}{r_0^2}(1 - 7l_\sigma^2) \Bigg\} \Bigg] \\
& + x_\sigma \big[\sum_\mu B_\mu \alpha_{\mu\sigma}^2 + \sum \sum_{\mu<\nu} C_{\mu\nu} \alpha_{\mu\sigma} \alpha_{\nu\sigma} \big] \\
& + x_\sigma^2 \big[\sum_\mu D_\mu \alpha_{\mu\sigma}^3 + \sum \sum_{\mu\neq\nu} E_{\mu\nu} \alpha_{\mu\sigma} \alpha_{\nu\sigma}^2 + F\alpha_{1\sigma}\alpha_{2\sigma}\alpha_{3\sigma} \big] .
\end{aligned}
\tag{138}
$$

In this case the A's, which determine the direct gravitational field of the environment on the test object, drop out, so that only the gravity gradient effect of the environment remains. On forming $\sum_\sigma \ddot{x}_\sigma / x_\sigma$, the whole Earth gravity gradient term drops out (including the effects of the higher harmonics that are not indicated in (138)), as does part of the term arising from the gravitational effects of the spacecraft itself. We shall then be neglecting only the truly negligible quadratic gravity gradient terms and the quartic terms of the spacecraft potential. We obtain

$$
\begin{aligned}
\sum_\sigma \ddot{x}_\sigma / x_\sigma = & -4\pi G\varrho - \sum_\sigma f^T_{G\sigma} / x_\sigma + 2\omega^2 \\
& + \sum_\sigma x_\sigma \big[\sum_\mu D_\mu \alpha_{\mu\sigma}^3 + \sum \sum_{\mu\neq\nu} E_{\mu\nu} \alpha_{\mu\sigma} \alpha_{\nu\sigma}^2 + F\alpha_{1\sigma}\alpha_{2\sigma}\alpha_{3\sigma} \big] .
\end{aligned}
\tag{139}
$$

Now suppose that the design has been such that no test object is extracted during a run and that the servomechanisms work well, so that each $\ddot{x}_\sigma$ remains small. Let us also suppose that at times $t_1, t_2, t_3, \ldots t_N$ we observe ω^2, each x_σ, and the orientation of the tunnel system relative to the inertial system, so that we know all the γ's at each instant. We may know the β's from measurements of the inertial orientation of the spacecraft. If not, but if the orbit trackers give us the $X_{c\mu}$'s and $\dot{X}_{c\mu}$'s at each instant, we can determine the β's, to a certain approximation, if the cylinder is kept property oriented. We can then determine the α's from $\alpha = \beta^{-1}\gamma$. Numerical differentiation can furnish the small terms $\ddot{x}_\sigma$.

In Equation (139) there are then the unknowns $G\varrho$, three D's, six E's, and F, or 11 unknowns in all, and all presumably constant, if everything in the spacecraft has been tied down during the run, and if the spacecraft has been kept properly oriented. We then have N equations to determine the 11 unknowns. By taking $N \gg 11$, we can increase the accuracy of determination of $G\varrho$, with use of statistical methods for reducing the highly redundant set of data.

18. Reduction of the Observations: Constrained Sphere

To reduce the observations when the sphere is constrained, we may use Equations

(64), (112) with the b_μ's zero, (118), (129), and (137). We obtain

$$
\begin{aligned}
\ddot{x}_\sigma/x_\sigma = & -\frac{4\pi}{3}\,G\varrho - f^{T}_{G\sigma}/x_\sigma + \omega^2 - \omega_\sigma^2 \\
& -\frac{\mu}{r_c^3}\,(1 - 3l_\sigma^2) + \frac{3\mu J_2 R_e^2}{2r_c^5}\Bigg[-1 + 5l_\sigma^2 - 2\gamma_{3\sigma}^2 + \frac{20\gamma_{3\sigma} l_\sigma Z_c}{r_c} \\
& + \frac{5Z_c^2}{r_c^2}\,(1 - 7l_\sigma^2)\Bigg] \\
& + x_\sigma^{-1}\sum_\sigma A_\mu \alpha_{\mu\sigma} + \sum_\mu B_\mu \alpha_{\mu\sigma}^2 + \sum\sum_{\mu<\nu} C_{\mu\nu}\alpha_{\mu\sigma}\alpha_{\nu\sigma} \\
& + x_\sigma\Bigg[\sum_\mu D_\mu \alpha_{\mu\sigma}^3 + \sum\sum_{\mu\neq\nu} E_{\mu\nu}\alpha_{\mu\sigma}\alpha_{\nu\sigma}^2 + F\alpha_{1\sigma}\alpha_{2\sigma}\alpha_{3\sigma}\Bigg] \\
& + \frac{\psi(t)}{x_\sigma}\,[(\dot{X}_c + \Omega_e Y_c)\,\gamma_{1\sigma} + (\dot{Y}_c - \Omega_e X_c)\,\gamma_{2\sigma} + \dot{Z}_c\gamma_{3\sigma}]\,, \\
& \qquad (\sigma = 1, 2, 3). \qquad (140)
\end{aligned}
$$

Here l_σ is given by (91.3), with the b_μ's placed equal to zero. The quantity $\psi(t)$, defined by

$$\psi(t) = \frac{C_D A}{2M}\,\varrho_a v_a\,, \qquad (141)$$

is a highly variable and largely unknown quantity, because of the atmospheric density ϱ_a.

In this case, the A's, which determine the direct gravitational field of the environment on the test object, are not eliminated. Thus imprecise determination of the spacecraft axes, $\mathbf{i}_0$, $\mathbf{j}_0$, $\mathbf{k}_0$, would have a more important effect than in the case of the unconstrained sphere.

Assume that the $X_{c\mu}$, $\dot{X}_{c\mu}$, x_σ, ω_σ, γ's, and α's are given, as before, at each time of measurement. With N times, Equation (140) then furnishes $3N$ equations for $19+N$ unknowns. These include $G\varrho$, three A's, three B's, three C's, three D's, six E's, and one F, which add to 20, diminished to 19, by the equation of condition $\sum_\mu B_\mu = 0$. They also include the N values of $\psi(t)$. If we choose N so large that

$$3N > N + 19\,, \quad i.e.,\quad N \geqq 10\,, \qquad (142)$$

we shall have enough data to do the reduction and thus find $G\varrho$.

We could not do the summing over σ in this case, because of the need to determine the values of the unknown $\psi(t)$. Larger values of N should give more accuracy, through greater redundancy, up to the point where the time becomes so large that things cannot be kept tied down in the spacecraft.

In this scheme, harmonics of the Earth's potential higher than J_2 are not included in the Earth gravity gradient field. They may give a field of the order $(3)\times 10^{-6}$ of the main field $4\pi G\varrho x_\sigma/3$. In the other method, with the sphere unconstrained, they did not enter the picture at all, since they disappeared on summing over σ.

19. The Computer Simulation

The above equations for the $\ddot{x}_\sigma$'s are sufficient for the reduction of the data. In this reduction we have given to us the x_σ, the $X_{c\mu}$, $\dot{X}_{c\mu}$, the γ's, and the ω's.

The purpose of a computer simulation is to find out whether any test object will be extracted from one of the tunnels and to check or aid the design of the servomechanisms. Now an experimental run is not expected to last longer than one orbital revolution, else we could not hope to keep things tied down in the spacecraft. For the rather qualitative purpose of computer simulation, it should thus suffice to use an elliptic (or circular) orbit as input for the $X_{c\mu}$, $\dot{X}_{c\mu}$. It would also be reasonable to omit the J_2 part of the earth gravity gradient field in (140), since it does not amount to more than about $\frac{1}{300}$ of the main field $4\pi G\varrho x_\sigma/3$.

The simulation begins with Equations (138) for the unconstrained sphere or Equations (140) for the constrained sphere. The other necessary equations will be those that tell us how the ω's and the $\gamma_{\mu\nu}$ vary with time. (This is on the assumption that the spacecraft maintains the orientation that has been described, so that the direction cosines $\beta_{\mu\nu}$ and $\alpha_{\mu\nu}$ can then be found from the $\gamma_{\mu\nu}$ and the $X_{c\mu}$ and $\dot{X}_{c\mu}$.)

I shall assume that the principal moments of inertia of the sphere, viz. A, B, and C, are those about the axes of the tunnels and that they have been measured. If $\mathbf{S}$ is the total spin angular momentum of the sphere, relative to its center of mass, then

$$\mathbf{S} = \mathbf{i}A\omega_1 + \mathbf{j}B\omega_2 + \mathbf{k}C\omega_3 . \tag{143}$$

The variation of the ω's is then given by

$$\dot{\mathbf{S}} = \mathbf{L}, \tag{144}$$

where the dependence of the torque $\mathbf{L}$ will be expressible by means of Equations (23) or something similar. Since $\dot{\mathbf{i}} = \boldsymbol{\omega} \times \mathbf{i}$, etc, Equations (144) and (23) result in

$$A\dot{\omega}_1 + (B - C)\,\omega_2\omega_3 = k_{11}x + k_{12}\dot{x}, \tag{145.1}$$

$$B\dot{\omega}_2 - (C - A)\,\omega_3\omega_1 = k_{21}y + k_{22}\dot{y}, \tag{145.2}$$

$$C\dot{\omega}_3 - (A - B)\,\omega_1\omega_2 = k_{31}z + k_{32}\dot{z}, \tag{145.3}$$

for the variation of the ω's. Here $A \approx B \approx C \approx 2MR^2/5$, where M and R are, respectively, the mass and the radius of the sphere. Thus the terms containing $\omega_2\omega_3$, etc, can be neglected.

To describe the orientation of the sphere we need either three differential equations for Eulerian angles or nine differential equations for the direction cosines $\gamma_{\mu\nu}$. Of these six will be redundant, since the $\gamma_{\mu\nu}$ obey six orthonormal relations. It appears simpler to use the direction cosines. The computer simulation will then involve integrating fifteen differential equations, with six equations of constraint on the $\gamma_{\mu\nu}$ that will serve as checks.

