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MEASUREMENT OF THE GRAVITATIONAL CONSTANT USING THE ATTRACTION BETWEEN TWO FREELY FALLING DISCS: A PROPOSAL

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The constant of gravitation, G, is the least well-known of the physical constants. A new, independent method of measurement, estimated as having a potential uncertainty at least as small as that achieved by existing methods, would be useful for an improvement in G determination. This experiment is based on the measurement of the relative motion of two freely falling test bodies (discs), caused by their gravitational attraction. The uncertainties are analyzed for two parallel tungsten discs with masses of about 30 kg. The use of test bodies with an incorporated optical system of multipass two-beam interferometers, as well as of multibeam interferometers, is proposed to measure their relative displacement. The estimations were made for laboratory experiment with free fall duration of 0.714 s. In this case, the relative displacement to be measured is about 0.1 μm. These estimates show that relative uncertainties lower than 5×10^{-5} can be obtained in G measurement in a single drop of the test bodies. The proposed experiment can be made in outer space. In space a lower uncertainty can be achieved because the time interval of the measurement of relative motion of the test bodies can be increased.

Keywords: Gravitational constant; laser interferometry; space experiments; metrology.

1. Introduction

Recent advances in laser interferometry allow for displacement measurements with subnanometer uncertainty. For example, laser displacement interferometers combined with X-ray interferometers are now being developed for the calibration of linear transducers with subnanometer uncertainty. Another scientific effort concerns the design of interferometers for the detection of gravitational waves that are extremely sensitive to the relative displacement of test bodies. In both cases the resolution of laser displacement interferometers, limited only by shot noise of photo detection, was reached.

Current experiments aimed at detecting gravitational waves (see for example Ref. 1) have reached the shot-noise-limited resolution of about $1 \times 10^{-19}\,\mathrm{m/Hz^{\frac{1}{2}}}$. Also, a shot-noise level of $1 \times 10^{-10}\,\mathrm{m/Hz^{\frac{1}{2}}}$ has been reached using the combined optical/X-ray interferometer.[2]

An essential experimental result for understanding the limitations of the accuracy of two-beam laser displacement interferometry was reported in Ref. 3. The residual nonlinearity of the interference signal of the heterodyne interferometer outlined in this paper was less than 0.02 nm.

Such developments in laser displacement interferometry will permit determination, with appropriate accuracy, of the gravitational constant G from direct measurements of the motion of two nearby free-moving test bodies caused by their proper gravitational attraction.

This new, independent method to measure the gravitational constant G will have an uncertainty, at least, not greater than that achieved in previous experiments.

In this paper we analyze the possibility of measuring G from the relative motion of two free-falling discs with parallel basal planes. We show that the gravitational attraction between two discs is greater than that between two spheres of the same mass and separation between the surfaces. Furthermore, the discs are simpler to incorporate within an optical system for measurement of their relative motion. It is worth noting that regularity of a disc can be one or two orders better than that of a sphere.[4]

The sources of uncertainties in the proposed ground-based experiment are analyzed for the reasonable characteristics of the experimental setup. This analysis allows the conclusion that if such an experiment is carried out on a spacecraft, where the time interval of the measurement of relative motion of the test bodies can be significantly increased, the uncertainty in the measurement of the gravitational constant can be diminished.

2. Basic Estimations

2.1. *Relative acceleration of test bodies due to mutual gravity attraction*

A relative acceleration of two identical spheres with the mass $M_1 = M_2 = M$ due to their gravitational attraction is described by

$$a_{GS} = 2G\frac{M}{L^2} = \frac{8\pi G R_s^3 \rho}{3(2R_s + d)^2}, \tag{1}$$

where L is the distance between the centers of the spheres, $G = 6.672 \times 10^{-11}\,\mathrm{Nm^2kg^{-2}}$, ρ is the density of the spheres and d is the distance between their surfaces. For tungsten spheres ($\rho = 19.3 \times 10^3\,\mathrm{kg/m^3}$) with $R = 0.072\,\mathrm{m}$, $d = 0.001\,\mathrm{m}$ and $M = 30.3\,\mathrm{kg}$, we obtain

$$a_{GS}(R = 0.072\,\mathrm{m},\ d = 0.001\,\mathrm{m}) = 1.9 \times 10^{-7}\,\mathrm{m\,s^{-2}}. \tag{2}$$

