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Newton's constant and the twenty-first century laboratory

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The main aim of this paper is to describe the problems that confront experimentalists who attempt to determine Newton's constant of gravitation, G. I will motivate this work by discussing the role of Newton's constant of gravitation in classical physics and recent ideas as to its role in quantum physics. I will then discuss some key aspects of a precision determination of G. This will include criteria for the selection of the detector of the gravitational torque from the point of view of random uncertainties due to read-out noise, thermal and vibrational noise. Another important factor in precise determinations of G is the control of systematic effects (type B uncertainties) such as those due to uncertainties in absolute calibration of the gravitational torque, density homogeneity of source masses and length metrology. I will illustrate the discussion using the determination of G currently underway at the International Bureau of Weights and Measures in France, and describe other experimental configurations that have been used in the past or are being currently developed.

Keywords: Newtonian constant of gravitation; experimental tests of gravity; gravitation in astrophysics; design of physics experiments

1. Introduction

In 1998, a conference was held in London (Speake & Quinn 1999) to celebrate the bicentenary of Cavendish's determination of the Newtonian constant of gravitation, G (Cavendish 1798). The accepted value of G at the time was the CODATA's 1986 value of $(6.67259\pm0.00085)\times10^{-11}\ \mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$, which was based largely on the determination of Luther & Towler (1982). There was a crisis of confidence in this value and the CODATA were about to increase its uncertainty from 128 p.p.m. to 0.15%! A group at the Physikalisch-Techniche Bundesanstalt (PTB) in Braunschweig in Germany had completed a determination of G that they had started in the 1980s. In 1996, they reported (Michaelis *et al.* 1996) a value with a claimed uncertainty of 68 p.p.m. but which was 0.6% larger than the 1986 value of CODATA. This group had broken with tradition and, unlike Luther and Towler, had not used a torsion balance with the usual material fibre but had developed an entirely new balance based on a mercury suspension. Further, Kuroda (1995) pointed out a possible systematic

One contribution of 14 to a Discussion Meeting 'The fundamental constants of physics, precision measurements and the base units of the SI'.

uncertainty in determinations of G that used torsion fibres. I will discuss both these developments in more detail below. Since 1998 considerable effort has been made worldwide to establish a reliable value for G. Thanks to this work the current CODATA value of 2002 is $(6.6742\pm0.001)\times10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$. The uncertainty amounts to 150 p.p.m., which is comparable with the 1986 value but the value itself has shifted by 1.2 times the combined uncertainty. For a discussion of the development of the CODATA value for G the reader should refer to Mohr & Taylor (2005).

The dimensions of G are (length^3 mass^{-1} s^{-2}). In SI, length is measured in terms of the fixed speed of light and the second, whereas the unit of mass remains defined in terms of the kilogram artefact. The time interval of the second can now be determined with an accuracy of parts in 10^{15}. On the other hand platinum kilogram mass standards can be inter-compared with an accuracy of parts in 10^8 (see Arias 2005 and Davis 2005, in this issue). This could lead one to suppose that the uncertainty in G should be at the level of tens of parts per billion. However, in practice, current coordinate measurement techniques allow mass separations to be determined with accuracies of only a fraction of 1μm whilst the masses of the type used in G determinations can be determined with accuracies in the range of a few p.p.m. Consequently, given that the mass separations are of the order of 10cm, we would expect the accuracy of G determinations to be limited by length metrology to the order of 10 p.p.m., and this is indeed the case. This very elementary and crude analysis indicates that future improvements to the accuracy of the determination of G may come with improvements in length metrology. The reader will appreciate, upon reading this paper, that the above analysis is also extremely naïve and that there are in fact a multitude of subtle problems that can prevent determinations of G agreeing at the 10 p.p.m. level.

(*a*) *The role of G in classical physics*

In classical physics the role of G can be summarized by the following two equations

$$\nabla^2\boldsymbol{\Phi} = 4\boldsymbol{\pi} G\boldsymbol{\rho}, \tag{1.1}$$

and

$$G_{\boldsymbol{\mu\nu}} = \frac{8\boldsymbol{\pi} G}{c^4} T_{\boldsymbol{\mu\nu}}. \tag{1.2}$$

The first equation is essentially Newton's inverse square law written in its differential form in terms of the gravitational potential, $\boldsymbol{\Phi}$, and the source density $\boldsymbol{\rho}$. The second is the equivalent, relativistic, equation relating Einstein's curvature tensor $G_{\boldsymbol{\mu\nu}}$ to the energy-momentum tensor $T_{\boldsymbol{\mu\nu}}$. The weakness of gravitation implies that its role in physics is limited to the scales of astrophysical objects and of the Universe itself. It is therefore perhaps paradoxical that our current value of G is not derived from phenomena in the astrophysical and cosmological domain. Indeed it could be argued that the observational evidence for missing matter in galaxies and of an accelerating Universe invalidate equations (1.1) and (1.2) on such large-scales. The favoured interpretations of these fascinating observations are, of course, the presence of dark matter and dark energy, respectively. However, interest in deriving values of G in various astrophysical contexts does appear to be increasing as discussed below.

Newton's constant cannot be determined by simple observations of the dynamics of gravitationally bound systems as it always appears in a product with a mass characterizing the system. This product is often referred to as the gravitational parameter, $\boldsymbol{\mu}$. For example, according to Kepler's third law we have

$$\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 = \left(\frac{2\boldsymbol{\pi}}{p}\right)^2 a^3. \tag{1.3}$$

Here $\boldsymbol{\mu}_i = GM_i$ and M_i are the masses of the two components of an isolated system of orbiting bodies with semimajor axis a and period p. It is impossible to determine the masses of astronomical objects by observation without an independent value for G. The value for the Earth's mass (Chambat & Valette 2001) is determined in terms of the laboratory-derived value of G and the gravitational parameter for the Earth, which is known with an uncertainty of about 10^{-8}. Determination of the masses of objects in our Solar System is an important motivation for seeking a precise value for G. It is also of note that the motivation behind the first determination of G by Cavendish was a determination of the mean density of the Earth. At the time of Cavendish there was no unit of force and the concept of the Newtonian constant per se arose only in the late nineteenth century notably with work of Boys (Clotfelder 1987).

In the general case, even the gravitational parameters of binary systems cannot be determined accurately due to the inherent uncertainty in the inclination of the orbit to the line of sight. In highly relativistic systems such as binary pulsars, the gravitational parameters of the individual objects can be determined with high accuracy due to relativistic observables (post-Keplerian parameters) such as the precession of their periastron and relativistic time delay. For example, the precessional rate of the periastron is given as (Shapiro & Teukolsky 1983)

$$\dot{\boldsymbol{\omega}} = 3\left(\frac{2\boldsymbol{\pi}}{P}\right)^{5/3} \frac{(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)^{2/3}}{(1-e^2)c^2}, \tag{1.4}$$

where e is the eccentricity of the orbit and again p is its period. The actual masses cannot be determined without knowledge of Newton's constant, or conversely, these systems cannot be used to determine a value for G. Relativistic binary systems do, however, afford us very powerful tests of general relativity (Taylor *et al.* 1992). It is also interesting to note that the gravitational spin–spin coupling, currently being measured by Gravity Probe B, is determined by the product of G and the moment of inertia of the Earth. It would appear that measurable general relativistic quantities do not offer a means of determining G.

The reason behind this problem is, firstly, that G appears in equations (1.1) and (1.2) only in the product with mass-energy density of the gravitating source and secondly, the equivalence principle ensures that inertial mass does not appear in the balance of forces.

This situation changes when we consider the balance of gravity with forces in nature other than those derived from inertia. The balance of gravitational and thermal pressures primarily determines the stability of main sequence stars, such as our Sun. Teller (1948) pointed out that the luminosity, L, of such a star varies as

$$L \propto G^7 M^5. \tag{1.5}$$

In principle, it is possible to determine a value for G by developing solar models that are constrained by the observed values of solar radius, luminosity, the product of GM from Kepler's third law and observed frequencies of solar oscillations from helioseismology. Unfortunately (Christensen-Dalsgaard *et al.* 2005), current models of the Sun are not in sufficiently good agreement with helioseismic observations to allow this method to determine G with an accuracy that is comparable with laboratory determinations.

