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Anelasticity in *G* experiments

Kazuaki Kuroda

ICRR, University of Tokyo, 3-2-1, Midoricho, Tanashi, Tokyo, 188-8502, Japan

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Abstract. We use the time-of-swing method to investigate the systematic error in the determination of the gravitational constant G, due to the anelasticity of torsion fibre.

Keywords: Newtonian gravitational constant, anelasticity, torsion balance, precision measurement

1. Introduction

The Newtonian gravitational constant has been measured precisely by using torsion balances with the time-of-swing method, which converts the weak gravitational force into a change of period. Since the resolution of time is much better than that of force, this method has been used widely in the determination of G. Although the magnitude of the gravitational force is extremely small compared with the electromagnetic force, one need not worry about the lack of sensitivity, provided that one adopts a torsion balance with a long swing period. To increase the torque sensitivity the torsion spring can be made weaker. However, this will sacrifice the long-term stability of the torsion fibre.

From the time of Cavendish, it had been assumed that the spring constant was exactly that—a constant. Cavendish measured the spring constant by knowing the resonance period of the balance in his static torque measurement. In 1995, this author carried out an intensive study on suspension fibres for laser-interferometric gravitational wave detectors and pointed out that this constancy might not be valid due to the anelasticity of the torsion fibre [1]. If the damping of the fibre obeys the law of internal structure damping, the estimation of thermal noise differs drastically from that of velocity damping.

Several experiments [2–5] support the structure-damping model. Two experiments confirmed the change of the spring constant of tungsten fibre [6, 7]. Although the magnitude of this anelasticity effect on the change of spring constant needs more precise investigation, one can estimate its effect by using the Kramers–Kronig relation and knowledge of the mechanical quality factor (Q value) of the torsion spring. After a brief review of the history of G determination, this paper discusses the effect of anelasticity on the measurement of the Newtonian gravitational constant.

2. The history of measuring the gravitational constant *G*

Fundamental physical constants are systematically adjusted and summarized, according to the recommendations of

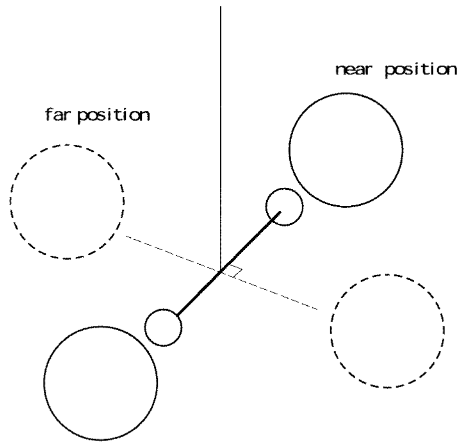

Figure 1. The time-of-swing method of determination of G using a torsion balance with two attracting masses. If the attracting masses are along the line of the axis of the balance dumbbell, the effective spring constant of the torsion fibre increases. However, if the two masses are orthogonal to the line of the axis, the effective spring constant decreases. The difference of the squares of these frequencies gives information on the magnitude of G.

CODATA. Since the gravitational constant G is unrelated to other physical constants, this adjustment does not contribute to an increase in accuracy. The 1990 CODATA recommendation retained the 1986 value, that is

$$G = (6.672\,59 \pm 85) \times 10^{-11}$$

(in SI units). This value was mainly due to the work of Luther and Towler [8], who adopted the dynamic method. As shown in figure 1, if the attracting masses are arranged parallel to the axis of the balance, an extra returning torque is produced, which increases the apparent torsion constant. If, however, those attracting masses are orthogonal to the balance axis, the extra torque applied to the balance tends to decelerate the swing, which reduces the torsion spring constant. The difference of the squares of these swing frequencies reflects the gravitational torque.

In this experiment, the error arising from the measurement of a 5 g dumbbell amounted to a 48 ppm error in the determination of G and the timing error produced

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Table 1. Measured data for six dumbbells.

Dumbbell	Moment of inertia (kg m^2)	Frequency (Hz)	Spring constant (10^{-7} N m rad^{-1})	Q
1	3.3217×10^{-5}	1.3591×10^{-2}	2.422	3200
2	5.3947×10^{-5}	1.0664×10^{-2}	2.422	3500
3	7.8836×10^{-5}	8.8213×10^{-3}	2.422	3200
4	1.1092×10^{-4}	7.4352×10^{-3}	2.421	3300
5	2.4712×10^{-4}	4.9798×10^{-3}	2.419	3100
6	4.4743×10^{-4}	3.7013×10^{-3}	2.420	3300