From

$$\boldsymbol{\omega} = \mathbf{i}\omega_1 + \mathbf{j}\omega_2 + \mathbf{k}\omega_3 , \tag{146}$$

$$\dot{\mathbf{i}} = \boldsymbol{\omega} \times \mathbf{i}, \qquad \dot{\mathbf{i}} = \boldsymbol{\omega} \times \mathbf{j}, \qquad \dot{\mathbf{k}} = \boldsymbol{\omega} \times \mathbf{k}, \tag{147}$$

$$\mathbf{i}_\nu = \sum_\mu \mathbf{I}_\mu \gamma_{\mu\nu} , \tag{148}$$

we readily deduce

$$\begin{aligned} \dot{\gamma}_{11} &= -\gamma_{13}\omega_2 + \gamma_{12}\omega_3 , & \dot{\gamma}_{12} &= \gamma_{13}\omega_1 - \gamma_{11}\omega_3 , & \dot{\gamma}_{13} &= -\gamma_{12}\omega_1 + \gamma_{11}\omega_2 , \\ \dot{\gamma}_{21} &= -\gamma_{23}\omega_2 + \gamma_{22}\omega_3 , & \dot{\gamma}_{22} &= \gamma_{23}\omega_1 - \gamma_{21}\omega_3 , & \dot{\gamma}_{23} &= -\gamma_{22}\omega_1 + \gamma_{21}\omega_2 , \\ \dot{\gamma}_{31} &= -\gamma_{33}\omega_2 + \gamma_{32}\omega_3 , & \dot{\gamma}_{32} &= \gamma_{33}\omega_1 - \gamma_{31}\omega_3 , & \dot{\gamma}_{33} &= -\gamma_{32}\omega_1 + \gamma_{31}\omega_2 . \end{aligned} \tag{149}$$

Since γ is an orthogonal matrix, satisfying $\gamma\tilde{\gamma}=1$, Equations (149) are subject to the six orthonormal relations

$$\sum_\nu \gamma_{\mu\nu}\gamma_{\sigma\nu} = \delta_{\mu\sigma} \quad (\mu, \sigma = 1, 2, 3), \tag{150}$$

as conditions of constraint.

In doing the computer simulation, one will have to guess various possible distributions of mass in the spacecraft, in order to estimate the coefficients in the Taylor's series (114). One will also have to guess various values for the $k_{\mu\nu}$ in (145) that describe the servos. Numerical integration of the system of 15 differential equations can be checked by seeing if Equations (150) are satisfied. The results can than be used to find the values of the $k_{\mu\nu}$ that are required to keep the $\ddot{x}_\sigma$ small (ideally, zero) and to prevent extraction of any test object from a tunnel.

Appendix A: Corrections to the Gravitational Field Produced by the Tunnels

A.1. The Effect of the Test Object's Own Tunnel

The notation in these appendices will be independent of that in the main body of the paper. Figure 1 shows a single tunnel in the sphere. It is not drawn to scale.

Let us consider a field point P on the axis of the tunnel, at distance s from the center of the sphere. With cylindrical coordinates r, ϕ, z, a volume element $r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$ at Q produces at P a potential $-G\varrho r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z/QP$. Here

$$QP = [r^2 + (s - z)^2]^{1/2} . \tag{1}$$

Differentiation of this potential with respect to s gives the attractive field at P produced by the volume element at Q. It is

$$\mathrm{d}f = G\varrho r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z\,(s - z)\,[r^2 + (s - z)^2]^{-3/2} . \tag{2}$$

Integration with respect to ϕ replaces $\mathrm{d}\phi$ by 2π and yields the attractive field at P produced by a ring of cross-section $\mathrm{d}r\,\mathrm{d}z$, radius r, and height z.

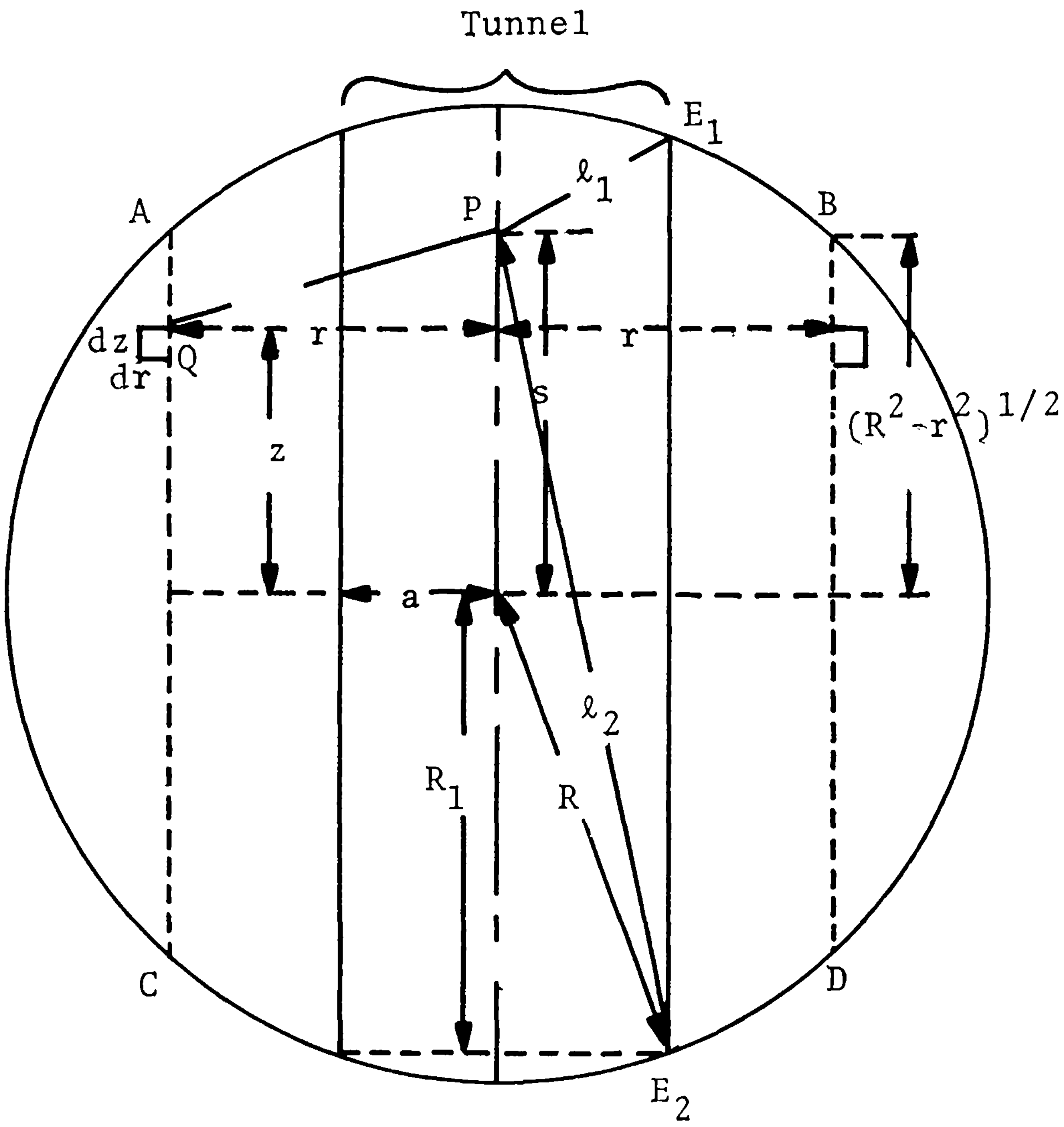

Fig. 1. $(\overline{QP})^2 = r^2 + (s-z)^2$; $R_1{}^2 = R^2 - a^2$; P = field point (position of test object); Q = source point; volume element $r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$ at Q; $ABCD$ = cylinder of radius r, length $2(R^2 - r^2)^{1/2}$, thickness $\mathrm{d}r$; $l_1 = PE_1$; $l_2 = PE_2$.

The total attractive field, produced at P by the solid material, is then

$$f_1 = 2\pi G\varrho \int_a^R r\,\mathrm{d}r \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} (s-z)\left[r^2 + (s-z)^2\right]^{-3/2} \mathrm{d}z\,. \tag{3}$$

Here the integral over z first gives the field produced at P by a cylinder of radius r, thickness $\mathrm{d}r$, and length $2\sqrt{(R^2 - r^2)}$. The subsequent integration over r adds up the contributions of all the thin cylinders to give the total attractive field at P.

On making a change of variable

$$(s-z)^2 = v\,, \tag{4}$$

in (3) we first find an indefinite integral

$$\int (r^2 + v^2)^{-3/2}\,\mathrm{d}v = -\,2\,(r^2 + v)^{-1/2}\,. \tag{5}$$

Insertion of this into (3), with use of the corresponding appropriate limits for v then shows that

$$\frac{f_1}{2\pi G\varrho} = \int_a^R r\,\mathrm{d}r\left[\left(R^2 + s^2 - 2s\sqrt{R^2 - r^2}\right)^{-1/2} - \right.$$
$$\left. -\left(R^2 + s^2 + 2s\sqrt{R^2 - r^2}\right)^{-1/2}\right]. \tag{6}$$

The change of variable

$$R^2 - r^2 = u^2 \tag{7}$$

then results in

$$\frac{f_1}{2\pi G\varrho} = \int_0^{R_1} u\,\mathrm{d}u\left[(R^2 + s^2 - 2us)^{-1/2} - (R^2 + s^2 + 2us)^{1/2}\right], \tag{8}$$

where

$$R_1 \equiv (R^2 - a^2)^{1/2}. \tag{9}$$

On placing

$$F_1 \equiv (R^2 + s^2 + 2us)^{1/2}, \qquad F_2 \equiv (R^2 + s^2 - 2us)^{1/2}, \tag{10}$$

and introducing another new variable

$$w \equiv F_1 - F_2, \tag{11}$$

we find

$$w^2\,\mathrm{d}w = \frac{s(F_1 - F_2)^2(F_1 + F_2)}{F_1F_2}\mathrm{d}u \tag{12.1}$$

$$= \frac{s(F_1^2 - F_2^2)(F_1 - F_2)}{F_1F_2}\mathrm{d}u \tag{12.2}$$

$$= 4s^2u(F_2^{-1} - F_1^{-1})\,\mathrm{d}u. \tag{12.3}$$

Comparing (12.3) with (8), we find

$$\frac{f_1}{2\pi G\varrho} = \frac{1}{4s^2}\int_0^{w'} w^2\,\mathrm{d}w = \frac{1}{12s^2}w'^3, \tag{13}$$

where

$$w' \equiv (R^2 + s^2 + 2R_1s)^{1/2} - (R^2 + s^2 - 2R_1s)^{1/2}. \tag{14}$$

Thus

$$f_1 = \frac{\pi G\varrho}{6s^2}\left(\sqrt{R^2 + s^2 + 2R_1s} - \sqrt{R^2 + s^2 - 2R_1s}\right)^3, \tag{15}$$

$$R_1 = (R^2 - a^2)^{1/2}. \tag{15.1}$$

To check this, note that if $a=0$, we obtain $R_1=R$, and then

$$f_1(a=0) = \frac{4\pi}{3}G\varrho s, \tag{16}$$

which is the correct result for the attractive field inside a homogeneous sphere of density ϱ. Thus f_1, as given by (15), minus $4\pi G\varrho s/3$ gives the correction for the tunnel in which the test object moves.

