A Method to Measure Newton's Gravitational Constant

M. P. Fitzgerald, T. R. Armstrong, R. B. Hurst and A. C. Corney

Abstract. This paper describes the design and testing of a method to measure the Newtonian gravitational constant G. The method uses a torsion balance in which a small cylindrical mass is suspended from a slender fibre. Two large cylindrical masses produce a gravitationally induced torque on the small mass. This induced torque is balanced by the electrostatic torque produced by an electrometer. A servo-control system adjusts the electrostatically induced torque so that the fibre does not twist during the measurement. The torque produced by the electrometer is calibrated by an acceleration method that uses torques much larger than those produced by the gravitational attraction of the large masses. Other features of the method include a reduction in sensitivity of the measurement to density variations in the suspended mass, a measurement strategy which reduces the effects of drift in the measurement and computer control of the measurement. Preliminary measurements of the terms used to calculate G showing an uncertainty of 2,5 parts in 10^4 can currently be achieved. The limiting factor at present is the variation in the gravitational potential seen as the apparatus accelerates. Indications are that further modifications and measurements will provide a value of G with an uncertainty of about 1 part in 10^4.

1. Introduction

Newton's law of gravity, formulated in 1683, states that the force of attraction F between two spherical masses M and m whose centres are separated by a distance r is given by

$$F = GMm/r^2, \tag{1}$$

where G is Newton's gravitational constant.

Measurements to determine the value of G have been carried out since the early 1700s. Probably the most famous measurement is that made by Cavendish in the eighteenth century [1]. The value of G obtained by Cavendish, using a torsion balance, was within 1 % of the value accepted today. Since then, the uncertainty in the measured value of G has decreased only slowly due to experimental difficulties in measuring the small gravitational forces produced in the laboratory. The 1986 recommended value of G is $6{,}672\,59(85) \times 10^{-11}$ $\mathrm{m^3{\cdot}kg^{-1}{\cdot}s^{-2}}$ [2]. This value, with an uncertainty of 128 parts in 10^6, is among the least well known of the fundamental constants. The recommended value is based on the result of just one measurement, the uncertainty of the value being twice that of the measurement [2].

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The history of G measurements has been fully covered in several review articles [3-7]. Most of the recent measurements of G have used a torsion balance. (An exception is the pendular Fabry-Perot cavity measurement of Schurr et al. [8].) A torsion balance is a small mass suspended by a slender fibre [6]. The gravitational attraction of large masses placed near the small mass produces a torque about the fibre axis. Angular movements of the small mass, in response to this torque, are typically detected using an optical lever. The torsion balance has the great advantage that the small mass is able to respond to the weak gravitationally induced torque in the presence of the very large gravitational attraction of the Earth. For example, the gravitational force between two closely spaced 1 kg spheres of a dense material such as platinum is about 3×10^{-8} N. This is 3×10^{-9} times the Earth's gravitational pull on each sphere.

High-precision measurements of G with torsion balances include the well-known experiments of Heyl [9], Heyl and Chrzanowski [10] and Luther and Towler [11]. These measurements used a torsion balance in an oscillating mode in which the gravitationally induced torque changes the period of oscillation of the balance. In these measurements the fibre twists continuously which may cause changes in the rest point of the balance. In addition, the small mass oscillates in a non-linear gravitational potential making it difficult to calculate the gravitational torque.

An alternative to the oscillating mode is to operate the torsion balance in a compensated mode in which the angular deflection of the fibre is actively nulled using a servo-control loop. The advantages of operating in this way are that the torsion fibre is not twisted and that the response of the torsion balance can be modified by the action of feed-back from the control loop. An example of this method is the recent measurement made at the PTB [12] where an electrostatically induced torque is used to balance the gravitational attraction on the small mass. In this measurement the fibre was replaced by a mercury bearing. Another example of a compensated method is that developed by Beams and his colleagues [13, 14] where the acceleration of the small mass due to the attracting masses is balanced and measured by accelerating the entire apparatus. In this last measurement, static gravitational gradients seen by the apparatus as it rotated proved to be a limiting factor. All methods are affected by vibration, stray gravitational attractions, changes in temperature and density variations in the masses.

The method described in this paper is designed to overcome most of the difficulties encountered in previous measurements of G. It is based on an electrostatically compensated torsion balance so that problems due to twisting of the fibre and oscillations of the small mass are eliminated. The electrometer that provides the electrostatic compensation is calibrated in a separate measurement by an acceleration method similar to that used by Beams [13, 14], but which uses higher accelerations to reduce the effect of static gravitational gradients. The aim of the measurement is to obtain a value of G with an uncertainty of 1 part in 10^4 or better.

The design of the method and the optimization of the dimensions is described in Section 2. The apparatus and method are described in more detail in Sections 3 and 4, and the results of measurements designed to test the performance of the method are discussed in Section 5.

2. Method

The starting point for the design was to base the measurement on a torsion balance using a fibre. To overcome the problems experienced in previous torsion balance measurements the following requirements were considered important. These are:

(a) The fibre must not twist during the measurement to reduce creep of the fibre rest point [15].

