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Electrostatic damping and its effect on precision mechanical experiments

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Abstract

We develop a model for damping of a mechanical oscillator when subjected to an electrostatic force applied via electrodes with resistive losses. The model predicts viscous damping that varies as the inverse of oscillation frequency and that is proportional to a dimensionless constant Δ_e, which depends on the surface properties of the electrodes. Preliminary experiments made using a torsion strip pendulum with aluminium electrodes establish that $2.2(5) \times 10^{-3} > \Delta_e > 5.9(3.4) \times 10^{-6}$ depending on the time spent under vacuum. Finally we calculate the magnitude of error that such losses produce in certain determinations of Newton's constant of gravitation where an electrostatic torque calibration is employed.

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1. Introduction

A common problem encountered in electrical metrology is the determination of capacitance values at low frequencies (below 1 Hz) from ac measurements at, typically, a few kHz. Thin, lossy, dielectric films on the surfaces of electrodes produce parasitic capacitances that appear in series with the capacitance between the electrodes and the value of the effective capacitance depends on the frequency of measurement. This problem had an impact [1] on the realisation of the farad until the invention of the calculable capacitor [2], where the effects of parasitic capacitances can be made to cancel to first order. Lossy capacitances have been studied by researchers attempting to evaluate the unit of the volt in terms of the kilogramme [3] and have been considered as a source of systematic error in determinations of the Newtonian gravitational constant, G [4]. Recently direct measurements of noise forces by Willemenot [5] on an electrostatically suspended torsion pendulum revealed a relatively large component which could be interpreted as a consequence of energy losses in such thin films. Importantly, this author noted that the spectrum of excess noise force had a $1/f$ character that was consistent with a thin film

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whose dielectric constant had an imaginary component independent of frequency over the range investigated. The importance of this source of noise in experimental gravitation has been emphasised by Vitale and colleagues [6]. Our interest in this topic has been stimulated by its relevance to noise in experiments measuring weak forces and, more specifically, by its relevance to the calibration of an electrostatic force transducer for use in our experimental determination of G [7]. In this paper we generalise the usual equivalent electrical circuit for a series lossy capacitance to calculate the damping and force generated by an electrostatic force transducer whose electrodes are coated with a thin film having a frequency-independent imaginary component of dielectric constant. We then describe measurements of electrostatic loss on aluminium capacitor plates attached to a torsion strip pendulum. We compare the results of these measurements with those in the literature and estimate the magnitude of a systematic error that could be introduced into a determination of G where the gravitational force is calibrated by an electrostatic torque.

2. Theory of electrostatic damping

2.1. Equivalent circuit for a lossy air-capacitor

According to the model given by Astin [1] the physical capacitor can be represented as a perfect capacitance in series with a much larger stray capacitance which is itself shunted with a resistor. The circuit is shown in Fig. 1 The impedance of this circuit is

$$Z=\frac{1}{j\omega C_{\mathrm{D}}}+\frac{1}{j\omega C_{\Delta}}\left\{\frac{\omega^2\tau^2}{1+\omega^2\tau^2}+j\frac{\omega\tau}{1+\omega^2\tau^2}\right\} \tag{2.1}$$

where $\tau=R_{\Delta}C_{\Delta}$. The problem of lossy dielectrics has been well studied (see [8–11] for a review) and it is well known that most dielectrics have dissipation factors (ratio of imaginary and real components of capacitance) which are independent of frequency over many decades. It seems reasonable to expect that a more realistic model would include a superposition of loss elements with a spectrum of time

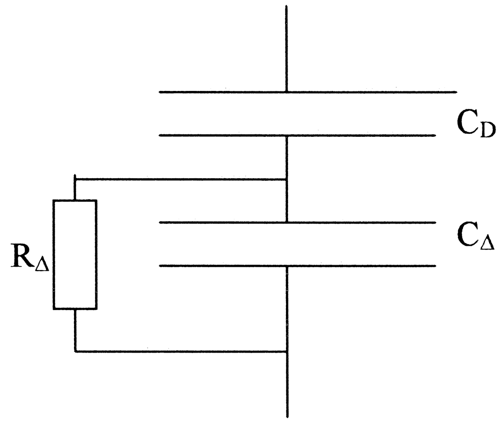

Fig. 1. Equivalent circuit for a lossy capacitor.