A numerical calculation of the relative acceleration a_{GS} of two parallel tungsten discs with the mass of 30.3 kg, the radius $R = 0.1\,\mathrm{m}$, the thickness $h = R/2 = 0.05\,\mathrm{m}$ and the distance $d = 0.001\,\mathrm{m}$ between the adjacent surfaces gives

$$a_{GS}(R = 0.1\,\mathrm{m},\ h = 0.05\,\mathrm{m},\ d = 0.001\,\mathrm{m}) = 3.97 \times 10^{-7}\,\mathrm{m\,s^{-2}}. \tag{3}$$

It was found that the optimal shape of the discs to give the maximum mutual attraction for a fixed mass is that with $h = R/2$.

2.2. *Equation of motion*

The gravitational constant may be evaluated from the relative motion equation of two free-falling discs,

$$\frac{\partial^2(\Delta z)}{\partial t^2} = \frac{\partial^2 z_2}{\partial t^2} - \frac{\partial^2 z_1}{\partial t^2} = \gamma_{zz}\Delta z + 2G\frac{M}{L(R,h,\Delta z)}, \tag{4}$$

where z_i is the z axis coordinate of the gravity center of the ith test body, γ_{zz} is the vertical gravity gradient of the Earth gravity field and $L(R, h, \Delta z)$ is a numerically calculated function of the dimensions of the discs and of the distance between the gravity centers.

For such an evaluation the distance between the discs and time intervals should be measured during a free fall. The distance–time intervals should be used for a least-squares evaluation of G. This task is somewhat similar to that of the measurement of a vertical gravity gradient using an absolute ballistic gravity gradiometer with two free-falling bodies.[5,6]

3. Required Uncertainty in Displacement Measurement

The uncertainty in the displacement measurement is crucial for the G measurement. For a fall height of 2.5 m, the free fall duration will be 0.714 s. The relative displacement of the discs with relative acceleration of $3.9 \times 10^{-7}\,\mathrm{m\,s^{-2}}$ is

$$\Delta z_G = \frac{1}{2}a_{GD}t^2 = 0.10\,\mu\mathrm{m}. \tag{5}$$

The optical path change Δs_G in a two-beam interferometer is twice Δz_G. In order to achieve relative uncertainty for a single measurement below 5 parts in 10^5, the change of the optical path should be measured with a relative uncertainty of 0.01 nm. If the multipass optical system with, for example, 48 double passes of the beam will be used in the interferometer, the total optical path change is

$$\Delta s_G = 9.5\,\mu\mathrm{m} = 18.8\lambda, \tag{6}$$

where the laser wavelength $\lambda = 0.515\,\mu\mathrm{m}$ (see Refs. 7 and 8). Use of the multipass optical system allows increased uncertainty in the measurement of the optical path change to 0.5 nm.

