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Status of measurement of the Newtonian gravitational constant G

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Abstract. A precise knowledge of the Newtonian gravitational constant G has an important place in physics and is of considerable metrological interest. Although G was the first physical constant to be introduced and measured in the history of science, it is still the least well known. The 1998 CODATA recommended value for G, $(6.673 \pm 0.010) \times 10^{-11}$ m^3 kg^{-1} s^{-2}, has an uncertainty of 1500 parts per million (ppm), which is much larger than that of all other fundamental constants. Here we review the status of our knowledge of the absolute value of G, nine experiments for measuring the absolute values of G within the last five years, the experiments in progress and being planned, and the systematic error due to the inelasticity, the nonlinearity and the thermoelasticity of torsion fibre.

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1. Introduction

The Newtonian gravitational constant is the multiplying factor G in the formula

$$F = G\frac{Mm}{r^2} \tag{1}$$

giving the force F of gravitational attraction between two point masses M and m, with a distance r between their centres.

An accurate knowledge G is not only important from the point of view of theoretical physics, but is also significant for practical purposes, particularly when finding the density and density distributions of the interiors of the Earth, Moon, planets and stars [1]. Further more, a precise knowledge of G is of considerable metrological interest, and it provides a unique as well as a valuable measurement challenge that sharpens and prepares experimental skill to better deal with a variety of precise and null experiments [2]. For those reasons, great efforts have been made over two centuries to obtain a reliable value. Since Cavendish reported the first experimental value of G [3], nearly 300 different measurements of G have been made over the years, including several in which the objective was to search for some type of variation in G [4]. In spite of these many strenuous efforts, G is the least precisely determined of all the fundamental constants of nature.

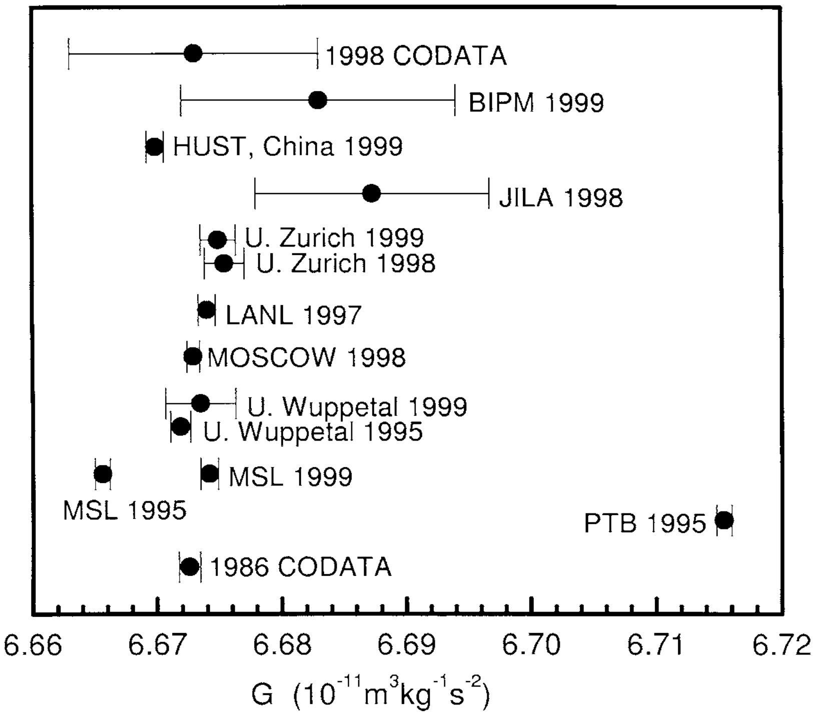

Figure 1. Recent experimental values of the Newtonian gravitational constant and their errors.

At the start of the 20th century, the accepted value of G was 6.66×10^{-11} m^3 kg^{-1} s^{-2}, which rested upon the independent work of Boys [5] and of Braun [6]. It was displaced by the result of Heyl and Chrazanowski [7, 8], who concluded that

$$G = (6.673 \pm 0.003) \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}. \tag{2}$$

