I. PROBLEM 1

This is a simple differential equation, we start by separating ρ_ϕ on one side and everything with t on the other:

$$\begin{aligned}
\dot{\rho}_\phi = \frac{d\rho_\phi}{dt} &= -3H(1+\omega_\phi(t)\ )\rho_\phi \\
\frac{d\rho_\phi}{\rho_\phi} &= -3H(1+\omega_\phi(t)\ )dt \\
\int_{\rho_{\phi,0}}^{\rho_\phi} \frac{d\rho'_\phi}{\rho'_\phi} &= \int_{t_0}^{t} -3H(1+\omega_\phi(t')\ )dt' \\
\ln\left(\frac{\rho_\phi}{\rho_{\phi,0}}\right) &= \int_{t_0}^{t} -3H(1+\omega_\phi(t')\ )dt'
\end{aligned}$$

Here I choose to substitute t with a and then z. I use the definition of H:

$$\begin{aligned}
H \equiv \frac{\dot{a}}{a} &= \frac{1}{a}\frac{da}{dt} \\
\Rightarrow Hdt &= \frac{da}{a}
\end{aligned}$$

We put this into our equation:

$$\ln\left(\frac{\rho_\phi}{\rho_{\phi,0}}\right) = \int_{a_0}^{a} -3(1+\omega_\phi)\frac{da'}{a'}$$

Then we use the definition of redshift z:

$$\begin{aligned}
1+z = \frac{a(t_0)}{a(t)} &= \frac{a_0}{a} \\
\Rightarrow a &= \frac{a_0}{1+z} \\
\Rightarrow da &= -\frac{a_0}{(1+z)^2}dz
\end{aligned}$$

We put these two preceding expressions into our equation:

$$\ln\left(\frac{\rho_\phi}{\rho_{\phi,0}}\right) = \int_0^z -3(1+\omega_\phi(z')\ )\left(-\frac{a_0}{(1+z')^2}dz'\right)\frac{1+z'}{a_0}$$

For the integral limits we choose no redshift at time $t = t_0$ ($z = 0$) and to integrate to a certain redshift z, we simplify the equation:

$$\ln\left(\frac{\rho_\phi}{\rho_{\phi,0}}\right) = \int_0^z \frac{3[1+\omega_\phi(z')]}{1+z'}dz'$$

$$\Rightarrow \rho_\phi(z) = \rho_{\phi,0}\exp\left\{\int_0^z \frac{3[1+\omega_\phi(z')]}{1+z'}dz'\right\} \qquad \text{(A)}$$

which is what we wanted to show.

II. PROBLEM 2

First we differentiate the project's equation (1) with regards to time, afterwards we put it into the continuity equation for the scalar field:

$$\begin{aligned}
\rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\
\Rightarrow \dot{\rho_\phi} &= \ddot{\phi}\dot{\phi} + \frac{dV}{dt} \\
&= \ddot{\phi}\dot{\phi} + \frac{dV}{d\phi}\frac{d\phi}{dt} \\
&= \ddot{\phi}\dot{\phi} + V'(\phi)\dot{\phi}
\end{aligned}$$

Now we put the above expression, along with the projects' (1) and (2) into the continuity equation for the scalar field:

$$\begin{aligned}
\dot{\rho_\phi} &= -3H(\phi_\phi + p_\phi) \\
\ddot{\phi}\dot{\phi} + V'(\phi)\dot{\phi} &= -3H(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}\dot{\phi}^2 - V(\phi)) \\
\ddot{\phi}\dot{\phi} + V'(\phi)\dot{\phi} &= -3H\dot{\phi}^2 \\
\ddot{\phi} + V'(\phi) &= -3H\dot{\phi} \\
\ddot{\phi} + 3H\dot{\phi} + V'(\phi) &= 0
\end{aligned}$$

which is what we wanted to show.

III. PROBLEM 3

First off we can differentiate H to see what it equals:

$$\begin{aligned}
\dot{H} &= \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) \\
&= \frac{\ddot{a}a - \dot{a}^2}{a^2} \\
&= \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2
\end{aligned}$$

which we recognize is the second Friedmann equation minus the first Friedmann equation which we can put in where we get:

$$\begin{aligned}
&= -\frac{4\pi G}{3}\left[(\rho_m + 3p_m) + (\rho_r + 3p_r) + (\rho_\phi + 3p_\phi)\right] \\
&\quad -\frac{8\pi G}{3}\left(\rho_m + \rho_r + \rho_\phi\right) \\
&= -4\pi G[\frac{1}{3}(\rho_m + 3p_m) + \frac{1}{3}(\rho_r + 3p_r) + \frac{1}{3}(\rho_\phi + 3p_\phi) \\
&\quad + \frac{2}{3}\rho_m + \frac{2}{3}\rho_\phi + \frac{2}{3}\rho_\phi]
\end{aligned}$$

Here we substitute in $\kappa^2 = 8\pi G$:

$$= -\frac{\kappa^2}{2}\left[(\rho_m + p_m) + (\rho_r + p_r) + (\rho_\phi + p_\phi)\right]$$

for the first two parentheses we use the equation of state to substitute the pressure $p_i = \omega_i\rho_i c^2 = \omega_i\rho_i$ (with c=1), then for the final parentheses we use the project's equation (1) and (2):

$$\begin{aligned}
&= -\frac{\kappa^2}{2}\left[\rho_m(1+\omega_m) + \rho_r(1+\omega_r) + (\frac{1}{2}\dot{\phi}^2 + V + \frac{1}{2}\dot{\phi}^2 - V)\right] \\
&= -\frac{\kappa^2}{2}\left[\rho_m(1+\omega_m) + \rho_r(1+\omega_r) + \dot{\phi}^2\right]
\end{aligned}$$

the final step is using that for non-relativistic matter we have $\omega_m = 0$:

$$\dot{H} = -\frac{\kappa^2}{2}\left[\rho_m + \rho_r(1+\omega_r) + \dot{\phi}^2\right]$$

which is what we were to show.

