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Higher precision gravitational experiments would be obtained by measuring in the transient state

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Abstract

Research on the thermal limitation on displacement measurements of an oscillator shows that the signal response of the oscillator reaches a maximum while the thermal noise response reaches a minimum during the transient state, and the precision of gravitational experiments could increase about two orders of magnitude in the transient state compared to that in the equilibrium state. © 1998 Elsevier Science B.V.

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Most gravitational experiments in the laboratory deal with measuring the constant forces acting on an oscillator, which could be a torsion balance, a simple pendulum or a spring oscillator [1], particularly the determination of the gravitational constant [2,3] and verification of the inverse square law [4,5], or the test of the presence of other forces [6]. In such experiments, the sensitivity of the oscillator is mainly limited by thermal noise fluctuations, local gravitational disturbances and microseismic noise, of which the thermal noise sets the most fundamental limit to laboratory experiments. Chen and Cook [7] have discussed the thermal noise limitation in detail and gave the smallest detectable acceleration of the oscillator acted on by a constant external force as follows,

$$A_{C\,\min} = 2\pi u_n/\tau_0, \tag{1}$$

where $u_n^2 = k_B T/m$, k_B is the Boltzmann constant and T the absolute temperature. m and τ_0 are the mass and the free period of oscillator, respectively. As we will show, the smallest detectable acceleration will be greatly reduced by measuring in the transient state.

Let x be the displacement of the oscillator, and $f(t)$ the force acting on it. The equation of motion governing the oscillator is

$$\ddot{x} + \beta\dot{x} + \omega_0^2 x = f(t)/m, \tag{2}$$

where ω_0 is the intrinsic frequency of the oscillator and β the damping factor. In most of the higher sensitivity gravitational experiments, the oscillator has a very high quality factor Q and a very long relaxation time τ^*. The following discussion is based on these experiments. First let $f(t)$ be a random force $F_r(t)$. The authors of Refs. [7–9] gave the mean-square root random displacement as

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$$\sqrt{\langle x_n^2(t)\rangle} = \frac{u_n}{\omega_0}\left(1 - \frac{\omega_0^2}{\omega_1^2}\mathrm{e}^{-\beta t}\sin^2(\omega_1 t + \alpha)\right)^{1/2}, \tag{3}$$

where

$$\omega_1^2 = \omega_0^2 - \beta^2/4, \quad \alpha = \arctan(2\omega_1/\beta). \tag{4}$$

This is the expression for the mean-square root displacement at any time when the motion is underdamped. To derive the smallest detectable acceleration in the presence of thermal noise, we have to compare this displacement with the one caused by a constant external force.

Now let $f(t)$ be mA, the constant external signal force to be detected. Then Eq. (2) becomes

$$\ddot{x} + \beta\dot{x} + \omega_0^2 x = A. \tag{5}$$

The solution of Eq. (5) is

$$x_s(t) = \frac{A}{\omega_0^2}\left(1 - \frac{\omega_0}{\omega_1}\mathrm{e}^{-\beta t/2}\sin(\omega_1 t + \alpha)\right). \tag{6}$$

We can now derive the smallest detectable acceleration in an experiment by measuring the displacement with a linear harmonic oscillator in the presence of thermal noise. We adopt the criterion used by Braginsky and Manukin [10], namely that a signal to be detected in any observation at time t should at least be as great as the random fluctuations,

$$x_s(t) \geqslant \sqrt{\langle x_n^2(t)\rangle}. \tag{7}$$

It is obvious that when the observation is made at the positive peak of the signal response, i.e. when the time t obeys the condition

$$\sin(\omega_1 t + \alpha) = -1, \tag{8}$$

$x_s(t)$ reaches the maximum amplitude while $\sqrt{\langle x_n^2(t)\rangle}$ reaches the minimum amplitude, and Eqs. (3) and (6) will be respectively replaced by

$$\sqrt{\langle x_n^2(t)\rangle} = \frac{u_n}{\omega_0}\left(1 - \frac{\omega_0^2}{\omega_1^2}\mathrm{e}^{-\beta t}\right)^{1/2}, \tag{9}$$

$$x_s(t) = \frac{A}{\omega_0^2}\left(1 + \frac{\omega_0}{\omega_1}\mathrm{e}^{-\beta t/2}\right). \tag{10}$$

If $\tau_0 < t \ll \tau^*$, which means that the measurement is taking place in the transient state, the smallest

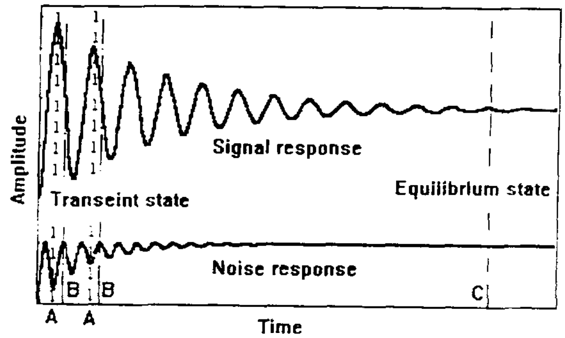

Fig. 1. The signal and noise responses of the oscillator. Point A is the measurement position with the highest SNR, and B is the measurement position chosen by Chen and Cook.

