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Torsion Vector and Variable G

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In the Einstein-Cartan theory coupled to a classical spin fluid only the traceless part of the torsion field can be related to spin. Here the theory is generalized by allowing intrinsic dilation currents which couple algebraically to the torsion trace. The intrinsic dilation current is interpreted as the relative change of intrinsic length scales relative to the gravitational length scale. By a dilation current conservation law, one finds the scale function between the atomic length scales and the gravitational length scales. This is in effect a dynamic unification of the atomic and gravitational units of measurement. The theory is applied to the Friedmann-Robertson-Walker cosmology, and physical effects in the solar system are considered.

1. GRAVITATION AND TORSION

1.1 Incompleteness of General Relativity

Space-time is the underlying structure of any physical theory. The properties of this structure could therefore be reflected also in non-gravitational physics. This is indeed the case: In quantum field theory the *microscopic causality* implied by the causal structure of space-time, leads to the celebrated spin-statistics theorem [1]. Yet more fundamentally, the space-time structure underpins the concepts of mass and spin, because these quantities can be identified as classification labels of the irreducible representations of the Poincaré group [2,3].

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On this background one suspects that General Relativity is incomplete, because of its asymmetric treatment of mass and spin. However, since spin is not an additive quantity like mass, spin-gravitation is less important than the mass-energy-gravitation in the macroscopic world. This explains the experimental success [4] enjoyed by Einstein's General Theory of Relativity.

1.2 Einstein-Cartan theory

Cartan [5–7] made the first attempts to include a coupling between geometry and spin-angular momentum. After the work of Cartan the problem of the dynamical rôle of spin in gravitational physics seems to have been more or less forgotten until Utiyama [8] put gravitation within a gauge theoretic framework. Later the gauge approach to gravity was developed by Kibble [9] and Sciama [10]. For a lucid presentation of Poincaré gauge theory, see Refs. 11,12. These gauge theory investigations indicated that gravity cannot be described only by the metric, but that torsion has independent dynamical importance. Hence, one was lead to the U_4; the Riemann-Cartan geometry.

Another strong argument in favour of the Riemann-Cartan geometry is that this is the most general geometry which is compatible with a field theoretic version of the equivalence principle [13–15]. Also other heuristic arguments can be given [15–17].

If the simplest possible curvature scalar, the Ricci-Cartan scalar, is taken as the gravitational Lagrangian density, and if one considers the metric and the torsion as fundamental fields, one gets the Einstein-Cartan theory [18–23].

The Einstein-Cartan theory predicts specific spin-effects such as gravitational repulsion [24], inertial dragging [25–27], and global causality violation [27]. Some of these effects are known also in connection with orbital angular momentum (Refs. 28–30, 31, Chs.21 and 33, 32–36). It is reassuring that these angular momentum effects appear also for spin.

1.3 The torsion trace problem and its solution

While the traceless part of torsion can be thought of as a geometrization of spin [37–39], the physical meaning of the torsion trace has been obscure [40]. Ray and Smalley noted a source-term in the energy-conservation law when a torsion vector is present, and they were thereby lead to associate it with non-conservation of particle number [38–40]. This was the only possible interpretation in a constant G framework.

In the Weyssenhoff-Raabe [41] spin-fluid approximation one applies the Frenkel [42] condition $S^{\alpha}{}_{\beta\alpha} = u^{\alpha} s_{\beta\alpha} = 0$. The Casimir operators $P \cdot P$ and $W \cdot W$ of the Poincaré group (Ref. 3, vol. 2, Ch. 15) correspond to mass squared and spin squared, respectively. From a group theoretic point of view, it is interesting that the Frenkel condition has an analogue among the 'square roots' of these Casimir operators: $P \cdot W = 0$. The momentum and spin operators are orthogonal. This fact strongly suggests that the proper spin tensor never has more than 20 independent components.

In the standard interpretation of the Einstein-Cartan theory one identifies proper spin, $S^{\alpha}{}_{\beta\gamma}$, as the source in the Cartan field equation for torsion. Then the Frenkel condition implies a vanishing torsion trace. This restriction of the U_4 geometry also has a geometric justification if torsion is understood as the effect of a local Lorentz rotation field. In this connection the torsion trace has been identified with a scale gauge field, and the Frenkel condition can be interpreted as a 'no dilation' constraint [43]. It could, however, be regarded as a problem that the theory does not use the full geometry. At any rate a restriction of this type should be motivated by physical considerations.

Here we propose to follow up the 'dilation' interpretation of the torsion trace, and relax the 'no dilation' constraint. It is shown how the torsion trace can be included in a self-consistent theory with the torsion vector as a physical field associated with a variable gravitational coupling. This link between $\dot{G}$ and the torsion vector was first noticed by Rauch [44]. The resulting theory completes the scale-covariant theory of Canuto et al. [45,46] by a dilation conservation law which fixes the evolution of the relative scale of gravitational and atomic units.

It could be objected that the problem of interpretation of the torsion trace is connected only with the use of a crude spin-fluid approximation, and that one should not base a general theory on statements which hold only for specific cases. However, as far as a *classical phenomenological theory of gravitation* is concerned, the spin tensor has no trace, and if the torsion trace is nonvanishing it must have another source.

2. EQUIVALENCE PRINCIPLES

2.1 Non-locality of the strong equivalence principle

The guiding principle in the construction of General Relativity was an equivalence principle. Einstein's theory satisfies the *Strong Equivalence Principle* which encompasses the universality of free fall of test-particles, the universality of gravitational redshift, and which demands that the result of any local test-experiment (gravitational or non-gravitational) in a freely falling inertial system does not depend on where and when it is done and not on the velocity of the inertial system [47–49].

The Strong Equivalence Principle is to some extent non-local. To require that all locally inertial test-experiments are independent of position elevates it to a principle of *global equivalence of space-time points*. Relaxing this principle opens the possibility of a variable gravitational coupling [50]. Also the concept of particles become non-local if matter is described by matter fields, because then the properties of particles such as mass and spin appear as results of integrations over the whole space. Thus the strong equivalence principle is incompatible with a field-theoretic description of nature.