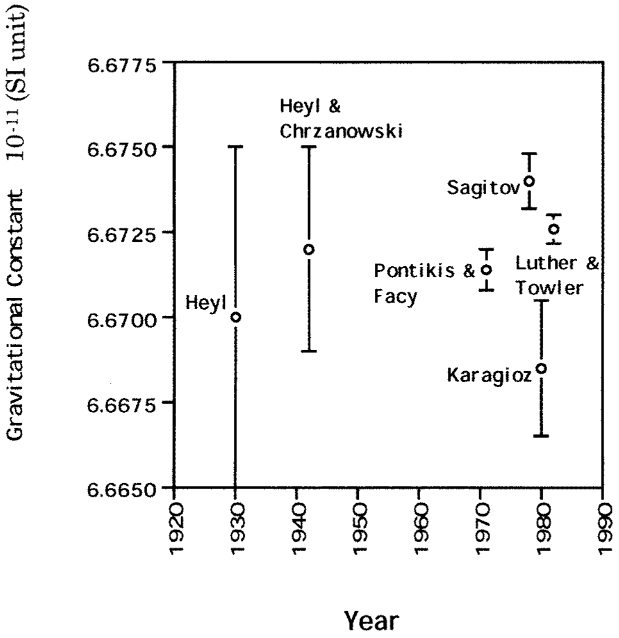

Figure 2. Experiments recommended for values of CODATA in 1973, 1986 and 1990. All experiments adopted the time-of-swing method apart from those by Pontikis and Facy.

a 40 ppm error in G. The results of several experiments that have contributed to the determination of G are plotted in figure 2. The CODATA committee had used the Heyl and Chrzanowski data [9] for their recommendation before 1973. As well as those of Luther and Towler [8], the experiments by both Pontikis and Sagitov were considered in 1986. However, the former had obtained a systematic change of values for three period measurements and the latter had insufficient analysis of systematic error. Therefore, these data were not used in the determination of G in 1986. In 1986, the experiment by Karagioz was also nominated, but was judged to contain unidentified systematic error(s) because its value was far from the average of the others. All of these experimenters [10] adopted the time-of-swing method, except Pontikis, who utilized a resonance technique.

3. Anelasticity of fibre

The time-of-swing method assumes constancy of the torsion spring constant. However, recent studies of suspension fibres in the field of gravitational wave detection show that this is not a valid assumption.

Quinn *et al* [2] measured the anelasticity of a Cu–Be strip, applying this to a rigid pendulum. They provided a schematic model of the anelasticity, shown in figure 3, in which an infinite number of Maxwell units, having a range of relaxation times, are placed in parallel with the main spring. The result was in good agreement with the anelasticity behaviour.

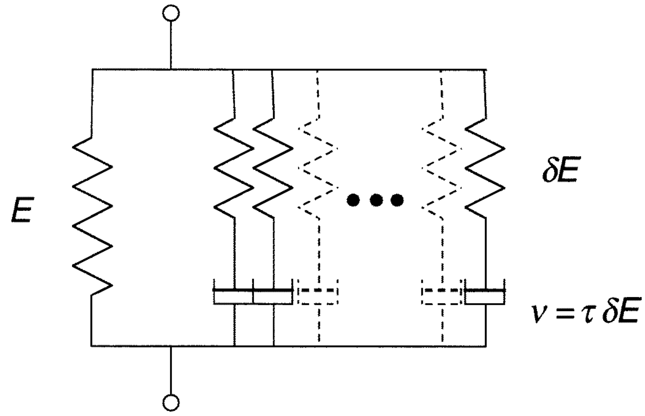

Figure 3. The anelasticity model of the spring constant of a fibre. There are many Maxwell units in parallel with an ideal spring as given by Quinn *et al*.

Saulson *et al* [3] also tested the nature of the anelasticity using an inverted pendulum and confirmed its correctness. This effect is commonly seen in materials science and the phenomenon is widely known in the field of engineering. Its empirical characteristic was reviewed and formalized by Saulson [11], who represented a spring constant by a complex number, $k_r(\omega) + \mathrm{i}k_i(\omega)$. These experiments suggest that the loss represented by the imaginary term is constant with frequency. Since measurements are limited to a finite frequency range, we do not know whether the imaginary part is constant outside the measured region. However, if we apply the Kramers–Kronig relation, we obtain a frequency change of its real part with frequency under the assumption of a constant imaginary part. To show this, we introduce the constancy of the imaginary part using the small positive numbers α and ϵ as follows. If $\omega \geqslant 0$,

$$k_i(\omega) = \epsilon\omega^\alpha$$

and, if $\omega \leqslant 0$,

$$k_i(\omega) = -\epsilon(-\omega)^\alpha.$$

It is practical to use the above equation. From the Kramers–Kronig relation, the real part $k_r(\omega)$ is $2\epsilon\omega^\alpha/(\pi\alpha)$. Since $k_i(\omega)/k_r(\omega)$ is the inverse of the mechanical quality factor Q, $\alpha = 2/(\pi Q)$. From this equation, one can calculate the frequency change of the spring constant knowing its mechanical loss, if the system obeys the

