Meas. Sci. Technol. **10** (1999) 499–507. Printed in the UK PII: S0957-0233(99)00171-X

Precision limits of the modern Cavendish device: thermal noise measurement regimes and strategies in the torsion pendulum

R C Ritter, L I Winkler and G T Gillies

University of Virginia, Department of Physics, McCormick Road, Charlottesville, VA 22901, USA

Received 16 December 1998, accepted for publication 30 March 1999

Abstract. The lightly damped torsion pendulum is among the most sensitive of mechanical force detectors. Major limits to its sensitivity arise from horizontal gravitational gradients, seismic disturbance and thermal fluctuations. Unlike the other sources, the more fundamental thermal noise can serve as a common theoretical 'standard' against which much of the pendulum's performance can be measured. Nevertheless, its 'pure' statistical character from molecular bombardment is not retained through the processes of pendulum action and those of its measurement, as will be shown.

Sensitivity limit studies using thermal fluctuation theory apply to most types of sensitive measurement, not just the torsion pendulum. This theory originated in the context of Brownian motion as developed by Einstein, and incorporates ideas involving random walks. The lightly damped pendulum, however, does not execute a random walk under practical observing conditions.

We present a brief history of noise theory, followed by its application to the torsion pendulum. A simple measurement strategy for the static mode pendulum is adopted to develop the subject. Intrinsic noise of the pendulum couple is discussed, in which a natural damping limit appears. Spectral behaviour is seen to be important in understanding the system noise. The spectral character is developed, along with a method of analysis used in precision frequency standards, which can be seen to have useful self-evaluation within it. The equilibrium situation of the lightly damped pendulum presents a practical difficulty in the application of noise theory to lightly damped pendula. Negative derivative feedback is seen as a means of handling this, and of optimizing the measurement conditions.

Keywords: torsion pendulum, thermal noise, Brownian motion, fluctuation theory, random walk, damped harmonic oscillator, sensitivity limits, phase fluctuations, power spectrum, Allan variance

1. Introduction

The torsion pendulum has played the central role in experimental gravitational physics throughout virtually all of the 19th and 20th centuries. During that span it has been used for determination of the Newtonian gravitational constant, G, in tests of the equivalence principle, in gravity gradiometers for geophysical exploration, in studies of the superposition principle, and in a wide variety of related measurements aimed at searching for gravitational anomalies and discovering weak, long-range forces. It has been built in versions having one millimetre diameter suspension fibres 100 m long and supporting test masses of 10 kg each, down to micron-sized threads of quartz or tungsten able to lift no more than a fraction of a gram. It has been operated in submerged tanks, evacuated chambers, atop mountains and deep in caves. Various versions have been carefully vibration isolated, rotated, fed back by magnetic and electrostatic forces, cooled to cryogenic temperatures, and heated to incandescence during the course of their use. Few other scientific instruments have been subjected to as much scrutiny or seen as much study over so long a period of time as has this simple mechanical device that still serves us well today.

However, the surprising scope of designs for this often robust instrument and the tremendous dynamic range over which it can be used belie the complexity of its performance limits as a force measuring device. Many attempts have been made to understand its noise floor, its response to microseisms and other perturbations, and its behaviour under the influence of feedback. Even so, the present substantial disagreement between several recent measurements of G using torsion pendula and the new insights into the effects of anelasticity of the suspension fibre, continually call into question our basic understanding of the physical behaviour of this class of instrument.

Therefore, on this bicentennial occasion of the first application of a torsion pendulum to a problem in gravitational physics, it is useful and interesting to revisit fundamental issues associated with these devices, some of which are still not uncontroversial today. We begin with the presentation of some background on the philosophy underlying the torsion pendulum's utility and continue with a discussion of its behaviour as a noise-limited harmonic oscillator, in a historical context. We then consider measurement strategies for its use in the detection of weak signals in the presence of fluctuations, and introduce an Allan variance based assessment of the noise in torsion pendulum measurement systems. Lastly, we analyse how the fluctuations of the torsion balance behave under the influence of feedback and in other nonequilibrium conditions. Our goal is to elucidate the differences in performance characteristics of a torsion pendulum operated in different regimes of dynamic limitation and to derive expressions that may help guide the thinking of those designing experimental searches for weak gravitational signals.

2. Background

In the gravitational context, the torsion pendulum is an extremely sensitive force measuring harmonic oscillator operating almost always in either the coercive (static) or in the oscillatory (dynamic) manner. Without fully realizing it, Cavendish [1] was the first to explore gravitational couplings (our weakest force) with it, although Coulomb and Mitchell are credited with the balance conception when it was used earlier to measure electrical forces. These scientists intuitively knew that the 'horizontal' balance (including the Earth's centrifugal force) removed the huge effect of the Earth's nearly but not perfectly static background gravity. They also knew that making the pendulum a sensitive integrator, with a long period and a long decay time constant, could remove much of the useless short-term information in its response. At that time, however, they did not know about the molecular origin of thermal noise and were unable to separate or evaluate that source in the presence of seismic or other somewhat random ambient interferences. Although the latter can have considerable significance in limiting the sensitivity of the torsion pendulum, we confine this paper to considerations of thermal noise.

From the earliest G experiments, strategies concerning the size of the pendulum were adopted. A related strategy, maximal loading of the fibre, was known to increase the sensitivity. Ultimately, from this the question of fibre material behaviour, at first simply a study of the damping added by it, has evolved to a consideration of the frequency-dependent anelastic effects discussed by others (see these proceedings) and will not be taken up here. When significant, they can affect spectral thermal noise studies and amplitudes of noise estimates.

An understanding of thermal noise itself came from the study of Brownian motion of particles in a liquid, not originally in relation to the harmonic oscillator. Van der Ziel, in his paper *History of Noise Research* [2], credits Einstein with the beginning of this field of research into fluctuations. It is ironic that this source of understanding, in which the fluctuating object was highly damped by a surrounding fluid, came from a physical system so different in its nature to the immensely sensitive, lightly damped torsion pendulum.