constants. Further justification for this is given by the results of Willemenot [5] as mentioned above. A natural extension, therefore, of the above model is to connect additional parallel resistor–capacitor networks in series with the one shown in Fig. 1. This is analogous to the development given in [12] in connection with anelasticity and has been also discussed in connection with the behaviour of dielectrics in [8–11]. If we follow the normal assumption that the number density of relaxation elements as a function of relaxation time varies as $1/\tau$ and that they all have the same value of capacitance, we can integrate over the distribution to find the new effective impedance

$$Z=\frac{1}{j\omega C_{\mathrm{D}}}+\frac{1}{j\omega C_{\Delta}\ln(\tau_{\mathrm{m}}/\tau_0)}\left\{\frac{1}{2}\ln\left(\frac{1+\omega^2\tau_{\mathrm{m}}^2}{1+\omega^2\tau_0^2}\right)+j\left(\tan^{-1}\omega\tau_{\mathrm{m}}-\tan^{-1}\omega\tau_0\right)\right\}, \tag{2.2}$$

where τ_0 and where τ_{m} are the shortest and longest time constants present in the ensemble. It is useful for what follows to define the values of resistance and capacitance for the series network that is equivalent to the lossy capacitor as shown in Fig. 2. We have

$$R_{\mathrm{e}}=n\frac{\left(\tan^{-1}\omega\tau_{\mathrm{m}}-\tan^{-1}\omega\tau_0\right)}{\omega C_{\Delta}\ln(\tau_{\mathrm{m}}/\tau_0)}, \tag{2.3a}$$

$$C_{\mathrm{e}}=\frac{C_{\Delta}2\ln(\tau_{\mathrm{m}}/\tau_0)}{\ln\left(\frac{1+\omega^2\tau_{\mathrm{m}}^2}{1+\omega^2\tau_0^2}\right)}. \tag{2.3b}$$

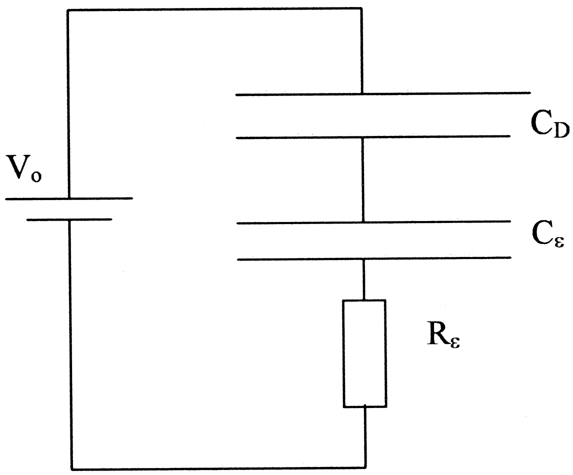

Fig. 2. Equivalent series circuit for a lossy capacitor.

2.2. Mechanical effects arising from parasitic capacitances

We can calculate the equations of motion of a torsion pendulum when a fixed voltage, V_0, is applied across the circuit shown in Fig. 2 following methods discussed by Chen and Tan [13]. If the voltages across the variable capacitor C_D and the lossy capacitor C_e are V_1 and V_2, respectively, the torque can be written as

$$\Gamma = \frac{1}{2}\frac{dC_D}{d\theta}V_1^2 \qquad (2.4a)$$

or

$$\Gamma = \frac{1}{2}\frac{dC_D}{d\theta}(V_0 - V_2)^2. \qquad (2.4b)$$

We can calculate V_2 using the values of R_e and C_e given in Eqs. (2.3a) and (2.3b): Using Kirchoff's law we find

$$V_2 = \dot{Q}R_e + \frac{Q}{C_e}, \qquad (2.5)$$

where Q is the common charge for the three series components R_e, C_e and C_D which can be written

$$Q = C_D V_1 = C_D(V_0 - V_2). \qquad (2.6a)$$

We also have

$$\dot{Q} = -C_D\dot{V}_2 + \frac{dC_D}{d\theta}\dot{\theta}(V_0 - V_2). \qquad (2.6b)$$

