Precise determination of the amplitude of signal with known frequency based on correlated noise model

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Based on Gauss correlated noise model, a method is suggested to determine the uncertainty of the amplitude for a sinusoidal signal with known frequency in each period, where the correlation analysis and the weighting statistical analysis are incorporated to improve the precision of determining the amplitude of the signal. The uncertainty of determining the amplitude with this method has been improved fourfolds than before in dealing with the experimental data in determining limits on the photon mass. © *2006 American Institute of Physics.*
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I. INTRODUCTION

Accurate determination the amplitude of a sinusoidal signal with known frequency is essential in some classical gravitational experiments, such as detections of upper bounds on the photon mass,[1,2] tests of weak equivalence principle,[3] tests of Newtonian inverse square law,[4] tests of Lorentz invariance in classical electrodynamics,[5,6] tests of cosmic spatial isotropy,[7] and so on. The most prominent character in these experiments is that the amplitude of the signal is very small or even zero, but the frequency of the signal is known accurately. In these null experimental tests, determination of the amplitude with high precision is always a formidable challenge because of the very small signal to noise ratio. The conventional methods of determining the amplitude of the sinusoidal signal with known frequency are the fast Fourier transform (FFT) method[8] and the correlation method, but the precision of determining the amplitude of the signal is limited. In the FFT method, for example, the "window" effect and the "phase" one lead to a serious deviation of the amplitude. Although Goldblum and Ritter have done some beneficial work on this problem, the two effects cannot be eliminated effectively. The correlation method which is one of the most efficient methods for subtle signal analysis[9] has been used frequently. It is clear that the correlation method is superior to the FFT method due to efficaciously avoiding window and phase effects in the FFT method. For comparison with the method proposed in this article, we need to introduce and discuss the correlation method here. For convenience in calculation, the series of sinusoidal sample data $\{y_i\}$ is expressed as

$$\begin{aligned}\{y_i\} &\equiv y(t_i)\\ &= A\cos(\omega t_i+\varphi)+kt_i+l+\varepsilon_i\\ &= a\cos(\omega t_i)+b\sin(\omega t_i)+kt_i+l+\varepsilon_i,\end{aligned} \tag{1}$$

where A, ω, and φ represent the amplitude, the frequency, and initial phase, respectively. k and l are the slope and the intercept of the monotonic drift, respectively. The coefficients a and b are the decomposed coefficients of A. The parameter ε_i represents the noise sequence.

To extract the amplitude of the sinusoidal signal with known frequency from the raw data with high precision, one should suppress the monotonic drift and the disturbances of low frequency noise by subtracting pairs of raw data points separated by π phase in the signal frequency. In this case, the filtered data sequence becomes

$$\begin{aligned}\tilde{y}(t_i) &= \frac{1}{2}\left[y(t_i)-y\left(t_i+\frac{T}{2}\right)\right]\\ &= A\cos(\omega t_i+\varphi)\\ &\quad -kT/4+\frac{1}{2}\left[\varepsilon(t_i)-\varepsilon\left(t_i+\frac{T}{2}\right)\right],\end{aligned} \tag{2}$$

where T is the period of the sinusoidal signal with known frequency. Here the signal to be determined, $A\cos(\omega t_i+\varphi)$, is unchanged. After the above data processing, the new expression of the noise can be rewritten as

$$\tilde{\varepsilon}_i \equiv \tilde{\varepsilon}(t_i) = \frac{1}{2}\left[\varepsilon(t_i)-\varepsilon\left(t_i+\frac{T}{2}\right)\right]. \tag{3}$$

Now, one can extract the amplitude A (or a and b) by correlation method. Comparing the filtered data $\tilde{y}(t_i)$ with the standard sinusoidal serials $\{\sin(\omega t_i)\}$ (named as reference signal), one can calculate the magnitudes of the coefficients a_j and b_j $(j=1,2,\ldots,m)$ at the jth period according the following equations:

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$$a_j = \frac{2}{T}\int_{(j-1)T}^{jT} \tilde{y}(t)\cos(\omega t)dt, \tag{4a}$$

$$b_j = \frac{2}{T}\int_{(j-1)T}^{jT} \tilde{y}(t)\sin(\omega t)dt, \tag{4b}$$

where $T=2\pi/\omega$ and m is the number of the periods including in the filtered data. For numerical computation, the integral should be replaced by the sum.