The production of light elements in the early Universe is sensitive to its expansion rate which is determined in part by G. Copi *et al.* (2004) have used recently observed values for the ratio of deuterium to hydrogen in Quasars together with new estimates of the ratio of the number of baryons to photons to constrain the value of G at this epoch (starting at about 1 s after the big bang) to within 20% or so of today's value. This analysis, with some assumptions, constrains the time rate of change of G to, approximately $|\dot{G}/G| < 4\times 10^{-13}\ \mathrm{yr}^{-1}$.

(*b*) *The role of G in quantum physics*

The masses of degenerate objects are determined by the balance of gravity and degeneracy pressure produced by the confinement of electrons or neutrons to volumes comparable with the cube of their Compton wavelengths. In the case of neutron stars we find that their mass, M_{NS}, depends on G in the following simple way (Shapiro & Teukolsky 1983)

$$M_{\mathrm{NS}} \approx m_{\mathrm{p}} \left(\frac{M_{\mathrm{p}}}{m_{\mathrm{p}}}\right)^3, \tag{1.6}$$

where m_{p} is the mass of the proton and M_{p} is the Planck mass; $M_{\mathrm{p}} = (\hbar c/G)^{1/2}$. The gravitational parameters of neutron stars in several binary systems have been accurately determined from their post-Keplerian parameters and an independent estimate of their gravitational parameters could be used to estimate G. Up until recently, it appeared that neutron stars in binary pulsar systems had masses in the range 1.33–1.44 $\mathrm{M}_{\mathrm{Solar}}$, however this picture has been now overturned by the discovery of the double pulsar with one body weighing only $1.250 \pm 0.005\ \mathrm{M}_{\mathrm{Solar}}$ (Kramer *et al.* 2004). There is also mounting evidence for neutron stars weighing up to 2 $\mathrm{M}_{\mathrm{Solar}}$ (Nice *et al.* 2003).

One of the greatest problems that confront physics at the turn of the twenty-first century is the incompatibility of Einstein's General Relativity and quantum mechanics. In the former, space and time are classical continua whose geometry is curved by the presence of matter. Singularities in the continuum are thought to be produced by black holes. In the quantum mechanical vacuum the uncertainty principle suggests that at short enough length-scales, the Planck length (10^{-35} m equivalent to an energy scale of 10^{19} GeV), virtual black holes can be produced. An understanding of the very early Universe, when matter is believed to have existed under these extreme conditions, can only be accomplished with a theory of gravitation that is consistent with quantum mechanics. Our understanding of particle physics (the Standard Model) is confined to the other three forces in nature, namely electromagnetism and the Weak and Strong nuclear forces. The origin of these forces is well understood in terms of the local symmetries that the forces are required to enforce. It is tempting to try to write a dimensionless coupling constant for gravitation in a way that is analogous to that for electromagnetism. If we take the mass of the proton as a fundamental unit of mass (analogous to the charge on the electron in the case of electromagnetism) we can write the Newtonian gravitational potential energy, V_G, of interaction between two objects containing n_1 and n_2 protons as

$$\frac{V_G}{\hbar c} = -\frac{G m_{\mathrm{p}}^2}{\hbar c}\,\frac{n_1 n_2}{r}. \tag{1.7a}$$

We can then define a dimensionless coupling constant, sometimes referred to as the gravitational fine structure constant, as

$$\alpha_G \approx \left(\frac{m_{\mathrm{p}}}{M_{\mathrm{p}}}\right)^2. \tag{1.7b}$$

The ratio of the gravitational and electromagnetic fine structure constant is 10^{-36}. It is well known (Amaldi *et al.* 1991) that the other fundamental forces have dimensionless coupling constants and that these vary with the energy of the interacting particles. The dimensionless coupling constants for the weak and electromagnetic interactions increase with particle energy while that of the strong force decreases. String theory predicts that the dimensionless coupling constants of all the forces are about 0.04 at a unification energy equivalent to the fundamental string mass which is thought to be 2×10^{16} GeV$/c^2$ (Damour 1999). A value for G can be calculated by setting the gravitational fine structure constant equal to this unified value and by using the fundamental string mass instead of the proton mass in equation (1.7b). This value turns out to be a factor of 100 or so too large. It is to be hoped that advances in our understanding of quantum gravity may enable us, in the future, to compare experimental values against a theoretical prediction.

String theory also asserts that there are possibly 10 spatial dimensions in our Universe: the quantum forces are constrained to exist in the normal three-dimensional world that we are familiar with, which is referred to as the 'brane', while gravitation is able to pervade all the dimensions. It is thought that the other dimensions are compact, or in some way curled up into cylindrical, toroidal or spherical topologies on scales that are small enough such that experimental observations are not contradicted. Arkani-Hamed and colleagues (1999) have suggested that n of these are compact on a macroscopic scale of, say, λ. If this is the case, gravitational forces between particles at scales less than λ would no longer vary with the inverse square of their separation. On this scale we have a new formulation for the potential energy of interaction between two masses, m_1 and m_2, separated by distance r

$$\frac{V_{Gn}}{\hbar c} = -\frac{m_1 m_2}{M_*^2}\left(\frac{\hbar}{M_* c}\right)^n \frac{1}{r^{1+n}}, \tag{1.8}$$

where M_* is a new mass scale that could be considerably lower than the Planck scale. For example, with $n=2$, the gravitational force will vary as the inverse fourth power of the mass spacing.

The gravitational constant at $r \ll \lambda$ becomes

$$G_n = \frac{\hbar c}{M_*^2}\left(\frac{\hbar}{M_* c}\right)^n. \tag{1.9}$$

The constant of gravitation that we determine at larger-scales is then seen to be a weakened, effective constant where

$$\frac{G}{G_n} = \left(\frac{M_*}{M_{\mathrm{p}}}\right)^2 \left(\frac{M_* c}{\hbar}\right)^n. \tag{1.10}$$

Arkani-Hamed *et al.* (1999) have suggested that the new mass scale could be in the range of future particle accelerators (of the order of few TeV) and this would imply that, at particle separations of about 2 mm (for $n=2$), Newton's inverse square law would be violated. Although this suggestion has been ruled out by recent experiments (Adelberger *et al.* 2003), a considerable effort is being invested worldwide in exploring gravitation at short distances both theoretically and experimentally.

All the methods of determining G to date have effectively measured the acceleration of a test object (albeit an atom) in the presence of an external attracting source mass. It is tempting to think that an understanding of the unification of gravity and quantum mechanics would enable us to determine the gravitational constant in new ways that are intrinsically quantum mechanical. One instance where G occurs naturally in combination with Planck's constant is in the Hawking–Bekenstein expression for the entropy of a black hole (Bekenstein 1973)

$$S_{\mathrm{BH}} \approx \frac{GM^2 k_{\mathrm{B}}}{\hbar c}, \tag{1.11}$$

where M is the mass of the black hole and k_{B} is Boltzmann's constant. That black holes possess macroscopic properties such as entropy and temperature points to them also having well-defined quantum states. Of course, to our knowledge, there is no possible means of using this physics as the basis for a determination of G.

2. Methods of determining G in the laboratory

In this section I will discuss some of the special problems that confront the experimentalist who wishes to determine a precise value of Newton's constant. In the near future atomic interferometry will make it possible to do this by using the apparent change in the resonant frequency of atoms in free-fall in the vicinity of source masses. This method has already been successfully used to measure the acceleration due to the Earth's gravitational field (McGuirk *et al.* 2002) and is now being applied in a determination of G by Fattori *et al.* (2003). At present these techniques are not competitive with the mechanical detectors on which I will focus.

