Experiments on gravitation

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Experiments on gravitation

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Abstract

The theoretical setting of the experimental study of gravitation is indicated and a brief account is given of experiments on gravitation from Newton onwards. The main content of the review comprises surveys of three groups of experiment: the investigation of the inverse square law, studies of the weak equivalence principle and determinations of the constant of gravitation. Attention is called to the reasons for the difficulties of carrying out experiments on gravitation. In summary, it is concluded that there is no evidence for deviations from general relativity and that gravitation is mainly a manifestation of the geometry of spacetime.

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1. Introduction

Experiments on gravitation have a rather unusual place in physics. Gravitation is one of the four forces that appear to account for the structure and change of the material world, but whereas the other three—electromagnetism, the weak force of particle physics and the strong force between nucleons—depend strongly on definite properties of the constituents of matter, as for example electrical charge, gravitation depends on mass, specifically inertial mass, in a way that is the same for all materials independently of all other properties. At once one asks, is that really so? Is mass, so far as gravitational interactions go, strictly proportional to inertial mass, or are there small deviations that have so far escaped experimental investigations? It has indeed been suggested recently—initially as a result of observations in kaon physics (Fishbach *et al* 1986a)—that there is a fifth force of nature that is similar to, but not identical with gravitation, is generally confused with it, but which might be revealed in certain geophysical observations and through the work of Eötvös *et al* (1922) on the weak equivalence principle. It is now understood that electromagnetic forces and the weak force are to be regarded as different manifestations of a common interaction, the electroweak force, just as electrical and magnetic interactions are not distinct but different aspects of the electromagnetic force. Can gravitation likewise be seen ultimately as one aspect of a more general interaction, linked to one or both of the electroweak and strong forces?, or is gravitation really entirely distinct and a manifestation of the geometrical structure of the world, as expressed by general relativity, and so strictly, and not just nearly, independent of all material properties save inertial mass?

There are two major experimental issues involved in those questions, of which the first is what limits can be placed on the weak equivalence principle (gravitational mass is proportional to inertial mass, independent of any other physical property). The second is what limits can be placed on deviations from the inverse square law. One interpretation of a deviation from the inverse square law is that the gravitational field itself has a rest energy or rest mass in a way that the electromagnetic field does not. The electrostatic Coulomb interaction satisfies the inverse square law to very tight limits, although the fact that the photon has zero mass can be established in other ways, as will be discussed further in the next section. Limits on deviations from the inverse square law of gravitation accordingly place restrictions on the nature of any proposal for a theory unifying gravitation and some other force (see Gibbons and Whiting 1981).

While experiments on the principle of equivalence and the inverse square law are fundamental, others are also of interest. In particular, despite some discrepant experimental results, shielding by third bodies has never been established: the gravitational force is a strictly linear superposition of two-body forces. Nor has any effect of temperature or velocity or anisotropy been found. In general, the experiments that test such matters have not been developed as fully as those on the equivalence principle and the inverse square law and so they are considered along with some historical material in the third section, whereas the experiments on the equivalence principle, the inverse square law and the determination of the constant of gravitation are discussed in separate sections.

One issue that has often been discussed is whether the constant of gravitation may be changing over time. Any such change is far too small to detect, and the significance of a change and why it is experimentally difficult to find it are discussed in connection with the account of measurements of the constant of gravitation.

This review is concerned with experiments in the strict sense, that is to say with physical observations under conditions that may be altered at the will of the experimenter, in contrast to the analysis of observations of celestial mechanics and geophysics where the conditions are given and cannot be changed by the observer. The conclusions to be drawn from celestial mechanics have been explained elsewhere (see Cook 1987a, b) and the discussion will not be repeated here, suffice to say that no deviation from the inverse square law has ever been confirmed save only in the neighbourhood of massive bodies (Mercury, Venus and the Earth near the Sun) where there is an inverse cube contribution in accordance with general relativity. Further, there is no evidence for any directional effect, that is, that the force between two bodies might depend on the direction of the line joining them, for example in relation to some preferred velocity vector. Will (1981) has summarised the magnitude of that and other possible effects in celestial mechanics in terms of limits on the parameters of the parametrised post-Newtonian (PPN) formulation of theories of gravitation. As for geophysics, the tidal variation of gravity at the surface of the Earth might contain a part corresponding to an effect of a preferred direction (Will 1981) but no such effect has been found so far. Geophysical evidence has been thought to imply deviations from the inverse square law (Stacey and Tuck 1981, Holding and Tuck 1984, Holding *et al* 1986, Fishbach *et al* 1986a, b) and that issue, though strictly not a matter of experiment as defined here, will be discussed at the end of § 4 on the inverse square law. (Stacey *et al* (1987) have recently reviewed the evidence in some detail.)

Experiments on the nature of gravitation are inherently difficult. The coupling constant of gravitation is very much less than that of electromagnetism, so that the gravitational attraction between two protons is about 10^{-39} of the electrical repulsion. The forces are therefore very small, and to be detectable, let alone measurable with any precision, large masses at small separations must be used and the attracting masses must be electrically and magnetically neutral. Even so, very delicate detectors of the forces are needed for experiments to be contained within a normal laboratory. Not only must the detector respond very sensitively to the gravitational forces, but other disturbances must be small: the detector must be well shielded from thermal and electromagnetic forces and, at the level of sensitivity approached nowadays, thought has to be given to quantum fluctuations.

The detector is almost always a torsion balance. A horizontal rod is suspended at its centre of mass by a vertical fibre. From one end of the rod is hung a test mass; the gravitational force exerted upon it by some other mass develops a torque which is balanced by some known force applied at the opposite end of the rod, for example, by the electrostatic force between an electrode fixed to the rod and one fixed to the laboratory. Such an arrangement can be made very sensitive if the ratio of breaking stress to shear modulus for the material of the fibre is high (see Boys (1895), who introduced fibres of fused silica for that reason). In early work, the rotation of the beam was observed by eye with an optical lever but nowadays other highly sensitive means are available, such as interferometers, capacitance bridges and optical levers with laser sources and position-sensitive photodetectors of various sorts, and with many of these detectors the angular sensitivity or the equivalent resolution of torque may greatly exceed common sources of noise, if not the limit of quantum fluctuations of the balance beam.

Attempts to detect gravitational waves are not included in this review. The existence of such waves carrying energy is predicted by general relativity and on current estimates the most sensitive interferometric detectors should respond to the strongest astronomical sources. Observations of the binary star with a pulsar component first detected in 1974 (Hulse and Taylor 1975, Taylor and McCullock 1980) show clearly that the system is losing orbital energy by gravitational radiation, so that the phenomenon of gravitational radiation is hardly beyond doubt, but attempts to detect radiation directly from the response of detectors upon the Earth are still fundamental, although they are not strictly experiments as defined here, and the whole matter is beyond the scope of this report. Thorne (1987) has given a most comprehensive account of the present state of the search for gravitational radiation.

Over the years, very many experiments on gravitation have been performed, and a complete bibliography is too long to give here. However, anyone who writes on the subject is in the debt of Gillies (1983) who has prepared a useful and comprehensive list of the literature on experiments on gravitation; I have drawn on it extensively in my preparation and those who wish to pursue any of the topics further should consult it.

This review is complementary to two others mentioned above, those of Stacey *et al* (1987) and Thorne (1987)

2. Theoretical considerations

An experimenter contemplating an experimental study of gravitation might proceed in one of two ways. On the one hand—the approach of the naive experimenter—he might simply ask what deviations from Newton's law of gravitation he might try to detect. By Newton's law, the force is Gm_1m_2/r^2. The experimenter might ask, is the dependence on mass just the product of masses, and further is it just the product of inertial masses; is there any dependence on location or velocity or temperature or direction of the line joining the masses, or on the presence of a third body or of any material between the two masses? Alternatively, the experimenter might look at the predictions of some theories and ask whether behaviour in accordance with them might be sought. In fact, one cannot divorce oneself completely from theory and work in an entirely *ad hoc* opportunist way, for the phenomenon of gravitation seems so basic to our understanding of the physical world at large that any experimental result has far reaching implications, if only in its relation with general relativity. Thus some sketch of theoretical ideas must precede a review of experiments, although it is to be understood that what follows is in no way complete or fundamental: it is intended to summarise the main concepts to which experiments are relevant.

The starting point must be general relativity. General relativity is a geometrical theory in that it is supposed that particles move along geodesic paths in the four-dimensional space of space and time. Once the metric of the four-dimensional space is known, so is the motion of any particle. The connection with physics is that the metric is determined by the energy in some region of space. Without going through the algebra that may be found in any standard text on general relativity, the metric in the neighbourhood of an isolated mass M (for example, the Sun) may be shown to be, in spherical coordinates,

$$ds^2 = (1-2GM/rc^2)(dx^0)^2 - (1-2GM/rc^2)^{-1}\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2$$

(the Schwarzschild metric). Here ds is the infinitesimal interval, x^0 is ct and θ and ϕ are the colatitude and azimuthal angle.

In free space the metric for cartesian coordinates is

$$g_{ij} = \mathrm{diag}\{1, -1, -1, -1\}.$$

As a first approximation, when GM/rc^2 is small, g_{00} may be taken to be $(1\text{-}2GM/rc^2)$. But the geodetic equation may be shown to reduce to

$$\mathrm{d}^2x^i/\mathrm{d}t^2 = -\delta_{ij}\partial_j[c^2(g_{00}-1)]$$

when $g_{00}-1$ is small. That is just Newton's equation for motion in a potential given by

$$V = \tfrac{1}{2}c^2(g_{00}-1).$$

Hence for the Schwarzschild metric in the first-order approximation

$$V = -GM/r.$$

It can readily be seen that in higher approximations, the factor $(1\text{-}2GM/rc^2)^{-1}$ that multiplies $\mathrm{d}r^2$ must be taken into account, and that leads to higher powers of r^{-1} in the potential, with the well known consequence of a contribution to the motion of the perihelion of Mercury and other planets because of the r^{-2} term in the potential.

The general theory of relativity has now satisfied so many observational tests—the deflection and slowing down of electromagnetic waves near the Sun, the precession of the perihelion of Mercury, the weak equivalence principle—that it seems most probable that the theory does provide the correct (or nearly correct) geometrical basis of large scale physics, and hence, that gravitation is indeed the expression of that geometry and nothing more. However, while metric geometry may be the basis of gravitation, it is also possible, as Fishbach *et al* (1986a, b) have argued, that there are other force fields that masquerade as gravitation, but deviate in some respects from it. Evidently they are weak, but careful experimentation might uncover them.

The first question is whether a mass is to be associated with the gravitational field.

So far as electromagnetic radiation is concerned, we know from astronomical observations of pulsars that there is no dispersion over a very wide range of frequency and so we may assert that the rest mass of the photon is indistinguishable from zero.

The potential, V, of a static field therefore obeys

$$\nabla^2 V = 0$$

which is satisfied by the potential $(1/r)$, so that the force between two particles is inverse square.

Further, since $\nabla^2 V$ vanishes, then the field inside a closed conductor is zero. Because electrical measurements are very sensitive, it has been possible to confirm that prediction with very great precision, so that any deviation from the value of 2 for the exponent in the law of force is less than 3×10^{-16} (Williams *et al* 1971).

There is an assumption, unstated so far, that $\nabla^2(1/r)$ is indeed zero. That is certainly true for many geometries, but it is conceivable that in the geometry of spacetime it might not be so. However, the fact that the electromagnetic force is inverse square, taken with the fact that electromagnetic waves are dispersionless and have no rest mass, implies that in our local geometry $\nabla^2(1/r)$ vanishes. Thus, in discussing the inverse square law of gravitation, we may take it that $\nabla^2(1/r)$ is zero for laboratory experiments.

Now suppose that a wavefield is associated with a finite rest energy, E_0. Then for static fields

$$\nabla^2 V = \mu^2 V$$

where $\mu = E_0/c\hbar$. A solution of this is

$$V = -(\mu^2/r)\,e^{-\mu r}.$$

It is therefore natural to look for a gravitational force that corresponds to a potential of that form, namely

$$-(1/r^2)(1+\mu r)\,e^{-\mu r}$$

instead of

$$-1/r^2.$$

Fujii (1971, 1972 (see also Scherk 1979)) has proposed such a form and in one particular model finds that the total potential should be

$$-(GM/r)(1+\tfrac{1}{3}\,e^{-\mu r}).$$

The rest energy corresponding to $\mu = 1\ \mathrm{m}^{-1}$ is extremely small, about 10^{-6} eV, compared with the rest energy of the electron of 0.5 MeV and an upper limit of about 15 eV for the neutrino. More generally, the factor $\frac{1}{3}$ is replaced by an arbitrary parameter.

Fishbach *et al* (1986a, b) have also proposed a similar contribution to the effective gravitational potential, except that they suppose that it is proportional to the product of baryon numbers, B_1 and B_2 say, of the attracting bodies. Now the baryon number is very nearly proportional to atomic mass, but not quite because of differences in the binding energies of nucleons, and so Fishbach *et al* (1986a, b) have anticipated both that the weak equivalence principle would be violated and also that the inverse square law would not be exactly obeyed.

Will and Nordvedt (see Will 1981) have looked in a general way at how to express deviations from general relativity and have developed their formulation as parametrised post-Newtonian (PPN) theories of gravitation. The theories are determined by the metric of the spacetime geometry, in that the path of a particle is a geodesic in the geometry. The PPN formulation cannot include theories which are not metric in that sense; but it does enable the metric to depend on a wider range of material properties than that of general relativity.

The metric coform is written with ten arbitrary parameters that have the following significance in quantifying departures from general relativity:

γ: the space-curvature produced by unit mass	1 in general relativity;
β: non-linearity in superposition of gravitational fields	1 in general relativity;
ξ: existence of a preferred location in space	0 in general relativity;
$\left.\begin{matrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{matrix}\right\}$ existence of preferred velocity	all 0 in general relativity;
α_3 and ζ_1–ζ_4: violation of conservation of momentum	all 0 in general relativity.

We now consider briefly the observable phenomena that have set limits on the PPN parameters and hence on the deviations from general relativity.

The first is the deflection of light near a massive body. The path of a photon is a geodesic of the PPN metric, but the path of photons from a single source is not observed by itself, rather two sources are observed, one always far from the Sun and one sometimes close to it, and the variation of the angle between the directions of the two sources according to the position of the Sun is recorded.

If m is the mass of the Sun, d is the distance of closest approach of a photon from a distant source and θ_0 is the angle between the two sources, then

$$\delta\theta = (m/d)(1+\gamma)(1+\cos\theta_0).$$

Deflections have been observed optically and with radio waves. The early optical observations were consistent with $\gamma = 1$ and general relativity, but the scatter was great and even the best recent results (Texas Mauritanian Eclipse Team 1976) were seriously affected by poor seeing. The radio results depend on the measurement of the direction of a far-distant small radio source by very long baseline radio interferometry, and the results are very consistent (Formalont and Sramek 1977).

The extra time taken by a radio signal when it passes close to the Sun can also be found from timing radio returns either from a planet (Mercury or Venus) seen close to the Sun, or from a spacecraft *en route* to or in orbit round a planet, Mars or Venus (Anderson *et al* 1975, Reasenberg *et al* 1979).

The conclusion from all recent observations is that γ must lie in the range 1 ± 0.005.

The perturbations of the orbit of a planet (Mercury), even one relatively close to the Sun, are small, as may be calculated from the Lagrangian equations for perturbations of planetary orbits. Let p be the semilatus rectum of the orbit.

Integration of the perturbation of the longitude of perihelion round the orbit gives for the change in one orbit

$$\Delta\tilde{\omega} = \frac{6\pi m}{3p}[(2+2\gamma-\beta)+\tfrac{1}{2}(2\alpha_1-\alpha_2+\alpha_3+2\zeta_2)\mu/m]$$

where m is the sum of the masses of the planet and Sun, and μ is the reduced mass:

$$m = m_1 + m_2 \qquad \mu = m_1 m_2/m.$$

There is in addition a perturbation of the orbit by any quadrupole moment of the Sun, and it produces a secular motion of the node of the orbit upon the ecliptic as well as the motion of the perihelion.

