AST3220 Project 3
Part 2

Candidate 21

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I. Miscellaneous problems

Problem 1

In problem 1 we assume that the universe is described by the Einstein-de Sitter model (EdS).

a)

In EdS the scale factor a is given by (Øystein Elgarøy, 2024, eq. (3.17))

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3}$$

which we use to show the given expression for the time

$$\Rightarrow \frac{t}{t_0} = \left(\frac{a(t)}{a_0}\right)^{3/2}$$
$$t = \frac{t_0}{(1+z)^{3/2}}$$

where I used the definition of redshift $1+z = a_0/a$, and this is what we were to show.

b)

For each redshift z we can find the time coordinate of when this light was emitted by the above formula. We find

$$\begin{aligned} t_1 &= \frac{t_0}{(1+z_1)^{3/2}} \\ &= \frac{t_0}{(1+3)^{3/2}} \\ &= \underline{\frac{t_0}{8}} \end{aligned}$$

$$\begin{aligned} t_2 &= \frac{t_0}{(1+z_2)^{3/2}} \\ &= \frac{t_0}{(1+8)^{3/2}} \\ &= \underline{\frac{t_0}{27}} \end{aligned}$$

Meaning the light from the first object was emitted at one eighth the age of the universe (in the EdS-model), and from the second object at one twenty-seventh the age of the universe (EdS).

c)

The expression for the comoving radial coordinate r is given as (Øystein Elgarøy, 2024, eq. (1.22))

$$r = \mathcal{S}_k\left[\int_{t_e}^{t_0} \frac{cdt'}{a(t')}\right]$$

where

$$\mathcal{S}_k(x) = \begin{cases} \sin x, & k=1 \\ x, & k=0 \\ \arcsin x, & k=-1 \end{cases}$$

The EdS model is a flat universe, meaning $k=0$, so the expression for the comoving radial coordinate

$$\begin{aligned} r &= \int_{t_e}^{t_0} \frac{cdt}{a(t)} \\ &= \int_{t_e}^{t_0} \frac{cdt}{a_0\left(\frac{t}{t_0}\right)^{2/3}} \\ &= \frac{ct_0^{2/3}}{a_0}\int_{t_e}^{t_0} t^{-2/3}dt \\ &= \frac{ct_0^{2/3}}{a_0}\left[3t^{1/3}\right]_{t_e}^{t_0} \\ &= \frac{ct_0^{2/3}}{a_0}3\left(t_0^{1/3} - {t_e}^{1/3}\right) \end{aligned}$$

$$\begin{aligned} r &= \frac{3ct_0}{a_0}\left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right] \\ &= \frac{3ct_0}{a_0}\left(1 - \frac{1}{\sqrt{1+z}}\right) \end{aligned} \tag{1}$$

For these two objects we get

$$\begin{aligned} r_1 &= \frac{3ct_0}{a_0}[1-(1/8)^{1/3}] \\ &= \underline{\frac{3ct_0}{2a_0}} \end{aligned}$$

$$\begin{aligned} r_2 &= \frac{3ct_0}{a_0}[1-(1/27)^{1/3}] \\ &= \underline{\frac{2ct_0}{a_0}} \end{aligned}$$

d)

Now we consider that the light from an object with $z = z_2 = 8$ to be emitted at a time we call $t_e = \frac{t_0}{27}$. We want to determine the comoving coordinate r for the light heading towards us from that object at an arbitrary later time t. For this we have to replace the upper integration limit in the expression for r with an arbitrary time t:

$$r = \int_{t_e}^{t} \frac{cdt'}{a(t')}$$

this derivation was shown in the previous problem c), so I skip a bit ahead here

$$= \frac{3ct_0^{2/3}}{a_0}\left(t^{1/3} - {t_e}^{1/3}\right)$$

We put in $t_e = \frac{t_0}{27}$:

$$\begin{aligned}
&= \frac{3ct_0^{2/3}}{a_0}\left(t^{1/3} - \left(\frac{t_0}{27}\right)^{1/3}\right) \\
&= \frac{3ct_0}{a_0}\left[\left(\frac{t}{t_0}\right)^{1/3} - \frac{1}{3}\right] \\
&= \frac{3ct_0}{a_0}\left[\left(\frac{t}{t_0}\right)^{1/3} - \frac{1}{3}\right]
\end{aligned}$$

e)

We want to calculate the redshift that an observer at the object with $z = z_1 = 3$ would measure for the light from the object at $z = z_2 = 8$. For this I simply change the reference frame for the redshift by replacing a_0 with a_1:

$$\begin{aligned}
1 + z_{12} &= \frac{a(t_1)}{a(t_2)} \\
&= \frac{a_1}{a_2} \\
&= \frac{a_0}{a_2}\frac{a_1}{a_0} \\
&= \frac{1+z_2}{1+z_1} \\
\Rightarrow z_{12} &= \frac{1+z_2}{1+z_1} - 1 \\
&= \frac{1+8}{1+3} - 1 \\
&= \underline{1.25}
\end{aligned}$$

Meaning the observer would measure the redshift $z_{12} = 1.25$.

This can also be done by simply using the expression for $a(t)$, perhaps this is simpler to read:

$$\begin{aligned}
1 + z_{12} &= \frac{a(t_1)}{a(t_2)} \\
&= \frac{a_0\left(\frac{t_1}{t_0}\right)^{2/3}}{a_0\left(\frac{t_2}{t_0}\right)^{2/3}} \\
z_{12} &= \left(\frac{t_1}{t_2}\right)^{2/3} - 1 \\
&= \left(\frac{t_0/8}{t_0/27}\right)^{2/3} - 1 \\
&= 1.25
\end{aligned}$$

Problem 2

a)

We are to explain why there has to be a time in the past where the scale factor vanished $a = 0$, in the models with the following requirements:

- $\rho + \frac{3p}{c^2} > 0$
- With respect to a: ρ decreases faster than $1/a^2$ does
- $H_0 > 0$