Combining (2.5) and (2.6) we obtain

$$\frac{C_D}{C_e}V_0 + R_e\frac{dC_D}{d\theta}V_0\dot{\theta} - V_2\left(1 + R_e\frac{dC_D}{d\theta}\dot{\theta} + \frac{C_D}{C_e}\right) - R_e C_D \dot{V}_2 = 0. \qquad (2.7)$$

If we consider cases where $C_e \gg C_D$ (which is consistent with our results reported below) then, from the above equation, we see that V_2 itself will be a small quantity compared to V_0. We can therefore ignore terms that vary as the product of V_2 and C_D/C_e and solve (2.7) for V_2:

$$V_2 = \frac{C_D}{C_e}V_0 + R_e\frac{dC_D}{d\theta}V_0\dot{\theta}. \qquad (2.8)$$

Substituting this equation into (2.4b) and keeping only first-order terms we find

$$\Gamma = \frac{1}{2}\frac{dC_D}{d\theta}V_0^2\left(1 - \frac{2C_D}{C_e}\right) - R_e\left(\frac{dC_D}{d\theta}\right)^2 V_0^2\dot{\theta}. \qquad (2.9)$$

The torque generated by the electrostatic transducer then becomes, using (2.3):

$$\Gamma = \frac{1}{2}\frac{dC_D}{d\theta}V_0^2\left(1 - \Delta_e\frac{1}{\pi}\ln\left(\frac{1+\omega^2\tau_m^2}{1+\omega^2\tau_0^2}\right)\right) - \Delta_e\frac{1}{\pi C_D}\left(\frac{dC_D}{d\theta}\right)^2 V_0^2\frac{\left(\tan^{-1}\omega\tau_m - \tan^{-1}\omega\tau_0\right)}{\omega}\dot{\theta} \qquad (2.10a)$$

where we have denoted the electrostatic defect as

$$\Delta_e = \frac{C_D}{C_\Delta}\frac{\pi}{\ln(\tau_m/\tau_0)}. \qquad (2.10b)$$

In the frequency regime $\tau_m^{-1} \ll \omega \ll \tau_0^{-1}$ the electrostatic damping torque resulting from electrodes with similar parasitic capacitances on both electrodes becomes simply

$$\Delta_e\frac{1}{C_D}\left(\frac{dC_D}{d\theta}\right)^2 V_0^2\frac{\dot{\theta}}{\omega}\ \mathrm{Nm/rad\,s^{-1}}. \qquad (2.11)$$

Eqs. (2.10a) and (2.10b) indicate that the lossy capacitor introduces both a frequency-dependent error in the magnitude of the torque applied to the pendulum and a damping torque. The electrostatic damping term has been discussed by other authors. In [13] a resistor, R, in parallel with the variable capacitor, C, is shown to give rise to damping which is proportional to $R(dC/d\theta)^2$. In [14] the presence of a frequency-independent imaginary component of capacitance gives rise to a resistance which varies inversely as frequency which in turn gives rise to $1/f$ noise. Eq. (2.10a) can be compared with that given by Kuroda [15] for the change in the stiffness with oscillation frequency of a torsion pendulum due to anelasticity. Importantly, it follows from Eqs. (2.10a) and (2.10b) that, given the validity of the model of the impedance of the lossy layer, the frequency-dependent torque error can be estimated from measurements of the damping coefficient.

3. Damping measurement

We have some experimental evidence of the electrostatic defect from measurement of damping times of a torsion-strip pendulum as a function of the electrostatic restoring torque applied to it. The torsion-strip pendulum employed here has been fully described elsewhere [7] as a prototype of an apparatus for determination of G. Voltages are applied to the parallel-plate electrodes attached to the pendulum. The amplitude during ring-down and oscillation period were measured for a range of applied voltages. No attempt was made to bake the vacuum chamber or to clean the electrodes other than the routine degreasing after manufacture. The first of two sets of data was taken after pumping down the vacuum system and the pendulum for the first time. A summary of the results is shown in Table 1a. A second data set was taken on the fourth evacuation of the system after the bore of the tube connecting the vacuum chamber to the flexible pipe had been significantly widened in order to lower the base pressure and shorten the pump-down time. This data set is shown in Table 1b.