2.2 The local equivalence principle

A new, strictly local equivalence principle has been proposed [13,14]. It states that *locally the properties of matter in a non-inertial frame of reference in flat space-time cannot be distinguished from their properties in a corresponding gravitational field.* By locally we mean that the extension of the system in consideration is restricted in space and time so that tidal effects cannot be measured within the actual measuring accuracy.

The equivalence principle is not a statement about properties of matter as such, because the question of equivalence of inertial mass and gravitational mass is a question which cannot be raised without reference to a theory of gravitation. Thus the equivalence principle is a statement about the coupling of matter to gravitation, and accordingly it is a specification of the gravitational theory itself. Both the equivalence of gravitation with acceleration which is a kinematic phenomenon and thereby directly linked to space-time properties, and the universality of gravitation implied by the equivalence principle, suggests that the equivalence principle should be interpreted as a statement about the geometry of space-time.

2.3 U_4-geometry and the local equivalence principle

Let a gravitational field in the point x be given by the metric $g_{\mu\nu}$ and the contortion tensor $K^{\alpha}{}_{\beta\gamma}$ relative to some frame. A transformation to an orthonormal[2] Lorentz frame is given by the matrix $e^{\alpha}{}_{\hat{\alpha}}$. In a fixed point we may choose 'initial conditions' corresponding to e and ∂e. This determines the 16 components of e and the 16×4 components of ∂e. If we require

$$g_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} \qquad \text{and} \qquad e^{\nu}{}_{\hat{\gamma}}\partial_{\nu} g_{\hat{\alpha}\hat{\beta}} = 0, \tag{1}$$

that is, if we require a locally constant Minkowski metric, we fix 10 components of e and 40 of its derivatives. To satisfy the equivalence principle it should be possible to find a Lorentz frame in which the metric is Minkowski *and* the connection vanishes.

[2] Indices with a hat refer to a (pseudo) orthonormal tetrad basis.

In a Riemann-Cartan space the connection is of the form

$$\Gamma_{\alpha\beta\gamma} = g_{\alpha\rho}\{^{\rho}{}_{\beta\gamma}\} + \Upsilon_{\alpha\beta\gamma} + K_{\alpha\beta\gamma}. \tag{2}$$

Here the terms on the right hand side are the Christoffel symbol, the object of anholonomy and the contorsion tensor, respectively.

In an orthonormal frame $\{^{\rho}{}_{\beta\gamma}\} = 0$. Thus for the connection to vanish in this frame one must have

$$\Upsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = -K_{\hat{\alpha}\hat{\beta}\hat{\gamma}}. \tag{3}$$

This condition fixes the remaining 24 components of ∂e. Thus the 'initial conditions' of the Lorentz frame are fixed up to 6 free parameters of e which are related to local validity of special relativity.

This shows that the Riemann-Cartan geometry is the most general geometry which is compatible with the existence of freely falling frames where gravitational effects locally cannot be felt. Conversely, if the equivalence principle is valid, it should also be possible to mimic any accelerated frame in Minkowski space by a gravitational field.

Consider therefore the matter Lagrangian $L = L(\phi, \partial\phi, \eta_{\alpha\beta})$. Let now the frame of reference be transformed as

$$e_{\alpha} = e_{\alpha}{}^{\hat{\alpha}} e_{\hat{\alpha}}. \tag{4}$$

Then the metric takes the form

$$g_{\alpha\beta} = e_{\alpha}{}^{\hat{\alpha}} e_{\beta}{}^{\hat{\beta}} \eta_{\hat{\alpha}\hat{\beta}}. \tag{5}$$

In Minkowski space-time, we may write the covariant derivative as

$$\nabla_{\alpha} = e_{\alpha}{}^{\hat{\beta}}[\partial_{\hat{\beta}} + e^{\mu}{}_{\hat{\rho}}(\partial_{\hat{\beta}} e_{\nu}{}^{\hat{\rho}}) f_{\mu}{}^{\nu}] \tag{6}$$

where $f_{\mu\nu}$ are Lorentz generators of the field ϕ. Since $e_{\hat{\beta}} = \partial/\partial x^{\hat{\beta}}$ where $x^{\hat{\beta}}$ are Cartesian *coordinates*, the expression $\partial_{\hat{\beta}} e_{\nu}{}^{\hat{\rho}}$ equals a connection $\Gamma^{\hat{\rho}}{}_{\hat{\alpha}\hat{\beta}} e_{\nu}{}^{\hat{\alpha}}$ which is metric preserving. Thus with respect to the frame e_{α} the Langrangian density of the matter field takes the form

$$L \rightarrow L(\phi, \nabla_{\mu}\phi, g_{\mu\nu})\sqrt{-g}\,. \tag{7}$$

If matter fields couple minimally to gravitation, that is, if gravitational effects appear through the substitution

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu} \qquad \text{and} \qquad \partial_{\mu} \rightarrow \nabla_{\mu}, \tag{8}$$

it is possible to mimic the behaviour of matter in an accelerated frame by a suitable gravitational field. We conclude that a theory of gravitation based on the Riemann-Cartan geometry, when minimally coupled, cannot be locally distinguished from noninertial effects in Minkowski space-time. By a 'maximalistic hypothesis' that Nature makes use of all possible degrees of freedom, the local equivalence principle is the strongest reason for taking the Riemann-Cartan geometry as the mathematical model of space-time [15].

2.4 Non-local aspect of the Riemannian constraint

The General Theory of Relativity is based on Riemannian geometry. Einstein (Ref. 51, p.70) justified the requirement of a symmetric connection by demanding that *"by the aid of a Euclidean system of local co-ordinates the same parallelogram will be described by the displacement of an element* $d^{(1)}x_\nu$ *along a second element* $d^{(2)}x_\nu$ *as by a displacement of* $d^{(2)}x_\nu$ *along* $d^{(1)}x_\nu$*."* It is, however, not in the spirit of the local character of the Equivalence Principle to make the assumption that the local reference frame is represented by a *holonomic* Euclidean frame, as this would force the local frames of reference to establish a global holonomic frame [13]. Thus the restriction to a Riemann geometry has no fundamental physical justification.

