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Experiment and theory in anelasticity

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Abstract. We briefly review the results of experimental and theoretical studies of anelasticity at low frequencies and we discuss the essential features of models that explain consistently the key aspects of this behaviour. In our original paper on this topic we reported that damping in a compound pendulum at frequencies in the range 10^{-3}–10^{-2} Hz was consistent with an imaginary component of Young's modulus for the Cu–Be suspension that was independent of frequency. Damping with this characteristic frequency dependence can also be described in terms of viscous damping whose magnitude varies as the inverse of frequency. With the hindsight provided by our further work we now realize that stick–slip processes can give rise to such losses, not only in the suspensions but also in the structures and clamping mechanism of the suspensions of long-period pendulums.

Keywords: anelasticity, dislocation damping, $1/f$ noise, weak-force measurement

1. Introduction

A key characteristic of all determinations of Newton's constant of gravitation G is that they involve slow movements of suspended masses. In torsion-balance experiments, for example, the cadence of the experiment is set by the response time or period of oscillation, which is usually many minutes. In other experiments the time taken to displace large source masses in order to modulate the gravitational signal may limit the data-collection rate to a similar time scale. Experimentalists in this field are therefore almost unique in having to deal with classical systems and their associated sources of noise at extremely low frequencies. In this paper we discuss advances in our understanding of some aspects of these problems. In particular we take the opportunity to review, in the light of our further work and the work of others, the paper of Quinn *et al* [1] on the study of damping in a compound pendulum suspended from Cu–2% Be flexure strips. We shall henceforth refer to this paper as QSB.

2. The Maxwell model

Mechanical measurements of weak forces invariably involve the use of a suspension that acts as a spring in some shape or form. Springs made of real materials fail to obey perfectly Hooke's law in that, after deflection, they do not return to their unstressed state immediately and, when they are stressed, they do not reach their final state of strain immediately. Such materials are said to exhibit anelasticity. Maxwell proposed the model shown in figure 1 to describe this behaviour. The ideal spring is shunted by a weaker spring in series with a dash-pot. The stiffness of the combination clearly depends on the speed at which stress is applied. A slowly applied stress

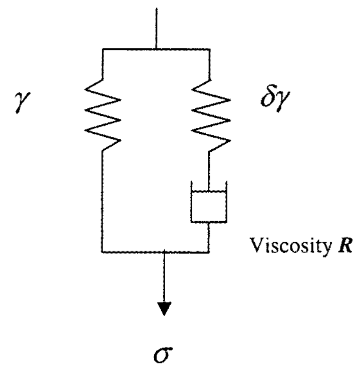

Figure 1. The Maxwell model of an anelastic solid.

will meet the elastic resistance of the main spring element (labelled γ in figure 1) and the viscous damping of the dash-pot. However, when a stress is rapidly applied, the dash-pot will lock and the stiffness of the combination will be given as the sum of the two spring constants $(\gamma + \delta\gamma)$. The response to an oscillatory stress can be written as (see [1], for example)

$$\gamma(\omega) = \frac{\sigma(\omega)}{e(\omega)} = \gamma + \delta\gamma\left(\frac{\omega^2\tau^2}{1+\omega^2\tau^2} + \mathrm{i}\frac{\omega\tau}{1+\omega^2\tau^2}\right) \quad (2.1)$$

where $\sigma(\omega)$ is the stress, $e(\omega)$ is the strain and $\tau = R/\delta\gamma$, where R is the viscosity of the dash-pot. This simple Maxwell's model reproduces the behaviour of the bulk elastic modulus of anelastic materials with a single relaxation process and has been used extensively to understand energy losses, primarily in crystalline materials [2]. It is of interest to note that the simple model predicts a phase lag between the strain and the applied stress and that this has two consequences: firstly, the material exhibits damping and,

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in the limit of $\omega\tau \ll 1$, this is proportional to ω and is therefore viscous. Secondly, the finite response time leads to an 'anelastic after-effect'. To demonstrate the anelastic after-effect, imagine applying a step change in strain to the spring and plotting the stress necessary to do this. Whilst the strain is rapidly increasing the stress can be calculated using the appropriate value for the modulus which, as discussed above, is given as $\gamma + \delta\gamma$. Once the strain stops increasing, the dash-pot can relax. The stress also relaxes to a value given by the product of the final strain and the relaxed modulus, γ. Equation (2.1) can be used to show that the stress relaxes exponentially with time constant τ.