The theory of noise has occupied a huge amount of literature concerning sensitive measurements in many other fields, but the torsion pendulum is a supreme example of its application. Studies of fluctuations have evolved to include concepts of random walk, the fluctuation–dissipation theorem and the use of the equipartition theorem as a scaling means. However, not all of these concepts can be applied to the lightly damped torsion pendulum without considerable care.

Understanding of the noise of the pendulum follows mostly from earlier experiments with the galvanometer, which was able to provide variable damping. Figure 1 shows the obviously different noise response of a galvanometer to three different electromagnetic damping conditions. Immediately, we must conclude that the response from the purely stochastic molecular bombardment has undergone a kind of filtering which must affect the strategy of measurement to a certain level of accuracy. McCombie [3] makes the point, following Zernicke [4], that correlation functions are important in analysing fluctuations in sensitive measurements, and we will see that aspect of correlated noise later. Similarly, Uhlenbeck and Goudsmit [5] Fourier analysed time series of the position of small mirrors hung on fibres and fitted them to the damped harmonic oscillator spectrum with correct gas and mirror properties.

3. 'Standard' noise theory of phase fluctuations of the harmonic oscillator

We will briefly summarize the evolution of a 'standard' theory of noise as it applies to the torsion pendulum, mostly in the context of the pendulum in the static or balance mode, as a basis for getting simply at the basics of noise in such sensitive instruments. This will lead to the incorporation of concepts of strategy in torsion pendulum measurements.

A particularly simple type of measurement will illustrate the necessary elements. We use a pendulum in the static mode and look for a significant difference in pendulum angle (phase) as a result of moving the attracting 'source' masses, the source of the signal. This will require an angle assessment at two different times, t_1 and t_2, bounding a time interval during which the pendulum must be able accurately to respond to the signal. Pendulum angle fluctuations will cause errors in the position measurement at each of the times. Time-honoured calculations (below) for the variance of pendulum position will give us an expectation of error for each of the two position measurements, at t_1 and t_2. This arises from ϕ_N^2, the intrinsic noise variance of the pendulum angle, but must also incorporate some consideration of the duration of measurement. An instantaneous position sampled at t_1 would have a statistically weighted chance of having any value ϕ through a range labelled by ϕ_N. Averaging samples over some time interval τ might improve on this, and we might expect the angular position given by this average to have an error $1/\tau^{1/2}$. Such an expectation is based on the

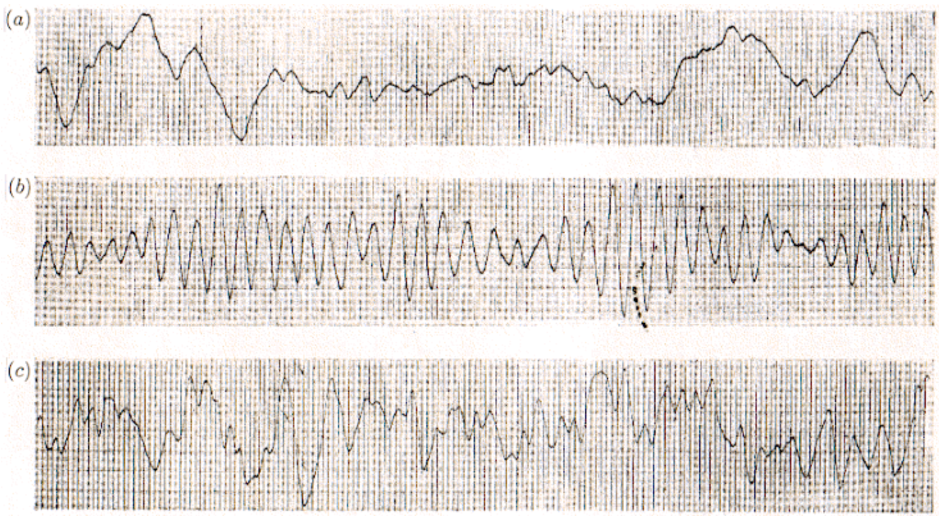

Figure 1. The response of a galvanometer [3] with three different levels of electromagnetic damping: (*a*) overdamped (short circuited); (*b*) underdamped (open circuit); (*c*) critically damped.

statistics of pure random noise. We will see that the response of a pendulum constitutes correlated noise, which can be of several types, and which takes a more careful strategy. Allan has shown [6], for example, that 'averaging down' does not always work.

First, the intrinsic noise. Einstein's (and similarly von Smoluchowski's [7]) theory of fluctuations leads directly to the essence of the fluctuation–dissipation theorem, the connection between the microscopic nature of the origin of fluctuations and the macroscopic dissipation resulting from them. We present these results, but use an independent variable appropriate to the primary angular mode of the idealized single mode torsion pendulum–the fluctuation in angle ϕ (the phase of the pendulum).

The differential equation of motion of the phase of the damped harmonic oscillator driven by torque $\Gamma(t)$ is

$$I\,\mathrm{d}^2\phi/\mathrm{d}t^2 + \beta\,\mathrm{d}\phi/\mathrm{d}t + C\phi = \Gamma(t). \tag{1}$$

In Einstein's theory of Brownian motion, the restoring coefficient C in (1) is zero. In three steps (see [2]) he arrived at what is equivalent to the random walk in phase ϕ over a time interval τ. His result, derived through the diffusion equation, is in our coordinates

$$\langle\phi^2(\tau)\rangle = 2(kT/\beta)\tau \tag{2}$$

at temperature T.

A variety of proofs of this result have appeared; that of Langevin [8] is well known. Our derivation of this random walk in a rotational system with $C = 0$, will parallel Langevin's approach. (This is a case in which we have studied the fluctuations of a rotor [9]. Rotor phase, the fluctuation around a nearly constant, slowly decaying speed of rotation ω_0 is the analogue of a pendulum with zero restoring torque.) Limiting noise attributes of this lower order system are treated in the same manner as the sensitive standard pendulum [2]. Its simplicity leads to a more lucid view of the concepts, and at the same time it is an analogue of the Brownian motion case Langevin originally studied.

