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FUNDAMENTAL PROBLEMS IN METROLOGY

CALCULATION OF THE GRAVITATIONAL CONSTANT USING AN ANALYTICAL FORMULA AND A DIFFERENTIAL EQUATION

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Various approaches are considered to calculating the gravitational constant using an analytical formula containing fourth-order coefficients of the angle of oscillation and using the differential equation of motion of a balance. A comparison of the results facilitates the elimination of possible sources of systematic error and makes it possible to estimate the correctness of all the assumptions made when implementing each method.

A measurement of the gravitational interaction force between test bodies using a torsion balance enables the gravitational constant G to be calculated after laborious calculations have been made. The use of spherical interacting bodies promotes the derivation of the simplest analytical formulas, but for a dynamic operating regime the calculations are made considerably more complex by the need to take account of nonlinear terms in the equations of motion. The nonlinearity in the case of free oscillations is above all determined by the form of the elastic characteristic which is functionally related to the oscillation amplitude. In the most general case nonlinearity can also be caused by damping forces. A mathematical pendulum is a classic example when one can obtain a rigorous solution in the presence of a nonlinear restoring force. In practice, the majority of systems are symmetric about the equilibrium position and so the equation of motion contains only odd powers of the angle of oscillation. In order to obtain periodic solutions the method of successive approximations is used which has been developed by various authors in connection with important astronomical problems requiring the solution of similar equations [1]. The frequency of anharmonic oscillations is determined from the condition for the absence of resonance which would lead to an unlimited growth in the amplitude of oscillations. This frequency is not constant since it depends on the amplitude. When making measurements of G, as the angle of deviation φ increases the moment of the forces acting to return the system to its original position increases more slowly than does φ. Therefore, the frequency of the system decreases as the oscillation amplitude φ_0 of the balance increases. Higher harmonics are imposed on the motion of the oscillator. As a rule, an analytical solution of the differential equation of free nonlinear oscillations of a system is laborious or even impossible. Moreover the use of graphical and numerical methods for calculations of G is of no practical interest. For very small amplitudes one could limit oneself to a term with the first power of the angle of oscillation φ_0 of the balance. However, experiments indicate that operation at extremely small amplitudes leads to a marked increase in the measurement error. This is due to the action of various destabilizing factors, above all microseisms and free fluctuations which limit the stability of operation of the balance, displace the equilibrium position, change the period of oscillation, and increase the error of measuring the time intervals between the pulses formed by the optical recording system.

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TABLE 1. Calculation of G in Four-Position Array Scheme 870303 from Analytical Formula and Differential Equation

Position of masses		$G \cdot 10^{11}$, $N \cdot m^2 \cdot kg^{-2}$ for φ_{01}, rad					
		0.03373		0.04253		0.06117	
i	j	Formula	Equation	Formula	Equation	Formula	Equation
1	2	6.67410	6.67421	6.67228	6.67234	6.67601	6.67579
2	3	6.66651	6.66658	6.66805	6.66808	6.67718	6.67708
1	3	6.67071	6.67080	6.67044	6.67048	6.67669	6.67654
3	4	6.67651	6.67651	6.67726	6.67724	6.67275	6.67271
2	4	6.67052	6.67056	6.67177	6.67179	6.67554	6.67548
1	4	6.67208	6.67215	6.67207	6.67211	6.67593	6.67580
4	3	6.67633	6.67633	6.67546	6.67545	6.67589	6.67586
3	2	6.67180	6.67186	6.67542	6.67545	6.67525	6.67516
4	2	6.67363	6.67366	6.67550	6.67551	6.67564	6.67557
2	1	6.67362	6.67373	6.67250	6.67255	6.67579	6.67556
3	1	6.67284	6.67292	6.67389	6.67393	6.67570	6.67553
4	1	6.67368	6.67374	6.67432	6.67435	6.67589	6.67575

TABLE 2. Calculation of G in Three-Position Array Scheme 890620

Position of masses		$G \cdot 10^{11}$, $N \cdot m^2 \cdot kg^{-2}$ for φ_{01}, rad					
		0.03173		0.06885		0.08777	
i	j	Formula	Equation	Formula	Equation	Formula	Equation
1	2	6.66961	6.66954	6.66576	6.66571	6.67320	6.67337
2	3	6.67883	6.67879	6.67138	6.67132	6.67603	6.67602
1	3	6.67293	6.67285	6.66781	6.66775	6.67425	6.67437
3	2	6.67588	6.67585	6.66802	6.66792	6.67611	6.67618
2	1	6.67513	6.67506	6.67437	6.67428	6.67365	6.67382
3	1	6.67540	6.67531	6.672072	6.671994	6.674570	6.674665