3. Early work at the BIPM

The simple picture described above did not describe observations of our flexure-strip balance that were made by us during the early 1980s [3]. We therefore investigated in detail the anelastic behaviour of Cu–Be flexure suspensions on a specially constructed compound pendulum. This work was eventually published in 1992, when we reported that, over frequencies in the range 10^{-3}–10^{-2} Hz, the logarithmic decrement (Q^{-1}) varied as the inverse of the square of the frequency. We interpreted this as evidence that, in Cu–Be over this frequency range, the imaginary component of Young's modulus was independent of frequency, contrary to that which was predicted by the Maxwell model. Furthermore, our measurements of an anelastic after-effect were better described as a superposition of exponential functions, a more complex model than that of Maxwell. We strove to make sense of these two observations. We noted that, if we added, in parallel with those shown in figure 1, an infinite number of spring and dash-pot systems with a range of time constants and a suitable weighting factor, we could explain both the damping and after-effect results. The physical interpretation of the weighting factor, $g(\tau)$, is the number of relaxation processes with time constant τ lying within the range τ and $\tau + \mathrm{d}\tau$. The modulus can be calculated as

$$\gamma(\omega) = \gamma + \frac{\delta\gamma}{\ln(\tau_\infty/\tau_0)} \int_{\tau_0}^{\tau_\infty} \left(\frac{\omega^2\tau^2}{1+\omega^2\tau^2} + \mathrm{i}\frac{\omega\tau}{1+\omega^2\tau^2} \right) \frac{\mathrm{d}\tau}{\tau}. \tag{3.1a}$$

The logarithm of the ratio of the time constants is included to normalize the integral of the density function over the range of time constants. The result is

$$\gamma(\omega) = \gamma + \frac{\delta\gamma}{\ln(\tau_\infty/\tau_0)} \left[\frac{1}{2} \ln\left(\frac{1+\omega^2\tau_\infty^2}{1+\omega^2\tau_0^2} \right) + \mathrm{i}[\tan^{-1}(\omega\tau_\infty) - \tan^{-1}(\omega\tau_0)] \right] \tag{3.1b}$$

In the range $\tau_\infty^{-1} \ll \omega \ll \tau_0^{-1}$ we find that the imaginary component of the modulus is independent of frequency, as required by our observations. We define the modulus defect as $\Delta\gamma = \pi\delta\gamma/[2\ln(\tau_\infty/\tau_0)]$ and note that, if $\delta\gamma \ll \gamma$ and $\tau_\infty^{-1} \ll \omega \ll \tau_0^{-1}$, the observed loss tangent (the ratio of the imaginary and real components of the modulus) is simply $\tan\phi = \delta\gamma/\gamma$. In QSB we evaluated only the imaginary component of the frequency-dependent modulus. The importance of the real component has recently been emphasized by Kuroda [4] in connection with determinations of G. The anelastic component of stress as a function of time, t, following the application of a unit step change in strain, is described in this model as

$$\sigma(t) = \frac{\delta\gamma}{\ln(\tau_\infty/\tau_0)} \int_{\tau_0}^{\tau_\infty} \frac{\mathrm{e}^{-t/\tau}}{\tau}\,\mathrm{d}\tau = \frac{2\delta\gamma}{\pi}[E_1(t/\tau_\infty) - E_1(t/\tau_0)] \tag{3.2a}$$

where

$$E_1(x) = \int_x^\infty \frac{\mathrm{e}^{-u}}{u}\,\mathrm{d}u \tag{3.2b}$$

is an exponential integral function [5]. The integral in equation (3.2a) can now be determined numerically using commonly available software. From measurements both of the decay of free oscillations and of the anelastic after-effect, we deduced that the loss tangent for Cu–Be was $(1.6\pm0.2)\times10^{-3}$. The after-effect measurements suggested that τ_∞ lay in the range 4000–6000 s and that τ_0 was less than 30 s and probably around 10 s.