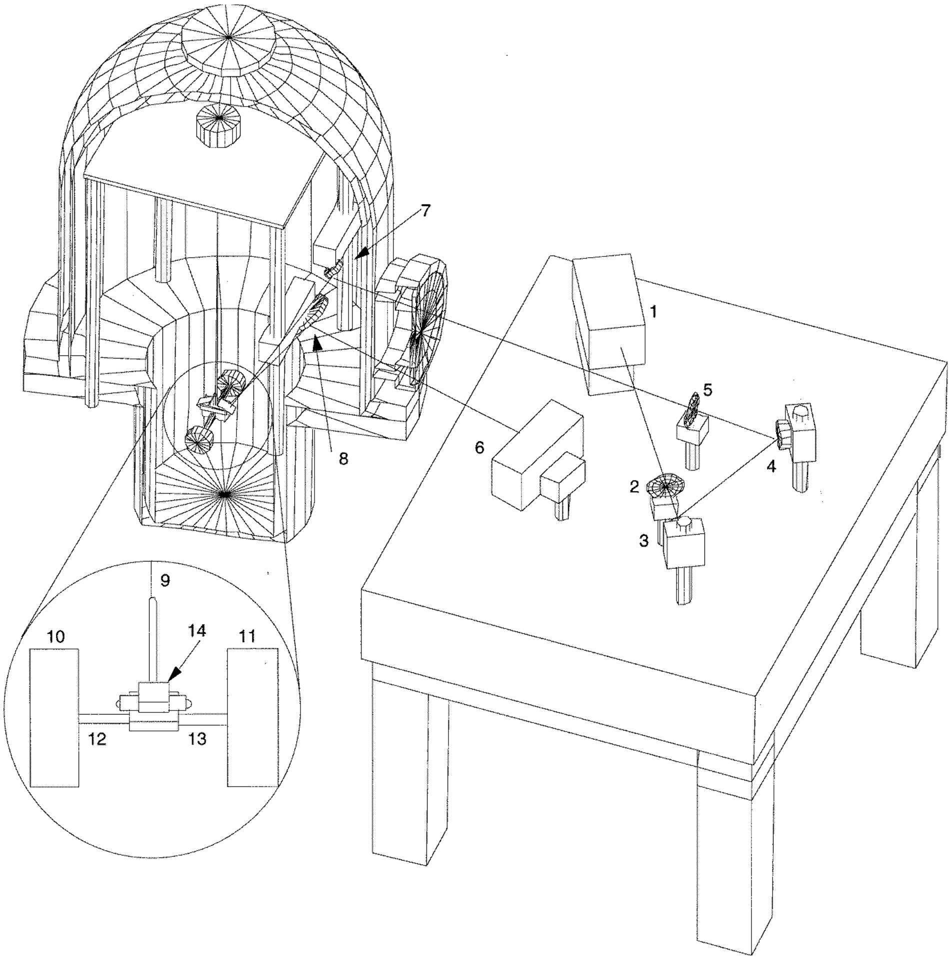

Figure 4. The torsion balance was housed in a high-vacuum chamber and the swing was measured by an optical lever: 1, laser source; 2 and 5, lenses; 3, 4, 7 and 8, mirrors; 6, photodiode array; 9, fibre; 10 and 11, masses; 12 and 13, horizontal shafts; and 14, right-angle prism. This figure is taken from [7]. (Reprinted from *Phys. Lett.* A **244** S Matsumara *et al*. A measurement of the frequency dependence of the spring constant, 4, © 1998 with permission from Elsevier Science.)

Kramers–Kronig relation. This is qualitatively true [7], but more experiments are needed in order to obtain more quantitative characteristics.

4. Measurement of the change of torsional stiffness

We know about the existence of structural damping in the fibre. Empirically, the mechanical Q of vibrations seems to be constant for a wide range of frequencies and this damping differs apparently from velocity damping. If we assume the constancy of the imaginary part of the complex spring constant, we have to accept the frequency change of the real part of the spring constant represented in the above equation. The Kramers–Kronig relation is given not in the time region but in the frequency region. Since we do not have complete knowledge of the frequency region which would allow us to check the relation for a practical system, this may not be applied to a real system. Therefore, we measured the direct change of the spring constant with frequency and found that it was qualitatively correct. We need further precise experiments in order to say more about the quantitative nature of anelasticity. This experiment was published in [7] and a brief summary follows.