The observed motion of the perihelion of Mercury is much greater than the general relativistic component, nearly 6000 arcsec/century, as compared with 43 arcsec/century; it comprises the general precession of the Earth's equinox, relative to which the longitude of the perihelion of Mercury is measured, and the precession that results from the attractions of other planets, particularly Venus and Jupiter. When these are removed, the optical observations of the direction of Mercury over three centuries give the remainder to about 1% (Morrison and Ward 1975) whilst the radar observations of the distance of Mercury give it to about 0.5% (Shapiro *et al* 1976).

There is still some uncertainty about the quadrupole moment of the Sun, but the general conclusion seems to be that the combination $\frac{1}{3}(2+2\gamma-\beta)$ is equal to 1 to within 0.5%.

Taken together, the light path observations and the motion of the perihelion of Mercury show that β and γ are indeed very close to 1 as entailed by general relativity.

The remaining classical test of general relativity is the red shift of photons in a gravitational field. Classical ideas lead to a shift half that predicted by general relativity, and the original observations of red shifts in solar and stellar spectra were too disturbed by other effects to be definitive, but observations of Mössbauer radiation from a source at different heights above the Earth, and even better, measurements of the frequencies of atomic oscillators in satellites in orbit about the Earth, have given extremely close agreement with the prediction of general relativity (Pound *et al* 1965, Vessot *et al* 1980).

Will (1981) shows that the red-shift result is to be interpreted as implying that the PPN metric is independent of any special location and consequently that the parameter ξ is effectively zero.

Other observations have sought deviations of other parameters from zero, and so far none has been established. One group concerns tidal variations of the force of gravity at the surface of the Earth. To a first approximation, the force of gravity at a point fixed to the surface of the Earth is the vector sum of the Newtonian attraction to the centre of mass and the centrifugal inertial force normal to the polar axis. If the attraction of gravity depends on a preferred direction, then as the Earth rotates the force of gravity at a point on the surface will vary with half the period of the Earth's rotation. However, such a possible periodic variation has to be sought in the presence of the actual variations due to tidal forces. The varying potential of the Moon and the Sun at a point on the surface of the Earth gives rise to a direct variation in the force of gravity with many harmonic terms, and also to a complex set of indirect variations, corresponding to the physical distortion of the Earth, the gravitational attraction of the oceans moved by the tides and the indirect effect of the varying pressure of the oceans upon the sea bed. The direct effect and the tidal deformation of the Earth can be calculated with considerable precision, but those of the oceans are not so surely known. The tidal effect on gravity is about 1 part in 10^7 overall, an effect that is easily measured with modern instruments (Warburton and Goodkind 1976), but the uncertain effects of the oceans mean that any possible effect of preferred direction on gravitation would be uncertain by about 1 part in 10^8. The PPN parameters α_1, α_2, α_3 that relate to preferred frames thus cannot be distinguished from zero.

In summary, none of the PPN parameters has so far been found to have a value other than that entailed by general relativity, but not all possible deviations of gravity from the predictions of general relativity are covered by the PPN formalism; in particular, it is not very helpful in the consideration of inverse square law dependence.

Various people, mostly notably Dirac (1937) (see also Dyson 1972) have suggested that there might be a variation of the constant of gravitation with time. The experimental problems in looking for such a change are discussed in § 6 but here some remarks are made on what might be implied by a change.

A 'change' in G can only mean in practice that the measured value alters relative to the standards of mass, length and time relative to which it is measured. In fact, the standard of length is now, by international agreement, replaced by a value for the speed of light. If it is thought that G changes, that means that the relations between G and the standards change, thus raising the question whether it may be expected that G would be likely to vary in relation to the atomic parameters that are involved in specifying energy levels of atoms by which the standard frequency is established, or relative to the mass of an arbitrary piece of platinum-iridium alloy. The latter would not be interesting, but it is now possible to establish a standard of force in terms of electrical, and therefore quantum, quantities (Petley 1985) so that, in principle, the ratio of G to the velocity of light and the constants of quantum physics ($\hbar$, e, m_e) might be examined. However, if a change were established, would general relativity or quantum physics be regarded as more fundamental?

Indeed, one could see that if G were to be measured much more precisely than it is at present, it might be taken to be, like the speed of light, one of the fundamental constants of the system of metrology, for it might well be argued that the basic geometrical structure of general relativity should be one of the elements of a system of measurement, just as is the velocity of light, the constant related to the basic geometrical structure of special relativity. In that case, $\hbar$ should be another constant in view of its relation to the anticommutation relations of quantum mechanics. However, such ideas and thoughts about the fourth constant are purely speculative in view of the poorly known value of G.

3. History and minor experiments

When he formulated the law of gravitation, Newton well understood how small the forces would be between objects in a laboratory, or indeed those between a mountain and a nearby plumb bob, but he was the first to carry out a laboratory experiment on gravitation in an examination of the weak equivalence principle. He describes his work in *Principia* (Newton 1687, book III, proposition VI, theory VI). Let m_i be the inertial mass of a pendulum and m_g the (passive) gravitational mass when subject to the attraction of the Earth. If ρ is the radius of gyration about the point of support and h the distance of the centre of mass below it, the period of the pendulum is

$$2\pi(m_i\rho^2/gm_g h)^{1/2}$$

where g is the gravitational acceleration in the field of the Earth. It is actually a combination of centrifugal and properly gravitational terms, but as the former are about 10^{-3} of the latter the approximate formula is good enough.

If two pendulums have the same geometrical structure, so that ρ^2/h is the same, then

$$T_1/T_2 = (m_i/m_g)_1^{1/2}/(m_i/m_g)_2^{1/2}.$$

Put

$$m_i/m_g = 1 + 2\mu.$$

Then if ω_i is the frequency of pendulum i,

$$\omega_1/\omega_2 = (1+\mu_2)/(1+\mu_1)$$

and the difference is

$$\omega_1 - \omega_2 = (\mu_2 - \mu_1)\omega$$

since μ is small.

According to the rather condensed account in *Principia*, Newton suspended two pendulums side by side and looked for a phase difference between them as they swung together for a long period. He does not say for how long he observed, nor what phase difference he thought he might detect, nor, an important point, whether the pendulums were in phase or antiphase—in the latter condition displacement of the support, which might otherwise disturb the results, would not occur. Thus it is now difficult to estimate how small a value of $(\omega_1 - \omega_2)$ Newton might have detected; if the pendulums were to swing for an hour together he might have seen 1 part in 10^4, but he himself claimed only that the different substances he used were equivalent to 1 part in 10^3.

The pendulums he used were hollow round wooden boxes, made externally the same so that the resistance of the air was the same, and suspended by a string 11 ft (3 m) long. The period would then be about 3.5 s and a pendulum would execute 1000 oscillations if it swung for 1 h. Within the boxes he suspended additional masses at the centre of oscillation, one pendulum always containing wood, whilst in the others there were placed successively gold, silver, lead, glass, sand, salt, water and wood, each of a mass equal to that of the wood in the reference pendulum.

Newton says that the two pendulums were hung close together and that they then 'played together, forward and backward for a long time with equal vibration'. The meaning of that phrase is presumably that they remained in phase as they oscillated over a long period of time. As already said, Newton estimated that any difference between wood and any of the materials he used was less than one in a thousand.

Bessel (1832) later carried out an experiment similar in principle to Newton's, using twelve different materials, and he found no difference greater than 1 in 6×10^4. Later experiments with pendulums were made by Potter (1923) who obtained similar limits to those set by Bessel.

A great improvement was made by Eötvös as a consequence of his studies of the gradients of the Earth's gravity field; the discussion of Eötvös's work is reserved for a subsequent section (§ 5) as is that of the more recent experiments of Roll *et al* (1964) and Braginski and Panov (1972).

Newton considered the possibility of measuring the constant of gravitation by the attraction of a mountain for a plumb bob, or by the velocity which two spheres, initially at rest, would acquire towards each other, but considered that neither effect would be detectable, although the attraction of a mountain was used (Maskelyne 1775) as was the somewhat analogous experiment of measuring the change of attraction of gravity down a mine (Airy 1856); the latter test has been involved in studies of the inverse square law (Stacey and Tuck 1981, Holding and Tuck 1984, Holding *et al* 1986). The direct measurement of one body towards another was also recently tried as the basis of a measurement of the constant of gravitation (Luther *et al* 1976), as is described in more detail in § 6. The measurement of the constant of gravitation, and other experiments on gravitation, became practical, however, when Mitchell introduced the torsion balance and Cavendish (1798) exploited it. Cavendish's own measurement of the constant of gravitation was followed by many others in the nineteenth century; the principles of the various methods are discussed in § 6 but the main developments are just summarised here. Boys' (1895) use of fused silica suspension fibres led to a great increase in sensitivity of the torsion fibre, while Braun (1897), by evacuating the vessel in which the torsion pendulum hung, improved the stability. Braun also observed the frequency of oscillation of a torsion balance in the presence of attracting masses, as subsequently followed by Heyl (1930) and others. Zahradnicek (1933) drove the torsion pendulum by the attraction of masses moved from side to side (on other torsional suspension) at the resonant frequency of the first pendulum. A quite different approach was followed by von Jolly (1881) and later by Poynting (1892), who used an ordinary equal-arm chemical balance to determine the attractions of a large mass placed under the balance upon a mass in one pan of the balance; the attraction was counterpoised by a small mass on the other pan.

It is sometimes thought that investigations of the inverse square law were undertaken in the nineteenth century, but it seems not to be so. Cavendish (1798) states that he checked the law in the course of his determination of G, but gives no details, and Baily (1843) made no reference to examination of the law, nor did Cornu and Baille (1873, 1878), in their respective determinations of the constant of gravitation, nor did any of those responsible for the later measurements already referred to. Some indication of the validity of the law on a laboratory scale can be obtained from the general agreement between determinations of the constant of gravitation made with the masses at different distances; thus the results of Braun and of Boys are very close although the dimensions of Boys' apparatus are less by a factor of about ten. No doubt during the latter part of the nineteenth century, it was considered that the inverse square law was quite securely established in celestial mechanics and there was no need to check it in the laboratory. The first explicit experiment on the law of which details seem to have been published was by Mackenzie (1895). He was investigating a possible dependence of the attraction of crystalline bodies upon the alignment of their respective crystallographic axes. He used apparatus very similar to that of Boys, having a torsion pendulum suspended by a fibre of fused silica in an enclosure which was not evacuated. He found no differences between attractions when the crystalline bodies were set in different orientations. In the course of his experiments, Mackenzie made observations of the deflection of the balance when the attracting masses were at different distances. His results are a little difficult to interpret because the geometry is not simple, but Mackenzie claimed that he had verified the inverse square law to 1 part in 500 over distances ranging from 36 to 74 mm. He also checked that the gravitational attraction was proportional to the product of the masses.

Poynting (1900) in a review of studies of gravitation made in the latter part of the nineteenth century, was mainly concerned with measurements of the constant and did not speak of any experimental studies of the inverse square law, apart from the work of Mackenzie; and in fact the inverse square law was not examined experimentally again before Long (1976).

In the latter part of the nineteenth century, various possible influences upon gravitation were studied experimentally. Poynting and Gray (1899) followed up the work of Mackenzie by looking for an effect of crystallographic orientation in the attraction of one quartz sphere upon another. One of the spheres was suspended by a torsion fibre and the other sphere was moved about it. If the alignment of the crystallographic axes were significant, the suspended sphere would have been set into forced rotational oscillations around the vertical axis of the suspension. The spheres were separated by 59 mm, and the authors concluded that they could detect no effect of alignment greater than 6 parts in 10^5. Heyl (1924) later found that any effect of alignment of crystallographic axes was less than 1 in 10^9.

Another possible effect, now by common consent quite discounted it seems, is that of temperature upon gravitational attraction. It was first investigated by Thompson (1873), and Poynting and Phillips (1905), using an equal-arm balance, seem to have made the most rigorous check. A mass was hung from one pan at some distance below it and was placed in an enclosure which could be heated by steam or cooled in liquid air.

Any difference of weight found over the range of almost 300 K corresponded to less than 1 part in 10^{10} per K.

Because electromagnetic forces depend on the properties of intervening matter, the screening of gravitation has also been investigated experimentally and in astronomical observations. Austin and Thwing (1897) used a torsion balance much like that of Boys (1895) and placed screens of different materials—lead, zinc, mercury, water, alcohol and glycerine—between the balance and attracting masses; no change greater than 1 part in 10^5 was found. On the other hand, Majorana (1957), who carried out an extensive series of experiments with a chemical balance surrounded by various screens, claimed to have detected a definite effect (see also Braginski *et al* 1963).

The effect of intervening material has usually been expressed by an experimental factor

$$\exp\left(-\lambda \int_L \sigma \, \mathrm{d}l\right)$$

where σ is the density of material along the straight line, L, joining two attracting masses, and λ is a constant. The expression assumes that gravitation is in some sense propagated in straight lines, and does not represent a solution of Laplace's equation in a medium of variable properties.

Majorana claimed that his experiments in which a sphere of lead weighing 1.3 kg was surrounded by screens of 114 kg of mercury or 9880 kg of lead, gave a value of $\lambda = 10^{-13}\ \mathrm{m^2\,kg^{-1}}$.

Saxl and Allen (1971) also argued that the attractions of the Sun and the Moon upon a torsional pendulum were not simply additive, but other geophysical and astronomical experiments have established very low limits for λ. Harrison (1963) analysed many observations of Earth tides and found no anomalies associated with the Sun being below the horizon, as might have been expected from screening by the intervening Earth. He set a limit of $10^{-16}\ \mathrm{m^2\,kg^{-1}}$ on λ. Subsequently, Slichter *et al* (1965) used observations of Earth tides during the solar eclipse of 1961 to place a similar limit on λ.

Caputo (1962), following a suggestion of Marussi, made use of the particularly favourable circumstances of the 1961 eclipse at Trieste, where two horizontal pendulums of very long period had been set up in the nearby Grotta Gigante, one sensitive to a N–S component of gravity, the other to an E–W component. At the maximum of the eclipse, the Sun and Moon were nearly due west of Trieste at an altitude of 13° 13′, and if there were any screening of the attraction of the Sun by the Moon, there should have been a clear difference between the responses of the two pendulums. None was found and a limit of $10^{-17}\ \mathrm{m^2\,kg^{-1}}$ was set on λ.

While some very tight limits have been set on λ the position is not wholly satisfactory. The astronomical and geophysical schemes are not equivalent to the laboratory experiments in which masses were completely surrounded by screens, and the latter have sometimes given anomalous results. There may be a case for further experimental investigation (see Bocchio 1971).

Experiments on the effects of magnetisation (Lloyd 1909) and electrification (Simons 1922) have given essentially negative results.

4. The inverse square law

4.1. Introduction

After the first laboratory experiment on the inverse square law by Mackenzie (1895), no further experimental work was done until that of Long (1976). Because gravitational forces are so weak, direct measurements of the forces between two bodies are restricted to separations not much greater than 1 m, and most precise studies have been at distances of about 0.1 m. Laboratory measurements have been supplemented by geophysical observations over distances of 10 km or so and there have been some measurements of gravity over full and empty oil tanks at a typical range of 10 m. The most reliable results are over the shortest range and most attention will be paid to four experiments: Long (1976), Spero *et al* (1980), Hoskins *et al* (1985) and Chen *et al* (1984), with briefer discussions of other laboratory work and of the geophysical observations. The later experiments were undertaken because Long (1976) appeared to get a significant deviation, which nonetheless corresponds to a very small mass of the graviton. Geophysical measurements might be thought more discriminating but are much more difficult.

