Does the Time-of-Swing Method Give a Correct Value of the Newtonian Gravitational Constant?

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A standard way of measuring the Newtonian gravitational constant has been the time-of-swing method using a torsion pendulum. A key assumption is that the spring constant of the torsion fiber is independent of frequency. This is likely to be true to a good approximation if any damping present is proportional to velocity. However, recent work on the elasticity of flexure hinges suggests that typically the damping at low frequency is best modeled by including a frequency-independent imaginary component in the spring constant. In this case, the real part of the spring constant must vary, leading to an upward bias in a measurement of G.

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Measurements of the Newtonian gravitational constant have commonly been performed using a torsion balance and the time-of-swing method, in which the time of swing is measured for two positions of the source masses, shown as "near" and "far" in Fig. 1 [1]. The accuracy of such measurements depends on constancy of the spring constant of the torsion fiber.

Recently, in the neighboring field of gravity wave detection, there has been a considerable amount of work done concerning the elasticity of materials at very low frequencies, motivated by the need for test mass suspensions with ultralow mechanical noise for use in laser interferometric gravity wave detectors. Attention has been focused on the frequency dependence of internal friction because of the connection with thermomechanical noise via the fluctuation dissipation theorem. Quinn *et al.* [2] obtained good agreement between the dissipation obtained experimentally and a model of an anelastic solid incorporating an infinite number of relaxation processes with a continuum of time constants from a minimum of τ_0 up to a maximum τ_∞, each with the same relaxation amplitude. A similar model has also been treated by Saulson [3] and an experimental method to probe such behavior has been devised by Saulson *et al.* [4]. It is thought that such a model may be applicable to a broad class of materials.

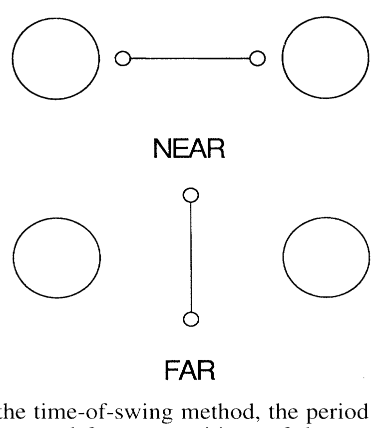

FIG. 1. In the time-of-swing method, the period of the torsion balance is measured for two positions of the attracting masses labeled "near" and "far."

A consequence of the anelastic solid model is that not only the dissipation but also the elasticity is frequency dependent. This is irrelevant compared to thermal noise in suspensions and has thus been neglected. However, in this paper I point out that it is highly relevant to high-precision quantitative applications such as time-of-swing measurements. Of course, anelasticity is only one of many possible error sources in measurements of G [5], and is not relevant to certain other recent measurements [6], which work on different principles. Nonetheless, if the fiber materials used in recent measurements of G have anelastic parameters similar to materials already measured, it could well explain some of the observed discrepancies [7].

Anelasticity can be represented by a generalized complex Young's modulus, the real part of which leads to the normal spring constant and the imaginary part of which represents damping. A simple system that shows anelasticity is the spring and dash-pot system of Fig. 2, which has the following relationship between stress σ and strain ϵ:

$$\epsilon E_R + \dot{\epsilon}(E_R + \delta E)\tau = \sigma + \dot{\sigma}\tau . \tag{1}$$

E_R is the relaxed Young's modulus, i.e., the effective value in the limit of low frequencies, and δE is the difference $E_U - E_R$ between the relaxed modulus and the high frequency, unrelaxed modulus. The relaxation time constant τ is the ratio of the dash-pot viscosity ν and δE. Since we are interested in the frequency response we take the Fourier transform of Eq. (1),

$$E(\omega) = \frac{\sigma(\omega)}{\epsilon(\omega)} = E_R + \delta E\left(\frac{\omega^2\tau^2}{1 + \omega^2\tau^2} + \frac{i\omega\tau}{1 + \omega^2\tau^2}\right). \tag{2}$$

Certain materials are well modeled by an ideal spring with a single Maxwell unit over certain frequency ranges, implying a relaxation process with a well-defined time scale. However, in order to explain experimental results, Quinn

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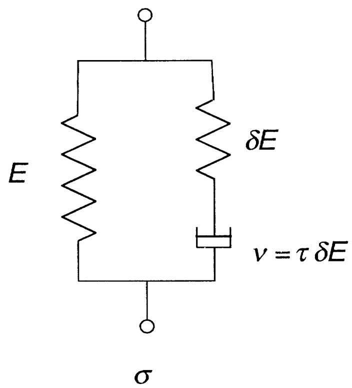

FIG. 2. An ideal spring in parallel with a Maxwell unit represents a single relaxation process. The materials considered in this paper are modeled as an infinite number of microdomains, each behaving as a Maxwell unit. The units have a common relaxation amplitude but a continuum of time constants.