IV. PROBLEM 4

We use the given definition of $\Omega_i(t)$ from the project's (9) together with (10) which gives:

$$\begin{aligned}
\Omega_\phi &= \frac{\rho_\phi(t)}{\rho_c(t)} \\
&= \frac{\frac{1}{2}\dot{\phi}^2 + V(\phi)}{} \\
&= \frac{\kappa^2}{3H^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]
\end{aligned}$$

Now for the right side of the equation we use (11) and (12):

$$\begin{aligned}
x_1^2 + x_2^2 &= \frac{\kappa^2\dot{\phi}^2}{6H^2} + \frac{\kappa^2 V(\phi)}{3H^2} \\
&= \frac{\kappa^2}{3H^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]
\end{aligned}$$

which proves that they are equal, i.e

$$\Omega_\phi = x_1^2 + x_2^2$$

We do the same for the second one:

$$\begin{aligned}
\Omega_r &= \frac{\rho_r(t)}{\rho_c(t)} \\
&= \frac{\kappa^2}{3H^2}\rho_r \\
&= x_3^2
\end{aligned}$$

by the project's equation (13).

The last one we can more easily show by using that

$$\begin{aligned}
\Omega_\phi + \Omega_r + \Omega_m &= 1 \\
\Rightarrow \Omega_m &= 1 - \Omega_\phi - \Omega_r \\
&= 1 - x_1^2 - x_2^2 - x_3^2
\end{aligned}$$

To show that these terms must be equal to one we can use the project's (7) at $t = t_0$ which gives no redshift $z = 0$

$$\begin{aligned}
H^2(t = t_0) = H_0^2 &= H_0^2\left[\Omega_{m0}(1+0)^3 + \Omega_{r0}(1+0)^4 + \Omega_{\phi 0}\exp(0)\right] \\
\Rightarrow 1 &= \Omega_{m0} + \Omega_{r0} + \Omega_{\phi 0}
\end{aligned}$$

this also applies at any time t so we have:

$$1 = \Omega_m + \Omega_r + \Omega_\phi$$

Therefore I have showed all three parts of the project's equation (14).

V. PROBLEM 5

We start from the project's equation (8):

$$\begin{aligned}
\frac{\dot{H}}{H^2} &= -\frac{\kappa^2}{2H^2}\left[\rho_m + \rho_r(1+\omega_r) + \dot{\phi}^2\right] \\
&= -\frac{1}{2}\left[\frac{\kappa^2}{H^2}\rho_m + \frac{\kappa^2}{H^2}\rho_r(1+\omega_r) + \frac{\kappa^2}{H^2}\dot{\phi}^2\right]
\end{aligned}$$

here we substitute in $\rho_c \Longrightarrow \frac{\kappa^2}{H^2} = 3\frac{1}{\rho_c}$ (eq. 10) together with $\omega_i = \frac{\rho_i}{\rho_c}$ (eq. 9):

$$= -\frac{1}{2}\left[3\Omega_m + 3\Omega_r(1+\omega_r) + \frac{\kappa^2}{H^2}\dot{\phi}^2\right]$$

also we substitute in for the relativistic matter/radiation we have $\omega_r = 1/3$:

$$\begin{aligned}
&= -\frac{1}{2}\left[3\Omega_m + 3\Omega_r(1+\frac{1}{3}) + \frac{\kappa^2}{H^2}\dot{\phi}^2\right] \\
&= -\frac{1}{2}\left[3\Omega_m + 4\Omega_r + \frac{\kappa^2}{H^2}\dot{\phi}^2\right] \\
&= -\frac{1}{2}\left[3\Omega_m + 4\Omega_r + 6\frac{\kappa^2\dot{\phi}^2}{6H^2}\right] \\
&= -\frac{1}{2}\left[3\Omega_m + 4\Omega_r + 6x_1^2\right]
\end{aligned}$$

now we use the previously shown equation (14) for Ω_r and Ω_m:

$$\begin{aligned}
&= -\frac{1}{2}\left[3(1 - x_1^2 - x_2^2 - x_3^2) + 4x_3^2 + 6x_1^2\right] \\
&= -\frac{1}{2}\left[3 + 3x_1^2 - 3x_2^2 + x_3^2\right]
\end{aligned}$$

which is what we were to show.

