ELSEVIER

16 February 1998

PHYSICS LETTERS A

Physics Letters A 238 (1998) 337-340

Optimum positioning of attracting masses to reduce the non-linear effects in the swing-time method of measuring the gravitational constant

Zhongkun Hu, Jun Luo [1]

Department of Physics, Huazhong University of Science and Technology, Wuhan 430074, China

Received 11 June 1997; revised manuscript received 1 December 1997; accepted for publication 3 December 1997
Communicated by J.P. Vigier

Abstract

Non-linear effects in the swing-time method of measuring the gravitational constant have been studied in both far and near positions. We found that the optimum positioning should be selected to observe the periods of a pendulum as the attracting masses move in and out of the far position, because the non-linear effect can be greatly reduced in this case. © 1998 Elsevier Science B.V.

PACS: 06.20.Jr; 04.90.+e
Keywords: Torsion pendulum; Non-linear effect; Measurement of gravitational constant

The gravitational constant G has an important place in physics. It has been investigated experimentally during the last three centuries and the precision of the measurement has improved steadily [1]. The most precise determination of G is possible by means of a Cavendish torsion balance [2]. Based on this method, the gravitational constant G is determined with a relative uncertainty of 130 parts per million (ppm), much larger than the uncertainty of all other fundamental constants [3]. Recently, three new values for G have been reported [4–6], differing dramatically from the CODAT value and also from one another. It seems clear that an uncertainty of about 100 ppm is at present adequate.

The gravitational constant can be measured by observing the swing period of a sensitive torsion pendulum when the attracting masses are in and out of positions [7] or in the near and far positions [8,9]. This method, suggested by Forbes, was first used by Reich [10], developed further by Braun [11], Eötvös [12], Heyl [8,9], Renner [13], Karagioz [14] and Sagitov [15], and was used to best effect by Luther and Towler [7]. The swing-time method, unlike the conventional Cavendish direct deflection method, is capable of avoiding errors caused by the drift of the equilibrium position of the torsion pendulum, which often occurs because of imperfections in the crystal structure of the torsion fibre, and fluctuations in the experimental environment, and it plays an important role in the determination of the gravitational constant G. Recently, Kuroda [16] pointed out that the swing-time method may have a systematic bias due to the inelasticity of the torsion fibre, and this upward fractional bias should be $1/\pi Q$. Bagley

[1] E-mail: jluo@blue.hust.edu.cn.

0375-9601/98/$19.00
PII S0375-9601(97)00924-9

and Luther [17] tested this hypothesis in experiment. It turned out that this effect should be considered carefully in the experiment of determining G with the swing-time method.

Because of the non-zero amplitude of the swing and the varying intensity of the gravitational attraction field, non-linear effects in the swing-time method are important. This may cause serious systematic errors in the determination of the gravitational constant at high precision, i.e. 100 ppm or higher. To reduce the non-linear effects, most experimenters selected the small amplitude θ_0 operation, but the noise contribution to the frequency measurement increases with amplitude as $1/\theta_0$, and sets a limitation to the precision of the experiments. Chen discussed the non-linear effect due to the θ^2 term in the swing-time method in Ref. [18]. In most experiments the system of the torsion pendulum is symmetrical, i.e. the attracting and attracted masses are equal in mass M and m, respectively. In this case, no non-linear effect due to the θ^2 term would be observed, but generally speaking, the non-linear effect due to the θ^3 term may exist. We studied this θ^3 term and found that the non-linear effect due to this term does not exist when the attracting masses are in the optimum positioning in the far position, but in the near position we can not find the optimum positioning.

Fig. 1 shows a typical experimental arrangement for the swing-time method. If the attracting masses are placed in line with the beam, i.e. in the near position, the period of oscillation is decreased. If they are placed on a line perpendicular to the beam, i.e. in the far position, the period is increased. The gravitational constant therefore can be determined from the difference of these periods.