(b) The small mass must remain stationary when measuring the gravitational attraction to eliminate the effects of static gravitational gradients.

(c) The uncertainty requirements for dimensional measurements should be minimized by choosing simple geometries and working near extrema in the induced gravitational and electrostatic torques.

(d) The measurement strategy should reduce the effects of drifts in the signal due to slow changes such as those caused by movement in the fibre rest point, dynamic gravitational gradients and temperature changes during the measurement.

(e) The measurement should be insensitive to density variations of the masses, especially for the small mass.

A schematic view of the apparatus is shown in Figure 1. Angular motions of the small mass system are detected by an autocollimator viewing the mirror attached to the small mass. A servo-control system adjusts the voltage on the electrometer to oppose any angular movement of the small mass system. The small mass system consists of the small mass and a mirror. The electrometer, a variation of the quadrant electrometer, was developed to allow electrostatic control of the small mass system (see Section 3.2). It has the feature that the moving "vane" is the small mass. This avoids extra electrode structures that load the fibre and complicate the calculation of the gravitational attraction.

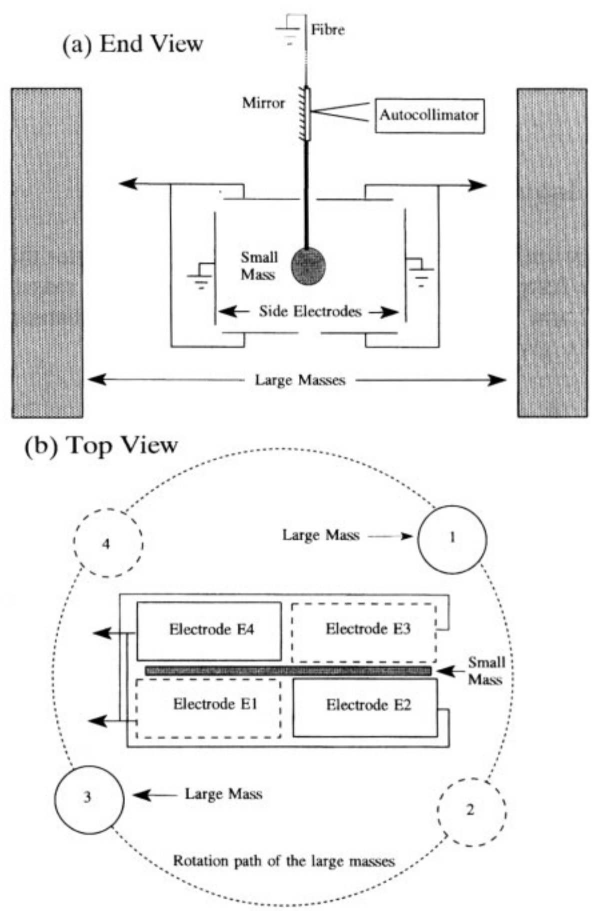

Figure 1. A schematic representation of the torsion balance and the electrometer.

Two large masses provide the gravitational attraction on the small mass system. These are positioned so that the small mass system is midway between the two large mass centres. By using two large masses, the gravitational torque induced on the small mass system depends mainly on the separation between the large masses and only weakly on the separation between each large mass and the small mass system. This holds true as long as the small system is approximately midway between the masses. The advantage is that the distance between the two large masses can be measured more accurately than the distance from each mass to the small mass system. Cylindrical forms were chosen for both the small and the large masses so that they could be manufactured to a high accuracy. The cylinder is also a good shape for accurate dimensional measurement.

The measurement of the gravitational signal involves stopping the large masses at each of four positions as they rotate about the fibre axis. These positions correspond to the angles at which the induced gravitational torque is at a maximum. Calculations show that at these angles the torque varies by 1 part in 10^5 for angular movements of ± 9 mrad. The control voltage on the electrometer electrodes is measured at each position. The induced gravitational torque Γ_G and electrometer voltage V_G are related by

$$\Gamma_G = G\,K = (\mathrm{d}C/\mathrm{d}\theta)\, V_G^2/2, \tag{2}$$

where K is the calculated gravitational torque between the two large masses and the small mass system divided by G, and $\mathrm{d}C/\mathrm{d}\theta$ is the change in electrometer capacitance with angular position of the small mass system. The time taken to measure V_G at each maximum position is chosen so that periodic torques introduced by coupling between the horizontal and torsional pendulum modes of the torsion balance [16] are averaged over an integral cycle. The data obtained from each of the four positions as the large masses rotate about the fibre are then averaged so that the effects of drifts in the signal are reduced. The frequent rotation of the large masses about the fibre axis results in a periodic reversal of the induced gravitational torque. This reduces errors in the measurement caused by background torques.