Table 1

Date	Bias voltage (V)	$Q/10^3$	Pressure (μTorr)	Period (s)
a. Data set from first evacuation				
13-15/6/98	0	5.826(75)	6.5(1.5)	84.8(2)
15-17/6/98	2.5	4.302(31)	4.5(5)	85.13(2)
17/6/98	5.0	2.199(20)	3.9(1)	87.2(2)
26/6/98	7.5	1.232(26)	2.2(1)	96.9(2)
18-20/6/98	0	9.778(73)	3.5(5)	84.8(2)
b. Data set from fourth evacuation				
28-29/7/98	0	21.73(45)	3.0(5)	84.8(2)
30/7-3/8/98	13.74	6.813(10)	1.7(1)	185.0(2)
3-7/8/98	13.08	10.37(10)	1.5(1)	154.5(2)
7-11/8/98	12.07	13.54(14)	1.35(5)	132.4(2)
11-15/8/98	0	37.84(28)	1.30(5)	84.8(2)
15-19/8/98	10.7	24.41(10)	1.20(5)	103.8(2)

4. Interpretation

We can write the complete equation of motion of the torsion strip pendulum as

$$I\ddot{\theta} + \left(\Delta_E \frac{k_E}{\omega} + \Delta_e \frac{\alpha |k_e|}{\omega} + b_g \right) \dot{\theta} + \left(k_e + k_g - |k_e| \right) \theta = 0, \tag{4.1a}$$

where k_E and k_g are the elastic and gravitational restoring torques of the torsion strip pendulum. The moment of inertia of the torsion balance is I. The quantity Δ_E is the elastic modulus defect of Cu–Be which we have determined in previous experiments [12], at this stress, to be $1.0(2) \times 10^{-4}$. Note that k_g is lossless as discussed in [16]. Viscous gas damping is represented by the coefficient b_g. The configuration of electrodes and applied voltages is such that the net electrostatic torque is zero; however, each pair of electrodes contributes a (negative) restoring torque given by

$$k_e = -\frac{1}{2} \frac{d^2 C_D}{d\theta^2} V_0^2. \tag{4.1b}$$

In the regime $\tau_m^{-1} \ll \omega \ll \tau_0^{-1}$, the dimensionless constant α can be written

$$\alpha = \frac{2}{C_D} \frac{\left(\dfrac{dC_D}{d\theta} \right)^2}{\left(\dfrac{d^2 C_D}{d\theta^2} \right)}. \tag{4.1c}$$

It must be remembered that C_D refers to the capacitance of a single pair of electrodes as shown in Fig. 1 and not to the total capacitance of the torsion balance to ground. In our geometry α is approximately unity.

We can express the quality factor to a good approximation as

$$\frac{1}{Q} = \frac{\Delta_E k_E + \alpha \Delta_e |k_e| + \omega b_g}{I\omega^2} \tag{4.2}$$

If we denote the oscillation period of the torsion pendulum in the absence of electrostatic rotational stiffness as t_0, and in the presence of an applied voltage V_i as t_i, we can rearrange (4.2) as follows:

$$\frac{1}{Q}\left(\frac{t_0}{t}\right)^2 = \left(\frac{\Delta_E k_E}{k_E + k_g} + \alpha \Delta_e\right) - \alpha \Delta_e \left(\frac{t_0}{t}\right)^2 + \frac{1}{Q_{gas}}\left(\frac{t_0}{t}\right) \tag{4.3}$$

where Q_{gas} is the quality factor which would arise purely from viscous gas damping at the oscillation period t_0. In practice the residual gas pressure could not be reduced to the extent where the last term was negligible and, further, the pressure varied between and during measurements of the damping time. In order to account for the change in gas damping we estimated the value of Q_{gas} by making two measurements of relaxation time in the absence of electrostatic forces at two different measured pressures as shown in the tables above. The gas damping term in (4.3) was then calculated at the pressure of the measurement assuming that the damping scaled linearly with pressure. The final equation for interpreting the data becomes

