AST3220 Project 3
Part 2

Candidate 21

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I. Miscellaneous problems

Problem 1

In problem 1 we assume that the universe is described by the Einstein-de Sitter model (EdS).

a)

In EdS the scale factor a is given by (Øystein Elgarøy, 2024, eq. (3.17))

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3}$$

which we use to show the given expression for the time

$$\Rightarrow \frac{t}{t_0} = \left(\frac{a(t)}{a_0}\right)^{3/2}$$
$$t = \frac{t_0}{(1+z)^{3/2}}$$

where I used the definition of redshift $1+z = a_0/a$, and this is what we were to show.

b)

For each redshift z we can find the time coordinate of when this light was emitted by the above formula. We find

$$\begin{aligned} t_1 &= \frac{t_0}{(1+z_1)^{3/2}} \\ &= \frac{t_0}{(1+3)^{3/2}} \\ &= \underline{\frac{t_0}{8}} \end{aligned}$$

$$\begin{aligned} t_2 &= \frac{t_0}{(1+z_2)^{3/2}} \\ &= \frac{t_0}{(1+8)^{3/2}} \\ &= \underline{\frac{t_0}{27}} \end{aligned}$$

Meaning the light from the first object was emitted at one eighth the age of the universe (in the EdS-model), and from the second object at one twenty-seventh the age of the universe (EdS).

c)

The expression for the comoving radial coordinate r is given as (Øystein Elgarøy, 2024, eq. (1.22))

$$r = \mathcal{S}_k \left[\int_t^{t_0} \frac{cdt'}{a(t')}\right]$$

where

$$\mathcal{S}_k(x) = \begin{cases} \sin x, & k=1 \\ x, & k=0 \\ \arcsin x, & k=-1 \end{cases}$$

The EdS model is a flat universe, meaning $k=0$, so the expression we use is

$$\begin{aligned} r &= \int_t^{t_0} \frac{cdt'}{a(t')} \\ &= \int_t^{t_0} \frac{cdt'}{a_0\left(\frac{t'}{t_0}\right)^{2/3}} \\ &= \frac{ct_0^{2/3}}{a_0}\int_t^{t_0} t'^{-2/3}dt' \\ &= \frac{ct_0^{2/3}}{a_0}\left[3t'^{1/3}\right]_t^{t_0} \\ &= \frac{ct_0^{2/3}}{a_0} 3\left(t_0^{1/3} - t^{1/3}\right) \end{aligned}$$

$$\begin{aligned} r &= \frac{3ct_0}{a_0}\left[1 - \left(\frac{t}{t_0}\right)^{1/3}\right] \\ &= \frac{3ct_0}{a_0}\left(1 - \frac{1}{\sqrt{1+z}}\right) \end{aligned} \tag{1}$$

For these two objects we get

$$\begin{aligned} r_1 &= \frac{3ct_0}{a_0}[1 - (1/8)^{1/3}] \\ &= \underline{\frac{3ct_0}{2a_0}} \end{aligned}$$

$$\begin{aligned} r_2 &= \frac{3ct_0}{a_0}[1 - (1/27)^{1/3}] \\ &= \underline{\frac{2ct_0}{a_0}} \end{aligned}$$

d)

Now we consider that the light from an object with $z = z_2 = 8$ to be emitted at a time we call $t_e = \frac{t_0}{27}$. We want to determine the comoving coordinate r for the light heading towards us from that object at an arbitrary later time t. This is determined by the previous equation (1).

e)

We want to calculate the redshift that an observer at the object with $z = z_1 = 3$ would measure for the light from the object at $z = z_2 = 8$. For this I simply change the reference frame for the redshift by replacing a_0 with a_1:

$$\begin{aligned} 1 + z &= \frac{a_1}{a_2} \\ &= \frac{a_0}{a_2}\frac{a_1}{a_0} \\ &= \frac{1+z_2}{1+z_1} \\ \Rightarrow z &= \frac{1+z_2}{1+z_1} - 1 \\ &= \frac{1+8}{1+3} - 1 \\ &= \underline{1.25} \end{aligned}$$

Meaning the observer would measure the redshift $z = 1.25$.