Langevin divided the torque (force in his case) acting on the object into slowly varying (the damping term) and fluctuating parts for a system at t_1 before the signal or at t_2 after it:

$$\Gamma = -\beta\,\mathrm{d}\phi/\mathrm{d}t + \Gamma'(t). \tag{3}$$

$\Gamma'(t)$ yields a fluctuating acceleration $\alpha(t)$ from molecular bombardment and the differential equation of fluctuating velocity about a slowly decreasing angular velocity ω_0 can be written

$$\mathrm{d}^2\phi/\mathrm{d}t^2 = -(1/\tau^*)\,\mathrm{d}\phi/\mathrm{d}t + \alpha(t) \tag{4}$$

with $\tau^* = I/\beta$. The method assumes a zero average for the stochastic motion.

We multiply through by ϕ, manipulate the resulting equation, and take an ensemble average of thermal molecular accelerations. The result is

$$\mathrm{d}^2\langle\phi^2\rangle/\mathrm{d}t^2 + (1/\tau^*)\,\mathrm{d}\langle\phi^2\rangle/\mathrm{d}t = 2\langle(\mathrm{d}\phi/\mathrm{d}t)^2\rangle. \tag{5}$$

Under the important assumption of thermal equilibrium, this equation can be solved using the autocovariance function [9]. This scales from $\langle(\mathrm{d}\phi/\mathrm{d}t)^2\rangle = kT/I$, the equipartition noise driving function, in equation (5). Taking ensemble averages over some time interval τ, and with simple initial conditions $\langle\phi^2\rangle = \mathrm{d}\langle\phi^2\rangle/\mathrm{d}t = 0$, the variance of phase is found:

$$\langle\phi^2(\tau)\rangle = \phi_N^2 = 2(\tau^* kT/\beta)[\tau/\tau^* - (1 - \mathrm{e}^{-t/\tau^*})] \tag{6}$$

A significant result of this derivation of phase noise of a torsionless pendulum is that phase fluctuations depend in a complex way on the observing time τ, a departure from the random walk of (2). The spectral character depends on the relationship between τ and τ^*. For $\tau \ll \tau^*$ equation (6) becomes

$$\langle\phi^2(\tau)\rangle_{\ll} = kT\tau^2/(\beta\tau^*) \tag{7a}$$

and for $\tau \gg \tau^*$ it becomes

$$\langle\phi^2(\tau)\rangle_{\gg} = 2kT\tau/\beta \tag{7b}$$

the same random walk as equation (2).

In the lightly damped case, (7*a*), the phase is reversible, $\phi_N \sim \tau$, not a random walk. In the heavily damped case, $\phi_N \sim \tau^{1/2}$, a random walk. Smoluchowski pointed out (as discussed in [10], a superb review of stochastic problems), as an explanation for the Brownian motion case, that processes only appear to us as irreversible due to the short time τ in which the system is observed. Zernicke [4] has formally shown this limiting behaviour.

Following Van der Ziel in the calculation for the harmonic oscillator [2], the differential equation (4) now retains the restoring term, and

$$I\,\mathrm{d}^2\phi/\mathrm{d}t^2 + C\phi = -\beta\,\mathrm{d}\phi/\mathrm{d}t + I\alpha(t). \tag{8}$$

As before, $(\mathrm{d}\phi/\mathrm{d}t)(0) = 0$ and $\phi(0) = 0$. Again the equation is multiplied by ϕ and an ensemble average taken for $t > 0$,

$$(1/2)I\,\mathrm{d}^2\langle\phi^2\rangle/\mathrm{d}t^2 + (1/2)\beta\,\mathrm{d}\langle\phi^2\rangle/\mathrm{d}t + C\langle\phi^2\rangle = I\langle(\mathrm{d}\phi/\mathrm{d}t)^2\rangle. \tag{9}$$

A particular solution of (9) results from $C\langle\phi^2\rangle = I\langle(\mathrm{d}\phi/\mathrm{d}t)^2\rangle$, and a transient solution is

$$\langle\phi^2(t)\rangle = A\,\mathrm{e}^{s_1 t} + B\,\mathrm{e}^{s_2 t} \tag{10}$$

where s_1 and s_2 are solutions to the indicial equation

$$(1/2)Is^2 + (1/2)\beta s + C = 0. \tag{11}$$

For the overdamped case, $\tau \gg 2I/\beta$, i.e. $\tau \gg \tau^*$. Then (10) approaches zero exponentially, as s_1 and s_2 both have negative real parts. At long observing times the particular solution holds, and

$$\langle\phi^2(\tau)\rangle = \phi_N^2 = (I/C)\langle(\mathrm{d}\phi/\mathrm{d}t)^2\rangle = kT/C \tag{12}$$

when the equipartition theorem, $I\langle\mathrm{d}\phi/\mathrm{d}t\rangle^2 = kT$, is applied.

No measuring time dependence is present, except for the assumption of a long observing time compared with damping time, i.e. equilibrium of the system. The alternative assumption implies a measurement in the transient regime, to be taken up later.

4. Measurement strategy for determining minimum detectable signal

The predominant use of a torsion pendulum in measuring G is to determine the acceleration of detector masses on the pendulum in response to placement or movement of external source masses. This acceleration can torque a static mode pendulum or it can change the period of a dynamic mode pendulum. The noise couple Γ_N is found from the phase fluctuations of (12) in Γ, i.e. $\langle\Gamma'^2(\tau)\rangle = \langle\Gamma_N^2\rangle = C^2\langle\phi^2(\tau)\rangle$ ($= CkT$ at equilibrium).

A very different practical effect appears in measurements with the lightly damped (not totally) static pendulum. It is driven by seismic or other ambient torques to have an ambient oscillation. When $\beta = 0$ the oscillations will grow without limit for the ideal harmonic oscillator. Braginsky and Manukin [11] went to great lengths to manually drive down this noise in their pendulum, $\tau^* \sim 10^9$ s, before a series of measurements. Once manually damped down, the long natural damping time kept the pendulum oscillations from growing significantly during the few days of measurement. An alternate solution is to accept the oscillations and measure the pendulum period, often called the 'time-of-swing method.' This method is used by many experimenters, the present authors included, and it has a number of favourable features.

But, staying with the static balance, we consider optimizing the measurement sensitivity by estimating the least change in amplitude that can be evidence of a signal acting on the oscillator in the presence of noise. In such cases, the measurement is usually made over one or an integral number of periods, unless the pendulum is well damped.