We presented a model of dislocation motion that could reproduce the observed behaviour of Cu–Be and found the following relationship

$$\frac{\delta\gamma}{\gamma} = \frac{b}{l}\frac{\sigma/\hat{\sigma}}{1-\sigma/\hat{\sigma}} \tag{3.3}$$

where b is Burgers vector for the dislocation of length l, σ is the stress and $\hat{\sigma}$ is the critical stress for the array of pinning sites. This relationship was developed from the basic idea that the pinning sites for dislocations (the precipitated platelets of Be) are destroyed by diffusion of point defects along the dislocation. Thus the lengths of the dislocations grow as they are pinned by a succession of obstacles further apart. The strain stored in the dislocations increases as their lengths increase, leading to a decrease in the modulus with time. The model was able to describe semi-quantitatively the observations, provided that the time taken for diffusion was proportional to some power of the length of the dislocation. In this picture the range of relaxation times is set by the range of possible lengths of the dislocation chains. Equation (3.3) predicts that the magnitude of the modulus defect should vary with load and that the dissipation should be weakly dependent on amplitude if the stresses are small in comparison with the yield stress. A further important conclusion was that the variation of logarithmic decrement with frequency would lead to $1/f$ noise and this was first pointed out in another paper [6].

4. Early BIPM work in the context of hindsight

At the time of the publication of the QSB paper there had been, to the best of our knowledge, no other measurements of anelasticity in a similar frequency regime, indeed even now there are very few. Saulson [7–9] independently realized the importance of the nature of internal losses in materials and was studying the thermal noise problem in the context of pendulum suspensions for gravitational wave detectors. (Further measurements of damping in simple-pendulum suspensions, at frequencies of about 1 Hz and above, that are aimed at development of gravitational wave detectors are reported in [23–25]). Saulson pointed out that Kimball and Lovell [10] had reported measurements of loss tangents in many materials and found them to be independent of frequency. These authors measured the transverse deflection of the end of a rod when it was rotated about a horizontal axis whilst supporting a radial load. Measurements in the range 2–200 Hz were made for 18 different materials and all exhibited a loss tangent that was independent of frequency. The loss tangent of phosphor-bronze, for example, was reported to be 1.17×10^{-4}.

We later also discovered that the superposition of relaxation processes with a density function varying as $1/\tau$ had already been proposed by du Pre [11] and van der Ziel [12] in connection with the problem of 1/f noise in various contexts associated with electrical phenomena. Both these authors pointed out the physical significance of this particular density function. The relaxation time of the microscopic elements of the distribution is usually limited by motion over an energy barrier. The relaxation time at temperature, T, for an energy barrier with activation energy, Q, is given by the Arrhenius relation

$$\tau^{-1} = \nu_0\,\mathrm{e}^{-Q/(kT)} \tag{4.1}$$

where ν_0 is the attempt frequency, which in the case of dislocations, for example, is of the order of magnitude of the Debye frequency (10^{13} Hz). Both du Pre and van der Ziel assumed that the relaxation elements should have activation energies uniformly distributed over a range of values. If the activation energies lie between Q_1 and Q_2, for example, the number of relaxation elements having activation energies lying between Q and $Q + \mathrm{d}Q$ is then assumed to be

$$g(Q)\,\mathrm{d}Q = \frac{\mathrm{d}Q}{Q_2 - Q_1}. \tag{4.2a}$$

We can use equation (4.1) to relate this number density to the number density as a function of the relaxation time. We find

$$g(\tau)\,\mathrm{d}\tau = g(Q)\,\mathrm{d}Q \tag{4.2b}$$

with

$$\mathrm{d}\tau = \frac{\tau}{kT}\,\mathrm{d}Q.$$

Therefore

$$g(\tau) = \frac{kT}{Q_2 - Q_1}\frac{1}{\tau}. \tag{4.2c}$$

Thus we find a general and natural explanation for the density-of-states function that they invoked to explain flicker noise in radio tubes and semiconductors and also after-effect phenomena in dielectrics.