The final term to be measured in (2) is the electrometer term $\mathrm{d}C/\mathrm{d}\theta$. Direct measurement of this term by angle and capacitance measurement appeared difficult for a practical electrometer design. Instead the method chosen is a variation on the acceleration method used by Beams [13, 14]. To measure $\mathrm{d}C/\mathrm{d}\theta$, the large masses are removed and the servo-control loop maintaining the small mass system at a constant angular position is turned off. A constant voltage V_A is then applied to the electrometer to produce a torque on the small mass system. The whole apparatus is accelerated so that the small mass system remains stationary with respect to the electrometer. A second servo-control loop, using the output of the autocollimator viewing the small mass system, controls the acceleration rate to ensure that this condition is maintained. A value for $\mathrm{d}C/\mathrm{d}\theta$ is obtained from the angular acceleration $\ddot{\alpha}$ and the moment of inertia I of the small mass system using

$$\mathrm{d}C/\mathrm{d}\theta = 2\,(I\ddot{\alpha})/V_A^2. \tag{3}$$

The big improvement in this method over that of Beams [13, 14] is that V_A is chosen to be greater than V_G so that the induced torque is many times larger than Γ_G (V_A^2 is typically 300 times V_G^2). This significantly reduces the effects of any static gravitational gradient during acceleration of the apparatus.

By combining (2) and (3), G is obtained from

$$G = V_G^2\,(\ddot{\alpha}/V_A^2)\,(I/K). \tag{4}$$

The ratio I/K in (4) provides further justification for using the acceleration method. It is shown in Section 4.1 that terms common to both I and K considerably reduce the requirements to measure the small mass system dimensions, mass and density uniformity. This point, while commented on in a discussion at the end of a paper by Towler et al. [14], does not appear to have been widely exploited. When using the oscillation method, such as that used by Luther and Towler [11], a similar ratio occurs in the calculation of G.

Once the experimental method was decided, the next step was to design the apparatus so as to optimize the signal due to the gravitational attraction. This involved choosing the dimensions and materials for the fibre and the small and large masses.

2.1 Fibre

The best fibre is one which has the following properties:

(a) It can carry a high load so that the mass of the small mass system and therefore the gravitationally induced torque is maximized.

(b) It has a low torsional stiffness to ensure that the torsion balance has sufficient sensitivity to detect the small torque signals.

(c) The creep rate is low to reduce drifts in the signal.

(d) It is electrically conducting so that the electrometer may function correctly.

The above properties are controlled by the fibre material, shape and dimensions. The most suitable material is one which has a high ratio of tensile strength to shear modulus and a low creep rate. A circular cross-section was chosen for the fibre shape because, although there are cross-sectional shapes that are torsionally weaker, such fibres were not available with small enough dimensions.

The choice of fibre diameter ϕ_f determines the maximum size of the suspended load and the angular control required for the small mass system position. The angular control is limited mainly by the resolution of the autocollimator used to monitor the angular position of the small mass system. The standard equation for a torque acting on a cylindrical fibre shows that the angle of twist produced is proportional to Γ_a/ϕ_f^{-4}, where Γ_a is the applied torque [17]. In addition, the maximum value of Γ_a is proportional to the maximum suspended load which scales as ϕ_f^2. Combining these two ratios shows that the angular resolution required for the autocollimator varies as ϕ_f^{-2}. The largest fibre diameter is therefore fixed by the resolution of the autocollimator that measures the angular displacement of the small mass system. The size of the fibre diameter also puts an upper limit on the mass of the small mass system.

2.2 Small mass system

The size of the gravitational torque that can be induced on a cylindrical small mass system, of constant mass, depends strongly on the length of the cylinder. To choose the small mass system length, the maximum torque produced by the gravitational attraction of the large masses as a function of small mass system length was calculated numerically. In these calculations the separation of the large masses is increased with increasing small mass length. This is to ensure that the large masses are just able to rotate around the outside of the vacuum chamber that surrounds the small mass system. The results of these calculations are shown in Figure 2. It can be seen that the gravitationally induced torque eventually reaches a plateau with increasing small mass system length. However this plateau occurs at a length that is excessively long. The final length of the small mass was therefore limited by the following factors:

(a) Cylinders with a large length to diameter ratio are difficult to manufacture accurately.

(b) The sensitivity to torques induced by background gravitational disturbances increases as the square of the small mass system length (see Appendix 1).

(c) The sensitivity to torques produced by small temperature differentials around the small mass system depends on the small mass system length to the power of 3/2. These temperature differentials cause pressure differences and molecular flow effects (such as the Knudsen effect) that produce forces which act on the cross-sectional area of the small mass system. The torque produced is therefore proportional to the product of the cross-sectional area (which for a cylinder of constant mass increases as length to the power of 1/2) and length of the small mass system.

The limiting factor for this measurement is the length to diameter ratio. The final length was chosen so that the length to diameter ratio was about 25. With this ratio, the torques on the small mass system produced by the Knudsen effect can be as large as 1 part in 10^4 of Γ_G (assuming a temperature stability of 10 mK and a base pressure of 3×10^{-5} Pa). The broken line in Figure 2 shows the size of the torque produced by the Knudsen effect. In practice these torques can be avoided by keeping the surrounding gas pressure high enough that the gas mean free path is smaller than the dimensions of the small mass system [8].

