AST3220 Project 2

Candidate 21

May 1, 2024

3 Big Bang Nucleosynthesis - Predicting the abundance of light elements

Disclaimer: My code is inspired by the skeleton code provided by Jakob Borg 2024, written on my own, and is provided in package form `bbn`. Each of the numerical problems have their own script for ease of running in the directory `problem_scripts`.

I. Problem a

We begin with $dY_i/d(\ln T)$ and convert the differential to something we can calculate:

$$\frac{dY_i}{d(\ln T)} = \frac{dY_i}{dt}\left(\frac{d(\ln T)}{dt}\right)^{-1} \tag{A}$$

It is easier to deal with each of these differentials one by one, we begin with the first one using the chain rule:

$$\begin{aligned}\frac{dY_i}{dt} &= \frac{d}{dt}\left(\frac{n_i}{n_b}\right)\\ &= \frac{n_b\frac{dn_i}{dt} - n_i\frac{dn_b}{dt}}{n_b^2}\\ &= \frac{\frac{dn_i}{dt} - Y_i\frac{dn_b}{dt}}{n_b}\end{aligned}$$

Here I calculate this second differential from above separately, then put in the value $n_b = n_{b0}a^{-3}$

$$\begin{aligned}\frac{dn_b}{dt} &= \frac{dn_b}{da}\frac{da}{dt}\\ &= \frac{d}{da}(n_{b0}a^{-3})\frac{da}{dt}\\ &= -3n_{b0}a^{-4}\frac{da}{dt}\\ &= -3n_b a^{-1}\frac{da}{dt}\end{aligned}$$

Here we recognize the Hubble parameter $H = a^{-1}\frac{da}{dt}$

$$= -3Hn_b$$

we now put this back into the previous differential:

$$\begin{aligned}\frac{dY_i}{dt} &= \frac{\frac{dn_i}{dt} - Y_i\frac{dn_b}{dt}}{n_b}\\ &= \frac{\frac{dn_i}{dt} - Y_i(-3Hn_b)}{n_b}\\ &= \frac{\frac{dn_i}{dt} + 3Hn_i}{n_b}\end{aligned}$$

where we recognize the numerator as the project's equations (8) and (9) (*AST3220, spring 2022, project 2*, 2022), so this can be rewritten as the two following expressions

$$\begin{aligned}\frac{dY_n}{dt} &= \frac{1}{n_b}\left(n_p\Gamma_{p\to n} - n_n\Gamma_{n\to p}\right)\\ &= Y_p\Gamma_{p\to n} - Y_n\Gamma_{n\to p}\end{aligned} \tag{B}$$

and

$$\begin{aligned}\frac{dY_p}{dt} &= \frac{1}{n_b}\left(n_n\Gamma_{n\to p} - n_p\Gamma_{p\to n}\right)\\ &= Y_n\Gamma_{n\to p} - Y_p\Gamma_{p\to n}\end{aligned} \tag{C}$$

Now we go back to eq. A and calculate the second differential

$$\begin{aligned}\frac{d(\ln T)}{dt} &= \frac{d(\ln T)}{da}\frac{da}{dt}\\ &= \frac{d}{da}\left(\ln\frac{T_0}{a}\right)\frac{da}{dt}\\ &= \frac{d}{da}\left(\ln T_0 - \ln a\right)\frac{da}{dt}\\ &= -a^{-1}\frac{da}{dt}\\ &= -H\end{aligned} \tag{D}$$

Now we combine all three eqs. B, C, D back into eq. A to show our two new equations

$$\begin{aligned}\frac{dY_n}{d(\ln T)} &= \frac{dY_n}{dt}\left(\frac{d(\ln T)}{dt}\right)^{-1}\\ &= \left(Y_p\Gamma_{p\to n} - Y_n\Gamma_{n\to p}\right)\left(-\frac{1}{H}\right)\\ &= -\frac{1}{H}\left(Y_p\Gamma_{p\to n} - Y_n\Gamma_{n\to p}\right)\end{aligned}$$

which is the project's eq. (10), and furthermore

$$\frac{dY_p}{d(\ln T)} = \frac{dY_p}{dt}\left(\frac{d(\ln T)}{dt}\right)^{-1}$$
$$= -\frac{1}{H}\left(Y_n\Gamma_{n\to p} - Y_p\Gamma_{p\to n}\right)$$

which is the project's eq. (11). I have now shown both expressions.