VI. PROBLEM 6

We start with the first expression (19) and use the fact we learned of $\frac{df}{dt} = H\frac{df}{dN} \Rightarrow \frac{df}{dN} = \frac{1}{H}\frac{df}{dt}$:

$$\begin{aligned}
\frac{dx_1}{dN} &= \frac{1}{H}\frac{dx_1}{dt} \\
&= \frac{1}{H}\frac{d}{dt}\left(\frac{\kappa\dot{\phi}}{\sqrt{6}H}\right) \\
&= \frac{\kappa}{\sqrt{6}H}\frac{d}{dt}\left(\frac{\dot{\phi}}{H}\right) \\
&= \frac{\kappa}{\sqrt{6}H}\left(\frac{\ddot{\phi}H - \dot{\phi}\dot{H}}{H^2}\right) \\
&= \frac{\kappa}{\sqrt{6}H}\left(\frac{\ddot{\phi}}{H} - \dot{\phi}\frac{\dot{H}}{H^2}\right)
\end{aligned}$$

here we substitute $\ddot{\phi}$ by using the continuity equation for the scalar field (6) which gives $\ddot{\phi} = -3H\dot{\phi} - V'(\phi)$:

$$
\begin{aligned}
&= \frac{\kappa}{\sqrt{6}H}\left(-3\dot{\phi} - \frac{V'}{H} - \dot{\phi}\frac{\dot{H}}{H^2}\right) \\
&= -3\frac{\kappa\dot{\phi}}{\sqrt{6}H} - \frac{\kappa}{\sqrt{6}H}\frac{V'}{H} - \frac{\kappa\dot{\phi}}{\sqrt{6}H}\frac{\dot{H}}{H^2} \\
&= -3x_1 + \frac{\sqrt{6}}{2}\left(-\frac{V'}{\kappa V}\right)\frac{\kappa^2 V}{3H^2} - x_1\frac{\dot{H}}{H^2}
\end{aligned}
$$

Now we just substitute in equations (12), (16), and (17) to finally show what we were to show:

$$
= -3x_1 + \frac{\sqrt{6}}{2}\lambda x_2^2 + \frac{1}{2}x_1\left[3 + 3x_1^2 - 3x_2^2 + x_3^2\right]
$$

For the second expression we begin the same:

$$
\begin{aligned}
\frac{dx_1}{dN} &= \frac{1}{H}\frac{dx_1}{dt} \\
&= \frac{1}{H}\frac{d}{dt}\left(\frac{\kappa\sqrt{V}}{\sqrt{3}H}\right) \\
&= \frac{\kappa}{\sqrt{3}H}\frac{d}{dt}\left(\frac{\sqrt{V}}{H}\right) \\
&= \frac{\kappa}{\sqrt{3}H}\left(\frac{\frac{1}{2\sqrt{V}}\frac{dV}{dt}H - \sqrt{V}\dot{H}}{H^2}\right) \\
&= \frac{\kappa}{\sqrt{3}H}\left(\frac{\frac{1}{2\sqrt{V}}\frac{dV}{d\phi}\frac{d\phi}{dt}H - \sqrt{V}\dot{H}}{H^2}\right) \\
&= \frac{\kappa}{\sqrt{3}H}\left(\frac{\frac{1}{2\sqrt{V}}V'\dot{\phi}H - \sqrt{V}\dot{H}}{H^2}\right) \\
&= \frac{\kappa}{\sqrt{3}H}\left(\frac{V'\dot{\phi}}{2\sqrt{V}H} - \frac{\sqrt{V}\dot{H}}{H^2}\right) \\
&= \frac{\sqrt{6}}{2}\frac{V'}{\kappa V}\frac{\kappa\dot{\phi}}{\sqrt{6}H}\frac{\kappa\sqrt{V}}{\sqrt{3}H} - \frac{\kappa\sqrt{V}}{\sqrt{3}H}\frac{\dot{H}}{H^2} \\
&= -\frac{\sqrt{6}}{2}\lambda x_1 x_2 + \frac{1}{2}x_2(3 + 3x_1^2 - 3x_2^2 + x_3^2)
\end{aligned}
$$

which is what we were to show.

For the last expression:

$$
\begin{aligned}
\frac{dx_3}{dN} &= \frac{1}{H}\frac{dx_3}{dt} \\
&= \frac{1}{H}\frac{d}{dt}\left(\frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H}\right) \\
&= \frac{\kappa}{\sqrt{3}H}\frac{d}{dt}\left(\frac{\sqrt{\rho_r}}{H}\right) \\
&= \frac{\kappa}{\sqrt{3}H}\left(\frac{\frac{1}{2\sqrt{\rho_r}}\frac{d\rho_r}{dt}H - \sqrt{\rho_r}\dot{H}}{H^2}\right) \\
&= \frac{\kappa}{\sqrt{3}H}\left(\frac{\dot{\rho}_c}{2\sqrt{\rho_r}H} - \sqrt{\rho_r}\frac{\dot{H}}{H^2}\right)
\end{aligned}
$$

For $\dot{\rho}_c$ we use the continuity equation given as $\rho + 3H(\rho + \frac{p}{c^2}) = 0$ which gives $\rho_r = -4H\rho_r$ when using $c = 1$, $p_r = \omega_r\rho_c$ with $\omega_r = 1/3$. We get:

$$
\begin{aligned}
&= \frac{\kappa}{\sqrt{3}H}\left(\frac{-4H\rho_r}{2\sqrt{\rho_r}H} - \sqrt{\rho_r}\frac{\dot{H}}{H^2}\right) \\
&= \frac{\kappa}{\sqrt{3}H}\left(-2\sqrt{\rho_r} - \sqrt{\rho_r}\frac{\dot{H}}{H^2}\right) \\
&= -2\frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H} - \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H}\frac{\dot{H}}{H^2} \\
&= -2x_3 + \frac{1}{2}x_3(3 + 3x_1^2 - 3x_2^2 + x_3^2)
\end{aligned}
$$

which is what we were to show.