3. PHYSICAL UNITS OF MEASUREMENT

3.1 Identification of units

To measure a length one needs a unit of measurement. In gravitational physics the two body system provides a clock and a length scale, the unit of time being the orbital period and the unit of length the perihelion distance. Similarly atomic physics contains clocks and length units. Because different interactions provide *independent* systems of units, it is logically conceivable that these different systems are related by a variable scaling factor [45,50].

In the Newtonian framework space was itself the cause of inertial properties of matter. Space was regarded as absolute; *"having a physical effect, but not itself influenced by physical conditions"* (Ref. 51, p.55). With the advent of General Relativity inertia and gravitation were unified and attributed to dynamical properties of space-time. In doing so, Einstein introduced a new absolute property, the gravitational coupling constant G — again a property having a physical effect, but itself not being influenced by physical conditions. Experience has shown that to a high precision, gravitational and atomic units define equal (up to a constant scale) units of measurement, but as Einstein (Ref. 51, p.56) noted (in connection with the equality of inertial and gravitational mass), *"science is fully justified in assigning such a numerical equality only after this numerical equality is reduced to an equality of the real nature of the two concepts."* Hence, the implicit assumption of a constant scale between gravitational and atomic units is unacceptable, firstly because it involves an unjustified equivalence of gravitational and atomic units, and secondly because it provides space-time with an absolute property which acts itself, but which cannot be acted upon.

Electromagnetism and gravitation have independent fundamental units of length, and consequently they define independent physical clocks and measuring rods [45,52]. These units are the unit of classical gravity Gm/c^2, the unit of quantum gravity $(G\hbar/c^3)^{1/2}$, the unit of classical electromagnetism e^2/mc^2 and finally the unit of quantum electrodynamics $\hbar^2/me^2$. The basic units depend on the parameters G, m, $\hbar$, e and c, which can be reduced to three fundamental dimensions: mass, length and time. Special Relativity, which henceforth will be assumed to be valid in all units, unifies length and time through the universal constant c. We are therefore left with two types of dimensional units, mass and length.

3.2 Variable units

A variation of a unit has physical meaning only as variation relative to some standard. This standard must be given by some physical unit of length, which we may *define* to be constant. Atomic physics and gravitational physics provide such standards with independent units of length and mass.

The scope of the theory presented here is to give field equations describing the relative change of gravitational and atomic units. These field equations must be coordinated with the physics of a specific unit of length, that is, the 'atomic units' must be identified. This identification is not an arbitrary coordinative definition, but a testable physical hypothesis. Here we make the assumption that the 'atomic units' of the field equations are equal to the *unit of quantum electrodynamics* which again is related by a constant scale to the unit of *classical electromagnetism*. Thus the theory is assumed to describe the relative scale of classical gravitation relative to electromagnetism.

A variety of 'fundamental principles' such as Dirac's Large Number Hypothesis, Mach's principle, Kaluza-Klein unification, and scale-free gravity, has lead to theories with variable fundamental parameters [53–73].

Canuto et al. [45,46,50] have generalized General Relativity to include the effect of a variable G. In their framework the mathematical concept of a scale transformation is associated with the physics of using different systems of units. By incorporating an arbitrary conformal factor, interpreted as the relative scale of different length units, into Einstein's theory, they were able to discuss physical effects of variable fundamental units in a self-consistent way.

However, the scale-covariant theory is not, nor is it meant to be, a true unification of units, and the scaling function has to be determined by imposing certain *ad hoc gauge conditions.* A complete gravitational theory should not only specify the geometry of space-time measured in gravitational units, it should also completly specify the coupling of gravity to matter so that atomic units can be used. Most of the variable-G theories mentioned in Refs. 53–73 suffer from significant problems [64,65,74], or they are incomplete.

4. FIELD EQUATIONS AND INTERPRETATION

4.1 The metricity condition

In order to accomodate conformal invariance, one conventionally assumes the underlying space-time to be of the Weyl type. But Obukhov [75] has pointed out that this is not the only possibility. Within the Riemann-Cartan framework the torsion trace may play the rôle of the Weyl gauge field. This close connection between conformal transformations, variable G, and the torsion trace has been recognized by several authors [37,44,75–77].

Both Einstein's theory and the Einstein-Cartan theory have metric compatible connections. This guarantees that length is preserved under parallel transport. This mathematical property is physically motivated by the observed independence of history of physical length scales. Making the assumption that this property is independent of units, the metricity condition should be a scale-invariant condition. This eliminates the Weyl geometry, and one has to follow Obukhov and identify the torsion trace with the scale gauge field.

4.2 Two types of conformal symmetry

In the Weyl geometry approach to conformal invariance, the non-metricity of the connection compensates for the non-invariance of the Christoffel symbols to make the total world affine connection invariant under the conformal transformations [78].

In the Riemann-Cartan geometry the local Lorentz connection is postulated to be invariant under the scale transformations of the metric [75]. Then the world affine connection components must transform like under the projective transformation (or the so-called Einstein λ-transformation, Ref. 79), which leave the world Riemann-Cartan curvature tensor invariant.

These combined conformal and projective (CCP) transformations[3] are of course different from the scale transformations of Weyl's geometry. Physically the CCP transformation represents a change of all proper lengths in such a way that the Lorentz connection 1-form with respect to an orthonormal tetrad reference remains unchanged. This is indeed the physical effect one should expect from a change of the unit of measurement: a change of lengths but no change of the Lorentz connection.

Because of these differences, one may group the conformal symmetries into two groups; an extrinsic symmetry with the Weyl vector as the gauge field, and an intrinsic symmetry with the torsion vector as the gauge field.

4.3 The source of the torsion trace

If the torsion vector is to be identified with the gauge field of an intrinsic conformal symmetry, its source has to be identified with the intrinsic dilation current of matter (similar to the way intrinsic spin is the source of the traceless torsion). Physically we interpret the intrinsic dilation current as an intrisic change of length scale. The scale breaking occurs by gravitational coupling to an energy-momentum tensor with a trace. While such a trace breaks conformal invariance, the coupled geometry-matter theory may still be invariant under scale-changes provided variations of the matter Lagrangian with respect to the conformal factor is proportional to the trace of the energy-momentum tensor. This requirement leads to a dilation conservation law.