Some general points are common to all laboratory experiments. First, there is the form in which a deviation from the inverse square law might be expressed. One way is to write the law as $r^{-(2+\delta)}$ where δ is small. The corresponding potential is $-GM/r^{1+\delta}$. Long (1976) took the force to be in the form

$$-(GM/r^2)(1+\varepsilon \ln r)$$

and the potential would then be

$$-(GM/r)(1+\varepsilon+\varepsilon \ln r).$$

If the deviation is supposed to reflect some massive field, then the potential would be expected to have the form

$$-(GM/r)(1+\alpha\, e^{-\mu r})$$

and the force would be

$$-(GM/r^2)[1+(\alpha+\mu r)\, e^{-\mu r}].$$

No other form has so far been used in stating the results of experiments. The three forms are related. It is suitable to consider the expressions for the force since that is what is generally measured. The exponential expression contains an obvious scale factor in μ; the scale factors in the other two expressions are implicit and it is desirable to make them evident. If r_0 is some characteristic local length, then the power law is written as

$$-(G_1 M/r_0^2)(r_0/r)^{2+\delta}$$

which is

$$-(G_1 M/r_0^2)(r_0/r)^2 \exp[-\delta \ln(r/r_0)]$$

or, taking r to be close to r_0 and δ small,

$$-(G_1 M/r_0^2)(r_0/r)^2[1-\delta \ln(r/r_0)].$$

Long's form of the law is rewritten as

$$-(G_2 M/r_0^2)(r_0/r)^2[1+\varepsilon \ln r_0+\varepsilon \ln(r/r_0)]$$

showing that ε is equal to $-\delta$, while

$$G_1 = G_2(1+\varepsilon \ln r_0).$$

Lastly, for the exponential form, the force is

$$-\frac{GM_3}{r_0^2}\left(\frac{r_0}{r}\right)^2\left\{1+\left[\alpha+\mu r_0\left(\frac{r}{r_0}\right)\right]\exp\left[-\mu r_0\left(\frac{r}{r_0}\right)\right]\right\}$$

which reduces to

$$-\frac{G_3 M}{r_0^2}\left(\frac{r_0}{r}\right)^2\left[1+\alpha+\mu r_0\, e^{-\mu r_0}+(\alpha+\mu r_0)\mu r_0\, e^{-\mu r_0}\left(\frac{\delta r}{r_0}\right)\right]$$

where

$$\delta r = r - r_0.$$

Since $\ln(r/r_0) = \delta r/r_0$ when $\delta r/r_0$ is small, it can be seen that

$$G_1 = G_3(1 + \alpha + \mu r_0 \, e^{-\mu r_0})$$

and

$$\varepsilon = (\alpha + \mu r_0)\mu r_0 \, e^{-\mu r_0}$$

which will generally be close to $\alpha\mu r_0$.

A more satisfactory way of expressing the experimental results was introduced by Spero *et al* (1980) and exploited by Gibbons and Whiting (1981). The exponential law contains two parameters, so that a single experiment in which gravitational forces are compared at two distances can only determine a set of pairs, α and μ, that are consistent with the observations, and that set is conveniently shown as a region of the $\alpha\mu$ plane.

While the fact that gravitational forces are weak has been particularly emphasised, the accuracy of the definition and measurement of distance is also important, for a small error in measuring a distance would lead to an apparent deviation from the inverse square law that might be quite large compared with the errors of the force measurement. The measurement of length is not the problem—it can be done with very high precision—the difficulty lies in defining experimentally parameters in the model of the experiment to which the calculations refer. The model of an experiment adopted for calculation may involve, for example, the distances between the centres of mass of two spheres—how is that distance to be identified on experimental objects? There are some obvious considerations. Suppose the force on a test mass on the arm of a torsion balance is being measured, and the distance from the test mass to an attracting mass is required. That would be a very difficult measurement to make because the test mass is not fixed in space. Accordingly, the design of all the experiments discussed here is such that no measurement to the test mass is needed. Another obvious consideration is that the centre of mass of an extended body is not at the geometrical centre of the figure unless the density of the material is uniform. That can never be assured to the precision involved, and so again the experiment is designed so that effects of unknown variations of density are reduced.

A more subtle question is the effect of a deviation from the inverse square law upon the nominal values of distances. An example is provided by the attraction of a ring as used by Long (1976). The attraction of a ring attains a maximum at a certain distance along its axis perpendicular to its plane, having the advantage that if a test mass is placed at the point of maximum force it is not necessary to know the distance from the test mass, but only that the force is a maximum. That result is for an inverse square law of force and in fact it also turns out to be true for an exponential law.

Let r be the radius and m the mass of a thin ring. Let z be the distance measured along the axis from the plane of the ring. By symmetry, the attraction is perpendicular to the plane of the ring and has the value

$$Gmz/(r^2 + z^2)^{3/2}.$$

The maximum is attained when $z = r/2^{1/2}$ and is $2GM/3^{1/2}r^2$. Provided μr is much less than 1, the maximum occurs at the same value of z with the exponential law.

No such simple result applies when the ring is thick, as it will be in practice. There will still be a position of maximum force (the force must be zero in the centre of the ring and also at a very great distance) but it is not easy to find it analytically and the value of the maximum must likewise be found numerically.

The force calculations for an experiment will usually then require two steps, the calculation of the inverse square law force and the calculation of a force according to some other law. The analytical work is often heavy. Heyl (1930) studied the fields of masses of cylindrical form such as he used in his work on the constant of gravitation, and obtained some lengthy expressions in series of spherical harmonics. An expansion in spherical harmonics turns out to have some advantages, especially in indicating that by rather simple forms of composite cylinder the potential can be made very close to that of a sphere, with almost negligible harmonics of higher order. Metherell *et al* (1988) made a systematic study of the fields of solid cylinders but they and others before them kept to the inverse square law. Chen (1982) (see also Cook and Chen 1982) establish closed analytic formulae for the attractions of hollow and solid cylinders according to the inverse square and the exponential laws. The present state of the theory seems adequate for the experiments that have been done so far, in which the effects of noise in the force measurements are roughly matched by uncertainties in dimensional specifications, but if in future the random scatter of the force measurement can be reduced, as indeed seems very likely, then more detailed attention would have to be given to the specification of the forms of attracting masses and of the effects they might have on the estimates of forces.

4.2. Long's experiment

The principle of the experiment performed by Long (1976) (figure 1) is that the dimensions and masses of two rings, one large and one small, are chosen so that the maximum force exerted by each is the same with the inverse square law. If the rings

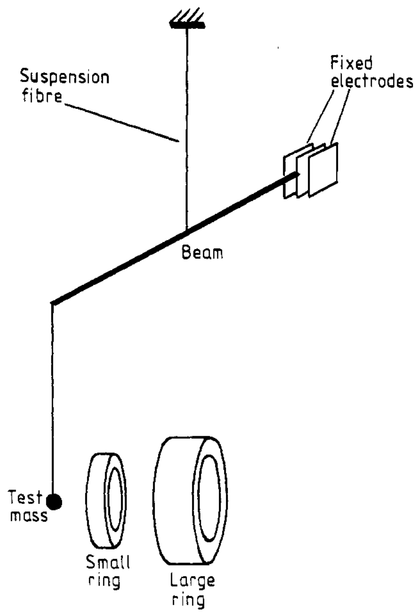

Figure 1. The inverse square law experiment of Long (1976). The beam of the torsion balance is kept in a constant position by a servo system using the attraction between the fixed electrodes and the electrode on the beam, the test mass being attracted either by the large or the small ring.

are then placed in turn to exert their maximum attractions upon a test mass on a torsion balance, the balancing force required to annul the deflection of the balance will be the same for both rings. The arrangement satisfies the requirement that no measurement should be made to the test mass, which is subject to the maximum attraction of the ring, a function just of the mass and diameter of the ring (and other dimensions of a thick ring). The large ring was made of brass and weighed 57.58 kg. It was 76.3 mm thick from back to front and the inner and outer radii were 215.9 mm and 271.1 mm respectively. The small ring was of tantalum and weighed 1.22 kg. Its thickness was 17.8 mm and the inner and outer radii were 27.5 mm and 45.5 mm respectively. The maximum force of the large ring was at 174.15 mm from the face nearer the test mass, while that of the small ring was at 26.1 mm. (These are the values given by Long, but Chen (1982) states that they are misprints.) Evidently neither ring is even approximately thin and the separation of the inner and outer cylindrical surfaces of the small ring is a large fraction of the inner radius. The calculation of the force and the analysis of the effects of errors in dimensions and density therefore requires considerable care. The test mass was a sphere of tantalum weighing 50 g and it was suspended 870 mm below the arm of the torsion balance.

The arm of the torsion balance was suspended by a tungsten wire 50 μm in diameter and 486 mm long, and was in an enclosure evacuated to 10^{-5} Pa with the wall temperature constant to within 0.01 K. The rotation of the arm was observed with an optical lever in which a mirror reflected a beam of light onto a pair of photodiodes.

Long states that great care was taken to reduce vibration of the torsion arm and the top of the tungsten fibre was suspended from a vibration damper.

In almost all applications of the torsion balance there is a residual force between attracting masses and the structure of the torsion balance. The latter is usually rather complex, and the estimation of the forces difficult. For that reason, the test mass is suspended far below the arm, but the forces are rarely negligible. The largest error in Long's experiment was attributed to the estimate of the force on the balance arm, and the second largest came from estimates of the dimensions of the large ring. The rings were reversed to average out effects of variations of density within them. Long expressed his results in terms of the relative difference of torques produced by the rings. If T_l is the torque exerted by the large ring and T_s that by the small ring, the relative difference is equal to

$$(T_l - T_s)/T_s.$$

When the departures from the nominal dimensions and masses of the rings are taken into account, and the forces of the rings on the arm of the balance are calculated, the relative difference of torque on the assumption of an inverse square law would be 0.038 07; the mean value found experimentally was 0.0417, so that the difference was 0.0037, to which an uncertainty of 0.0007 was assigned.

Long himself gave his result as a value of ε in the logarithmic law, namely 0.0002 for a reference distance of 0.01 m. Chen (1982) reanalysed the data and found that if α is taken to be $\frac{1}{3}$, μ^{-1} lies between 1.8 and 2.2 m.

4.3. *The experiments of Spero et al (1980) and Hoskins et al (1985)*

Spero *et al* (1980) carried out an experiment based on an analogue of the Faraday cage in electromagnetism, in which the effective range of distance was less than 100 mm. The same group (Hoskins *et al* 1985) later supplemented it with experiments at a greater distance which, as mentioned above, might be more sensitive to a very small rest energy of a graviton. This paper also contains more details of the earlier experiment which was described only rather briefly by Spero *et al.*

If the inverse square law holds, then the force inside a closed cavity, and in particular that inside an infinitely long cylinder, should be zero. It is, of course, not possible to make a completely closed cavity or a truly infinitely long cylinder because the test mass of a torsion balance has to be placed inside, but Spero *et al* (1980) (figure 2) used a very long hollow cylinder with the test mass of the torsion balance in its mid-level plane. The principle of the experiment was to look for deflections of the balance, corresponding to variations of force, as the cylinder was moved from side to side. Analytical formulae for the inverse square and exponential laws of force for a finite cylinder were derived by Chen (1982). The cylinder in the experiment of Spero *et al* (1980) was made of non-magnetic stainless steel 600 mm long with an inside diameter of 600 mm and a mass of 10.44 g. It was examined very thoroughly for any magnetic polarisation or residual magnetisation, and by γ-radiography for irregularities of density. Its dimensions were measured at many places. The test mass suspended inside it in the mid-plane was a copper cylinder, 44 mm long and weighing 20 g. It was hung 830 mm below the beam of the torsion balance, which was 600 mm long and suspended by a tungsten fibre 300 mm long and 75 μm in diameter. The period of the balance was 400 s and the damping time 5 d. The deflection of the balance was calibrated by the attraction of a ring on the test mass, the test mass being at the position of maximum attraction by the ring, which was of copper with a radius of 121 mm and a mass of 133 g.

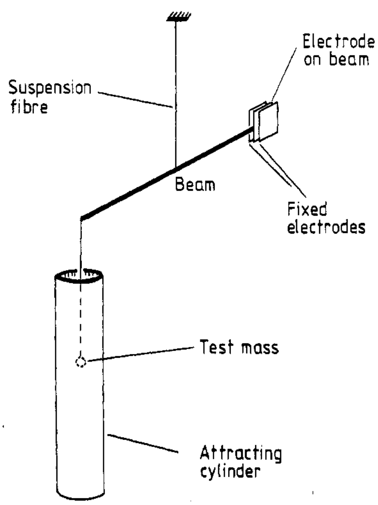

Figure 2. The inverse square law experiment of Spero *et al* (1980). The beam of the torsion balance is kept in a constant position by a servo mechanism as the attracting cylinder is moved from side.

The deflection of the torsion balance was measured with an optical lever, illuminated by an infrared-emitting diode. The reflected light fell upon a photocell divided into quadrants, the signals from which were amplified in a differential amplifier. The torsion balance was in an enclosure evacuated to 4×10^{-6} Pa.

The cylinder was placed with its axis vertical upon a trolley that could be moved automatically horizontally back and forth so that the test mass was alternately on the axis of the cylinder and near the inner surface, and the recorded deflection of the torsion balance was examined for a signal of the same period as the displacement of the cylinder.

The test mass was also attracted by the trolley and other moving parts, as was the arm of the torsion balance, and the attraction of the cylinder upon the balance arm also changed when the cylinder was moved. All those effects needed careful calculation.

In terms of the law of force adopted by Long (1976) the value of the exponent ε derived from the observations over a range from 20 to 50 mm is

$$\varepsilon = (1 \pm 7) \times 10^{-5}.$$

A problem to which considerable attention was given was that of tilts caused by the movement of the trolley and cylinder. Ideally the rotation of a torsion pendulum about its axis should be independent of oscillations of the bar as a simple pendulum about the point of support of the torsion fibre, and also of the bar about horizontal axes through its point of attachment to the fibre. However, any mechanical asymmetry will lead to some coupling of these motions, and, in addition, the optical lever may be sensitive to other purely rotational motions.

In the other experiment of the group (Hoskins *et al* 1985), the beam of a torsion balance (not a test mass hung from it) was attracted by a small mass close to it and two large ones further away, so that with the inverse square law of force the torques upon the beam were almost equal (figure 3). The three attracting masses were then placed in positions antisymmetric with respect to the beam where the torques should again balance. A change of deflection of the balance on that interchange, corresponding to an inequality of the two residual torques, would indicate a departure from the conditions of equality under the inverse square law or a failure of that law. As with the first experiment, the Newtonian attractions of the carriages upon which the attracting masses move have to be calculated carefully to determine accurately the residual that might be ascribed to a deviation from the inverse square law.

The experiment is arranged so that it is not necessary to measure distances to the torsion arm. Let the distances of the small mass m from the torsion arm in the two

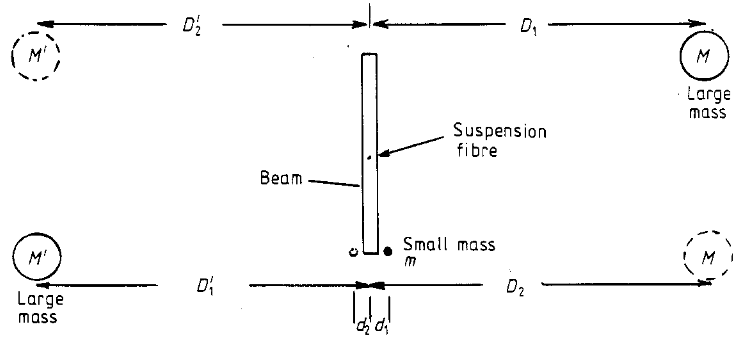

Figure 3. The inverse square law experiment of Hoskins *et al* (1985). The full and broken circles show the alternative positions of the large masses, M and M', and the small mass, m. The net torque from the attractions of the small and large masses should be nearly zero in each configuration.

positions be d_1 and d_2 and let the distances of the large masses M and M' be D_1 and D'_1 and D_2 and D'_2 respectively. Then the balance conditions are

$$m/d_1^2 = M/D_1^2 + M'/D_1'^2$$

and

$$m/d_2^2 = M/D_2^2 + M'/D_2'^2$$

for the inverse square law.

If the masses M and M' are nearly equal, and the distances D_1 and D'_1 are also nearly equal, it can be shown that the error in writing

$$d_1 = [m/(M+M')]^{1/2}(D_1 + D'_2)$$

and

$$d_2 = [m/(M+M')]^{1/2}(D_2 + D'_1)$$

is of order $[(M-M')/M]^2$ and $[(D_1 - D'_1)/D_1]^2, [(D_2 - D'_2)/D_2]^2$ at most. Thus, to a close approximation

$$d_1 + d_2 = [m/(M+M')]^{1/2}[(D'_2 + D_1) + (D'_1 + D_2)]$$

in which expression the individual values of D_1, D_2, D'_1, D'_2 and d_1, d_2 do not enter.