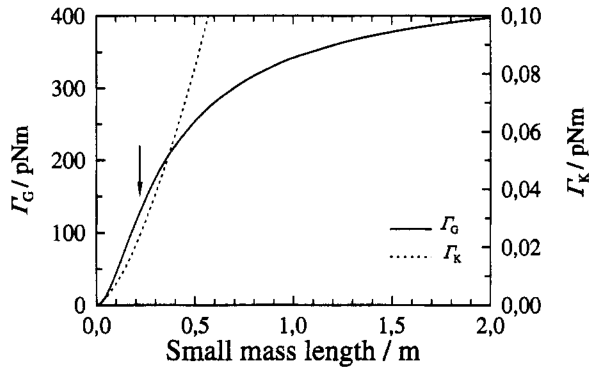

Figure 2. Maximum calculated gravitationally induced torque Γ_G and torque due to Knudsen forces Γ_K versus small mass length. The arrow indicates the chosen small mass length.

2.3 Large masses

The large mass spacing is determined by the length of the small mass so the remaining dimensions to choose are the large mass diameter and length. While it might be expected from (1) that the torque would be proportional to the large mass diameter squared, calculations show that the torque is approximately proportional to the diameter. This is because increasing the diameter also increases the distance between the large masses. For constant large mass diameter, the torque approaches a constant value for increasing large mass length. Therefore the large mass length and diameter were chosen so that the torque is at a plateau for increasing length and the mass is near 30 kg for ease of handling. The size is also sufficiently small to ensure good uniformity of the density.

3. Apparatus

The previous section shows how the gravitational signal acting on the torsion balance was optimized. In practice, the design required several iterations of the procedure to allow such practical considerations as the availability of materials and equipment to be introduced. The present section gives a detailed description of the final apparatus and its dimensions.

3.1 Torsion balance

A summary of the dimensions of the current torsion balance is shown in Table 1. These dimensions are the optimum values calculated using the method of Section 2 except that the torsion fibre diameter is currently twice the optimum value for the autocollimator. This value was chosen to make the apparatus more robust during the setting-up period.

Tungsten was chosen for the fibre as it is a readily available material with a high ratio of load to torsional stiffness [18]. It also has a proven low creep characteristic when treated correctly [15]. Copper (99,99 % pure) was selected for the small mass as a common material with good uniformity of density, low magnetic susceptibility and good electrical conductivity [18].

Table 1. The dimensions of the torsion balance components. Also shown is the relative standard uncertainty (1 σ) for each component.

Component	Final value	10^6 × relative standard uncertainty (1 σ)
Cu small mass		
Length/mm	220,042	1,5
Diameter/mm	7,929	1
Density (kg·m^{-3})	8 913,8	5
Mass/g	96,747	*
SS large mass		
Length/mm	437,80 437,73	10
Diameter/mm	101,039 101,036	10
Density (kg·m^{-3})	7 947,8	4
Mass/g	27 894,9 27 891,8	5
Spacing/mm	435,128	20
Angle/rad	0,953	30
W Fibre		
Diameter/mm	0,05	*
Length/mm	1 011	*
Autocollimator		
Resolution/µrad	0,2	*

* Negligible effect on the final value of G.

The large masses are made from a single bar of austenitic 316 stainless steel. This material was selected for its low cost, ready availability and low magnetic susceptibility. The masses were heat treated before final finishing to reduce their magnetic susceptibility. The large masses are kinematically mounted on a solid aluminium turntable centred on the fibre axis. There are two sets of mounting positions so that the large masses can be shifted with respect to the turntable. This allows any density variations in the turntable to the averaged.

The whole apparatus, excluding the two large masses, is enclosed in an aluminium vacuum chamber and the pressure can be maintained at less than 10^{-4} Pa. The apparatus is also mounted on an air bearing, with a circular run-out of less than 1 µm, to enable the apparatus to rotate. During the electrometer calibration, the apparatus is driven by a 50,000 step per revolution micro-stepping motor through a 360:1 worm wheel drive. Each step of the motor represents an angular step of the apparatus of 0,4 µrad. Measurements of the rotation angle of the apparatus α versus time are made using an eight-sided polygon attached to the rotating part of the apparatus. The polygon faces are detected by an autocollimator [19] which triggers a counter connected to the computer to obtain the timing measurements.

3.2 Electrometer

The electrometer is shown in Figure 1. It differs from a quadrant electrometer in that earthed side electrodes have been incorporated and the moving "vane" of the electrometer is the small mass. The electrometer has two sides each with two pairs of top and bottom electrodes (i.e. electrode pairs E1 and E3, E2 and E4 as indicated in Figure 1b). A torque can be produced in either direction by applying voltages to the appropriate side while connecting the unused side to earth. The entire electrode system is fully enclosed in an earthed metal box. This geometry was arrived at experimentally by building models of the electrometer and measuring the change in capacitance with angle.