After a choice of the overall conformal factor ($\psi =$ constant) has been done, which means the choice of some definite scale of the theory, we get the usual Einstein-Cartan action. The intrinsic scale freedom, represented by the torsion trace, is interpreted as follows: *the gravitational field equations with the torsion trace are identified with the field equations as expressed in atomic units and those with zero torsion trace and a constant G are the same field equations expressed in gravitational units.* The dilation conservation law is a *gauge condition* which determines the relative size of the two possible scales of the theory. It is therefore a gauge condition in the sense of Canuto et al. [45].

4.4 Action principle

The Einstein-Cartan theory is derived from the action

$$S = \int (R + 16\pi L)\sqrt{-g}\, d^4x \tag{9}$$

[3] See section 3 in Baker's paper [77] for a detailed discussion of the geometrical meaning of the CCP transformations.

where R is the Riemann-Cartan curvature scalar and the gravitational coupling G is absorbed in the matter term L.

By assumption there is a material current t_α which couples to the torsion trace, so that

$$\frac{\partial L}{\partial T_\alpha} = -Gt^\alpha. \tag{10}$$

$-t^\alpha$ is identified with an intrinsic dilation current.

The total dilation current W^α, which is the sum of an orbital part and an intrinsic part [80]

$$W^\alpha = \Sigma^{\alpha\beta} x_\beta - t^\alpha, \tag{11}$$

is conserved in the Minkowski limit if

$$t^\alpha{}_{,\alpha} = \Sigma. \tag{12}$$

We note that the above conservation law is of the same form as the conservation law for spin

$$S^\alpha{}_{\mu\nu,\alpha} = \Sigma_{[\mu\nu]}. \tag{13}$$

Hence, just as the antisymmetric part of the energy-momentum tensor is a torque for the spin, the energy-momentum trace is a force generating intrinsic dilation currents.

Now, since T^α geometrically corresponds to a volume change in the direction x^α, and the interpretation of t^α is the same, one should expect that

$$T^\alpha{}_{;\alpha} = -8\pi G\Sigma. \tag{14}$$

Note that without further assumptions concerning the scaling properties of the matter fields, this equation is not derivable from the action (9). Yet, the assumption that the total dilation current is conserved in the Minkowski limit, is a reasonable gauge condition.

4.5 Field equations and conservation laws

The Einstein-Cartan action (9) leads to the Einstein-Cartan equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \Sigma_{\mu\nu} \tag{15}$$

$$\frac{1}{2} T^\mu{}_{\alpha\beta} + g^\mu{}_{[\alpha} T^\sigma{}_{\beta]\sigma} = 8\pi G t^\mu{}_{\alpha\beta}. \tag{16}$$

The source of the U_4-Einstein tensor is the (generally asymmetric) energy-momentum tensor $\Sigma_{\mu\nu}$. One may decompose the torsion source, $t^{\mu}{}_{\nu\gamma}$, in a traceless part, $S^{\mu}{}_{\nu\gamma}$, and a trace, $t_{\mu} = t^{\alpha}{}_{\mu\alpha}$, by

$$t^{\mu}{}_{\nu\gamma} = S^{\mu}{}_{\nu\gamma} - \frac{2}{3} g^{\mu}{}_{[\nu} t_{\gamma]}. \tag{17}$$

The traceless part is naturally identified with the spin-tensor, whereas the trace here is attributed to *the intrinsic dilation current.*

From Cartan's equation (16) we find that the traceless torsion ${}^*T^{\alpha}{}_{\beta\gamma}$ and the trace T_{α} is given by

$${}^*T^{\alpha}{}_{\beta\gamma} = 16\pi G S^{\alpha}{}_{\beta\gamma} \qquad \text{and} \qquad T_{\alpha} = -8\pi G t_{\alpha}. \tag{18}$$

In the conservation laws of the Einstein-Cartan theory it is convenient [20] to introduce a modified covariant derivative ${}^*\nabla$, defined by use of the traceless contortion ${}^*K^{\alpha}{}_{\beta\gamma}$. Since the modified covariant divergence of a vector is equal to the Christoffel divergence, the dilation conservation law (14) takes the form

$$T^{\alpha}{}_{;\alpha} = {}^*\nabla_{\alpha} T^{\alpha} = -8\pi G \Sigma. \tag{19}$$

Together with the hypothesis that these field equations (15), (16), (19) refer to the atomic units, the theory is complete.

The energy and spin conservation laws follows from the Bianchi identities. The first Bianchi identity yields

$${}^*\nabla_{\alpha} G t^{\alpha}{}_{\beta\gamma} = G \Sigma_{[\beta\gamma]}. \tag{20}$$

Inserting from eq. (17), we get

$${}^*\nabla_{\alpha} \left(G S^{\alpha}{}_{\beta\gamma} - \frac{2}{3} G g^{\alpha}{}_{[\beta} t_{\gamma]} \right) = G \Sigma_{[\beta\gamma]}. \tag{21}$$

Inserting for t_{γ} using eq. (18) yields

$${}^*\nabla_{\alpha} \left(G S^{\alpha}{}_{\beta\gamma} + \frac{2}{3} \frac{1}{8\pi} g^{\alpha}{}_{[\beta} T_{\gamma]} \right) = G \Sigma_{[\beta\gamma]}. \tag{22}$$

By our hypothesis that the torsion trace is a measure of the rate of change of G, it follows that the torsion trace is proportional to the derivative of the scalar G. This implies that $T_{[\alpha;\beta]} = 0$, so that only the antisymmetric part of the traceless contortion gives a contribution to the second term of the lefthand side of eq. (22). Since

$$K_{\alpha\beta\gamma} = \frac{1}{2}\left(T_{\beta\alpha\gamma} + T_{\gamma\alpha\beta} - T_{\alpha\beta\gamma}\right), \tag{23}$$

$$^{*}K^{\alpha}{}_{[\beta\gamma]} = -\frac{1}{2}\,{}^{*}T^{\alpha}_{\ \beta\gamma}. \tag{24}$$