VII. PROBLEM 7

From the project's equation (17) we see that for λ to be constant means that:

$$
\frac{V'}{V} = constant = C_0
$$

which we can solve as a differential equation:

$$
\frac{dV}{V} = C_0 d\phi
$$

here I'm not that interested in the exact values of the constants, so I differentiate indefinitely:

$$
\begin{aligned}
&\Rightarrow \ln V = C_0\phi \\
&\Rightarrow V(\phi) = C_1\exp(\phi) \text{ or rewritten: } V(\phi) = C_2\exp(C_3\phi)
\end{aligned}
$$

which is the form that V must have, where C_i are some constants. We differentiate this form to find V' and V'':

$$
\begin{aligned}
V' &= C_1\exp(\phi) = V \\
V'' &= V
\end{aligned}
$$

From the project's (18) we can find the value of Γ:

$$
\begin{aligned}
\Gamma &= \frac{VV''}{V'^2} \\
&= \frac{VV}{V^2} \\
&= 1
\end{aligned}
$$

so we see that when λ is constant, Γ is equal to one.

VIII. PROBLEM 8

We start by rewriting the differential, then we put the project's expression (17) in and differentiate that:

$$
\begin{aligned}
\frac{d\lambda}{dN} &= \frac{1}{H}\frac{d\lambda}{dt} \\
&= \frac{1}{H}\frac{d\phi}{dt}\frac{d\lambda}{\phi} \\
&= \frac{\dot{\phi}}{H}\frac{d}{d\phi}\left(-\frac{V'}{\kappa V}\right) \\
&= -\frac{\dot{\phi}}{H\kappa}\frac{V''V - V'^2}{V^2} \\
&= -\frac{\dot{\phi}}{H\kappa}\left[\frac{V''V}{V^2} - \left(\frac{V'}{V}\right)^2\right]
\end{aligned}
$$

now I put in the expression for Γ (18) and afterwards again for λ (17), then lastly for x_1 (11):

$$
\begin{aligned}
&= -\frac{\dot{\phi}}{H\kappa}\left[\Gamma\frac{V'^2}{V''V}\frac{V''V}{V^2} - \left(\frac{V'}{V}\right)^2\right] \\
&= -\frac{\dot{\phi}}{H\kappa}\left[\Gamma\left(\frac{V'}{V}\right)^2 - \left(\frac{V'}{V}\right)^2\right] \\
&= -\frac{\dot{\phi}}{H\kappa}\left(\frac{V'}{V}\right)^2(\Gamma-1) \\
&= -\frac{\dot{\phi}\kappa}{H}\left(\frac{V'}{\kappa V}\right)^2(\Gamma-1) \\
&= -\frac{\dot{\phi}\kappa}{H}\lambda^2(\Gamma-1) \\
&= -\frac{\dot{\phi}\kappa}{H}x_1\frac{\sqrt{6}H}{\kappa\dot{\phi}}\lambda^2(\Gamma-1) \\
&= \underline{-\sqrt{6}\lambda^2(\Gamma-1)\,x_1}
\end{aligned}
$$

which is what we wanted to show.

IX. PROBLEM 9

First off I start by differentiating the two potentials:

$$
\begin{aligned}
V_{\text{power}}(\phi) &= M^{4+\alpha}\phi^{-\alpha} && \text{(B)} \\
V'_{\text{power}}(\phi) &= -\alpha M^{4+\alpha}\phi^{-\alpha-1} && \text{(C)} \\
V''_{\text{power}}(\phi) &= -\alpha(-\alpha-1)M^{4+\alpha}\phi^{-\alpha-2} \\
&= \alpha(\alpha+1)M^{4+\alpha}\phi^{-\alpha-2} && \text{(D)}
\end{aligned}
$$

Which gives the following values:

$$
\begin{aligned}
\lambda_{\text{power}} &= -\frac{V'_{\text{power}}}{\kappa V_{\text{power}}} \\
&= -\frac{-\alpha M^{4+\alpha}\phi^{-\alpha-1}}{\kappa M^{4+\alpha}\phi^{-\alpha}} \\
&= \frac{\alpha}{\kappa}\phi^{-1} && \text{(E)} \\
\Gamma_{\text{power}} &= \frac{V_{\text{power}}V''_{\text{power}}}{(V'_{\text{power}})^2} \\
&= \frac{M^{4+\alpha}\phi^{-\alpha}\alpha(\alpha+1)M^{4+\alpha}\phi^{-\alpha-2}}{(-\alpha M^{4+\alpha}\phi^{-\alpha-1})^2} \\
&= \frac{\phi^{-\alpha-\alpha-2+2\alpha+2}}{\alpha} \\
&= \frac{\alpha+1}{\alpha} && \text{(F)}
\end{aligned}
$$

And for the exponential potential we recognize that λexp should be constant, from our work in problem 7, and therefore Γ_{exp} should also be equal to one. We check these quickly:

$$
\begin{aligned}
V_{\text{exp}}(\phi) &= V_0\exp(-\kappa\xi\phi) && \text{(G)} \\
V'_{\text{exp}}(\phi) &= -\kappa\xi V_0\exp(-\kappa\xi\phi) = -\kappa\xi V_{\text{exp}} && \text{(H)} \\
V''_{\text{exp}}(\phi) &= \kappa^2\xi^2 V_0\exp(-\kappa\xi\phi) = \kappa^2\xi^2 V_{\text{exp}} && \text{(I)}
\end{aligned}
$$