II. Bonus question

To estimate the baryon mass density at the time of BBN ($T \sim 10^9 K$), I will start with the baryon-to-proton ratio at a general time/temperature T, using that they both scale over time/redshift in identical manner:

$$\frac{n_b}{n_\gamma} = \frac{n_{b0}(1+z)^3}{n_{\gamma 0}(1+z)^3}$$
$$= \frac{n_{b0}}{n_{\gamma 0}}$$
$$\Rightarrow n_b = n_{b0}\frac{n_\gamma}{n_{\gamma 0}}$$
$$= \frac{\Omega_{b0}\rho_{c0}}{m_n}\frac{\frac{2.404}{\pi^2}\left(\frac{k_b T}{\hbar c}\right)^3}{\frac{2.404}{\pi^2}\left(\frac{k_b T_0}{\hbar c}\right)^3}$$
$$= \frac{\Omega_{b0}\rho_{c0}}{m_n}\left(\frac{T}{T_0}\right)^3$$
$$\Rightarrow \rho_b = \Omega_{b0}\rho_{c0}\left(\frac{T}{T_0}\right)^3$$

This used the expression for the proton ratio from the lecture notes, however I realized this could be done easier by simply doing this method:

$$\rho_b = \rho_{b0}(1+z)^3$$
$$= \rho_{b0}a^{-3}$$
$$= \rho_{b0}\left(\frac{T}{T_0}\right)^3$$
$$= \Omega_{b0}\rho_{c0}\left(\frac{T}{T_0}\right)^3$$

when using $a_0 = 1$.

To find the baryon mass density at the BBN we just put in $T \sim 10^9\ K$ and today's values of $\Omega_{b0} = 0.05, \rho_{c0} \approx 9.2\cdot 10^{-27}\ kg/m^3, T_0 = 2.725\ K$ to get:

$$\rho_b(T\sim 10^9\ K) \sim 0.05\cdot 9.2\cdot 10^{-27}\ kg/m^3 \cdot \left(\frac{10^9\ K}{2.725\ K}\right)^3$$
$$\sim \underline{10^{-2}\ kg/m^3}$$

Comparing this to the rough mean density of the Sun we get

$$\frac{\rho_b}{\bar{\rho}_\odot} \sim \frac{10^{-2}\ kg/m^3}{10^3\ kg/m^3}$$
$$= 10^{-5}$$

which is incredibly small compared.

To estimate the baryon and radiation energy density ratio we can simply do the same as before

$$\frac{\rho_b}{\rho_c} = \frac{\rho_{b0}a^{-3}}{\rho_{ro}a^{-4}}$$
$$= \frac{\rho_{b0}}{\rho_{r0}}a$$
$$= \frac{\Omega_{b0}\rho_{c0}}{\Omega_{r0}\rho_{c0}}\left(\frac{T}{T_0}\right)^{-1}$$
$$= \frac{\Omega_{b0}}{\Omega_{r0}}\left(\frac{T}{T_0}\right)^{-1}$$
$$\sim \frac{0.05}{10^{-4}}\left(\frac{10^9\ K}{2.725\ K}\right)^{-1}$$
$$\sim 10^{-6}$$

III. Problem b

The total entropy of the universe can be calculated by

$$S = g_{*s}(aT)^3 \tag{E}$$

where g_{*s} is the effective number of relativistic degrees of freedom related to the entropy given by

$$g_{*s} = \sum_{i=bosons} g_i\left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{i=fermions} g_i\left(\frac{T_i}{T}\right)^3 \tag{F}$$

where g_i is the number of degrees of freedom for the given relativistic boson/fermion i, and T_i is an effective temperature for that boson/fermion.