Firstly, here it is best to begin as simple as possible. In this problem we specifically consider the past time interval where $a \geq 0$, because of $a_0 > 0$ we know for a fact that there has to be a time where $a = 0$ again to be able to reach $a_0 > 0$ if it ever were $a < 0$. So we simply consider the time interval from (potentially) the final time it crossed zero $a = 0$ up until today $t = t_0$:

$$a(t \leq t_0) \geq 0$$

The first we note is that the last requirement gives us the following:

$$H_0 = \frac{\dot{a}_0}{a_0} > 0$$

where the subscripts $_0$ always means at the time $t = t_0$, for example $\dot{a}_0 = \dot{a}(t_0)$. Since we know for a fact that $a_0 > 0$, this gives

$$\dot{a}_0 > 0$$

The first requirement helps us with Friedmann second equation (F2):

$$\begin{aligned}
\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + \frac{3p}{c^2}) \\
&> -\frac{4\pi G}{3} \\
&< 0 \\
\Rightarrow \ddot{a}(t < t_0) &< 0
\end{aligned}$$

This directly means that $\dot{a}$ is always decreasing (in our chosen time interval), but since today $\dot{a}_0 > 0$, this proves that the following has to be true for any time:

$$\dot{a}(t \geq t_0) > 0$$

Since if it weren't, then $\ddot{a}$ would have to become positive for $\dot{a}$ increase back up to positive because $\dot{a}_0 > 0$, but we already showed $\ddot{a} < 0$. Therefore, the above has to be true.

In other, simpler words; when we go backwards in time, $\dot{a}$ increases and therefore its impossible for $\dot{a} \leq 0$.

To be safe, I want to still consider the first Friedmann equation (F1), I want to be sure that $\dot{a}$ does not reach zero before a potentially reaches zero (when going backwards). F1 is:

$$\frac{\dot{a}}{a} = \pm\sqrt{\frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}}$$

The important fact we need is that today $\dot{a}_0$ is clearly real and positive, meaning today we have:

$$\frac{8\pi G}{3}\rho_0 > \frac{kc^2}{a_0^2}$$

So we choose the positive sign for the square root so that $\dot{a}_0 > 0$. For $\dot{a}$ to become zero in our chosen time interval, the first term in the square root must become equal than the second term if $k = 1$. For $k = -1, 0$ this is not an issue, so we are in the clear there.

The second requirement tells us that ρ falls faster than $1/a^2$ does, with respect to a. This means that there is a time t where $a(t)$ is sufficiently large such that the second term becomes greater than the first term. However, this is guaranteed to be in the future instead of the past, since we know that today the first term is greater! Meaning, in our chosen time interval of the past the first term is always greater than the second. So we can now continue with the proven fact $\dot{a}(t \leq t_0) > 0$.

The above fact directly means that $a(t \to 0) \to 0$. But, since $\ddot{a}(t < t_0) < 0$; the graph for $a(t)$ can not flatten out to guarantees to never reach zero! In other words, a has to increase over time, but at the same time this change decelerates over time. This is illustrated in the below sketch (figure 1), where a couple examples of graphs in red are shown to be impossible. The only possible graph of $a(t)$ is the blue one, which guarantees that $a(t < t_0)$ has to equal zero at some point in the past, which is exactly what we were to show.

b)

Today we seem to be living in an accelerating expansion

$$\dot{a}_0 > 0$$
$$\ddot{a}_0 > 0$$

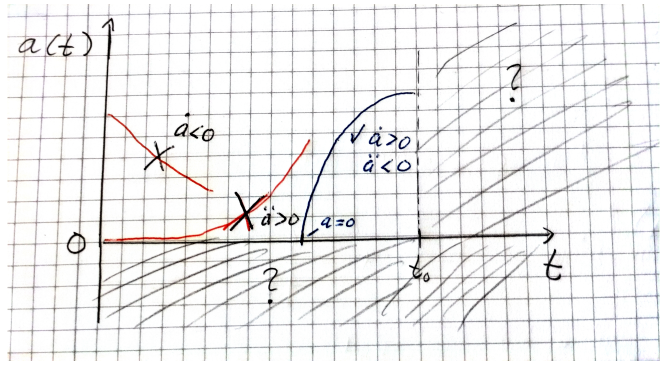

Figure 1: Problem 2 sketch of $a(t)$ showing that the only possible graph is the one that guarantees it equals zero at some time in the past (shown in blue). This is because I showed that $\dot{a}(t \leq t_0) > 0$ and $\ddot{a}(t < t_0) < 0$.

and with

$$\Omega_{m0} \simeq 0.3$$
$$\Omega_{\Lambda 0} \simeq 0.7$$

Of course, we still have $a_0 > 0, H_0 > 0$, in addition we still look at the same time interval where $a(t < t_0) \geq 0$. In the widely used ΛCDM-model F1 can be written as (Øystein Elgarøy, 2024, p. 40-42):

$$\frac{\dot{a}}{a} = \pm H_0\sqrt{\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{k0}(1+z)^2 + \Omega_{\Lambda 0}}$$

Here we also choose the positive sign for the square root, because that is required for $\dot{a}_0 > 0$. The special case of this is the simplified spatially flat ΛCDM-model, where we can neglect both the curvature term $\Omega_{k0} \simeq 0$ and the radiation term $\Omega_{r0} \simeq 0$ which simplifies F1 to

$$\begin{aligned}\frac{\dot{a}}{a} &= H_0\sqrt{\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0}} \\ &= H_0\sqrt{0.3(1+z)^3 + 0.7}\end{aligned}$$

This simplified model is always non-negative, so

$$\dot{a}(t < t_0) \geq 0$$

In this model F2 can be written as (Øystein Elgarøy, 2024, p. 44)

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{H_0}{2}\left(\Omega_{m0}(1+z)^3 - 2\Omega_{\Lambda 0}\right) \\ &= -\frac{H_0}{2}\left(0.3(1+z)^3 - 1.4\right) \\ &\begin{cases} < 0, & z \gtrsim 0.6711 \\ > 0, & z \lesssim 0.6711 \end{cases}\end{aligned}$$

We see that up to a certain point in time the expansion went from decelerating $\ddot{a} < 0$ to accelerating $\ddot{a} > 0$, before this point the cosmological constant had not yet "caught up" with the matter density.