An example of this is the work of Boynton [12], who integrates the harmonic oscillator spectrum. He employs a trick which uses a filter within the integral [13] to average over (ultimately) N periods summing to a time τ, arriving at

$$\langle\phi^2(\tau)\rangle = 3kT\beta/C^2\tau = 3kT\tau^*/\tau. \tag{13}$$

Here $\tau^* = \beta/C$ for a pendulum in the coercive mode and $\phi_N \sim 1/\tau$, not a random walk in phase. Noise of the couple is given by

$$\langle\Gamma_N^2\rangle = \Gamma_N^2 = C^2\langle\phi^2(\tau)\rangle = 3kT\beta/\tau = 3kTI/\tau^*\tau. \tag{14}$$

This result is lower than the equipartition value CkT by the factor $3/(\omega_0^2\tau^*\tau) = 3/(\omega_0 Q\tau)$ for a measurement duration τ, where $Q = \omega_0\tau^*$ is the pendulum quality factor. McCombie has, by a variety of different methods, arrived at essentially this same result.

5. Noise spectra and the Allan variance

Commonly the couple noise is expressed as a power spectral density, i.e. a fluctuation torque squared per hertz, found by dividing by $\omega_0/2\pi$

$$\Pi_N^2(\tau) = 6\pi kT\beta/\omega_0\tau \tag{15a}$$

$$= 6\pi kT\beta\tau^*/Q\tau. \tag{15b}$$

Results (13) to (15) are integrals over the frequency response of a pendulum and do not give much insight into the self-analysis of measurement mentioned at the start of this paper. The power spectrum of a noise-driven damped harmonic oscillator is

$$\langle\Pi_N^2(\tau)\rangle = 4kT\beta/[(1-y^2)^2 + y^2/Q^2] \tag{16}$$

where $y = \omega/\omega_0$. Figure 2 shows this spectrum for several values of Q, given a pendulum with $\tau^* = 10^6$ s, $I = 400$ g cm^2, and at a temperature of 300 K. This theoretical pendulum is given constant damping β and Q is changed by other means such as derivative feedback.

At low frequency, $y \sim 0$, the spectrum is 'white', i.e. flat, signifying pure stochastic noise, corresponding to the dominance of the first term in the denominator of equation (16). At high frequency the term in y^4 dominates, yielding a slope of -4 in the log–log plot. Finally in the intermediate region about $\omega = \omega_0$, there is a straight-line region of slope -2, corresponding to dominance by the

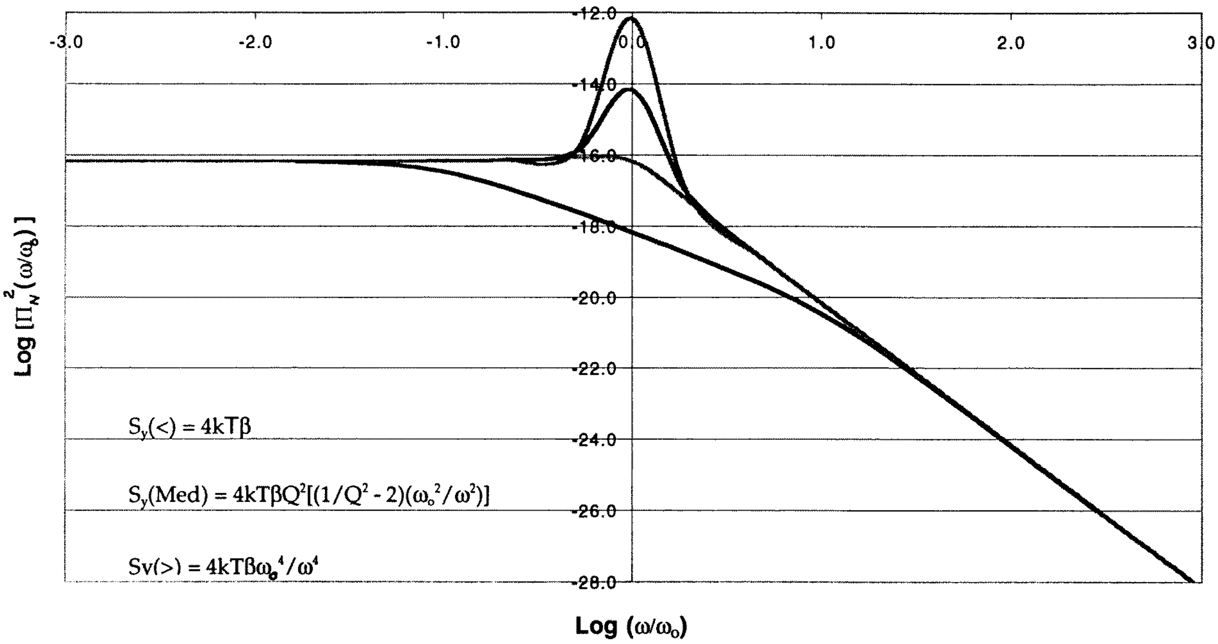

Figure 2. Theoretical spectra of harmonic oscillators having four values of Q but with fixed damping β (most easily arranged with derivative feedback). Power spectra $S_y(\omega)$ are listed for three frequency regimes. $\tau^* = 10^6$ s; $I = 400$ g cm^{-2}; $T = 300$ K; from the bottom to the top curve $Q = 0.1$, 1, 10 and 100.