The beginning of the modern era of measurements of the absolute value of G is usually associated with the appearance of their results. Much of the experimental work on the measurement of G that was carried out during the 1960s and 1970s came to fruition with the 1986 adjustment of the fundamental constant, when the recommended value of G became $G = (6.672\,59 \pm 0.000\,85) \times 10^{-11}$ m^3 kg^{-1} s^{-2} [9]. The relative uncertainty in the 1986 CODATA value, 128 parts per million (ppm), is 10^2 to 10^5 times larger than that of most of the constants which arise in atomic and nuclear physics [10]. Moreover, the nine recent measurements of G [2, 11–18] have produced values that differ wildly from each other and the 1986 CODATA value, which are listed both in table 1 and figure 1. This situation suggests a new value for G should be recommended. In 1998, the CODATA recommended value for G was $(6.673 \pm 0.010) \times 10^{-11}$ m^3 kg^{-1} s^{-2}†. It is noted that the new recommended CODATA value for G is essentially the same as the 1986 value but the relative uncertainty placed on this value has been expanded to 1500 ppm. This situation, with a disagreement far in excess of the estimate, suggests the presence of unknown systematic problems. It seems clear that investigations on all kinds of systematic errors of the different methods are desirable.

† See http://physics.nist.gov/cuu/Constants/index.html

Table 1. Recent results of the experimental measurements of G.

Source	G value ($\times 10^{-11}$ m^3 kg^{-1} s^{-2})	Error estimate (ppm)	Deviation from 1986 CODATA value (ppm)
1986 CODATA value [9]	6.6726	128	0
1998 CODATA value[a]	6.673	1500	+61
Michaelis *et al* (1995) [11]	6.7154	83	+6410
Fitzgerald *et al* (1995) [12]	6.6656	95	−1050
Fitzgerald *et al* (1999) [27]	6.6742	90	+240
Walesch *et al* (1995) [13]	6.6719	83	−105
Kleinevoß *et al* (1999) [35]	6.6735	432	+135
Karagioz *et al* (1998) [14, 15]	6.6729	78	+45
Bagley *et al* (1997) [16]	6.6740	102	+210
Schurr *et al* (1998) [17]	6.6754	240	+420
Nolting *et al* (1999) [36]	6.6749	220	+345
Schwarz *et al* (1998) [2]	6.6873	1400	+2200
Luo *et al* (1999) [18]	6.6699	105	−405
Richman *et al* (1999) [29]	6.683	1700	+1560

[a] See http://physics.nist.gov/cuu/Constants/index.html.

The difficulties in determining G are well known [19]. First, the gravitational force is extremely weak and is ubiquitous. The Newtonian attraction between fundamental particles is about 10^{40} times weaker than the electromagnetic or nuclear force between them. Second, the stray gravitational fields due to other bodies, particularly the Earth, cannot be screened out, since the gravitational force cannot be shielded in the experiment, and therefore significant efforts are made to isolate such experiments from cultural and natural sources of gravity gradients and time-varying fields. Third, G has no known confirmed dependence on any other fundamental constant. Hence, its value cannot be estimated in terms of other quantities and can be determined according to the Newtonian inverse square law. Fourth, the instruments chosen for measuring G such as torsion pendulums or torsion balances are subject to a variety of parasitic couplings and systematic effects which ultimately limit their usefulness as transducers of the gravitational force. Beam balances, vertical and horizontal pendulums, and other sensitive mechanical devices are also pressed to be limits of their performance capabilities when employed for this purpose. Finally, the absolute measurements increase the difficulties. G is defined by three fundamental quantities—time, length and mass. The measurement of G requires that absolute values be measured for the masses of attracted bodies and attracting bodies, the separation, the period of motion of the mechanical oscillator and so on, all of which may give rise to considerable experimental difficulties.

Many reviews are of interest in pursuing the measurement of G. Earlier, Poynting [20] and Mackenzie [21] summarized the contemporary knowledge of G in 1894 and 1900, respectively. Recently, de Boer [22] made a survey of recent major experiments and Gillies [23] published a very comprehensive report giving an index of measurement of G, containing over 1200 references. Chen and Cook [1] give an extensive discussion of laboratory techniques employed in measurements of G. Gillies [4] summarized the recent measurements of G and the search for variation in G. In this paper, we will review the status of the recent measurement of G and the systematic errors due to the inelasticity, the nonlinearity and the thermoelasticity of torsion fibres in the measurement of G.

2. Recent measurements

Experiments determining G carried out hitherto can be roughly grouped as follows:

(a) measurements with geophysical methods;
(b) measurements in space;
(c) measurements with torsion balances;
(d) measurements with beam balances;
(e) measurements with a gravimeter; and
(f) measurements with a Fabry–Perot microwave resonator.

The geophysical methods use the Earth and mountains or parts of the Earth's crust as the attracting masses, and were commonly used in the early experiments. Measurements in space are in progress. Sanders and Gillies [10] reviewed the various proposals for experiments to measure G in space, such as the NEWTON proposal of the University of Pisa [24], the satellite energy exchange (SEE) proposal of the University of Tennessee [25], the G/ISL test proposed for incorporation into the STEP mission [26]. The other methods were employed in recent experiments.