Over the two centuries since Cavendish's experiment many variations on his experimental method have been developed (see Gillies 1999 and references therein). Cavendish's work on the Newtonian constant is widely known, however, I will briefly outline the principles of his method (see figure 1) and two other methods that have been used in the past. In Cavendish's method the attraction of a pair of source masses on a pair of test masses, that are suspended on a torsion balance, produces a torque, $\boldsymbol{\Gamma}_G$, and an angular deflection, $\boldsymbol{\theta}$, of the balance. We can write the torque in terms of the values of the source and test masses and their mutual separations in a general way as

$$\boldsymbol{\Gamma}_G = GK(x_{\mathrm{s}}, x_{\mathrm{t}}, \boldsymbol{\rho}_{\mathrm{s}}, \boldsymbol{\rho}_{\mathrm{t}}), \tag{2.1}$$

where x_{s}, x_{t} represent the coordinates of the source and test mass distributions with densities $\boldsymbol{\rho}_{\mathrm{s}}$, $\boldsymbol{\rho}_{\mathrm{t}}$. The constant K is usually derived by integration over the mass distributions and the measured values of the mass separations. It has physical dimensions of (mass^2.length^{-1}). The deflection can be written in terms of the rotational stiffness of the torsion fibre, $\boldsymbol{\kappa}(\boldsymbol{\omega}_m)$,

$$\boldsymbol{\Gamma}_G = \boldsymbol{\kappa}(\boldsymbol{\omega}_m)\boldsymbol{\theta}, \tag{2.2}$$

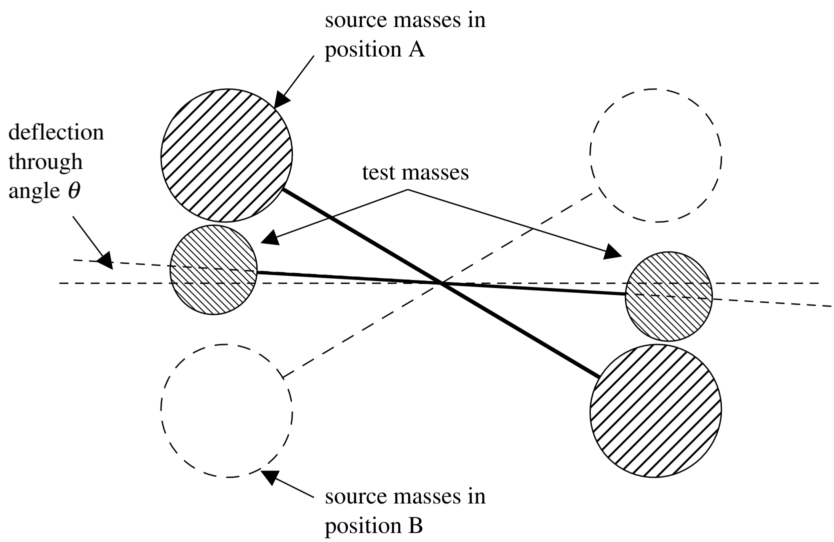

Figure 1. Schematic plan view illustrating Cavendish's method of determining G. The torque due to the gravitational attraction of the source masses in position A produces an angular deflection of the torsion balance of magnitude $\boldsymbol{\theta}$. The torque can be reversed by rotating the source masses to position B.

where $\boldsymbol{\kappa}(\boldsymbol{\omega}_m)$ refers to the angular stiffness of the torsion balance at the angular frequency, $\boldsymbol{\omega}_m = 2\pi/T_m$, at which the gravitational signal is measured, i.e. the angular frequency corresponding to the periodic reversal of the applied gravitational torque due to the source masses. Cavendish and subsequent experimenters evaluated $\boldsymbol{\kappa}(\boldsymbol{\omega}_m)$ by measuring the period of simple harmonic motion of the torsion balance, T, and computing its moment of inertia, $I(x_t, \boldsymbol{\rho}_t)$ from length and mass metrology

$$\boldsymbol{\kappa}(\boldsymbol{\omega}_m) = I(x_t, \boldsymbol{\rho}_t)\left(\frac{2\boldsymbol{\pi}}{T}\right)^2. \tag{2.3}$$

The assumptions leading to this equation are discussed in §2. Finally G can be calculated using

$$G = \frac{I(x_t, \boldsymbol{\rho}_t)}{K(x_s, x_t, \boldsymbol{\rho}_s, \boldsymbol{\rho}_t)}\left(\frac{2\boldsymbol{\pi}}{T}\right)^2 \boldsymbol{\theta}. \tag{2.4}$$

Clearly in this scheme the accuracy of the result depends on the accuracy of the evaluation of the factors I, K, T and $\boldsymbol{\theta}$.

An important variation on Cavendish's work was perfected by Heyl & Chrzanowski (1942) and used by Luther & Towler (1982). In this method (commonly referred to as the time-of-swing method), the change in the natural frequency of oscillation of the torsion balance due to the source masses is determined with the masses in the 'near' and 'far' positions (see figure 2). In the near positions the Newtonian attraction of the sources reduces the period of oscillation and the period is extended with the masses in the far positions. We can follow through a similar analysis to that given above to find an expression for

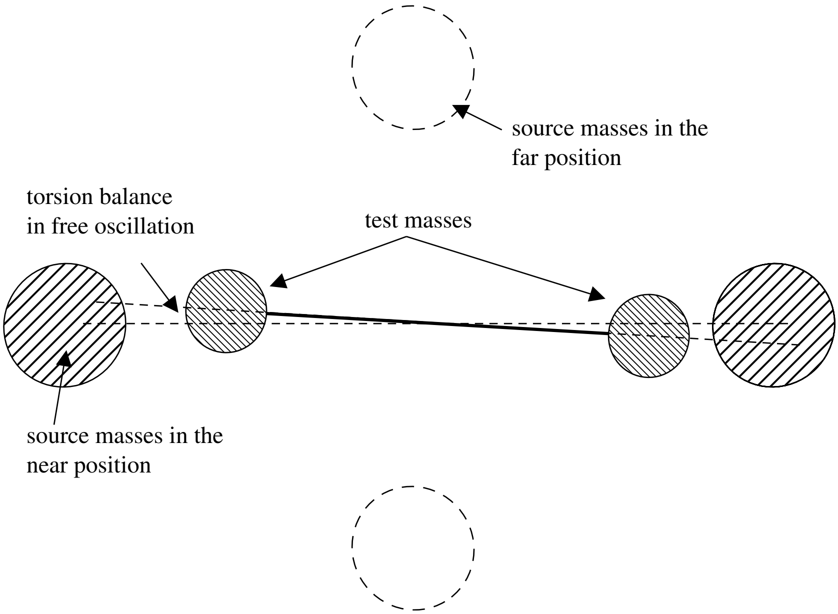

Figure 2. Schematic plan view of the time-of-swing method. Source masses in the near and far positions reduce and increase, respectively, the period of oscillation of the torsion balance.

G in terms of the geometry of the apparatus

$$G=\frac{\boldsymbol{\kappa}(\boldsymbol{\omega}_2)-\boldsymbol{\kappa}(\boldsymbol{\omega}_1)-I(x_{\mathrm{t}},\boldsymbol{\rho}_{\mathrm{t}})(\boldsymbol{\omega}_2^2-\boldsymbol{\omega}_1^2)}{K_2'+K_1'}, \tag{2.5}$$

where $\boldsymbol{\kappa}(\boldsymbol{\omega}_{1,2})$ is the stiffness of the torsion fibre at angular frequencies $\boldsymbol{\omega}_{1,2}$ and $K_{1,2}'(x_{\mathrm{t}},x_{\mathrm{s}1,2},\boldsymbol{\rho}_{\mathrm{t}},\boldsymbol{\rho}_{\mathrm{s}})$ are the geometrical factors giving the gravitational contribution to the restoring torque. Notice that K'_2 is taken to be a positive number, i.e. it is the magnitude of the negative gravitational stiffness. In each case the subscripts indicate the near or the far position of the source masses. In Heyl's analysis, the stiffness of the torsion fibre was considered to be constant for each source mass position, however, as discussed below in §2*b*, this is generally not the case.

A final method that I will discuss was that adopted by Michaelis *et al.* (1996) and involved determining G in terms of electrostatic forces. I will refer to this as the electrostatic force-balance method and is discussed in more detail in §2*b* below. The torque generated between electrodes with potential difference V_{E} with cross-capacitance C (defined in §1*e* below) is given approximately by

$$\boldsymbol{\Gamma}_{\mathrm{E}}=\frac{1}{2}\frac{\mathrm{d}C}{\mathrm{d}\boldsymbol{\theta}}V_{\mathrm{E}}^2. \tag{2.6}$$

The change in capacitance versus angle is measured using a capacitance bridge and an autocollimator or interferometer. The voltage, V_{E}, required to balance the Newtonian torque is determined using a torsion balance as a 'null-detector'. Newton's constant can then be found to be

$$G=\frac{1}{2}\frac{\mathrm{d}C}{\mathrm{d}\boldsymbol{\theta}}\frac{V_{\mathrm{E}}^2}{K}, \tag{2.7}$$

where the factor K is determined as before (equation (2.1)).