Then by use of eq. (18), one finds that eq. (22) yields

$$^{*}\nabla_{\alpha} G S^{\alpha}{}_{\beta\gamma} = G\Sigma_{[\beta\gamma]} + \frac{2}{3} G S^{\alpha}{}_{\beta\gamma} T_{\alpha}. \tag{25}$$

5. COSMOLOGY WITH VARIABLE G

5.1 Geometry

Motivated by the observed isotropy of the universe, and the Cosmological Principle, we assume a Friedmann-Robertson-Walker line element

$$ds^2 = -dt^2 + R^2\left(\frac{1}{1-kr^2}\,dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right). \tag{26}$$

We assume that the traceless part of the torsion is zero (vanishing spin density). Then due to the symmetry of the model, torsion is restricted to a torsion vector pointing in the time direction.

An orthonormal tetrad frame is defined as

$$\omega^0 = dt \quad \omega^1 = \frac{R}{\sqrt{1-kr^2}}\,dr \quad \omega^2 = Rrd\theta \quad \omega^3 = Rr\sin\theta d\phi. \tag{27}$$

In form language the torsion is given by

$$\mathcal{T}^0 = 0 \qquad \mathcal{T}^i = \frac{1}{3}\,\sigma\omega^0\wedge\omega^i. \tag{28}$$

Here the index i refers to the spatial tetrad indices. The torsion form is related as $\mathcal{T}^{\mu} = 1/2\, T^{\mu}{}_{\alpha\beta}\omega^{\alpha}\wedge\omega^{\beta}$ to the torsion tensor. Cartan's equations of structure determines the connection forms $\Omega^{\mu}{}_{\nu}$

$$d\omega^{\mu} + \Omega^{\mu}{}_{\alpha}\wedge\omega^{\alpha} = \mathcal{T}^{\mu} \tag{29}$$

and the curvature forms $\mathcal{R}^{\mu}{}_{\nu}$

$$\mathcal{R}^{\alpha}{}_{\beta} = d\Omega^{\alpha}{}_{\beta} + \Omega^{\alpha}{}_{\nu}\wedge\Omega^{\nu}{}_{\beta}\,. \tag{30}$$

The Riemann-Cartan tensor $R^{\mu}{}_{\nu\alpha\beta}$ can be found by use of the relation $\mathcal{R}^{\mu}{}_{\nu} = 1/2\, R^{\mu}{}_{\nu\alpha\beta}\omega^{\alpha} \wedge \omega^{\beta}$.

5.2 Cosmological field equations

In the perfect fluid approximation the energy-momentum tensor is

$$\Sigma_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + p g_{\mu\nu}. \tag{31}$$

By use of the tetrad (27), the torsion form (28), and the equations of structure (29) and (30), we calculate the Einstein-Cartan tensor $R_{\mu\nu} - 1/2\, g_{\mu\nu} R$ in the physical tetrad frame of a comoving cosmic observer. Using that the torsion vector is of the form $T^{\alpha} = -\delta^{\alpha}{}_{t}\sigma$, the Einstein equations (15) and the dilation conservation law (19) gives the following complete set of field equations

$$3H^2 + \frac{1}{3}\sigma^2 - 2\sigma H + \frac{3k}{R^2} = \quad 8\pi G\rho, \tag{32}$$

$$2\dot{H} + 3H^2 - \frac{2}{3}\dot{\sigma} - \frac{4}{3}\sigma H + \frac{1}{9}\sigma^2 + \frac{k}{R^2} = -8\pi G p, \tag{33}$$

$$\dot{\sigma} + 3H\sigma = -8\pi G(\rho - 3p). \tag{34}$$

Here $H \equiv \dot{R}/R$ is a Hubble factor. If one makes the identification[4]

$$\sigma \equiv -3\frac{\dot{\beta}}{\beta}, \tag{35}$$

the field equations (32) and (33) are equivalent to those of the scale-covariant theory [45].

G scales as $G \sim \beta^{-g}$ where g is a constant parameter determining the gauge[5] . Before we proceed, a discussion of the freedom of G-scaling represented by the parameter g is appropriate.

5.3 The gauge parameter g and self-consistence

From a consideration of the dynamics of charged particles, Canuto and Goldman [46] concluded that the scale-covariant theory is self-consistent only for $g = 2$, because only for this value of g do the Maxwell equations reduce to their standard form in atomic units. To study the dynamics of gravitational and atomic clocks in general units, Canuto and Goldman

[4] The gauge function β of Canuto et al. [45] represents the scale factor between gravitational and atomic units of length ($L_E = \beta_a L_a$).

[5] In the sequel the symbol $\sim$ should be read 'varies as'.

[46] proposed a scale-covariant action. This action must be of zero power. The relevant scale-powers are: $\pi(ds) = 1$, $\pi(m) = 1 - g$, $\pi(\hbar) = 2 - g$, $\pi(\sqrt{-g}\, d^4x) = 4$, $\pi(F_{\alpha\beta}F^{\alpha\beta}) = -2 - g$, $\pi(A_\alpha u^\alpha) = -g/2$, $\pi(e) = 1 - g/2$. A possible scale-covariant action is

$$I = \int \beta^{2-g} m ds + \frac{1}{16\pi} \int \beta^{2-g} F_{\alpha\beta} F^{\alpha\beta} \sqrt{-g}\, d^4x + \int \beta^{2-g} e A_\alpha u^\alpha ds. \tag{36}$$

The equations of motion deduced from this action are

$$(\sqrt{-g}\, F^{\lambda\nu} \beta^{1-g/2})_{,\nu} = 4\pi \int e \beta^{1-g/2} \delta^4(x - z) dz^\lambda \tag{37}$$

$$u^\alpha_{;\nu} u^\nu + \frac{\beta_{,\nu}}{\beta} h^{\nu\alpha} = \frac{e}{m} u^\nu F_\nu{}^\alpha. \tag{38}$$

The first equation reduces to the standard Maxwell equation only if $g = 2$. The second equation, which essentially is Newton's second law combined with the Lorentz force, contains a *cosmic force term* related to a variability of the inertial mass. For realistic atomic clocks this *cosmic force* is so much smaller than the electromagnetic force on the righthand side that it can be neglected.