Equation (15) can be given a geometrical interpretation. By (9) we have $R^2 = a^2 + R_1^2$, so that

$$R^2 + s^2 + 2R_1 s = a^2 + (R_1 + s)^2 = l_2^2, \tag{17.1}$$

$$R^2 + s^2 - 2R_1 s = a^2 + (R_1 - s)^2 = l_1^2. \tag{17.2}$$

The lengths l_1 and l_2 can be identified in Figure 1, l_1 and l_2 being respectively the distances from the test object to the intersections with the sphere of the near and the far ends of the tunnel. Then

$$f_1 = \frac{\pi G \varrho}{6s^2}(l_2 - l_1)^3. \tag{18}$$

From (17)

$$(l_2 - l_1)^2 = 2(R^2 + s^2) - 2[(R^2 - s^2)^2 + 4a^2 s^2]^{1/2} \tag{19}$$

$$< 4s^2. \tag{20}$$

Thus

$$l_2 - l_1 < 2s,$$

so that if

$$f_{\text{main}} \equiv \frac{4\pi}{3} G\varrho s, \tag{21}$$

then

$$f_1 < f_{\text{main}}, \tag{22}$$

as expected. Here f_{main} is the attractive field at distance s from the center of an intact sphere of density ϱ.

A.2. Intersection of the Three Tunnels

Having corrected the field on the test object for the effect of its own tunnel, we now have to correct for the effects of the two perpendicular tunnels, also of radius a. It is easy to see that the effect of these tunnels is just twice that of one. The intersection of the three tunnels, however, is complicated, being a dodecahedron with curved faces. Moreover, after correcting for all three complete tunnels, we shall have counted the dodecahedron three times as a cavity. This means that we have to finish the calculation by finally adding to the above corrections twice the field produced by a solid dodecahedron of density ϱ*.

The radius of a sphere inscribed in the dodecahedron is a and that of a sphere circumscribed about it is $a\sqrt{\frac{3}{2}}$.

A.3. Effects of the Perpendicular Tunnels on the Field

Figure 2 shows a tunnel T_1 perpendicular to the tunnel σ, the axis of σ being along OP, with P a field point on its axis, at distance s from O, the center of the sphere. The tunnel T_1 consists of the empty cylinder $ABA'B'$ and of the empty spherical segments $AA'C$ and $BB'C$. There is no need to draw the other perpendicular tunnel T_2.

* See note added in proof, p. 253.

The axial field correction at P, produced by the perpendicular tunnels T_1 and T_2, is then four times the sum of the axial fields produced at P by the cylinder $AA'DD'$ and the spherical segment $AA'C$, both of density $-\varrho$. (Such a correction does not include the effect of the dodecahedral intersection.)

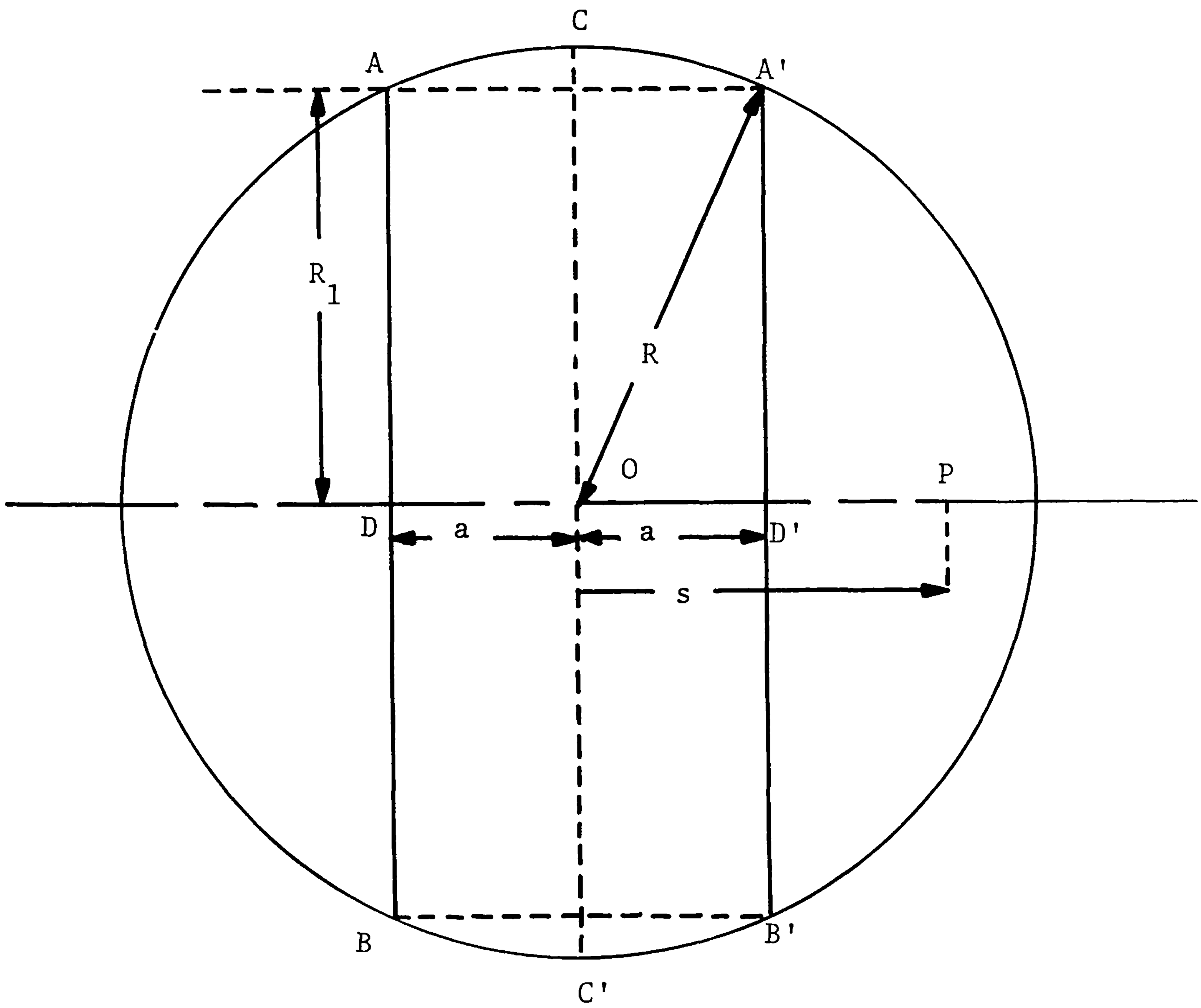

Fig. 2. A perpendicular tunnel T_1; cylinder $ABA'B'$ empty; spherical segments $AA'C$, $BB'C'$ empty; P = field point on axis OP of tunnel σ; $R_1{}^2 = R^2 - a^2$.

A straighforward attempt to evaluate the field produced at P, either by the cylinder $AA'DD'$ or by the spherical segment $AA'C$ would involve the calculation of a triple integral over three cylindrical coordinates. One finds at once that the integration over the angular coordinate yields an elliptic integral, of large modulus, which would then have to be integrated twice more. Such a straightforward attack is therefore to be shunned.

A.4. The field of a disk

To avoid the above difficulty, I shall regard both the cylinder $AA'DD'$ and the spherical segment ACA' in Figure 2 as composed of elementary disks. The procedure will then be to calculate the potential produced by each disk at P, take its directional derivative along the axis of the σ tunnel to find the field produced by each disk, and then integrate over all the disks to find the field corrections produced by $AA'DD'$ and ACA'.

Since the center of each disk is farther from the field point P than from any source point, the potential at P can be expressed by means of a converging series of zonal harmonics, similar to that for an axially symmetric planet. In $AA'DD'$ the disks all have the same radius a, but in ACA' the radius varies.

Let $\mathrm{d}z$ be the thickness of such a disk, r_d its radius, r_p the distance from its center O_d to the field point P in Figure 3, and θ the angle from O_dC to $\mathbf{r}_p \equiv \mathbf{O_dP}$. If

$$\mu_d = - G\pi r_d^2 \varrho \,\mathrm{d}z\,, \tag{23}$$

the zonal expansion for the potential V_d produced by the disk at P takes the form

$$V_d = -\frac{\mu_d}{r_P}\left[1 - \sum_{k=1}^{\infty}\left(\frac{r_d}{r_P}\right)^{2k} J_{2k}P_{2k}(\cos\theta)\right]. \tag{24}$$