The aim in the electrometer design was to minimize variations of the electrostatic torque constant for small movements of the small mass system. The constant is relatively insensitive to translations (1 part in 10^5 for ±0,5 mm movements) but as the moving electrode is a cylinder the constant $\mathrm{d}C/\mathrm{d}\theta$ depends strongly on angle. However there is a region in the electrometer where $\mathrm{d}C/\mathrm{d}\theta$ reaches a maximum and therefore varies less rapidly with angle. This maximum position can be controlled by adjusting the earthed side electrodes. These electrodes have to be adjusted so that the position of the maximum in $\mathrm{d}C/\mathrm{d}\theta$ occurs at the same angle for both sides of the electrometer. Currently the electrometer is adjusted so that $\mathrm{d}C/\mathrm{d}\theta$ varies by 1 part in 10^5 for angular movements of ±0,5 µrad.

The final dimensions of the electrometer, determined experimentally, are 80 mm between top and bottom electrodes and 100 mm between side electrodes. The resulting torque constant of about 1 pF·rad^{-1} requires V_G to be about 10 V, a convenient signal to measure and one that is large with respect to the differences in the electrometer electrode contact potentials.

3.3 Experimental control

A block diagram of the control and measurement equipment in shown in Figure 3. All aspects of the measurement are controlled by a PC computer. This eliminates gravitational disturbances due to the presence of an operator. The position of the small mass system is controlled by a proportional, integral and differential (PID) control loop. This control loop is carried out in software to allow for the optimization of its performance in each phase of the measurement. The PID loop differential time constants used in the measurement range between 20 s and 400 s while the integral time constants range from 15 s to 130 s. The output of the autocollimator is sampled with an analogue-to-digital convertor every 0,2 s. By averaging the readings for an integral number of simple pendulum cycles (about 2,2 s) an angular error signal independent of small mass system pendulum modes is obtained.

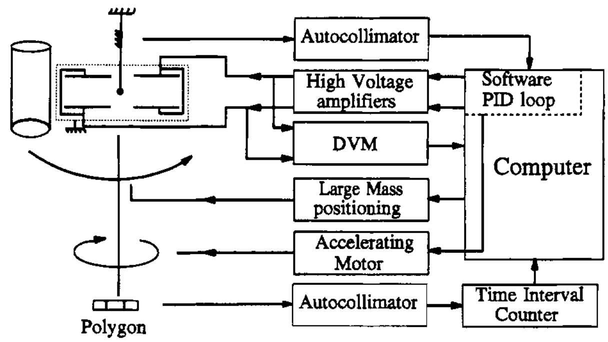

Figure 3. Block diagram of the control and measurement equipment.

In the measurement of Γ_G any required corrections to the electrometer voltage are fed to the appropriate digital-to-analogue (D-A) convertor that drives high-voltage amplifiers connected to the appropriate electrodes of the electrometer. Two 12-bit D-A convertors have been combined to obtain a resolution of 25 μV in the voltages applied to the electrometer. The applied voltage is measured using a digital voltmeter. Although not shown in the diagram, computer driven relays are used to ensure that the unused electrodes of the electrometer are held at the same potential as the small mass system.

In the electrometer calibration phase of the measurement, the computer applies the appropriate constant voltage (up to 270 V) to the electrometer electrodes. The signal from the autocollimator, after processing by a PID control loop, is fed to a D-A convertor which drives a voltage-to-frequency convertor. The output from the voltage-to-frequency convertor provides pulses to the 50,000 steps per revolution micro-stepping motor that accelerates the apparatus.

4. Measurement of G

The measurement of G involves determining the three quantities (I/K), V_G^2 and $(\ddot{\alpha}/V_A^2)$ defined in (4).

4.1 Ratio I/K

The ratio I/K is calculated numerically from length, mass and position measurements on the small mass system and the large masses. The moment of inertia I of the small mass system is dominated by the copper cylinder, with the mirror system contributing only 0,1 % to the total. The term K is calculated using the results of measurements made on the small mass system and the large masses. The calculation is a volume integration over the masses using Heyl's [9] formula for the gravitational attraction of a finite cylinder at any external point. As a check on the calculations, a point to point numerical integration over the volume of the masses has also been made.

To determine the sensitivity of I/K to the accuracy of the dimensional and mass measurements it is convenient to treat the small mass and large masses separately.

4.1.1 Sensitivity of I/K to measurements of the small mass

The sensitivity of I/K to errors in measurements of the small mass was determined by calculations in which the small mass dimensions and density were varied. The constraint for these calculations was that the mass of the small mass remained constant as each component was varied. It was found that although both I and K depend strongly on the small mass length, form, mass and uniformity of density the ratio I/K does not. For example, the length of the small mass must be known to about 1 μm to determine I or K to 1 part in 10^5. However the ratio I/K remains constant to 1 part in 10^5 for variations in length of 20 μm.

A similar effect was found for variations in the density of the small mass. Modelled variations of density of 1 part in 10^4 along the small mass axis change I or K by up to 3 parts in 10^5. The ratio I/K however remains constant to better than 2 parts in 10^6. Measurements made on samples cut from the ends of the copper bar from which the small mass was made failed to detect differences in density at a measurement uncertainty of 3 parts in 10^5.

4.1.2 Sensitivity of I/K to measurements of the large masses

The errors in the measurement of the large mass dimensions and mass affect the calculation of K. There are two dominant factors. These are the large mass spacing and the uniformity of the density.

