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Time-varying G

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ABSTRACT

We formulate and study the problem of varying G in Newtonian gravitation and cosmology. Exact solutions and all asymptotic cosmological behaviours are found for universes with $G \propto t^{-n}$. Generalizations of Meshcherskii's theorem are proved for universes containing a perfect-fluid equation of state. The inequivalence of a positive cosmological constant and a $p=-\rho$ stress is investigated when G varies and limits are placed on the rate of decrease of $G(t)$ in order for de Sitter expansion to be attained as $t \to \infty$. The conditions for more general forms of inflationary universes to occur are determined and new forms of inflation are found. These results are related to the more complicated systems of equations governing scalar–tensor gravity theories, their solutions and conformal properties.

Key words: cosmology: theory.

1 INTRODUCTION

The study of gravitation theories in which Newton's gravitational constant varies in space and time has many motivations. It began with the proposal by Dirac (1937a,b, 1938) that the ubiquity of certain large dimensionless numbers, $O(10^{39})$, which arise in combinations of physical constants and cosmological quantities (Weyl 1919; Zwicky 1939; Eddington 1923) was not a coincidence but a consequence of an underlying relationship between them (Barrow & Tipler 1986; Barrow 1990a). This relationship requires a linear time variation to occur in the combination $e^2G^{-1}m_{\rm N}$ (where e is the electron charge, $m_{\rm N}$ the proton mass and G the Newtonian gravitation constant) and Dirac proposed that it was carried by $G \propto t^{-1}$, (Chandrasekhar 1937; Kothari 1938). This led to a range of new geological and paleontological arguments being brought to bear on gravitation theories and cosmological models (Jordan 1938, 1952; Teller 1948; Dicke 1957, 1964; Gamow 1967a,b). Brans & Dicke (1961) refined the scalar–tensor theories of gravity first formulated by Jordan and, motivated by apparent discrepancies between observations and the weak-field predictions of general relativity in the Solar system, proposed a generalization of general relativity that became known as the Brans–Dicke theory. As the Solar system and binary pulsar observations have come into close accord with the predictions of general relativity so the scope for a theory of the Brans–Dicke type to make a significant difference to general relativity in other contexts, notably the cosmological, has been squeezed into the very early Universe. However, more general theories with varying G exist, in which the Brans–Dicke parameter is no longer constant (Barrow 1993a). These theories possess cosmological solutions which are compatible with Solar-system gravitation tests (Hellings 1984; Reasenberg 1983; Shapiro 1990; Will 1993), gravitational lensing (Krauss & White 1992), and the constraints from white-dwarf cooling (Vila 1976; García-Berro et al. 1995). The crucial role that scalar fields may have played in the very early Universe has been highlighted by the inflationary universe picture of its evolution. A scalar field, ϕ, which acts as the source of the gravitational coupling, $G \sim \phi^{-1}$, is a possible source for inflation and would modify the form of any inflation that occurs as a result of the Universe containing weakly coupled self-interacting scalar fields of particle physics origin. There have been brief periods when experimental evidence was claimed to exist for a non-Newtonian variation in the Newtonian inverse-square law of gravitation at low energies over laboratory dimensions (Fischbach et al. 1986) and speculations that non-Newtonian gravitational behaviour in the weak-field limit might explain the flatness of galaxy rotation curves (Milgrom 1983; Bekenstein & Milgrom 1984; Bekenstein & Meisels 1980; Bekenstein & Sanders 1994) usually cited as evidence of non-luminous gravitating matter in the Universe. Most recently, particle physicists have discovered that space–times with more than four dimensions have special mathematical properties which make them compelling and possibly essential arenas for self-consistent, finite, anomaly-free, fully-unified theories of four fundamental forces of nature (Green & Schwarz 1984). Our observation of only three large dimensions of space means that some dimensional segregation must have occurred in the early moments of the expansion of the Universe with the results that all but three dimensions of space became static and confined to very small dimensions $\sim 10^{-33}$ cm. Any time evolution in the mean size of any extra (>3) space dimensions will be manifested as a time evolution in the observed three-dimensional coupling constants (Freund 1982; Marciano 1984; Kolb, Perry & Walker 1986; Barrow 1987). The effect of this dimensional reduction process is to create a scalar–tensor gravity theory in which the mean size of the extra dimensions behaves like a scalar field. In particular, low-energy bosonic superstring theory bears a close relation to a particular limit of Brans–Dicke theory (see Section 6).

However, despite these interconnections with modern ideas in the cosmology of the early universe, the theoretical investigation of gravity theories with time-varying G is still far from complete and, aside from the Solar system and binary pulsar observations (Will 1993), there are few general observational restrictions on scalar–tensor theories which are clear-cut. In this paper we shall investigate the Newtonian form of gravity theories with varying G, pointing out the relationships that these simple solutions have to the more complicated solutions of scalar–tensor gravity theories. In the past there has been very little discussion of the Newtonian case. The exceptions are the rediscoveries of Meshcherskii's theorem (1893, 1949): for example, by Batyrev (1941, 1949), Vinti (1974), Savedoff & Vila (1964), Duval, Gibbons & Horváthy (1991), McVittie (1978) and Lynden-Bell (1982). These authors all recognized the equivalence of the Newtonian gravitational problem with time-varying G to the problem with constant G and varying masses.