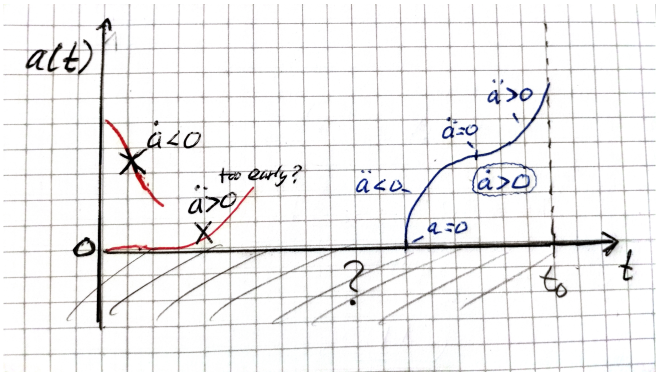

Figure 2: Problem 2 sketch of $a(t)$ for the simplified flat ΛCDM-model with $\Omega_{m0} \simeq 0.3, \Omega_{\Lambda 0} \simeq 0.7$. This shows that the only possible graph is the one that guarantees it equals zero at some time in the past (shown in blue). The expansion $\dot{a}(t)$ first decelerates ($\ddot{a} < 0$), then it starts accelerating ($\ddot{a} > 0$) at some point in time.

We can apply the same argumentation as in 2a; since the early universe's expansion must decelerate it guarantees that a must have been equal to zero at the beginning. This is again illustrated in the below sketch (figure 2), where the graph $a(t)$ must always increase, but in the early universe the change must decelerate, after a certain point in time it begins to accelerate (shown in blue). This is what we were to show.

Problem 3

a)

We want to substitute the expression for the proper distance to the particle horizon given by

$$d_{P,PH}(t) = a(t) \int_0^t \frac{cdt'}{a(t')}$$

to the redshift z. First we can substitute $a(t)$ for z by its definition

$$1 + z = \frac{a_0}{a} \Rightarrow a = \frac{a_0}{1+z}$$

Then for the integration variable we can write

$$\begin{aligned} dt &= dt\frac{dz}{dz} \\ &= dz\frac{dt}{dz} \\ &= dz\left(\frac{dz}{dt}\right)^{-1} \end{aligned}$$

For this differential we again begin with the definition

$$\begin{aligned} \frac{dz}{dt} &= \frac{d}{dt}\left(\frac{a_0}{a} - 1\right) \\ &= -\frac{a_0}{a^2}\frac{da}{dt} \\ &= -\frac{a_0}{a}\frac{\dot{a}}{a} \\ &= -\frac{a_0}{a}H \end{aligned}$$

where we used that the Hubble parameter is defined as $H = \frac{\dot{a}}{a}$. We then get

$$\begin{aligned} dt &= dz\left(\frac{dz}{dt}\right)^{-1} \\ &= -\frac{adz}{a_0 H} \end{aligned}$$

We can now do the substitution to z:

$$\begin{aligned} d_{P,PH}(t) &= a(t)\int_0^t \frac{cdt'}{a('t)} \\ d_{P,PH}(z) &= \frac{a_0}{1+z}\int_\infty^z -\frac{c\frac{a(z')dz'}{a_0 H(z')}}{a(z')} \\ &= -\frac{a_0}{1+z}c\frac{1}{a_0}\int_\infty^z \frac{a(z')dz'}{a(z')H(z')} \\ &= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H(z')} \end{aligned}$$

which is what we were to show.

b)

For a matter-dominated universe the Hubble parameter can be expressed as

$$H(z) = H_0\sqrt{\Omega_{m0}}(1+z)^{3/2}$$

We can use this to calculate the proper distance to the particle horizon:

$$\begin{aligned} d_{P,PH}(z) &= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H(z')} \\ &= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H_0\sqrt{\Omega_{m0}}(1+z')^{3/2}} \\ &= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{m0}}}\int_z^\infty \frac{dz'}{(1+z')^{3/2}} \\ &= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{m0}}}\left[-2\frac{1}{\sqrt{1+z'}}\right]_z^\infty \\ &= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{m0}}}\frac{2}{\sqrt{1+z}} \\ &= 2c\frac{1}{H_0\sqrt{\Omega_{m0}}(1+z)^{3/2}} \\ &= \frac{2c}{H_0} \\ &\sim \frac{c}{H_0} \end{aligned}$$

For a radiation-dominated universe the Hubble parameter can instead be expressed as

$$H(z) = H_0\sqrt{\Omega_{r0}}(1+z)^2$$

We do the same calculation:

$$\begin{aligned}
d_{P,PH}(z) &= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H(z')} \\
&= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{r0}}}\int_z^\infty \frac{dz'}{(1+z')^2} \\
&= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{r0}}}\left[-\frac{1}{1+z'}\right]_z^\infty \\
&= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{r0}}}\frac{1}{1+z} \\
&= c\frac{1}{H_0\sqrt{\Omega_{r0}}}\frac{1}{(1+z)^2} \\
&= \frac{c}{H_0}
\end{aligned}$$

Which is also what we were to show.

c)

We want to find the redshift that corresponds to the radius of the observable universe (proper distance to the particle horizon) to be equal that of a typical neutron star with a radius of around 10 km and a mass around 1.5 solar masses. This will be in the incredibly early universe, when the universe should have been radiation-dominated. We then can put these values into the expression for $d_{P,PH}$ which was shown above, and solve for the redshift z

$$\begin{aligned}
d_{P,PH} &= \frac{c}{H(z)} \\
&= \frac{c}{H_0\sqrt{\Omega_{r0}}(1+z)^2} \\
\Rightarrow (1+z)^2 &= \frac{c}{H_0\sqrt{\Omega_{r0}}d_{P,PH}} \\
z &= \sqrt{\frac{c}{H_0\sqrt{\Omega_{r0}}d_{P,PH}}} - 1 \\
&= \sqrt{\frac{2.998\times10^8\ m/s}{2.269\times10^{-18}\ s^{-1}\ \sqrt{10^{-4}}10^4\ m}} - 1 \\
&\simeq \underline{1.150\times10^{12}}
\end{aligned}$$

where I used the values $h = 7$ for the Hubble constant, and $\Omega_{r0} = 10^{-4}$.

If one instead calculated for a matter-dominated universe at this point one would instead get $z = 8.350\times10^{14}$, but the universe should not be matter dominated at that point in time. We will come back to this at the end of this subproblem.