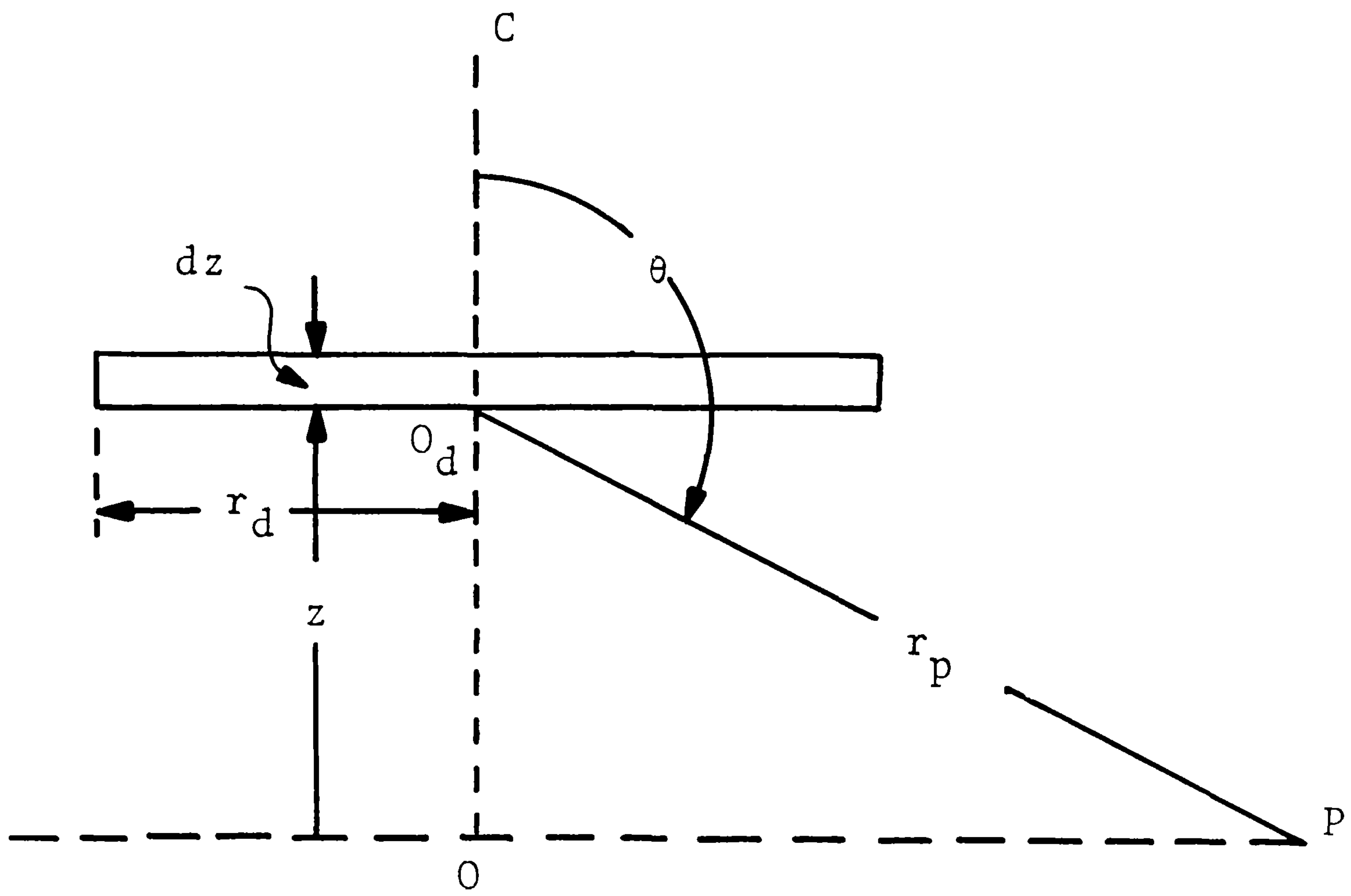

Fig. 3. An elementary disk of $AA'DD'$ or of $AA'C$.

The odd zonals drop out, because of the symmetry with respect to the plane of the disk. If the mass of the disk is

$$M_d = - \pi r_d^2 \varrho \,\mathrm{d}z\,, \tag{25}$$

the J's are given by

$$M_d J_{2k} = - \int (r/r_d)^{2k} P_{2k}(\cos\theta')\,\sigma\,\mathrm{d}S\,, \tag{26}$$

integrated over the disk. Here $\theta' = \pi/2$, $\sigma = -\varrho\,\mathrm{d}z$, and $\mathrm{d}S = r\,\mathrm{d}r\,\mathrm{d}\phi$. The result is

$$J_{2k} = -\frac{P_{2k}(0)}{k+1}\,, \tag{26.1}$$

where

$$P_{2k}(0) = \frac{(-1)^k (2k)!}{2^{2k}(k!)^2}. \tag{26.2}$$

From (26.1) and (26.2), the first few J_{2k}'s are $J_2 = \frac{1}{4}$, $J_4 = -\frac{1}{8}$, $J_6 = \frac{5}{64}$, and $J_8 = -\frac{7}{128}$.
With $OO_d = z$, we now have in (24)

$$r_P^2 = z^2 + s^2, \tag{27.1}$$

$$\cos\theta = -z/r_P, \tag{27.2}$$

so that

$$V_d = -\frac{\mu_d}{r_P} + \mu_d \sum_{k=1}^{\infty} \frac{r_d^{2k}}{r_P^{2k+1}} J_{2k} P_{2k}(z/r_P), \tag{28}$$

since $P_{2k}(-\xi) = P_{2k}(\xi)$. The attractive field produced by the disk along the σ tunnel is given by

$$f_d = \frac{\partial V_d}{\partial s} = \frac{\mu_d s}{r_P^3} + \mu_d \sum_{k=1}^{\infty} J_{2k} r_d^{2k} \frac{\partial}{\partial s}\left[r_P^{-2k-1} P_{2k}(z/r_P)\right], \tag{29}$$

or

$$f_d = \frac{\mu_d s}{r_P^3} - \frac{\mu_d s}{r_P^3} \sum_{k=1}^{\infty} J_{2k} \left(\frac{r_d}{r_P}\right)^{2k} \left[(2k+1) P_{2k}\left(\frac{z}{r_P}\right) + \frac{z}{r_P} P'_{2k}\left(\frac{z}{r_P}\right)\right]. \tag{30}$$

A.5. Corrections from the Empty Cylinders

The attractive field $f_{AA'DD'}$ produced by the cylinder $AA'DD'$ is then given by placing $r_d = a$ and $\mu_d = -G\pi a^2 \varrho \, dz$ in (30) and integrating it from $z = 0$ to $z = R_1 = \sqrt{(R^2 - a^2)}$. On using (27.1) and placing

$$\lambda \equiv z/r_P = z(z^2 + s^2)^{-1/2}, \tag{31}$$

we thus find

$$f_{AA'DD'} = -G\pi a^2 \varrho s \int_0^{R_1} (z^2 + s^2)^{-3/2} \, dz + G\pi a^2 \varrho s \sum_{k=1}^{\infty} J_{2k} a^{2k} \int_0^{R_1} (z^2 + s^2)^{-k-3/2} Q_k(\lambda) \, dz, \tag{32.1}$$

where

$$Q_k(\lambda) \equiv (2k+1) P_{2k}(\lambda) + \lambda P'_{2k}(\lambda). \tag{32.2}$$

With use of the identities

$$(4k+1) P_{2k}(\lambda) = P'_{2k+1}(\lambda) - P'_{2k-1}(\lambda), \tag{33}$$

and

$$2k P_{2k}(\lambda) = \lambda P'_{2k}(\lambda) - P'_{2k-1}(\lambda), \tag{34}$$

subtraction of (34) from (33) yields

$$Q_k(\lambda) \equiv P'_{2k+1}(\lambda). \tag{35}$$

On placing $z = s \tan\alpha$ in (32.1), we find

$$(z^2 + s^2)^{-3/2}\, \mathrm{d}z = s^{-2} \cos\alpha\, \mathrm{d}\alpha = s^{-2}\, \mathrm{d} \sin\alpha = s^{-2}\, \mathrm{d}\lambda, \tag{36}$$

and

$$(z^2 + s^2)^{-k-3/2}\, \mathrm{d}z = s^{-2k-2}(1 - \lambda^2)^k\, \mathrm{d}\lambda. \tag{37}$$

Insertion of (35) and (37) into (32.1) then yields

$$\frac{(f_{AA'DD'})\, s}{G\pi a^2 \varrho} = -\lambda_1 + \sum_{k=1}^{\infty} J_{2k} \left(\frac{a}{s}\right)^{2k} \int_0^{\lambda_1} (1 - \lambda^2)^k P'_{2k+1}(\lambda)\, \mathrm{d}\lambda, \tag{38}$$

where

$$\lambda_1 \equiv R_1 (R_1^2 + s^2)^{-1/2}. \tag{38.1}$$

Before we proceed further, it is interestign to check (38). If we put $b = 0$ and $R_1 = \infty$, the field $f_{AA'DD'}$ should be half that produced at a distance s from its axis by an infinite cylinder of radius a and density $-\varrho$. In such a case $\lambda_1 = 1$. The integrals in (38) then take the form

$$\int_0^1 (1 - \lambda^2)^k P'_{2k+1}(\lambda)\, \mathrm{d}\lambda = \tfrac{1}{2} \int_{-1}^1 (1 - \lambda^2)^k P'_{2k+1}(\lambda)\, \mathrm{d}\lambda$$

$$= \tfrac{1}{2}(1 - \lambda^2)^k P_{2k+1}(\lambda)]_{\lambda=-1}^1 + k \int_{-1}^1 \lambda (1 - \lambda^2)^{k-1} P_{2k+1}(\lambda)\, \mathrm{d}\lambda, \tag{39}$$

on integration by parts. The first term vanishes and so does the second, since $\lambda(1 - \lambda^2)^{k-1}$ is a polynomial in λ of degree $2k - 1$. Its expansion in Legendre polynomials does not include $P_{2k+1}(\lambda)$, so that orthogonality proves the vanishing.

Thus (38) reduces to

$$2 f_{AA'DD'} = -\frac{2G\pi a^2 \varrho}{s}. \tag{40}$$

Suppose one places a pill box of radius s around an infinite cylinder of radius a and density $-\varrho$ and applies Gauss's theorem to it. The latter states that the attractive flux (inward) through it is equal to the product of $4\pi G$ and the mass in the pill box. The result is (40). Thus Equation (38) survives this check.

The correction from the four empty cylinders is thus

$$4 f_{AA'DD'} = \frac{4G\pi a^2 \varrho}{s} \left[-\lambda_1 + \sum_{k=1}^{\infty} J_{2k} \left(\frac{a}{s}\right)^{2k} I_k \right], \tag{41}$$

where λ_1 is given by (38.1), the J_{2k} by (26.1), and

$$I_k \equiv \int_0^{\lambda_1} (1-\lambda^2)^k P'_{2k+1}(\lambda)\,\mathrm{d}\lambda. \tag{42}$$

It is simple to work out the I_k. We find

$$I_1 = \tfrac{3}{2}\int_0^{\lambda_1} (1-\lambda^2)(5\lambda^2-1)\,\mathrm{d}\lambda$$

$$= \tfrac{3}{2}(-\lambda_1^5 + 2\lambda_1^3 - \lambda_1), \tag{43.1}$$

$$I_2 = \tfrac{1}{8}\int_0^{\lambda_1} (1-\lambda^2)^2(315\lambda^4 - 210\lambda^2 + 15)\,\mathrm{d}\lambda$$

$$= \tfrac{5}{8}(7\lambda_1^9 - 24\lambda_1^7 + 30\lambda_1^5 - 16\lambda_1^3 + 3\lambda_1), \tag{43.2}$$

$$I_3 = \tfrac{1}{16}\int_0^{\lambda_1} (1-\lambda^2)^3(3003\lambda^6 - 3465\lambda^4 + 945\lambda^2 - 35)\,\mathrm{d}\lambda$$

$$= \tfrac{7}{16}(-33\lambda_1^{13} + 162\lambda_1^{11} - 323\lambda_1^9 + 332\lambda_1^7 - 183\lambda_1^5 + 50\lambda_1^3 - 5\lambda_1). \tag{43.3}$$

By means of an integration by parts, like that of (39), it is easy to show that $|I_k| < k+1$. Also, by (26.1) and (26.2)

$$|J_{2k}| = (k+1)^{-1}\,\frac{1\cdot 3\cdot 5\ldots(2k-1)}{2\cdot 4\cdot 6\ldots 2k}, \tag{43.4}$$

so that

$$|J_{2k}I_k| < \frac{1\cdot 3\cdot 5\ldots 2k-1}{2\cdot 4\cdot 6\ldots 2k} < \frac{35}{128}, \quad \text{if} \quad k \geqq 4. \tag{43.5}$$

The ratio of the k'th term in (41) to the main term $4\pi G\varrho s/3$ is thus less than 105/128 $(a/s)^{10} < (10^{-10})$ for $a = 0.5$ cm and $s = 5$ cm. It thus appears reasonable to stop at $k = 3$.