The beam of the torsion balance was a copper rod 600 mm long and weighing 523 g. The torsion fibre was tungsten, of 90 μm diameter and 200 mm long. The deflection was again observed with an optical lever, an infrared-emitting diode and a quadrant photodiode, followed by a differential amplifier. The torque exerted by the electrostatic force in an electrode at the end of the beam opposite to the test mass kept the angular position constant. The pendulum modes of the balance were damped with eddy currents and the balance was in an enclosure evacuated to 4×10^{-6} Pa, the whole apparatus being in a well-insulated pit with remote control of the operations.

The single attracting mass was a copper cylinder weighing 43 g and placed 50 mm from one end of the bar. It was moved from one side of the bar to the other by a remotely controlled mechanism. The two distant masses were cylinders of copper that each weighed 7.3 kg and were set 1.05 m from opposite ends of the bar, so that the torques they exerted on the bar were in the same sense but in the opposite sense to that exerted by the small mass. The large masses were on carriages, enabling them to be moved by remote control to the antisymmetric positions. As with the cylinder experiment, the tilts produced by moving the large masses were examined very carefully.

The result of experiments extending over three days was that the ratio of the torque, $T(1050)$, exerted by the two masses at 1.05 m, to the torque, $T(50)$, exerted by a single mass at 50 mm, was

$$T(1050)/T(50) = 1.018\,37$$

while the calculated value on the basis of inverse square law attraction for all parts to each other was

$$T(1050)/T(50) = 1.018\,25.$$

The corresponding value of the exponent ε is $(5 \pm 27) \times 10^{-5}$.

The regions in the $\alpha\mu$ plane delineated by the experiment are shown in figure 4.

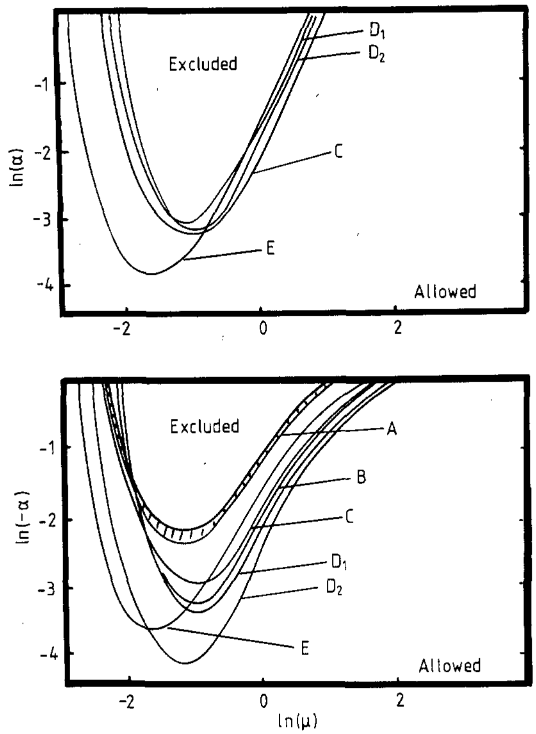

Figure 4. Plot of allowed and excluded regions of the $\alpha\mu$ plane according to experiments on the laboratory scale. Curves: A, Long (1976), the permitted band; B, Milyukov (1985); C, Hoskins *et al* (1985); D_1, Chen *et al* (1984), null; D_2, Chen *et al* (1984), non-null; E, Spero *et al* (1980).

4.4. *The experiment of Chen et al (1984)*

The design of this experiment (figure 5) was again different from each of the three described earlier in this section. A test mass on a torsion balance was attracted by two masses on opposite sides of the balance arm. One of the two masses was always left in the same position, the other was either a mass equal to the first at the same nominal distance from the test mass, or a larger mass at a greater distance. The net torque on the balance arm should be the same in the two configurations if the inverse square law of force is followed. As with the other experiments described above, no measurements have to be made to the test mass. Let the masses be labelled 1, 2, 3, with 1 and 2 being similar. Let their masses be m_1, m_2, m_3 and their distances from the test mass r_1, r_2, r_3. Suppose that mass 1 is always left in place. At equilibrium under the inverse square law

$$m_1/r_1^2 = m_2/r_2^2$$

and

$$m_1/r_1^2 = m_3/r_3^2.$$

Let

$$r_1 + r_2 = d_2 \qquad r_1 + r_3 = d_3.$$

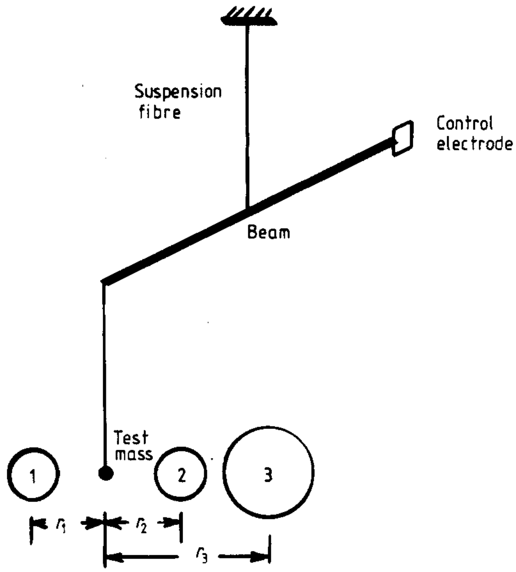

Figure 5. The inverse square law experiment of Chen *et al* (1984). Mass 1 is kept in a fixed position at r_1 from the nominal position of the test mass and its attraction is balanced alternately by masses 2 and 3.

Then

$$d_2 = r_1[1 + (m_2/m_1)^{1/2}]$$

$$d_3 = r_1[1 + (m_3/m_1)^{1/2}]$$

and since mass 1 and the angular position of the beam are always kept the same

$$d_2/d_3 = (m_1^{1/2} + m_2^{1/2})/(m_1^{1/2} + m_3^{1/2}).$$

The separations d_2 and d_3 of masses 2 and 3 from 1 are straightforward to measure and any deviation from the above ratio will arise from attractions on the beam of the balance, errors in measurement and a deviation from the inverse square law.

The beam of the torsion balance was 600 mm long and was suspended from a tungsten fibre 75 μm in diameter and 800 mm long. A feature of the experiment was the care to improve the stability of the balance by removing dislocations from the wire using a procedure that was an elaboration of that of Braginsky and Manukin (1977); a drift rate of less than 10^{-6} rad in 24 h was achieved. The balance was in an enclosure evacuated to 10^{-5} Pa and was carefully insulated from temperature changes and magnetic fields. The deflection of the balance was observed with the optical lever with a helium-neon laser as source and a position-sensitive photocell as detector, followed by a differential amplifier. Negative feedback with differentiation was applied by means of electrostatic forces on the beam to keep the deflection constant. The smallest measurable acceleration as determined from the sensitivity of the detector system was 5×10^{-14} m s^{-2}, while the acceleration produced on the test mass by one of the attracting masses was 10^{-8} m s^{-2}, so that in principle a sensitivity of 10^{-6} in the measurement of gravitational forces was attainable, but a poorer limit was set in practice by noise and drift of the zero.

The three large masses had the following dimensions and masses:

	Length (mm)	Diameter (mm)	Mass (kg)
1	100	100	6.25
2	100	100	6.25
3	400	100	25.00

They were made of non-magnetic stainless steel; they and all other parts of the apparatus were carefully checked for residual magnetisation. The test mass was a sphere weighing 41 g.

Two experiments were performed. In the first there was a net attraction on the test mass balanced by the applied electrostatic force; that arrangement was intended to meet the objection of Long to the Spero experiment that there might be a vacuum polarisation effect not seen in a null experiment.

The masses 1 and 2 were 50 mm apart and 1 and 3, 91 mm apart. Experimentally the relative difference of the observed from the nominal force was

$$(\Delta F/F)_{e} = (6.69 \pm 0.01) \times 10^{-2}$$

whilst that calculated from residual attractions with an inverse square law was

$$(\Delta F/F)_{c} = (6.68 \pm 0.008) \times 10^{-2}.$$

The difference $(\Delta F/F)_{e} - (\Delta F/F)_{c}$ was $(1.1 \pm 1.35) \times 10^{-4}$.

In the other arrangement, the masses were placed so that the net field at the test mass was zero when the masses 1 and 2 were 120 mm apart and again when 1 and 3 were 195 mm apart. The distances were made larger in this experiment to extend the range of the study. The results were now

$$(\Delta F/F)_{e} = (1.436 \pm 0.018) \times 10^{-2}$$

$$(\Delta F/F)_{c} = (1.430 \pm 0.009) \times 10^{-2}$$

$$(\Delta F/F)_{e} - (\Delta F/F)_{c} = (0.6 \pm 2.0) \times 10^{-4}.$$

The values of the exponent ε were

non-null experiment $\quad \varepsilon = -(4.3 \pm 5.1) \times 10^{-5}$
null experiment $\quad \varepsilon = -(2.0 \pm 6.6) \times 10^{-5}$.

The boundaries in the $\alpha\mu$ plane are shown in figure 4. The results of Chen *et al* (1984) are generally consistent with those of Spero *et al* (1980) and of Hoskins *et al* (1985) but not with Long's (1976) result.

4.5. Analysis of the determination of the constant of gravitation at the Shternberg Astronomical Institute

A determination of the constant of gravitation at the Shternberg Astronomical Institute (Sagitov *et al* 1979, see § 6.3), which followed the scheme of Heyl and Chzranowski (1942), involved a set of four distances between masses over a range of 100–210 mm. The original reduction of the measurements was confined to a calculation of the value of the constant of gravitation, but subsequently Milyukov (1985) has analysed the data to derive an estimate of the constant ε in the logarithmic law as used by Long.

Milyukov's results are shown in table 1. The systematic variation of ε with the value of the larger separation is not consistent with the simple logarithmic law, but Milyukov considered that the data all belonged to one statistically consistent set and that the estimate of ε was $(9 \pm 10) \times 10^{-5}$. The boundary in the $\alpha\mu$ plane is shown in figure 4.

4.6. Other experiments

The experiments summarised in this section are of lower precision than those just considered, but are useful as they were done in different ways and at different distances. The most direct experiment was that of Panov and Frontov (1979) who in effect determined the constant of gravitation when the attracting masses were at different distances. A torsion balance was deflected by a single large mass which was moved from side to side with the period of the balance. The torsion balance was made of fused silica, it was 400 mm long and had a test mass of 10 g at each end. Its free period was 910 s. It was suspended by a tungsten wire and placed in an evacuated enclosure and its deflection was observed with a capacitance sensor.

The value of G was measured at separations of the attracting and test mass of 421, 2958 and 9840 mm and the following ratios of the value of G at the greater distances to that at the shortest (421 mm) were obtained:

$$2958 \text{ mm} \qquad G/G_{421} = 1.003 \pm 0.006$$
$$9840 \text{ mm} \qquad G/G_{421} = 0.998 \pm 0.013.$$

The authors express their result in the form

$$\alpha\mu^2 r_0^2/(1+\alpha) = (0.8 \pm 5.0) \times 10^{-5}$$

where r_0 is the shortest distance, 421 mm.

Table 1. Milyukov's (1985) analysis of the determination of Sagitov *et al* (1979). The values given are of $\varepsilon \times 10^4$ in $G(r) = G_0[1 + \varepsilon/g(r/r_0)]$.

	Configuration		
Observation	a	b	c
1	0.669	0.233	−5.270
2	7.730	−1.613	−3.693
3	14.674	7.536	0.272
4	4.282	−2.701	−6.930
5	−2.184	−4.371	−7.474
6	4.816	3.891	0.540
7	−4.601	−0.149	−3.813
8	12.093	4.852	−1.496
9	10.020	−2.489	0.952
10	4.490	5.407	−5.940
Mean	+5.22	+1.12	−3.28

a: attracting masses at 112.5 mm and 132.5 mm.
b: attracting masses at 112.5 mm and 162.5 mm.
c: attracting masses at 112.5 mm and 212.5 mm.

A further laboratory experiment was performed by Ogawa *et al* (1982). A mechanical oscillator was driven into forced oscillations by the attraction of a bar rotating at half the resonant frequency of the oscillator. The bar was 0.9 m long, had a diameter of 0.27 m and a mass of 401 kg and was driven to rotate at 30.4 Hz. The detector was a square aluminium plate of mass 1400 kg and of dimensions $1.65\times1.65\times0.19$ m. Cuts from the corners to the centre ensured that it was sensitive to the quadrupole potential of the spinning rod. Its resonant frequency was 60.8 Hz and it had a Q of 5.5×10^4.

The results were that over a range from 2.6 to 10.7 m, the exponent δ in the potential $GM/r^{1+\delta}$ is $(2.1\pm6.2)\times10^{-3}$, or, alternatively, the least value of α is 0.012 when μ^{-1} is 0.63 m.

An experiment on a larger scale was that of Yu *et al* (1979) who measured the values of the acceleration of gravity across a large oil tank when it was full and empty. With the mass of oil and the dimensions of the tank known, the following results were obtained:

	Distance from rim of tank (m)			
	2	4	6	8
$\Delta g(\mathrm{obs})/\Delta g(N)$	1.01 ±0.13	1.02 ±0.22	1.09 ±0.29	1.35 ±0.44

Δg(obs) is the observed difference in gravity and $\Delta g(N)$ that to be expected for the inverse square law. The results are not useful.

The fourth experiment was of a different nature, an attempt to determine the value of $\nabla^2 V$, where V is the gravitational potential in free space. By the inverse square law it would be zero, by the Proca equation it would be $-k^2 V$. Chan *et al* (1982) used a superconducting gravity gradiometer (see Warburton and Goodkind 1976) to measure the gradients of gravity in three orthogonal directions; the sum of those gradients is the divergence of $\boldsymbol{g}$, that is $\nabla\cdot\boldsymbol{g}$ or $\nabla^2 V$. To distinguish a particular source, the oscillating field produced by a heavy pendulum was studied; the mass of the pendulum was 1600 kg, its frequency 0.228 Hz and the gradiometer was placed 2.3 m away. The amplitudes of the three contributions to the divergence ranged from $+10$ to $-6\times10^{-9}\ \mathrm{s}^{-2}$ (that is, $10^{-10}\ g\ \mathrm{m}^{-1}$, where g is the ordinary acceleration due to gravity, $\simeq10\ \mathrm{m\ s}^{-2}$), and the sum was $(0.15\pm0.23)\times10^{-9}\ \mathrm{s}^{-2}$. The weakness of the experiment was that the three components could not be determined simultaneously but had to be measured separately. The lowest value of α corresponding to the observed sum occurred at $\mu^{-1}=1$ m and had the value $|\alpha|_{\min}=0.024\pm0.036$.

4.7. *Summary: the status of the inverse square law*

The principal results are shown in table 2, which gives the values of ε in the law of force, $GMr^{-2}[1+\varepsilon\ln(r/r_0)]$, the minimum value of $|\alpha|$ and the value of μ^{-1} at which it occurs for the boundaries in the $\alpha\mu$ plane. The boundaries themselves are shown in figure 4; the curves for Long's (1976) experiment show the range of (α,μ) corresponding to his result.

To summarise, Long's result is inconsistent with the other experiments, which show no significant deviation from the inverse square law, but the sensitivity of all experiments is very poor compared with that of electrical tests of the inverse square law.

Table 2. Results of some experiments on the inverse square law. The value of μ^{-1} is that at which $|\alpha|$ attains its least value, $|\alpha|_{min}$.