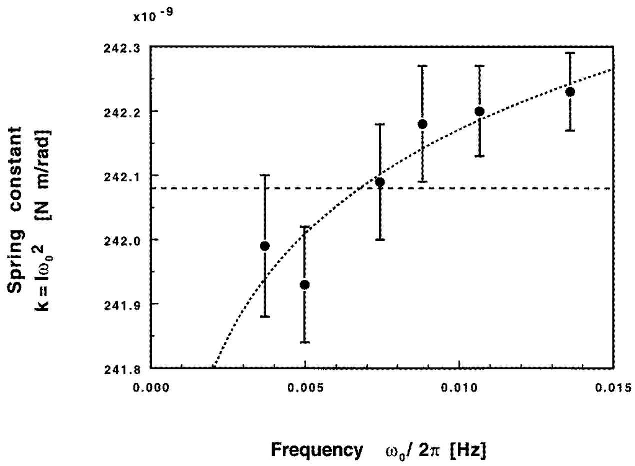

Figure 5. The frequency dependence of the real part of the torsional spring constant. The dotted line shows the result of fitting with the real part of the spring constant given in section 3 in this paper. This plot is also taken from [7]. (Reprinted from *Phys. Lett.* A **244** S Matsumara *et al*. A measurement of the frequency dependence of the spring constant, 4, © 1998 with permission from Elsevier Science.)

We adopted a dumbbell with a moment of inertia changeable by displacing the position of weights as in

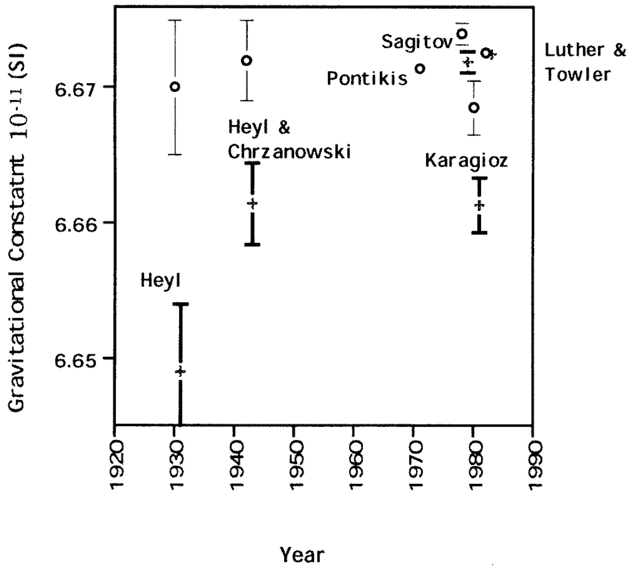

Figure 6. The historical bias of the determination of G. Circular marks represent the original results given in figure 2 in the same order. Plus marks show the estimated shifts due to fibre anelasticity. Q values are 100 for Heyl, 200 for Heyl and Chrzanowski, 2000 for Luther and Towler and 100 both for Karagioz and for Sagitov. (The last two were arbitrarily set by this author.)

figure 4, which its resonance torsion period was changed from 3.7 to 13.6 mHz. The period measurement had sufficient accuracy but the geometrical measurement of the dumbbell was not adequate to provide a satisfactory result. Thus, we produced a set of dumbbells with well-defined arm lengths and determined their inertial moments by measuring their pendulum periods. This increased the determination of the relative accuracy of the inertial moments by about one order of magnitude. Table 1 summarizes the results of this experiment. The change of the spring constant is plotted in figure 5. Apparently the spring constant was not constant in frequency and Q was practically constant. We could say that the anelasticity model is qualitatively proved, i.e. its structure damping does not require the spring constant to be constant with respect to frequency.

5. Systematic errors of historical measurements

Since we believe that there is a systematic error in the time-of-swing method for the determination of G, we estimated a historical bias in the determination of G, shown in figure 6. To estimate this, only the torsion Q of the balance, which can be obtained from the literature [9, 10] is necessary. The effect of this systematic error is comparable to the relative difference among experiments in the case of the Heyl, Heyl and Chrzanowski, and Karagioz experiments. Note that this is only a hypothetical estimation, assuming the validity of the Kramers–Kronig relation in the spring system and attributing the mechanical loss to the torsion fibre only.

6. Conclusion

Although the mathematical model of anelasticity is not complete and quantitative testing of anelasticity needs further investigation, the assumption of the constancy of a torsion spring is not evident. It is surprising to discover that no-one has doubted this point for two centuries since Cavendish. I hope that this fact stimulates young scientists to tackle this ‘old but new’ frontier of science.

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