The quality of the result of any determination depends on the ability of the sensor to detect the gravitational torque or gradient of torque in the presence of random noise sources and the ability of the experimentalist to determine the factors such as K and K' from the physical characteristics of his experiment and Newton's inverse square law. Read-out noise, mechanical vibrations, thermal noise, floor tilt and changes in local gravity gradients will, in general introduce random, or type A, uncertainties, in the measured values of $\boldsymbol{\theta}$, $\boldsymbol{\omega}_{1,2}$ and V_{E} in the methods described above. These quantities are also subject to calibration uncertainties, which are regarded as uncertainties of type B or systematic uncertainties. Random uncertainties, such as scatter in the measurements of mass separations, fluctuations in measured capacitances etc, are also present in the factors K, $K'_{1,2}$ and $\mathrm{d}C/\mathrm{d}\boldsymbol{\theta}$. From the general description given above of the subset of the methods employed for determining G, it can be appreciated that both types of noise source enter differently into each method. An obvious example of this is that, in the Cavendish and time-of-swing methods, the test mass values appear in the numerator and denominator in equations (2.4) and (2.5) and therefore the uncertainty in their values does not contribute to the value of G. On the other hand, the electrostatic force-balance method does not rely in any way on the elastic quality of the torsion fibre. It is important that a diversity of methods are used to ensure that a final adopted value will be as free as is possible from uncertainty. The author has collaborated with colleagues at International Bureau of Weights and Measures (BIPM) for a number of years on a determination of G (Quinn *et al.* 2001). A key aspect of this work is that we have decided to employ the three methods described above in an attempt to produce a robust value for G. Ideally the different methods should be completely independent. The degree of independence between measurements can be calculated using the cross-correlation of the uncertainties. In a particular method the value of G_i is calculated from a number of measured parameters, x_j, using equations such as (2.4), (2.5) and (2.7)

$$G_i = f_i(x_j). \tag{2.8}$$

The variation of G with the parameters can be calculated by partial differentiation

$$\Delta G_i = \sum_j \frac{\partial f_i}{\partial x_j} \Delta x_j = \sum_j \partial G_{ij}. \tag{2.9}$$

Now suppose that we wish to calculate a final value of G from a weighted combination of the G_is, for example

$$G = \sum_i \alpha_i G_i. \tag{2.10}$$

The uncertainty in the final value can be calculated using

$$\Delta G^2 = \left(\sum_i \boldsymbol{\alpha}_i \sum_j \partial G_{ij} \right)^2. \tag{2.11}$$

It is convenient to write this as

$$\Delta G^2 = \underline{\boldsymbol{\alpha}}^{\mathrm{t}} \underline{\underline{S}} \underline{\boldsymbol{\alpha}}, \tag{2.12}$$

where $\underline{\boldsymbol{\alpha}}$ is the vector representing the weights, $\underline{\boldsymbol{\alpha}}^{\mathrm{t}}$ is its transpose and $\underline{\underline{S}}$ is the variance–covariance matrix. The diagonal elements of $\underline{\underline{S}}$ are the variances of each experimental method

$$S_{ii}=\sum_j(\partial G_{ij})^2, \tag{2.13}$$

whereas the off-diagonal terms are the covariances of methods i and j, for example

$$S_{ij}=\sum_l \partial G_{il}\sum_m \partial G_{jm}\equiv\sum_k \partial G_{ik}\partial G_{jk}. \tag{2.14}$$

In writing the above equation we have assumed that only systematic and random uncertainties that are common to each measurement method are correlated and lead to a finite covariance term. We can quantify the degree of cross-correlation between any two methods as follows

$$CC(i,j)=\frac{S_{ij}}{\sqrt{S_{ii}S_{jj}}}. \tag{2.15}$$

The smaller the value of $CC(i,j)$ between two measurements, the more convincing the final result for G will be. We have employed this method of estimating the correlations in our result that was published in 2001 (*op. cit.*). Here we found values of G using the Cavendish and electrostatic force-balance methods. The positions of the source and test masses were not measured independently in each determination and therefore the uncertainties in these parameters were correlated. It should be noted that, even if the positions of the masses were measured separately for each determination, the systematic component of the uncertainty in the length metrology would correlate assuming that the same coordinate measurement machine was employed in each case. It is also interesting to note that a systematic uncertainty in the calibration of the autocollimator, used to determine the angular displacement in the Cavendish determination and the gradient of the capacitance in the electrostatic force-balance method, resulted in type B uncertainties in G that were anti-correlated. This can be easily understood as the angle appears in the numerator of equation (2.4) but in the denominator in equation (2.7). This anti-correlation was not exact because of the range of angle employed for the measurement of the capacitance gradient was significantly larger than the angular deflection generated in the Cavendish method and different corrections for nonlinearity were applied in each case. The degree of correlation between the measurements was calculated using equation (2.5) to be -0.15 (i.e. anti-correlated).

(*a*) *Random uncertainities*

The key characteristic of a mechanical sensor is the signal to noise ratio. I will assume that the sensor is some sort of harmonic oscillator, such as Cavendish's balance, with restoring torque, $\boldsymbol{\kappa}$, damping constant, β, and moment of inertia I and described by the following differential equation

$$I\ddot{\boldsymbol{\theta}}+\beta\dot{\boldsymbol{\theta}}+\boldsymbol{\kappa\theta}=\boldsymbol{\Gamma}. \tag{2.16}$$

The gravitational signal-torque is here indicated by $\boldsymbol{\Gamma}$. In the case of a classical torsion fibre the elastic stiffness is given as

$$\boldsymbol{\kappa}=\frac{\boldsymbol{\pi\gamma}d^4}{32L}, \tag{2.17}$$

where $\boldsymbol{\gamma}$ is the shear modulus of the fibre, d is its diameter and L its length.

Boys (1895) argued famously that the critical characteristic of a mechanical detector was its period of free oscillation ($T=2\boldsymbol{\pi}\sqrt{I/\boldsymbol{\kappa}}$). He pointed out that, as the stiffness of the torsion fibre scaled as the fourth power of its radius and the supportable load scaled as its square, the deflection of the balance, for a given gravitational torque, was maximized by using the finest fibre that was practical and the smallest possible suspended masses. Boys' innovation was the development of extremely fine torsion fibres of quartz, and he was able to use these to great advantage in a torsion balance that was less than 1 inch long. This compared with Cavendish's beam that had a length of 73 inches. Boys' concerns over the displacement sensitivity of his instrument can be restated in modern terminology as the need for the intrinsic noise in the sensor of angular motion to be sufficiently small that the gravitational signal can be resolved in a given integration time. If the read-out noise is assumed to have a white spectrum given by $\boldsymbol{\delta\theta}$ rad Hz$^{-1/2}$, the signal torque that is resolvable in an averaging time of $\boldsymbol{\tau}$ seconds is

$$\boldsymbol{\Delta\Gamma} = \frac{\boldsymbol{\kappa\delta\theta}}{\sqrt{\boldsymbol{\tau}}}\ \text{Nm}. \tag{2.18}$$

This also assumes that the signal is modulated at a frequency lower than the natural frequency of oscillation ($1/T$). The signal to noise ratio is simply $\boldsymbol{\Gamma}/\boldsymbol{\Delta\Gamma}$. The signal to noise ratio given in equation (2.18) is of importance for Cavendish's method and the electrostatic force-balance method. Clearly, sensor noise is also a limiting factor in the time-of-swing method however I will not discuss this here. In order to optimize the signal to noise ratio there is a need to employ a mechanical detector with a small $\boldsymbol{\kappa}$. However, this can be mitigated by the use of modern displacement sensors such as the superconducting quantum interference device, capacitive sensors, interferometers and autocollimators. In conjunction with a balance beam the length of Boys', an interferometer can achieve shot-noise-limited angular sensitivities of order 10^{-12} rad Hz$^{-1/2}$ with a micro-Watt of optical power. The commercial autocollimator used at BIPM has a sensitivity of about $5n$ rad with an integration time of 0.05 s. This compares with the sensitivity of Boys' optical lever of 4μ rad, which was a significant achievement at the time. A possible advantage of using a stiffer suspension and a higher sensitivity angular sensor would be that more mass could be supported on the thicker fibre, to increase the signal torque and the signal to thermal noise ratio (see below). Further the cadence of the experiment, which is determined by the modulation period of the gravitational torque $T_{\rm m}$, could be increased (given that the settling time $T_{\rm m}> T\propto\boldsymbol{\kappa}^{-1/2}$) in order to reduce the effect of drift in the equilibrium angle of the fibre, thermally induced drifts, gravitational gradient noise and any source of $1/f$ noise. It should be noted that, in practice, modulation of the Newtonian torque requires the motion of fairly massive sources and this gives, typically, $T_{\rm m}\approx 1000$ s.