Authors	Range (mm)	$\varepsilon \times 10^5$	$\lvert\alpha\rvert_{min}$	μ^{-1} (m^{-1})
Long (1976)	45-300	200 ± 40	8×10^{-3}	0.08
Spero *et al* (1980)	20-50	1 ± 7	10^{-4}	0.02
Ogawa *et al* (1982)	2600-10700	-210 ± 620	0.012	0.63
Chen *et al* (1984)				
non-null	25-45	-4.3 ± 5.1	10^{-4}	0.1
null	60-100	-2.0 ± 6.6	2×10^{-4}	0.06
Hoskins *et al* (1985)	100-1000	5 ± 27	10^{-3}	0.1
Milyukov (1985)	110-210	9 ± 10	4×10^{-3}	0.1

In recent years, attention has increasingly been called to geophysical tests of the inverse square law. The question is, given that laboratory experiments exclude values of μ^{-1} of the order of 1 m or less but that the range of laboratory measurements is greatly restricted, is it possible that much larger values of μ^{-1} and smaller values of $|\alpha|$ might be revealed by experiments over greater distances although of poorer precision? Celestial mechanics excludes deviations with ranges greater than 10^6 km, and comparisons between the mean values of gravity at the surface of the Earth and at the heights of artificial satellites show that the inverse square law is valid over ranges of 500 km and more. Thus there remains a gap which might be fulfilled in part by geophysical measurements.

Stacey and Tuck (1981) considered possible geophysical evidence and concluded that observations in mines and on and under the waters of the Gulf of Mexico suggested that the value of the constant of gravitation calculated from those observations was greater than that found in laboratory experiments (see § 6). Holding and Tuck (1984) and Holding *et al* (1986) have discussed in great detail measurements of gravity made in and around two mines, Mt Isa and Hilton, in Queensland (Australia) and their analysis indicates that the effective value of the constant of gravitation is 6.712×10^{11} N m^2 kg^{-2} compared with a value of 6.6726×10^{-11} N m kg^{-2} estimated from recent laboratory determinations (see § 6). The estimated systematic errors are 0.089 to -0.022×10^{-11} N m^2 kg^{-2}, so the discrepancy, 0.039×10^{-11} N m^2 kg^{-2}, is scarcely significant. If the discrepancy is real, it corresponds to an α of -0.0107 and a μ^{-1} of 1360 m, while acceptable values of α and μ would cover the ranges of

$$\alpha = -0.0075 \qquad \mu^{-1} < 200 \text{ m}$$

to

$$\alpha = -0.014 \qquad \mu^{-1} > 500 \text{ m}.$$

The comparison of surface gravity measurements with satellite motions seems to exclude $\alpha < 0.010$, $\mu^{-1} > 800$ m. The latest review is that of Stacey *et al* (1987).

Fishbach and others (1986a, b) have related the geophysical results to what they see as inconsistencies in the experiments of Eötvös *et al* (1922) on the weak equivalence principle, a matter to be discussed further in § 5.

Experiments in the laboratory, nonetheless, give no support to any suggested deviation from the inverse square law.

5. The principle of weak equivalence

5.1. Introduction

The essence of the principle of equivalence goes back to Newton who asserted that the weight of a body, the force acting on it in a gravitational field, was proportional to its mass, the quantity of matter in it, irrespective of its constitution and, as described in § 3, he made experiments with pendulums to demonstrate the principle for a diverse set of substances. Later work by Bessel and Potter was also mentioned in § 3. Nowadays in the light of general relativity, the principle is formulated in some such terms as: all small uncharged bodies, placed at the same initial point in spacetime, and given the same initial velocity, follow the same trajectory in space and time, irrespective of their internal constitutions. The principle refers to small, uncharged bodies so that electrical forces and self-gravitational energy can be ignored. In general relativity the paths of small test objects are geodesics of the spacetime metric (determined by energy sources) and so the path followed is determined by the spacetime geometry and not by the nature of the test object.

Experimental tests of weak equivalence depend on showing that the ratio of inertial mass to (passive) gravitational mass is independent of constitution, and the various experiments that have been performed differ in the ways in which the accelerations are applied and the source of the gravitational field.

In the pendulum experiments of Newton, Bessel and Potter, the acceleration was that of the pendulum rotating about its point of support, while the gravitational field was that of the Earth. Eötvös developed an experiment in which the rotation of the Earth about its polar axis produced the acceleration and the gravitational field was again that of the Earth (the terrestrial experiment). He also suggested a solar experiment, using the acceleration of the Earth in its orbit about the Sun and comparing it with the gravitational attraction of the Sun (Eötvös *et al* 1922). The Eötvös schemes have the advantage over the pendulum that the relevant components of the forces are horizontal and so can be compared by a torsion balance, while the solar scheme has the advantage over that using the attraction and rotation of the Earth that the apparatus does not have to be moved to distinguish between the behaviours of different test masses. Eötvös made some preliminary tests using the solar attraction, but it is to Roll *et al* (1964) and to Braginski and Panov (1972) that the major applications are due.

5.2. The experiments of Eötvös and his successors: terrestrial experiments

The principle of terrestrial experiments of the type introduced by Eötvös is shown in figures 6 and 7. The forces acting on a unit test mass fixed on the surface of the Earth, supposed spherical, at a colatitude θ are $GM_\oplus/r^2$ to the centre of the Earth and $\omega^2 r \sin\theta$ perpendicular to the polar axis. $M_\oplus$ is the mass of the Earth and ω the spin angular velocity about the polar axis.

A torsion balance placed at that point on the surface of the Earth and consisting of a horizontal arm suspended by a fibre will align itself so that the fibre is in the direction of the resultant of the gravitational and centrifugal forces acting upon it.

Now the gravitational acceleration is

$$-(GM_\oplus/r^3)\boldsymbol{r}$$

where $\boldsymbol{r}$ is the radius vector from the centre of the Earth, while the centrifugal

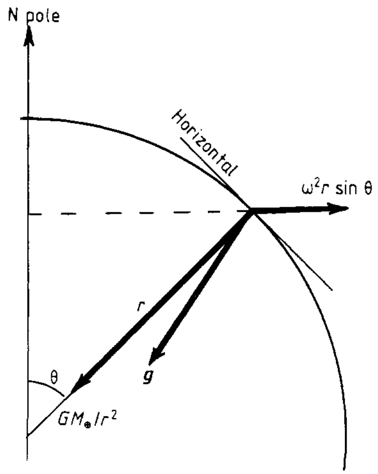

Figure 6. The net acceleration due to gravity, g, and its components $GM_\oplus/r^2$ and $\omega^2 r \sin\theta$.

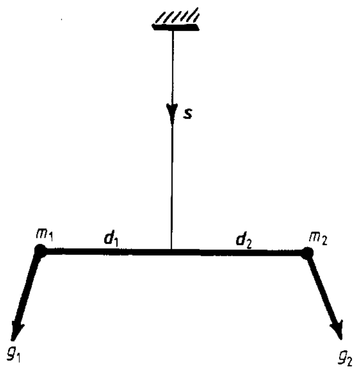

Figure 7. The net accelerations acting on masses of different chemical composition in the terrestrial experiment of Eötvös (1891). $\boldsymbol{s}$ is the unit vector along the suspension fibre and $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are the position vectors of the test masses from the point of support.

acceleration is

$$\omega^2 \boldsymbol{p}$$

where $\boldsymbol{p}$ is the vector perpendicular to the polar axis and of magnitude $r \sin\theta$.

The resultant is

$$-(GM_\oplus/r^3)\boldsymbol{r} + \omega^2 \boldsymbol{p}.$$

Suppose a test object of mass m_1 is placed at one end of the balance arm and an object of mass m_2 is placed at the other end. Nominally m_1 and m_2 are equal, but if the materials differ, then the ratios of the passive gravitational to inertial masses, $(m_g/m_I)_1$ and $(m_g/m_I)_2$, may also differ. The resultant force on mass 1 will be

$$-m_{g1}(GM_\oplus/r^3)\boldsymbol{r} + m_{I1}\omega^2 \boldsymbol{p} = \boldsymbol{g}_1$$

with a similar expression for mass 2. If the ratios of gravitational to inertial mass are unequal, the resultant forces on the two masses will be neither equal nor parallel.

Let the masses be at the vector distances $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ measured from the point of suspension (figure 5). The torque about the suspension fibre by the force on mass 1 is

$$(\boldsymbol{g}_1 \wedge \boldsymbol{d}_1) \cdot \boldsymbol{s}$$

where $\boldsymbol{s}$ is a unit vector directed along the fibre.

The sum of forces acting on the two masses is $(\boldsymbol{g}_1 + \boldsymbol{g}_2)$ and so the vector $\boldsymbol{s}$ is

$$(\boldsymbol{g}_1 + \boldsymbol{g}_2)/|\boldsymbol{g}_1 + \boldsymbol{g}_2|.$$

Suppose first that, as is very nearly the case, $\boldsymbol{d}_1 = -\boldsymbol{d}_2$ and denote $\boldsymbol{d}_1$ and $-\boldsymbol{d}_2$ by $\boldsymbol{d}$. Then the net torque is

$$\frac{(\boldsymbol{g}_1 \wedge \boldsymbol{d}_1) \cdot (\boldsymbol{g}_1 + \boldsymbol{g}_2) + (\boldsymbol{g}_2 \wedge \boldsymbol{d}_2) \cdot (\boldsymbol{g}_1 + \boldsymbol{g}_2)}{|\boldsymbol{g}_1 + \boldsymbol{g}_2|} = \frac{(\boldsymbol{g}_1 \wedge \boldsymbol{d}_1) \cdot \boldsymbol{g}_2 + (\boldsymbol{g}_2 \wedge \boldsymbol{d}_2) \cdot \boldsymbol{g}_1}{|\boldsymbol{g}_1 + \boldsymbol{g}_2|}$$

or

$$\frac{2(\boldsymbol{g}_1 \wedge \boldsymbol{g}_2) \cdot \boldsymbol{d}}{|\boldsymbol{g}_1 + \boldsymbol{g}_2|}.$$

If $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are parallel, the net torque vanishes. Generally,

$$\boldsymbol{g}_1 \wedge \boldsymbol{g}_2 = -(\mu\omega^2/r^3)(m_{g1}m_{I2} - m_{g2}m_{I1})\boldsymbol{r} \wedge \boldsymbol{p}.$$

where μ is $GM_{\oplus}$. $\boldsymbol{r} \wedge \boldsymbol{p}$ is the vector of magnitude $r^2 \sin\theta \cos\theta$ directed perpendicular to the meridian plane through the point of observation, which shows that the beam of the torsion balance should also be perpendicular to the meridian to obtain the maximum effect.

Evidently there is no torque if θ is 0 or $\frac{1}{2}\pi$ and the best sites for the experiment will be at latitude 45°.

If $\rho = m_g/m_I$, the net torque when the beam is perpendicular to the meridian is

$$-\frac{2\mu\omega^2}{r} d \frac{m_{I1}m_{I2}}{(m_{I1} + m_{I2})g}(\rho_1 - \rho_2) \sin\theta \cos\theta$$

where g is the acceleration due to gravity as usually understood.

Roll *et al* (1964) introduced the factor $\eta(1, 2)$, equal to $2(\rho_1 - \rho_2)/(\rho_1 + \rho_2)$, in terms of which the net torque is

$$-\frac{\eta m_1 m_2}{m_1 + m_2} \frac{\mu\omega^2 d}{rg} \sin\theta \cos\theta$$

since ρ_1, ρ_2 are very nearly 1.

If the lengths of the arms of the balance are unequal, $2d$ is replaced by $(d_1 - d_2)$.

The problem of the scheme of Eötvös is that it is not possible to identify the zero from which the torque is measured, and the only possibility is to turn the whole apparatus through 180° in the horizontal plane so that masses 1 and 2 are interchanged. Thus, relative to the frame to which the suspension is attached, the torque is reversed, and the arm of the beam will be deflected through an angle proportional to twice the net torque. It is rather undesirable to move delicate apparatus.

Eötvös gave a preliminary account of his investigations in 1890 (see Eötvös 1891) and the main work seems to have been completed within the next fifteen years, although a final account was only published three years after his death (Eötvös *et al* 1922, see also Eötvös *et al* 1935). Eötvös introduced the measurement of gradients of gravity for geophysical exploration—the measurement of gravity itself by pendulums being too imprecise and time-consuming in those days—and he developed a torsion balance sensitive to gradients by hanging the masses from the ends of the beam at different heights. Roll *et al* (1964) reproduced a diagram of such a balance (figure 8) from the collected papers of Eötvös. It seems that Eötvös used a balance originally built for geophysical measurements in his studies of equivalence, although, as Roll *et al* point out, the fact that it was sensitive to tidal gradients of gravity was a disadvantage in the equivalence experiments. Done at a time when methods of observation were coarse compared with those now available, with apparatus not designed for the specific purpose, and with some defects of procedure, it would seem that there is little point in spending much time on the work of Eötvös when the experiments of Roll *et al* (1964) and of Braginski and Panov (1972) far surpass it in precision. However, because the attracting mass (the Earth) in the terrestrial scheme of Eötvös is effectively much closer than that (the Sun) in solar experiments, the Eötvös terrestrial experiment has been subject to close attention in recent years (Fishbach *et al* 1986a, b, Keyser *et al* 1986), and in consequence deserves more attention insofar as it may be sensitive to different forces than the solar experiments because part of the attracting mass of the Earth is within the scale distance of the fifth force postulated by Fishbach *et al.*

One torsion balance used by Eötvös had a beam 0.4 m long with masses of 30 g at the two ends. One mass was fixed directly to the end of the arm, the other was suspended 212 mm below the opposite end. The suspension wire was about 0.8 m long with a torsional constant of 0.5×10^7 rad N^{-1} m^{-1}. The arm of the balance carried a mirror and the deflection of the arm was observed with a telescope viewing an image of a scale in the mirror. The arm length of the optical lever was about 0.6 m. Roll *et al* estimated that if η were 10^{-11}, the deflection would be 1.9×10^{-8} rad, which corresponds to 1/20 000 of the smallest division of the scale, and they concluded from the

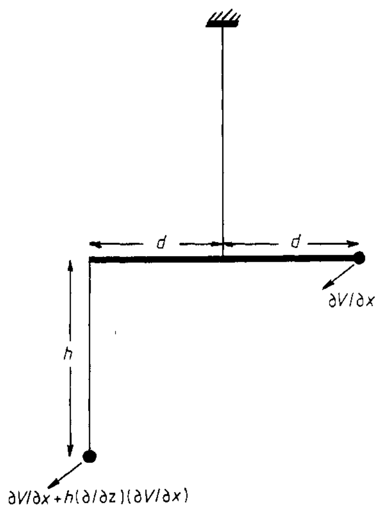

Figure 8. Principle of the torsion balance of Eötvös used to measure gradients of gravity. Since $(\partial/\partial z)(\partial V/\partial x) = (\partial/\partial x)(\partial V/\partial z)$ the difference between the forces on the upper and lower masses is $h(\partial/\partial x)(\partial V/\partial z) = h\partial g/\partial z$.

scatter of the observations that Eötvös and his colleagues were estimating to about 1/20 of a division. Experienced observers commonly achieve such a discrimination when the magnification and the index mark are suitably chosen.

The beam and suspension were in a vessel with air at normal pressure and temperature. There were three metal walls with air between them which equalised the temperature across the enclosure to reduce convection currents around the beam. The air also damped the oscillations of the beam, so that a steady reading was obtained about an hour or so after adjusting the apparatus.

The mass fixed directly to the beam was kept the same in a series of experiments, but the suspended one was changed.

Many observations were made with the 'double gravity variometer', which comprised two similar beams and suspensions in the same enclosure, but with the changed masses at opposite ends.

The net torque on the beam was found by turning the whole balance in its enclosure through 180°, when the difference of deflection would correspond to twice the net torque. In the double variometer the angle between the two beams would change by an amount corresponding to four times the net torque.

In all, the experiments of Eötvös *et al* (1922, 1935) were done in three configurations. In the first, a single torsion balance was used, with a platinum mass always on one end (b), while the mass at the other end (a) was changed. The balance was turned so that the a end pointed successively N, E, S and W. Suppose the masses on end a are labelled 1 and 2. Let θ_1 be the rotation of the beam relative to the enclosure when the whole balance is turned from E to W with mass 1 on end a. Similarly, let θ_2 be the rotation with mass 2. Then $\eta(1,2)$ is proportional to $\theta_1-\theta_2$. This scheme, by placing masses 1 and 2 in the same gravitational fields, is insensitive to constant gradients of gravity ($\partial g/\partial x$, $\partial g/\partial y$), but sensitive to changes in them and in the torsional constant of the suspension. The differences between platinum and magnalium and platinum and snakewood were measured by the procedure.