$$\frac{1}{Q(p,t)}\left(\frac{t_0}{t}\right)^2 - \frac{1}{Q_{gas}\, p_0}\left(\frac{p t_0}{t}\right) = \left(\frac{\Delta_E k_E}{k_E + k_g} + \alpha \Delta_e\right) - \alpha \Delta_e \left(\frac{t_0}{t}\right)^2 \tag{4.4}$$

where $Q_{gas}\ p_0$ is the mean of the products of the pressure and the quality factor from gas damping at pressure p_0 and period t_0. The expression on the left of (4.4) is then plotted against $(t_0/t)^2$. This is shown using the data from both evacuations in Fig. 3a and Fig. 3b, respectively. The data were fitted to a straight line and the results for both data sets are shown in Table 2 below.

Table 2

	Slope	Intercept
1st evacuation	$-2.2(5)\times 10^{-3}$	$2.3(5)\times 10^{-3}$
4th evacuation	$-5.9(3.4)\times 10^{-6}$	$2.54(20)\times 10^{-5}$

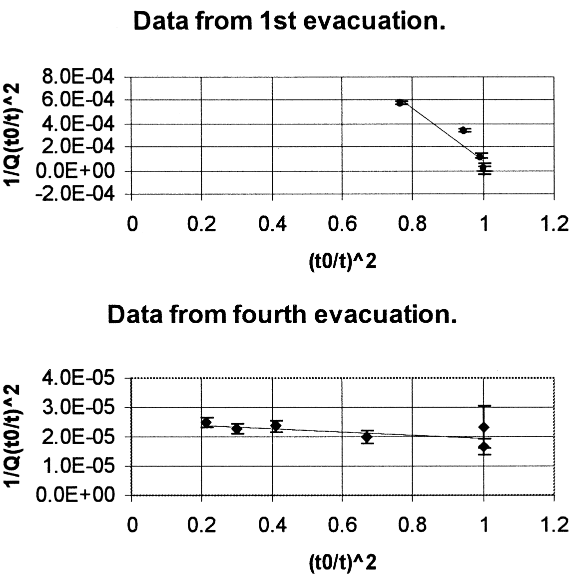

Fig. 3. (a) Plot of data from first evacuation using (4.4). (b) Plot of data taken during fourth evacuation using (4.4).

5. Conclusions

The results from the first evacuation indicate an electrostatic defect $\Delta_e = 2.2(5) \times 10^{-3}$. The value of the reduced Chi-squared for 4 degrees of freedom is 0.6. The data from the fourth evacuation give an electrostatic defect of $\Delta_e = 5.9(3.4) \times 10^{-6}$ with a reduced Chi-squared of 0.7. A conservative interpretation of this latter result is that it represents an upper limit to the electrostatic defect of 10^{-5}. The uncertainties in our values are dominated by the uncertainty in the elastic damping term, Δ_E, and gas damping. In order to improve the accuracy of both these parameters further experiments are necessary at considerably lower pressures or with more accurate gas pressure gauges. In both cases the intercept and the slope are consistent with the values of the electrostatic and elastic defects within the uncertainties.

Table 3

Experiment	Electrostatic damping coeff. multiplied by electrode gap (mm)
Uncleaned Al, 1st evacuation (current work)	2.3×10^{-3}
Uncleaned Al, 4th evacuation (current work)	10^{-5} (upper limit)
Ultrasonically cleaned Gold coated Al [3]	6×10^{-6} (upper limit)
Gold coated ULE [5]	3×10^{-4}

It is most likely that the parasitic capacitance on the electrodes during the first evacuation was reducing in time and this would explain the slightly unsatisfactory fit to the data. Given the arguments in Section 2, we would expect the electrostatic defect to scale inversely as the spacing between the capacitor plates. In Table 3 we multiply our values for Δ_e by the spacing (1 mm) and compare our results with those obtained by others. The data in the table show a spread in the damping to be expected from different surfaces. Our data show that electrostatic damping of relatively dirty aluminium depends strongly on the vacuum history of the electrodes. The result obtained after the fourth evacuation is consistent with that measured by Bego et al. The result from Willemenot is unexpectedly large, indicating perhaps that the scaling law we have assumed here is invalid at spacings of 30 μm, as was the case in that experiment. It should be noted that, in the Willemenot experiment, the electric field strengths were significantly higher than those employed in this work and in that of Bego et al.