The matter density at this redshift is be calculated from

$$\begin{aligned}
\rho_m &= \rho_{m0}(1+z)^3 \\
&= \rho_{c0}\Omega_{m0}(1+z)^3 \\
&= \underline{4.195\times10^9\ kg/m^3} \\
&= \underline{5.891\times10^{-9}\rho_{\text{neutron}*}}
\end{aligned}$$

Where $\rho_{\text{neutron}*} = 7.120\times10^{17}\ kg/m^3$ is the average mass density of a typical neutron star, calculated from the given values for the typical radius and typical mass. And where $\rho_{c0} = \frac{3H_0^2}{8\pi G}$ is the critical density.

The radiation density is be calculated in the same manner

$$\begin{aligned}
\rho_r &= \rho_{r0}(1+z)^4 \\
&= \rho_{c0}\Omega_{r0}(1+z)^4 \\
&= \underline{1.607\times10^{18}\ kg/m^3} \\
&= \underline{2.267\rho_{\text{neutron}*}}
\end{aligned}$$

Which is **significantly** (nine magnitudes) greater than the matter density, at this redshift/point in time.

As a final note, this conclusion also supports my previous assumption that the universe was radiation dominated at this time, because we see that the radiation density's proportionality is greater for radiation than matter (the fourth power vs. the third power). Meaning, even if I used the redshift z I calculated assuming matter domination, the ratios would be even greater because that redshift was higher!

d)

We can use the same approach as previously, the photon temperature (CMB temperature) can be calculated from

$$\begin{aligned}
T &= T_0(1+z) \\
&= 2.725\ K\ (1 + 1.150\times10^{12}) \\
&= \underline{3.133\times10^{12}\ K}
\end{aligned}$$

Meaning the CMB would have had the above temperature when the radius of the observable universe was equal to the typical neutron star radius.

e)

We can derive the expression for the age of the universe for a given redshift z by first integrating the definition of the Hubble parameter:

$$\begin{aligned}
H &\equiv \frac{1}{a}\frac{da}{dt} \\
dt &= \frac{da}{aH(a)}
\end{aligned}$$

We integrate from the beginning of the universe at $t = 0$, where we image the scale factor will somehow be equal to zero $a = 0$:

$$\int_0^t dt' = \int_0^a \frac{da'}{a'H(a')}$$
$$t = \int_0^a \frac{da'}{a'H(a')}$$

Now we substitute to the redshift z, again using that

$$a = \frac{a_0}{1+z}$$

here I choose to set $a_0 = 1$ such that

$$a = \frac{1}{1+z}$$

which gives

$$da = -\frac{dz}{(1+z)^2}$$

We now do the substitution

$$t(a) = \int_0^a \frac{da'}{a'H(a')}$$
$$t(z) = \int_\infty^z -\frac{\frac{dz'}{(1+z')^2}}{(1+z')H(z')}$$
$$= \int_z^\infty \frac{dz'}{(1+z')H(z')}$$

which is what we were to show.

For the radiation-dominated universe we can put in its expression for the Hubble parameter:

$$\begin{aligned}
t(z) &= \int_z^\infty \frac{dz'}{(1+z')H(z')} \\
&= \int_z^\infty \frac{dz'}{(1+z')H_0\sqrt{\Omega_{r0}}(1+z')^2} \\
&= \frac{1}{H_0\sqrt{\Omega_{r0}}}\int_z^\infty \frac{dz'}{(1+z')^3} \\
&= \frac{1}{H_0\sqrt{\Omega_{r0}}}\left[-\frac{1}{2}\frac{1}{(1+z')^2}\right]_z^\infty \\
&= \frac{1}{2}\frac{1}{H_0\sqrt{\Omega_{r0}}}\frac{1}{(1+z)^2} \\
&= \frac{1}{2H(z)}
\end{aligned}$$

which we were to show. Using the redshift found in c) $z = 1.150 \times 10^{12}$ we get

$$\begin{aligned}
t(z = 1.150 \times 10^{12}) &= \frac{1}{2}\frac{1}{H_0\sqrt{\Omega_{r0}}}\frac{1}{(1+z)^2} \\
&= \underline{1.668 \times 10^{-5}\ s}
\end{aligned}$$

Meaning that at the redshift from c), the age of the universe (calculated as radiation-dominated) was 0.167 μs.

II. On inflation

In this section we use $\hbar = c = 1$.

Problem 4

The horizon problem is the formulated problem to as how the entire universe looks basically homogeneous in every direction (isotropic) if the typical processes which can smooth out the temperatures after the last scattering only seem to be able to cover around 1° of angular size in the sky today. In other words, how could the temperatures look basically the same in every direction with angles up to 180° away from each other, if separated regions of space seemingly couldn't interact?

The flatness problem is refers to the problem of why the universe looks basically flat with very-close-to zero curvature, because of the fact that the curvature should increase with time. The density of matter and energy today is very close to the critical density ρ_{c0} which would imply zero curvature, and more so this means that the densities must have been even closer at the early universe. The problem asks why there seems to have been some sort of fine-tuning or even 'coincidence' for why the universe started out this way. Inflation can resolve this problem by explaining how the rapid expansion could drive the curvature parameter so close to zero, that even today it would looks almost flat.

Problem 5

We assume inflation is driven by a scalar field with the potential

$$V(\phi) = \lambda\phi^p$$

where λ is a positive constant, $p \geq 2$, and the field is only a function of time $\phi = \phi(t)$. We also assume that the slow-roll conditions are fulfilled, which are the following

$$\epsilon \equiv \frac{E_p^2}{16\pi}\left(\frac{V'}{V}\right)^2 \ll 1$$
$$|\eta| \equiv \frac{E_p^2}{8\pi}\frac{|V''|}{V} \ll 1$$

where $E_p^2 = \frac{1}{G}$ is the Planck energy (again, this is with $\hbar = c = 1$). To calculate the number of e-foldings during inflation, we calculate the number of e-foldings at the start of the inflation $N(t_i)$, because we model that $N(t)$ decreases with time until the end of inflation where $N(t_{end}) = 0$ (Øystein Elgarøy, 2024, p. 114-115). We can use the lecture notes' equation (6.10) to calculate $N(t)$ and therefore $N_{tot} = N(t_i)$:

$$N_{tot} = N(t_i) = \frac{8\pi}{E_p^2}\int_{t_{end}}^{t_i} \frac{V}{V'}\dot{\phi}dt$$

which also can be written as

$$N_{tot} = N(\phi_i) = \frac{8\pi}{E_p^2}\int_{\phi_{end}}^{\phi_i} \frac{V}{V'}d\phi$$

where $\phi_{end} = \phi(t_{end})$ which is found from the criterion $\epsilon(\phi_{end}) = 1$.