TABLE 3. Calculation of G in Three-Position Array Scheme 970930

Position of masses		$G \cdot 10^{11}$, $N \cdot m^2 \cdot kg^{-2}$ for φ_{01}, rad					
		0.03757		0.07830		0.09025	
i	j	Formula	Equation	Formula	Equation	Formula	Equation
1	2	6.67315	6.67320	6.67328	6.67324	6.67360	6.67354
2	3	6.67680	6.67682	6.66376	6.66376	6.67309	6.67310
1	3	6.67476	6.67480	6.66909	6.66907	6.67340	6.67338
3	2	6.67148	6.67150	6.67944	6.67945	6.67254	6.67255
2	1	6.67526	6.67531	6.67080	6.67076	6.67523	6.67517
3	1	6.67360	6.67364	6.67464	6.67462	6.67407	6.67404

The gravitational constant G is calculated from the analytical formula by expanding the moments of the forces of attraction between the interacting masses in a series in powers of the oscillation amplitude φ_0 of the balance with successive account being taken of the influence of terms for the different powers of φ_0. For an analysis of the influence on the shift in the period of anharmonic oscillations of the term containing φ_0^4 expressions are given in [2–5] which are difficult to compare but give a roughly identical numerical value. The term with φ_0^4 was not considered in [3] but by utilizing the method proposed in this work it was possible to determine the coefficient for this term and this made it possible to obtain a formula for calculating the gravitational constant [6]. A similar result was published in [2] where the method put forward in [4] was used to obtain an estimate of the influence of terms with φ_0^4 and φ_0^6 on the period of anharmonic oscillations.

A specialized program for calculating the gravitational constant [6] enabled a comparison to be made between experimental values of G calculated from an analytical formula and a differential equation. Analytical formulas work successfully up to an amplitude of the order of 0.09 rad. For a further increase in φ_0 the error becomes appreciable and it is necessary to utilize only the results obtained from the differential equation. Striking evidence of this is given by data given in Tables 1–3. A comparison of the results is made for three different oscillation amplitudes using the example of the data array 870303 obtained for a four-position scheme (see Table 1) and also the arrays 890620 and 970930 in which the masses were fixed in only three positions (see Tables 2 and 3). The four-position scheme gives six combinations G_{ij} for each cycle of measurements (where i and j are the positions of the attracting masses). Three of these combinations are independent while the other three are derivatives of them. In the three-position scheme two combinations give new independent values of G_{ij} while the third combination is functionally related to them. Nevertheless, each combination G_{ij} is of interest in itself and so the program calculates all these combinations. In addition, direct and inverse measurement cycles are individually considered. In the case of the direct cycles the attracting masses are placed remotely from the balance beam weights and for the inverse cycles they are placed close to them. The tables give two adjacent cycles and so the four-position scheme contains twelve values of G_{ij} while the three-position scheme contains six. A comparison of the data obtained individually for the direct and inverse cycles is of practical significance. It was found that even for a large number of measurements the direct and inverse cycles lead to different values of G_{ij}. This provides indirect evidence of a systematic measurement error. For amplitudes exceeding 0.1 rad the analytical formulas give underestimates of the G values. It is necessary to introduce following subsequent terms of higher power in φ_0 in order further to broaden the range.

Calculation using the differential equation involve a number of assumptions making it possible with a minimum expenditure of time to select a value of G for which the differences of the inverse squares of the experimental and calculated values of the periods of the anharmonic oscillations coincide for the two positions of the attracting masses. Here it is necessary to know the value of the oscillation period T_0 in the absence of the attracting masses. This is not recorded during measurements of the gravitational constant. In view of this an additional error arises associated with an inaccurate knowledge of this quantity. In order to determine the period of the anharmonic oscillations it is necessary to find the time interval between two successive zero values of the oscillation angle of the balance. The oscillation period is exactly twice this time interval. The widely known Runge–Kutt method was used when determining the time interval between the zero values. Its principal advantage is that integration can be performed having only the initial conditions as the initial data. Another important advantage of this method is the possibility of changing the integration step during the calculations. This makes it possible to make multiple order-of-magnitude changes in the integration step and in its sign after the first intersection of the zero value of the function. In the final analysis it is possible to find the calculated time interval with an error which is lower than the initial step by many orders of magnitude. The accuracy can be increased by reducing the initial integration step. Disadvantages of the method must be considered to include its laborious nature and the impossibility of estimating the accuracy of the solution. The correctness of the choice of the step can be estimated by carrying out control calculations with the initial step reduced by an order of magnitude. If the error obtained for the original step is below the required level this step can be decided on.