Canuto and Goldman [46] argued that because the righthand side of eq. (38) is spherically symmetric (in an atomic clock), it does not affect the angular momentum conservation law, and that it consequently cannot affect the ratio of periods as measured in different units. This argument is, however, not valid. The atomic clock has a period determined by

$$T \sim m^2 (\ell/m)^3 / e^4 \tag{39}$$

where m is mass, ℓ is angular momentum and e is electric charge. In the case $g = 2$ both e and ℓ are *independent of units*, so in this case T (in different units) is determined exclusively by the (inertial) mass.

If we neglect all kinds of curved space-time effects, including β-forces, the period, T, of the electromagnetic clock in different units, is given only by the physical acceleration, a, in these units:

$$T \sim a^{-1}. \tag{40}$$

Because the force (in physical units), is of power -1, we find a ratio of accelerations of $a_E/a_a = \beta_a^{-1}$.

To summarize:

Gravitational clock

This clock is governed by eq. (38) with $e = 0$.

i. Einstein Units:
Here $\beta = 1$ so that the equation of motion is $u^{\alpha}_{;\nu}u^{\nu} = 0$, which for the Schwarzschild metric has the solution
$P_E =$ constant.

ii. Atomic Units:
In this case $\beta = \beta_a$, giving $u^{\alpha}_{;\nu}u^{\nu} + \beta_{,\nu}/\beta h^{\nu\alpha} = 0$. Using the conformally transformed Schwarzschild metric, we get
$P_a = \beta_a^{-1} P_E$.

Electromagnetic clock

In both units eq. (37) has the same form. Different periods appear because of the scaling of the Lorentz force of eq. (38).

i. Atomic Units:
T_a=constant

ii. Einstein Units:
$T_E = \beta_a T_a$

The $g = 2$ version of the theory gives the correct Maxwell equations in atomic units. Newton's second law gets a cosmological correction term, but the correction is even smaller than corrections due to local gravitational forces. Therefore the period of an electromagnetic clock is predominantly governed by the interaction strength of electromagnetism.

We conclude that for $g = 2$ the dynamical equations of gravitation and electromagnetism are consistent with the fundamental assumptions of the scale-covariant theory.

5.4 Conservation laws

By defining the scalar $\phi \equiv -3 \ln \beta/\beta_0$, the dilation conservation law takes the form

$$\ddot{\phi} + 3H\dot{\phi} = -8\pi G_0 \rho e^{g\phi/3}. \tag{41}$$

This is equivalent to the equation of motion of a classical particle with position ϕ in an effective exponential potential $V(\phi)$ under influence of the friction force $-3H\dot{\phi}$. Since $\rho > 0$ the potential drives ϕ towards $-\infty$, but as long as $H > 0$ the friction stabilizes the motion of ϕ.

The 'energy-conservation' law in scale-covariant form reads,

$$\dot{\rho} + 3\frac{\dot{R}}{R}(\rho + p) = (\rho - 3p)\frac{\dot{\beta}}{\beta} + (g - 2)\rho\frac{\dot{\beta}}{\beta}\,. \tag{42}$$

For dust we find

$$\rho \sim R^{-3}\beta^{g-1}. \tag{43}$$

For radiation, $p = \rho/3$, eq. (42) has the the solution

$$\rho_\gamma = n\epsilon \sim \beta^{g-2} R^{-4}. \tag{44}$$

Assuming that the photon number is conserved $n \sim R^{-3}$, we get

$$\epsilon = h\nu \sim \beta^{g-2} R^{-1}. \tag{45}$$

In section 6.2 it will be shown that the orbital angular momentum of planets J (defined as $J = mvR$) changes as $J \sim \beta^{g-2}$ when β varies. This is what should be expected from dimensional reasons: $[J] = ML$. This should be compared with the spin conservation law (25), which in the case of spinning dust with the spin density current $S^\alpha{}_{\mu\nu} = u^\alpha s_{\mu\nu}$, takes the form

$$\dot{s}_{\mu\nu} + 3H s_{\mu\nu} = (g-2)\frac{\dot{\beta}}{\beta} s_{\mu\nu}. \tag{46}$$

Hence, the spin density varies as

$$s_{\mu\nu} \sim \beta^{g-2} R^{-3}. \tag{47}$$

5.5 Particle number conservation

To give a physical interpretation of eq. (43), one should first note that the Planck mass $M_{\text{Pl}} = (G\hbar_g)^{1/2}$ varies as

$$M_{\text{Pl}} \sim \beta^{g-1}. \tag{48}$$

Multiplying eq. (43) with the volume factor R^3, we find that

$$M_g \equiv \rho R^3 = N_{\text{Pl}} M_{\text{Pl}} \sim \beta^{g-1}. \tag{49}$$

Thus the number of Planck-mass units in a volume is constant.

Similarly the Planck unit of (gravitational) spin $\hbar_g$ varies as

$$\hbar_g \sim \beta^{g-2}. \tag{50}$$

Then eq. (47) multiplied by the volume factor R^3, gives

$$J_g \equiv sR^3 = N_{\text{Pl}} \hbar_g \sim \beta^{g-2}. \tag{51}$$

The number of Planck units of (gravitational) spin in a volume is constant. For $g = 2$ the Planck unit of spin $\hbar_g$ is equal to the atomic unit $\hbar_a$, but the inertial mass varies as β relative to the atomic mass. We note that the photon-energy law takes the standard form when $g = 2$. If photon frequency is defined by the propagation of successive signals, the theory predicts the redshift law $\lambda \sim R$. It has been proved that the theory is compatible with the observed 3° K black-body cosmic spectrum [81].