We consider the conservation of total entropy before S_0 and after electrons and positrons annihilate S_1. Applying eq. E to the conservation of entropy we can write it as

$$S_0 = S_1$$
$$S_{\text{rest},0} + S_{\nu,0} = S_{\text{rest},1} + S_{\nu,1}$$

which we split into the entropy part from the neutrinos ν, and the rest which includes the photons and before the annihilation also the electrons and positrons. Since the electrons and positrons annihilate we only need to consider them for the initial state before the event. After this event, the neutrinos will be thermally decoupled from the photons, which means that the entropy contributions from the neutrinos will be the same before and after this event:

$$S_{\nu,0} = S_{\nu,1}$$

Therefore we can drop these terms and only consider the rest:

$$S_{\text{rest},0} = S_{\text{rest},1}$$
$$\left[g_{*s}(aT)^3\right]_0 = \left[g_{*s}(aT)^3\right]_1$$

Before the electrons and positrons become non-relativistic and annihilate we will have the following relativistic particles: photons (massless) and the electrons and positrons (the temperature has not dropped below $k_b T = m_e c^2 \approx 0.511\ MeV$) [no more since we don't consider the neutrinos]. The effective degrees of freedom using $T_i = T$ will then be

$$\begin{aligned} g_{*s,0} &= g_\gamma + \frac{7}{8}(g_{e^+} + g_{e^-}) \\ &= g_\gamma + \frac{7}{8}(2g_e) \\ &= 2 + \frac{7}{8}(2 \cdot 2) \\ &= \frac{11}{2} \end{aligned}$$

After the annihilation the only relativistic particles/degrees of freedom are then the photons which gives

$$g_{*s,1} = g_\gamma = 2$$

Using these values into the conservation we get

$$\frac{11}{2}(aT)_0^3 = 2(aT)_1^3$$

Here we use the assumption that this annihilation event happens so quickly that scale factor approximately does not change $a_0 \approx a_1$, this gives

$$\begin{aligned} &\frac{11}{2}T_0^3 = 2T_1^3 \\ &\Rightarrow T_0 = \left(\frac{4}{11}\right)^{1/3} T_1 \end{aligned}$$

The next connection we have to make is that because the temperature of the neutrinos before the annihilation $T_{\nu,0}$ is still thermally coupled to the temperature of the cosmic plasma T_0, then we obviously have the following relation:

$$T_{\nu,0} = T_0$$

This shows that

$$T_0 = T_{\nu,0} = \left(\frac{4}{11}\right)^{1/3} T_1$$

Finally the last connection is that since the neutrinos now are decoupled, both T and T_ν will fall off in the same fashion because of the expansion, that is to say that

$$T, T_\nu \propto a^{-1}$$

after the decoupling. This means that this relation I have shown will always be correct after this annihilation and the decoupling, I have therefore shown the general relation (for after the decoupling):

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T$$

IV. Problem c

To show this we first recognize the density parameter

$$\Omega_{r0} = \frac{\rho_{r0}}{\rho_{c0}}$$

For the numerator we use the lecture note's result (4.21) at $T = T_0$, and for the denominator we use from the project

$$\rho_{c0} = \frac{3H_0^2}{8\pi G}$$

For the effective number of relativistic degrees, assuming that all of the radiation is made up by photons and N_{eff} number of neutrino species, we can write the degrees of freedom using $T_\nu = (4/11)^{1/3} T$ as:

$$\begin{aligned} g_* &= g_\gamma + \frac{7}{8} g_\nu \left(\frac{T_\nu}{T}\right)^4 \\ &= 2 + \frac{7}{8} N_{eff} \cdot 2 \left(\frac{4}{11}\right)^{4/3} \\ &= 2\left[1 + N_{eff}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right] \end{aligned}$$

We then can write the radiation density parameter as

$$\begin{aligned} \Omega_{r0} &= \frac{\frac{\pi^2}{30}\frac{(k_B T_0)^4}{(\hbar c)^3} g_* \ \frac{1}{c^2}}{\frac{3H_0^2}{8\pi G}} \\ &= \frac{1}{3}\frac{8\pi^3}{30}\frac{G}{H_0^2}\frac{(k_B T_0)^4}{\hbar^3 c^5} 2\left[1 + N_{eff}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right] \\ &= \frac{8\pi^3}{45}\frac{G}{H_0^2}\frac{(k_B T_0)^4}{\hbar^3 c^5}\left[1 + N_{eff}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right] \end{aligned}$$

which is what we were to show.