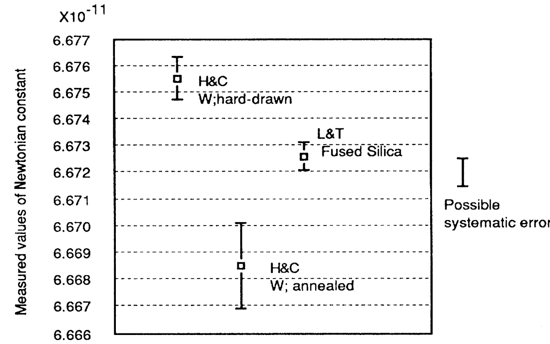

FIG. 3. Values of the Newtonian gravitational constant determined using torsion balances are plotted with their estimated systematic errors as derived in the respective papers. Heyl and Chrzanowski showed two values in their paper [1] which were averaged to give the pre-1986 CODATA value [9].

et al. [2] used a more complex model where the above relaxation process was ascribed to an infinite number of microscopic domains with a continuum spread of relaxation times but a common value of relaxation strength, δE. By integrating this against τ from τ_0 to τ_∞, they obtained a semiempirical formula for Young's modulus. In Ref. [2], the real part was not considered in detail because the experimental data were modeled adequately by the imaginary part. However, if we suppose that the continuum Maxwell model is basically correct and wish to apply the result given in Ref. [2] to the torsion balance, the real part becomes of considerable interest. Because of the proportional relationship between Young's modulus and shear modulus, the above result can readily be adapted to the shear modulus, with the result that the period of the torsion balance depends on the anelastic characteristics of the fiber material. It is not difficult to obtain the real part of the shear modulus, γ, and to find the frequency dependence of the torsion fiber spring constant in terms of the corresponding parameters τ_0, τ_∞, and $\delta\gamma$. The angular frequency of the resonance of a torsion pendulum is greater than the value predicted from the zero frequency elastic constant by a factor

$$1 + \frac{(\delta\gamma/\gamma)}{2\ln(\tau_\infty/\tau_0)} \ln\left(\frac{1 + \omega^2\tau_\infty^2}{1 + \omega^2\tau_0^2}\right). \tag{3}$$

Let us apply Eq. (3) to measurements of the Newtonian gravitational constant G from the literature. In the absence of specific information concerning the particular fiber materials used, we take the parameter values $\tau_0 = 10$ s and $\tau_\infty = 5000$ s for Cu-Be alloy from Ref. [2]. Since these parameters affect ω^2 through a logarithm function, slight errors in these numbers will not produce a large difference in ω^2. The parameter to which ω^2 is most sensitive is $\delta\gamma/\gamma$. Since $\delta E/E$ was 0.02 in Ref. [2] and $\delta\gamma/\gamma = \delta E/E$ for isotropic materials, we expect $\delta\gamma/\gamma$ also to be around 0.02. To estimate this value from existing data of G, I compare the values of G determined by Heyl and Chrzanowski [1], which are plotted in Fig. 3, with the recent value by Luther and Towler [8]. The measurements by Heyl and Chrzanowski used different torsion fibers, specifically, one of hard-drawn tungsten wire, 250 μm in diameter and 1 m in length, and one of annealed tungsten with the same diameter and length. The resulting relative difference in G was 0.1%. If $\delta\gamma/\gamma$ is exactly 0.02 for both wires, it is difficult to find appropriate values of τ_0 and τ_∞ to explain the difference of the G values. If, on the other hand, the difference is attributed wholly to variations in $\delta\gamma/\gamma$, the difference in G can be simply explained by a difference in $\delta\gamma/\gamma$ between the two wires, roughly equal to 0.02. Since $\delta\gamma/\gamma$ is always positive, the larger of the two values must therefore be greater than 0.02. This is still compatible with the spread of values of $\delta E/E$ considered in Ref. [2].

From the existing data it is impossible to determine $\delta\gamma/\gamma$ for each wire. However, the true value of G must necessarily be less than both values in Ref. [1], and this is not compatible with the value by Luther and Towler [8]. Although the parameters of the fused silica used by Luther and Towler are likely to differ substantially from those of the tungsten of Heyl and Chrzanowski, we naturally expect a similar anelastic effect at some level. There are not enough experimental data to be more precise; however, note that, because the difference of frequency in the determination by Luther and Towler was smaller than for Heyl and Chrzanowski, for a given set of parameters the bias will be less. If we again assume the same values of τ_0, τ_∞, and $\delta\gamma/\gamma$, the true value of G is lower by 150 ppm, which is far larger than the error estimated in [9].

If, in fact, the value of $\delta\gamma/\gamma$ for fused silica is far greater than that in tungsten, it is conceivable that all values of G given in Fig. 3 might converge to one point below all of them.

The above discussion assumes the relatively specific parametrization and numerical values of Quinn *et al.* [2] However, it is possible to generalize in a useful way by using a well-known special case of the Kramers-Kronig relations, Bode's phase-gain law. If over some frequency range the imaginary part of the spring constant is a fixed fraction ϕ of the real part, then the magnitude of the spring constant must be proportional to $\omega^{2\phi/\pi}$ [4]. Accordingly, the measurement by the time-of-swing method gives a value for G too high by a fraction ϕ/π. If the wire is the principal source of damping, then the bias is equal to $1/\pi Q$, where Q is the quality factor of the main torsional mode. This gives a simple way to estimate the systematic error in G caused by anelasticity. This can immediately be applied to the experiment of Luther and Towler [8] who quote an approximate damping factor of 10^{-4}. Considering the lack of significant figures, I take Q to be in the range from 2.5×10^3 to 1.6×10^4. The bias is then from 20 ppm to 130 ppm. This is smaller than the estimation using Eq. (3), but, in fact, quite close considering that parameters for a different material were used. Strictly, this is a maximum value, because in addition to the intrinsic damping of the wire the measured Q may include a number of other possible losses. If the dominant components are velocity dependent, then the anelastic bias will be negligible.

In conclusion, if the torsion fibers used to determine G by the time-of-swing method are subject to anelasticity, as described by a continuum Maxwell model with plausible parameters, then there is upward bias in the determined values that could explain some of the spread in recent experimental results. The technique is not completely invalidated by this bias, but experimenters considering using it for new G measurements should try to reduce, and/or control for, anelasticity in the wire, clamps, and support.

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