However, the key component of the modern design of an experimental determination of G that was missing from Boys' argument is the need to avoid the intrinsic thermal noise and accompanying anelasticity in the suspension itself. It is now well known (see Speake *et al.* 1999*a*,*b* for a review) that most metals exhibit internal damping that increases with decreasing frequency. This can be expressed in terms of an imaginary elastic modulus

$$\gamma = \gamma_{\rm r}(1 + \Delta\{f(\boldsymbol{\omega}) + {\rm i}g(\boldsymbol{\omega})\}), \tag{2.19a}$$

where $\varDelta$ is referred to as the modulus defect. Experiments have shown that $g(\boldsymbol{\omega})$ is independent of frequency and this is consistent with a distribution of relaxation processes in the material that is constant in strength as a function of relaxation time between two extremes, $\boldsymbol{\tau}_0$ and $\boldsymbol{\tau}_\infty$. Anelastic behaviour can then be accurately parametrized as

$$f(\boldsymbol{\omega}) = \frac{1}{2}\ln\left(\frac{1+\boldsymbol{\omega}^2\boldsymbol{\tau}_\infty^2}{1+\boldsymbol{\omega}^2\boldsymbol{\tau}_0^2}\right), \tag{2.19b}$$

and

$$g(\boldsymbol{\omega}) = \tan^{-1}(\boldsymbol{\omega}\boldsymbol{\tau}_\infty) - \tan^{-1}(\boldsymbol{\omega}\boldsymbol{\tau}_0). \tag{2.19c}$$

The imaginary component of the stiffness given in equation (2.19*a*) can be interpreted as a new source of dissipation. In a regime where the $1/\boldsymbol{\tau}_0 \gg \boldsymbol{\omega} \gg 1/\boldsymbol{\tau}_\infty$ the effective damping parameter due to anelasticity in equation (2.16) becomes

$$\beta_{\mathrm{an}} \approx \varDelta\frac{\boldsymbol{\kappa}}{\boldsymbol{\omega}}. \tag{2.20}$$

The values of $\boldsymbol{\tau}_0$ and $\boldsymbol{\tau}_\infty$ are material dependent typically with $\boldsymbol{\tau}_0 \ll T$ for metals (Kimball & Lovell 1927). To our knowledge, a value of $\boldsymbol{\tau}_\infty$ has not been reliably determined for any material, although all measurements of dissipation in polycrystalline metals are consistent with equations (2.19). It is reasonable to assume that $\boldsymbol{\tau}_\infty \gg T_{\mathrm{m}}$. The fluctuation–dissipation theorem can be used to calculate the power spectral density of torque fluctuations due to thermal noise that is produced, presumably by, dislocation motion in the material

$$\boldsymbol{\delta\Gamma}^2 = 4k_{\mathrm{B}}T_{\mathrm{A}}\boldsymbol{\varDelta}\frac{\boldsymbol{\kappa}}{\boldsymbol{\omega}} \quad \mathrm{N^2\,m^2\,Hz^{-1}}, \tag{2.21}$$

where T_{A} refers to the ambient temperature. This equation illustrates, firstly, the important point that mechanical springs exhibit $1/f$ noise and, secondly, that the thermal noise is proportional to the stiffness of the suspension. Thermal noise in the torsion balance suspension limits the sensitivity of the time-of-swing method as discussed by Chen & Cook (1993). Importantly, internal losses in the fibre lead to anelasticity where the stresses in the fibre decay away over long periods (a problem of which Boys and Cavendish were well aware). This problem can be reduced by the use of servo-control, which can minimize the angular motion of the fibre in response to any signal. Note that the presence of a servo-controller does not significantly modify equations (2.19) or (2.21). Anelasticity is undesirable as the need to allow the stresses to dissipate imposes a minimum value on T_{m}. So Boys' intuition for the design of G experiments is still valid today but not only for the reasons he stated! Quartz (fused silica), due to its low modulus defect, is the material of choice for the suspensions of mirrors in gravitational wave antennae (Cagnoli *et al.* 2000). Tungsten filament is generally preferred for torsion balance suspensions as it ensures that the test masses are electrostatically grounded.

It is interesting to point out here that torsion fibres that are not of circular cross-section violate Boys' scaling rule. The rotational stiffness of a fibre that has a rectangular cross-section of width, b, which is much larger than its thickness, t, is given as

$$k_s = \frac{\gamma b t^3}{3L} + \frac{W b^2}{12L}\left[1 + \left(\frac{t}{b}\right)^2\right], \tag{2.22}$$

where W is the weight of the suspended test mass assembly (Quinn *et al.* 1997). There are evidently two components to the rotational stiffness, one due to elasticity that gives rise to anelasticity and one due to the gravitational field of the Earth that is lossless and free of thermal noise. At the BIPM we have employed a, so-called, torsion strip to great advantage. In this work, a copper–beryllium torsion fibre of $t=30\ \mu\text{m}$, $b=2.5$ mm and $L=160$ mm is used to suspend a torsion bob of total mass of about 6 kg. The resulting elastic torsional stiffness is about 8×10^{-6} Nm rad^{-1}, the gravitational stiffness is 2×10^{-4} Nm rad^{-1} and the pendulum has a period of oscillation of approximately 125 s. The gravitational torque resulting from the attraction of four source masses of 12 kg on four test masses of 1.2 kg amounts to about 1.7×10^{-8} Nm. This is a factor of about 3000 larger than that used by Boys. In conjunction with the autocollimator, the signal to noise as defined by equation (2.18) is approximately 8.5×10^4 for an averaging time of 1 s. The modulus defect for copper–beryllium alloy has been measured to be about 5×10^{-5} (Quinn *et al.* 1997). As the lossy restoring torque is dominated by the gravitational term, the ring-down time (i.e. the time for an oscillation amplitude to decay by a factor $1/e$) is about five months. The device therefore has a mechanical quality factor Q of 3×10^5. The thermal noise (equation (2.21)) amounts to 2×10^{-15} Nm Hz$^{-1/2}$.

Another serious source of noise is the effect of ground vibrations. Man-made, or cultural, noise produces ground vibrations with an acceleration spectral density of around 10^{-5} ms^{-2} Hz$^{-1/2}$ upward from about 1 Hz, although at the quietest sites this is reduced by a factor of 10 or so. The so-called microseismic peak is at a frequency of about 0.1 Hz and this is due to travelling waves generated by the action of ocean waves on continental shelves. The result of these disturbances is to produce vertical, horizontal and tilt displacements of the laboratory. Asymmetric solar heating of the outer walls of laboratories can generate low-frequency tilt. One problem with the torsion balance is its susceptibility to vibrational noise. In particular horizontal motion of the point of attachment of the fibre results in simple pendulum oscillation of the balance beam. This motion can generate torques on the torsion balance. This can be understood by considering the fibre to be rigid and connected to its point of suspension with a perfect gimbal. Simple pendulum oscillations about horizontal axes through the gimbal (x and y) couple strongly to torques about the z-axis, in the case where the moments of inertia about x and y axes are different (Speake *et al.* 2001). It also follows that this difference in the moments of inertia is proportional to the spherical quadrupole moment of the test mass assembly, q_{22}. Luther & Towler (1982) made a significant contribution when they were able to reduce this effect by attaching a copper disk to a short length of thick fibre from which a longer length of normal fibre and the test mass assembly was suspended. The outer edge of the disk sat in the field of a permanent magnet, which damped the simple pendulum motion of the torsion balance. Another problem with conventional torsion balances is that vertical ground vibrations cause extensions in the fibre (Adelberger *et al.* 2003), which in turn can generate rotation of the suspended test mass assembly due to helicity in the fibre's longitudinal elasticity. In the above I have assumed that ground vibrations cannot be eliminated using isolators of the sort now commonly used for atomic force microscopes. It is difficult to apply this technology to effectively vibrationally isolate torsion balances because they are sensitive to low-frequency tilt. This is due to slight asymmetries in the cross-section of the torsion fibre. The BIPM torsion strip balance generates a rotation of the suspended test mass assembly of the order of 0.4% of the ground tilt. While it is possible to construct anti-vibration mounts that reduce vibrations at frequencies above 1 Hz or so, it is not easy to extend this frequency range to the resonant frequency of torsion balances and to combine this with a high degree of tilt stability.