2.1. Nine experiments within the last five years

There have been nine experiments within the last five years that have reported absolute values of G. These results differ wildly from each other as shown in table 1 and figure 1. All of the experiments have been on-going for several years, and some of these efforts are still in progress. The torsion pendulum was employed by six of them, and one Fabry–Perot resonator, one mass comparator beam balance and one FG-5 gravimeter were employed as the detectors in the others.

A compensated torsion balance was employed by Fitzgerald and colleagues to measure G at the Measurement Standards Laboratory of Industrial Ltd, in New Zealand [12]. A cylindrical high-purity copper attracted mass system of about 97 g was suspended from a 1 m length of 50 μm diameter tungsten fibre. Two large cylindrical stainless steel attracting masses were used to produce a gravitationally induced torque acting on the attracted mass. The angular position of the attracted mass system was measured by an autocollimator viewing the mirror. The output of the autocollimator is fed through a servo-control loop to adjust the voltage on each set of electrodes that surround the attracted mass system. This electrostatically induced torque produced by the electrometer balances the gravitational torque acting on the small mass to maintain it at constant angular position, and was calibrated by using an acceleration method. Finally, they obtained the value for G as

$$G = (6.6656 \pm 0.0006) \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}} \tag{3}$$

with a combined standard uncertainty of 95 ppm. The main features of their measurement are: a compensated method was chosen so that the torsion fibre does not twist during the measurement and the suspended mass remains stationary—this was done to reduce the effects of the inelastic behaviour of the fibre and the local gravitational gradient on the measurement. They made a repeat measurement with an improved apparatus, in which the cylindrical attracted mass was increased to about 500 g and was suspended from a tungsten fibre with a rectangular cross section of 17 μm $\times$ 340 μm. They also discovered some second-order corrections that were omitted from the 1995 measurement [27]. Their latest values are

$$G = (6.6746 \pm 0.0010) \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}} \tag{4}$$

and

$$G = (6.6742 \pm 0.0007) \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}} \tag{5}$$

for the recalculated 1995 value and 1998 measurement, respectively.

A thin and heavily loaded strip suspended torsion balance was used to determine G by Quinn *et al* in BIPM [28, 29]. They used four 1.2 kg test masses and four 15.5 kg source masses to produce a gravitational torque of 2×10^{-8} N m, and measured the deflection angle of the balance under the influence of the source masses by an autocollimator. A preliminary result for their measurement of G is

$$G = (6.683 \pm 0.011) \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}} \tag{6}$$

with a relative combined standard uncertainty of 1700 ppm. They plan to employ a servo-controlled configuration, in which the gravitational torque will be balanced by an electrostatic torque applied between fixed thin cylindrical electrodes and test masses. Their goal is to determine G with a relative uncertainty of 100 ppm or better.

Perhaps the most enigmatic result to emerge from any of the modern experimental measurements of G is that which has been found by workers at the Physikalisch-Technische Bundesanstact (PTB) in Germany [11]. The experiment was carried out by means of an apparatus consisting of a fibreless torsion pendulum carried by the buoyancy of a floater in mercury. The motion of the pendulum is sensed by a differential laser interferometer. Movable attracting masses can be used to apply a calculable alternating torque to the pendulum via their gravitational interaction with the attracted masses fixed to it. Control signals applied to the vanes of a quadrant electrometer provide an electrostatic restoring torque used to counteract the gravitational torque and the null motion of the pendulum. The difference in the electrostatic torques measured when the attracting masses are moved from one of their test positions to another, will be equal to the associated change in the gravitational torque acting on the pendulum. This work began in 1986 and the first result obtained with this apparatus was [30]

$$G = (6.667 \pm 0.005) \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}. \tag{7}$$

With an improved apparatus and with the detection of systematic errors, they finally obtained the value of G in 1994 of

$$G = (6.7154 \pm 0.0006) \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}} \tag{8}$$

with a combined standard uncertainty of 83 ppm, which differs from the 1986 CODATA value by about 0.006 in relative value. The main features of their experiment are: the torsion pendulum is supported by a mercury bearing and the gravitational torque is compensated by a counter torque.