Large mass spacing. The critical measurement is the separation of the two large masses at the height of the suspended small mass. This separation, nominally 435 mm, must be known to within ± 2 μm to keep the uncertainty in K below 1 part in 10^5.

To achieve this uncertainty the length comparator gauge shown in Figure 4 was developed. The large aluminium plate is suspended by three wires from a movable frame. The comparator is positioned so that the cylindrical stops and probe ends are just in contact with the large masses. The probe positions are then adjusted as indicated by the dashed arrows in the figure so that they are at the necessary maximum and minimum positions. A variable gauging force is applied to determine the spacing for no load. The gauge setting is then compared with a combination of calibrated length bars and the final spacing is determined after subtracting half the large mass diameters. The large mass diameters are measured by comparison with ring gauges using a computer-controlled three-axis measuring machine. To ensure that the large spacing is constant during the measurement, the temperature of the aluminium turntable must remain constant to around 0,2 K. At present the laboratory air-conditioning maintains the turntable temperature to better than ±20 mK over a 24 hour period.

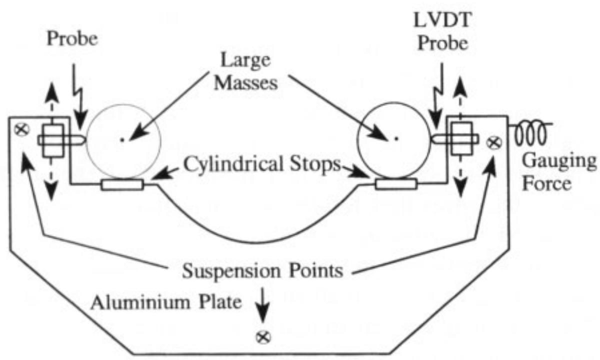

Figure 4. Schematic diagram of the length comparator gauge used to measure the large mass spacing. The length comparison is made using a linear variable differential transformer (LVDT) displacement transducer.

Large mass uniformity of density. The value of K also depends on the uniformity of density throughout the large cylindrical masses. To calculate K to 1 part in 10^5 the density distributions must be known to a few parts in 10^5. The sensitivity of K to angular density variations is reduced by rotating the large masses on their kinematic mounts. However, longitudinal variations in density would be significant. To determine the variation of density in the large masses, samples in the shape of wedges and rings were cut at three different locations along the stock bar used to manufacture the large masses (see Figure 5). The density of these samples was measured, by weighing the samples in air and in distilled water, to assess the radial, angular and longitudinal density variations. Average values for the density are shown in Figure 6. The only significant variations found were radial, with the density decreasing towards the centre of the cylinder. Fortunately K is largely insensitive to this kind of density variation.

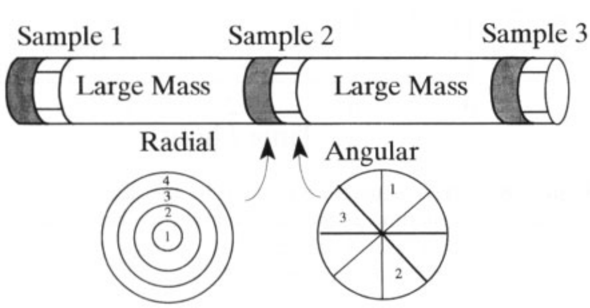

Figure 5. Diagram of the stainless steel bar used to make the large masses. Also shown is the location and type of samples cut for the density measurements.

4.2 Electrometer voltage V_G^2

The term V_G^2 represents the average of the square of the voltage measured during the gravitational attraction phase of the measurement. At each large mass position the average of the voltage and the average of the square of the voltage are recorded for both positive and negative polarities. These measurements are first analysed to determine the contact potentials V_c in the electrometer. The contact potentials are determined using (2) and, since Γ_G stays constant during polarity reversal,

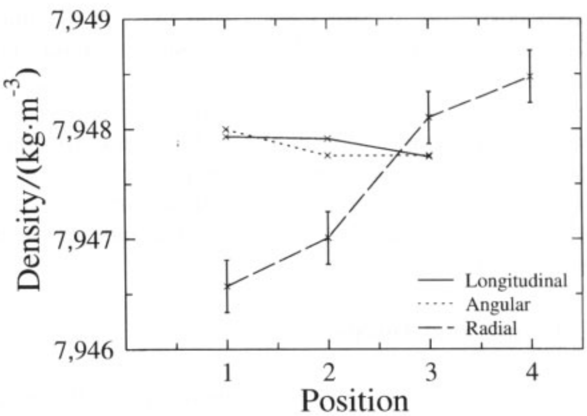

Figure 6. Graph showing the measured density variations of samples from the stainless steel bar used for the large masses.

$$\begin{aligned}\Gamma_G &= (\mathrm{d}C/\mathrm{d}\theta)\,(V_+ + V_c)^2/2 \\ &= (\mathrm{d}C/\mathrm{d}\theta)\,(V_- + V_c)^2/2, \qquad (5)\end{aligned}$$

where the + and − subscripts indicate the polarity of the applied voltages. Rearranging (5) gives

$$V_c = (V_+^2 - V_-^2)/2\,(V_- - V_+). \qquad (6)$$

An average value for V_G^2 is then calculated by combining the measurements for each 2π rotation of the large masses using

$$V_G^2 = [(V_1^2 + V_3^2) + (V_2^2 + V_4^2)]/4, \qquad (7)$$

where the subscripts 1 to 4 indicate the angular position of the large masses around the fibre axis. Positions 1 and 2 are π from positions 3 and 4 respectively, as indicated in Figure 1b. In practice, the $\mathrm{d}C/\mathrm{d}\theta$ values for each side of the electrometer differ by about 1 part in 10^4 and this is taken into account in the calculation of V_G^2.