In the BIPM work, we have recently installed a knife-edge gimbal to decouple the torsion balance from tilt and to damp the simple pendulum mode of oscillation. Magnetic dampers are attached to the gimbal to damp the simple pendulum mode of the torsion balance. In addition we use a torsion balance bob that has a fourfold symmetry which eliminates to a high precision the spherical quadrupole and thus minimizes the dynamical coupling described above.

A further source of spurious torques comes from changes in the gravitational torques generated by masses in the vicinity of the experiment. The torsion balance cannot be shielded from these effects. If we have a torsion balance test mass assembly with an l-fold symmetry supporting an array of l point masses of value m (with the first mass of the pattern being located at $\boldsymbol{\phi}=0$) at a radius r from the axis of rotation of the torsion balance, the gravitational torque acting on it due to a point mass M at distance R and at polar angle $\boldsymbol{\theta}_{\mathrm{s}}$ from the centre of the test mass assembly, is given as

$$\boldsymbol{\Gamma}_{gg} = Gl\frac{(2l-1)!!}{(2l-2)!!}\mathrm{mM}\frac{r^l}{R^{l+1}}(\sin\boldsymbol{\theta}_{\mathrm{s}})^l \sin l\boldsymbol{\phi}, \tag{2.23}$$

where the !! symbol indicates the double factorial, i.e. $n!!=n(n-2)(n-4)\ldots$, etc. For the BIPM work, $r=120$ mm. This formula can be used to calculate the approximate magnitude of the torque generated by one source mass by taking $R=214$ mm as the distance of the source mass from the torsion axis. All determinations of G, except that of the BIPM, have employed only test mass assemblies with twofold symmetry. The torques due to external masses then decay as (r^2/R^3). This has led to significant sensitivity to 'so-called' gravity gradient noise but which is effectively eliminated in the BIPM determination with $l=4$, even though the diameter of the torsion balance bob is considerably larger than that employed in determinations that use conventional fibres with circular sections and less massive test mass assemblies.

It is tempting to conceive of alternatives to the torsion fibre as a means of suspending the test mass assembly. The main motivation for this endeavour has been to avoid the dependency of the stiffness of the suspension on its load. The fruit of our own search for such a suspension led us the rediscovery of the torsion strip.

In the early days beam balances and vertical pendulums were used with modest success (see Gillies 1997 and references therein). It should be noted, however, that Schlamminger *et al.* (2002) recently successfully used a beam balance. Faller and his colleagues have constructed detectors using floats in water (Keyser *et al.* 1984) and active magnetic suspensions at room temperature (Koldewyn 1976, unpublished dissertation). As noted in §1 Michaelis *et al.* have constructed a torsion balance that was suspended in mercury pool. In general, these ideas have been successful in being able to increase the magnitude of the gravitational torque while maintaining low stiffness, and these instruments have coincidentally been less sensitive to the effects of ground vibration. However, they have fallen foul of problems associated with thermal noise and stability in the zero point of the suspension (analogues to anelasticity). An example of this is the vertical pendulum (Quinn *et al.* 1992). This device comprises a vertical beam, with test masses at either end, suspended just above its centre of mass by a flexure pivot. As the cross-section of the flexure increases in order to support more mass, the stiffness also increases and so too does the thermal noise and the magnitude of the anelastic after-effect. The net stiffness can be reduced by moving the centre of mass of the vertical pendulum above the pivot. This lengthens the period but also increases the anelastic damping and leads to excessive tilt sensitivity. The torsional stiffness, about a horizontal axis, due to the elasticity of a simple vertical strip flexure pivot of length Λ, thickness t and width b supporting a total weight W is

$$\kappa_{\mathrm{f}} = \frac{(WE\sigma)^{1/2}}{2}\left\{\coth \alpha\Lambda + \frac{\alpha\Lambda}{\sinh^2\alpha\Lambda}\right\}, \tag{2.24}$$

where E is Young's modulus, σ is the second moment of area of the strip ($\sigma = bt^3/12$) and $\alpha = (W/E\sigma)^{1/2}$ (Quinn *et al.* 1995). It is apparent from equation (2.24) that the losses in the suspension in fact scale as the square root of the load. An additional problem, that is difficult to solve in practice, is that the torques acting at the point of attachment of the flexure due to the oscillation of the pendulum also increase with the load and flexure stiffness. Losses in the support of the flexure due to stick–slip mechanisms are then difficult to avoid. This is also true with the torsion balance, however, again with low mass systems with low stiffness fibres the resulting stresses in the suspension can be made relatively free from dissipation. In the BIPM balance, the ends of the torsion strip are flared out in order to ensure energy loss at the attachment point is minimized.

As mentioned above, there is now considerable interest in instruments that can detect gravitational forces at ever decreasing mass separations. This makes it important to find ways of eliminating the motion of the test masses due to parasitic modes of oscillation that can be excited by ground vibrations. In order to study gravitation at ranges of the order of 10 μm requires test mass configurations of similar dimensions. A new instrument (Hammond *et al.* 2004) is being developed at the University of Birmingham that is based on a superconducting levitation system. A persistent magnetic field is generated by a superconducting coil that is wound on a substrate that has the form of part of a sphere. The resulting magnetic pressure levitates a float carrying a test mass assembly. The float is in the form of a part of spherical shell. In principle, the levitation system should behave as a perfect gimbal in that rotations of the coil assembly about any axis do not couple torques to the float. In addition, at low frequencies, the float is effectively pivoted at its centre of magnetic buoyancy, which is its centre of figure. The centre of mass of the suspended test mass assembly is a few mm below the centre of buoyancy. This arrangement is immune to a high degree to horizontal accelerations of the bearing. We believe that this device is well suited to the measurement of weak forces acting over short ranges such as the Casimir force and new interactions suggested by string theory as mentioned in §1*b*.

(b) Systematic uncertainties

The sources of systematic uncertainties in determinations of G are legion and usually prove more problematic than the sources of random noise discussed above. Of particular concern is the density homogeneity of the source and test masses, accurate determination of mass separations and calibration of the gravitational torque in units of the SI.

In most experimental configurations averaging the torques generated with the masses in different orientations can eliminate the effect of their inhomogeneity. In the BIPM determination of G, gradients in the density across the diameters of the source mass cylinders of between one and two parts in 10^4 result in a change in torque of 90 p.p.m. in the worst case. These effects are easily eliminated by determining torques with the masses rotated about their axes into 3 positions. Such density gradients generate dipolar spherical multipole moments in the source masses. The effect that these have on the gravitational force that they produce on the test masses can be conveniently calculated using the formalism described by D'Urso & Adelberger (1997). Length metrology can nowadays be carried out using coordinate measuring machines, as mentioned in §1. Under the best conditions these devices can achieve accuracies of better than 1 μm over distances of about 10 cm. Thermal expansion of most metals would render such measurements inaccurate at these levels if the temperature was not controlled to better than a fraction of a Kelvin. At BIPM the G experiment is actually set-up on a coordinate measuring machine and housed in a cabin with temperature control of a few tenths of a Kelvin.

Following recent improvements in our understanding of anelasticity, it is now clear that Cavendish's and Boys' result would have been in error due to anelasticity. In Cavendish's method the magnitude of the gravitational torque is determined, as shown in equation (2.2) using the elastic stiffness of the fibre. A systematic error arises because the stiffness of the fibre is frequency dependent due to anelasticity. As explained in §2a, a frequency independent imaginary component of the modulus defect will be accompanied with a real component that varies with frequency (equation (2.19b)). As a result the stiffness of the fibre at the low frequency at which the experiment takes place (i.e. the frequency of the reversal of the gravitational torque) is less than at the higher frequency at which the torsion balance oscillates ($T_\mathrm{m} \gg T$). Thus the gravitational torque is over estimated, as is the value of G. The fractional error in the value for G is hard to estimate as the actual stiffness or damping of the fibre cannot be easily measured at the frequency of the experiment. Assuming the validity of the model described in §1d the fractional overestimate in G is $\boldsymbol{\varDelta}\ln(T_\mathrm{m}/T)$. Anelasticity introduces an uncertainty of only about 2 p.p.m. in the BIPM determination of G employing Cavendish's method using the torsion strip due to the dominance of the gravitational component in the rotational stiffness (equation (2.22)).