V. Problem d

First I integrate the Friedmann eq. (14) of the project to get $a(t)$:

$$\begin{aligned} H = \frac{1}{a}\frac{da}{dt} &= H_0\sqrt{\Omega_{r0}}a^{-2} \\ a da &= H_0\sqrt{\Omega_{r0}}dt \\ \int_0^a a' da' &= \int_0^t H_0\sqrt{\Omega_{r0}}dt' \\ \frac{1}{2}a^2 &= H_0\sqrt{\Omega_{r0}}t \end{aligned}$$

$$a(t) = \sqrt{2H_0\sqrt{\Omega_{r0}}t} \tag{G}$$

Using the temperature $T = T_0/a$ we can rewrite this to get $t(T)$

$$\begin{aligned} a = \frac{T_0}{T} &= \sqrt{2H_0\sqrt{\Omega_{r0}}t} \\ 2H_0\sqrt{\Omega_{r0}}t &= \left(\frac{T_0}{T}\right)^2 \end{aligned}$$

$$t(T) = \frac{1}{2H_0\sqrt{\Omega_{r0}}}\left(\frac{T_0}{T}\right)^2 \tag{H}$$

I now want to solve this expression for the given temperatures. First I calculate the value for $H_0\sqrt{\Omega_{r0}}$ with $N_{eff} = 3$:

$$H_0\sqrt{\Omega_{r0}} = \sqrt{\frac{8\pi^3}{45}G\frac{(k_BT_0)^4}{\hbar^3c^5}\left[1+3\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right]}$$
$$\approx 2.0889\cdot 10^{20}$$

Which we can use to calculate the different times as

$$t(10^{10}\ K) = 1.7774\ s$$
$$t(10^{9}\ K) = 1.7774\cdot 10^{2}\ s$$
$$t(10^{8}\ K) = 1.7774\cdot 10^{4}\ s$$

VI. Problem e

At the initial temperature T_i we can use equation (4) of the project, which gives us the initial equilibrium relative number density

$$\frac{n_n^{(0)}}{n_p^{(0)}} = \frac{m_p}{m_n}^{3/2}\exp\left[\frac{-(m_n-m_p)c^2}{k_BT}\right]$$

We then write

$$Y_n(T_i) = Y_n^{(0)} = \frac{n_n^{(0)}}{n_b^{(0)}}$$
$$= \frac{n_n^{(0)}}{n_n^{(0)}+n_b^{(0)}}$$
$$= \frac{1}{1+\left(\frac{n_n^{(0)}}{n_b^{(0)}}\right)^{-1}}$$
$$= \left[1+\left(\frac{m_p}{m_n}^{3/2}\exp\left[\frac{-(m_n-m_p)c^2}{k_BT_i}\right]\right)^{-1}\right]^{-1}$$

here we use $m_n \approx m_p \Rightarrow m_p/m_n \approx 1$

$$\approx \left[1+\left(\exp\left[\frac{-(m_n-m_p)c^2}{k_BT_i}\right]\right)^{-1}\right]^{-1}$$
$$Y_n(T_i) = \left[1+\exp\left[\frac{(m_n-m_p)c^2}{k_BT_i}\right]\right)^{-1}$$

which is what we were to show.

For the other relative number density we can simply infer

$$Y_p(T_i) = \frac{n_p}{n_b}$$
$$= \frac{n_b-n_n}{n_b}$$
$$= \frac{n_b}{n_b}-\frac{n_n}{n_b}$$
$$= 1-Y_n(T_i)$$

(because $1 = n_b/n_b = n_n/n_b + n_p/n_b = Y_n + Y_p$).