2 NEWTONIAN GRAVITATION

Newtonian gravitation is a potential theory that is derived from the axiom that the external gravitational potential due to a sphere of mass M, be equal to that of a point of mass M. This fixes the potential to be equal to

$$\Phi(r)=\frac{A}{r}+Br^2 \tag{1}$$

where A and B are constants; $A=-GM$ and $B=\frac{1}{6}\Lambda$, where Λ is the cosmological constant of Einstein. This argument shows how the cosmological constant arises naturally in Newtonian theory, as it does in general relativity. In Sections 2 and 3 we shall set the cosmological constant term to zero ($B=0=\Lambda$). In Section 4 we shall discuss how its interpretation differs from a $p=-\rho$ fluid when G is not constant and prove some restricted cosmic no-hair theorems

Consider the Newtonian N-body problem with a time-varying gravitational 'constant' $G(t)$. If the N bodies have masses m_j and position vectors $\boldsymbol{r}_j$ then

$$\frac{\mathrm{d}^2\boldsymbol{r}_j}{\mathrm{d}t^2}=-\sum G(t)m_k\frac{\boldsymbol{r}_j-\boldsymbol{r}_k}{|\boldsymbol{r}_j-\boldsymbol{r}_k|^3}. \tag{2}$$

Now, if we have a solution, $\hat{\boldsymbol{r}}_j(\hat{t})$, of these equations with $G=G_0$ independent of time, then (Meshcherskii 1893; Savedoff & Vila 1964; Vinti 1974; Lynden-Bell 1982; Duval, Gibbons & Horváthy 1991),

$$\boldsymbol{r}_j(t)=\left(\frac{t+b}{t_0}\right)\hat{\boldsymbol{r}}_j\left(-\frac{t_0^2}{t+b}+c\right) \tag{3}$$

is an exact solution of the equations (1) with

$$G(t)=G_0\times\left(\frac{t_0}{t-c}\right) \tag{4}$$

where b, c and t_0 are constants with $t_0\neq 0$. Thus, given any solution of a gravitational problem with constant G we can immediately write down an exact solution in which G varies inversely with time. For example, suppose we take the simplest Newtonian cosmological model with zero total energy, when $G=G_0$ is constant. Then, the expansion scalefactor of the universe is

$$\hat{r}\propto\hat{t}^{2/3}. \tag{5}$$

By the theorem we have that

$$r(t)=\left(\frac{t+b}{t_0}\right)\left(-\frac{t_0^2}{t+b}+c\right)^{2/3} \tag{6}$$

when $G(t)$ varies as

$$G(t)=G_0\times\left(\frac{t_0}{t-c}\right); \qquad t_0\neq 0 \tag{7}$$

and so $r(t)\propto t^{1/3}$ as $t\to\infty$.

The result (4) is also useful for modelling small variations in G over short time-scales. If we expand an arbitrary analytic form for $G(t)$ to first order in t then

$$G(t)=G_0+\dot{G}_0t+\ldots O(t^2)\approx G_0(1-t\dot{G}_0/G_0)^{-1} \tag{8}$$

and this has the form (4).

This result, a consequence of the scale invariance of the inverse-square law of force, was first found by Meshcherskii (1893). It has often been rediscovered and elaborated. Duval, Gibbons & Horváthy (1991) have explored its existence in a wider context and displayed similar invariances of the non-relativistic time-dependent Schrödinger equation with Coulomb potential (see also Barrow & Tipler 1986) which enables solutions with time-varying electron charge ($e^2\propto t$) to be generated by transformation of known exact solutions with constant values of e. In the next section we shall prove a generalization of Meshcherskii's theorem for cases where the pressure is non-zero and the equation of state has perfect-fluid form.

3 NEWTONIAN COSMOLOGIES WITH $G(t)\propto t^{-n}$

We adopt the standard generalization of Newtonian cosmology (Milne & McCrea 1934; Heckmann & Schücking 1955, 1959) to include matter with non-zero pressure and a perfect-fluid equation of state. We shall confine our attention to isotropic Newtonian solutions. This is of particular interest for the real Universe in the recent past but we also know that anisotropic Newtonian cosmological models are not well posed, in the sense that there are insufficient Newtonian field equations to fix the evolution of all the degrees of freedom (there are no propagation equations for the shear anisotropies, Barrow & Götz 1989a) and this incompleteness must be repaired by supplementing the theory with extra boundary conditions or by importing shear propagation equations from a complete relativistic theory, like general relativity (Evans 1974, 1978; Shikin 1971, 1972), or by ignoring the evolution of the shear anisotropy (Narlikar 1963; Davidson & Evans 1973, 1977; Narlikar & Kembhavi 1980).

Consider a homogeneous and isotropic universe with expansion scalefactor $r(t)$. The material content of the universe is a perfect fluid with pressure, p, and density ρ, obeying an equation of state (where the velocity of light has been set equal to unity)

$$p=(\gamma-1)\rho; \qquad 0\leq\gamma\leq 2, \tag{9}$$

with γ constant. If $G=G(t)$ then the equation of motion for $r(t)$ is

$$\ddot{r}(t)=-\frac{G(t)M}{r^2}=-\frac{4\pi G(t)(\rho+3p)r}{3}. \tag{10}$$

The mass conservation equation is

$$\dot{\rho}+3\frac{\dot{r}}{r}(\rho+p)=0. \tag{11}$$

Hence, we have

$$\rho=\frac{\Gamma}{r^{3\gamma}}; \qquad \Gamma\geq 0 \text{ and constant.} \tag{12}$$

We shall initially be interested in power-law variations of $G(t)$ of the form

$$G(t)=G_0\left(\frac{t_0}{t}\right)^n \tag{13}$$

so we have

$$\ddot{r}=-\lambda t^{-n}r^{1-3\gamma} \tag{14}$$

where λ is a constant defined by

$$\lambda=\frac{4\pi G_0 t_0^n(3\gamma-2)\Gamma}{3} \tag{15}$$

so the sign of λ is determined by the sign of $3\gamma-2$, as in isotropic general relativistic cosmologies. Hence, accelerating universes ($\ddot{r}>0$) arise when $3\gamma<2$ regardless of whether G varies or not. However, these accelerating universes need not solve the horizon and flatness problems in the way that conventional inflationary universes do; that depends upon the value of n.