First I find the potentials derivative

$$V' = p\lambda\phi^{p-1} = pV/\phi$$

Secondly I find the ϕ_{end} parameter

$$\begin{aligned}
\epsilon &= 1 \\
\frac{E_p^2}{16\pi}\left(\frac{V'}{V}\right)^2 &= 1 \\
\frac{E_p^2}{16\pi}\left(\frac{pV/\phi_{end}}{V}\right)^2 &= 1 \\
\frac{E_p^2}{16\pi}\frac{p^2}{\phi_{end}^2} &= 1 \\
\phi_{end} &= \frac{pE_p}{4\sqrt{\pi}}
\end{aligned}$$

We can also note here that the first slow-roll condition become

$$\begin{aligned}
\epsilon = \frac{E_p^2}{16\pi}\left(\frac{V'}{V}\right)^2 &\ll 1 \\
\epsilon = \frac{E_p^2}{16\pi}\left(\frac{pV/\phi}{V}\right)^2 &\ll 1 \\
\epsilon = \frac{E_p^2}{16\pi}\frac{p^2}{\phi^2} &\ll 1 \\
\phi \gg \frac{pE_p}{4\sqrt{\pi}} &= \phi_{end}
\end{aligned}$$

Finally we can calculate the number of e-foldings

$$\begin{aligned}
N_{tot} = N(\phi_i) &= 8\pi G \int_{\phi_{end}}^{\phi_i} \frac{V}{V'} d\phi \\
&= \frac{8\pi}{E_p^2}\int_{\phi_{end}}^{\phi_i} \frac{V}{V'} d\phi \\
&= \frac{8\pi}{E_p^2}\int_{\phi_{end}}^{\phi_i} \frac{V}{pV/\phi} d\phi \\
&= \frac{8\pi}{E_p^2 p}\int_{\phi_{end}}^{\phi_i} \phi d\phi \\
&= \frac{p}{2}\frac{16\pi}{E_p^2 p^2}\left[\frac{1}{2}\phi^2\right]_{\phi_{end}}^{\phi_i} \\
&= \frac{p}{4}\left(\frac{16\pi}{E_p^2 p^2}\phi_i^2 - \frac{16\pi}{E_p^2 p^2}\phi_{end}^2\right) \\
&= \frac{p}{4}\left(\frac{1}{\epsilon} - \frac{16\pi}{E_p^2 p^2}\frac{p^2 E_p^2}{16\pi}\right) \\
N_{tot} &= \frac{p}{4}\left(\frac{1}{\epsilon} - 1\right) \gg 0
\end{aligned}$$

Since $\epsilon \ll 1 \Rightarrow 1/\epsilon - 1 \gg 0$, which is definitely true for the initial field value ϕ_i, because the slow-roll condition is valid in the beginning of inflation. Therefore, the number of e-foldings during inflation will be guaranteed to be large.

Problem 6

a)

Inflation is defined as the scale factor's rate of change accelerating

$$\ddot{a} > 0$$

We apply the second Friedmann equation (F2) to analyze this;

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3E_p^2}(\rho + 3p)$$

For scalar fields we have the following expressions for the density and pressure (project 1):

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \frac{1}{2}\dot{\phi}^2$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) = \frac{1}{2}\dot{\phi}^2 = \rho_\phi$$

when $V(\phi) = 0$. Assuming at inflation that the density and pressure is dominated by the scalar field ϕ we can write F2 as:

$$\begin{aligned}
\frac{\ddot{a}}{a} &\simeq -\frac{4\pi}{3E_p^2}(\rho_\phi + 3p_\phi) \\
&= -\frac{4\pi}{3E_p^2}4\rho_\phi \\
&= -\frac{4\pi}{3E_p^2}2\dot{\phi}^2 \\
&= -\frac{8\pi}{3E_p^2}\dot{\phi}^2 < 0
\end{aligned}$$

So, the double derivative is negative meaning inflation (which we defined as positive double derivative) is not possible: the answer is no. This is assuming that the field ϕ is not complex such that the square can become negative.

b)

With the condition $\dot{\phi}^2 = 2V(\phi)$, the expressions for the density and pressure will then be

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) = 2V(\phi) = \dot{\phi}^2$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) = 0$$

Doing the same analysis for $\ddot{a} > 0$ as in problem a), F2 will in this case look like this

$$\begin{aligned}
\frac{\ddot{a}}{a} &\approx -\frac{4\pi}{3E_p^2}(\rho_\phi + 3p_\phi) \\
&= -\frac{4\pi}{3E_p^2}\dot{\phi}^2 < 0
\end{aligned}$$

Thus, inflation is also not possible in this model, just like in problem a).