4.3 Electrometer calibration $\ddot{\alpha}/V_A^2$

As the apparatus accelerates, its angular position α versus time is measured using the polygon. A quadratic fit to the angle versus time data gives the average acceleration $\ddot{\alpha}$ of the apparatus. Following two complete revolutions the voltage is then applied to the opposite side of the electrometer to decelerate the apparatus.

After correcting V_A^2 for the residual twist in the fibre, for contact potentials and for the stray gravitational torques in the room, a value for $\ddot{\alpha}/V_A$ is obtained from (3). A voltage of 190 V gives an acceleration of about 40 μrad·s^{-2}. With this voltage, the apparatus takes approximately 9 minutes for the first revolution and 4 minutes for the second (this compares with 30 minutes for the first rotation in the measurements made by Beams [13, 14]).

5. Results and Discussion

Preliminary values have been obtained for the three quantities (I/K), V_G^2 and $(\ddot{\alpha}/V_A^2)$ used to calculate G.

A typical calibration of the electrometer is shown in Figure 7 where the angular displacement is plotted versus time. In this example the applied voltage was 190 V (corresponding approximately to 160 times the torque due to a single large mass). Also shown are the residuals after the fitted curve is subtracted. The average acceleration measured for an individual measurement typically has a standard deviation of about 5 parts in 10^5 but acceleration measurements made over a period of a month give values for $\ddot{\alpha}/V_A^2$ that differ by around 2 parts in 10^4. During each acceleration measurement, the average angular position of the small mass system is within ±0,5 μrad of the set point. Variations in the gravitational potential seen by the small mass system as it rotates have been determined by measuring the rest point voltage at eight angular positions. The acceleration data were then corrected for these variations in rest point. These variations are quite large, around 3 parts in 10^3 of the electrostatically induced torque, due to the presence of a large hill nearby.

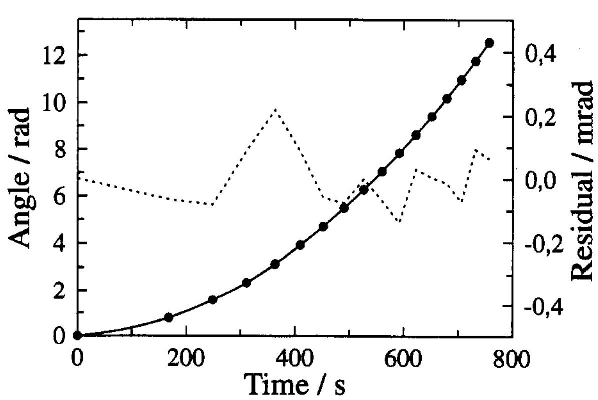

Figure 7. Angular displacement versus time for a typical calibration of the electrometer. The broken line shows the residuals after the fitted curve is subtracted from the data.

The large mass spacing, which is the dominant factor in the ratio I/K, has been measured using the length comparator described in Section 4.1.2 with a reproducibility of better than 1 μm. Overall, the total uncertainty in the centre-to-centre spacing is ±4 μm due largely to the uncertainty in the large mass diameters. This contributes an uncertainty of 2 parts in 10^5 to the term K.

Prior to the measurement of V_G^2 the angular position of the large masses was adjusted so that Γ_G was at a maximum. This was achieved by stepping the large masses around the fibre axis in 30 mrad steps and measuring V_G^2 at each step. The V_G^2 values versus angle were then fitted to the calculated shape of Γ_G. The large masses were then positioned at the angle where the measured torque was at a maximum. As a check on the angular positioning of the large masses the angle between the positions of maximum torque on each side of the small mass system was measured. The measured and calculated changes in angle agree to within 3 mrad implying an uncertainty of 3 parts in 10^5 for the calculated torque.

A typical example of the signal obtained while measuring V_G^2 is shown in Figure 8. The direction of the applied torque changes each time the large masses are moved to the next position and this is indicated by the sign of V^2. The steps during each 34 minute measurement period are caused by changing the electrometer voltage polarity. Values of between 0,2 V and 0,3 V for the contact potential difference have been measured. The exact value seems to be related to the amount of time the apparatus has been evacuated. The contact potentials are stable at the millivolt level over a 24 hour period. To obtain V_G^2 to better than 1 part in 10^4 the value of these potentials must be known to better than 1 mV.