In the gravitational field of the attracting masses, the torsion pendulum has the Lagrangian function

$$L = \tfrac{1}{2} I\dot{\theta}^2 - \tfrac{1}{2} K\theta^2 - \sum_{i,j=1}^{2} m_i V_j, \tag{1}$$

where θ is the angle of displacement of the pendulum, I is the momentum of inertia of the whole torsion pendulum system, $\frac{1}{2}K\theta^2$ is the potential energy of the suspension fibre, K is the torsion constant, and $\sum m_i V_j$ is the potential energy of the attracted masses in the gravitational field of the attracting masses. From Eq. (1) the equation of motion of the pendulum is

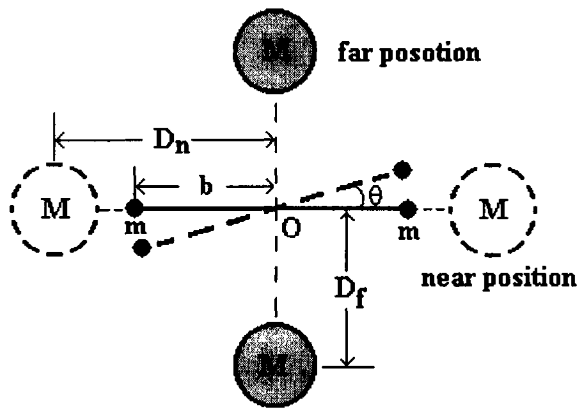

Fig. 1. Illustration of the typical arrangement of the swing-time method to measure the gravitational constant. m: attracted masses; M: is attracting masses (full circles: far position; broken circles: near position). D_f: distance of the attracting mass from the centre of suspension in the far position; D_n: distance of the attracting mass from the centre of suspension in the near position; b: half-length of the torsion beam; θ: angle of the torsion pendulum.

$$I\ddot{\theta} + K\theta + \sum m_i \frac{\partial V_i}{\partial \theta} = 0. \tag{2}$$

If damping of the torsion pendulum can not be ignored as the effect of the non-linear term varies with amplitude, the above equation will not be complete. Hence, over a given interval of time the change of the period due to non-linearity will depend upon the damping factor. We therefore study the following equation,

$$I\ddot{\theta} + K\theta + \sum m_i \frac{\partial V_i}{\partial \theta} + 2\gamma\dot{\theta} = 0, \tag{3}$$

where γ is the damping factor. For simplicity all masses are taken to be point masses. In the case of the far position, we expand the series $\sum m_i V_j$ to third order in θ,

$$\sum m_i V_j = K_{gf}\theta + C_{f3} K_{gf} \theta^3, \tag{4}$$

where K_{gf} is the equivalent torsion constant of gravitation, the dimensionless constant C_{f3} indicates the non-linear term due to θ^3, and K_{gf} and C_{f3} are defined as follows,

$$K_{gf} = -\frac{12 D_f^2 b^2}{(D_f^2 + b^2)^{5/2}} GMm, \tag{5a}$$

$$C_{f3} = \frac{1}{6}\left(\frac{35\lambda_f^2}{(1+\lambda_f^2)^2} - 4\right), \tag{5b}$$

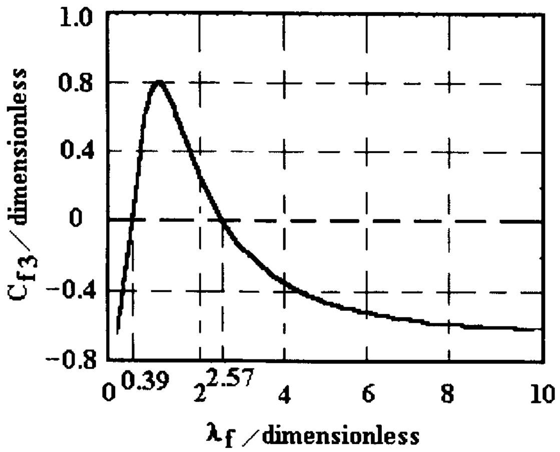

Fig. 2. The non-linear term in the far position.