Problem 7

a)

In the slow-roll approximations we have the two following equations (6.7) and (6.8) from the lecture notes (Øystein Elgarøy, 2024):

$$H^2 \approx \frac{1}{3M_p^2} V(\phi)$$

$$3H\dot{\phi} \approx -V'(\phi)$$

With the given potential

$$V(\phi) = V_0 e^{-\lambda\phi}$$

$$\Rightarrow V'(\phi) = -\lambda V(\phi)$$

these become

$$H^2 \approx \frac{1}{3M_p^2} V_0 e^{-\lambda\phi}$$

$$3H\dot{\phi} \approx \lambda V(\phi) = \lambda V_0 e^{-\lambda\phi}$$

b)

To solve for $\phi(t)$ we take the square root of the first expression and put it into the second one:

$$3\left(\frac{1}{3M_p^2} V_0 e^{-\lambda\phi}\right)^{1/2} \dot{\phi} \approx \lambda V_0 e^{-\lambda\phi}$$
$$\dot{\phi} = \frac{d\phi}{dt} \approx \lambda M_p \sqrt{\frac{V_0}{3}} e^{-\lambda\phi/2}$$
$$e^{\lambda\phi/2} d\phi \approx \lambda M_p \sqrt{\frac{V_0}{3}} dt$$

Here I choose integrate from the start time and approximate it to be $t \approx 0$, to get rid of t_i. Because of our slow-roll approximations we know that the field should start with a big value so it "rolls" slowly, so I could approximate when $t \to 0$ then $\phi \to -\infty$, so we integrate from negative infinity:

$$\int_{-\infty}^{\phi} e^{\lambda\phi'/2} d\phi' \approx \lambda M_p \sqrt{\frac{V_0}{3}} \int_0^t dt'$$
$$\frac{2}{\lambda}\left[e^{\lambda\phi'/2}\right]_{-\infty}^{\phi} \approx \lambda M_p \sqrt{\frac{V_0}{3}} t$$
$$\frac{2}{\lambda} e^{\lambda\phi/2} \approx \lambda M_p \sqrt{\frac{V_0}{3}} t$$
$$\frac{\lambda\phi}{2} \approx \ln\left[\frac{\lambda^2 M_p}{2}\sqrt{\frac{V_0}{3}} t\right]$$
$$\phi(t) \approx \frac{2}{\lambda} \ln\left[\frac{\lambda^2 M_p}{2}\sqrt{\frac{V_0}{3}} t\right]$$

To solve for $a(t)$ we integrate the first expression:

$$\frac{1}{a}\frac{da}{dt} = H \approx \frac{1}{M_p}\sqrt{\frac{V_0}{3}} e^{-\lambda\phi/2}$$
$$\frac{da}{a} \approx \frac{1}{M_p}\sqrt{\frac{V_0}{3}} e^{-\lambda\phi(t)/2} dt$$
$$\int_{a_0}^{a} \frac{da'}{a'} \approx \frac{1}{M_p}\sqrt{\frac{V_0}{3}} \int_{t_0}^{t} e^{-\ln(\frac{\lambda^2 M_p}{2}\sqrt{\frac{V_0}{3}} t')} dt'$$
$$\int_{a_0}^{a} \frac{da'}{a'} \approx \frac{1}{M_p}\sqrt{\frac{V_0}{3}} \int_{t_0}^{t} \left(\frac{\lambda^2 M_p}{2}\sqrt{\frac{V_0}{3}} t'\right)^{-1} dt'$$
$$\int_{a_0}^{a} \frac{da'}{a'} \approx \frac{2}{\lambda^2 M_p^2} \int_{t_0}^{t} t'^{-1} dt'$$
$$\ln\frac{a(t)}{a_0} \approx \frac{2}{\lambda^2 M_p^2} \ln\frac{t}{t_0}$$
$$a(t) \approx a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{\lambda^2 M_p^2}}$$

c)

The full equations, without the slow-roll approximation, are

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$
$$H^2 = \frac{1}{3M_P^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$

The solution has the ansatz

$$a(t) = Ct^{\alpha}$$
$$\phi(t) = \frac{2}{\lambda}\ln(Bt)$$

I am to determine the constants α and B. First I calculate the expressions and derivatives I will need:

$$\dot{\phi} = \frac{2}{\lambda} t^{-1}$$
$$\ddot{\phi} = -\frac{2}{\lambda} t^{-2}$$
$$\dot{a}(t) = \alpha C t^{\alpha-1} = \frac{\alpha}{t} a(t)$$
$$V(\phi(t)) = V_0 e^{-\lambda\phi(t)}$$
$$V(t) = \frac{V_0}{B^2} t^{-2}$$
$$V'(\phi) = -\lambda V(\phi) = -\lambda V(t)$$
$$= -\frac{V_0 \lambda}{B^2} t^{-2}$$

First I consider the above expression for the Hubble parameter, and similarly as in c) I integrate to find $a(t)$:

$$\begin{aligned} H = a^{-1}\frac{da}{dt} &= \frac{1}{\sqrt{3}M_p}\sqrt{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \\ a^{-1}\frac{\alpha}{t}a &= \frac{1}{\sqrt{3}M_p}\sqrt{\frac{1}{2}\frac{4}{\lambda^2}t^{-2} + \frac{V_0}{B^2}t^{-2}} \\ \frac{\alpha}{t} &= \frac{1}{\sqrt{3}M_p}\sqrt{\frac{2}{\lambda^2} + \frac{V_0}{B^2}}t^{-1} \\ \Rightarrow \alpha &= \frac{1}{\sqrt{3}M_p}\sqrt{\frac{2}{\lambda^2} + \frac{V_0}{B^2}} \end{aligned}$$

Which is the solution for this constant (when choosing the positive square root). We can substitute out B after we've solved for it.

To solve for B we have to consider the other equation:

$$\begin{aligned} \ddot{\phi} + 3H\dot{\phi} + V'(\phi) &= 0 \\ -\frac{2}{\lambda}t^{-2} + 3\alpha t^{-1}\frac{2}{\lambda}t^{-1} - \frac{V_0\lambda}{B^2}t^{-2} &= 0 \\ t^{-2}\left(3\alpha\frac{2}{\lambda} - \frac{2}{\lambda} - \frac{V_0\lambda}{B^2}\right) &= 0 \end{aligned}$$

The first solution is $t = 0$, but this is not very interesting, so we consider the case $t \neq 0$:

$$\begin{aligned} 3\alpha\frac{2}{\lambda} - \frac{2}{\lambda} - \frac{V_0\lambda}{B^2} &= 0 \\ 3\frac{1}{\sqrt{3}M_p}\sqrt{\frac{2}{\lambda^2} + \frac{V_0}{B^2}}\frac{2}{\lambda} - \frac{2}{\lambda} - \frac{V_0\lambda}{B^2} &= 0 \\ \frac{2\sqrt{3}}{M_p\lambda}\sqrt{\frac{2}{\lambda^2} + \frac{V_0}{B^2}} - \lambda\left(\frac{2}{\lambda^2} + \frac{V_0}{B^2}\right) &= 0 \\ \frac{2\sqrt{3}}{M_p\lambda}x - \lambda x^2 &= 0 \end{aligned}$$

where I introduced the variable $x = \sqrt{\frac{2}{\lambda^2} + \frac{V_0}{B^2}}$. Let us now first solve this for x, the first solution is $x = 0$:

$$\begin{aligned} x &= 0 \\ \sqrt{\frac{2}{\lambda^2} + \frac{V_0}{B^2}} &= 0 \\ \Rightarrow B &= \pm\lambda\sqrt{-\frac{V_0}{2}} \end{aligned}$$