The other three recent torsion pendulum experiments employed the time-of-swing method and measured the influence of source masses on the oscillation period. Recently, Karagioz and his colleagues reported their experimental results of 10 years of measurements of G in Moscow [14, 15]. Three sets of torsion pendulums with periods of about 2077, 2731 and 1783 s were used to detect the gravitational force. The measurements were carried out in a thermostatic enclosure that maintained the apparatus at a temperature of $(23.0 \pm 0.1)\,^\circ$C. The main feature of their experiment is that the instrumentation was employed in one of two fundamentally different modes. In the first variant only one attracting mass is used (they called this the 'asymmetric' mode). In the other variant two attracting masses are placed on opposite sides of the pendulum (they called this the 'symmetric' mode). They have taken about 6900 measurements over the period 1985–1996, and the final result is

$$G = (6.6729 \pm 0.0005) \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}} \tag{9}$$

with a combined standard uncertainty of 78 ppm. A drift in the value of G with time over the range $\pm 0.05\%$ was observed in their experiment. They think that this drift is most likely to be due to some as yet unknown destabilizing factor, probably due to microseisms affecting the oscillation period and the equilibrium position of the pendulum.

In [16], Bagley and Luther reported their preliminary determinations of G employing low-Q torsion pendulums and using the time-of-swing method at Los Alamos National Laboratory in New Mexico. Their experiment was a redesigned version of that of Luther and Towler [31], and it incorporated some of the same components. The pendulum had an oscillation period of 210 s when suspended from either of two 12 μm tungsten wires and the attracting masses were positioned alternately in the near and far positions each half an hour. The pendulum was enclosed in a stainless vacuum system maintained at 10^{-7} Torr and a constant temperature within 1 °C per day. A very interesting feature of the experiment was that two different tungsten fibres were used to suspend the same pendulum in the same apparatus. Both fibres had a diameter of about 12 μm, and one fibre had a gold coating which made it more lossy and yielded a quality factor Q of 490, while another uncoated fibre yielded a high Q of 950. By making measurements at two different quality factors, they were able to test the hypothesis advanced by Kuroda [32] that the inelasticity of the suspension fibre produced an upward bias of $1/\pi Q$. (Newman and Bantel [33] had further demonstrated that the upward bias is bounded by $0 \leqslant \delta G/G \leqslant 1/2Q$.) The values for G as determined by the two experiments with different Q were 6.6784 ± 0.0008 and 6.6761 ± 0.0011 ($\times 10^{-11}$ m^3 kg^{-1} s^{-2}), disagreeing by 345 ppm. They applied a correction as indicated by Kuroda to correct the value for G to 6.6739 ± 0.0011 and 6.6741 ± 0.0008 ($\times 10^{-11}$ m^3 kg^{-1} s^{-2}). Combining these two values, they finally obtained a value of G of

$$G = (6.6740 \pm 0.0007) \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}. \tag{10}$$

We determined G by means of a high-Q and long-period torsion pendulum in the time-of-swing method in Huazhong University of Science and Technology (HUST), China [18]. Two 6.25 kg cylindrical attracting masses of the copper spherical test mass hang from one end of a 400.00 mm length aluminium beam by a 25 μm tungsten wire which is about 435 mm long. The counterweight mass is fixed on the other end of the beam. The torsion pendulum is suspended by a 25 μm tungsten wire which is about 513 mm long, and enclosed in a chamber at a vacuum of 2×10^{-5} Pa. The apparatus was located in a cave laboratory in our campus. The least thickness of the cover on the laboratory is more than 40 m, and the exit is 150 m away from the laboratory. The daily change of temperature in the laboratory is less than 0.005 °C. The periods of the pendulum were about 4441 and 3484 s with and without the attracting masses in the far position, respectively, and the period change was about 27%. The result of our experiment is

$$G = (6.6699 \pm 0.0007) \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}} \tag{11}$$

with a combined standard uncertainty of 105 ppm. The main feature of our experiment is that the systematic bias associated with the fibre inelasticity, which may have afflicted other previous experiments, can be avoided due to a very high Q in the pendulum.

A very novel type of experiment has been carried out by a team at the University of Wuppertal, Germany [13]. Started in the 1980s [34], the focus of the project has been the development of a gravimeter that uses a Fabry–Perot microwave cavity as a position sensor. Two mirrors of a Fabry–Perot resonator are suspended as pendulums of equal length separated at a distance by a suspension platform. The gravitational force of a laboratory test mass M acting on this resonator changes the mirror separation which is measured by means of a shift in the resonator frequency. The measured shift Δf of the resonance frequency can be converted into a shift of the mirror separation between the test masses of the resonator. Different separations between the test mass and the resonator in the range 0.6–2.1 m have been chosen to measure the gravitational force. No significant deviations from the inverse square law are observed on a level of a few parts in 10^4 in the near range, and finally the result for G is

$$G = (6.6719 \pm 0.0008) \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2} \tag{12}$$

with a combined standard uncertainty of 110 ppm. In [35], they reported a new preliminary value for G from 1998 measurements by the improved experiment:

$$G = (6.6735 \pm 0.0011 \pm 0.0026) \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2} \tag{13}$$

where the first uncertainty is the statistical one and the second value is the systematic one.