A generalization of Meshcherskii's theorem (1893) can be proved for the case with $p=(\gamma-1)\rho$. If $\hat{r}(\hat{t})$ is a solution with $G=G_0$ constant, then

$$r(t)=\left(\frac{t+b}{t_0}\right)\hat{r}\left(-\frac{t_0^2}{t+b}+c\right) \tag{16}$$

with b, c and $t_0\neq 0$, constants, is an exact solution of (14) with

$$G(t)=G_0\times\left(\frac{t_0}{t-c}\right)^{4-3\gamma}; \qquad t_0\neq 0. \tag{17}$$

These results provide a Newtonian analogue to the conformal properties of relativistic scalar–tensor theories. We can draw a number of general conclusions from them. As $t\to\infty$ we have

$$r(t)\to t, \qquad \text{if } c\neq 0,\ \forall\gamma \tag{18}$$

$$r(t)\to\frac{t}{t_0}\hat{r}\left(\frac{-t_0^2}{t}\right)^{2/3\gamma}, \qquad \text{if } c=0 \text{ and } \gamma\neq 0. \tag{19}$$

In particular, if we take the solutions with constant $G=G_0$ to be the zero-curvature Friedmann solutions then, when $c\neq 0$, we have

$$\hat{r}(\hat{t})\propto\hat{t}^{2/3\gamma}, \qquad \text{if } \gamma\neq 0 \tag{20}$$

$$\hat{r}(\hat{t})\propto\exp[H_0\hat{t}], \qquad H_0 \text{ constant, if } \gamma=0, \tag{21}$$

and the solutions with $G(t)\propto t^{3\gamma-4}$ at large time have the form

$$r(t)\propto t^{(3\gamma-2)/3\gamma}, \qquad \text{if } \gamma\neq 0\neq c,$$

$$r(t)\propto t, \qquad \text{if } \gamma\neq 0,\ c=0, \tag{22}$$

$$r(t)\propto t\exp\left[H_0\left(c-\frac{t_0^2}{t}\right)\right]\to t, \qquad \text{if } \gamma=0,\ \forall c. \tag{23}$$

These are particular solutions only, of course, and their properties need not be shared by the general solutions for a given value of n or γ. The $\gamma=0$ solution, (23), does not exhibit inflation and is asymptotic to the solution of the equation $\ddot{r}=0$. This is a result of the very rapid decay of $G(t)\propto t^{-4}$. We shall investigate this feature more systematically in Section 4. When $\gamma=4/3$ there is no possible time-variation of G which preserves the scaling invariance and for other positive values of γ the expansion is slower than in universes with constant G; power-law inflation does not occur in the varying-G solutions when $0<\gamma<2/3$.

Equation (14) describes motion under a time-dependent force for which there need exist no time-independent energy integral. Therefore we cannot write down a Friedmann equation for $\dot{r}$ in the usual way. However, there exists a class of particular exact solution with simple power-law form:

$$r(t)\propto t^{(2-n)/3\gamma}; \qquad \gamma\neq 0 \tag{24}$$

$$\rho(t)=\frac{(2-n)(3\gamma-2+n)}{12\pi G_0\gamma^2 t_0^n(3\gamma-2)t^{2-n}}=\frac{(2-n)(3\gamma-2+n)}{12\pi\gamma^2(3\gamma-2)G(t)t^2} \tag{25}$$

so $\rho\geq 0$ requires that

$$\frac{(2-n)(3\gamma-2+n)}{(3\gamma-2)}\geq 0. \tag{26}$$

Clearly, when $n=0$, these solutions reduce to the familiar zero-curvature (zero-energy) solutions of general relativistic (Newtonian) cosmology with constant G. They describe expanding universes so long as $n<2$. They are particular solutions because they do not possess the full complement of arbitrary constants of integration that specify the general solution. Solutions of this sort suggest that there may exist more general solutions that behave at early times like a solution of the form (24) with one value of $n=n_1$ for $t\leq t_1$ and with another value $n=n_2$ for $t\geq t_1$. A 'bouncing' solution would have $n_1>2$ and $n_2<2$, for example, with $t_1\ll t_0$. There are many examples of scalar–tensor gravity theories with cosmological models that display this early-time behaviour (Barrow 1993b; Barrow & Parsons 1996).

In the case of a radiation-dominated universe ($\gamma=4/3$) we have the specific form

$$\rho_r=\frac{3(1-n^2/4)}{32\pi G(t)t^2}=\frac{3(1-n^2/4)}{32\pi G_0 t_0^n t^{2-n}} \tag{27}$$

$$r(t)\propto t^{1/2-n/4}. \tag{28}$$

If the universe contains blackbody radiation ($\rho_r\propto T^4$) then the temperature–time relation which determines the primordial nucleosynthesis of helium-4 has the form

$$t\propto\frac{1}{T^{4/(2-n)}}. \tag{29}$$

Very strong bounds on non-zero values of n in this model can be obtained by considering the abundance of helium-4 produced in the early Universe. In the standard model of a radiation-dominated early Universe with constant G the neutron–proton ratio (n/p) at the time when the age of the Universe t equals the weak interaction time, $t_w=(10^{10}\,\mathrm{K}/T)^5$ s, is fixed by the temperature, $T_f\sim 10^{10}$ K, at that time. When $n\neq 0$ we have for the freeze-out temperature ($t_0=10^{17}$ s)

$$T_f=T_f(n=0)\times\frac{10^{17n(2-n)/(6-5n)}}{(1-n^2/4)} \tag{30}$$

and so for small $|n|\simeq 0$, we have

$$T_f\simeq 10^{10}\,\mathrm{K}\times 10^{17n/3}\left(1+\frac{n^2}{4}\right) \tag{31}$$

and the neutron–proton ratio is

$$\left(\frac{n}{p}\right)_{n\neq 0}=\left[\left(\frac{n}{p}\right)_{n=0}\right]^{10^{-17n/3}}. \tag{32}$$

Since the observed helium-4 mass fraction is close to ~ 0.22 we have $(n/p)_{n=0}\simeq 0.28$. Hence, if $n>0$ we have an increase in the helium-4 fraction, but if $n<0$ we have a decrease. The agreement between the standard model predictions of nucleosynthesis and observation constrain $|n|\ll 1$ but we shall not give the details here. We simply note that whilst it is easy to create increases in helium by changing the standard picture of nucleosynthesis, no other changes have been found which lead to a decrease in the helium-4 abundance.