The second configuration was like the first, but the observations in the four orientations, N, E, S, W, were made at equal intervals of time, and any linear change of torsional constant with time could be allowed for, but again, the gradients of gravity must not change. The difference between platinum and copper was measured, and also the change that occurred with the chemical reaction between silver sulphate and iron sulphate—a reaction for which changes of weight had earlier been claimed by Landolt (1983).

In the third configuration, a double balance was used. One mass (1) was hung from end a of one arm and the other (2) from end b of the second arm and observations were made in the four orientations, N, E, S, W. The masses were then exchanged between the balance arms (still at ends a) and the observations were repeated. In that way, changes of both torsional constant and of gravity gradients were eliminated. The substances compared in this series were water and copper, copper and copper sulphate, copper and copper sulphate solution, asbestos and copper, and tallow and copper.

Renner (1935) made a further series of observations in the third configuration, with some improvements to the apparatus and conditions; he examined the differences between platinum, brass, glastreane, paraffin, ammonium fluoride, manganese-copper alloy and bismuth.

The problem with the observational procedure that Roll *et al* (1964) considered in their analysis of the results of Eötvös *et al* (1922) and of Renner (1935), was why the gravity field of the observer at the telescope did not grossly disturb the results, and they conjectured that the observations were made quickly enough that the balance did not have time to respond to the attraction of the observer.

The results of Eötvös *et al* (1922) and of Renner (1935), as given by Roll *et al* (1964) in terms of the parameter η, are listed in table 3. Some of the recent discussion has turned on the results of Renner and whether or not they should be ignored in looking for evidence for or against a fifth force. Roll *et al* (1964) discussed the statistics of the earlier as well as of the later experiments, and a summary of their findings follows.

They were able to obtain some original records and particulars of the reductions from Renner, and it appears that for any one disposition of the weights, Renner made 24 observations with the balance arm in one direction (N) and 24 in the opposite direction (S). He obtained 90 differences (N − S), took the mean, and assigned to the mean a standard deviation equal to the standard deviation of a single observation divided by $90^{1/2}$. Evidently the 90 differences are correlated and the factor $90^{1/2}$ is too great. When they reduced the data in a way that avoided correlations, Roll *et al* found that the standard deviations of the means should be about three times those quoted by Renner. Some of the difference must come from a larger estimate of the standard deviation of a single difference when properly calculated, for a factor such as $(90/24)^{1/2}$

Table 3. Results of terrestrial experiments of (*a*) Eötvös *et al* (1922, 1935) and (*b*) Renner (1935).

(*a*)

Materials		$\eta(A, B)$	SD of η	
A	B	(10^{-9})	(10^{-9})	χ^2
Magnalium	Platinum	+4	1	16
Snakewood	Platinum	−1	2	—
Copper	Platinum	+4	2	4
$Ag_2SO_4 + FeSO_4$ in glass and brass	Products of $Ag_2SO_4 + FeSO_4$ in glass and brass	0	1	—
Water in brass	Copper	−5	1	25
$CuSO_4$ crystals, brass	Copper	−3	1	9
$CuSO_4$ solution, brass	Copper	−4	1	16
Asbestos, brass	Copper	−2	1	4
Talc, brass	Copper	−3	1	9
		Mean = −1		Sum = 83

(*b*)

Materials		$\eta(A, B)$	SD of η	
A	B	(10^{-9})	(10^{-9})	χ^2
Platinum	Brass	+0.45	+0.65	0.5
Batavian glass drops	Brass	−0.06	0.67	—
Powdered glass drops	Brass	+0.21	0.65	0.09
Paraffin in brass	Brass	+0.24	0.26	0.9
NH_4 in brass	Brass	+0.06	0.25	0.06
Copper	Manganese alloy	−0.08	0.20	0.16
Copper	Manganese alloy	−0.12	0.22	0.3
Bismuth	Brass	0.14	0.74	0.04
		Mean = +0.10		Sum = 2.05

is not as great as 3. Roll *et al* do not say if the actual mean values changed when the observations were correctly reduced. It is probable that they would not, for the general experience of the results of mishandling correlated observations is not that a mean result is greatly affected but that the uncertainty is considerably underestimated. In table 3 Renner's results are given with his own estimates of the standard deviation. Roll *et al* point out a consequent peculiarity of the Renner results. Even with Renner's own standard deviations there is no mean value that exceeds the standard deviation, whereas if the observations were drawn from a purely random set of zero mean, it would be expected that four of the eight means would be greater than 0.6 times the standard deviation, and there are just two which are greater than that amount. The value of χ^2 for the Renner set with his standard deviations is 2.05 on eight degrees of freedom. That is abnormally low and suggests the presence of some systematic correlation; with the Roll *et al* standard deviations χ^2 would be 0.22. The situation with Eötvös's own data is quite different. Roll *et al* consider that the standard deviations of the mean values of η are much as would be expected from the uncertainties in the scale reading and see no reason to increase them. The value of χ^2 is however then unduly great, namely 83 on 9 degrees of freedom, and again inconsistent, but in a different way, with samples taken from a random set with zero mean.

Renner used the original apparatus of Eötvös, with some improvements to the torsion suspension and to the method of reading. It is at first sight surprising that the same apparatus should yield results of such different statistical character as those of the original series and Renner's, and the obvious conjecture is that besides the improvements to stability and sensitivity there was some difference between the procedure used in the observations. Roll *et al*, as already mentioned, drew attention to the apparent lack of effect of the observer, and it is possible that Eötvös and his colleagues took slightly longer over the observations than did Renner. Again, in the original work, one hour elapsed between rotating the enclosure through 180° and making a reading. If the beam did not come quite to equilibrium, a difference here might show up.

Can either the Eötvös or Renner results be used in evidence for or against a fifth force with a characteristic scale length? It seems rather rash to do so in view of the peculiarities of the statistics; at the same time, nothing in the results seems to justify rejecting Renner while using Eötvös. So it appears that the experiments give no support to the hypothesis of a fifth force.

Roll *et al* (1964) summarised their discussion of the Eötvös and Renner work as follows.

At 95% confidence limits, η is less than 9×10^{-9} according to Eötvös and less than 4.2×10^{-9} according to Renner. The criticisms of the scheme of Eötvös are principally the unknown disturbances produced by turning the apparatus, the gravity field of the observer and gradients from the tides, and possible magnetic torques.

An experiment according to the Eötvös terrestrial scheme was undertaken by Cook because the work of Heyl on the determination of the constant of gravitation (Heyl 1930) appeared to show a difference between platinum and optical glass. The data obtained by Cook (quoted by Heyl 1930) lead to

$$\eta(\mathrm{Pt, glass}) = (3.6 \pm 22)\times10^{-8}$$

a notably less discriminating result than that of Eötvös *et al* or of Renner.

So far, no further experiment using the attraction of the Earth has been performed—no doubt on account of the greater delicacy of the solar scheme—but some work is in preparation: for example, Faller and Keiser of the Joint Institute of Laboratory Astrophysics (Boulder, CO) are developing an experiment in which a frame carrying the masses floats on water instead of being suspended from a fibre (see Keiser and Faller 1979).

5.3. The solar experiments of Eötvös et al (1922, 1935)

The beam of a single torsion balance was set in the meridian, in which position the rotational acceleration and gravity field of the Earth exert no net torque upon the two masses, but the attraction of the Sun and the acceleration about it may do so.

Let $\boldsymbol{D}$ (figure 9) be the vector distance of the Sun from the beam and let n be the angular velocity of the Earth about the Sun. The net force on mass i is then

$$m_{\mathrm{I}i}n^2\boldsymbol{D} - Gm_{\mathrm{g}i}M_{\odot}\boldsymbol{D}/D^3$$

where $M_{\odot}$ is the mass of the Sun.

By Kepler's law

$$n^2 = GM_{\odot}/D^3$$

and so the net force is

$$n^2(m_{\mathrm{I}i} - m_{\mathrm{g}i})\boldsymbol{D}.$$

Let $\boldsymbol{a}_i$ be the position vector of mass i from the centre of suspension of the beam and let $\boldsymbol{s}$ be a unit vector parallel to the suspension fibre. The torque exerted on the suspension is then

$$n^2(m_{\mathrm{I}i} - m_{\mathrm{g}i})(\boldsymbol{D} \wedge \boldsymbol{a}_i) \cdot \boldsymbol{s}.$$

To evaluate $\boldsymbol{D} \wedge \boldsymbol{a}_i$ consider first the components of $\boldsymbol{a}_i$ in a framework in which ξ is drawn horizontally northwards, η horizontally eastwards and ζ vertically parallel to the suspension. $\boldsymbol{a}_i$ is then $(\xi_i, \eta_i, 0)$.

It should be noted here that the gravitational and inertial forces are both in the plane of the ecliptic (at a variable angle), and consequently the torque exerted by the forces about the suspension is given by the components of the forces in the horizontal

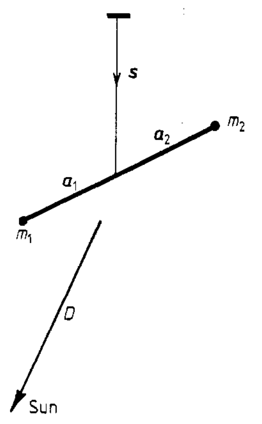

Figure 9. Geometry of experiments on the weak equivalence principle using the attraction of the Sun. s is the unit vector along the direction of the suspension fibre. $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are the position vectors of the masses m_1 and m_2 respectively.

plane. The gravitational and inertial forces are exactly equal only at the centre of mass of the Earth, and so there is a net tidal force upon the masses.

If $\boldsymbol{a}_1$ makes an angle ϕ with true North its components are

$$a_1(\cos\phi, \sin\phi, 0)$$

those of a_2 making an angle $(\phi+\pi)$ with true North are

$$a_2(-\cos\phi, -\sin\phi, 0).$$

The evaluation of the vector product $(\boldsymbol{D}\wedge\boldsymbol{a})$ is most simply done with components parallel to the equatorial plane and in the meridian of the site (x), parallel to the equatorial plane and perpendicular to the meridian (y), and parallel to the polar axis (z). Then if the latitude is λ

$$a_{1x} = -a_1\cos\phi\sin\lambda$$
$$a_{1y} = a_1\sin\phi$$
$$a_{1z} = a_1\cos\phi\cos\lambda$$

while

$$a_{2x} = a_2\cos\phi\sin\lambda$$
$$a_{2y} = -a_2\sin\phi$$
$$a_{2z} = -a_2\cos\phi\cos\lambda.$$

Let the solar declination be δ and the true solar time, counted from noon, be t. Then the components of $\boldsymbol{D}$ in and perpendicular to the meridian of the site are

$$D_x = D\cos\delta\cos t$$
$$D_y = D\cos\delta\sin t$$
$$D_z = D\sin\delta.$$

The value of the triple product is

$$(a_yD_x - a_xD_y)\sin\lambda + (a_zD_y - a_yD_z)\cos\lambda$$

which is

for a_1: $\quad a_1D[\cos\phi\cos\delta\sin t + (\cos\delta\sin\lambda\cos t - \sin\delta\cos\lambda)\sin\phi]$

for a_2: $\quad -a_2D[\cos\phi\cos\delta\sin t + (\cos\delta\sin\lambda\cos t - \sin\delta\cos\lambda)\sin\phi].$

If ϕ is made zero, or nearly so, the net torque becomes

$$n^2aD[(m_{\text{I}1} - m_{\text{g}1}) - (m_{\text{I}2} - m_{\text{g}2})]\cos\phi\cos\delta\sin t$$

where a is the mean of a_1 and a_2, supposed to be nearly equal.

If, similarly, m is the mean of the masses 1 and 2, the net torque is finally

$$\eta(1, 2)n^2maD\cos\phi\cos\delta\sin t$$

a result that is independent of the latitude and which has a period of 24 h.

Eötvös *et al* (1922) used a single torsion balance to look for a difference between magnalium and platinum. A platinum mass was attached to the b end and hourly observations of the orientation of the beam were made for two periods of two weeks, the first with a platinum mass on the a end (pointing North) and the second with a mass of magnalium. The value of η was calculated from the observations around sunrise and sunset, rather than from a Fourier analysis of all the observations.

The result for platinum and magnalium was

$$\eta = 6\times10^{-9}.$$

No uncertainty was assigned.

5.4. *The experiment of Roll et al (1964)*

The principle of this experiment is the same as that of the solar experiment of Eötvös *et al* (§ 5.3), except that instead of a beam with two masses, a triangular framework supported three masses, one of gold and two of aluminium. The report by Roll *et al* is one of the most thorough and deep accounts of any fundamental experiment of great delicacy, not just in gravitation but in other fields as well, and repays detailed study for the analyses and results it presents, as well as for the approach to a fundamental experiment.

As shown in figure 10, a horizontal triangular frame carrying the three masses, one at each corner, was suspended from its centre of mass by a torsion fibre.

With the same notation as in § 5.3, the net torque on the suspension is

$$n^2 \sum_i (m_{1i} - m_{gi})(\boldsymbol{D} \wedge \boldsymbol{a}_i) \cdot \boldsymbol{s}$$

where now the summation extends over the one gold and two aluminium masses.

The components of the position vectors in the horizontal plane are

$$\begin{array}{ll} \boldsymbol{a}_1(\text{gold}) & a_1\{\cos\phi, \sin\phi\} \\ \boldsymbol{a}_2(\text{aluminium}) & a_2\{\cos(\phi+\tfrac{2}{3}\pi), \sin(\phi+\tfrac{2}{3}\pi)\} \\ \boldsymbol{a}_3(\text{aluminium}) & a_3\{\cos(\phi-\tfrac{2}{3}\pi), \sin(\phi-\tfrac{2}{3}\pi)\}. \end{array}$$

The (x, y, z) components referred to the equator and the meridian are

for $\boldsymbol{a}_1$:

$$a_{1x} = -a_1 \cos\phi \sin\lambda$$
$$a_{1y} = a_1 \sin\phi$$
$$a_{1z} = a_1 \cos\phi \cos\lambda$$

for $\boldsymbol{a}_2$:

$$a_{2x} = \tfrac{1}{2}a_2(\cos\phi + \sqrt{3}\sin\phi)\sin\lambda$$
$$a_{2y} = -\tfrac{1}{2}a_2(\sin\phi - \sqrt{3}\cos\phi)$$
$$a_{2z} = -\tfrac{1}{2}a_2(\cos\phi + \sqrt{3}\sin\phi)\cos\lambda$$

for $\boldsymbol{a}_3$:

$$a_{3x} = \tfrac{1}{2}a_3(\cos\phi - \sqrt{3}\sin\phi)\sin\lambda$$
$$a_{3y} = -\tfrac{1}{2}a_3(\sin\phi + \sqrt{3}\cos\phi)$$
$$a_{3z} = -\tfrac{1}{2}a_3(\cos\phi - \sqrt{3}\sin\phi)\cos\lambda.$$

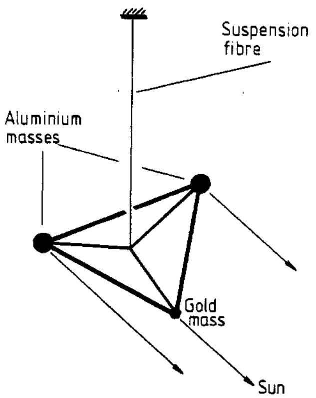

Figure 10. The experiment of Roll *et al.* (1964), showing the position of aluminium and gold masses on the triangular framework.