Schurr *et al* proposed a new method to measure G with a beam balance at the University of Zurich, Switzerland [17]. A beam balance used in the detection compares the weight of two 1 kg test masses and measures the gravitational force of two source masses with a statistical uncertainty of 10 ng. Two stainless steel tanks filled with water were employed for the source masses. The principle of their experiment is as follows: two large source masses are moved in the vertical direction and alternately positioned in one of two states. Their gravitational field acts on two small test masses which are separately suspended on wires. The suspension devices of the test masses are alternately connected to a single-pan beam balance which measures the weight change due to the gravitational force. The finally result of their experiment with water-filled tanks is

$$G_{water} = (6.6754 \pm 0.0005_{stat} \pm 0.0015_{sys}) \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2} \tag{14}$$

with a statistical uncertainty of 75 ppm and a systematic uncertainty of 230 ppm. The main features of their experiment are: the arrangement of two test and source masses is symmetrical and enables a differential measurement such that many disturbing forces and drift cancel each other out; the source masses have a cylindrical shape, and the test masses are located at an extremum of the gravitational field of source masses. In [36], they gave a preliminary result with mercury-filled tanks,

$$G_{mercury} = (6.6749 \pm 0.0014_{sys}) \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2} \tag{15}$$

where the statistical uncertainty is negligible and the systematic uncertainty is 217 ppm. They planned to make some improvements and investigations and hope to reach the design uncertainty of 10 ppm.

A research group at the Joint Institute for Laboratory Astrophysics in Boulder, Colorado, reported a free-fall determination of G [2]. Their method involves using a laser interferometer system to track the motion of a test mass that is repeatedly dropped in the presence of a locally induced perturbing gravity field. This gravitational field is produced by an 500 kg tungsten ring-shaped source mass located alternately above and below the dropping region. The experiment was concluded in a differential mode in which the acceleration of the test mass was alternately increased and then decreased by the source masses. The acceleration of the test mass was measured with a FG-5 absolute gravimeter, and the local background gravity concurrent with the G experiment was measured by a superconducting relative gravimeter. Their final result is

$$G = (6.6873 \pm 0.0094) \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2} \tag{16}$$

with a relative uncertainty of 1400 ppm.

2.2. *Experiments in progress and being planned*

Most of the above nine groups are going forward with refined versions of their experiments with the aim of decreasing the measurement uncertainties. There are also several interesting determinations of G that are in progress and being planned. Many experimenters have attempted to develop new approaches to the challenge of determining G and weak-force measurement. Gillies and Ritter [37] reviewed this activity and summarized the characteristics of torsion balances.

Newman and Bantel [33] prepared a measurement of G using a cryogenic torsion pendulum in the time-of-swing method and hope to determine G at the level of a few ppm. Features of the design include: first, operation at cryogenic temperature (2 K) to reduce thermal noise and increase frequency stability and for ease of magnetic shielding; second, large pendulum oscillation amplitudes to increase the signal-to-noise ratio and reduce the effect of the amplitude-determination error; third, use of a pair of source mass rings to produce an extremely uniform field gradient; lastly, use of a thin quartz plate as a torsion pendulum to minimize the sensitivity to the pendulum density in homogeneity and dimensional uncertainties. Ni *et al* [38] have obtained a preliminary measurement of G using a high-finesse Fabry–Perot optical cavity and hope to measure G with high precision. Speake *et al* [39] designed a torsion balance based on a spherical superconductive suspension. Gundlach and co-workers [40, 41] proposed a rotating torsion balance method, in which a torsion balance is placed on a slowly and continuously rotating turntable and the attracting masses are placed on a separate turntable that rotates at a constant rate. The twist due to the gravitational attraction is sensed and a feedback is turned on that changes the balance turntable rotation rate to minimize the twist, and the angular acceleration of the balance turntable compensates the gravitational angular acceleration of the balance. This method severely reduces the sensitivity to the leading sources of error in previous torsion balance measurements: it is mostly free of uncertainties due to inelastic torsion fibre properties; the torsion balance mass density distribution does not have to be known accurately; gravitational effects due to mass distribution changes in the vicinity are reduced. They hope to determine G to 10 ppm. Kolosnitsyn [42] suggested a novel laboratory measurement of G and hopes to measure G at a level of 10 ppm: a small dumb-bell-shaped torsional pendulum is planned in a spherical cavity cut out from a uniaxial oblate ellipsoid. The gravitational field within the ellipsoid is homogeneous with respect to the second derivatives of the gravitational potential, the third and higher derivatives vanish. The main difficulty in realizing the experiment is the preparation of the ellipsoid, especially the manufacture of its inner spherical and outer ellipsoidal surfaces.