It is interesting that, in the theory developed in QSB, the activation energy is considered to be a constant and the variation in time constants arises from the range in diffusion times across the distribution of lengths of dislocation. Both these models result in a density of states varying as $1/\tau$.

Nowick and Berry [2, p 92] describe methods whereby the anelastic behaviour of crystalline materials can be modelled using a superposition of relaxation processes. They do, in fact, describe a box distribution that is exactly equivalent to the one we employed and find an expression for the real and imaginary components of the frequency-dependent modulus which is identical to equation (3.1*b*).

With hindsight, the required distribution was already in the literature, although we are not aware of any other authors employing it to explain the behaviour of polycrystalline materials at low frequencies. It is interesting to note that the relationship $g(\tau) \approx 1/\tau$ is ubiquitous. It appears in many contexts in which the macroscopic response of a system of microscopic elements is described, in the linear regime, by a susceptibility. In fact, it is now known that elastic, magnetic [12] and dielectric materials exhibit $1/f$ noise. The frequency dependence of these generalized susceptibilities is governed by the Kronig–Kramers relations. An obvious result of this analysis is that the dissipative component of the susceptibility should have zero magnitude at zero frequency. It came as a surprise to us that we could not treat 10^{-3} Hz as zero frequency! This was a key reason for us delaying the publication of the 1985 work until 1992. A possible resolution of the apparent conflict between experimental observations and the Kronig–Kramers relations lies in the amplitude dependence of the damping or nonlinearity proposed in the models of QSB and Cagnoli *et al* [15]†. This nonlinearity would also, presumably, produce an amplitude dependence in the real component of the effective elastic modulus in equation (3.1*b*). It should be pointed out, however, that Gonzalez and Saulson [14] have directly measured the torque noise present in a torsion balance that was suspended from a nylon fibre. They reported that the spectrum of thermal noise torques varied as $1/f$ in accordance with the fluctuation-dissipation theorem and their measurements of damping in the fibre.

Cagnoli *et al* extended the model of dislocation motion that was presented in QSB to one which included self-organizing criticality. These authors invoke a stick–slip mechanism to explain the residual losses at extremely low frequency. They also argue that such a model, which has the same quantitative results as those in QSB, differs from the well-known model of Granato and Lucke [16]. In the latter model there is a critical stress that separates dynamic and hysteretic loss mechanisms. The dynamic losses are frequency dependent but hysteretic losses occur only above a critical stress at which the dislocation lengths become larger than the so-called network length. The hysteretic losses are frequency independent. There are, therefore, important differences between the theories of Cagnoli *et al* and QSB and the theory of Granato and Lucke. Crucially, the theory of Granato and Lucke predicts zero dislocation damping at low frequency and at low stresses, which appears to be contradicted by the experimental results of QSB and more recent results (see below). According to QSB and Cagnoli *et al*, dislocation damping is hysteretic even at low stresses and remains finite at low frequency.