A.6. Corrections from the empty spherical segments

The corrective field $f_{ACA'}$ produced by a spherical segment is to be obtained by considering a disk in it of variable radius $r_a = \xi$, thickness dz, and

$$\mu_d = -\,G\pi\xi^2\varrho\,\mathrm{d}z. \tag{44}$$

We make these substitutions in (30) and integrate the latter over z from $z = R_1$ to $z = R$, where R is the radius of the sphere and $R_1 = \sqrt{(R^2 - a^2)}$. The result is then

$$f_{ACA'} = \int_{R_1}^{R} f_d, \tag{45}$$

where f_d contains dz as a factor, by (44).

Here, however

$$\xi^2 = R^2 - z^2, \tag{46}$$

so that the integration is more difficult. We have for the attractive field produced by the disk on the test object

$$f_d = -\frac{G\pi\varrho(R^2 - z^2)\, s\, \mathrm{d}z}{r_P^3} + G\pi\varrho s(R^2 - z^2)\,\mathrm{d}z \sum_{k=1}^{\infty} J_{2k} \frac{(R^2 - z^2)^k}{r_P^{2k+3}} Q_k(\lambda). \tag{47}$$

This follows from (30), (44), (46), and (32.2), where $Q_k(\lambda)$ is given by (35) and $\lambda \equiv z/r_P$. The J_{2k} are still given by (26.1).

Then

$$\begin{aligned} 4f_{ACA'} = &- 4G\pi\varrho s \int_{R_1}^{R} \frac{R^2 - z^2}{r_P^3}\,\mathrm{d}z \\ &+ 4G\pi\varrho s \sum_{k=1}^{\infty} J_{2k} \int_{R_1}^{R} \frac{(R^2 - z^2)^{k+1}}{r_P^{2k+3}} Q_k(\lambda)\,\mathrm{d}z. \end{aligned} \tag{48}$$

From (27.1) and (46), we now have

$$r_P^2 = R^2 + s^2 - \xi^2 = h_0^2 - \xi^2, \tag{49.1}$$

$$h_0^2 \equiv R^2 + s^2. \tag{49.2}$$

Consider the first integral in (48). If we change to ξ as independent variable, it becomes

$$\int_{R_1}^{R} \frac{R^2 - z^2}{r_P^3}\,\mathrm{d}z = \int_0^a \xi^3 (R^2 - \xi^2)^{-1/2} (h_0^2 - \xi^2)^{-3/2}\,\mathrm{d}\xi \tag{50}$$

$$= \frac{1}{Rh_0^3} \int_0^a \xi^3 \left(1 + \frac{1}{2}\frac{\xi^2}{R^2} + \cdots\right)\left(1 + \frac{3}{2}\frac{\xi^2}{h_0^2} + \cdots\right)\mathrm{d}\xi \tag{50.1}$$

$$= \frac{1}{Rh_0^3}\left[\frac{a^4}{4} + \frac{1}{6}\left(\frac{1}{2R^2} + \frac{3}{2h_0^2}\right)a^6 + \cdots\right]. \tag{50.2}$$

Next consider the integral multiplying J_2. It is

$$N_1 = \int_{R_1}^{R} \frac{(R^2 - z^2)^2}{r_P^5} Q_1(\lambda)\,\mathrm{d}z = \int_{R_1}^{R} \frac{(R^2 - z^2)^2}{r_P^5}\left(\frac{15}{2}\frac{z^2}{r_P^2} - \frac{3}{2}\right)\mathrm{d}z \tag{51}$$

$$= \int_0^a \frac{\xi^5\,\mathrm{d}\xi}{\sqrt{R^2 - \xi^2}\,(h_0^2 - \xi^2)^{5/2}}\left[\frac{15}{2}\frac{(R^2 - \xi^2)}{h_0^2 - \xi^2} - \frac{3}{2}\right]. \tag{51.1}$$

To obtain the term in a^6, we may drop ξ in this integral everywhere except in the main ξ^5 factor. Then

$$N_1 \approx \frac{1}{Rh_0^5}\left(\frac{15}{2}\frac{R^2}{h_0^2} - \frac{3}{2}\right)\frac{a^6}{6} \tag{51.2}$$

$$\approx \frac{a^6}{4}\left[\frac{5R}{(R^2+s^2)^{7/2}} - \frac{1}{R(R^2+s^2)^{5/2}}\right] < \left(\frac{a}{R}\right)^6. \tag{51.3}$$

Here $R \approx 10$ cm, so that if the tunnel diameter is less than 1 cm, $(a/R)^6 < \frac{1}{64}\,10^{-6}$. Also, since $J_2 = \frac{1}{4}$, the factor $4\pi G\varrho s J_2$ is $\frac{3}{4}$ of the main field $4\pi G\varrho s/3$ on the test object. It appears that all the terms containing the J_{2k}'s can be neglected. Also, in (50.2), the term in a^6 is less than $\frac{1}{3}\,(a/R)^6$, so that we can neglect it also.

Thus, closely enough

$$4f_{ACA'} = -\frac{G\pi\varrho a^4}{R(R^2+s^2)^{3/2}}\, s. \tag{52}$$

A.7. Correction arising from the dodecahedral intersection

We have previously counted the dodecahedron three times as a cavity. To correct for this error, we now need to count it twice, as a solid body of density ϱ. To do so, we need to express its gravitational potential in spherical harmonics and then to take its derivative along the axis of a tunnel. If we let the tunnel axes be Ox, Oy, and Oz, the corresponding spherical coordinates r, θ, ϕ are given by

$$x + iy = r\sin\theta\exp i\phi, \tag{53.1}$$

$$z = r\cos\theta. \tag{53.2}$$

If we take the axis of the tunnel to be Oz, the tesseral and sectorial harmonics will have factors $\sin^m\theta\,(m \geqq 1)$, $\sin\theta$ vanishing on the axis. Differentiation of the potential with respect to $z = r$ leaves these factors intact. Thus we need consider only zonal harmonics, so that

$$V_{\text{dod}} = -\frac{\mu}{r}\left[1 - \sum_{k=1}^{\infty}\frac{C_{2k}}{r^{2k}}P_{2k}\left(\frac{z}{r}\right)\right]. \tag{54}$$

Here

$$\mu = GM_{\text{dod}}, \tag{55}$$

where M_{dod} is the mass of the solid dodecahedron and

$$M_{\text{dod}}C_{2k} = -\int r^{2k}\varrho P_{2k}\left(\frac{z}{r}\right)d\tau, \tag{56}$$

integrated over the dodecahedron. Odd zonal harmonics do not appear in (54), because of symmetry with respect to the plane $z = 0$.

By (54), the z-component of the attractive field

$$f_z = \frac{\partial V_{\text{dod}}}{\partial z}, \tag{57}$$

is given along the axis of the tunnel by

$$f_z = \frac{\mu z}{r^3} - \mu \sum_1^\infty C_{2k} \left[(2k+1)\, r^{-2k-3} z \right]. \tag{58}$$

Along the axis of the tunnel $z = r = s$ and the attractive field becomes

$$f_{\text{dod}} = \frac{\mu}{s^2} \left[1 - \sum_1^\infty (2k+1) \frac{C_{2k}}{s^{2k}} \right] \tag{59.1}$$

$$= \frac{\mu}{s^2} \left[1 - \frac{3C_2}{s^2} - \frac{5C_4}{s^4} - \frac{7C_6}{s^6} + \cdots \right]. \tag{59.2}$$

To evaluate M_{dod} and the C's, we need to evaluate the integrals (56), where $C_0 = -1$. With the aid of Figure 4 (and a lathe-produced model of the dodecahedron!), one sees that the volume Ω of the dodecahedron is 24 times the volume of the solid bounded by the planes OXV, OYV, OXY, and the cylinder $x^2 + y^2 = a^2$. Or, it is 48 times the volume bounded by the planes OXV, OXQ, OVQ, and the cylinder $x^2 + y^2 = a^2$.