VII. Problem f

The script `problem_f.py` creates the plot shown in figure 1. We see that basically all neutrons decay to protons over time (temperature), which is expected because free neutrons are unstable while protons are believed to be stable (definitely stable on this timescale).

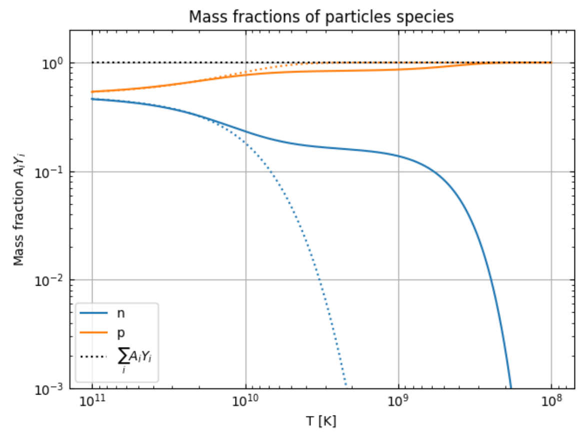

Figure 1: Problem f: recreation of the projects figure 1. The solution of the mass fractions Y_n and Y_p from eqs. (10) and (11), shown in solid lines, as well as the equilibrium values from eqs. (16) and (17), shown in dotted lines. Solved from temperature $T_i = 100\cdot 10^9\ K$ to $T_f = 0.1\cdot 10^9 K$.

VIII. Problem g

To show this (19) we do the exact same as we did for problem a. The intermediate result expression I calculated after A was:

$$\frac{dY_i}{d(\ln T)} = \frac{dY_i}{dt}\left(\frac{d(\ln T)}{dt}\right)^{-1}$$
$$= \frac{\frac{dn_i}{dt}+3Hn_i}{n_b}(-H)^{-1}$$

Using the general Boltzmann equation (18) from the project, we write this as

$$= -\frac{1}{Hn_b}\left[\sum_{j\neq i}(n_j\Gamma_{j\to i}-n_i\Gamma_{i\to j})+\sum_{jkl}(n_kn_l\gamma_{kl\to ij}-n_in_j\gamma_{ij\to kl})\right]$$
$$= -\frac{1}{H}\left[\sum_{j\neq i}(Y_j\Gamma_{j\to i}-Y_i\Gamma_{i\to j})+\sum_{jkl}(Y_kn_l\gamma_{kl\to ij}-Y_in_j\gamma_{ij\to kl})\right]$$
$$= -\frac{1}{H}\left[\sum_{j\neq i}(Y_j\Gamma_{j\to i}-Y_i\Gamma_{i\to j})+\sum_{jkl}(Y_kY_l\Gamma_{kl\to ij}-Y_iY_j\Gamma_{ij\to kl})\right]$$

which is what we were to show. For clarification; above I used

$$n_l\gamma_{kl\to ij} = \frac{n_l}{n_b}n_b\gamma_{kl\to ij} = Y_l\Gamma_{kl\to ij}$$

IX. Problem h

The script `problem_h.py` creates the plot shown in figure 2. As the temperature decreases to a certain threshold (around $T \sim 10^9$ K, which basically matches the given value $T \approx 9 \times 10^8$ K) the deuterium fraction rises massively, while the neutron fraction falls massively. At a certain point this deuterium fraction "plateaus" and remains constant while the temperature continues to decrease. The neutron fraction has at this point fallen to basically zero. At the same time the proton fraction also dropped before "plateauing" at the same time (same temperature).

This is clearly the added reaction $p + n \rightarrow D + \gamma$ (b.1 table 2 in Wagoner et al. (1967)), that begins to show when the temperature has dropped such that there are no more photons with enough energy to do this process in reverse. This continues until there are no more neutrons, at which point the ratios remain constant since the process stops.