This is complex, which possibly is undesired. If we put this into α we see that we get

$$\alpha = 0$$

which is not a very interesting solution where $a(t)$ is constant, so I discard these solutions to B. Let us now consider the solutions with $x \neq 0$ and solve the equation we got:

$$\begin{aligned} \frac{2\sqrt{3}}{M_p\lambda}x - \lambda x^2 &= 0 \\ \Rightarrow x &= \frac{2\sqrt{3}}{\lambda^2 M_p} \\ \sqrt{\frac{2}{\lambda^2} + \frac{V_0}{B^2}} &= \frac{2\sqrt{3}}{\lambda^2 M_p} \\ \Rightarrow B^2 &= \frac{V_0\lambda^4 M_p^2}{12 - 2M_p^2} \\ &= \frac{\lambda^4 M_p^2}{4}\frac{2V_0}{6 - M_p^2\lambda^2} \\ B &= \underline{\frac{\lambda^2 M_p}{2}\sqrt{\frac{2V_0}{6 - M_p^2\lambda^2}}} \end{aligned}$$

Which is the general solution for B (if we choose the positive square root).

By putting the above solution into α we get

$$\begin{aligned} \alpha &= \frac{1}{\sqrt{3}M_p}\sqrt{\frac{2}{\lambda^2} + \frac{V_0}{B^2}} \\ &= \frac{1}{\sqrt{3}M_p}\sqrt{\frac{2}{\lambda^2} + \frac{V_0}{\frac{\lambda^4 M_p^2}{4}\frac{2V_0}{6 - M_p^2\lambda^2}}} \\ &= \frac{1}{\sqrt{3}M_p}\sqrt{\frac{2}{\lambda^2} + \frac{2}{\lambda^4 M_p^2}(6 - M_p^2\lambda^2)} \\ &= \frac{1}{\sqrt{3}M_p}\sqrt{\frac{2}{\lambda^2}\left[1 + \frac{6}{\lambda^2 M_p^2} - 1\right]} \\ &= \frac{1}{\sqrt{3}M_p}\sqrt{\frac{12}{\lambda^4 M_p^2}} \\ &= \frac{1}{\sqrt{3}M_p}\frac{2\sqrt{3}}{\lambda^2 M_p} \\ &= \underline{\frac{2}{\lambda^2 M_p^2}} \end{aligned}$$

which is the general solution to α, and also exactly the slow-roll solution I showed in problem 7b.

The slow-roll approximation had the following criteria:

$$\begin{aligned} \epsilon &\ll 1 \\ \frac{M_p^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 &\ll 1 \\ \frac{M_p^2}{2}\left(\frac{-\lambda V(\phi)}{V(\phi)}\right)^2 &\ll 1 \\ \frac{M_p^2\lambda^2}{2} &\ll 1 \\ M_p^2\lambda^2 &\ll 2 < 6 \end{aligned}$$

Applying this approximation to the general solution for

B we get

$$B = \frac{\lambda^2 M_p}{2}\sqrt{\frac{2V_0}{6 - M_p^2\lambda^2}}$$
$$\approx \frac{\lambda^2 M_p}{2}\sqrt{\frac{2V_0}{6}}$$
$$= \frac{\lambda^2 M_p}{2}\sqrt{\frac{V_0}{3}}$$

Which is exactly the slow-roll solution I showed in problem 7b. Now I have shown the general solutions, and also that we regain the slow-roll solution in the appropriate limit.

d)

Looking at the two slow-roll parameters ϵ and η, they both cancel out any field value ϕ to become the same expression:

$$\frac{M_p^2\lambda^2}{2} \ll 1$$

which is a constant, it can never change, and because of this it can never equal to one $\epsilon \neq 1$. Since we defined inflation as stopping when $\epsilon = 1$, I believe this shows the problem with this potential is that the inflation period never stops, or it never began in the first place. This shows that this potential can not model an inflation period which starts or stops, it must always be or not be.

Problem 8

In this problem we assume $\dot{\phi} > 0$ during inflation.

a)

Using the projects equations (1) and (2), I am to show that

$$\dot{\phi} = -2M_p^2 H'(\phi)$$

I begin by differentiating (2) with respect to ϕ:

$$H^2 = \frac{1}{3M_p^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$
$$2HH'(\phi) = \frac{1}{3M_p^2}\left[\dot{\phi}\frac{d\dot{\phi}}{d\phi} + V'(\phi)\right]$$
$$2HH'(\phi)3M_p^2 = \frac{d\phi}{dt}\frac{d}{d\phi}\left(\frac{d\phi}{dt}\right) + V'(\phi)$$
$$2HH'(\phi)3M_p^2 = \frac{1}{dt}\left(\frac{d\phi}{dt}\right) + V'(\phi)$$
$$2HH'(\phi)3M_p^2 = \ddot{\phi} + V'(\phi)$$

The right hand side we recognize from equation (2), such that we can substitute this using $\ddot{\phi} + V'(\phi) = -3H\dot{\phi}$:

$$2HH'(\phi)3M_p^2 = -3H\dot{\phi}$$
$$-2M_p^2 H'(\phi) = \dot{\phi}$$

which is what we were to show.

b)