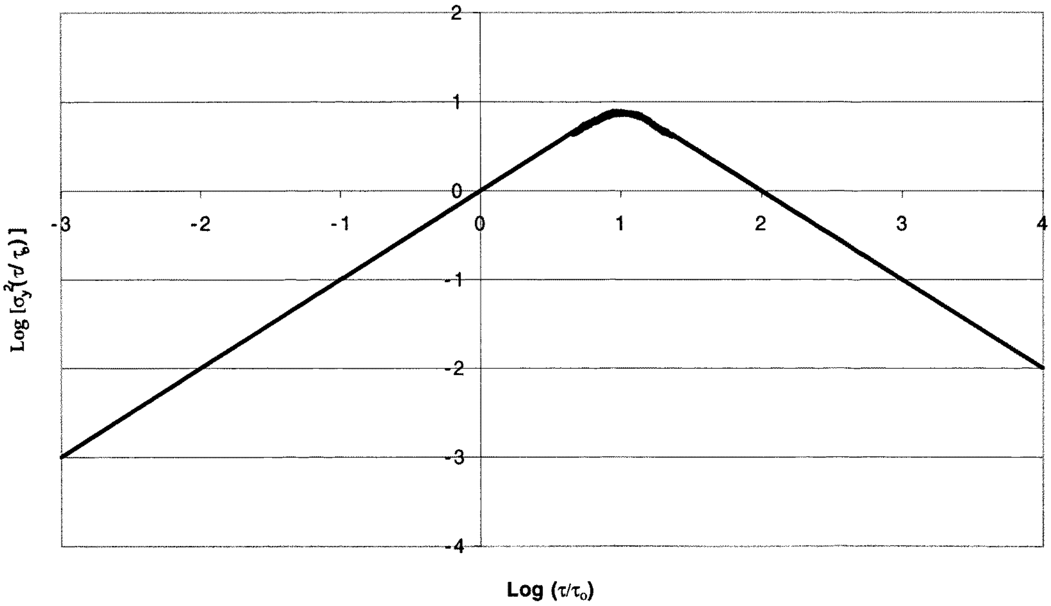

Figure 3. Calculated Allan variance, renormalized, for the pendulum of figure 2. $Q = 0.1$; low- and medium-frequency regimes.

y^2 term, for low Q. The low- and intermediate-frequency regions have the same spectral character as the full torsionless pendulum.

Spectral analysis alone can be cumbersome, and a more efficient statistical tool is available, the 'Allan variance.' This is a formal way of incorporating systematic sampling-time dependence directly into graphical interpretation in the analysis. The method uses a phase variable $x = \phi/\omega_0$ and a dimensionless 'frequency variable' $y = \mathrm{d}(\phi/\omega_0)/\mathrm{d}t = \omega/\omega_0$, the instantaneous fractional angular velocity deviation of the oscillator. Time is proportional to the phase, hence the integral of the frequency, so

$$S_x(\omega) = S_y(\omega)/\omega^2 \tag{17}$$

relates the power spectrum of the variable y to S_x, the power spectrum of the phase deviations.

An estimate of the Allan variance of a measured discrete time series in y (i.e. not the phase) is calculated from

$$\sigma_y^2(\tau) = \{1/[2(M-1)]\} \sum_{k=1}^{N-1} (\bar{y}_{k+1} - \bar{y}_k)^2 \tag{18}$$

where the bar over the y terms indicates an average over integral numbers of y values. (This can also be calculated directly from the x variables [6].) For each increasing number of points taken into an average y_k, there is a new sum associated with a new multiple of the base sample rate, and a

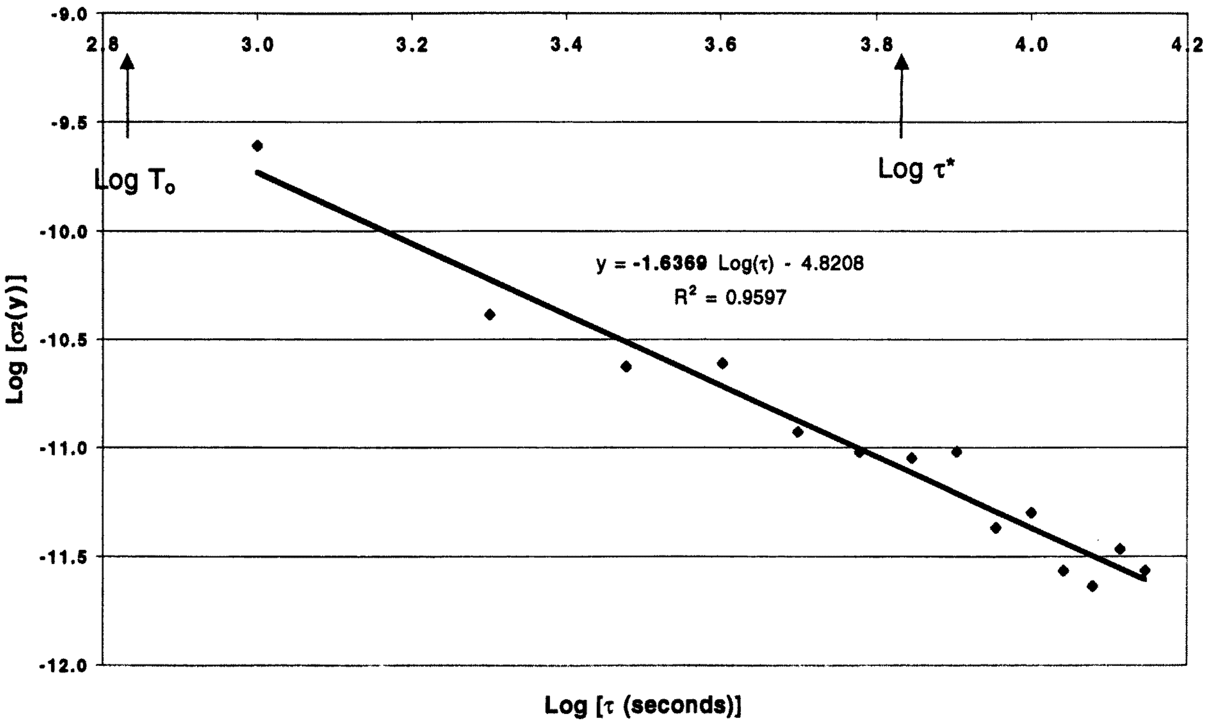

Figure 4. Allan variance of torsion pendulum in static mode, without feedback. $Q = 60$, pendulum period $T_0 = 645$ s, $\tau^* = 7900$ s, sampling interval 1000 s, $C = 0.09$ dyne cm rad^{-1}. $\sigma = 0.038 =$ standard deviation of phase ϕ about the mean; $\tau =$ averaging time; $y = \phi/\omega_0$; $\langle\phi\rangle = 0.0013$. Analogue filtering applied.

new term in a sequence. The progression of these terms will yield information about the correlative character of the noise of the series. Log–log plots of the sequence, with abscissae labelled 'averaging time' τ, are a major analytical tool, with slopes having interpretive significance.