4 INFLATIONARY UNIVERSE MODELS WITH $p=-\rho$

Scalar–tensor gravity theories have provided an arena in which to explore variants of the inflationary universe theory first proposed by Guth (1981) in which inflation is driven by the slow evolution of some weakly-coupled scalar field. The scalar field from which the gravitational coupling is derived can, in principle, be the scalar field that drives the inflationary expansion, or it can influence the form of inflation produced by some other explicit scalar matter field. A number of studies have been made of the behaviour of inflation in scalar–tensor gravity theories (Mathiazhagen & Johri 1984; La & Steinhardt 1989; Barrow & Maeda 1990; Steinhardt & Accetta 1990; García-Bellido, Linde & Linde 1994; Barrow & Mimoso 1994; Barrow 1995). The non-linear master equation, (14), governing the evolution for $r(t)$ has interesting behaviour in the inflationary cases where $\ddot r>0$.

The particular power-law solutions (24)–(26) with $2\geq\gamma>0$ expand when $n<2$. Although they accelerate with time ($\ddot r>0$) whenever $3\gamma-2<0$, the expansion only provides a possible solution of the horizon problem when

$$2-n>\gamma>0. \tag{33}$$

So, in the case of radiation ($\gamma=4/3$) the horizon problem can be solved if $n<-2$ in the early stages of the expansion.

In the most interesting case, when $\gamma=0$, and ρ is constant, the perfect-fluid matter source mimics the behaviour of a slowly rolling scalar field the evolution of which is dominated by its self-interaction potential. When $\gamma=0$ the master evolution equation (14) is linear in r

$$\ddot r=-\lambda t^{-n}r \tag{34}$$

with $\lambda<0$. We are interested in determining the asymptotic behaviour of this equation as $t\to\infty$ for all values of n in order to determine when there is asymptotic approach to the usual de Sitter solution that obtains when $n=0$. The solutions fall into three classes according to the value of n. For $n<2$, the solutions asymptote towards the WKB approximation as $t\to\infty$

$$r(t)\sim t^{n/4}\exp\left\{\frac{2H_0 t^{(2-n)/2}}{2-n}\right\};\qquad n<2 \tag{35}$$

where the constant Hubble parameter, H_0, is given by

$$H_0^2\equiv-\lambda=-\frac{4\pi G_0 t_0^n(3\gamma-2)\Gamma}{3} \tag{36}$$

which is positive for $3\gamma-2<0$.

For $n>2$ the asymptote is

$$r(t)\sim t;\qquad n>2. \tag{37}$$

In fact, this is a particular case of a stronger result that does not assume that $G(t)$ is a power law. The asymptote $r\sim t$ results whenever $G(t)$ falls fast enough to satisfy (Cesari 1963)

$$\int^{\infty} t|G(t)|\,\mathrm{d}t<\infty. \tag{38}$$

Whenever $n=2$ the asymptote is

$$r(t) \sim t^{\alpha} \tag{39}$$

$$\alpha = \frac{1}{2}(1+\sqrt{1+4A})$$

$$A \equiv \frac{8\pi G_0 \rho}{3t_0^n} > 0.$$

Thus we see that if $G(t)$ falls off faster than t^{-2} the solutions of equation (34) approach those of $\ddot{r}=0$ and no inflation occurs. By contrast, if $G(t)$ falls off more slowly that t^{-2}, grows ($n<0$), or remains unchanged ($n=0$), then inflationary solutions of the form (35) arise. We notice that in the absence of G-variation ($n=0$) this solution reduces to the well known de Sitter expansion $[r(t) \propto \exp(H_0 t)]$ familiar in general relativistic models of inflation with a constant vacuum energy density or positive cosmological constant. When $0<n<2$ it produces a form of subexponential inflation which is familiar from studies of scalar–tensor gravity theories with varying $G(t)$ and models of intermediate inflation studied in general relativistic cosmologies containing a wide range of scalar fields, (Barrow 1990b; Barrow & Saich 1990; Barrow & Liddle 1993). If $n<0$ there is superexponential inflation.

There is a Newtonian 'no-hair' theorem in the general case where no particular form is assumed for the time-variation of $G(t)$ because we can make use of the general asymptotic properties of the evolution equation. We have already given this result for cases where $G(t)$ falls off faster than t^{-2} as $t \to \infty$ in equations (37) and (38), and as t^{-2} in equation (39). In the case where the fall-off is slower than t^{-2} we have a WKB approximation

$$r(t) \sim c[G(t)]^{-1/4} \exp\left\{\omega \int [G(t)]^{1/2} \, dt\right\} \tag{40}$$

as $t \to \infty$, where $\omega^3=1$ and c is a constant, so long as

$$|t^2 G(t)| \to \infty \qquad \text{as } t \to \infty. \tag{41}$$

Clearly, equation (35) gives this asymptote in the special case that $G(t) \propto t^{-n}$ with $n<2$ when we choose the $\omega=1$ part of the linear combination of solutions.

When G is constant in general relativity and in Newtonian gravitation the presence of a perfect fluid with an equation of state $p=-\rho$ is equivalent to the addition of a cosmological constant to the field equations. However, in Newtonian gravitation with varying G and in scalar–tensor generalizations of general relativity, these two are no longer equivalent. If we had included a cosmological constant term in the Newtonian gravitational equation (10), it would become

$$\ddot{r}(t) = -\frac{4\pi G(t)(\rho+3p)r}{3} + \frac{\Lambda r}{3} \tag{42}$$

and we can see that the choice $p=-\rho=$constant, using (11), only makes the first term on the right-hand side of (42) equivalent to the term $\propto \Lambda r$ if $G(t)$ is constant. Clearly, the late-time may not be dominated by the cosmological constant term when $G(t) \propto t^{-n}$ and $n<0$. We will not investigate the behaviour for general γ. In the most interesting case, that of $\gamma=0$, $\Lambda \neq 0$ equation (42) has the general linear form

$$\ddot{r} = Q(t) r \tag{43}$$

with

$$Q(t) = \frac{8\pi G(t)\rho + \Lambda}{3} \tag{44}$$

and the asymptotic properties can now be deduced. Let us consider the case of $\Lambda>0$ in more detail to discover the circumstances under which the de Sitter solution $[r(t) \propto \exp(t\sqrt{\Lambda/3})]$ is approached as $t \to \infty$ when $G(t)$ varies and $\gamma=0$.