3. Systematic error in measurement of G

The current situation, where the measured values of G disagree to a level far in excess of their error estimates, suggests that there are some systematic errors in the experiments measuring G. de Boer [45] reviewed the influences on the measurement uncertainty, such as inconstancy of the torsional moment of suspension, disturbances of the ground, influence of the ambient temperature, inhomogeneity of the masses, gradients in the gravitational field, changes of the surrounding field, and magnetic and electrical influences. We will review the systematic errors due to the inelasticity, the nonlinearity and the thermoelasticity of the torsion fibre itself.

3.1. Inelasticity of torsion fibre

Kuroda [32] has pointed out that in determination of G by the time-of-swing method with a torsion pendulum, the frequency dependence of the effective torsion spring constant can introduce a systematic error. Kuroda evaluated this error for the case discussed by Quinn *et al* [43] of an inelastic material characterized by a continuum Maxwell model [44] with a particular assumed distribution of relaxation strengths. Kuroda also showed that if the imaginary part of the torsion constant is a fixed fraction ϕ of the real part, then the upward fractional bias in G will be equal to $1/\pi Q$ where Q is the pendulum's quality factor. Bagley and Luther [16] performed two determinations using oscillation quality factors which differed by a factor of 2 to test the Kuroda hypothesis. Different tungsten wires with and without a coating were used to suspend the same pendulum in the same apparatus. As stated in section 2, their experimental results supported this conjecture.

Recently, Kuroda and co-workers measured the frequency dependence of the torsional spring constant of a tungsten fibre based on the systematic change in the inertial moment of a suspended pendulum [46]. However, the pendulum they used was dumb-bell shaped, and its frequency will be affected by the background gravitational field due to the change of position of the test masses. So their experimental result may include this effect. We are designing a new experiment to test the background gravitation field and to measure the inelasticity of the torsion fibre.

Newman and Bantel [33] demonstrated that if the fibre inelasticity is described by a continuum Maxwell model with any distribution of relaxation strengths, the resulting error $\delta G/G$ will be bounded between 0 and $1/2Q$. They assume the complex torsion constant to be expressed by $k(\omega) = k_0 + k_1(\omega) + \mathrm{i}k_2(\omega)$, where k_0, the 'relaxed' (zero-frequency) limit of k, is the dominant term. In the dynamic method of G measurement, the presence of the source masses effectively changes k_0. The resulting effective change in k, which determines the oscillation frequency, includes an indirect contribution from $k_1(\omega)$ due to the frequency shift. If this contribution is neglected, the resulting error in G determination is

$$\frac{\delta G}{G} = \left(\frac{\partial\,\mathrm{Re}\,k}{\partial k_0} - 1\right) = \frac{\partial k_1}{\partial \omega^2}\frac{\partial \omega^2}{\partial k_0} \approx \frac{\partial k_1}{\partial \omega^2}\frac{\omega^2}{k_0}. \tag{17}$$

In the continuum Maxwell model, k_1 and k_2 are expressed in terms of a distribution of relaxation strengths $F(\tau)$ ($F(\tau) \equiv \delta E f(\tau)$ in the notation of Quinn *et al* [43]):

$$k_1(\omega) = \int_0^\infty F(\tau)\frac{\omega^2\tau^2}{1+\omega^2\tau^2}\,\mathrm{d}\tau, \tag{18}$$

$$k_2(\omega) = \int_0^\infty F(\tau)\frac{\omega\tau}{1+\omega^2\tau^2}\,\mathrm{d}\tau. \tag{19}$$

From equations (17) and (18) they find

$$\frac{\delta G}{G} = \frac{1}{k_0}\int_0^\infty F(\tau)\frac{\omega^2\tau^2}{\left(1+\omega^2\tau^2\right)^2}\,\mathrm{d}\tau. \tag{20}$$

The Q of the pendulum is given approximately by k_0/k_2, so

$$\frac{1}{Q} = \frac{1}{k_0}\int_0^\infty F(\tau)\frac{\omega\tau}{1+\omega^2\tau^2}\,\mathrm{d}\tau. \tag{21}$$

Here ω is the oscillation frequency of the pendulum. Equation (20) differs from equation (21) only by an extra factor $\omega\tau/\left(1+\omega^2\tau^2\right)$ in the integrand which has a value $\leqslant \frac{1}{2}$ for all values of

τ. As the other factors in the integrands are all positive, they find the following bound on the G error in the general model discussed by Kuroda: $0 \leqslant \delta G/G \leqslant 1/2Q$.