A value for G obtained using the time of swing method will also be in error due to anelasticity. The fibre stiffnesses at the two frequencies are not equal. According to the equations (2.19b), G is overestimated by a factor $\boldsymbol{\varDelta}/\boldsymbol{\pi}$. This was pointed out by Kuroda (1995) and was one of the key reasons why the CODATA increased the uncertainty on G in 1998.

Gundlach & Merkowitz (2000) performed a determination of G with the smallest ever reported uncertainty and avoided anelastic effects by developing a method first devised by Beams (Rose *et al.* 1969). In the experiment of Rose *et al.* (1969) the torsion balance, carrying a test mass assembly with a quadrupole geometry, and two source masses are mounted on a rotating table. As the torsion balance twists under the gravitational attraction of the source masses, the turntable is driven to produce an angular rotational acceleration so as to maintain the angular deflection of the beam constant within the rotating frame. A value for G is measured in terms of the constant angular acceleration of the rotating table. Gundlach and colleagues (Gundlach *et al.* 1996) mounted the balance and masses on independent turntables. The turntable supporting the torsion balance was rotated at a nominally constant rotation rate. The rotation rate is controlled by a servo system in such a way that the torsion balance remained aligned with an autocollimator in the rotating frame. This results in a sinusoidal modulation of the rotational acceleration: as the test mass comes into alignment with the source masses, the turntable accelerates and subsequently decelerates as the test masses move past the aligned configuration. The product of the amplitude of the sinusoidal acceleration of the torsion balance and the moment of inertia of the source mass assembly gives the Newtonian torque. A value for G can then be derived by dividing this product by a factor that depends on the geometry of the test masses and source masses. As the torsion balance is used as a null detector for the turntable drive control system, the torsional stress in the fibre is kept to a minimum and so, therefore, are the effects of anelasticity.

Gundlach *et al.* noticed that for test masses in the shape of thin plates, the spherical quadrupole and the moment of inertia are proportional. Thus with this arrangement the value of G did not depend critically on the geometry of the test mass assembly. Torques due to gravity gradients coupling to the test mass assembly are averaged out by also rotating the source masses.

Another class of G determinations employs some kind of external torque to balance the gravitational torque. The experiments that have used beam balances fall into this category where the gravitational couple is calibrated in terms of calibration weights (Schlamminger *et al.* 2002). The electrostatic force-balance method described in §2 also falls into this category. The traditional configuration of electrodes used to apply electrostatic torques to the torsion balance was developed by Lord Kelvin (Maxwell 1892) and is known as a quadrant electrometer. The basic design comprises a cylindrical cavity that is divided into quadrants (electrodes 1 and 2 in figure 3). Electrode 3 is butterfly shaped, is suspended from the torsion balance and lies at the centre of the cylindrical cavity as shown. Voltages V_1 and V_2 are applied to opposite quadrants of the fixed plates while a voltage V_3 is applied to the butterfly electrode. The electrometer can be, in principle, calibrated by measuring the change in capacitance of the electrodes with angle. The energy in the electrostatic field can be written in general as (Smythe 1989)

$$W = \tfrac{1}{2}\underline{V}^{\mathrm{t}}\,\underline{\underline{C}}\,\underline{V}, \tag{2.25}$$

where $\underline{V}$ is a vector of potentials applied to the various electrodes, $\underline{V}^{\mathrm{t}}$ is its transpose and $\underline{\underline{C}}$ is a matrix of self- and mutual capacitances. It is interesting to note that the expression given by Maxwell is in terms of the self-capacitances of the electrodes 1 and 2

$$W = \tfrac{1}{2}(V_1 - V_2)\{(V_1 - V_3)C_{11} - (V_2 - V_3)C_{22}\}. \tag{2.26}$$

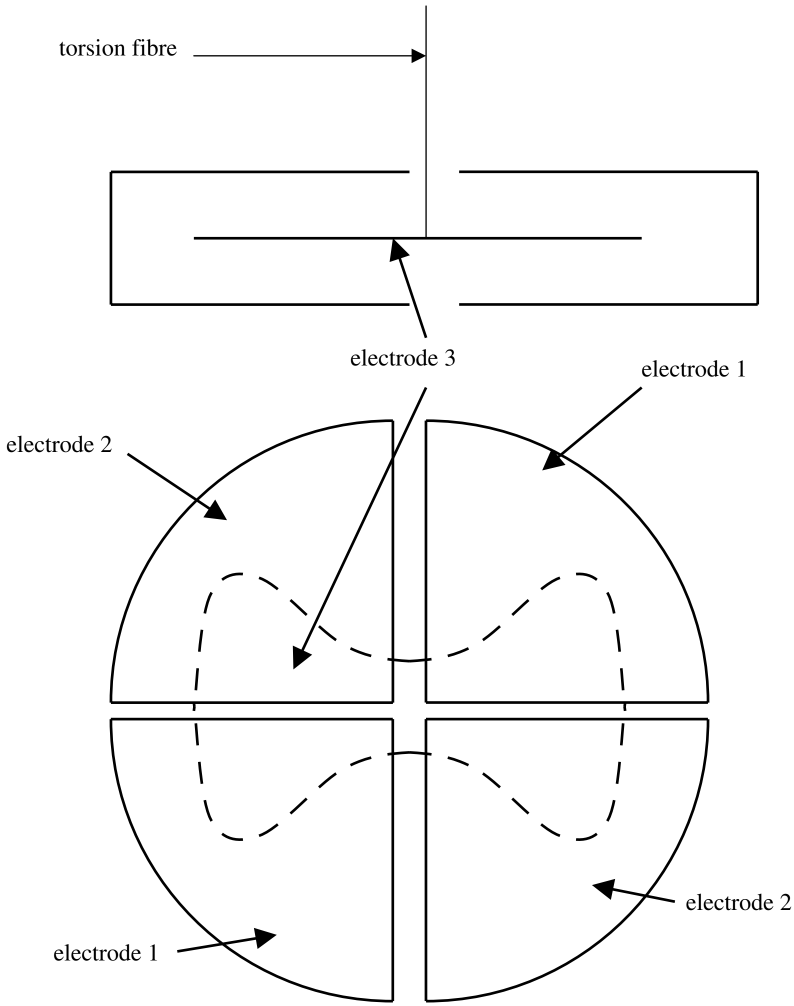

Figure 3. Schematic of Kelvin's quadrant electrometer.

The torque generated is

$$\boldsymbol{\Gamma} = \frac{\partial W}{\partial \boldsymbol{\theta}}. \tag{2.27}$$

The self-capacitances cannot be easily measured. Modern measurements of capacitance use the, so-called, three terminal technique (Leslie 1961) where the 'cross-capacitance', C_{cij}, between two electrodes i and j can be determined accurately provided that they are surrounded by a shield electrode which is connected to bridge ground. In this case the sum of all the charges within the shielded volume must be zero and we can write the self-capacitances in terms of the mutual capacitances

$$C_{ii} = -\sum_{j \neq i} C_{ij}. \tag{2.28}$$

From the definition of the cross-capacitance and mutual capacitance we have

$$C_{ij} = -C_{cij}, \tag{2.29}$$

and the electrostatic energy (2.25) can be written in terms of cross-capacitances, after some manipulation, as

$$W = \frac{1}{2}\sum_{i,j} C_{ij}(V_i - V_j)^2. \tag{2.30}$$

If the capacitance of the electrodes are not influenced by other nearby conductors it can be shown, with some assumptions about the symmetries of the electrometer, that equation (2.26) is equivalent to equation (2.30). Note from figure 3 that the design of the classical electrometer was such that the moving inner electrode was almost completely surrounded by the fixed electrodes. Any capacitance between the fixed electrodes and any other fixed spurious conductor would not make a contribution to equation (2.27).