The formula describing the interaction of the spherical masses with the balance beam has terms for which the numerator contains the difference of two similar numbers (for small oscillation angles it vanishes as $\sin^2\varphi$) while the denominator contains a factor $\sin\varphi$ in explicit form. Despite the fact that the balance beam only weakly influences the results, the calculations of the moment associated with it must be made with a very small error, and this requires computer calculations having twice the accuracy. This mainly results from the fact that during the search for the second zero value the angle φ is steadily approaching zero. The calculations are continued to a step of 0.01 ms and this makes it possible to calculate the period of the anharmonic oscillations with an error of less than 0.1 ms. But even with such a short step one cannot allow a strict zero to be obtained in the denominator since this would result in the disruption of the calculations. Such a situation can arise with an insufficient number of computer digits on account of the rounding off of small quantities. In addition, the formula is obtained on the assumption of a negligibly small diameter of the balance beam, and this introduces an additional error. The course of the calculations using the differential equation has practically nothing in common with the calculations using the analytical formula. One must therefore compare the results from differing points of view and above all from the positions of eliminating errors when deriving the formulas and programming them.

Each approach to the calculations possesses specific errors, and this requires an analysis of the validity of certain assumptions adopted when implementing them. The analytical formula is limited by the maximum oscillation amplitude which is difficult to estimate accurately. In the case when the attracting masses are located remotely from the rotational axis of the balance the role of the nonlinear terms is reduced and the upper limit is displaced toward larger amplitudes φ_0. Difficulties arise when processing data arrays using an analytical formula containing a large range of φ_0. The point is that the formula has a limit imposed on the largest permissible amplitude φ_0 and φ_0 exceeds this limit in a number of rows of the array (see Tables 1–3). The inclusion in certain arrays of measurements for large amplitudes φ_0 is associated with a refinement of the accuracy of the constants of the optical indicating system which is used for calculating φ_0. An alternative approach to the calculation made it possible to circumvent this difficulty without eliminating such rows from this array. When a given value of φ_0 is exceeded the program automatically changes over to a calculation using the differential equation of motion and this eliminates any disruptions for calculations using the analytical formula. Thus, the two approaches to a solution not only make it possible to eliminate any errors when they are implemented but also mutually supplement each other.

Conclusions. A comparison of data obtained using the analytical formula with the results of calculating G using the differential equation of motion of the balance assists in estimating the errors of the two possible approaches and also in convincing oneself of the validity of the various assumptions adopted when implementing them. The analytical formula enables the time taken for the calculation to be shortened. There is no need to know or refine the oscillation period T_0 of the balance when the attracting masses are absent. The possibilities of the analytical method were considerably increased by utilizing a term with the fourth power of the oscillation amplitude φ_0 of the balance. A comparison of the data from the two methods of calculation facilitates the elimination of potential sources of systematic error.

Both approaches show promise for further development. The inclusion of terms of even higher order will lead to an increase in the efficiency of the analytical formula and will broaden the range of amplitudes for which calculations can be performed with a small error without using the equation of motion. The choice of the optimal time step or the use of new approaches to integrating the equation of motion can lead to a reduction in the noise when determining the period of the anharmonic oscillations. This will make it possible with considerable success to utilize this approach to monitor the results of calculations using the analytical formula containing high powers of the angle of oscillation.

The method of calculating the gravitational constant which we have developed, using the equation of motion, can also be extended to other experiments in which the moments of the attractive forces are described by incomparably more complex analytical expressions. In a number of experiments carried out using bodies of complex shape [7–9] such calculations failed to be performed. The presence of an additional alternative method of calculation can prove to be useful for a deeper analysis of such experiments.

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