We have pursued the measurement of anelasticity in Cu–Be in an attempt to corroborate our earlier results [17, 18]. All our measurements to date have reproduced the variation in damping with frequency that is characteristic of anelasticity due to a frequency-independent modulus defect. Significantly, however, we have become aware that losses associated with clamping the flexure and losses within

† This was pointed out by R Newman during the conference.

the structure of the suspended pendulum can also exhibit a similar damping characteristic and that such losses are generally amplitude dependent [17]. In contrast, intrinsic damping within the material seems to be, to within a good approximation, amplitude independent. It seemed reasonable to assume that the losses at the clamp of a flexure or suspension would be due to stick–slip motion at the interface between the clamp and the flexure. According to Cagnoli *et al*, such losses would have the same frequency dependence as losses due to dislocation motion, exactly as we had observed. We had, however, to explain the large amplitude dependency of the stick–slip motion at the clamp which is absent from the dislocation loss. Our hypothesis for the amplitude dependency is that, at clamped surfaces, there is always a clamping pressure gradient leading away from the screw or clamp. As the amplitude of the flexure increases, not only does the shear stress at any point increase, leading to stick–slip motion at that point, but also the total area over which the stick–slip motion will take place will increase as the shear stress 'climbs up' the clamping pressure gradient. With constant pressure over the clamping surfaces the stick–slip motion would not be significantly amplitude dependent and this is the analogous situation to dislocation damping, in which the dislocations are uniformly distributed throughout the material. It is the presence of the clamping pressure gradient resulting in an angle-dependent increase in the total area over which the stick–slip motion occurs that leads to the amplitude dependency. What is clear from our work described in [17] on various configurations of clamp is that, in the absence of stick–slip motion at the interface of the clamp and flexure, the amplitude dependence disappears. A further observation was that, in the case of a vertical pendulum, it is possible to have losses that originate in the structure of the pendulum.

Newman and Bantel [19] have investigated in detail the properties of Cu–Be at low frequencies and at low temperatures. They found that there were in fact two sources of damping. One component was independent of amplitude but varied with temperature whilst the other was linearly dependent on amplitude, exhibited a frequency dependence consistent with equation (3.1*b*) and was independent of temperature. They concluded that the latter component was consistent with a stick–slip loss mechanism. This observation seems to be consistent with our model of the amplitude dependence.

We know now that the function that we used to calculate the modulus defect from the observations in QSB did not take into consideration the gravitational potential energy stored in the pendulum test mass due to the shortening of the flexure strip (see the appendix of [17]). A revised value, using the observations from QSB, for $\Delta\gamma/\gamma$ is 1.1×10^{-3}. In [17] we showed that, when the stick–slip losses in the flexure-strip mount and pendulum structure were accounted for, losses in compound pendulums which were suspended from Cu–Be flexure strips were consistent with those in torsion pendulums which were suspended from Cu–Be strips (torsion strips). From these observations and further measurements with Cu–Be torsion strips [18] we believe $\Delta\gamma/\gamma$ to be 5×10^{-5} under conditions of negligible stress. We can conclude that, in the light of this further work, the original measurements were most probably dominated by losses in the pendulum support and flexure mount, although we had, at the time, taken what we thought to be appropriate precautions to avoid this.

5. Conclusion

We have now been studying anelastic properties of Cu–Be in various types of oscillating systems for a considerable period of time. Our findings can be summarized as follows. Anelasticity in Cu–Be may be described mathematically as a superposition of Maxwell units with a range of relaxation times, τ, such that the number density, $g(\tau)$, of relaxation processes varies as $1/\tau$. The physical explanation for this behaviour lies in the destruction of pinning sites for dislocations in the polycrystalline structure. In our model $g(\tau)$ varying as $1/\tau$ arises due to the time required for point defects to diffuse along dislocations of varying lengths. There is a single activation energy in this model. This is in contrast to the models given by van der Ziel and du Pre, in which $g(\tau)$ has the same form but is due to a range of activation energies. Measurements of damping in Cu–Be at extremely low frequencies are consistent with this model but not with that of Granato and Lucke. The analogues of anelasticity and its resultant $1/f$ noise are seen in a wide range of other processes (for example, dielectric and magnetic ones) described in terms of frequency-dependent susceptibilities.