It is most transparent to change the time coordiante from t to τ, where

$$\tau = t\sqrt{\frac{\Lambda}{3}} \tag{45}$$

and to scale out the inessential positive constants by defining

$$\theta = \frac{8\pi G(t)\rho}{\Lambda} \propto G(\tau) \tag{46}$$

so that the scalefactor obeys the second-order linear equation

$$\frac{d^2 r}{d\tau^2} = [1+\theta(\tau)] r. \tag{47}$$

We can obtain four results of increasing strength about the behaviour of the solutions of this equation as τ, $t \to \infty$ (Bellman 1950). We are interested in the conditions on $\theta(\tau)$ under which the de Sitter solution

$$r(\tau) \propto \exp[\tau] \tag{48}$$

is approached.

(i) If $\theta \to 0$ and $\int |\theta(\tau)| \, d\tau < \infty$, then

$$r(\tau) \to \exp[\tau] \tag{49}$$

as τ, $t \to \infty$.

(ii) If $\theta(\tau) \sim \sum_{k=2}^{\infty} c_k \tau^{-k}$, with c_k arbitrary constants, then an asymptotic series for $r(\tau)$ is

$$r(\tau) \to \exp[\tau] \times \sum_{k=0}^{\infty} a_k \tau^{-k} \tag{50}$$

with a_k constants, as $\tau \to \infty$. Recall that an asymptotic series need not be a convergent series (Bellman 1953), although it can be used to calculate $r(\tau)$ numerically. In fact, here the a_n will diverge with $a_k \geq (k-1)!$

(iii) If $\theta \to 0$ as $\tau \to \infty$ and $\int^{\infty} |\theta'(\tau)| \, d\tau < \infty$, then

$$r(\tau) \to \exp\left[\int_0^{\tau} \sqrt{1+\theta(\tau)} \, d\tau\right] \tag{51}$$

as $\tau \to \infty$.

(iv) If $\theta \to 0$ as $\tau \to \infty$ and $\int_0^\infty \theta^2(\tau)\,\mathrm{d}\tau < \infty$, then

$$r(\tau) \to \exp\left[\tau + \frac{1}{2}\int_0^\tau \theta(\tau)\,\mathrm{d}\tau + o(1)\right] \tag{52}$$

as $\tau \to \infty$.

These expressions allow us to determine when the expectations of the cosmic no-hair theorem are borne out and de Sitter expansion is achieved asymptotically when $G(t)$ evolves with time. Similar techniques were used by Barrow & Götz (1989b) to prove cosmic no-hair theorems for inflation in general relativistic cosmologies.

5 THE GENERAL BEHAVIOUR OF $\ddot{r} = -\lambda t^{-n} r^{1-3\gamma}$ WHEN $\gamma \neq 0$

After an appropriate transformation of variables, the evolution equation for $r(t)$ when $G \propto t^{-n}$ has the Fowler–Lane–Emden form (Lane 1870; Ritter 1878; Emden 1907; Kelvin 1911; Fowler 1931) found in the study of stellar models. To see this, we recall that the equilibrium of a spherical cloud of gas with a polytropic equation of state $p = k\rho^{\nu}$ and a gravitational potential, $\Phi(r)$, obeys Poisson's equation (G is constant)

$$\nabla^2\Phi(r) = -4\pi G\rho \tag{53}$$

with $\Phi = 0$ at the surface where $\rho = 0$, and $\mathrm{d}p = \rho\,\mathrm{d}\Phi$. Hence,

$$\rho = \mu\Phi^m \tag{54}$$

and

$$\nabla^2\Phi(r) = \Phi_{rr} + \left(\frac{2}{r}\right)\Phi_r = -A^2\Phi^m \tag{55}$$

where m and A are constants. Normalizing the variables to remove these constants allows the Poisson equation to be expressed as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + y^m = 0. \tag{56}$$

Now, if we substitute

$$y = \frac{r}{x} \tag{57}$$

this becomes

$$r'' = -x^{1-m} r^m. \tag{58}$$

This has the same general form as equation (14) although we are interested in quite general combinations of powers on the right-hand side, which can also have either sign.

Equations of the general form (14) have been studied in some detail (Fowler 1931; Bellman 1953; Chandrasekhar 1957; Ramnath 1971; Wong 1975) because they are the asymptotic forms for any second-order differential equation of the form

$$\ddot{r} = \frac{P(r,t)}{S(r,t)} \tag{59}$$

where P and S are polynomials in their arguments.

We can determine an analytic approximation to the solutions of this equation for any choice of n for the physically relevant range of $\gamma \in (0, 2]$. We have

$$r(t) \simeq \left[\beta_0 \mp \frac{3\gamma\lambda t^{2-n}}{(1-n)(2-n)} - \beta_1 t\right]^{1/3\gamma} + t\left[\alpha_0 \mp \frac{3\gamma\lambda t^{2-n-3\gamma}}{(2-n-3\gamma)(3-n-3\gamma)} - \alpha_1 t^{-1}\right]^{1/3\gamma}, \tag{60}$$

when

$$\gamma \neq 0; \qquad n \neq 1,\ 2,\ 2-3\gamma,\ 3-3\gamma \tag{61}$$

and where β_0, β_1, α_0 and α_1 are constants. For orientation, we can see how the expression (60) delivers the usual Friedmann open-universe solutions when G is constant ($n = 0$) as $t \to \infty$, with

$$r(t) \propto t^{2/3\gamma} + \alpha t; \qquad \alpha > 0 \text{ and constant.} \tag{62}$$

We have already dealt with the $\gamma = 0$ case in Section 4. The excluded values of n lead to the following analytic approximations.