With cylindrical coordinates R, z, ϕ, we have

$$x = R \cos \varphi, \qquad y = R \sin \varphi, \tag{60}$$

$$d\tau = R \, dR \, d\phi \, dz. \tag{61}$$

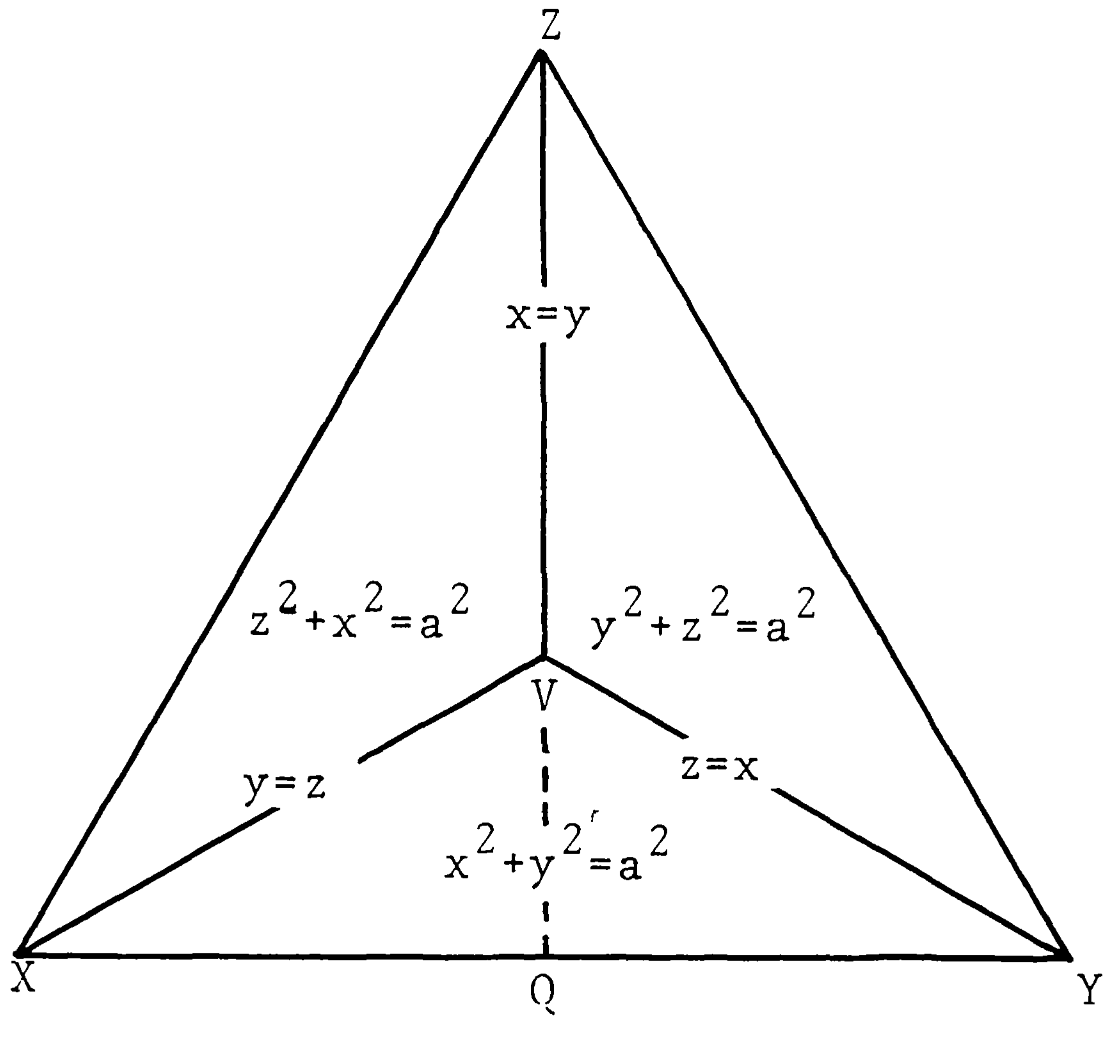

Fig. 4. Plane projection of octant of dodecahedral intersection, looking along VO, where O is the center of the sphere and V, at the center of XYZ, is the vertex in the octant. OX, OY, OZ are respectively along the axes of the x, y, and z tunnels, each being of length a. $|OV| = a\sqrt{\frac{3}{2}}$; VZX = Plane projection of part of cylinder $z^2 + x^2 = a^2$; VXY = Plane projection of part of cylinder $x^2 + y^2 = a^2$; VYZ = Plane projection of part of cylinder $y^2 + z^2 = a^2$.

The limits of integration are then:

$$z=0 \quad \text{to the plane} \quad y=z \quad \text{i.e.} \quad z=0 \quad \text{to} \quad z=R\sin\phi\,, \tag{62.1}$$

$$\phi=0 \quad \text{to} \quad \pi/4\,, \tag{62.2}$$

$$R=0 \quad \text{to} \quad a\,. \tag{62.3}$$

Thus

$$\Omega = 48\int_0^a R\,\mathrm{d}R\int_0^{\pi/4}\mathrm{d}\phi\int_0^{R\sin\phi}\mathrm{d}z = 16\left(1-\tfrac{1}{2}\sqrt{2}\right)a^3\,. \tag{63}*$$

Thus

$$\mu = G\Omega\varrho = 16G\varrho\left(1-\tfrac{1}{2}\sqrt{2}\right)a^3\,. \tag{64}$$

To evaluate the coefficients C_{2k}, note that

$$-\frac{M_{\mathrm{dod}}C_{2k}}{\varrho} = \int r^{2k}P_{2k}\left(\frac{x}{r}\right)\mathrm{d}\tau = \int r^{2k}P_{2k}\left(\frac{y}{r}\right)\mathrm{d}\tau = \int r^{2k}P_{2k}\left(\frac{z}{r}\right), \tag{65}$$

because of the symmetry of the dodecahedron with respect to the three tunnel axes. Thus

$$-\frac{3M_{\mathrm{dod}}C_{2k}}{\varrho} = \int r^{2k}\left[P_{2k}\left(\frac{x}{r}\right)+P_{2k}\left(\frac{y}{r}\right)+P_{2k}\left(\frac{z}{r}\right)\right]\mathrm{d}\tau\,. \tag{66}$$

The value of this integral is 8 times that over one octant or 24 times that over the solid $OXVY$. If we put

$$Q \equiv r^{2k}\left[P_{2k}\left(\frac{x}{r}\right)+P_{2k}\left(\frac{y}{r}\right)+P_{2k}\left(\frac{z}{r}\right)\right]. \tag{67}$$

then

$$-\,3M_{\mathrm{dod}}C_{2k}/\varrho = 24\,(I_1+I_2)\,, \tag{68}$$

where

$$I_1 = \int_0^{\pi/4}\mathrm{d}\phi\int_0^a R\,\mathrm{d}R\int_0^{R\sin\phi} Q\,\mathrm{d}z\,, \tag{69.1}$$

$$I_2 = \int_{\pi/4}^{\pi/2}\mathrm{d}\phi\int_0^a R\,\mathrm{d}R\int_0^{R\cos\phi} Q\,\mathrm{d}z\,. \tag{69.2}$$

In (69.2) z goes from 0 to the plane $z=x=R\cos\phi$. Now in (69.2), if we introduce the new variable

$$\phi' = \frac{\pi}{2}-\phi\,, \tag{70}$$

then

$$I_2 = \int_0^{\pi/4}\mathrm{d}\phi'\int_0^a R\,\mathrm{d}R\int_0^{R\sin\phi'} Q'\,\mathrm{d}z\,. \tag{71}$$

* See note added in proof, p. 253.

Here, with $r^2 = R^2 + z^2$,

$$Q' = r^{2k}\left[P_{2k}\left(\frac{R\sin\phi'}{r}\right) + P_{2k}\left(\frac{R\cos\phi'}{r}\right) + P_{2k}\left(\frac{z}{r}\right)\right]. \tag{72}$$

Since ϕ' is a dummy variable, we may replace it by ϕ. Then

$$Q' = Q, \tag{73}$$

and

$$I_2 = I_1. \tag{74}$$

Then, from (67), (68), (69.1), (71), and (74), it follows that

$$-\frac{M_{\text{dod}}C_{2k}}{\varrho} = 16\int_0^{\pi/4} d\phi \int_0^a R\,dR \int_0^{R\sin\phi} r^{2k} \times \left[P_{2k}\left(\frac{x}{r}\right) + P_{2k}\left(\frac{y}{r}\right) + P_{2k}\left(\frac{z}{r}\right)\right]_1 dz. \tag{75}$$

The bracket vanishes for $k=1$. For C_4, we have $k=2$. With use of

$$8P_4(\lambda) = 35\lambda^4 - 30\lambda^2 + 3, \tag{76}$$

and (60), we then obtain

$$\tfrac{8}{7}r^4[\quad]_1 = 5R^4(\cos^4\phi + \sin^4\phi) + 5z^2 - 3(R^2 + z^2)^2 \tag{77}$$

$$= 2R^4 + 2z^4 - 6R^2z^2 - 10R^4\sin^2\phi + 10R^4\sin^4\phi, \tag{78}$$

so that, finally,

$$M_{\text{dod}}C_4 = \frac{\varrho a^7}{75}(41\sqrt{2} - 68). \tag{79}$$

From (59.2), (64), and (79) it follows that the correction $2f_{\text{dod}}$ for the intersection is

$$2f_{\text{dod}} = 32G\varrho(1 - \tfrac{1}{2}\sqrt{2})\frac{a^3}{s^2} - \frac{2G\varrho}{15}(41\sqrt{2} - 68)\frac{a^7}{s^6} + \cdots. \tag{80}$$

The ratio of this to

$$f_{\text{main}} = \frac{4\pi}{3}G\varrho s \tag{21}$$

is

$$2f_{\text{dod}}/f_{\text{main}} = 2.237\,54\left(\frac{a}{s}\right)^3 + 0.318\,859\left(\frac{a}{s}\right)^7. \tag{81}$$

For $a=0.5$ cm and $s=5$ cm, the monopole part amounts to more than 2 parts in a thousand. The second or 16-pole part amounts to only 3 parts in 10^8.

A.8. THE TOTAL ATTRACTIVE FIELD ON A TEST OBJECT IN A TUNNEL

From Equations (18), (41), (52), and (80), the total attractive field f_a on a test object on the axis of a tunnel, at distance s from the center of the sphere, is given with sufficient accuracy by

$$f_a = \frac{\pi G\varrho}{6s^2}(l_2 - l_1)^3 + \frac{4G\pi a^2\varrho}{s}\left[-\lambda_1 + \sum_{k=1}^{3} J_{2k} I_k \left(\frac{a}{s}\right)^k\right]$$
$$- \frac{G\pi a^4 \varrho s}{R(R^2+s^2)^{3/2}} + 32G\varrho\left(1 - \tfrac{1}{2}\sqrt{2}\right)\frac{a^3}{s^2} - \frac{2G\varrho}{16}\left(41\sqrt{2} - 68\right)\frac{a^7}{s^6}. \tag{82}*$$

Here a is the radius of a tunnel, ϱ and R respectively the density and radius of the sphere, and l_1 and l_2 are given by (17), where $R_1^2 = R^2 - a^2$. Also $\lambda_1 = R_1(R_1^2 + s^2)^{-1/2}$, J_{2k} is given by (26), and the I_k, $k = 1 \ldots 3$, by (43).

The correction $f_{G\sigma}^T$ in Equations (137) through (140) in the main text is then given by

$$f_{G\sigma}^T = f_a - \frac{4\pi}{3} G\varrho s. \tag{83}$$

Appendix B: Vestigial Tunnels

Instead of drilling three perpendicular tunnels in the sphere, one might proceed as follows in achieving the balance of gravitational force against centrifugal force that has been described for measuring G.

Suppose we are given the completed sphere, accurately machined to be a good geometrical sphere. Since there will be some inhomogeneities, its center of mass will not quite coincide with its geometrical center. Determine the position of its center of mass, perhaps by means of physical pendulum experiments, and indicate this position with reference to some fiducial marks on the sphere.

Since the sphere will not be dynamically perfect, it will have unique principal axes. Determine these principal axes, perhaps by means of torsion pendulum experiments, and mark the six points at which they intersect the sphere. Also determine the three principal moments of inertia, viz. $A < B < C$. (The center of mass will have to be at the intersection of the principal axes.)

Instead of drilling three tunnels, simply attach six short thin hollow cylinders, each perhaps an inch long, on the outside of the sphere, at the exit points of the principal axes. These cylinders may be of some material much less dense than tungsten; they would contain the suspension systems.

Use the inertia reaction wheels, servomechanisms, and suspension systems, so that on rotation of the sphere, test objects in three of these 'vestigial' tunnels will remain fixed (or almost so) on the axes. The servos will now not only have to prevent extraction but also to prevent collision of the test objects with the sphere. Since the 'tunnels' are now much more accessible, however, it may be easier to design the necessary suspension systems.

* See note added in proof, p. 253.