Which gives

$$
\begin{aligned}
\lambda_{\text{exp}} &= -\frac{V'_{\text{exp}}}{\kappa V_{\text{exp}}} \\
&= -\frac{-\kappa\xi V_{\text{exp}}}{\kappa V_{\text{exp}}} = \xi && \text{(J)} \\
\Gamma_{\text{exp}} &= \frac{V_{\text{exp}}V''_{\text{exp}}}{(V'_{\text{exp}})^2} \\
&= \frac{V\kappa^2\xi^2 V}{(-\kappa\xi V)^2} = 1 && \text{(K)}
\end{aligned}
$$

which is what we expected.

Integrating the equations of motion numerically, we are to use the interval $0 \le z \le 2\cdot 10^7$ with initial condition $z_i = 2\cdot 10^7$. I want to turn this into N by using the project's equation (15):

$$
\begin{aligned}
N &\equiv \ln\left(\frac{a}{a_0}\right) \\
&= \ln\left(\frac{1}{1+z}\right)
\end{aligned}
$$

So that our z-interval turns into the following N-interval $\ln\left(\frac{1}{1+2\cdot 10^7}\right) \approx -16.81 \le N \le 0$ with initial condition $N_i = \ln\left(\frac{1}{1+2\cdot 10^7}\right) \approx -16.81$.

Firstly I plot the density parameters shown for the inverse power-law potential in figure 1, and the exponential potential in figure 2. In both figures the sum

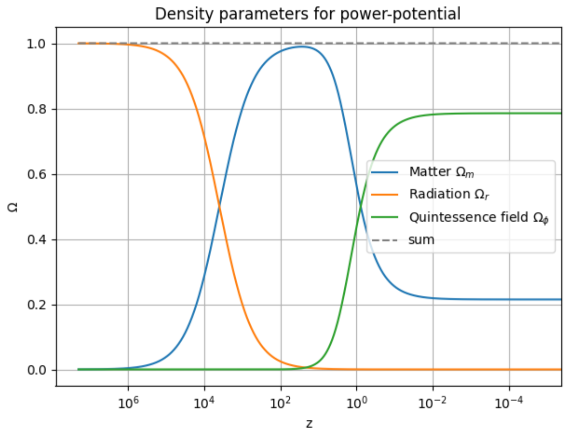

Figure 1: The density parameters Ω_i for matter $i = m$, radiation $i = r$, and the quintessence field $i = \phi$ as a function of the redshift z using the inverse power-law potential V_{power} (eq. B).

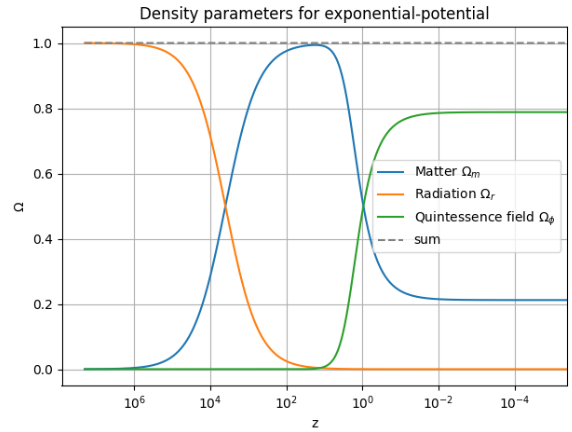

Figure 2: The density parameters Ω_i for matter $i = m$, radiation $i = r$, and the quintessence field $i = \phi$ as a function of the redshift z, using the inverse power-law potential V_{exp} (eq. G).

of the density parameters always add up to one for any time/redshift z which is expected as explained in problem 4. The graphs look almost identical for both potentials, but not exactly the same. Today's values (at $z = 0$) for the power-law potential are given as

$$\begin{aligned}
\Omega_{m0} &\approx 0.2145 \\
\Omega_{r0} &\approx 5.3616 \cdot 10^{-5} \\
\Omega_{\phi 0} &\approx 0.7855
\end{aligned}$$

and for the exponential potential as

$$\begin{aligned}
\Omega_{m0} &\approx 0.2119 \\
\Omega_{r0} &\approx 5.2963 \cdot 10^{-5} \\
\Omega_{\phi 0} &\approx 0.7881
\end{aligned}$$

Secondly I wish to plot the EoS-parameter ω_ϕ. To do this I first need to express ω_ϕ as a function of the parameters x_1, x_2, x_3 which I have solved numerically. We start at the project's equation (3) together with (1) and (2):

$$\begin{aligned}
\omega_\phi &= \frac{p_\phi}{\rho_\phi} \\
&= \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} \\
&= \frac{\frac{3H^2}{\kappa^2}\frac{\kappa^2\dot{\phi}^2}{6H^2} - \frac{3H^2}{\kappa^2}\frac{\kappa^2 V}{3H^2}}{\frac{3H^2}{\kappa^2}\frac{\kappa^2\dot{\phi}^2}{6H^2} + \frac{3H^2}{\kappa^2}\frac{\kappa^2 V}{3H^2}} \\
&= \frac{\frac{\kappa^2\dot{\phi}^2}{6H^2} - \frac{\kappa^2 V}{3H^2}}{\frac{\kappa^2\dot{\phi}^2}{6H^2} + \frac{\kappa^2 V}{3H^2}} \\
&= \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} \qquad \text{(L)}
\end{aligned}$$

which we use to plot the EoS parameter for the inverse power-law potential together with the exponential potential in figure 3. Today's values are given as

$$\begin{aligned}
\text{Power-law}: \quad \omega_{\phi 0} &= -0.7845 \\
\text{Exponential}: \quad \omega_{\phi 0} &= -0.5461
\end{aligned}$$