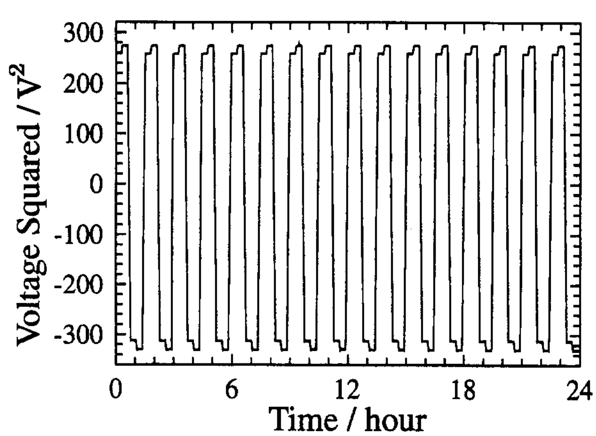

Figure 8. Voltage squared, measured by the digital voltmeter, versus time as the large masses are shifted around the small mass system. The sign of the squared voltage indicates the direction of the induced torque.

The value of V_G^2 calculated from the measurement shown in Figure 8 is 293,79 V^2 with a standard deviation of 1,7 parts in 10^4. Several measurements made over a period of two months give a mean value with a standard deviation of around 1,3 parts in 10^4. During the measurement shown in Figure 8, the average angular position of the small mass system remained within ±0,3 μrad of the angular set point. The data shown represent a 24 hour section of data selected out of a measurement. Useful data can only be collected during the weekends as data collected at other times are too noisy. This noise appears to be due to human activity but the exact cause has not yet been identified.

The magnetic susceptibility of the measurement has been tested by constructing a pair of Helmholtz coils 2 m in diameter. These coils produce a magnetic field of up to twice the Earth's magnetic field on the small mass system. No detectable change in the small mass angular position was observed when the direction of the Earth's magnetic field was effectively reversed.

Although problems with the horizontal pendulum modes of the small mass system were anticipated, in practice they have not been a problem. After operating the apparatus under servo-control for several days, the amplitude of the pendulum modes is around 5 μm peak-to-peak. While oscillations at this amplitude introduce a slowly varying torque signal of about 0,2 V^2 in the gravitational signal the integral cycle averaging of the data over 34 minute periods makes this effect negligible.

Table 2. The current measured uncertainties in the three quantities determining G.

Quantity	$10^6 \times$ standard relative uncertainty in G ($1\,\sigma$)	
	Type A	Type B
(I/K)	12	36
V_G^2	200	
$(\ddot{\alpha}/V_G^2)$	130	
Final value of G	238	36

A summary of the current uncertainties of the three quantities used to calculate G is shown in Table 2. To reduce the uncertainty in the final value of G it is planned to concentrate on reducing the uncertainties in $\ddot{\alpha}/V_A^2$ and V_G^2. The uncertainty in $\ddot{\alpha}/V_A^2$ will be reduced by (i) placing compensating masses around the apparatus to reduce the variations in gravitational potential seen by the small mass system; and (ii) increasing V_A^2 so that effects due to the changes in the gravitational potential are less significant.

The uncertainty in V_G^2 will be reduced by increasing the number of measurements of this quantity. It is also planned to investigate the effects of rotating the large masses about their axes and changing their mounting positions on the turntable. Current indications are that by making these changes an uncertainty of 1 part in 10^4 or better in the value of G will be obtained.

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Appendix 1

Sensitivity of a torsion balance to background gravitational disturbances

The calculation is made by first making the following approximations. The small mass system, of length $2l_0$ and mass m, can be represented by a one-dimensional mass distribution of density ρ. The large mass, of mass M, causing the disturbance can be represented as a point mass at a distance r_0 from the small mass system such that $r_0 \gg l_0$. Under these conditions the gravitational field $g\,(r)$ produced by M at the small mass system is given by

$$g\,(r) = g\,(r_0) + \Delta r\,\mathrm{d}g/\mathrm{d}r,$$

where $g\,(r_0) = GM/r^2$ and $\mathrm{d}g/\mathrm{d}r = 2GM/r_0^3$.

The torque Γ on the small mass system, rotated by an angle θ with respect to a line between M and the centre of the small mass system, is found by integrating the change in gravitational field along the length of the small mass system, so that

$$\mathrm{d}\Gamma = \mathrm{d}r\,(\mathrm{d}g/\mathrm{d}r)\,\mathrm{d}m\,l\,\sin(\theta).$$

By substituting $\mathrm{d}m = \rho\,\mathrm{d}l$ and $\mathrm{d}r = l\,\cos\theta$ the torque on the small mass system is given by

$$\Gamma = \int_{-l_0}^{l_0} l\,\cos\,\theta\,(2GM/r^3)\,\rho\,\mathrm{d}l\,l\,\sin\theta$$

$$= (\sin\,2\theta\,G\,M\rho/r^3)\int_{-l_0}^{l_0} l^2\,\mathrm{d}l$$

$$= \sin\,2\theta\,G\,M\rho\,2\,l_0^3/3\,r^3.$$

Finally, after substituting $m = 2\rho\,l_0$ and the small mass system length $L = 2\,l_0$, the torque is given by

$$\Gamma = \sin\,2\theta\,G\,MmL^2/12\,r^3.$$

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