The $g = 2$ version of the theory has generally been interpreted as a theory with varying particle number [46,50]. This interpretation follows from eq. (43), if one assumes that the atomic mass is equal to the gravitational mass appearing in the gravitational field equations. But it is possible, and indeed plausible, that gravitational or inertial mass varies with respect to atomic mass in a variable G framework [82]. Thus with

$$M_g = \beta_a m_a \qquad (g = 2) \tag{52}$$

particle number conservation is restored.

The change of gravitational mass as $M \sim \beta$ in atomic units can be understood as follows. According to our assumptions the intrinsic length scales of matter varies relative to the gravitational scales. Then if rest mass has a field origin, the matter waves inside the particle must suffer a redshift if the inner space of a particle expands. The intrinsic scale L_a, which is the scale of the particle, varies as β^{-1}. By a redshift law of the same type as the Hubble law, one should then expect that the intrinsic energy would change as

$$M_g \sim 1/L_a \sim \beta. \tag{53}$$

Note that this result is consistent with eq. (49) only if $g = 2$. We conclude that $g = 2$ is the only value which is compatible both with a conserved particle number and the (intrinsic redshift) variable mass hypothesis.

Another argument in favour of $g = 2$ is the fact that angular momentum conservation is due to spatial rotational symmetry. Since this symmetry is maintained both in atomic and gravitational units (in the cosmic rest frame), the spin angular momentum conservation law should take the same form in both units.

6. LOCAL CONSEQUENCES OF A VARIABLE G

6.1 The Solar system in a cosmic background

In the Einstein-Cartan theory torsion does not propagate in vacuum, and the torsion interaction is a contact force. Here we interpret the 'contact force' as a force which does not extend beyond the characteristic scale of the source. In cosmology the characteristic scale is the Hubble length which is much longer than the intergalactic separation. Thus galaxies are in 'contact', and should feel Einstein-Cartan contact forces. In the solar system the characteristic scale is the Schwarzschild radius. Since the separation distances in the solar system are much larger than the charactersistic Schwarzschild radii, the solar system is free of local contact forces. Thus there are no locally generated torsion fields in the solar system.

In the cosmological model matter has been modelled as a pressureless perfect fluid with a timelike dilation current vector. To embed the local gravitational field of a star in this background, we consider a spherically symmetric comoving volume of the universe as collapsed to a point mass. We demand that this region has a constant mass both in cosmological Einstein and atomic units. This can only be obtained if the spherical region contains the cosmological torsion vector. This implies a cosmological coupling to local gravitational phenomena as measured in atomic units. The solar system feels cosmological contact forces.

6.2 Experimental limits on variation of units

In scale-covariant form [45] the vacuum Einstein equations read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + 6 \left(\beta_{,\mu}\beta_{,\nu} - \frac{1}{2} g_{\mu\nu} \beta^{,\alpha}\beta_{,\alpha} \right) / \beta^2$$
$$-(\beta^2_{,\mu;\nu} - g_{\mu\nu}\Box\beta^2)/\beta^2 = 0. \tag{54}$$

These field equations have the solution

$$ds^2_{\mathrm{A}} = \beta^{-2} ds^2_{\mathrm{E}} \tag{55}$$

where ds^2_{E} is the Schwarzschild metric in gravitational units

$$ds^2_{\mathrm{E}} = (G_{\mathrm{E}} M_{\mathrm{E}})^2 [-(1 - 2/r)dt^2 + (1 - 2/r)^{-1} dr^2 + r^2 d\Omega^2]. \tag{56}$$

For purely gravitational experiments we expect predictions in accordance with Einstein's theory. In particular the perihelion advance per revolution as well as the deflection of light rays measured in radians are scale invariant. However, for experiments where one applies atomic units of length, one expects predictions different from Einstein's theory. The consequences of variable relative scales between gravitational and atomic units are discussed in detail by Canuto et al. [45]. Here we give only the specific predictions imposed by the cosmological model of the previous section.

For spherical orbits the revolution frequency n ($\equiv \dot{\varphi}$) and radius, R, measured in atomic units changes as [45]

$$\frac{\dot{n}}{n} = \frac{\dot{\beta}}{\beta} \qquad \text{and} \qquad \frac{\dot{R}}{R} = -\frac{\dot{\beta}}{\beta}. \tag{57}$$

Defining the orbital angular momentum J as $J = MvR = MnR^2$, and remembering that inertial mass varies as $M \sim \beta^{g-1}$, we get

$$\frac{\dot{J}}{J} = \frac{\dot{M}}{M} + \frac{\dot{n}}{n} + 2\frac{\dot{R}}{R} = (g-2)\frac{\dot{\beta}}{\beta}, \tag{58}$$

which agrees with the result of section 5.4.

Within quantum electrodynamics (QED) one may define $\hbar^2/e^2m$ as a unit of length, and m_e as a unit of mass. In these units QED takes its usual form, but when mixed with gravitational dynamics in the sense that if we use gravitational clocks to measure the dynamics of QED or vice versa, it is possible to find "clock drift" effects.

Thus a gravitational clock, represented by a planetary system, has a period $P \sim n^{-1}$ which according to eq. (57), drifts relative to atomic clocks as

$$\frac{\dot{P}}{P} = -\frac{\dot{\beta}}{\beta}. \tag{59}$$

Viking lander ranging data [83–85] and binary pulsar data [86] have put a limit on the relative change of gravitational clock periods relative to quantum atomic time (caesium clocks) of the order $|d\ln(P)/dt| < 10^{-11}\mathrm{yr}^{-1}$, which gives an experimental limit $|\dot{\beta}/\beta| < 10^{-11}\mathrm{yr}^{-1}$.