The triple products

$$(\boldsymbol{D} \wedge \boldsymbol{a}_i) \cdot \boldsymbol{s}$$

are then

for a_1: $$a_1 D[\cos\phi \cos\delta \sin t + (\cos\delta \cos t \sin\lambda - \sin\delta \cos\lambda) \sin\phi]$$

for a_2: $$-\tfrac{1}{2} a_2 D[(\cos\phi + \sqrt{3}\sin\phi) \cos\delta \sin t + (\sin\phi - \sqrt{3}\cos\phi)(\cos\delta \cos t \sin\lambda - \sin\delta \cos\lambda)]$$

for a_3: $$-\tfrac{1}{2} a_3 D[(\cos\phi - \sqrt{3}\sin\phi) \cos\delta \sin t + (\sin\phi + \sqrt{3}\cos\phi)(\cos\delta \cos t \sin\lambda - \sin\delta \cos\lambda)].$$

If a_2 and a_3 are very nearly equal, the sum of the torques exerted on the aluminium masses is proportional to

$$-\tfrac{1}{2}(a_2 + a_3) D[\cos\phi \cos\delta \sin t + (\cos\delta \cos t \sin\lambda - \sin\delta \cos\lambda) \sin\phi]$$

and the net torque exerted on the aluminium and gold masses is equal to

$$n^2 m\eta(\mathrm{Au, Al}) a D[\cos\phi \cos\delta \sin t + (\cos\delta \cos t \sin\lambda - \sin\delta \cos\lambda) \sin\phi]$$

where a is the mean of a_1 and $\tfrac{1}{2}(a_2 + a_3)$ and m is the mean of the masses.

Evidently there are some small corrections for the small differences of a_1, a_2, a_3 from the mean value of a.

If ϕ is adjusted to be zero, the net torque becomes

$$n^2 m\eta(\mathrm{Au, Al}) a D \cos\delta \sin t$$

which is independent of the latitude and has a period of 24 h.

The masses in the experiment of Roll *et al* (1964) were, in the final form, a gold cylinder 32 mm long and 7.8 mm in diameter and two aluminium cylinders 32 mm long and 21 mm in diameter; all weighed 30 g. The triangular frame from which they were suspended was of fused silica and the distance of each mass from the axis was 33 mm. In initial experiments (with masses of lead chloride and copper instead of gold and aluminium) the suspension was a tungsten ribbon with a torsional constant of 7×10^{-9} N m rad^{-1}, but in the final form of the apparatus, a fibre of fused silica was used with the somewhat larger constant of 2.4×10^{-8} N m rad^{-1}. Its breaking strength was 2 N. The balance was enclosed in a stainless-steel vessel with metallic seals which was pumped out and then left sealed. In the preliminary work the pressure was 10^{-4} Pa at the end of the observations, while in the final apparatus it was 10^{-6} Pa at the end. The oscillations of the beam were damped with eddy currents. The upper end of the suspension fibre was fixed to a screw fitment which provided up and down movement and rotation, both controlled by discs of soft iron moved from outside the vacuum enclosure by magnets.

The rotation of the triangular frame was observed with an optical lever. A slit illuminated by a small tungsten filament was reflected in a mirror attached to the frame opposite the gold cylinder and was imaged upon a wire that oscillated from side to side at 3 kHz. The light passing the wire fell upon a photomultiplier. When the image of the slit was symmetrically placed with respect to the wire, there was no photocurrent at the fundamental frequency of 3 kHz and the phase and amplitude of the fundamental gave the displacement of the image of the slit and its direction. That signal controlled a voltage applied to electrodes by which a torque was applied to return the frame to the symmetrical position. Consideration of the radiation pressure and heating by the light source led to the 6 V filament lamp being run at 5 V from a stabilised source; not only were the possible disturbances greatly reduced but also the lamp ran for a very long time. There was a steady drift of the electrostatic torque needed to return the frame to the zero position, and consequently a potentiometer, manually adjusted, was used to compensate for it. The whole apparatus was placed in a well-insulated pit, and thermocouples and thermistors were provided to measure temperatures in the pit generally as well as close to the vacuum vessel.

In the design of the experiment, great attention was paid to a number of possible sources of error. The effects of variable gradients of gravity were reduced by remote operation, as well as by making the moment arms short (33 mm). Because the fundamental period of the experiment is 24 h, the diurnal variation of the Earth's magnetic field would cause a spurious effect unless very great care was taken to eliminate all magnetic substances from the frame and masses.

The initial experiments with the lead chloride and copper masses were done under good conditions and some long observations were made that could be submitted to Fourier analysis, which showed that in addition to the expected components of 24 h period, there were others with periods of 12 and 27.5 h. The latter were not correlated with temperature, but with the rate of change of temperature, and their presence shows how difficult it is in such delicate experiments to remove unwanted disturbances.

The amplitude of the 24 h period gave

$$\eta = (0 \pm 1.6) \times 10^{-10}$$

for lead chloride and copper.

Unfortunately when the final observations came to be made, construction work was under way near the instrument pit, disturbances from which interrupted the runs of observations, with the result that none was long enough for a Fourier analysis to be made. It was, however, possible to make a regression of the observed torque against the rates of change of temperature as measured by two of the thermistors, T_2 and T_5, placed close to the beam. The whole apparatus was rotated by 180° between runs (telescope to the North or to the South of the beam) and the most consistent values for η were obtained when the regression against T_5 was subtracted. The results are tabulated below.

Telescope orientation	No regression allowance (10^{-11})		Regression against T_2 subtracted (10^{-11})		Regression against T_5 subtracted (10^{-11})	
	η	SD	η	SD	η	SD
N	2.22	1.42	1.38	1.27	1.30	1.25
S	−2.53	1.94	−0.25	2.70	1.33	1.73
N+S	0.09	1.20	0.59	1.45	1.32	1.04

From the last column,

$$\eta = (1.3 \pm 1.0) \times 10^{-11}$$

or, at 95% confidence,

$$|\eta| < 3 \times 10^{-11}.$$

5.5. The experiment of Braginski and Panov (1972)

In this experiment also, the differential solar attraction on dissimilar materials was looked for, but it differed in a number of points of design and technique from the two earlier ones. The suspended system was a horizontal star-like assembly of eight masses (figure 11), four of platinum and four of aluminium; the eight masses weighed 3.9 g altogether, and the beam assembly as a whole 4.4 g. The system was suspended by a torsion fibre, and the period of oscillation about this vertical axis was adjusted in accordance with an earlier analysis by Braginski (1968) of the least forced periodic acceleration that could be detected by observations of an oscillating system at a temperature T (see also Braginski and Manukin 1977).

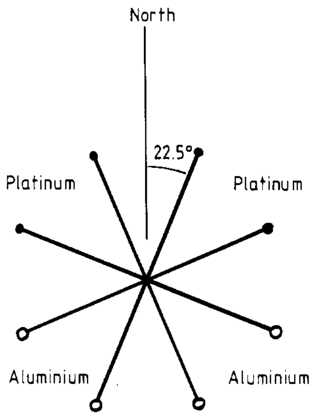

Figure 11. Plan of the arrangement of platinum and aluminium masses on their supports in the experiment of Braginski and Panov (1972).

If $\hat{\tau}$ is the time of measurement, τ^* the relaxation time of the oscillator, m the mass and θ a coefficient representing the confidence in the estimated acceleration, then

$$F/m = \theta(8\pi T/m\hat{\tau}\tau^*).$$

The losses in the suspension wire were estimated to correspond to a relaxation time of 8×10^9 s and those in the residual air around the star assembly to 3×10^9 s; experimentally τ^* was found to exceed 6×10^7 s (2 yr). The period of free oscillations was 1.92×10^4 s (5 h 20 min).

The angles between adjacent rods are $\frac{1}{4}\pi$. If the rod closest to North makes an angle ϕ with the meridian, the vector sums of the ends of the four northerly rods are

$$\{[(2^{1/2}+1)\cos\phi+\sin\phi], [-\cos\phi+(2^{1/2}+1)\sin\phi], 0\}$$

and of the four southerly rods,

$$\{-[(2^{1/2}+1)\cos\phi+\sin\phi], [\cos\phi-(2^{1/2}+1)\sin\phi], 0\}.$$

The net torque on the eight rods is therefore

$$\eta(\mathrm{Pt}, \mathrm{Al})n^2maD\{[(2^{1/2}+1)\cos\phi+\sin\phi]\cos\delta\sin t$$
$$+[\cos\phi-(2^{1/2}+1)\sin\phi](\cos\delta\cos t\sin\lambda-\sin\delta\cos\lambda)\}.$$

The dependences on ϕ and λ are eliminated if $\sin\phi$ is taken to be $(2^{1/2}+1)^{-1}\cos\phi$ ($\phi=\frac{1}{8}\pi$ instead of α) and then the net torque is

$$\eta(\mathrm{Pt}, \mathrm{Al})n^2 maD\alpha \cos\delta \sin t$$

when α is the factor $[(4+2.2^{1/2})/(3+2.2^{1/2})]^{1/2}$, which is about 1.09, and a, the length of an arm of the beam, is 100 mm.

In this experiment, the deflection of the beam was measured, whereas Roll *et al* measured the torque required to keep the beam in a constant position. If ω_0 is the free period of oscillation (nearly five times the 24 h period) and I is the moment of inertia of the beam assembly (equal to 8 ma^2) the amplitude of the 24 h forced oscillation of the beam is

$$\eta man^2 D\alpha \cos\delta / I\omega_0^2$$

or

$$\eta g\alpha \cos\delta / 8a\omega_0^2$$

where g is the value of the *solar* attraction of gravity at the Earth, namely $GM_{\odot}/D^2$ or $n^2 D$.

Braginski and Panov give a factor 3.07 instead of $8/\alpha\cos\delta$. They calculated that if η were 10^{-12}, $\Delta\phi$ would be 1.8×10^{-7} rad.

The rods that supported the masses were of dural, and were suspended by a tungsten fibre 5 μm in diameter and 290 mm long, which was pre-annealed by electrical heating to reduce the drift of the orientation of the beam assembly to 4×10^{-6} rad d^{-1}.

The star and suspension were in a glass vacuum vessel at a pressure of less than 10^{-6} Pa and were thoroughly shielded against magnetic and thermal disturbance.

The recording system was quite different from that of Roll *et al* (1964). Light from a laser was reflected from a mirror at the centre of the star onto a photographic film on a rotating drum. To amplify the excursions of the reflected beam on the surface of the drum, the axis of the drum was set to make an angle of 0.2 rad to the reflected beam. The amplitude of the 24 h component of the rotation of the star was measured directly from the film. The duration of a single record was rather short, the longest being 3 d, so that only two or three 24 h oscillations occurred in any one run. Altogether, seven runs were recorded, as shown in the table below.

Date February 1971	$\Delta\phi$ (10^{-7} rad)
1-4	+2.35
8-10	−0.40
10-12	−1.70
12, 13	+0.76
14, 15	−2.96
15-17	−0.10
19-21	−1.77

If each run is given equal weight, the mean value is -0.54×10^{-7} rad and the corresponding value of η is -0.3×10^{-12} with a standard deviation of 1.0×10^{-12}. (If, however, the values are weighted according to the length of run, the mean value is about -0.35×10^{-7} rad, corresponding to a value of η of about -0.2×10^{-12}, with a standard deviation of 0.25×10^{-12} and a 90% confidence limit of $\pm0.9\times10^{-12}$. Those figures are somewhat different from the weighted estimates just given but neither do they agree with the unweighted estimates; some intermediate scheme of weighting must have been used.)

In discussing their results, Braginski and Panov consider, as among the principal sources of error, the diurnal variations of the magnetic field, radiometric pressure on the elements of the star structure, perturbations of gravity gradients from movements of masses in the vicinity, variations of the pressure of the laser light, and seismic motions. The use of an annealed tungsten fibre with a small drift is an improvement upon the technique of Roll *et al*, whereas the use of a laser source for the optical lever is more questionable. Roll *et al* ran their lamp at a low voltage so that it should last for a long time and not have to be replaced; Braginski and Panov restricted their observations to three weeks, and the photographic record is almost independent of the intensity of the laser source. The laser does however exert a radiation pressure and generates heat within the vacuum enclosure, and it is more difficult to make a thorough analysis of the results from the chart recording.

5.6. Summary

The experiments on weak equivalence are by far the most delicate mechanical measurements ever made. They owe their success to careful design, sensitive apparatus and thorough tests and observation. Some of the experimental developments have proved valuable in other work—in particular the treatment of tungsten fibres to obtain very low drift rates. At the same time, it seems clear that yet more delicate observations could be made. The apparatus of Roll *et al* (1964) in its final form was not fully exploited because of disturbances from building works, and the observations of Braginski and Panov (1972) lasted for periods of only a few days. Again, the work of Eötvös and his successors, advanced and careful as it was in its day, was limited by the methods of observation that were available, and could no doubt be greatly improved today, something of considerable interest in consequence of the study of Fishbach *et al* (1986).

When a gravitational field has its source in a very distance body (the Sun) the principle of weak equivalence holds to within 1 part in 10^{12}, but if the field has some of its sources within a few hundred metres (the Earth) the principle is established to within a few parts in 10^9.

6. The constant of gravitation

6.1. Introduction

The constant of gravitation occupies a somewhat anomalous position in experimental physics. It seems at first sight to have as important a place in the structure of physics as other constants such as the velocity of light, or Planck's constant or the electronic charge, but it is in fact rather isolated from much of physics. First, it cannot at present be measured to comparable precision; all other constants of physics are known to better than 1 part in 10^6, while the precision of the best value of G is about 100 times poorer. Secondly, knowledge of the value of G is almost irrelevant to any other field of physical knowledge. It is not the basis of any system of measurement, as the velocity of light is the basis of the measurement of length in terms of the transit times of light signals, or as the ratio h/e is the basis of electrical standards related to frequency through the Josephson effect, nor does ignorance of it affect celestial mechanics or planetary physics, for in the absence of independent measurements of masses of celestial bodies, the only significant quantities are the products GM. In terrestrial laboratories masses are defined by their relation to the arbitrary standard, the kilogramme, while in celestial mechanics, the product GM is determined from the accelerations of bodies, and apart from the measured value of G there is no connection between those two ways of establishing mass. The argument is not affected by the possibility that mass might be redefined in terms of electrical quantities—when current can be related to frequency as conveniently as voltage, then energy or force can also be related to frequency through constants such as $\hbar$, e, c, which would afford the possibility of relating mass also to frequency through atomic constants. But so far as concerns connections with celestial mechanics, astrophysics or planetary physics, such a scheme would not affect them at all. A value is needed for the study of the internal physics of the Earth in order to compare the equations of state of terrestrial material, as found from seismology and the gravity field, with those determined experimentally in the laboratory. For that, an uncertainty of about 1 part in 1000 is at present adequate. In solar physics, also, a value of G is required to relate the density of the Sun derived from celestial mechanics to the masses of the nuclides (the proton in particular) that are its constituents.

The situation would no doubt be different if there were some theoretical prediction of a relation of G to the other principal constants of physics, but so far there is none, and if indeed it is true that gravitation is a purely geometrical phenomenon, then it may be doubted if there would be any meaning in such a possible relation. At present, the constant of gravitation seems to stand apart from the rest of physics.

In parallel with the independence from other physics, the constant of gravitation is difficult to measure and poorly known. For this there are two reasons. One is that the gravitational force is weak compared with disturbing forces, fundamental noise and the sensitivity of most detectors. The other, to some extent dependent on the first, is that because the masses needed to produce measurable effects are rather large relative to the distance between them, the determination of the centres of mass and the distance between them is not straightforward. Evidently, if the sensitivity of measurements of acceleration were to be improved, the attracting masses could be placed further apart and the relative definition of the centres of attraction would be better.

Newton himself speculated upon ways of measuring G. He estimated the horizontal attraction of a mountain upon a plumb bob and also the acceleration of one mass towards another in the laboratory, and hence the time it would take for two masses to come together from some distance. Many observations along the lines of the first suggestion and of the variation of gravity in a mine were made in the 18th and 19th centuries, being seen also as a way of estimating the mean density of the Earth given the acceleration due to gravity at the surface (see, for example, Airy 1856, Bouguer 1754, James 1857, Maskelyne 1775, Whewell and Airy 1827), and interest in that type of study has been revived by Stacey and his colleagues because of the possibility of detecting a deviation from the inverse square law (Stacey 1978, Stacey and Tuck 1981, Stacey *et al* 1981, Holding and Tuck 1984, Holding *et al* 1986).

Newton's other suggestion, the direct measurement of the free motion of two bodies towards each other, while hardly practical in an ordinary laboratory (although Luther *et al* (1976) attempted a determination based on that idea), may become so in a spacecraft, and some studies of such a scheme have been undertaken (Berman and Forward 1968, Vinti 1972). So far, however, no experiments have been performed.