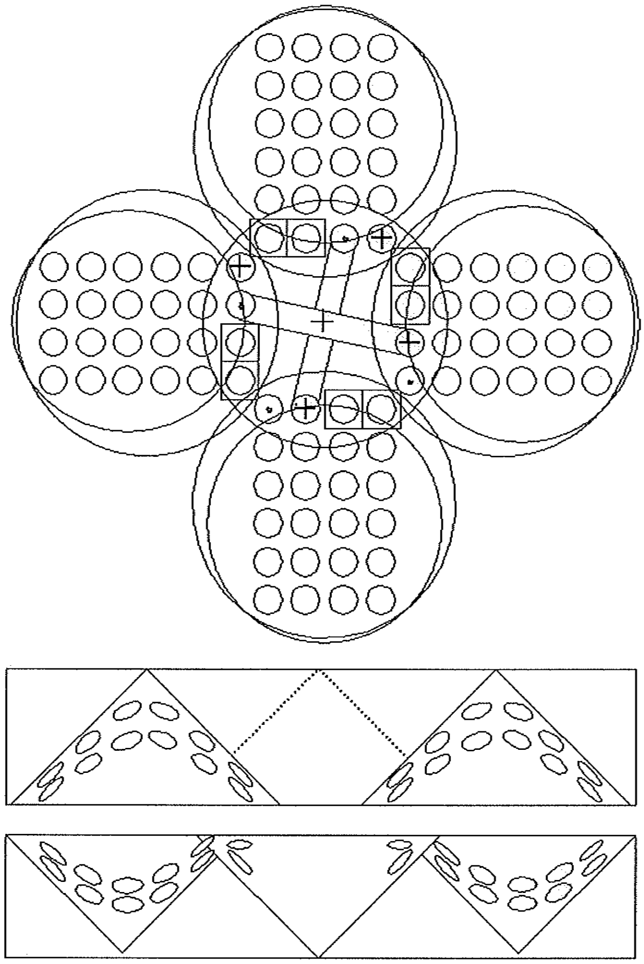

Fig. 1. Diagram of the propagating light beams in two independent multipass optical cells with 48 double passes based on the use of the shifted conical reflectors.

4. Laser Displacement Interferometer

Two independent multipass optical cells[9] based on the use of conical reflectors[10] are proposed for the interferometer (Fig. 1) used in the G measurement. This arrangement allows one to monitor relative tilts of the discs during the fall. The basic idea of the optical arrangement is described in Ref. 9. Instead of using traditional corner cubes, we propose using conical reflectors in order to avoid the deformation of the wave front on the edges of the corner cubes.

The main parts of the interferometer, including the optical elements of the reference arm, a beamsplitter and a photodetector board, can be fixed on the moving cart. This cart will be used to fix the test bodies at their initial positions and also to drop, catch and lift them back. An output beam from a laser installed on the pillar is separated by a beamsplitter in the reference and measuring arms of the interferometer.

5. Sources of the Uncertainty in the Measurement

A long list of possible measurement uncertainties, including that in the measurement of the relative displacements, time intervals, mass and dimensions of the test bodies, was analyzed.

Some of the disturbing factors acting on the test bodies are the same for the two test bodies and will therefore have no effect on their relative motion. They include tidal variations of the gravitational field, electromagnetic interactions between the test bodies and the chamber, interaction of the test bodies with the magnetic fields of the Earth, etc.

The ultimate sensitivity to absolute displacements of the mirror in a two-beam interferometer, limited solely by the shot noise of the photodetector, is given by the formula

$$\delta l_{\mathrm{min}} = 1.5 \times 10^{-6}\sqrt{\Delta f}\,\mathrm{nm}, \tag{7}$$

for the laser power $P = 1\,\mathrm{mW}$, the quantum efficiency of the photodetector $q = 0.3$ and the wavelength $\lambda = 515\,\mathrm{nm}$. In this formula Δf is the signal frequency band. It is seen that δl_{min} is much lower than the uncertainty required in the G measurement.

The uncertainty in the measurement of displacement by a two-beam interferometer with the difference ΔL of the length of the interferometer arms is limited by the laser frequency instability. For the laser with $\Delta f/f = 1 \times 10^{-12}$ and $\Delta L = 1\,\mathrm{m}$, we find the estimation of this uncertainty to be $\delta l_\lambda = 1 \times 10^{-4}\,\mathrm{nm}$.

One of the principal sources of uncertainties in the G measurements is the inhomogeneous gravity field in the laboratory. The relative displacement of two discs caused by the vertical gravity gradient may be obtained from the formula[7]

$$\Delta z_\gamma = \frac{1}{2}\gamma_{zz}\Delta z_0(\Delta t)^2, \tag{8}$$

where Δz_0 is the initial distance between the gravity centers of the discs and M is the free fall time. For the mentioned dimensions of the discs, their initial separation of 1 mm and the vertical gravity gradient of the normal gravity field of the Earth $\gamma_{zz} = 3.086 \times 10^{-6}\,\mathrm{s}^{-2}$, we obtain $\Delta z_\gamma = 0.79\,\mu\mathrm{m}$.