Using the above relation, I am to show that the first Friedmann equation (2) can be written as

$$[H'(\phi)]^2 - \frac{3}{2M_p^2}H^2(\phi) = -\frac{1}{2M_p^4}V(\phi)$$

I begin with simply putting the above relation into the first Friedmann equation (2):

$$H^2 = \frac{1}{3M_p^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$
$$3M_p^2 H^2(\phi) = \frac{1}{2}\left[-2M_p^2 H'(\phi)\right]^2 + V(\phi)$$
$$3M_p^2 H^2(\phi) = 2M_p^4[H'(\phi)]^2 + V(\phi)$$
$$\frac{3}{2M_p^2}H^2(\phi) = [H'(\phi)]^2 + \frac{1}{2M_p^4}V(\phi)$$
$$\Rightarrow [H'(\phi)]^2 - \frac{3}{2M_p^2}H^2(\phi) = -\frac{1}{2M_p^4}V(\phi)$$

which is what we were to show.

c)

We consider a linear perturbation around a solution $H_0(t\phi)$ (not the Hubble constant) of the above equation we showed in problem b, also shown as equation (3) in the project:

$$H(\phi) = H_0(\phi) + \delta H(\phi)$$

Assuming $H = H(\phi)$ also is a solution to (3), I am to show that to the first order in the perturbation we have

$$H_0'\delta H' = \frac{3}{2M_p^2}H_0\delta H$$

where $'$ denotes derivatives with respect to ϕ.

I begin by using (3) on this perturbation H:

$$H'H_0' + \delta H'$$
$$H'^2 = (H_0' + \delta H')^2$$

For ease of writing I will simply write H'^2 instead of $[H'(\phi)]^2$ for all the expressions.

$$H'^2 = H_0'^2 + \delta H'^2 + 2H_0'\delta H'$$
$$\Rightarrow 2H_0'\delta H' = H'^2 - H_0'^2 - \delta H'^2$$

We apply (3) to the both H'^2 and $H_0'^2$:

$$= \left(\frac{3}{2M_p^2}H^2 - \frac{1}{2M_p^4}V\right) - \left(\frac{3}{2M_p^2}H_0^2 - \frac{1}{2M_p^4}V\right) - \delta H'^2$$
$$= \frac{3}{2M_p^2}(H^2 - H_0^2) - \delta H'^2$$
$$= \frac{3}{2M_p^2}(\delta H^2 + 2H_0\delta H) - \delta H'^2$$
$$2H_0'\delta H' = \frac{3}{M_p^2}H_0\delta H + \frac{3}{2M_p^2}\delta H^2 - \delta H'^2$$
$$H_0'\delta H' = \frac{3}{2M_p^2}H_0\delta H + \frac{1}{2}\left(\frac{3}{2M_p^2}\delta H^2 - \delta H'^2\right)$$

To the first order this is just

$$H_0'\delta H' = \frac{3}{2M_p^2}H_0\delta H$$

which is what we were to show.

d)

I am to show the general solution to the above equation is

$$\delta H(\phi) = \delta H(\phi_i)\exp\left[\frac{3}{2M_p^2}\int_{\phi_i}^{\phi}\frac{H_0(\varphi)}{H_0'(\varphi)}d\varphi\right]$$

where φ is a dummy integration variable. I am then to use this result to explain why the perturbation δH quickly dies out.

Firstly I begin by integrating the expression we got from problem c:

$$\begin{aligned}
H_0'(\phi)\frac{d\delta H}{d\phi} &= \frac{3}{2M_p^2}H_0(\phi)\delta H(\phi)\\
\frac{d\delta H}{\delta H} &= \frac{3}{2M_p^2}\frac{H_0(\phi)}{H_0'(\phi)}d\phi
\end{aligned}$$

I integrate from the fields value at the start of inflation:

$$\begin{aligned}
\int_{\delta H(\phi_i)}^{\delta H(\phi)}\frac{d\delta H(\phi)}{\delta H(\phi)} &= \frac{3}{2M_p^2}\int_{\phi_i}^{\phi}\frac{H_0(\phi)}{H_0'(\phi)}d\phi\\
\ln\left(\frac{\delta H(\phi)}{\delta H(\phi_i)}\right) &= \frac{3}{2M_p^2}\int_{\phi_i}^{\phi}\frac{H_0(\varphi)}{H_0'(\varphi)}d\varphi\\
\Rightarrow \delta H(\phi) &= \delta H(\phi_i)\exp\left[\frac{3}{2M_p^2}\int_{\phi_i}^{\phi}\frac{H_0(\varphi)}{H_0'(\varphi)}d\varphi\right]
\end{aligned}$$

which is what we were to show.

Secondly, I am to use this to explain why the perturbation δH quickly dies out. This will be when $\partial H \to 0$ with time, and this will be when the exponential factor goes to zero. We can analyze this exponent closer by substituting the expression from 8a):

$$\begin{aligned}
\frac{3}{2M_p^2}\int_{\phi_i}^{\phi}\frac{H_0(\varphi)}{H_0'(\varphi)}d\varphi &= \frac{3}{2M_p^2}\int_{\phi_i}^{\phi}\frac{H_0(\phi')}{-\frac{\dot{\phi}'}{2M_p^2}}d\phi'\\
&= -3\int_{\phi_i}^{\phi}\frac{H_0(\phi')}{\dot{\phi}'}d\phi'\\
&= -3\int_{\phi_i}^{\phi}\frac{H_0(\phi')}{\frac{d\phi'}{dt}}d\phi'\\
&= -3\int_{t_i}^{t}H_0(t)dt\\
&= -3\int_{t_i}^{t}\frac{1}{a_0(t)}\frac{da_0}{dt}dt\\
&= -3\int_{a_{0i}}^{a_0}\frac{da_0'}{a_0'}\\
&= -3\ln\frac{a_0(t)}{a_{0i}}\\
\Rightarrow \delta H(\phi) &= \delta H(\phi_i)\exp\left(-3\ln\frac{a_0(t)}{a_{0i}}\right)\\
&= \delta H(\phi_i)\left(\frac{a_{0i}}{a_0(t)}\right)^3
\end{aligned}$$

where $a_0(t)$ is from the solution $H_0(t)$, not today's value. This value a/a_i is the factor the the scale factor increases by at time t during/after inflation, which we already know and define as being large. We have seen that we need at least 60 total e-foldings (e^{60}) through inflation. We therefore recognize this factor blowing up very quickly during inflation, and because of the negative sign, the exponential factor in the equation for $\partial H(\phi)$ quickly approaches zero when the universe expands exponentially. Therefore, the pertubation $\partial H(\phi)$ quickly dies out through inflation!

References

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