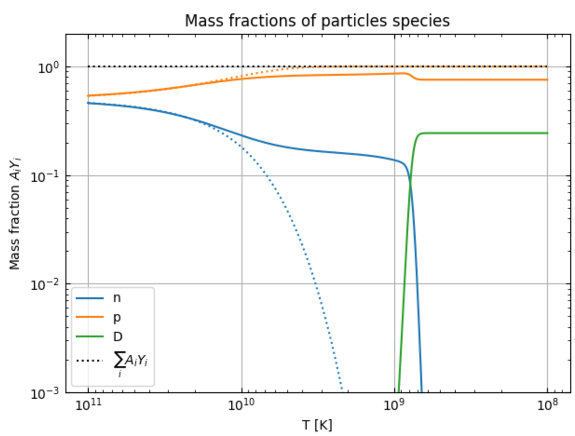

Figure 2: Problem h: recreation of the projects figure 2. The mass fractions Y_n, Y_p, and $2Y_D$ from eqs. (20), (21), and (22), shown in solid, as well as the equilibrium values for Y_n and Y_p from eqs. (16) and (17), shown in dotted. Solved from temperature $T_i = 100 \cdot 10^9$ K to $T_f = 0.1 \cdot 10^9 K$.

X. Problem i

The script `problem_i.py` creates the plot shown in figure 3, which shows all the species up to Be7, from $T_i = 100 \cdot 10^9$ K to $T_f = 0.01 \cdot 10^9$ K. We can from this see the final fractions of the different masses when the reactions, the nucleosynthesis, stop. As expected, this is mostly hydrogen-1 (proton) and Helium-4.

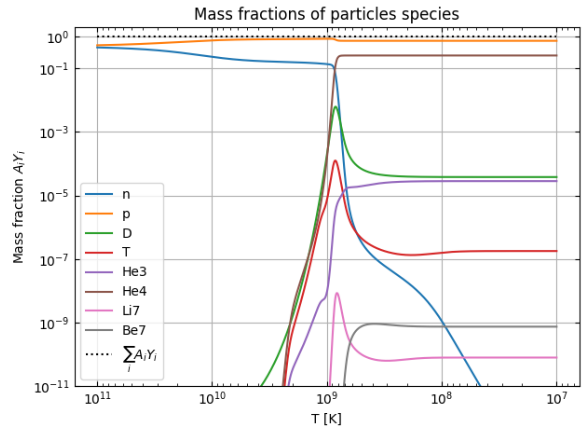

Figure 3: Problem i: recreation of the projects figure 3. The mass fractions A_iY_i are shown in solid. Solved from temperature $T_i = 100 \cdot 10^9$ K to $T_f = 0.01 \cdot 10^9 K$.

XI. Problem j

The script `problem_j.py` creates the plot shown in figure 4, which interpolates the relic abundances to find the optimal Ω_{b0} value. These calculations are again done from $T_i = 100 \cdot 10^9$ K to $T_f = 0.01 \cdot 10^9$ K.

The optimal value I calculated, and its corresponding χ^2-value, was the following:

$$\Omega_{b0} = 0.04986$$
$$\chi^2 = 0.636$$

A baryonic matter content of only $\approx 5\%$, where the total matter content is $\approx 30\%$ shows us that there is quite a lot more dark matter than regular baryonic matter. The dark matter content should then be $\approx 25\%$. This is a significant percent, which rules out certain proposals as complete contributions. Neutrinos as dark matter would only contribute a maximum content of 5%, so there has to be something else contributing.

One more likely proposal is that dark matter mostly consists of primordial black holes, which fits in the sense that they would be incredible hard to detect because of their small nature ($M \gtrsim 10^{11}$ $kg \approx 5 \cdot 10^{-20}$ $M_\odot$).

Other hypothetical proposals for dark matter would be WIMP's, Axion's, which describe undiscovered elemental particles which could interact in such ways to produce the current dark matter content we observe today. One such popular particle is the lighest supersymmetric particle.