Laboratory experiments depend on a sensitive way of detecting the forces between two masses. Cavendish (1798) first used the torsion balance to measure a force by the steady deflection of the arm. It was subsequently realised (Braun 1897) that the change of the period of free oscillation of a torsion balance was a more sensitive measure of the gravitational torque exerted upon it, and in a further development, the gravitational torque was varied periodically in resonance with the oscillation of the balance (Kunz 1930, Zahradnicek 1933). The chemical balance has been used as an alternative to the torsion balance to measure constant forces (von Jolly 1878, 1881, Poynting 1892).

The results of some laboratory determinations, from that of Cavendish onwards, are collected in table 4. There is little point in discussing them all in detail, but some comments on a number of salient points follow in the next section, while six determinations (Heyl 1930, Heyl and Chrzanowski 1942, Facy and Pontikis 1970, 1971, Pontikis 1972, Sagitov *et al* 1977, Luther and Towler 1982) are considered subsequently in somewhat more detail.

Table 4. Some historical determinations of the constant of gravitation.

Author	Method	G (10^{-11} N m^2 kg^{-2})
Cavendish (1798)	Torsion balance deflection	6.754
Poynting (1891)	Chemical balance	6.698
Boys (1895)	Torsion balance deflection	6.658
Braun (1897)	Torsion balance deflection	6.658
	Torsion balance period	6.658
Heyl (1930)	Torsion balance period	6.669
Zahradnicek (1933)	Torsion balance in resonance	6.659

6.2. Principles

Following the work of Braun (1897) who determined the constant of gravitation both from the static deflection of a torsion balance and from a change in its period of free oscillation, most experiments have used the latter method. Let the moment of inertia of the beam of a torsion balance be I and let the torsional constant of the fibre be κ.

If the deflection is θ, the Lagrangian is

$$\tfrac{1}{2}I\dot{\theta}^2-\tfrac{1}{2}\kappa\theta^2.$$

If the additional potential energy of the beam in the field of attracting masses is V, the Lagrangian becomes

$$\tfrac{1}{2}I\dot{\theta}^2-\tfrac{1}{2}\kappa\theta^2-V$$

and the equation of motion is

$$I\ddot{\theta}+\kappa\theta+\partial V/\partial\theta=0.$$

Suppose the beam consists of two masses m separated by $2l$ and suppose the attracting masses to be each M and placed in line with the beam and at a distance D from the centre of the beam (figure 12(a)). Then the potential V, when the deflection is θ, is

$$-(2GMm/D)[1+(l/D)\cos\theta+\ldots]$$

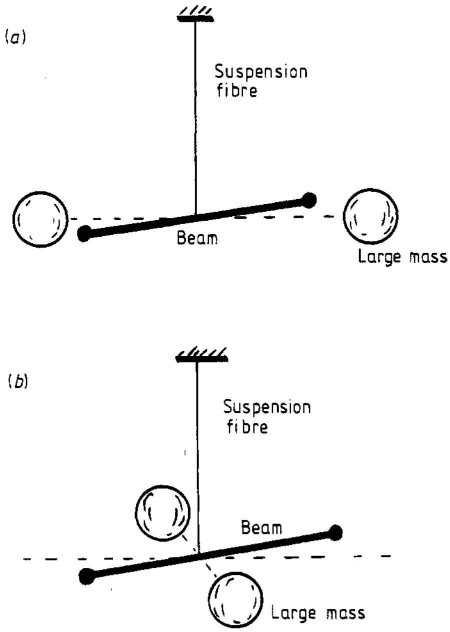

Figure 12. Scheme of determinations of the constant of gravitation with a torsion balance attracted by large masses in near (a) positions and far (b) positions.

and

$$\partial V/\partial\theta = [(2GMml/D^2)\sin\theta + \ldots].$$

The equation of motion is then

$$I\ddot{\theta} + \kappa\theta + (2GMml/D^2)\sin\theta + \ldots = 0.$$

If θ is small so that $\sin\theta \simeq \theta$, the motion is harmonic with a frequency

$$\omega = (1/I^{1/2})(\kappa + 2GMml/D^2)^{1/2}.$$

One way of making a determination of G is to observe the motion with the masses M present and absent (Luther and Towler 1982). The change in frequency is then given by

$$\Delta(\omega^2) = 2GMml/ID^2.$$

Alternatively, the frequency may be observed when the large masses are at different distances D (Sagitov *et al* 1977).

The large masses may also be placed in positions to the side of the beam (figure 12(b)). If, again, the distance of each from the centre of the beam is D, the potential is

$$-(2GMm/D)[1 + 3(l^2/D^2)\sin^2\theta + \ldots]$$

and

$$\partial V/\partial\theta = -(12GM/D)(l^2/D^2)\sin\theta\cos\theta$$

so that the equation of motion for small oscillations is

$$I\ddot{\theta} + \kappa\theta - (12GMml^2/D^3)\theta = 0$$

and the frequency is reduced to

$$(1/I^{1/2})(\kappa - 12GMml^2/D^3)^{1/2}.$$

On the face of it, the motion appears no more non-linear than that of a simple pendulum, but if the large masses are not spheres a further non-linearity is introduced because the potential of the large mass at the distance r of the small one is not just $-GM/r$ but a more complex expression. In general, therefore, the motion of the pendulum requires careful analysis, the more so if there is appreciable damping.

Heyl (1930) and Heyl and Chrzanowski (1942) placed the large masses in both the 'near' and 'far' positions and observed the change of period, given by

$$\Delta(\omega^2) = (2GMm/I)(l/D_n^2 + 6l^2/D_f^3)$$

where D_n is the distance of the large masses from the centre of the beam in the 'near' position, and D_f that in the 'far' position.

In practice, naturally, the two arms of the beam are not exactly equal, the masses m are not equal nor are the two large masses M, and the distances of the latter from the torsion fibre are not the same. More detailed analysis shows, however, that if the respective mean values are inserted in the foregoing expressions, the errors are of second order and generally negligible. In practice, a more serious doubt is how constant the torsion constant is—how well is it eliminated from the expression for $\Delta(\omega^2)$ since it usually varies considerably with temperature? A further troublesome matter is that the attraction of the beam of the balance has to be estimated, as does the moment of inertia. Those could both be eliminated if the masses m were placed at different separations l, but no successful experiment along those lines has been done; a disadvantage is that the accuracy will be determined by the lesser separation of the masses, not the greater.

The other method that has been used recently is the resonance scheme introduced by Kunz (1930) and Zahradincek (1933). Large masses placed close to a torsion balance are caused to oscillate about the axis of the balance at a period close to that of the free period of the balance. Let θ be the angular displacement of the arm of the torsion balance from its equilibrium position. Then the equation of motion is

$$I\ddot{\theta} + \lambda\dot{\theta} + \kappa\theta = F$$

where I is the moment of inertia of the beam, λ the damping coefficient and κ the torsional constant. F is the forcing function, which may in a general way, be written as a trigonometrical function of the angle between the line joining the large masses and the balance arm, namely $(\alpha \sin \omega t - \theta)$, where ω is the frequency of oscillation of the large masses and α the amplitude of their swing. Thus F may be written as a Fourier series:

$$F = \sum [A_n \cos n(\omega t - \theta) + B_n \sin n(\omega t - \theta)].$$

If $2r$ is the length of the balance arm, $2R$ the separation of the large masses, and M and m are respectively the values of the large masses and the masses on the balance arm, then the couple exerted on the beam is

$$2GMr \sin \psi / d^2$$

where d is the separation of the large and small masses and ψ is the angle between the beam and the join of the large and small masses (see figure 13).

But

$$(1/R) \sin \psi = (1/d) \sin(\alpha \sin \omega t - \theta)$$

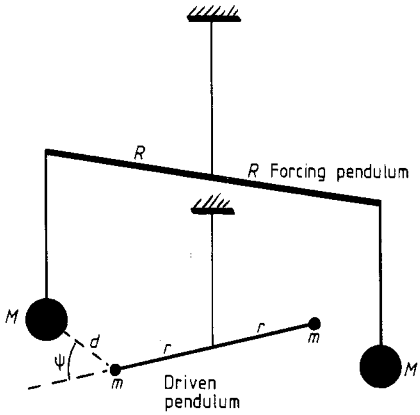

Figure 13. Determination of the constant of gravitation by means of a torsion pendulum driven by the attraction of another, more massive pendulum.

whence

$$\sin \psi / d^2 = (R/d^3) \sin(\alpha \sin \omega t - \theta).$$

But

$$d^2 = R^2 + r^2 - 2Rr \cos(\alpha \sin \omega t - \theta)$$

giving

$$\sin \psi / d^2 = (1/R^2)[1 + (3r/R) \cos(\alpha \sin \omega t - \theta) + \ldots]$$

from which the Fourier coefficients may be calculated.

The equation of motion is damped and non-linear so that the solution is not purely harmonic. Facy and Pontikis (1970) set out four ways of solving the equation, two by approximation, taking θ to be small, and two by numerical methods, using values of θ recorded frequently throughout the cycle.

6.3. Recent experiments

The results of the determinations of Heyl (1930) and later workers are collected in table 5. The scatter is larger than would be expected from the errors assigned in the respective publications, but there are also features which emphasise the difficulties of experiments on gravitation. Heyl found clear differences between his results according to the materials he used for his small masses, yet it is clear from the experiments of Eötvös *et al* (1922) as well as from the experiments done at the National Bureau of Standards by Cook and reported by Heyl (see § 5.2) that any such differences must arise from some systematic error in Heyl's work, and one possibility must be the effect of temperature on the torsional constant of the suspension. When the determination was repeated in a somewhat improved form by Heyl and Chrzanowski (1942), two suspension fibres were used, one a standard hard-drawn tungsten wire which was originally coiled up and took some months to straighten under load, and the other an annealed wire received uncoiled. Both showed appreciable drift of zero. The annealed wire had the greater diameter (35 μm instead of 30 μm), hence a greater torsional constant and a smaller difference between the periods of oscillation in the two positions, 240 s as against 720 s. The the annealed fibre gave a less sensitive system, as indeed the scatter of the results seems to indicate. The mean results were, for the hard-drawn wire $(6.6755 \pm 0.0005) \times 10^{-11}$ N m^2 kg^{-2} and for the annealed wire $(6.6685 \pm 0.0008) \times 10^{-11}$ N m^2 kg^{-2}. The difference, 0.0070×10^{-11} N m^2 kg^{-2}, is well outside the standard deviation of the difference, about 0.0010×10^{-11} N m^2 kg^{-2}; again thermal effects might be a reason.

Luther and Towler (1982) used large masses that were spheres of sintered tungsten, and the suspended system was a tungsten rod to the ends of which were fixed discs of tungsten. The torsion fibre was of fused silica. The free period of the torsion balance was 6 min and it changed by a few per cent when the large masses were brought up to the beam. Again, there was an apparently systematic effect. The alignment of the torsion balance was changed by 90° to eliminate any effect of the Earth's magnetic field, although tests showed that any such effect should be very small. The mean results were, for the differences of the squares of the frequencies:

orientation NE–SW	$1.244\,54 \times 10^{-5}$ rad^2 s^{-2}
NW–SE	$1.244\,43 \times 10^{-5}$ rad^2 s^{-2}.

The mean corresponds to a value of G equal to $(6.6726 \pm 0.0005) \times 10^{-11}$ N m^2 kg^{-2}, while the difference of 11×10^{-5} rad^2 s^{-2} corresponds to 0.0006×10^{-11} N m^2 kg^{-2}. The observations at the NE–SW orientation show a drift from 1.244 63 to $1.244\,45 \times 10^{-5}$ rad^2 s^{-2} over the duration of the experiment, corresponding to a drift of 0.001×10^{-11} N m^2 kg^{-2}.

In the determination of Sagitov *et al* (1979) the large attracting masses were placed at four distances from the torsion arm. As has already been explained, Milyukov (1985) has used the data in an examination of the inverse square law, while the mean value of the constant of gravitation was $(6.6745 \pm 0.0008) \times 10^{-11}$ N m^2 kg^{-2}.

Facy and Pontikis (1970, 1971) reported preliminary results of the resonance determination, and a more definitive result was given by Pontikis (1972). The large driving masses in the latter work were of silver, copper, bronze or lead and gave somewhat different results (in 10^{-11} N m^2 kg^{-2}):

silver	$6.671\,62 \pm 0.000\,51$
copper	$6.671\,54 \pm 0.000\,51$
bronze	$6.671\,22 \pm 0.000\,61$
lead	$6.671\,26 \pm 0.000\,66$.

Each was the mean of ten values, and the mean of the four is

$$(6.671\,42 \pm 0.000\,10) \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}.$$

The difference between materials is not significant but Pontikis states that there was some change of value with time, as has also been found in the experiments of Heyl (1930) and of Luther and Towler (1982).

The four latest results (table 5) seem more reliable than those of Heyl, but the discrepancies between them are considerable, as a calculation of χ^2 shows.

The standard deviation assigned to the result of Pontikis, 3×10^{-15} N m^2 kg^{-2}, is that entailed by the standard deviations assigned to values for different materials which, however, lead to a value of χ^2 of 0.4 on three degrees of freedom for Pontikis's data. If the overall standard deviation is calculated from the differences between materials

Table 5. Recent determinations of the constant of gravitation.

Authors	G (10^{-11} N m^2 kg^{-2})	SD (10^{-15} N m^2 kg^{-2})	δG (10^{-15} N m^2 kg^{-2})	χ^2
Heyl and Chrzanowski (1942)				
Hard-drawn wire	6.6755	5	+30	36
Annealed wire	6.6685	8	−40	25
Pontikis (1972)	6.6714	3	−11	14
Sagitov *et al* (1979)	6.6745	8	+20	6
Luther and Towler (1982)	6.6726	5	+1	—
Mean =	6.6725			Sum = 81

it is found to be 1.2×10^{-15} N m^2 kg^{-2}. If that value were used instead of 3×10^{-15} N m^2 kg^{-2} in calculating χ^2 for the different experiments, χ^2 would be nearly 200 instead of 80. Evidently there are considerable inconsistencies between the results, and some of the possible reasons have been indicated in the comments on the separate experiments. The best value of G at present would appear to be

$$6.6725 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$$

with a standard deviation, calculated from the differences of the separate experiments, of 12×10^{-15} N m^2 kg^{-2}.

If instead of separating the results of Heyl and Chrzanowski (1942) with the different wires, a mean value of 6.6720×10^{-11} N m^2 kg^{-2} were used, χ^2 would be reduced to 20, still too large on three degrees of freedom, and the overall mean would be 6.6726×10^{-11} N m^2 kg^{-2} with a standard deviation of 7×10^{-15} N m^2 kg^{-2}.

The mean value Heyl (1930) obtained from the glass and platinum balls was 6.669×10^{-11} N m^2 kg^{-2}.

If changes in G in time are to be sought experimentally, G must be compared with some physical quantities that do not change with time; by definition, those are now standards of time, the value of the speed of light and the standard of mass. The only way of detecting a change of G with time experimentally is to compare G with those standards at different epochs, that is, to make absolute determinations of G separated by suitable intervals of time. From what has just been said about the precision of measurements of G, it may be seen that the least that could be detected would be a change of 1 part in 10^6 per year if observations were repeated after ten years, and that is four orders of magnitude greater than the usual predictions of 10^{-10} per year.

7. Conclusion

The main points of this review can be summarised quite briefly. First, experiments on gravitation are difficult and almost all reviewed here seem to have unexplained systematic errors. Even so, the validity of the weak principle of equivalence has been confirmed to as high precision as almost any matter in physics, the limit of about 1 in 10^{12} being exceeded only by that with which the standards of frequency are established. All other experimental results (as distinct from those of celestial mechanics) have far greater uncertainties, of the order of 1 part in 10^5 or worse, and there seems no hope of investigating experimentally a possible change of G with time.

So far as experiment goes, no deviation from the predictions of general relativity has been established. In particular, weak equivalence is very firm, and the inverse square law, which is a consequence of the Schwarzschild metric for not too massive bodies, is well supported over long distances and short distances and, even though open to doubt at intermediate distances, no definite deviation has been established. The main conclusion is that, in line with all other empirical evidence (Will 1981), general relativity is the description of spacetime in the large and gravitation is an aspect of its geometry.

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