Problem 2

a)

We want to substitute the expression for the proper distance to the particle horizon given by

$$d_{P,PH}(t) = a(t)\int_0^t \frac{cdt'}{a(t')}$$

to the redshift z. First we can substitute $a(t)$ for z by its definition

$$1 + z = \frac{a_0}{a} \Rightarrow a = \frac{a_0}{1+z}$$

Then for the integral we can write

$$\begin{aligned} dt &= dt\frac{dz}{dz} \\ &= dz\frac{dt}{dz} \\ &= dz\left(\frac{dz}{dt}\right)^{-1} \end{aligned}$$

For this differential we again begin with the definition

$$\begin{aligned} \frac{dz}{dt} &= \frac{d}{dt}\left(\frac{a_0}{a} - 1\right) \\ &= -\frac{a_0}{a^2}\frac{da}{dt} \\ &= -\frac{a_0}{a}\frac{\dot{a}}{a} \\ &= -\frac{a_0}{a}H \end{aligned}$$

where we used that the Hubble parameter is defined as $H = \frac{\dot{a}}{a}$. We then get

$$\begin{aligned} dt &= dz\left(\frac{dz}{dt}\right)^{-1} \\ &= -\frac{adz}{a_0 H} \end{aligned}$$

We can now do the substitution to z:

$$\begin{aligned} d_{P,PH}(t) &= a(t)\int_0^t \frac{cdt'}{a('t)} \\ d_{P,PH}(z) &= \frac{a_0}{1+z}\int_\infty^z -\frac{c\frac{a(z')dz'}{a_0 H(z')}}{a(z')} \\ &= -\frac{a_0}{1+z}c\frac{1}{a_0}\int_\infty^z \frac{a(z')dz'}{a(z')H(z')} \\ &= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H(z')} \end{aligned}$$

which is what we were to show.

b)

For a matter-dominated universe the Hubble parameter can be expressed as

$$H(z) = H_0\sqrt{\Omega_{m0}}(1+z)^{3/2}$$

We can use this to calculate the proper distance to the particle horizon:

$$\begin{aligned} d_{P,PH}(z) &= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H(z')} \\ &= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H_0\sqrt{\Omega_{m0}}(1+z')^{3/2}} \\ &= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{m0}}}\int_z^\infty \frac{dz'}{(1+z')^{3/2}} \\ &= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{m0}}}\left[-2\frac{1}{\sqrt{1+z'}}\right]_z^\infty \\ &= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{m0}}}\frac{2}{\sqrt{1+z}} \\ &= 2c\frac{1}{H_0\sqrt{\Omega_{m0}}(1+z)^{3/2}} \\ &= \frac{2c}{H_0} \\ &\sim \frac{c}{H_0} \end{aligned}$$

For a radiation-dominated universe the Hubble parameter can instead be expressed as

$$H(z) = H_0\sqrt{\Omega_{r0}}(1+z)^2$$

We do the same calculation:

$$\begin{aligned}
d_{P,PH}(z) &= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H(z')} \\
&= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{r0}}}\int_z^\infty \frac{dz'}{(1+z')^2} \\
&= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{r0}}}\left[-\frac{1}{1+z'}\right]_z^\infty \\
&= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{r0}}}\frac{1}{1+z} \\
&= c\frac{1}{H_0\sqrt{\Omega_{r0}}}\frac{1}{(1+z)^2} \\
&= \frac{c}{H_0}
\end{aligned}$$

Which is also what we were to show.

c)

References

Øystein Elgarøy. (2024). *Ast3220 - cosmology i (lecture notes).* https://www.uio.no/studier/emner/matnat/astro/AST3220/v24/undervisningsmateriale/lectures_ast3220.pdf. ([Online; accessed 07-May-2024])