Formally, the Allan variance $\sigma_y^2(2, \tau)$ is defined as the two-sample variance [6, 9, 13], which is a subset of the more general N-sample variance $\sigma_y^2(N, \tau)$ that is an integral over a filtered power spectrum $S_y(\omega)$. The spectral connection of the Allan two-sample variance ($N = 2$) takes the form

$$\sigma_y^2(2, \tau) = 2\int \mathrm{d}\omega\, S_y(\omega)[\sin^2(\omega\tau/2)/(\omega\tau/2)^2] \times\{1 - \sin^2(\omega\tau)/4\sin^2(\omega\tau/2)\}. \qquad (19)$$

This equation is used extensively to relate the spectral and temporal responses of clocks in a more useful way than the simple Fourier transform. It can be seen [9] that $\sigma_y^2(N, \tau)$ and $\sigma_y^2(2, \tau)$ are equal only in the low-frequency regime and N therefore serves as a high-frequency filter.

Regime identification and evaluation starts with power spectra in the variable y, via (19), and the assumption that the power spectrum $S_y(\omega)$ can be represented by a power series in ω

$$S_y(\omega) = \sum h_a\omega^a \qquad (20)$$

in which only a few terms survive, corresponding to the only significant modes of noise in the oscillator under study. A well known term in this series is the flat spectrum, h_0 'white frequency' noise. From (17) there is a power-of-two difference between frequency noise and phase noise designations, i.e. between the power spectra of frequency ω and of phase ϕ (not discussed here). Although measured noise spectra or Allan variances typically follow series having terms with integral powers, sensible cases have been found where $\sigma_y^2(\tau)$ has terms in fractional powers [14].

Table 1. Types of noise [13, 9].

	S_y	σ_y^2
Random walk frequency	h_{-2}/ω^2	$j_1(2\pi)^2\tau/6$
Flicker frequency	h_{-1}/ω	j_0 (ln 2)
White frequency (random walk phase)	h_0	$j_{-1}/(2\tau)$
Flicker phase	$h_1\omega$	$j_{-2}[1.038 + 3\ln(\omega_h\tau)]/(4\pi^2\tau^2)$[a]
White phase	$h_2\omega^2$	$j_{-2}3\omega_h/(4\pi^2\tau^2)$[b]

[a] ω_h: high-frequency cutoff.
[b] This case fails the $\mu = -\alpha - 1$ rule [13].

Inspection of (19) shows that a power series in ω corresponds to another power series in τ

$$\sigma_y^2(\tau) = \sum j_\mu\tau^\mu. \qquad (21)$$

The correspondence between (20) and (21) is $\mu = -\alpha - 1$, except for $\alpha = 2$, white phase noise [13]. Table 1 presents an abbreviated version of published noise categories [13, 9]. These relationships are reasonably well met when noise processes can be separated for specific regimes.

A large and mathematically sophisticated set of publications in the field of precision time references have used the Allan variance to evaluate the physical origin and nature of noise regimes in a clock, as an essential part of understanding their performance. This might also be true to some extent for the torsion pendulum, but the natural time

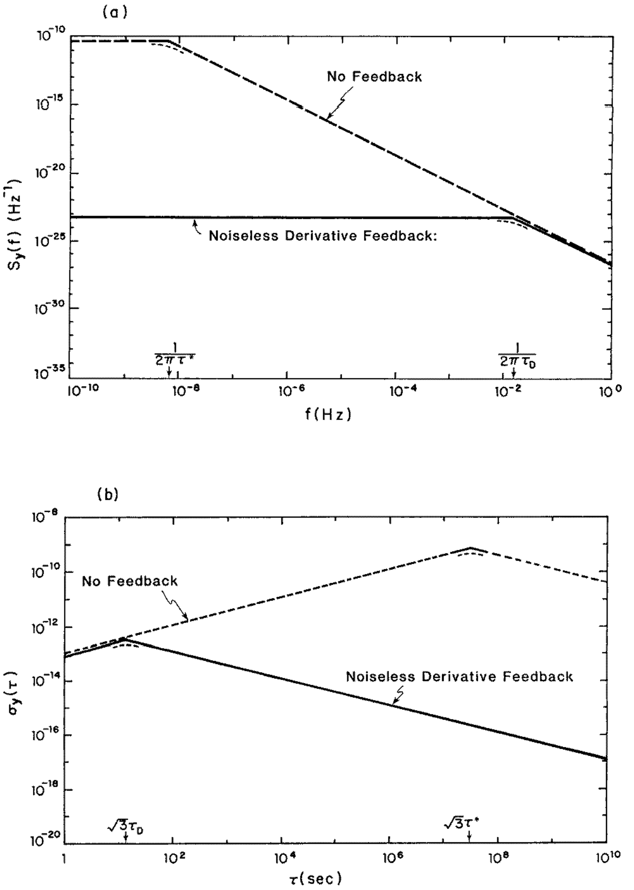

Figure 5. Power spectrum (*a*) and Allan variance (*b*) of low- and medium-frequency regimes of a damped harmonic oscillator, showing the effect of noiseless feedback.

series is that of phase rather than frequency, so equation (17) and its consequences must be applied to use table 1.

In a useful review[15], Vessot shows feedback analysis using both spectra and Allan variance in clocks. This work provides techniques for noise-free filtering, illustrated by Allan variance plots with feedback in the caesium beam resonator. For example, two clocks of different types are locked in a feedback loop in a way to provide overall behaviour having the best temporal qualities of each.

The three regimes of the log–log plot of figure 2 predict three terms in the power spectrum (proportional to $S_y(\omega)$) of a harmonic oscillator of interest to us: at low frequency, $\alpha = 0\,(\mu = -1)$; at intermediate frequency (low Q), $\alpha = -2$ ($\mu = 1$); and at high frequency $\alpha = -4$. The intermediate regime can only provide a linear region if the effective Q is $\ll 1$. Figure 3 is a theoretical Allan variance plot for the low- and medium-frequency regimes of the spectrum, shown for the case $Q = 0.1$. The crossing point of the two regimes is at $1/\tau^2 = (1/Q^2 - 1) \sim 0.01$, or $\tau = 10$. (This scaling uses a normalized $\tau = \tau/\tau_0$ where $\omega_0 = 2\phi/\omega_0$ and the magnitude $4kT\beta$ is taken to be unity.)