***n* = 1:**

$$r(t) \simeq [\beta_0 \mp 3\gamma(\lambda t \ln t + \beta_1 t)]^{1/3\gamma} + t\left[\alpha_0 \mp \frac{3\gamma\lambda t^{1-3\gamma}}{(1-3\gamma)(2-3\gamma)} - \alpha_1 t^{-1}\right]^{1/3\gamma}. \tag{63}$$

***n* = 2:**

$$r(t) \simeq [\beta_0 \mp 3\gamma(\lambda \ln t + \beta_1 t)]^{1/3\gamma} + t\left[\alpha_0 \mp \frac{\lambda t^{-3\gamma}}{(1-3\gamma)} - \alpha_1 t^{-1}\right]^{1/3\gamma}. \tag{64}$$

***n* = 2 − 3γ:**

$$r(t) \simeq \left[\beta_0 \mp \frac{3\gamma\lambda t^{2-n}}{(1-n)(2-n)} - \beta_1 t\right]^{1/3\gamma} + t[\alpha_0 - (\ln t + \alpha_1 t^{-1})]^{1/3\gamma}. \tag{65}$$

***n* = 3 − 3γ:**

$$r(t) \simeq \left[\beta_0 \mp \frac{\gamma\lambda t^{3+3\gamma}}{(2+3\gamma)(1+\gamma)} - \beta_1 t\right]^{1/3\gamma} + t\left[\alpha_0 \mp \frac{\gamma\lambda t^2}{2} - \alpha_1 t^{-1}\right]^{1/3\gamma}. \tag{66}$$

From these approximations we can disentangle the asymptotic forms describing universes that expand forever. These approximations hold as $t \to 0$ and as $t \to \infty$. As $t \to 0$ solutions with $r = 0$ at $t = 0$ and $3\gamma < 2$ have

$$r(t) \sim t^{2/3\gamma}. \tag{67}$$

As $t \to \infty$ the approximate expressions above lead to the following asymptotes for models containing physically realistic perfect fluids with $0 < \gamma \leq 2$.

$n > 2 - 3\gamma$:

$$r(t) \sim \frac{3\gamma\lambda t^{(2-n)/3\gamma}}{(1-n)(2-n)}, \tag{68}$$

or

$$r(t) \sim c_1 + c_2 t + \frac{t^{3-3\gamma-n}}{(3-3\gamma-n)(2-3\gamma-n)}, \tag{69}$$

or

$$r(t) \sim c_2 + \frac{c_2^n t^{2-n}}{(3-3\gamma-n)(2-3\gamma-n)}. \tag{70}$$

$n < 2 - 3\gamma$:

$$r(t) \sim c_1 + c_2 t + \frac{t^{3-3\gamma-n}}{(3-3\gamma-n)(2-3\gamma-n)}, \tag{71}$$

$$r(t) \sim \frac{t^{(2-n)/3\gamma}}{(3-3\gamma-n)(2-3\gamma-n)}. \tag{72}$$

$n = 2 - 3\gamma$:

$$r(t) \sim t^{(2-n)/3\gamma}; \qquad n < 2. \tag{73}$$

Asymptotes (68), (72) and (73) describe approach to the special power-law solutions found earlier in (24) and (25).

6 RELATIVISTIC SCALAR–TENSOR THEORIES: SOME COMPARISONS

Newtonian gravitation permits us to 'write in' an explicit time variation of G without the need to satisfy any further constraint. However, in general relativity the geometrical structure of space–time is determined by the sources of mass–energy it contains and so there are further constraints to be satisfied. Suppose that we take Einstein's equation in the form ($c \equiv 1$)

$$\tilde{G}^a_b = 8\pi G T^a_b \tag{74}$$

where $\tilde{G}^a_b$ and T^a_b are the Einstein and energy–momentum tensors, as usual, but imagine that $G = G(t)$. If we take a covariant divergence $_{;a}$ of this equation the left-hand side vanishes because of the Bianchi identities, $T^a_{b;a} = 0$, if energy–momentum conservation is assumed to hold, hence $\partial G/\partial x^a = 0$ always. In order to introduce a time-variation of G we need to derive the space and time variations from some scalar field, ψ which then contributes a stress tensor $\tilde{T}^a_b(\psi)$ to the right-hand side of the gravitational field equations.

We can express this structure by a choice of lagrangian, linear in the curvature scalar R, that generalizes the Einstein–Hilbert lagrangian of general relativity with,

$$L = -f(\psi)R + \frac{1}{2}\partial_a\psi\partial^a\psi + 16\pi L_{\mathrm{m}} \tag{75}$$

where L_{m} is the lagrangian of the matter fields. The choice of the function $f(\psi)$ defines the theory. When ψ is constant this reduces, after rescaling of coordinates, to the Einstein–Hilbert lagrangian of general relativity. (We ignore, for simplicity the possibility of including a cosmological 'constant' term which can now be a function of the scalar field ψ.) For historical reasons scalar–tensor theories have not been written with a gravitational lagrangian of this simple form (Bergmann 1968; Steinhardt & Accetta 1990; Holman et al. 1991) but have followed the formulation introduced by Brans & Dicke (1961). This can be obtained from (75) by a non-linear transformation of ψ and f. Define a new scalar field ϕ, and a new coupling function $\omega(\phi)$ by

$$\phi = f(\psi), \tag{76}$$

$$\omega(\phi) = \frac{f}{2f'^2}$$

then (75) becomes

$$L = -\phi R + \frac{\omega(\phi)}{\phi}\partial_a\phi\partial^a\phi + 16\pi L_{\mathrm{m}}. \tag{77}$$

This has the Brans–Dicke form. The Brans–Dicke theory arises as the special case

$$\text{Brans–Dicke: } \omega(\phi) = \text{constant}; \qquad f(\psi) \propto \psi^2. \tag{78}$$

However, in general, there exists an infinite number of these theories defined by the choice of $\omega(\phi)$. One of the reasons for renewed interest in scalar–tensor gravity theories of this form is the relationship that exists between the gravitational part of the lagrangian (i.e. excluding L_{m}) for Brans–Dicke theory and the low-energy effective action for bosonic string theory, which can be written as (Callan et al. 1985)

$$L_{\mathrm{sst}} = \exp(-2\chi)\left(\mathrm{R} + 4\chi^{,a}\chi_{,a} - \frac{1}{12}H^2\right) \tag{79}$$

where χ is the dilation field and $H^2 \equiv H_{abc}H^{abc}$, where H_{abc} is the totally antisymmetric 3-form field. If we identify $\phi = \exp(-2\chi)$ then L_{sst} is identical to the Brans–Dicke lagrangian, (77), when $\omega = -1$. However, differences arise in the couplings of the scalar fields to other forms of matter in the two theories.