Then, the central values and the uncertainties of the two coefficients are obtained in two different ways: weighting all coefficients a_j and b_j equally or weighting the coefficients by the inverse square of the fit errors.[10] It is obvious that the method weighting the coefficients by the inverse square of the fit errors is superior to the one weighting all coefficients equally in extracting the amplitude of the sinusoidal signal with known frequency, especially to the unstable experimental data due to the fluctuation of temperature and external noise. In order to acquire the fit errors, the all coefficients a_j or b_j $(j=1,2,\ldots,m)$ are then cut into segments containing several periods. However, we do not know how many periods should be considered in one segment. That is to say there exists an optimization question. In this article, we can show that the uncertainty of the amplitude for the sinusoidal signal with known frequency in each period can be determined based on a Gauss correlated noise model. Weighting the coefficients by the inverse square of their uncertainties correspondingly, we could acquire the amplitude and its uncertainty of the signal with high precision.

II. OBTAINING THE UNCERTAINTY OF EACH COEFFICIENT BASED ON GAUSS CORRELATED NOISE MODEL

Here we consider the uncertainty of the amplitude A for the sinusoidal signal with known frequency arising from the noise $\tilde{\varepsilon}_i$ of filtered data in Eq. (3), which usually have a definite correlation time and submit to Gauss distribution. Generally, for torsion balance experiments, there are fundamental noise sources that are present in all experiments such as Brownian motion of the pendulum or shot noise in the photodiode, and there are sources which are unique to a specific experiment. Although the physical origin of noise $\tilde{\varepsilon}_i$ is as yet unknown, we set Gauss correlated noise as follows:[11,12]

$$\langle \tilde{\varepsilon}(t) \rangle = 0, \tag{5a}$$

$$\langle \tilde{\varepsilon}(t_1)\tilde{\varepsilon}(t_2) \rangle = \sigma^2 \exp(-\alpha|t_1 - t_2|), \tag{5b}$$

where $\langle\,\rangle$ is the expectation operator representing population mean, $1/\alpha$ is the length of correlation time, and σ is the root mean square of the correlated noise. Generally, we have $1/\alpha \ll T$.

Based on Gauss correlated noise model, we can obtain the uncertainties Δa_j and Δb_j $(j=1,2,\ldots,m)$ at the jth period according the following equations:

$$(\Delta a_j)^2 = \left\langle \frac{2}{T}\int_{(j-1)T}^{jT} \tilde{\varepsilon}(t_1)\cos(\omega t_1)dt_1 \times \frac{2}{T}\int_{(j-1)T}^{jT} \tilde{\varepsilon}(t_2)\cos(\omega t_2)dt_2 \right\rangle, \tag{6a}$$

$$(\Delta b_j)^2 = \left\langle \frac{2}{T}\int_{(j-1)T}^{jT} \tilde{\varepsilon}(t_1)\sin(\omega t_1)dt_1 \times \frac{2}{T}\int_{(j-1)T}^{jT} \tilde{\varepsilon}(t_2)\sin(\omega t_2)dt_2 \right\rangle. \tag{6b}$$

If the magnitude of the noise $\tilde{\varepsilon}$ is σ_j and the length of correlation time is $1/\alpha_j$ at the jth period with $\langle \tilde{\varepsilon}(t_1)\tilde{\varepsilon}(t_2)\rangle_j = \sigma_j^2 \exp(-\alpha_j|t_1-t_2|)$, the above formulas can be reduced as

$$(\Delta a_j)^2 = (\Delta b_j)^2 = \frac{4\sigma_j^2 \alpha_j T}{4\pi^2 + \alpha_j^2 T^2}. \tag{7}$$