detectable acceleration can be obtained by inserting Eqs. (9) and (10) into Eq. (7),

$$A_{T\,\mathrm{min}} = \frac{\pi u_n}{\tau_0}\sqrt{\frac{2t}{\tau^*}}. \tag{11a}$$

In general, the observation time t is about several times the free period τ_0 if the measurement is taking place in the transient state, and $A_{T\,\mathrm{min}}$ can be rewritten as

$$A_{T\,\mathrm{min}} \approx 2\pi u_n/\sqrt{\tau_0\tau^*}. \tag{11b}$$

If $t \gg \tau^*$, i.e. the measurement is taking place in the equilibrium state, we can then obtain the smallest detectable acceleration as

$$A_{T\,\mathrm{min}} = 2\pi u_n/\tau_0. \tag{12}$$

Comparing Eqs. (11a) and (11b) with Eq. (12), we see that the smallest detectable acceleration is directly proportional to t, the duration of the measurement, and its value in the transient state is about $\sqrt{\tau_0/\tau^*}$ times smaller than that in the equilibrium state. The physical reason is quite clear, as shown in Fig. 1. Both the signal and noise response of a linear harmonic oscillator subjected to a constant external force oscillate in the transient state, and the signal response $x_s(t)$ reaches the maximum amplitude while the noise response $\sqrt{\langle x_n^2(t)\rangle}$ reaches the minimum amplitude at the same time (as marked by A in Fig. 1), so the signal-to-noise ratio (SNR) of the measurement can greatly increase at the positive peak of the signal response. When $t \gg \tau^*$, the motion of the oscillator reaches the equilibrium state (as marked by C in Fig. 1), the signal and noise responses reach two stable values, respectively, and the SNR becomes stable but larger than that in point A. In Ref. [7], the observation time t was chosen at the positive peak of the noise response (as marked by B in Fig. 1) instead of at the positive peak of the signal response.

From the continuous measurements, Chen and Cook [7] gave the smallest detectable acceleration as

$$A_{\min} \approx 2\pi u_n \sqrt{n_1 + n_2} / \sqrt{\tau_0 \tau^*}, \tag{13}$$

where integers n_1 and n_2 are related to the starting time t_1 and end time t_2 for continuous measurement. It means that the smallest detectable acceleration of continuous measurement is about $\sqrt{n_1 + n_2}$ times larger than that of the discrete measurement in the transient state.

For the typical experimental parameters $k_B = 1.380 \times 10^{-23}$ J/K, $m = 0.05$ kg, $T = 300$ K, $\tau_0 = 600$ s, $\tau^* = 6.0 \times 10^6$ s, we get the smallest detectable acceleration to be about 10^{-12} m/s^2 when the measurement is taken in the equilibrium state. If the measurement is taking place in the transient state, the smallest detectable acceleration is about 10^{-14} m/s^2, which is about two orders of magnitude higher than that in the equilibrium state. For the case of cold damping, the relaxation time τ^* is only a few times larger than the free period τ_0, so the smallest detectable accelerations both in the equilibrium state and in the transient state are approximately equal. If we select the measurement time in the transient state, we should avoid employing some sort of force feedback and keep the oscillator with a high Q.

In conclusion, the best observation time for an experiment in the case of a constant force acting on a linear harmonic oscillator is in the transient state. It is not only the higher precision that can be obtained, but also the time of measurement would be greatly reduced. The above discussion, of course, is based on the condition that the constant force to be detected is much larger than the noise level, which is usually the case in the experiments. If the force is very small compared with the noise level, we can not determine the positive peak of the response of the oscillator, so then the conclusion stated above does not hold.

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References

[1] A.H. Cook, Rep. Prog. Phys. 51 (1988) 707.

[2] H. de Boer, Metrologia 24 (1987) 171.

[3] W. Michaelis, Metrologia 32 (1996) 267.

[4] M.W. Moore et al., Class. Quant. Grav. 11 (1994) A97.

[5] B.H. Ubler et al., Phys. Rev. D 51 (1995) 4005.

[6] C.W. Stubbs et al., Phys. Rev. Lett. 58 (1987) 1070.

[7] Y.T. Chen, A.H. Cook, Gravitational Experiments in the Laboratory (Cambridge University Press, Cambridge, 1993).

[8] G.E. Uhlenbeck, L.S. Ornstein, Phys. Rev. 36 (1936) 823.

[9] M.C. Wang, G.E. Uhlenbeck, Rev. Mod. Phys. 17 (1945) 323.

[10] V.B. Braginsky, A.B. Manukin, Measurement of Weak Forces in Physics Experiments (University of Chicago Press, Chicago, 1977).