The field equations that arise by varying the action associated with L in (77) with respect to the metric, g_{ab}, and ϕ separately, are

$$\tilde{G}_{ab} = \frac{-8\pi}{\phi}T_{ab} - \frac{\omega(\phi)}{\phi^2}\left[\phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi_{,i}\phi^{,i}\right] - \phi^{-1}[\phi_{,a;b} - g_{ab}\Box\phi] \tag{80}$$

$$[3 + 2\omega(\phi)]\Box\phi = 8\pi T^a_a - \omega'(\phi)\phi_{,i}\phi^{,i} \tag{81}$$

where the energy–momentum tensor of the matter sources obeys the conservation equation

$$T^{ab}_{;a} = 0. \tag{82}$$

These field equations reduce to those of general relativity when ϕ [and hence $\omega(\phi)$] is constant, in which case the Newtonian gravitational constant is defined by $G = \phi^{-1}$. An interesting feature of these equations is clear by inspection: when the trace of the energy–momentum tensor vanishes (this includes vacuum and radiation-dominated solutions as important particular cases) any solution of general relativity is a particular solution of the scalar–tensor theory with ϕ (and hence G) constant.

These equations are also conformally related to general relativity when $T^a_a=0$. If we conformally transform the metric

$$g_{ab}\to\Omega^{-2}g_{ab} \tag{83}$$

with $\Omega=\phi^{1/2}$ and define Ψ by

$$\Psi=\int\sqrt{\frac{2\omega(\phi)+3}{2}}\,\frac{\mathrm{d}\phi}{\phi} \tag{84}$$

then the conformally transformed theory is general relativity with a matter source consisting of a scalar field Ψ with a potential $V(\Psi)$. This conformal invariance can be exploited to produce power solution-generating procedures for scalar–tensor theories (Barrow 1993a; Barrow & Mimoso 1994; Barrow & Parsons 1996). One can regard Meshcherskii's theorem and its generalization proved above in (16) and (17) as Newtonian analogues of these conformal invariances.

Although the coupling function $\omega(\phi)$ is unconstrained by the structure of scalar–tensor gravity theories, its choice determines the form of the cosmological models in the theory, and the form of the weak-field limit. Typically, the weak-field Solar-system predictions of scalar–tensor gravity theories have the following relationship to those of general relativity (GR)

$$\omega(\phi)\text{ weak field result}\approx(\text{GR result})\times\left[1+O\left(\frac{\omega'}{\omega^3}\right)\right]. \tag{85}$$

Specifically, the perihelion procession of Mercury, $\Delta\vartheta$, and the time-variation of G are predicted at second order to be (Nordtvedt 1970; Wagoner 1970; Will 1993)

$$\Delta\vartheta\simeq 43\text{ arcsec}\times\left[1-\frac{1}{12+6\omega}\left(4+\frac{\omega'}{(3+2\omega)^2}\right)\right]\text{ per 100 yr} \tag{86}$$

$$\frac{\dot G}{G}=-\left(\frac{3+2\omega}{4+3\omega}\right)\left[\frac{G(t)}{G_0}+\frac{2\omega'(\phi)}{(3+2\omega)^2}\right]\dot\phi \tag{87}$$

where G_0 is the present measured value of $G(t)$. There is even a special theory for which the term in [...] brackets vanishes in (87) and $G(t)=G_0$ is constant to this weak-field order (Barker 1978). In Brans–Dicke theory we have $G\propto\phi^{-1}$. The general relativistic limit of an $\omega(\phi)$ scalar–tensor theory is obtained (if it exists) by taking the two limits

$$\omega\to\infty\quad\text{and}\quad\frac{\omega'}{\omega^3}\to 0. \tag{88}$$

The observational limits on ω require $\omega>500$ but the limit on ω' is weak, $\omega'<O(1)$, see Will (1993). Similar limits are obtained from the binary pulsar (Damour, Gibbons & Tayler 1988) but are more model dependent.

In order to examine some of the counterparts to the Newtonian solutions discussed above we give the field equations for an isotropic and homogeneous Friedmann universe with scalefactor $a(t)$, curvature parameter k, and Hubble rate $H=\dot a/a$,

$$H^2=\frac{8\pi\rho}{3\phi}-\frac{H\dot\phi}{\phi}+\frac{\omega(\phi)\dot\phi^2}{6\phi^2}-\frac{k}{a^2} \tag{89}$$

$$\ddot\phi+3H\dot\phi+\frac{\omega'\dot\phi^2}{2\omega+3}=\frac{8\pi\rho(4-3\gamma)}{2\omega+3} \tag{90}$$

$$\rho\propto a^{-3\gamma}. \tag{91}$$

We will now focus attention upon solutions of these equations with $k=0$ in the case of Brans–Dicke theory (ω constant). Scalar–tensor theories with varying G differ from general relativity in that they admit vacuum solutions when $k=0$ (O'Hanlon & Tupper 1970)

$$G\propto\phi^{-1}\propto t^{-d/(1+d)} \tag{92}$$

$$a(t)\propto t^{1/3(1+d)}$$

$$d\equiv\omega^{-1}\left(1+\sqrt{1+\frac{2\omega}{3}}\right).$$

They also possess a class of special power-law solutions for perfect-fluid universes (Nariai 1969),