Obtaining $\Delta a_j, \Delta b_j$, and the magnitudes of the coefficients a_j and b_j $(j=1,2,\ldots,m)$ according to Eqs. (7), (4a), and (4b), respectively, we can acquire the coefficients a and b by weighting the coefficients a_j and b_j according to their inverse square of the uncertainties, which could be expressed in the following:

$$a = \sum_{j=1}^{m} \rho_{a,j} a_j, \tag{8a}$$

$$b = \sum_{j=1}^{m} \rho_{b,j} b_j \quad (j = 1,2,3,\ldots,m), \tag{8b}$$

and the weighting factors are

$$\rho_{a,j} = (\Delta a_j)^{-2} / \sum_{j=1}^{m} (\Delta a_j)^{-2}, \tag{9a}$$

$$\rho_{b,j} = (\Delta b_j)^{-2} / \sum_{j=1}^{m} (\Delta b_j)^{-2}. \tag{9b}$$

Then the uncertainties of a and b are obtained as follows:

$$(\Delta a)^2 = \left[\sum_{j=1}^{m} (\Delta a_j)^{-2} \right]^{-1}, \tag{10a}$$

$$(\Delta b)^2 = \left[\sum_{i=1}^{k} (\Delta b_i)^{-2} \right]^{-1}. \tag{10b}$$

Therefore, the amplitude and its uncertainty are finally determined as

$$A = \sqrt{a^2 + b^2}, \tag{11a}$$

$$\Delta A = \sqrt{(\Delta a)^2 + (\Delta b)^2}. \tag{11b}$$

In addition, by the computer simulation, we have showed the validity of Eq. (7) and found that our method is better than the conventional methods to determine the amplitude and its uncertainty for the sinusoidal signal with known frequency. In the next section, we will deal with experimental data by our method.

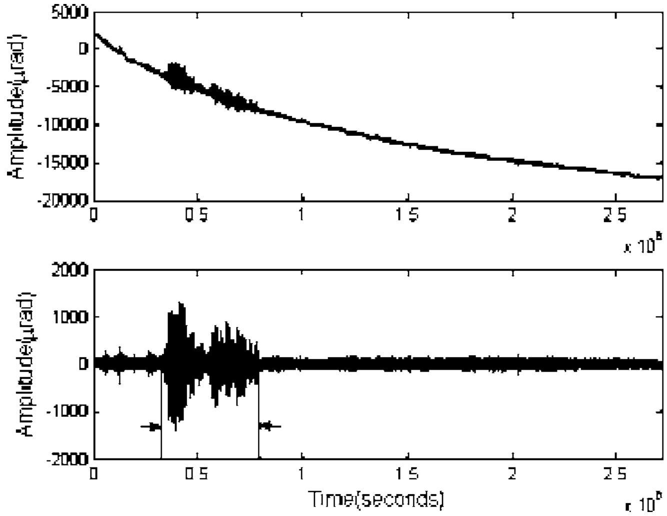

FIG. 1. Top panel: the raw data of determining the photon mass with a torsion pendulum. The most obvious is that the raw data have a monotonic drift. Bottom panel: the data after suppressing monotonic drift according to Eq. (2). The domain between two arrows could be caused by the seismic noise. The data are collected continuously from 26 March 2005 to 11 April 2005. It is over 15 days.