Michaelis *et al.* (1996) employed a quadrant electrometer and effectively used the expression given by Maxwell to calibrate their G signal. To their credit, the PTB team has been active in searching for an explanation for the discrepancy between their value and those obtained by other groups (Michaelis *et al.* 2005). It has come to light that this discrepancy arose due to a grounded cylindrical shield surrounded their electrometer: the cross-capacitance between the fixed electrodes and this shield was modified by the rotation of the central moving electrode that was fixed to their mercury suspension. The contribution[1] to the total energy due to the cross-capacitances to this fourth electrode was not taken into consideration and would have been responsible for a shift of $+0.71\%$ in the value of G. The PTB authors also state that they underestimated the uncertainty in their calibration of the variation of capacitance with angle by an unknown factor that could have reasonably increased their overall uncertainty to a few 100 p.p.m.

We have developed a special electrometer for the G determination at BIPM. It comprises two electrodes mounted adjacent to each test mass. Each rod is positioned so that the capacitance gradient between it and the cylindrical mass is a maximum which ensures that the variation of capacitance with angle is linear to first-order. Calibration of the actuator demands that we measure the variation of the three capacitances with angle. The vacuum can is connected to the torsion balance so that any change in capacitance between the rods and vacuum can produced by motion of the torsion balance is included in the equation (2.25). A schematic of the arrangement of electrodes is shown in figure 4.

Another problem with electrostatic calibration arises if there is dissipation in the capacitances being measured, due to the presence of plastic cable insulators, for example. This will result in a frequency dependence of the capacitance and, if the electrostatic forces required to balance the gravitational torque are not generated by a.c. voltages at the operating frequency of the bridge, equation (2.25) cannot be used to accurately determine the gravitational torque. This has been discussed elsewhere (Speake *et al.* 1999*a*,*b*). At the Measurements Standards Laboratory in New Zealand, Armstrong & Fitzgerald (2003) determined G using a torsion strip balance. They used an electrostatic torque to balance their gravitational torque. However, they calibrated the electrostatic actuator with the inertial force generated by an angular acceleration of the turntable upon which the balance was mounted.

[1] This possibility was pointed out to Michaelis *et al.* the author.

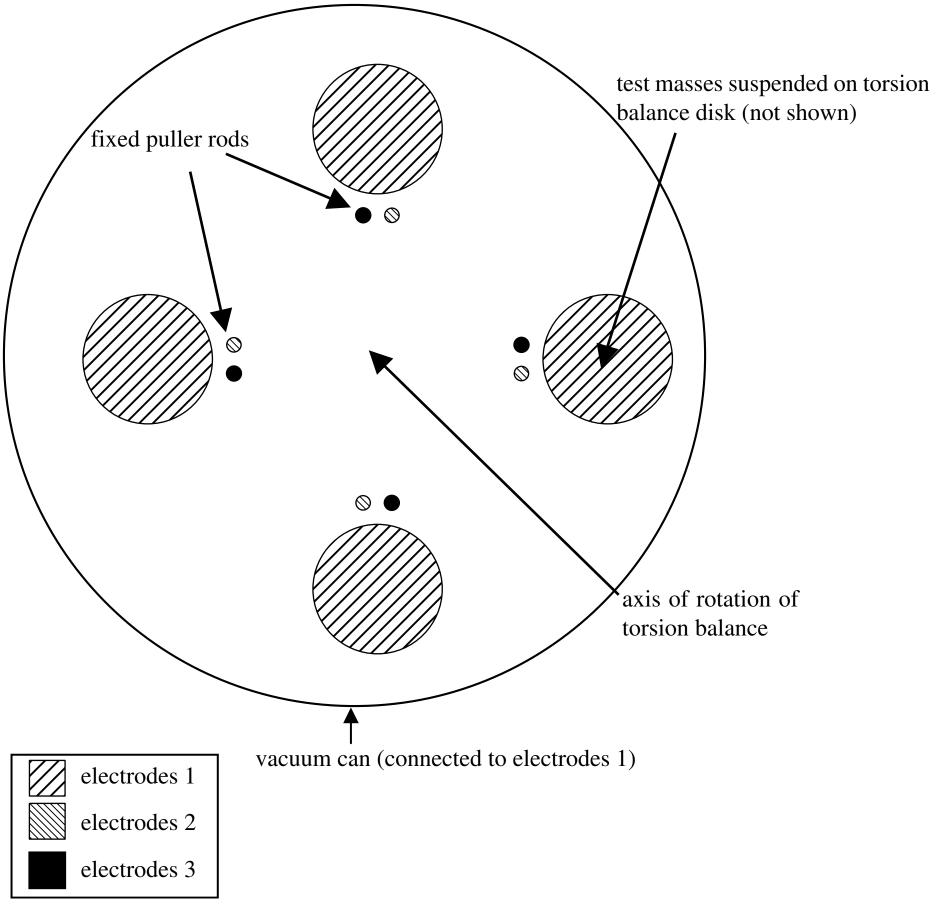

Figure 4. Schematic plan-view of arrangement of electrodes that comprise the electrostatic actuator in the BIPM torsion strip balance. The assignment of electrodes in equation (2.30) is also shown.

3. Conclusions and the future

I have discussed the design principles of experiments that determine Newton's constant of gravitation and the resolution of the controversies of 1998. In doing this I have alluded to some of the key historical and contemporary experiments that have contributed to this field. The most recent experimental values are listed in table 1 and plotted in figure 5 together with the CODATA values for 1986 and 2002. There is good agreement between Gundlach & Merkowitz (2000), Schlamminger *et al.* (2002) and Armstrong & Fitzgerald (2003). The BIPM value (Quinn *et al.* 2001) is 200 p.p.m. larger than the CODATA value. At BIPM work has continued and the apparatus has been improved, notably by the addition of the gimbal damper mentioned in §2*a*. We intend to determine G with three methods: we are repeating the work published in 2001 using the electrostatic force-balance method and Cavendish's method and we are also pursuing the time-of-swing method. We believe that performing three quasi-independent determinations of G will give a robust result as discussed in §2.

Table 1. *Recent values of Newton's constant of gravitation with uncertainties less than 50 p.p.m. and the CODATA values*

author	value of G ($\times 10^{11}$ kg^{-1} m^3 s^{-2})	uncertainty (p.p.m.)
CODATA (1986)	6.672 59 ± 0.000 85	128
CODATA (2002)	6.6742 ± 0.001	150
Gundlach & Merkowitz (2000), University of Seattle	6.674 215 ± 0.000 099	14
Quinn *et al.* (2001), BIPM	6.675 59 ± 0.000 27	41
Schlamminger *et al.* (2002), Paul Scherrer Institute, Zurich	6.674 07 ± 0.000 22	22
Armstrong & Fitzgerald (2003), Measurements Science Laboratory (MSL), New Zealand	6.673 87 ± 0.000 27	27

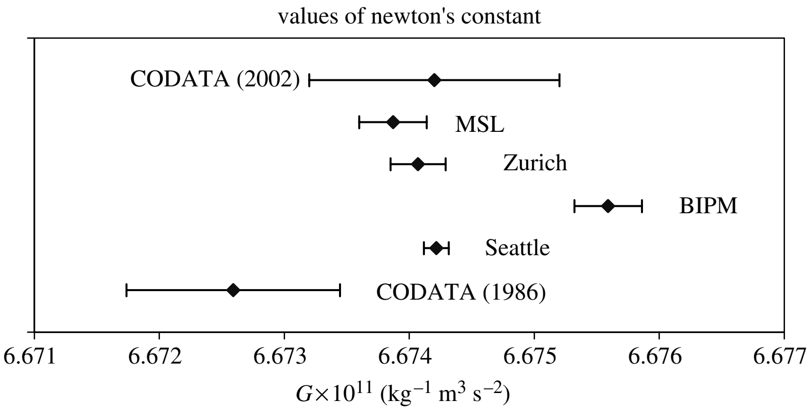

Figure 5. Values for G derived from determinations with uncertainties less that 50 p.p.m. and the CODATA values for 1986 and 2002.

There are a small number of other groups that are engaged in G determinations all aiming at uncertainties of 10s of p.p.m.: Newman & Bantel (1999) and colleagues are using a time-of-swing method and a cryogenic torsion balance; Faller and Parks are completing a determination that uses the deflection of a pair of pendulums that comprise a Fabry–Perot cavity in the presence of source masses; Luo *et al.* (1998) are pursuing a torsion balance time-of-swing method.

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