This displacement is practically the same as that caused by mutual gravitational attraction of the test bodies. The relative displacement due to mutual gravitational attraction and that caused by the gravity gradient are opposite in sign. The gravity gradient should be measured for use in the evaluation of motion equations. This measurement can be done with the same system, but with an increased initial separation between the discs to diminish the mutual gravitational attraction. It will also increase the sensitivity of such a vertical gravity gradiometer.

If the measuring system is located on a spacecraft, the influence of the higher derivatives of gravity potential is dramatically diminished, particularly with a proper design of the mechanical and optical assembly (supporting bench), which provides the minimal inhomogeneity of the background gravity field on the axis of the measuring system. The relative tilts of the discs can cause changes in the gravitational attraction and changes of the optical path of the beam. Preliminary estimations show that relative tilts below 0.1 arcsec cause a relative uncertainty in the G measurement below 3×10^{-6}.

In order to have a relative uncertainty in the G measurement below 1×10^{-6}, the pressure of residual gas in the chamber should be below 3×10^{-9} Torr. Precautions should be taken against the electrostatic charging of the discs to avoid disturbing the relative acceleration of the test bodies. The potential difference between the discs should be not more than a few microvolts, to allow a relative uncertainty of 1×10^{-5} in the G measurement. The analysis of other known sources of uncertainty leads to the conclusion that the relative uncertainty of the G measurement below 5×10^{-5}, in a single drop of the test bodies, is possible in the setup with the discs and the above-mentioned parameters.

Those other sources of uncertainty would be the following:

- angular instability of laser radiation,
- diffraction effects,
- inhomogeneity of the material of the discs,
- uncertainties in the evaluation of gravity attraction between the test bodies in the form of the discs with incorporated optical elements,
- uncertainties in the measurement of the dimensions and mass (currently, that can be done for the dimensions and masses used in our estimations at the level below 1×10^{-6}),
- thermal instabilities,
- negligible influence of the Casimir effect,
- uncertainty in the time interval measurements.

The following improvements would diminish the uncertainty in the measurement:

- the use of larger test bodies,
- a shorter wavelength of laser radiation,
- longer time intervals of the free fall of test bodies,
- simultaneous (or subsequent) measurements using a laser radiation at different wavelengths (e.g. at 515 nm and 532 nm),
- special design of test bodies that makes it possible to reinstall the discs after rotation at 180° on a vertical axis to diminish the influence of inhomogeneity of the material of the discs,
- the use of different materials for the discs.

A longer free motion of test bodies could be obtained in an experiment on a spacecraft. If this time interval were to be increased to 10 s, the measured relative displacement of the test bodies would be increased by a factor of 200, making it