III. APPLICATION

As a practical example, we will apply our method to process the experimental data in determining the photon mass.[1] In the experiment, the oscillating period of the pendulum which has been suppressed is 228.6 ± 0.4 s with a quality factor of $Q = 1031 \pm 1$, and the motion of the torsion pendulum is modulated by a turntable with known frequency (modulation period is 2003.5 s, because Luo and co-workers find that the spectrum of power of the noise is the lowest in the frequency nearly 4.99×10^{-4} Hz). Since the oscillating period of the pendulum has been suppressed, we are subtracting the data from itself at one-half modulation period later in order to remove low frequency drifts and not changing the modulating signal, which is different from the Eot-Wash group of Adelberger and co-workers at the University of Washington. Their method is adding the data to itself at one-half oscillation period later in order to remove most of the free oscillation. The top panel of Fig. 1 shows the time-domain figure of the raw signal with the sample interval of $\Delta t = 0.5$ s. The length of the raw data is over 15 days, and the number of total modulated period is 677. It is clear that the original signal has an obvious monotonic drift. The bottom panel of Fig. 1 gives the time-domain figure of the signal by suppressing the disturbance of the low frequency noise and the monotonic drift according to Eq. (2). The corresponding spectra of the amplitude for them are shown in Fig. 2. We only draw a section from 10^{-5} to 10^{-4} Hz due to the modulating frequency of 4.99×10^{-4} Hz. Comparing with the two panels of the Fig. 2, we can see that the low frequency noise has been suppressed nearly two orders besides getting rid of the monotonic drift. Figure 3 shows an autocorrelation function $R(\tau)$ of the noise of the filtered data at 1st, 340th, and 677th periods, respectively, where τ is the lag time and $R(\tau)$ is defined as

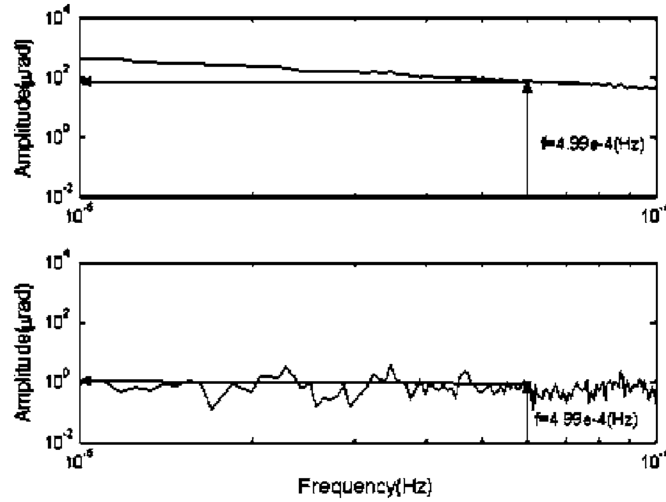

FIG. 2. Top panel: amplitude spectrum of the raw sampling data. Bottom panel: amplitude spectrum of the data after suppressing monotonic drift. The arrow shows the modulating frequency $f = 4.99 \times 10^{-4}$ Hz, and we find no expected signal.

$$R(\tau) = \begin{cases} \dfrac{1}{N-n} \displaystyle\sum_{i=0}^{N-n-1} \tilde{\varepsilon}_i \tilde{\varepsilon}_{i+n}, & n \geqslant 0 \\ \dfrac{1}{N+n} \displaystyle\sum_{i=0}^{N+n-1} \tilde{\varepsilon}_i \tilde{\varepsilon}_{i-n}, & n < 0 \end{cases} \qquad \text{with } n = \tau/\Delta t,$$

where $\{\tilde{\varepsilon}_i\}$ are the noise sequence of the filtered data at 1st period, 340th period, and 677th period, respectively. Δt is the interval in sample time. The figure of the autocorrelation function of the noise in each period is similar to the figures displaying in Fig. 3. It is clear that the noise of filtered data can be regarded as the Gauss correlated noise model with $\langle \tilde{\varepsilon}(t_1)\tilde{\varepsilon}(t_2)\rangle = \sigma^2 \exp(-\alpha|t_1 - t_2|)(\tau = t_1 - t_2)$. The top panel of Fig. 4 shows the coefficient a_j and its uncertainty Δa_j of the sinusoidal signal with known frequency correspondingly in different periods. The bottom panel of Fig. 4 shows the co-