The expression for the gravitational field f_σ on a test particle is now much simpler. With origin at the center of mass and x, y, z taken along the principal axes, let s be the distance from the center of mass and θ and λ be the corresponding colatitude and longitude. Then

$$x = s \sin\theta \cos\lambda, \tag{1.1}$$

$$y = s \sin\theta \sin\lambda, \tag{1.2}$$

$$z = s \cos\theta. \tag{1.3}$$

If M is the mass of the sphere and R its radius, the gravitational potential V outside the sphere is given by

$$V = -\frac{GM}{s}\Bigg[1 - \sum_{n=1}^{\infty}\left(\frac{R}{s}\right)^n J_n P_n(\cos\theta) + \sum_{n=1}^{\infty}\left(\frac{R}{s}\right)^n P_n^m(\cos\theta)\,(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda)\Bigg]. \tag{2}$$

Suppose we truncate this series at $n=2$. With the origin at the center of mass, we have

$$J_1 = C_{1,1} = S_{1,1} = 0. \tag{3}$$

Since the axes are principal axes, we also have

$$C_{2,1} = S_{2,1} = S_{2,2} = 0. \tag{4}$$

We then obtain

$$V = -\frac{GM}{s}\left[1 - \frac{R^2}{s^2} J_2\left(\frac{3}{2}\frac{z^2}{s^2} + \frac{1}{2}\right) + \frac{3R^2}{s^2}\sin^2\theta C_{2,2}\cos 2\lambda + \cdots\right]. \tag{5}$$

Using

$$\sin^2\theta \cos 2\lambda = \frac{x^2 - y^2}{s^2}, \tag{6}$$

we find

$$V = -\frac{GM}{s} - \frac{GMR^2 J_2}{2s^3} + \frac{3}{2}\frac{GMR^2 J_2 z^2}{s^5} - \frac{3GMR^2 C_{2,2}}{s^5}(x^2 - y^2) + \cdots. \tag{7}$$

In the x-tunnel the attractive field $f_x = \partial V/\partial x$ with $y=z=0$; in the y tunnel $f_y = \partial V/\partial y$ with $z=x=0$; in the z-tunnel $f_z = \partial V/\partial z$ with $x=y=0$.

On carrying out this procedure and using

$$J_2 = \frac{C - \frac{1}{2}(A+B)}{MR^2}, \tag{8}$$

$$C_{2,2} = \frac{B - A}{4MR^2}, \tag{9}$$

we find

$$f_x = \frac{GM}{x^2} + \frac{3}{2}\frac{G(B+C-2A)}{x^4} + \cdots \tag{10.1}$$

$$f_y = \frac{GM}{y^2} + \frac{3}{2}\frac{G(C+A-2B)}{y^4} + \cdots \tag{10.2}$$

$$f_z = \frac{GM}{z^2} + \frac{3}{2}\frac{G(A+B-2C)}{z^4} + \cdots \tag{10.3}$$

there being a cyclic symmetry, as expected.

For a perfectly homogeneous sphere, of course, $A = B = C$, so that the above quadrupole terms, which can be measured, would give some correction for inhomogeneities. A good-sized hole near the surface and near a tunnel would still cause trouble, but perhaps could be detected during construction of the sphere. Equations (10) are about as far as one can go and still determine coefficients in (2) experimentally.

The principal axes as thus determined may not be exact diameters of the sphere, since the sphere's center of mass may not exactly coincide with its geometrical center. In estimating the corrective field produced by the short external tunnels, however, I shall neglect this difficulty and assume that the axis of a tunnel lies along a diameter.

Another way of handling the matter would be to place the tunnels at the ends of three geometrically true diameters, not necessarily very near the principal axes. One could then use (2), truncated at $n = 2$, in place of (10) for the main field on a test object. One would then not place the coefficients J_1, $C_{1,1}$, $S_{1,1}$, $C_{2,1}$, and $S_{2,2}$ equal to zero, but would determine both these and $C_{2,2}$ from the above mentioned laboratory experiments.

Let each external tunnel be a very thin hollow cylinder, without end walls, of length $2b$, radius a, and surface density σ. The attractive field on a test object on the axis of a tunnel, produced by its own tunnel, is then

$$f_1 = 2\pi G a\sigma\left[\{a^2 + (2b + R - s)^2\}^{-1/2} - \{a^2 + (s - R)^2\}^{-1/2}\right]. \tag{11}$$

Here R is the radius of the sphere and s is the distance of the test object from its center. If $s = R + b$, the field f_1 vanishes. If $s = R + 2b$, however,

$$f_{1_{\max}} = 2\pi G a\sigma\left[a^{-1} - (a^2 + 4b^2)^{-1/2}\right]. \tag{12}$$

Let $\sigma = \varrho' t'$, where ϱ' is the density of the wall and t' its thickness. The main field

$$f_{\text{main}} \approx \frac{4\pi}{3} G\varrho R^3/s^2 \approx \frac{4\pi}{3} G\varrho R. \tag{13}$$

The ratio of $f_{1_{\max}}$ to f_{main} is then, approximately,

$$\frac{f_{1_{\max}}}{f_{\text{main}}} = \frac{3}{2}\frac{\varrho'}{\varrho}\frac{t'}{R}\left[1 - a(a^2 + 4b^2)^{-1/2}\right]. \tag{14}$$

If $\varrho' = 1$ gm cm^{-3}, $\varrho = 20$ gm cm^{-3}, $t' = 1$ mm, $R = 10$ cm, $a = 0.5$ cm, and $b = 1$ cm, then

$$\frac{f_{1\max}}{f_{\text{main}}} \approx (0.57) \times 10^{-3}, \tag{15}$$

or about 6 parts in ten thousand. This is a large correction. To keep it small, it would be desirable to servo the test object to remain close to the center of the vestigial tunnel.

The attractive field f_2, produced on a test object by the coaxial tunnel on the other side of the sphere, is given by

$$f_2 = 2\pi G a\sigma \left[\{a^2 + (R + s)^2\}^{-1/2} - \{a^2 + (2b + R + s)^2\}^{-1/2}\right] \tag{16}$$

$$\approx \frac{\pi G a b \sigma}{R^2}. \tag{16.1}$$

The ratio of this to $f_{\text{main}} \approx (4\pi/3)\, G\varrho R$ is

$$\frac{f_2}{f_{\text{main}}} \approx \frac{3}{4} \frac{bat'}{R^3} \frac{\varrho'}{\varrho}. \tag{17}$$

With $b = 1$ cm, $a = 0.5$ cm, $t' = 0.1$ cm, $R = 10$ cm, and $\varrho'/\varrho = \frac{1}{20}$, this amounts to 2 parts in a million, so that the effect is marginal.

The attractive field f_3, produced by the four perpendicular tunnels, is four times the attractive axial field produced at P in Figure 5 by the tunnel $ABCD$. To calculate this field, we may expand the potential of $ABCD$ in zonal harmonics and then take its derivative along the axis of T_1.

If we take an origin at the center of one of these other cylindrical tunnels, with z-axis along the axis of such a cylinder, then if a is its radius, $2b$ its length, and σ its surface density, we have

$$\mu = 4\pi a b \sigma G. \tag{18}$$

The potential at distance r from the center and coordinate z is then expressible in zonal harmonics as

$$V = -\frac{\mu}{r} + \mu \sum_{k=1}^{\infty} C_{2k} r^{-2k-1} P_{2k}\left(\frac{z}{r}\right). \tag{19}$$

Odd harmonics do not appear. The attractive field along the axis is given by calculating $f = \partial V/\partial z$ and putting $z = r$, the result being

$$f = \frac{\mu}{r^2} - \mu \sum_{k=1}^{\infty} (2k + 1)\, C_{2k} r^{-2k-2}. \tag{20}$$

To find the coefficients C_{2k} we resort to a direct integration for the field along the axis

$$f = 2\pi G a\sigma \int_{-b}^{b} \frac{(r - z)\, \mathrm{d}z}{[a^2 + (r - z)^2]^{3/2}}. \tag{21}$$

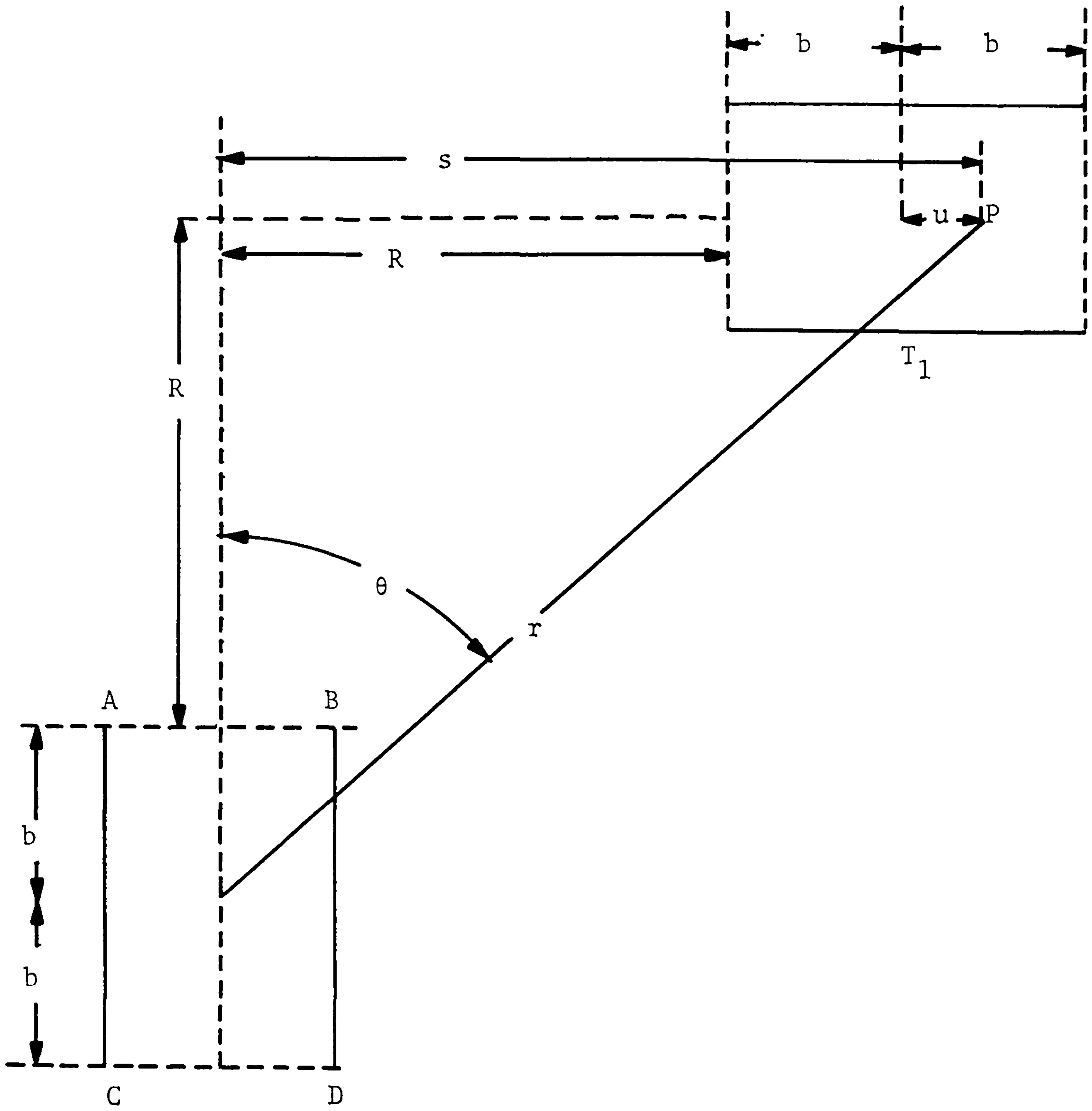

Fig. 5. P = test object in a vestigial tunnel; $s = R + b + u$, Attractive field on P, produced by the four perpendicular tunnels, is four times that produced by the hollow tunnel $ABCD$.