The dimensionless structure constant of the torsion pendulum λ_f is defined as

$$\lambda_f = D_f/b, \tag{5c}$$

where D_f is the distance of the attracting mass from the centre of suspension in the far position, and b is the half-length of the beam.

Inserting Eq. (4) into (3), we get the equation of motion of the pendulum in the case where the attracting masses are in the far position,

$$I\ddot{\theta} + (K + K_{gf})\theta + C_{f3}K_{gf}\theta^3 + 2\gamma\dot{\theta} = 0. \tag{6}$$

The non-linear effect will vary with C_{f3}, which is a function of λ_f. Fig. 2 shows the dependence of C_{f3} upon λ_f. From Fig. 2 and Eq. (5) we see that C_{f3} will be equal to zero when $\lambda_f = 0.389$ or $\lambda_f = 2.569$. The optimum positioning of the attracting masses can be chosen as $D_f = 0.389b$ or $D_f = 2.569b$ in the far position, and in this case, the non-linear effect due to θ^3 will disappear. So the whole non-linear effect of the motion of the pendulum will only be fifth or higher order in θ and will hence be greatly reduced. Furthermore, in the case of optimum positioning, we can operate the torsion pendulum at a large amplitude because of the smaller non-linear effect. A large amplitude operation is also favoured due to the fact that noise is contributing as $1/\theta_0$; comparison of G determinations carried out at different amplitudes provides a powerful check for the effects of fibre nonlinearities and background torques produced by a local gravity gradient, which varies greatly with the amplitude of the torsion pendulum.

Furthermore, from Eq. (5) we can see that D_f can be chosen to maximize K_{gf}. Speake [19] found that the maximum value of K_{gf} occurs at $D_f = 0.816b$. If this positioning is selected, the requirement of accuracy in measuring the distance between the attracting masses can be greatly reduced. As stated above, there are two kinds of advantages in the far position. First, it is capable of reducing the non-linear effect due to the θ^3 term, and secondly it is capable of reducing the requirement of accuracy in measuring the distance between the attracting masses. If we make use of the above two advantages at the same time, we can find a real optimum positioning for the attracting masses in the far position to obtain a higher precision in the experiments for determining G.

Now, let us consider the equation of motion of the pendulum when the attracting masses are in the near position,

$$I\ddot{\theta} + (K + K_{gn})\theta + C_{n3}K_{gn}\theta^3 + 2\gamma\dot{\theta} = 0, \tag{7}$$

where

$$K_{gn} = \frac{4b^2D_n\left(b^2 + 3D_n^2\right)}{(D_n^2 - b^2)^3}GMm, \tag{8a}$$

$$C_{n3} = -\frac{12\lambda_n^6 + 85\lambda_n^4 + 46\lambda_n^2 + 1}{6(3\lambda_n^6 - 5\lambda_n^4 + \lambda_n^2 + 1)}, \tag{8b}$$

and

$$\lambda_n = D_n/b. \tag{8c}$$

The non-linear effect will vary with C_{n3} in the near position. Fig. 3 shows the dependence of C_{n3} upon λ_f. From Fig. 3 we see that $|C_{n3}|$ is a monotonically decreasing function of λ_f, and there is no optimum positioning. At the same time, from Eq. (8a), K_{gn} does not have a maximum value either. The larger λ_f, the smaller the non-linear effect will be; but, at the same time, the smaller the relative difference of periods. Conversely, the greater the difference, and the more effective for the determination of G, the more serious the non-linear effect will be. For the typical experimental parameters $M = 10$ kg, $m = 50$ g, $b = 100$ mm, $\lambda_f = 1.5$, $K = 3 \times 10^{-8}$ Nm/rad and $\theta_0 = 0.01$ rad, the value of K_{gn} is 0.79×10^{-8} Nm/rad,

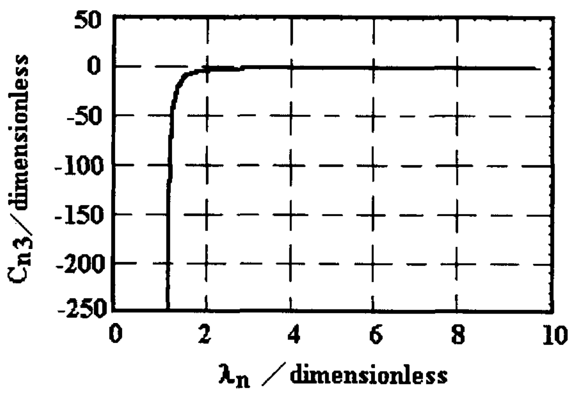

Fig. 3. The non-linear term in the near position.