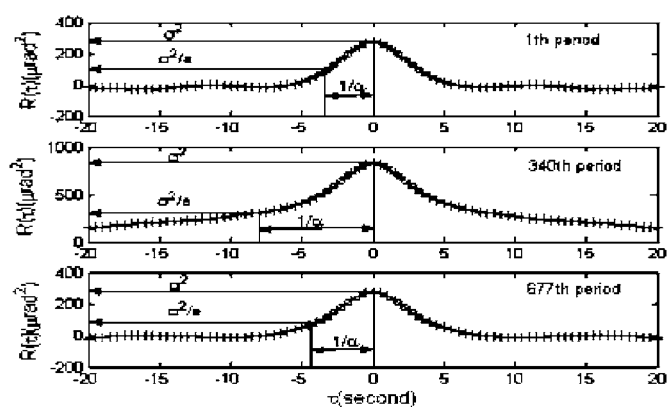

FIG. 3. The autocorrelation function $R(\tau)$ of the random correlated noise $\{\tilde{\varepsilon}_i\}$ of the filtered data at 1st, 340th, and 677th periods, respectively (the total period number is 677). The noise sequence $\{\tilde{\varepsilon}_i\}$ is calculated by a short section (about 80 samples) in each period. For different periods, σ is different, it varies in the range of 15.5–302.6 μrad, but it is sufficiently large compared to the thermal noise, which is about 0.1 μrad in the experiment at 300 K. The length of correlation time $1/\alpha$ is also different, it varies in the range of 2.94–8.33 s. The magnitude of the noise σ and the length of correlation time $1/\alpha$ in each period could be acquired by fitting the respective autocorrelation function with Eq. (5b).

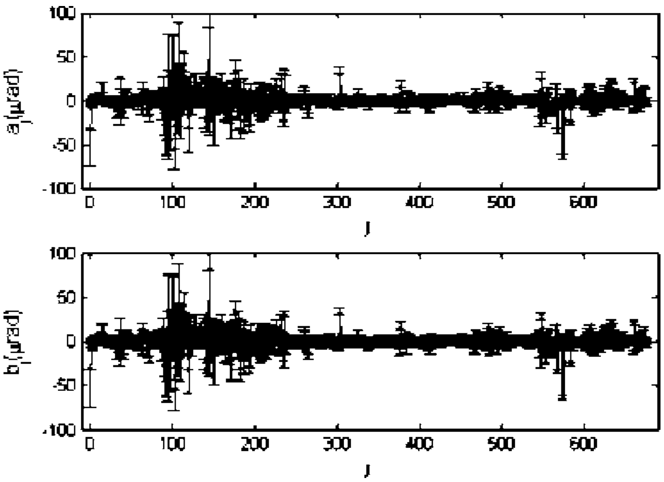

FIG. 4. Top panel: the coefficient a_j and its uncertainty Δa_j of the sinusoidal signal with known frequency in different periods. Bottom panel: the coefficient b_j and its uncertainty Δb_j in different ones, which can be obtained according to Eqs. (4a), (4b), and (7).

efficient b_j and its uncertainty Δb_j in different ones. With the new method, we can obtain $a=-0.001\pm 0.085$ μrad and $b=0.041\pm 0.085$ μrad, and then the amplitude and its uncertainty are 0.04 ± 0.12 μrad correspondingly. For comparison with the conventional method, the central values and the uncertainties of the two coefficients can be obtained by weighting all coefficients equally as 0.27 ± 0.53 μrad corresponding to $a=0.26\pm 0.44$ μrad and $b=0.07\pm 0.29$ μrad. As a conclusion, the experimental result is improved about fourfolds with this new method.

In addition, we also check the chi-square calculation of the data using the new, weighted analysis method, where the sum of the squares of the differences between individual period fits and overall means divided by the uncertainty for that period give the value of 1452. This value is nearly equal to the degrees of the data (there are two components and 677 periods of data with two fitted means giving $2\times 677-2=1352$ degrees of freedom). The result proves the correlated Gauss noise model is reasonable for this data.

IV. SUMMARY

The result of processing experimental data shows that the method based on Gauss correlated noise model can improve the precision of determining the amplitude of the period signal with known frequency. Especially, if the number of the period observed in the experiment is limited, the advantage of this method will be more prominent comparatively.

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