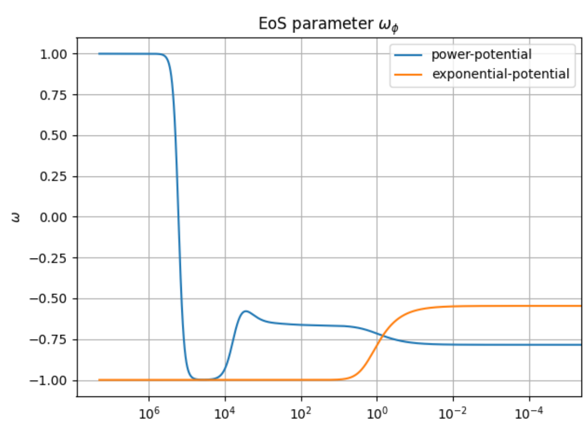

Figure 3: The EoS parameter ω_ϕ for the quintessence field as a function of the redshift z.

X. PROBLEM 10

To plot the Hubble parameter for the quintessence models we use the project's equation (7) together with the previously numerically solved equations which give us the density parameters for today. The equation has an integral expression which we can simplify by substituting z for N. We have

$$N = \ln\left(\frac{1}{1+z}\right)$$
$$\Rightarrow z = e^{-N} - 1 \tag{M}$$
$$\Rightarrow dz = -e^{-N}\ dN \tag{N}$$

Then we substitute this into the integral:

$$\begin{aligned} I &= \int_0^z dz' \frac{3[1+\omega_\phi(z')]}{1+z'} \\ &= \int_0^N (-e^{-N}\ dN) \frac{3[1+\omega_\phi(N)]}{e^{-N}} \\ &= \int_N^0 3[1+\omega_\phi(N)]\ dN \end{aligned}$$

which we can solve numerically. I use the same z-interval from problem 9 to plot H/H_0 (eq. 7) for both the inverse power-law potential, the exponential potential, and finally the spatially flat Λ-CDM model which uses the following values:

$$\begin{aligned} k &= 0 \\ \Rightarrow \Omega_{k0} &= 0 \\ \Omega_{m0} &\simeq 0.3 \\ \Omega_{r0} &= 0 \\ \Omega_{\Lambda 0} &\simeq 0.7 \end{aligned}$$

and the Hubble parameter for Λ-CDM can then be simplified to

$$H^2 = H_0^2 \left[\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0}\right] \tag{O}$$

All three models are characteristically plotted in figure 4. The edge parameters are as followed:

$$\begin{aligned} \text{Power-law}: \ \frac{H_i}{H_0} &= 1, \ \frac{H_f}{H_0} \approx 2.9292 \cdot 10^{12} \\ \text{Exponential}: \ \frac{H_i}{H_0} &= 1, \ \frac{H_f}{H_0} \approx 2.9113 \cdot 10^{12} \\ \Lambda\text{-CDM}: \ \frac{H_i}{H_0} &= 1, \ \frac{H_f}{H_0} \approx 4.8990 \cdot 10^{10} \end{aligned}$$

where $H_i = H(z=0)$ is today's value, and $H_f = H(z = 2 \cdot 10^7)$. We see that all three models correctly show $H_i/H_0 = 1$, and they are all quite similar at low redshift, but as it increases (longer back in time) they do differ. The two quintessence models are as expected quite similar the entire way, and at $z = 2 \cdot 10^7$ they both are two orders of magnitudes higher than the Λ-CDM model thanks to the extra terms in the Hubble parameter.

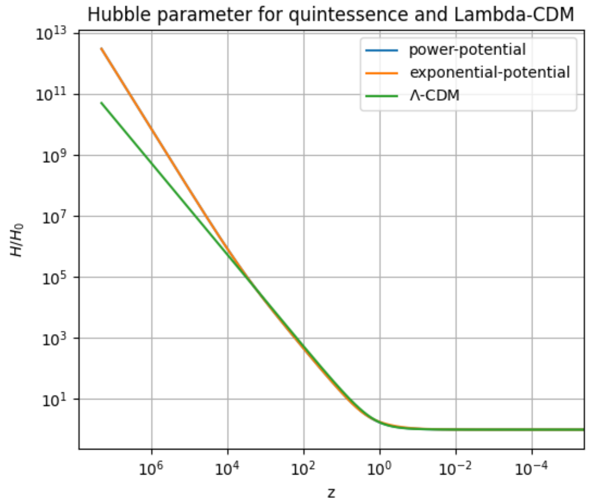

Figure 4: The characteristic Hubble parameter H/H_0 plotted as function of the redshift z for the two quintessence models together with the Λ-CDM model.