C_{n3} is -9.24, the relative change of period due to the non-linear effect is [20]

$$\frac{3C_{n3}K_{gn}}{8(K+K_{gn})}\theta_0^2 = -7\times10^{-5}, \tag{9}$$

and the corresponding relative change of G is about 1.4×10^{-4}. If the aim is to measure G to 100 ppm, non-linear effects must be considered in this positioning.

Eq. (5) will not be strictly correct for non-spherical masses. However, it is evident that, for the far position, there exists indeed an optimum positioning of the attracting masses, which will greatly reduce the non-linear effect due to the θ^3 term in the determination of the gravitational constant. Furthermore, there also exists another optimum positioning in the far position, in which the requirement of accuracy in measuring the distance between the attracting masses can be greatly reduced.

As stated above, there are two kinds of advantages in the far position, i.e. it is capable of reducing the non-linear effect, and also it is capable of reducing the requirement on position accuracy. If we make use of the above two advantages at the same time, we can find a real optimum positioning for the attracting masses in the far position to obtain a higher precision for experiments. Besides, the above discussion also implies that the optimum positioning in the swing-time method of measuring the gravitational constant should be selected to observe the periods of the pendulum as the attracting masses move in and out of the far position, and the near position should be avoided.

We are grateful to the National Natural Science Foundation of China for financial support under Grant 19425008.

References

[1] A. Cook, Rep. Prog. Phys. 51 (1988) 707.
[2] G.T. Gillies, T.C. Ritter, Rev. Sci. Instrum. 64 (1993) 283.
[3] E.R. Cohen, B.N. Talor, Rev. Mod. Phys. 59 (1987) 1121.
[4] M.P. Fizgerald, T.R. Armstrong, IEEE Trans. on Instrum. and Measur. 44 (1995) 494.
[5] H. Walesch, H. Meyer, H. Piel, J. Schurr, IEEE Trans. on Instrum. and Measur. 44 (1995) 491.
[6] W. Michaelis, H. Haars, R. Augustin, Metrologia 32 (1995/96) 267.
[7] G.G. Luther, W.R. Towler, Phys. Rev. Lett. 48 (1982) 21.
[8] P.R. Heyl, J. Res. Nat. Bur. Stds. 5 (1930) 1243.
[9] P.R. Heyl, P. Chrzanowski, J. Res. Nat. Bur. Stds. 29 (1942) 1.
[10] F. Reich, Leipz. Abh. Math. Phys. 1 (1852) 385.
[11] C. Braun, Denkschr. Akad. Wiss. Wien. Math. Naturwiss. 64 (1896) 187.
[12] R.V. Eö tvö s, Ann. Phys. Chemie. 59 (1896) 354.
[13] J. Renner, NASA Tech. Transl. F-15 (1974) 722.
[14] O.V. Karagioz et al., Izvest. Acad. Sci.USSR Phys. Solid. Earth 5 (1976) 351.
[15] M.U. Sagitov et al., Dokl. Akad. Nauk.SSSR 245 (1977) 256.
[16] K. Kuroda, Phys. Rev. Lett. 75 (1995) 2796.
[17] C.H. Bagley, G.G. Luther, Phys. Rev. Lett. 78 (1997) 3047.
[18] Y.T. Chen, Phys. Lett. A 106 (1984) 19.
[19] C.C. Speake, Metrologia 24 (1987) 97.
[20] Y.T. Chen, A.H. Cook, Gravitational Experiments in the Laboratory (Cambridge University Press, Cambridge, 1993).