It is apparent how to use this equation in the time-of-swing mode and the results from one example of such a measurement [16] show frequency fluctuations $(S_y(\tau))^{1/2}$ with a spectral slope of -1 on a log–log plot, a random walk in frequency, the expected behaviour for such measurements at low sampling rates.

While there is a large body of literature showing successful behaviour of these analyses, failure of a pendulum to exhibit integral slopes on log–log plots can signify other processes beside thermal noise. Figure 4 is an example of an Allan variance plot by the authors, for a static mode open loop pendulum, having a Q of about 60 and period of 645 s. For basic experimental reasons the sampling interval was 1000 s, so all data were aliased and a 1000 s analogue low pass filter used. Correspondingly, the frequency spectrum $S_y(\omega)$ ends well below ω_0, with an uninteresting slope of -0.2. The slope of -1.6 in the Allan variance plot, where $\mu = -1.0$ was expected, is evidence of distortion from our sampling–filtering method.

6. The nonequilibrium torsion pendulum

A torsion pendulum will operate in nonequilibrium states if it is consistently in a transient state when observed. If the damping is very light, statistical inference must deal with failure to be in equilibrium. The lightly damped pendulum will be found in equilibrium for a vanishing fraction of the measurements, when practical values of measuring time interval τ are used. For example, Braginsky and Manukin [11] carried out an equivalence principle experiment with τ^* estimated to be greater than 10^9 s, and Braginsky *et al* [17] considered future pendulums having $\tau^* \sim 10^{13}$ s.

Chen and Cook [18] recognized the nonequilibrium condition of the lightly damped pendulum and produced analyses of the statistically treated transient situation for such oscillators. These analyses add to earlier work by Chandrasekhar on the noisy harmonic oscillator [10] and carry those results towards use with the torsion pendulum. For the least detectable signal in discrete measurements, in our variables, their result is

$$\langle\phi^2(\tau)\rangle = (kT/C)\{1 - [\cos\omega_1\tau + (1/2\tau^*\omega_1))\sin\omega_1\tau]\,\mathrm{e}^{-\tau/\tau^*}\}. \quad (22)$$

This is the Einstein result (12) multiplied by a transient factor, which can be quite significant. As Chen and Cook point out, in the related resonant case, the calculation increases the variance by three orders of magnitude over an original estimate [17]. (Regarding other types of measurements considered in their paper we mention the example of frequency or period measurement, in which case the amplitude of the pendulum swing ω_0 is important in the evaluation of a limit of sensitivity.) In our static case, and for the limit of low damping with $\omega_1 t \ll 1$, $\langle\phi^2(\tau)\rangle \sim (kT\omega_0^2/2C)\tau^2$ from which

$$\sigma_y^2(\tau) = kT/2C \quad (23)$$

and $S_y \sim 1/\omega$, i.e. flicker frequency noise, in agreement with (15).

7. Derivative feedback

If a lightly damped pendulum position variance increases with observing time (random walk frequency), how are we to improve a measurement? We consider a pendulum with negative derivative feedback. This is consistent with positive

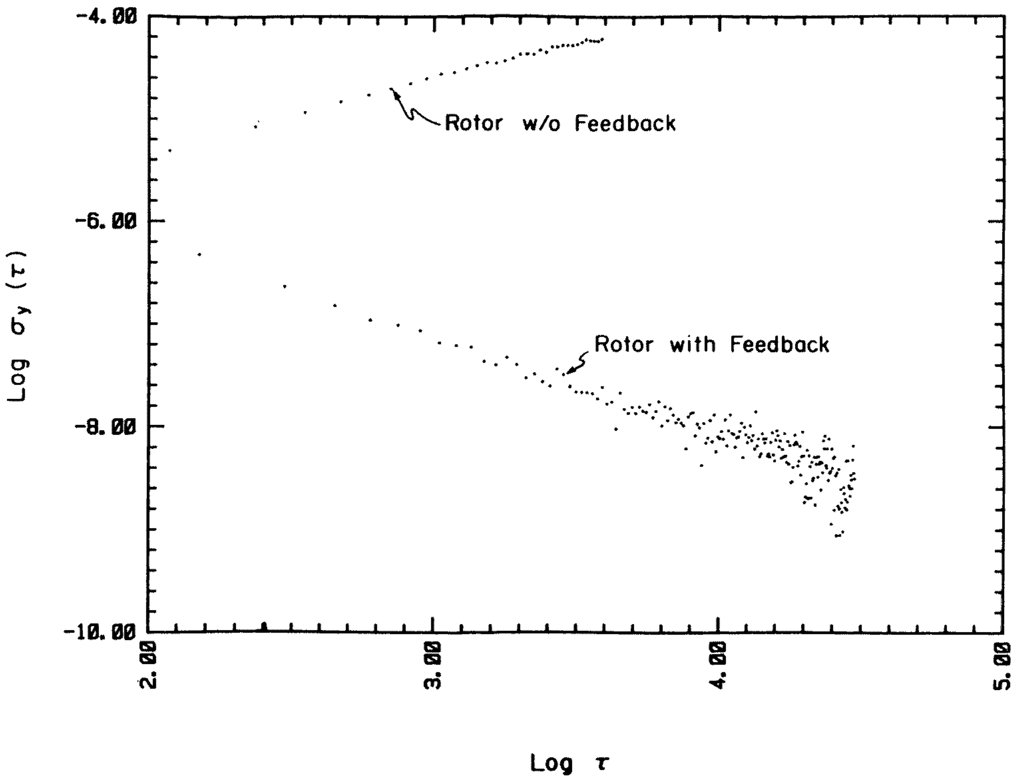

Figure 6. Allan variance of data from a precision rotor, with and without negative derivative feedback.

damping. We consider ideal noiseless derivative feedback in the analysis, using the method of Kittel [19]. If the feedback is given a coefficient g, then the differential equation of motion (1) becomes

$$I\,\mathrm{d}^2\phi/\mathrm{d}t^2 + \beta\,\mathrm{d}\phi/\mathrm{d}t + C\phi = G(\omega) - g\,\mathrm{d}\phi/\mathrm{d}t \qquad (24)$$

for a noise spectrum $G(\omega)$ driving the pendulum.