XI. PROBLEM 11

For this we will use the following expression for the dimensionless age of the universe:

$$H_0 t_0 = \int_0^{a_0} \frac{da}{ah(a)}$$

where we can substitute a for z, then for N:

$$1 + z = \frac{a_0}{a}$$
$$\Rightarrow a = \frac{a_0}{1+z} \tag{P}$$
$$\Rightarrow da = -\frac{a_0}{(1+z)^2} dz \tag{Q}$$

which turns the integral into

$$\begin{aligned} H_0 t_0 &= \int_\infty^0 \frac{-\frac{a_0}{(1+z)^2} dz}{\frac{a_0}{1+z} h(z)} \\ &= \int_0^\infty \frac{dz}{(1+z)h(z)} \end{aligned}$$

Here I will approximate this integral to the upper limit of $z = 2 \cdot 10^7$ such that

$$H_0 t_0 \approx \int_0^{2\cdot 10^7} \frac{dz}{(1+z)h(z)}$$

where we can again substitute for N using eq. M and eq. N:

$$\begin{aligned} H_0 t_0 &\approx \int_0^{\ln\left(\frac{1}{1+2\cdot 10^7}\right)} \frac{-e^{-N} dN}{e^{-N} h(N)} \\ &= \int_{\ln\left(\frac{1}{1+2\cdot 10^7}\right)}^0 \frac{dN}{h(N)} \end{aligned} \tag{R}$$

Solving this integral numerically for the two quintessence models and for the Λ-CDM model we get the following values:

$$\begin{aligned}\text{Power-law}: &\ H_0 t_0 \approx 1.4648 \cdot 10^{12} \\ \text{Exponential}: &\ H_0 t_0 \approx 1.4558 \cdot 10^{12} \\ \Lambda\text{-CDM}: &\ H_0 t_0 \approx 3.2660 \cdot 10^{10}\end{aligned}$$

Therefore, for a given H_0, the model which gives the oldest universe is the inverse power-law quintessence model. The exponential potential model is not far behind, with a relative difference of only 0.6%. We see that the Λ-CDM model is two orders of magnitudes younger than the quintessence universes.

XII. PROBLEM 12

The definition of the luminosity distance for flat space $k = 0$ is:

$$d_L = a_0(1+z)\int_{t_e}^{t_0} \frac{cdt}{a(t)}$$

where t_e is the time where the observed light departed. We do not have expressions for $a(t)$, so for this integral I will substitute t for a, then to z, and finally again to N for numerical stability when solving it numerically:

$$\begin{aligned} I &\equiv \int_{t_e}^{t_0} \frac{cdt}{a(t)} \\ \dot{a} &= \frac{da}{dt} \\ \Rightarrow dt &= \frac{da}{\dot{a}} \\ \Rightarrow I &= \int_{a_e}^{a_0} \frac{cda}{a\dot{a}} \end{aligned}$$

Here we substitute $H \equiv \frac{\dot{a}}{a} \Rightarrow \dot{a} = aH$ together with $h = H/H_0 \Rightarrow H = hH_0$

$$\begin{aligned} &= \int_{a_e}^{a_0} \frac{cda}{a^2 H(a)} \\ &= \int_{a_e}^{a_0} \frac{cda}{a^2 h(a) H_0} \\ &= \frac{c}{H_0}\int_{a_e}^{a_0} \frac{da}{a^2 h(a)} \end{aligned}$$

Then we substitute a for z using eq. P and eq. Q. For the lower limit $a = a_e$ we use the general observed redshift z, and for $a = a_0$ we have no redshift $z = 0$:

$$\begin{aligned} &= \frac{c}{H_0}\int_{z}^{0} \frac{-\frac{a_0}{(1+z')^2}dz}{\left(\frac{a_0}{1+z'}\right)^2 h(z)} \\ &= \frac{c}{H_0}\int_{0}^{z} \frac{dz}{a_0 h(z)} \\ &= \frac{c}{a_0 H_0}\int_{0}^{z} \frac{dz}{h(z)} \end{aligned}$$

Then substitute z for N using again eq. M and eq. N:

$$\begin{aligned} &= \frac{c}{a_0 H_0}\int_{0}^{ln\left(\frac{1}{1+z}\right)} \frac{-e^{-N}dN}{h(N)} \\ &= \frac{c}{a_0 H_0}\int_{\ln\left(\frac{1}{1+z}\right)}^{0} \frac{e^{-N}dN}{h(N)} \end{aligned}$$

which we can then use to find the dimensionless luminosity distance:

$$\begin{aligned} \frac{H_0}{c} d_L(N) &= \frac{H_0}{c} a_0 (1+z) I \\ &= \frac{H_0}{c} a_0 e^{-N} \frac{c}{a_0 H_0}\int_{\ln\left(\frac{1}{1+z}\right)}^{0} \frac{e^{-N}dN}{h(N)} \\ &= e^{-N}\int_{\ln\left(\frac{1}{1+z}\right)}^{0} \frac{e^{-N}dN}{h(N)} \end{aligned} \tag{S}$$