The result is

$$f = \frac{GM}{2b}\left[\{a^2 + (r-b)^2\}^{-1/2} - \{a^2 + (r+b)^2\}^{-1/2}\right]. \tag{22}$$

If we put

$$h \equiv (a^2 + b^2)^{1/2} r^{-1} \qquad \lambda = b\,(a^2 + b^2)^{-1/2}, \tag{23}$$

we obtain

$$f = \frac{GM}{2br}\left[(1 - 2\lambda h + h^2)^{-1/2} - (1 + 2\lambda h + h^2)^{-1/2}\right] \tag{24}$$

$$= \frac{GM}{br} \sum_{k=0}^{\infty} h^{2k+1} P_{2k+1}(\lambda), \tag{25}$$

or

$$f = \frac{\mu}{r^2} \sum_{k=0}^{\infty} \frac{(a^2 + b^2)^{k+1/2}}{br^{2k}} P_{2k+1}\left(\frac{b}{\sqrt{a^2 + b^2}}\right). \tag{26}$$

Comparison of (20) and (26) then yields

$$C_{2k} = -\frac{(a^2+b^2)^{k+1/2}}{(2k+1)\,b} P_{2k+1}\left(\frac{b}{\sqrt{a^2+b^2}}\right). \tag{27}$$

As a check, note that

$$C_2 = a^2/2 - b^2/3 = a^2/2,$$

for a ring of radius a, a correct result.

To find the attractive field f_3, refer to Figure 5. If V is the potential produced by the hollow cylinder $ABCD$, we see that

$$f_3 = 4\frac{\partial V}{\partial s}. \tag{28}$$

where we must place

$$z = R + b \tag{29}$$

at the end of the calculation. On applying (19) for V and using (18) and

$$r^2 = z^2 + s^2, \tag{30}$$

we find

$$f_3 = \frac{4\mu s}{r^3} - \frac{4\mu s}{r^3}\sum_{k=1}^{\infty}\frac{C_{2k}}{r^{2k}}\left[(2k+1)\,P_{2k}\left(\frac{z}{r}\right) + \frac{z}{r}P'_{2k}\left(\frac{z}{r}\right)\right] \tag{31}$$

$$= \frac{4\mu s}{r^3} - \frac{4\mu s}{r^3}\sum_{k=1}^{\infty}\frac{C_{2k}}{r^{2k}} P'_{2k+1}\left(\frac{z}{r}\right), \tag{32}$$

by (32.2) and (35) of Appendix A.

On inserting (29) and (27) into (32), we then obtain

$$f_3 = \frac{4\mu s}{r}\left[1 - \frac{\sqrt{a^2+b^2}}{b}\sum_{k=1}^{\infty}\frac{(a^2+b^2)^{2k}}{(2k+1)\,r^{2k}} P_{2k+1}\times\left(\frac{b}{\sqrt{a^2+b^2}}\right)\right.$$
$$\left. P'_{2k+1}\left(\frac{R+b}{r}\right)\right], \tag{33}$$

where

$$r = [(R+b)^2 + s^2]^{1/2}.$$

Comparison with

$$f_{\text{main}} = \frac{4\pi}{3}\varrho\frac{GR^3}{s^2} \approx \frac{4\pi}{3}\varrho G s \tag{34}$$

gives

$$\frac{f_3}{f_{\text{main}}} \approx \frac{12ab\sigma}{\varrho r^3}\left[1 - \frac{(a^2+b^2)^{3/2}}{3br^2} P_3\left(\frac{b}{\sqrt{a^2+b^2}}\right) P_3'\left(\frac{R+b}{r}\right) + \cdots\right]. \tag{35}$$

Here

$$r = [(R+b)^2 + s^2]^{1/2} \approx R\sqrt{2}, \tag{36}$$

$$\sigma = \varrho' t', \tag{37}$$

so that

$$\frac{12ab\sigma}{\varrho r^3} \approx \frac{12abt'}{2\sqrt{2}R^3}\frac{\varrho'}{\varrho} \tag{38}$$

$$\approx \frac{3\,(0.5)\,(1)\,(0.1)\sqrt{2}}{10^3}\frac{1}{20} \approx 10^{-5}. \tag{39}$$

Thus the first term is appreciable. The second term inside the bracket is

$$\frac{a^2+b^2}{3br^2} P_3\left(\frac{b}{\sqrt{a^2+b^2}}\right) P_3'\left(\frac{R+b}{r}\right) = \frac{2a^2-3b^2}{4r^2}\left[5\left(\frac{R+b}{r}\right)^2 - 1\right]. \tag{40}$$

With $r \approx R\sqrt{2}$, $a=0.5$ cm, $b=1$ cm, and $R=10$ cm, it becomes of order $-5(10^{-3})$. Thus the ratio of the full second term in (35) to the main field is of order $5(10^{-8})$ and is thus negligible. Effectively, then,

$$f_3 = \frac{4\mu s}{r^3} = \frac{16\pi ab\sigma Gs}{[(R+b)^2+s^2]^{3/2}}. \tag{41}$$

On assembling Equations (10), (11), (16), and (41), we find for the total attractive field on a test object in the vestigial tunnel along OX:

$$\begin{aligned} f_x = {} & \frac{GM}{x^2} + \frac{3}{2}\frac{B+C-2A}{x^4} \\ & + 2\pi Ga\sigma\left[\frac{1}{\sqrt{a^2+(2b+R-x)^2}} - \frac{1}{\sqrt{a^2+(x-R)^2}}\right] \\ & + 2\pi Ga\sigma\left[\frac{1}{\sqrt{a^2+(R+x)^2}} - \frac{1}{\sqrt{a^2+(2b+R+x)^2}}\right] \\ & + \frac{16\pi Gab\sigma x}{[(R+b)^2+x^2]^{3/2}}. \end{aligned} \tag{42}$$

For the y-tunnel, replace $B+C-2A$ by $C+A-2B$ and for the z-tunnel by $A+B-2C$.

In (42) the first two terms arise from the (imperfect) sphere, the third term from the test object's own vestigial tunnel, the fourth term from the distant tunnel on the same axis, and the fifth term from the four perpendicular vestigial tunnels.

With the method of vestigial tunnels, the above f_x is to replace the sum $4\pi G\varrho x + f^T_{Gx}$ in Equations (138), (139), and (140) of the main part of the paper.

This method of 'vestigial tunnels' would appear to be much simpler than the method first considered, both in construction of the equipment and in reduction of the data. It also preserves the advantage of Wilk's original proposal in those cases where it pays to sum over the tunnels, i.e., when the sphere is in free fall. This follows because summing over the three tunnels always eliminates most of the gravity gradient effects. On summation over the tunnels, these depend, to a high accuracy, only on the Laplacian of the gravitational potential of matter external to the sphere, which always vanishes at the sphere's center of mass.

Finally, the field given by (42) is independent of thermal expansion and contraction of the sphere. This statement does not hold for Equation (A82) for the field inside an interior tunnel, unless one knows how much to correct the density ϱ in that expression. Another way of saying the same thing is that, except for the weak dependence of the moment of inertia differences in (42) on the temperature, the fundamental unknown to be determined in this method of vestigial tunnels is G and not $G\varrho$.

Note Added in Proof

Dr. Reasenberg has pointed out that the complete intersection of the three cylinders is not simply the region common to all three cylinders, but must also include the three regions common to only two of them. Suppose Ω' is the volume of intersection of two of the cylinders of radius a and Ω the volume common to all three. Then, as one sees most easily from a Venn diagram, the volume $3(\Omega' - \Omega)$ has been counted twice as a cavity and the volume Ω three times.

To correct the error, one must count $3(\Omega' - \Omega)$ once as a solid body and Ω twice. Thus one must replace 2Ω in (63) of Appendix A by

$$V = 3(\Omega' - \Omega) + 2\Omega = 3\Omega' - \Omega$$

Now

$$\Omega' = \int_0^a \mathrm{d}z \int_0^{\sqrt{a^2 - z^2}} \mathrm{d}y \int_0^{\sqrt{a^2 - z^2}} \mathrm{d}x = \frac{16}{3} a^3 .$$

With $\Omega = 16\,(1 - \frac{1}{2}\sqrt{2})\,a^3$, we then find that in (82) of Appendix A, the term

$$32G\rho\,(1 - \tfrac{1}{2}\sqrt{2})\frac{a^3}{s^2}2$$

is to be replaced by

$$8\sqrt{2}G\rho\frac{a^3}{s^2}2$$

We may presumably neglect the corresponding change in the term in a^7/s^6, corresponding to the fourth zonal harmonic of the intersection.

References

Berman, D. and Forward, R. L.: 1968, *Exploitation of Space for Experimental Research*, Vol. 24. Advances in the Astronautical Sciences, American Astronautical Society.

Hildebrand, F. B.: 1964, *Advanced Calculus for Applications*, Prentice-Hall, Inc., Englewood Cliffs, N.J., p. 348.

Jacchia, L. G.: 1969, *Annals of the IQSY*, Vol. 5, Solar-Terrestrial Physics: Terrestrial Aspects, MIT Press, Cambridge, Mass., pp. 323–39.

United States Standard Atmospheric Supplements 1966, Sponsored by ESSA, NASA, and the U.S. Air Force, pp. 220–89.

Wilk, L. S.: 1969, 'A Gravitational Experiment in Space', unpublished communication.