For a single Fourier component of the noise, $G(\omega)$, given by $A\,\mathrm{e}^{\mathrm{i}\omega t}$, this is

$$I\,\mathrm{d}^2\phi/\mathrm{d}t^2 + (\beta + g)\,\mathrm{d}\phi/\mathrm{d}t + C\phi = A\,\mathrm{e}^{\mathrm{i}\omega t}. \qquad (25)$$

The steady state solution to (25) yields a transfer function $Z(\omega)$ written

$$\phi(\omega) = A\,\mathrm{e}^{\mathrm{i}\omega t}/[C - I\omega^2 + \mathrm{i}\omega(\beta + g)] = Z(\omega)A\,\mathrm{e}^{\mathrm{i}\omega t} \quad (26)$$

and

$$Z(\omega) = 1/[C - I\omega^2 + \mathrm{i}\omega(\beta + g)]. \qquad (27)$$

The output variance arising from an input spectrum G_N, for a linear system, is therefore

$$\langle\phi^2\rangle = \int G_N|Z(\omega)|^2\,\mathrm{d}\omega \qquad (28)$$

taking all noise components to be independent in phase and amplitude.

Two particular input spectra are of interest here. If G_N is white frequency noise and independent of frequency, i.e. constant G, then (28) becomes

$$\langle\phi^2\rangle = G/[4(\beta + g)C]. \qquad (29)$$

This can be normalized from the equipartition theorem $(1/2)C\langle\phi^2\rangle = (1/2)\ kT$, and we arrive at the Nyquist relation:

$$G = 4\beta kT \qquad (30)$$

(The feedback g is noiseless and makes no contribution to β.)

Finally, the deflection noise variance with feedback is

$$\langle\phi^2\rangle = (kT/C)[\beta/(\beta + g)] = kT'/C. \qquad (31)$$

$T' = \beta/(\beta + g)$ is an effective temperature lower than T, called the electronic cooling effect. Figure 5 shows the effect of derivative feedback in theoretical spectra and Allan variance plots for the low- and middle-frequency regimes of a pendulum. Figure 6 shows a plot of σ_y from data taken with a precision rotor, which has the same spectral behaviour as the low- and medium-frequency regimes of the damped harmonic oscillator.

The quantity G is not the magnitude of noise on a lightly damped harmonic oscillator, which instead is a narrow band about a frequency $\omega_0 = (C/I)^{1/2}$. In that case, the integral (28) gives us the variance

$$\langle\phi^2\rangle = G(\omega_0)/[4(\beta + g)C] \qquad (32)$$

since the response is nearly a delta function at ω_0.

The above reduction of the effective temperature is shown to lower the noise calculated by, for example, (13) or (15), according to T' just as if the actual temperature were lower. There is no conceptual limit to this reduction of the noise, but there is a limit to the signal-to-noise ratio. As the noise is reduced, the response becomes so sluggish that a signal cannot cause a response for a time much greater than that required without the feedback, or in particular, greater than some allowed measuring time τ. The limit of usefulness is reached at very nearly the point where the system operates effectively as if critically damped [20]. If we increase g beyond the point where the signal rise time is greater than that for critical damping, the noise will go down but a signal cannot be read in that optimal time τ. This combination of effects is the origin of the McCombie natural damping limit. McCombie has shown this more generally (but not specifically with feedback) and we can take it to be a useful absolute limit. It leaves us with the choice of using feedback as a filter to match the spectrum of the signal.

We conclude by pointing out that many of the careful strategies stated in the literature, e.g. [11], can be subsumed into this concept. Noiseless, negative derivative feedback gives us the opportunity to optimize our measurement, even with small natural damping, but we can only use as much of it as needed to retain our necessary response time τ.

References

[1] Cavendish H 1798 *Phil. Trans. R. Soc.* **88** 469

[2] Van der Ziel A 1980 *Adv. Electr. Electr. Phys.* **50** 351

[3] McCombie C W 1953 *Rep. Prog. Phys.* **16** 266
Jones R V 1951 *Proc. Phys. Soc.* B **64** 469

[4] Zernicke F 1928 *Handbuch der Physik* vol 3 (Berlin: Springer) p 419

[5] Kappler E 1931 *Ann. Phys., Lpz.* **11** 233
Uhlenbeck G E and Goudsmit S 1929 *Phys. Rev.* **34** 145

[6] Allan D W 1987 *IEEE Trans. Instrum. Meas.* **36** 646
Allan D W 1966 *Proc. IEEE* **54** 221

[7] von Smoluchowski M 1906 *Ann. Phys., Lpz* **21** 756

[8] Langevin P 1908 *C. R. Acad. Sci.* **146** 530

[9] Bernard B E, Winkler L I and Ritter R C 1985 *Metrologia* **21** 115

[10] Chandrasekhar S 1943 *Rev. Mod. Phys.* **15** 1

[11] Braginsky V B and Manukin A B 1997 *Measurement of Weak Forces in Physics Experiments* (Chicago Press) p 135

[12] Boynton P 1988 *5th Force-Neutrino Physics* ed O Fackler and J Tran (Gif-sur-Yvette: Frontieres) p 431

[13] Barnes J A *et al* 1971 *IEEE Trans. Instrum. Meas.* **20** 105

[14] Allan D W 1985 personal communication

[15] Vessot R F C 1973 *Experimental Gravitation* LVI Corso (New York: Academic) p 111

[16] Gillies G T and Ritter R C 1993 *Rev. Sci. Instrum.* **64** 283

[17] Braginsky V B, Caves C M and Thorne K S 1977 *Phys. Rev.* D **15** 2047

[18] Chen Y T and Cook A 1990 *Class. Quant. Grav.* **7** 1225

[19] Kittel C 1958 *Elementary Statistical Physics* (New York: Wiley) p 147

[20] Ritter R C and Gillies G T 1985 *Phys. Rev.* A **31** 995