When making mechanical measurements, it is particularly difficult to isolate intrinsic losses in the flexure material from those due to the support and structure of the suspended object. The losses at the interface between the support and the structure, stick–slip losses, have the same frequency dependence as the intrinsic losses and can be distinguished from the latter only by their stronger amplitude dependence. The most positive outcome from our experiences is that we have learnt that the best way of avoiding anelastic effects (thermal noise and drift) is to use a torsion strip. This is not only because the stress is distributed uniformly along the torsion strip and its shape is independent of load, in contrast to a flexure strip, but also because, in a wide, heavily loaded torsion strip, practically all the restoring torque comes from gravity. This feature can be exploited to increase the signal-to-noise ratio in determinations of G, for example [20, 21]. We are currently investigating the losses due to charge motion on the surface of electrodes of capacitors [22]. This may be an important limitation to the thermal noise in mechanical systems whose displacement is detected electrically by means of a capacitance bridge, for example. Furthermore, such loss mechanisms may give rise to errors in the calibration of G signals using electrostatic forces.

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References

[1] Quinn T J, Speake C C and Brown L M 1992 *Phil. Mag.* A **65** 261–76
Quinn T J and Speake C C 1987 *Rapport* BIPM-87/3
[2] Nowick A S and Berry B S 1972 *Anelastic Relaxation in Crystalline Solids* (New York: Academic)
[3] Quinn T J, Speake C C and Davis R S 1986 *Metrologia* **23** 87
[4] Kuroda K 1995 *Phys. Rev. Lett.* **75** 2796–8
[5] Press W H, Teukolsky S A, Vettering W T and Flannery B P 1996 *Numerical Recipes in Fortran 77* vol 1, 2nd edn, p 215
[6] Speake C C and Quinn T J 1987 *Gravitational Measurements, Fundamental Metrology and Constants* ed V de Sabbata and V N Melnikov (Dordrecht: Kluwer) p 443
[7] Saulson P R 1990 *Phys. Rev.* D **42** 2437
[8] Saulson P R, Stebbins R T, Dumont F D and Mock S E 1994 *Rev. Sci. Instrum.* **65** 182
[9] Saulson P R 1994 *Fundamentals of Interferometric Gravitational Wave Detectors* (Singapore: World Scientific)
[10] Kimball A L and Lovell D E 1927 *Phys. Rev.* **30** 948
[11] du Pre F K 1950 *Phys. Rev.* **78** 615
[12] van der Ziel A 1950 *Physica* **16** 359
[13] Vitale S, Cerdonio M, Prodi G A, Cavalleri A, Falferi P and Maraner A 1995 *Quantum Tunnelling of Magnetization-QTM'94'* ed L Gunther and B Barbara (Dordrecht: Kluwer) p 157
[14] Gonzalez G I and Saulson P R 1995 *Phys. Lett.* A **201** 12
[15] Cagnoli G, Gammaitoni L, Marchesoni F and Segoloni D 1993 *Phil. Mag.* A **68** 865
[16] Granato A V and Lucke K 1966 *Physical Acoustics, Principles and Methods* vol 9, part A, ed W P Mason (New York: Academic) p 238
[17] Quinn T J, Speake C C, Davis R S and Tew W 1995 *Phys. Lett.* A **197** 197
[18] Quinn T J, Davis R S, Speake C C and Brown L M 1997 *Phys. Lett.* A **228** 36
[19] Bantel M K 1998 *PhD Dissertation* University of California, Irvine
[20] Quinn T J, Speake C C and Davis R S 1997 *Metrologia* **34** 245
[21] Richman S J, Quinn T J, Speake C C and Davis R S 1999 *Meas. Sci. Technol.* **10** 460
[22] Speake C C, Davis R S, Quinn T J and Richman S J 1999 *Phys. Lett.* A submitted
[23] Braginskii V B, Mitrafanov V P and Okhrimenko O A 1992 *JETP Lett.* **55** 432
[24] Rowan S, Hutchins R, McLaren A, Robertson N A, Twyford S M and Hough J 1997 *Phys. Lett.* A **227** 153
[25] Kovalik J and Saulson P R 1993 *Rev. Sci. Instrum.* **64** 2942