The inelasticity of a torsion fibre would introduce a systematic error in measuring G. An effective way to overcome it is to operate the torsion pendulum system with very high Q as in our experiment [18] or to measure it experimentally for final correction.

3.2. Nonlinearity properties of the torsion fibre

The inelasticity indicates that the spring constant of the fibre is dependent on the oscillation frequency. While the nonlinearity of the oscillation indicates that the period of the pendulum oscillation varies with its amplitude, and the typical equation of the oscillation can be written as

$$I\ddot{\theta} + \gamma\dot{\theta} + K_1\theta + K_3\theta^3 = 0, \tag{22}$$

where I, θ and γ are the inertial moment, the angle displacement and the damping factor of the pendulum, respectively. The coefficient K_1 is the total equivalent torsion constant of the pendulum system, which usually includes three parts: the torsion constant of the fibre K_{1f}, the gravitational torsion constant of the attracting masses K_{1a} and the torsion constant of the background gravitational field K_{1b} [18]. The coefficient K_3 represents the nonlinearity of the pendulum system, which can also be classified into three parts K_{3f}, K_{3a} and K_{3b} correspondingly. Using the harmonic balance method [47], we can obtain the approximate solution of equation (22) at small-amplitude oscillation as follows:

$$\theta(t) \approx \theta_0 \mathrm{e}^{-\beta t}\cos\omega t + \frac{K_3}{32K_1}\theta_0^3\mathrm{e}^{-3\beta t}\cos 3\omega t, \tag{23}$$

where $\beta = \gamma/2I$ and θ_0 is the initial amplitude of the oscillation. The frequency ω^2 is shifted from its unperturbed value $\omega_0^2 = K_1/I$ to

$$\omega^2(A) = \omega_0^2 - \beta^2 + \frac{3K_3}{4I}A^2, \tag{24}$$

where $A = \theta_0\mathrm{e}^{-\beta t}$ is the amplitude of the oscillation at time t. Correspondingly, the period of the oscillation can be written as

$$T(A) = T_0\left[1 - \frac{I\beta^2}{K_1} + \frac{3K_3}{4K_1}A^2\right]^{-1/2}, \tag{25}$$

where $T_0 = 2\pi/\omega_0$ is the unperturbed period of the pendulum. In the time-of-swing method, the gravitational constant G is determined by

$$G = \frac{I\Delta(\omega_0^2)}{\Delta k_1}, \tag{26}$$

where $\Delta(\omega_0^2)$ is the difference of the square of the frequencies measured at two configurations, $\Delta K_1 = G\Delta k_1$ is the difference of the equivalent torsion constants in the two cases. From equations (24) and (26), we can obtain the systematic error due to the nonlinearity of the system as follows:

$$\frac{\delta G}{G} = \frac{\delta\left[\Delta(\omega_0^2)\right]}{\Delta(\omega_0^2)} = \frac{3A(A\delta K_3 + 2K_3\delta A)}{4I\Delta(\omega_0^2)}. \tag{27}$$

To reduce these effects, we should select small-amplitude operation. However, the noise contribution to the period measurement will increase with the decrease of the amplitude [1], and it sets a limit on the precision of the experiment. In [48], we found that the nonlinear term due to the attracting masses K_{3a} can approach zero in an optimum design.

We recently designed a symmetric-disc aluminium torsion pendulum of 57.415 mm in diameter and 9.988 mm in thickness to measure the fibre nonlinearity [49]. The symmetry of the pendulum can eliminate the influence of the background gravitational field on the period of the oscillation, so the first term in equation (27) can be neglected. The pendulum is suspended by a tungsten fibre, with diameter and length of 25 μm and 513 mm, respectively, and which is located in a stainless vacuum vessel maintained at a vacuum of 1.5×10^{-5} Pa by an ion pump. The preliminary results show that the mean value of K_3/K_1 is only about -0.026 rad^{-2} with a standard uncertainty of ± 0.008 rad^{-2}. For a typical experiment, the difference of the frequency squares $\Delta(\omega_0^2)$ is about 40% times ω_0^2. If we assume the oscillation amplitude $A = 1.0 \times 10^{-2}$ rad and $\delta A = 1.0 \times 10^{-3}$ rad as well as $|K_3/K_1| = 0.026$ rad^{-2} at the same time, the uncertainty in the determination of G due to the nonlinearity of the tungsten fibre would be less than 1 ppm according to equation (27). When we study the thermoelasticity of a tungsten torsion fibre [50], we find that this result includes the thermoelastic effect. After considering the variation of the ambient temperature, the corrected value of K_3/K_1 is about (-0.0011 ± 0.0018) rad^{-2}. This means that the effect of the nonlinearity of the torsion fibre can be omitted in small-amplitude operation.