Before a data-fit has been made for the new cosmological theory, it is difficult to give a prediction for the present rate of change of G. Assuming a 'near' standard relativistic model with $\sigma^2 \ll H^2$ and $|\dot{\sigma}| \ll |H\sigma|$, eq. (34) gives

$$3H\sigma = -8\pi G\rho = -3\Omega H^2. \tag{60}$$

Defining $H = h \cdot 100$ km/s $\mathrm{Mpc}^{-1} = h \cdot 1.0 \cdot 10^{-10}\ \mathrm{yr}^{-1}$, the variation of β is given as

$$\frac{\dot{\beta}}{\beta} = \frac{1}{3}\Omega h\, 1.0 \cdot 10^{-10}\ \mathrm{yr}^{-1}. \tag{61}$$

The best-fit cosmological model according to Rowan-Robinson (Ref. 87, p.281) has $\Omega_0 = 0.05$ and $h = 0.67$. This gives

$$\frac{\dot{\beta}}{\beta} = 1.2 \cdot 10^{-12}\ \mathrm{yr}^{-1}, \tag{62}$$

which is a factor of 8 below the experimental limit. While the majority of cosmological tests do give a low value for the density parameter Ω, even a critical density cannot be excluded. It is, however, worth saying that contrary to widespread belief, a low value of the density parameter is not necessarily incompatible with the inflation hypothesis [88]. Clearly further study of the cosmologies of the proposed theory is necessary to pin down the best-fit parameters Ω_0 and σ_0 (the present rate of change of G), and thereby predict the present value of $\dot{G}/G$ from cosmological observations.

6.3 Radial variation of G inside massive objects

According to the new theory there should also be a radial variation of G inside the earth and the sun. If G is a slowly varying function of r, G varies as

$$\frac{G'}{G} \simeq -\frac{m(r)G}{r^2} \tag{63}$$

inside a sphere of constant density. Integrating once more we find

$$G(r) \simeq G_0 \exp\left(\frac{MG_0}{2R} - \frac{m(r)G_0}{2r}\right). \tag{64}$$

where G_0 is the cosmic value of G, M is the total mass, and R is the radius of the sphere. G decreases with increasing radius and reaches the cosmic value at the surface of the sphere. It is through this parameter that cosmos couples to local physics. For constant density models of the earth and the sun the relative change of G from the center to the surface is $\Delta G/G \simeq 10^{-10}$ for the earth and $\Delta G/G \simeq 10^{-7}$ for the sun. In the surrounding vacuum the torsion trace does not propagate. Hence, we expect no seasonal variation of G as the earth orbits the sun in its elliptic orbit.

7. COMPARISION WITH PREVIOUS SCHEMES

Usually torsion has been attributed to spin, only. Lately it has been realized that the trace part, if present, must be given a different interpretation [38–40]. At the same time the Einstein-Cartan theory has been extended to include non-metricity [89]. Here the Weyl vector is associated with the intrinsic dilation current, whereas the traceless part of the non-metricity is attributed to a yet unknown shear current. The Weyl vector and the torsion trace are closely related, and in the present theory one identifies the dilation current as the source of the torsion trace, thereby resolving the problems of interpretation met by the Einstein-Cartan theory [38–40]. In this way the dilation current is introduced without the cost of a shear current.

The interpretation of the torsion trace as a measure of the rate of gravitational length relative to atomic length is adopted from the scale-covariant theory [45]. A dilation conservation law determines the rate of change of relative length scale. Because of this interpretation the theory belongs to the class of variable-G theories which are identical to Einstein's in gravitational units. Due to the non-propagating nature of torsion one expects no locally induced variations in the ratio of units in the vacuum. Thus the theory is not affected by the experimental restriction ($\omega_{BD} > 500$) on possible massless scalar components [4].

By allowing atomic rest mass to vary relative to inertial or gravitational mass, one avoids the conceptually troublesome violation of baryon number conservation which earlier was associated with the $g = 2$ scale-covariant theory [46,81].

Earlier variable mass theories are represented by Hoyle-Narlikar theory [57–59], the Brans-Dicke theory [60–62], the Bekenstein-Meisels theory [63], and Wesson's theory [70–73]. The Hoyle-Narlikar theory predicts a too fast variation of the scale of units. The Brans-Dicke theory is viable if $\omega_{BD} > 500$, but the theory lacks a fundamental motivation for this choice of the parameter. Also the Bekenstein-Meisels variable mass theory is viable, but it contains two free parameters and is therefore less attractive. Wesson attributed the rest-mass to a non-compactified fifth dimension, so that the "velocity" in this direction describes the change of mass. While the idea may be viable, its present mathematical formulation is not [74], since the field equations are equivalent to the Brans-Dicke theory with $\omega = 0$, a theory which is excluded by solar system experiments[6] .

It has earlier been suggested that both G and inertial mass are *"created"* by spontaneous breakdown of scale invariance [91,92]. A theory of this type has been constructed by Fujii [92–94]. Here all dimensional quantities in nature are determined by the cosmic background value of a scalar field. In the simplest version (one scalar) the gravitational and atomic units coincide [94]. The scope of the theory presented here is narrower, because no attempt is made to explain the origin of the atomic dimensional quantities. But given that conformal invariance is broken in atomic physics, the magnitude of the gravitational dimensional parameters such as G and gravitational masses, are determined by a field equation with the trace of the energy-momentum tensor as a source. Therefore one may interpret the theory as follows, *the dimensional properties of space-time are induced by a breakdown of conformal invariance in the matter fields.* The fundamental difference of the two theories is that Fujii's theory gives a unified description of matter and gravitation, whereas the present theory is strictly a theory of space-time and its coupling to matter. The nature of matter is only restricted by the Bianchi identities — the conservation

[6] See, however, Wesson's reply [90] to the criticism of Ref. 74.

laws of energy and spin — and the dilation conservation law.

8. CONCLUSIONS

We have proposed a new interpretation of the torsion trace of the Einstein-Cartan theory, thereby obtaining a new variable-G theory which unifies the units of measurement of the gravitational and atomic interactions. The theory implies that the Newtonian gravitational coupling varies on cosmological time-scales as measured in atomic units of measurement. This leads to measureable secular variations of the planetary orbits as measured in these units. For a best-fit cosmological model the predicted rate of change of G is within the experimental bounds by a factor of 8.

In atomic units, a radial variation of G inside massive objects is predicted. Inside the earth this effect is very small; the relative change of G inside the earth being only $\simeq 10^{-10}$.

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