XII. Problem k

The script `problem_k.py` creates the plot shown in figure 5, which interpolates the relic abundances to find the optimal N_{eff} value. Again we use the interval $T_i = 100 \cdot 10^9$ K to $T_f = 0.01 \cdot 10^9$ K.

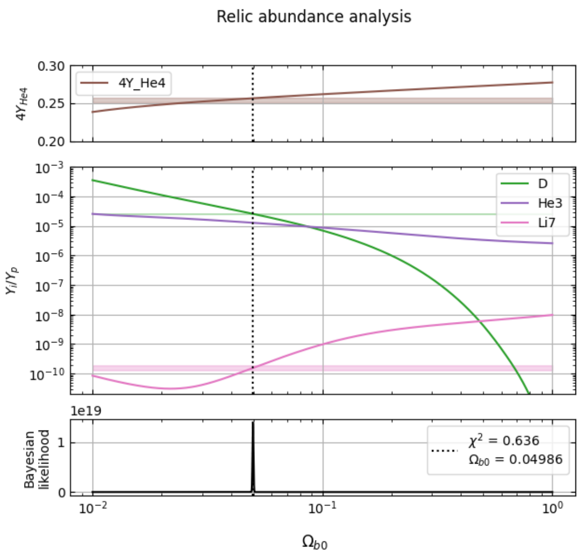

Figure 4: Problem j: recreation of the projects figure 4. The relic abundance of elements are shown as a function of the baryon density Ω_{b0}, along with measurements (23), (24), and (25) (horizontal shaded regions). In the lower plot the normalized probability eq. (28) is shown. The best-fit value of Ω_{b0} is indicated by the dotted line. Solved from temperature $T_i = 100 \cdot 10^9\ K$ to $T_f = 0.01 \cdot 10^9 K$.

My calculated optimal value, and its corresponding χ^2-value, was the following:

$$N_{eff} = 3.00760$$
$$\chi^2 = 0.768$$

This approximates to three neutrino species, which is exactly what I expected, since this is what the standard model of particle physics currently teaches us. This is what we use for most, if not all, of our calculations.

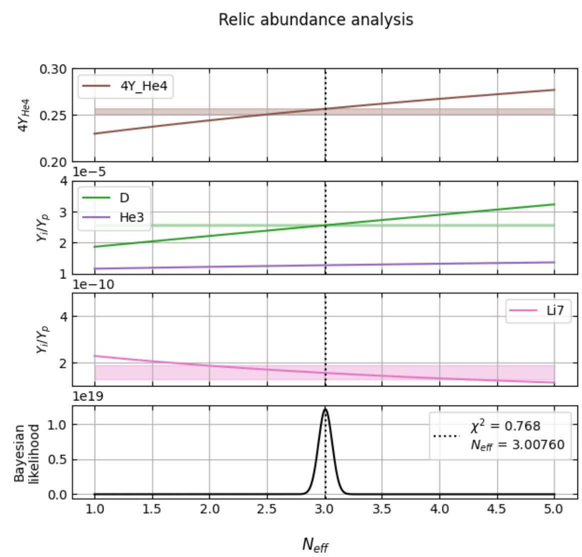

Figure 5: Problem k: recreation of the projects figure 5. The relic abundance of elements are shown as a function of the effective number of neutrinos N_{eff}, along with measurements (23), (24), and (25) (horizontal shaded regions). In the lower plot the normalized probability eq. (28) is shown. The best-fit value of N_{eff} is indicated by the dotted line. Solved from temperature $T_i = 100 \cdot 10^9\ K$ to $T_f = 0.01 \cdot 10^9 K$.

References

Ast3220, spring 2022, project 2. (2022). https://www.uio.no/studier/emner/matnat/astro/AST3220/v24/undervisningsmateriale/prosjekt-2-ast3220-2024.pdf. ([Online; accessed 25-April-2024])

Wagoner, R. V., Fowler, W. A., & Hoyle, F. (1967, April). On the Synthesis of Elements at Very High Temperatures. , *148*, 3. doi: 10.1086/149126