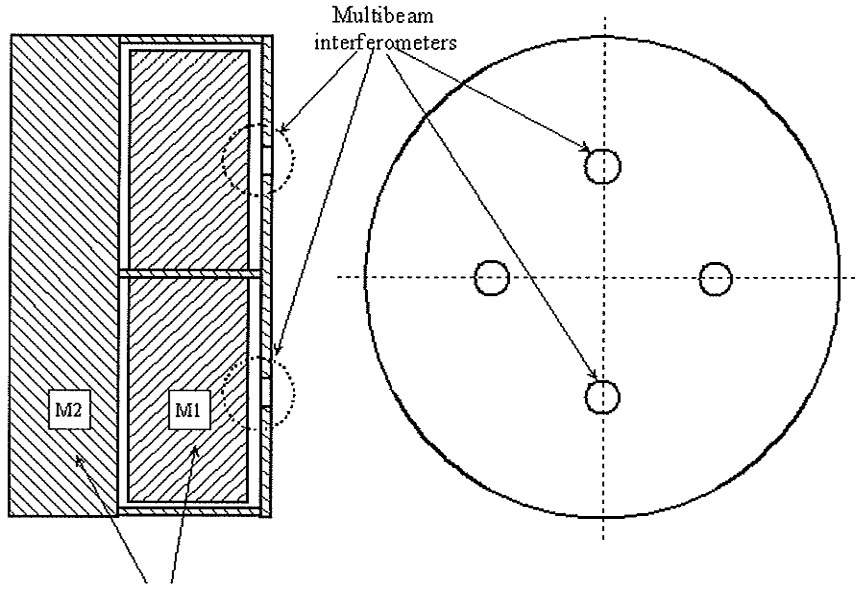

Fig. 2. Schematic configuration of the nested test bodies with four independent multibeam interferometers.

possible to diminish the masses of test bodies and to simplify the optical system. In addition, the influence of the inhomogeneity of the Earth's gravity field would be decreased.

A longer free motion time interval would also make it possible to use multibeam interferometers, of the Fabry–Perot type, for the measurement of the relative displacement of the test bodies. The relative displacement of $0.1\,\mu$m, estimated above for the experiment on the ground, is within one interference fringe corresponding to a displacement of $\lambda/2$ and could not be measured with the required uncertainty. In contrast, on a spacecraft with a possible relative displacement of $20\,\mu$m corresponding to about 78 interference fringes, multibeam interferometers can be used. A special design of the nested test bodies somewhat similar to that proposed for a vertical ballistic gravity gradiometer[5] can be used in the G measurement. In such a configuration (Fig. 2), the multibeam interferometers in reflection are formed by the reflecting surface of one test body and the optical mirrors incorporated in the thin plate fixed to the second test body. Use of four independent interferometers allows the control of relative tilts. The various tests, for example with the use of various wavelengths of laser radiation, are also possible with the independent interferometers.

6. Conclusions

We have proposed a new experiment on the measurement of the gravitational constant. The experiment is based on the measurement of the relative motion due to mutual gravity attraction of two freely falling discs with incorporated components of the optical laser interferometer for the displacement measurement. Precise measurements of the dimensions and mass of the test bodies, as well as precise computation of the function which describes their gravity attraction, are required. Currently this can be done with the relative uncertainty below 1×10^{-6} for the estimated parameters of the test bodies. The gravitational constant is then evaluated from the measured intervals of time and distance using the equation of the motion of freely falling test bodies. The vertical gravity gradient should be measured using the same experimental setup with the increased initial separation between the discs (for example, from 1 mm to 10 cm). Then the measured gravity gradient value should be included in the motion equation.

The theoretical estimations made using realistic parameters of the experimental setup (for example, for the height of free fall of 2.5 m) and based on the practical results showed that the uncertainty in the G measurement may be below 5×10^{-5}. Only actual experimentation can show how much lower the obtained uncertainty would be. Recommended in 2002 by the CODATA value $6.6742(10) \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$ of the gravitational constant is given with the relative standard uncertainty of 1.5×10^{-4} (see Ref. 11). The smallest assigned uncertainty of 1.5×10^{-5} was reported in Ref. 12. The experiments, used in the CODATA adjustment usually based on the torsion balance or on some measurements with the suspended masses. It would be interesting to perform a new, independent determination of the gravitational constant with a potential uncertainty at the same level of uncertainty that was obtained by existing methods.

The advantage of a new method is that it is one of a few which can be performed on a spacecraft where the uncertainty in the G measurement can be diminished at least by a factor 10. In such an experiment the observation time in a single measurement, determined by the length of the relative displacement of the test bodies, can be significantly increased.

The uncertainty in G determination on a spacecraft will mainly be limited by the following factors:

- the uncertainty in mass and dimension measurement,
- the inhomogeneity of the disc material,
- misalignments in the optical system,
- misalignments in the mechanical system.

It is worth noting that the technologies developed for the realization of the proposed experiment on the ground and in space will also stimulate progress in the absolute ballistic gravity gradiometry and in the study of gravitation at short distances.

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