Using the interval $0 \leq z \leq 2$ we solve this numerically. I had issues getting the right results with this integral, so I opted to instead integrate over z because our integral is very limited anyways. Because of this I believe substituting to the logarithmic N-interval would not provide a big improvement in the numerical accuracy. Reverting the final substitution the dimensionless luminosity distance then has the following expression:

$$\begin{aligned} \frac{H_0}{c} d_(z) &= \frac{H_0}{c} a_0(1+z) I \\ &= \frac{H_0}{c} a_0 (1+z) \frac{c}{a_0 H_0}\int_0^z \frac{dz}{h(z)} \\ &= (1+z)\int_0^z \frac{dz}{h(z)} \end{aligned} \tag{T}$$

I solved this numerically which is plotted in figure 5 for the inverse power-law potential and for the exponential potential. They look identical such that I can not distinguish their graphs. We see that the luminosity distance correctly increases with the redshift. The edge values are as following:

$$\begin{aligned} \text{Power-law}: &\ \frac{H_0}{c} d_L(z=0) = 0, \frac{H_0}{c} d_L(z=2) \approx 2.8891 \\ \text{Exponential}: &\ \frac{H_0}{c} d_L(z=0) = 0, \frac{H_0}{c} d_L(z=2) \approx 2.8891 \end{aligned}$$

XIII. PROBLEM 13

Using $h = 0.7$ we get the Hubble constant

$$\begin{aligned} H_0 &\equiv 100h\ km\ s^{-1}\ Mpc^{-1} \\ &= 70 \cdot 10^3\ m\ s^{-1}\ Mpc^{-1} \\ &= 70 \cdot 10^3\ m\ s^{-1}\ 10^3\ Gpc^{-1} \\ &= 70 \cdot 10^6\ m/s\ Gpc^{-1} \end{aligned}$$

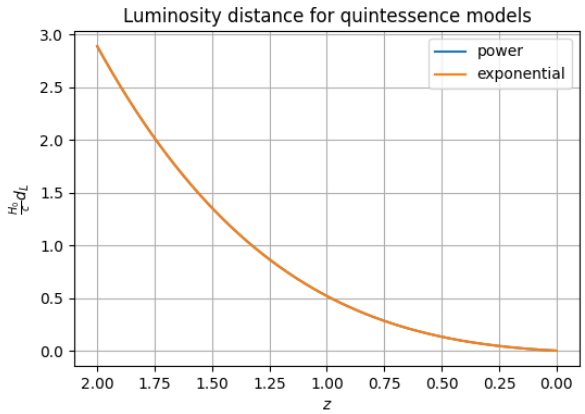

Figure 5: The dimensionless luminosity distance $\frac{H_0}{c}d_L$ plotted as function of the redshift z for the two quintessence potentials.

To convert the luminosity distances from problem 12 into the unit Gpc we just need to multiply with the constant

$$\frac{c}{H_0} = \frac{3 \cdot 10^8 \ m/s}{70 \cdot 10^6 \ m/s \ Gpc^{-1}}$$
$$\approx 4.2857 \ Gpc$$

Using the project's expression for the χ^2-value by comparing with the data values I get

$$\text{Power-law} : \chi^2 \approx 5355.21877$$
$$\text{Exponential} = \chi^2 \approx 5355.21874$$

which also is plotted in the figure 6.

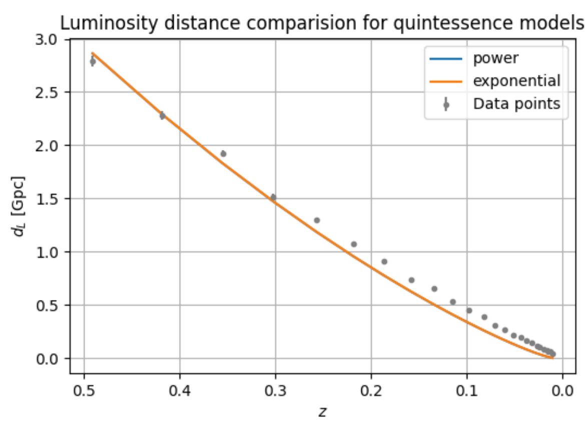

Figure 6: The luminosity distance in units of Gpc plotted as function of the redshift for the two quintessence models in addition to the project's data points.

XIV. PROBLEM 14

Using the same formula T; in figure 7 I plot the χ^2-values for different values of Ω_{m0} to find the optimal, which turns out to be

$$\Omega_{m0} = 0.0$$
$$\text{with } \chi^2 = 1539.48019$$

This is not what I expected, as we are comparing with real measurements I expected something closer to what we use in our models $\Omega_{m0} \sim 0.3$. At the very least a non-zero result, for I suspect the Λ-CDM model should not be very realistic with no matter in it. For this reason I suspect this could be wrong, however I am not sure where my potential code bug is.

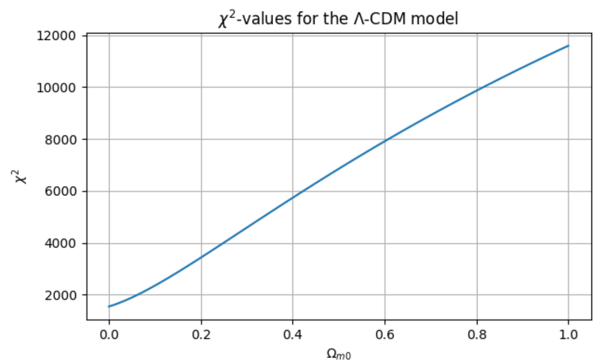

Figure 7: Plotted χ^2 as a function of the matter density parameter Ω_{m0} for the Λ-CDM model, comparing to the project's data points.