$$G\propto\phi^{-1}\propto t^{-B} \tag{93}$$

$$a(t)\propto t^A$$

$$p=(\gamma-1)\rho$$

where

$$A=\frac{2+2\omega(2-\gamma)}{4+3\omega\gamma(2-\gamma)}, \tag{94}$$

$$B=\frac{2(4-3\gamma)}{4+3\omega\gamma(2-\gamma)}. \tag{95}$$

The general solutions can be found for all γ but are rather cumbersome and opaque; exact solutions for $k=0$ have been found by Gurevich, Finkelstein & Ruban (1973) and for all k by Barrow (1993a); a phase-plane analysis has been performed by Kolitch & Eardley (1993) which includes the $k\neq 0$ models. Their typical properties are as follows. As $t\to 0$ they approach the vacuum solutions (92), while as $t\to\infty$ they approach the matter-dominated solutions (93)–(95). As $\omega\to\infty$ these matter-dominated solutions approach the general relativistic results, $a(t)\propto t^{2/3\gamma}$, G constant. There is a smooth transition between these simple early- and late-time behaviours. Thus the power-law matter-dominated solutions (93)–(95) are unstable as $t\to 0$. The general solutions are dominated by the Brans–Dicke scalar field. A similar early-time behaviour occurs in the $\omega(\phi)$ theories although the form of the early vacuum-dominated phase depends on the detailed functional form of $\omega(\phi)$, (see Barrow 1993a,b, 1995, Barrow & Mimoso 1994, Damour & Nordtvedt 1993, Serna & Alimi 1996, and Barrow & Parsons 1996 for details).

It is now possible to consider some of the similarities and differences that exist between the Newtonian cosmologies with varying G and their curved space–time counterparts. We recall that the special Newtonian power-law solutions (24) have the form $G \propto t^{-n}$ with scalefactor $r \propto t^{(2-n)/3\gamma}$. This is identical to the Brans–Dicke solution when $p=0$ and

$$n=\frac{2}{4+3\omega}. \tag{96}$$

In fact, for any choice of $\omega(\phi)$ dust universes have

$$\phi a^3 \rightarrow t^2 \tag{97}$$

which corresponds to Newtonian solutions with $G \propto t^{-n}$ and scalefactor $r \propto t^{(2-n)/3}$.

The vacuum solutions have a slightly different structure. If we identify

$$n=\frac{d}{d+1} \tag{98}$$

then we have

$$a(t) \propto t^{1/3(1+d)} \propto t^{(1-n)/3}. \tag{99}$$

However, there is no general correspondence for other equations of state. Most notably, Brans–Dicke radiation-dominated universes ($\gamma=4/3$) are the same as in general relativity, $G=$constant and $a(t) \propto t^{1/2}$, and differ from the Newtonian solutions. In general, the Newtonian solutions are the same as the Brans–Dicke matter-dominated solutions only if we make the choice

$$n=\frac{2(4-3\gamma)}{4+3\omega\gamma(2-\gamma)}. \tag{100}$$

Also, as can be seen from the right-hand side of the $\phi(t)$ evolution equation, (90), the effective sign of the gravitational coupling changes sign from negative to positive when γ becomes greater than 4/3.

The Newtonian inflationary solutions with $\gamma=0$, given in equation (35), do not have direct counterparts in Brans–Dicke gravity theories. However, scalar–tensor theories with

$$2\omega+3 \propto \phi^h \tag{101}$$

have solutions of similar form as $t \rightarrow \infty$ with (Barrow & Mimoso 1994; Barrow & Parsons 1996)

$$G \propto \phi^{-1} \propto t^{-2/(2h+1)} \tag{102}$$

and the scalefactor evolves as

$$a(t) \propto t^{(h-1)/3(2h+1)} \exp\{At^{2h/(2h+1)}\}. \tag{103}$$

Thus, with $2h+1>0$, we have $\omega \rightarrow \infty$ and $\omega'/\omega^3 \rightarrow 0$ as $t \rightarrow \infty$ and general relativity is approached in the weak-field limit. These solutions are not of identical form to the Newtonian solutions with $G \propto t^{-n}$ and $n=2/(2h+1)$.

7 CONCLUSIONS

We have formulated the problem of varying G in Newtonian gravitation theory in a simple and natural way. The resulting generalization of Newton's second law of motion is non-conservative and no energy integral is therefore guaranteed. Earlier results of Meshcherskii and others are generalized to Newtonian cosmologies with perfect-fluid equations of state $p=(\gamma-1)\rho$. These show how solutions of Newtonian gravitation problems where G varies in time with particular power-law forms can be obtained by direct scaling of the solutions to the equations in which G is constant. Exact power-law solutions were found which act as attractors for the general ever-expanding solutions for all perfect-fluid equations of state. We investigated the asymptotic behaviour of Newtonian varying-G cosmologies in the presence of a $p=-\rho$ stress, and with and without a cosmological-constant stress, showing how these two stresses differ when G varies, and establishing conditions on the rate of decrease of $G(t)$ in order for asymptotic approach to de Sitter expansion to occur when $\Lambda>0$.

Relativistic theories of gravitation with varying G are more complicated in structure than Newtonian theories because of their geometrical form. We have examined some of the inter-relationships between cosmological solutions of Newtonian theories with varying G and their relativistic counterparts which arise in scalar–tensor theories of gravity. Whilst there is no guarantee that any Newtonian solution with varying G will have a counterpart in a scalar–tensor gravity theory (and vice versa), the identification of simple Newtonian analogues of solutions to complicated scalar–tensor theories of gravity is of cosmological interest. It allows the cosmological consequences of varying G to be investigated in a very simple manner, just as the use of Newtonian cosmological solutions with exact general relativistic counterparts is a well-established practice of considerable methodological and didactic importance in the study and exposition of general relativistic cosmological models. It is intended that the simple Newtonian solutions discussed in this paper should aid the astronomical testing of the hypothesis that G is slowly varying in time.

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