6. Implications for *G* experiments

In determinations of G such as those described by Michaelis et al. [4] the gravitational signal is balanced using electrostatic torques. The transducer that applies these torques is calibrated by measuring its capacitance as a function of the rotation angle of the torsion pendulum. This capacitance measurement is performed using an ac bridge operating at around 1 kHz. Using (2.2), assuming that $\tau_0 \gg 1$ ms and that there are parasitic capacitances on the surfaces of both electrodes, we would deduce a value of the gradient of capacitance to be

$$\frac{dC_{\mathrm{ac}}}{d\theta} = \frac{dC_{\mathrm{D}}}{d\theta}\left(1 - \frac{4}{\pi}\Delta_{\mathrm{e}} \ln \tau_{\mathrm{m}}/\tau_0\right). \tag{6.1}$$

However, the actual measurements of the gravitational torque, Γ_{G}, will take place at a much lower frequency, $\omega_{\mathrm{G}}/2\pi$, and the torque generated by the transducer will be given by (2.10):

$$\Gamma_{\mathrm{G}} = \frac{V_{\mathrm{G}}^2}{2}\frac{dC_{\mathrm{D}}}{d\theta}\left\{1 - \frac{4}{\pi}\Delta_{\mathrm{e}}\frac{1}{2}\ln\left\{\frac{1+\omega_{\mathrm{G}}^2\tau_{\mathrm{m}}^2}{1+\omega_{\mathrm{G}}^2\tau_0^2}\right\}\right\} \tag{6.2}$$

where V_{G} is the voltage required to balance the gravitational torque. The calibration procedure will lead to an erroneous apparent value for the couple:

$$\Gamma_{\mathrm{A}} = \frac{V_{\mathrm{G}}^2}{2}\frac{dC_{\mathrm{ac}}}{d\theta}. \tag{6.3}$$

The resulting estimated value of the gravitational constant, G_{A}, becomes

$$G_{\mathrm{A}} = G\frac{\left(1 - \frac{4}{\pi}\Delta_{\mathrm{e}}\ln \tau_{\mathrm{m}}/\tau_0\right)}{\left\{1 - \frac{4}{\pi}\Delta_{\mathrm{e}}\cdot\frac{1}{2}\ln\left\{\frac{1+\omega_{\mathrm{G}}^2\tau_{\mathrm{m}}^2}{1+\omega_{\mathrm{G}}^2\tau_0^2}\right\}\right\}}. \tag{6.4a}$$

If we assume $\tau_m^{-1} \ll \omega_G \ll \tau_0^{-1}$, we can write

$$G_{\mathrm{A}} \approx G\left(1 + \frac{4}{\pi}\Delta_{\mathrm{e}}\ln\,\omega_{\mathrm{G}}\tau_0\right). \tag{6.4b}$$

The systematic uncertainty that is incurred from this source will therefore result in an fractional *underestimate* of G as $\omega_G\tau_0 \ll 1$. The fractional error will be of the order Δ_e. This source of error is discussed by the authors in [4] but they made no direct measurements of the electrostatic defect in their apparatus. It seems certain, however, that this error source cannot explain their high value for G. Further, the results reported here taken with those of Bego et al. [3] strongly suggest that, with some care, determinations of G using electrostatic calibration should be capable of accuracies of better than 10 ppm. Finally we point out that in our model we have assumed that the magnitude of the real and imaginary components of the parasitic capacitance are equal, having been led to this from the simple model shown in Fig. 1. Clearly, it is possible for the parasitic capacitance to contain a component which is a pure capacitance. If this extra term is included in the analysis we would expect that the damping measurements would lead to an upper limit for the force error. Evidently further work is desirable to establish experimentally the quantitative relationship between the force error and damping.

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