3.3. Thermoelastic properties of the torsion fibre

In the determination of G, the torsion pendulum works in two common modes: a static mode and a dynamic mode. For the static mode, such as the direct deflection method, the gravitational constant G is determined by

$$G = K\Delta\theta/C, \tag{28}$$

where K is the torsion spring constant, C is a constant determined by the masses and the geometries of the attracting masses and the torsion pendulum, $\Delta\theta$ is the deflection angle of the pendulum due to the attracting masses. For the dynamic method, such as the time-of-swing method, G is determined by

$$G = \Delta K/(C_2 - C_1) = I\Delta(\omega_0^2)/(C_2 - C_1), \tag{29}$$

where ΔK and $\Delta(\omega_0^2)$ are the differences in the torsion spring constant and the frequency squares measured at two configurations of attracting masses, C_1 and C_2 are two constants determined by the masses and the geometries of the attracting masses and the torsion pendulum in each case, respectively, and I is the inertial moment of the pendulum.

The accuracy of these experiments depends on the constancy of the torsion spring constant of the torsion fibre. While the research work on the physical properties of the materials indicates that the length and shear moduli are temperature dependent, i.e. the torsion spring constant of the torsion fibre varies with the temperature, and this will lead directly to a systematic error in the uncertainty of determining G using a torsion pendulum.

The torsion spring constant K change with temperature can be expressed as

$$K = K_0(1 + \alpha_K \Delta t). \tag{30}$$

So the uncertainty of measuring G due to the thermoelastic properties of the fibre is

$$\delta G/G = \delta K/K = \alpha_K \Delta t. \tag{31}$$

Usually, the value of α_K is at least one order larger than that of α_l [51], and should be determined for a precise measurement of G by a torsion pendulum. Therefore, we designed an experiment to measure the period variation with the fluctuation of room temperature and then determined α_K of the torsion pendulum correspondingly.

We also employed the same apparatus used in measuring the nonlinearity of the torsion fibre, and modulated and measured the surrounding temperature of the torsion pendulum [50]. The preliminary results indicate that $\alpha_K = -(165 \pm 6) \times 10^{-6}\,{}^\circ\mathrm{C}^{-1}$, which means that the thermoelastic effect of the fibre would introduce a systematic error of about ∓ 165 ppm to the uncertainty of measuring G if the ambient temperature has a fluctuation with $\pm 1\,{}^\circ$C. Some experiments also indicate the existence of this phenomenon, such as in [16], where the authors observed a diurnal variation of the torsion pendulum frequency of about ∓ 70 ppm as shown in figure 2 of [16], which may be due to the ambient temperature variation of about $\pm 0.5\,{}^\circ$C over the course of a day. Of course, the systematic error due to the fluctuation of the torsion spring constant with the temperature will be less than 70 ppm because the time span of measuring $\Delta(\omega_0^2)$ was limited to 1 h every time. In [12], there is a downward bias of about 33 ppm in the measured value of G due to the temperature of the electrometer calibrations being $0.2\,{}^\circ$C higher than that of the measurement of V_G^2. In [14, 18], the temperature fluctuations are ± 0.1 and $\pm 0.005\,{}^\circ$C and the systematic errors due to these are about 17 ppm and less than 1 ppm, respectively.

To determine G with an uncertainty of 100 ppm or less, we should control the ambient temperature to within $\pm 0.1\,{}^\circ$C and monitor the temperature variation to correct the experimental result. To reduce the temperature coefficient of the torsion spring constant, we can let the torsion pendulum operate at a cryogenic temperature such as in [33], or select a better constant elastic alloy material as the torsion fibre. However, the thermoelastic property of the torsion fibres should be tested in a high-precision gravitational experiment with the torsion pendulum both at room temperature and at cryogenic temperatures.

4. Conclusion

The experimental values of the Newtonian gravitational constant G disagree far in excess of the estimate. It seems clear that some new accurate experiments are desirable. For an accurate torsion pendulum experiment, the inelasticity, the nonlinearity and the thermoelastic properties should be studied and measured. Some new technology such as operating the torsion pendulum at cryogenic temperatures is recommended.

We hope that due to the effort of the experimenters the experimental values for G will agree.

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