AST3220 Project 3

Part 2

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I. Miscellaneous problems

Problem 1

In problem 1 we assume that the universe is described by the Einstein-de Sitter model (EdS).

a)

In EdS the scale factor a is given by (Øystein Elgarøy, 2024, eq. (3.17))

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3}$$

which we use to show the given expression for the time

$$\begin{aligned} \Rightarrow \frac{t}{t_0} &= \left(\frac{a(t)}{a_0}\right)^{3/2} \\ t &= \frac{t_0}{(1+z)^{3/2}} \end{aligned}$$

where I used the definition of redshift $1 + z = a_0/a$, and this is what we were to show.

b)

For each redshift z we can find the time coordinate of when this light was emitted by the above formula. We find

$$\begin{aligned} t_1 &= \frac{t_0}{(1+z_1)^{3/2}} \\ &= \frac{t_0}{(1+3)^{3/2}} \\ &= \underline{\frac{t_0}{8}} \end{aligned}$$

$$\begin{aligned} t_2 &= \frac{t_0}{(1+z_2)^{3/2}} \\ &= \frac{t_0}{(1+8)^{3/2}} \\ &= \underline{\frac{t_0}{27}} \end{aligned}$$

Meaning the light from the first object was emitted at one eighth the age of the universe (in the EdS-model), and from the second object at one twenty-seventh the age of the universe (EdS).

c)

The expression for the comoving radial coordinate r is given as (Øystein Elgarøy, 2024, eq. (1.22))

$$r = \mathcal{S}_k \left[\int_t^{t_0} \frac{c dt'}{a(t')}\right]$$

where

$$\mathcal{S}_k(x) = \begin{cases} \sin x, & k = 1 \\ x, & k = 0 \\ \arcsin x, & k = -1 \end{cases}$$

The EdS model is a flat universe, meaning $k = 0$, so the expression we use is

$$\begin{aligned} r &= \int_t^{t_0} \frac{c dt'}{a(t')} \\ &= \int_t^{t_0} \frac{c dt'}{a_0 \left(\frac{t'}{t_0}\right)^{2/3}} \\ &= \frac{c t_0^{2/3}}{a_0} \int_t^{t_0} t'^{-2/3} dt' \\ &= \frac{c t_0^{2/3}}{a_0} \left[3 t'^{1/3}\right]_t^{t_0} \\ &= \frac{c t_0^{2/3}}{a_0} 3 \left(t_0^{1/3} - t^{1/3}\right) \end{aligned}$$

$$\begin{aligned} r &= \frac{3ct_0}{a_0}\left[1 - \left(\frac{t}{t_0}\right)^{1/3}\right] \\ &= \frac{3ct_0}{a_0}\left(1 - \frac{1}{\sqrt{1+z}}\right) \end{aligned} \tag{1}$$

For these two objects we get

$$\begin{aligned} r_1 &= \frac{3ct_0}{a_0}[1 - (1/8)^{1/3}] \\ &= \underline{\frac{3ct_0}{2a_0}} \end{aligned}$$

$$\begin{aligned} r_2 &= \frac{3ct_0}{a_0}[1 - (1/27)^{1/3}] \\ &= \underline{\frac{2ct_0}{a_0}} \end{aligned}$$

d)

Now we consider that the light from an object with $z = z_2 = 8$ to be emitted at a time we call $t_e = \frac{t_0}{27}$. We want to determine the comoving coordinate r for the light heading towards us from that object at an arbitrary later time t. This is determined by the previous equation (1).

e)

We want to calculate the redshift that an observer at the object with $z = z_1 = 3$ would measure for the light from the object at $z = z_2 = 8$. For this I simply change the reference frame for the redshift by replacing a_0 with a_1:

$$\begin{aligned}
1 + z &= \frac{a_1}{a_2} \\
&= \frac{a_0}{a_2}\frac{a_1}{a_0} \\
&= \frac{1+z_2}{1+z_1} \\
\Rightarrow z &= \frac{1+z_2}{1+z_1} - 1 \\
&= \frac{1+8}{1+3} - 1 \\
&= \underline{1.25}
\end{aligned}$$

Meaning the observer would measure the redshift $z = 1.25$.

Problem 2

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Problem 3

a)

We want to substitute the expression for the proper distance to the particle horizon given by

$$d_{P,PH}(t) = a(t)\int_0^t \frac{cdt'}{a(t')}$$

to the redshift z. First we can substitute $a(t)$ for z by its definition

$$1 + z = \frac{a_0}{a} \Rightarrow a = \frac{a_0}{1+z}$$

Then for the integral we can write

$$\begin{aligned}
dt &= dt\frac{dz}{dz} \\
&= dz\frac{dt}{dz} \\
&= dz\left(\frac{dz}{dt}\right)^{-1}
\end{aligned}$$

For this differential we again begin with the definition

$$\begin{aligned}
\frac{dz}{dt} &= \frac{d}{dt}\left(\frac{a_0}{a} - 1\right) \\
&= -\frac{a_0}{a^2}\frac{da}{dt} \\
&= -\frac{a_0}{a}\frac{\dot{a}}{a} \\
&= -\frac{a_0}{a}H
\end{aligned}$$

where we used that the Hubble parameter is defined as $H = \frac{\dot{a}}{a}$. We then get

$$\begin{aligned}
dt &= dz\left(\frac{dz}{dt}\right)^{-1} \\
&= -\frac{adz}{a_0 H}
\end{aligned}$$

We can now do the substitution to z:

$$\begin{aligned}
d_{P,PH}(t) &= a(t)\int_0^t \frac{cdt'}{a('t)} \\
d_{P,PH}(z) &= \frac{a_0}{1+z}\int_\infty^z -\frac{c\frac{a(z')dz'}{a_0 H(z')}}{a(z')} \\
&= -\frac{a_0}{1+z}c\frac{1}{a_0}\int_\infty^z \frac{a(z')dz'}{a(z')H(z')} \\
&= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H(z')}
\end{aligned}$$

which is what we were to show.

b)

For a matter-dominated universe the Hubble parameter can be expressed as

$$H(z) = H_0\sqrt{\Omega_{m0}}(1+z)^{3/2}$$

We can use this to calculate the proper distance to the particle horizon:

$$\begin{aligned}
d_{P,PH}(z) &= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H(z')} \\
&= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H_0\sqrt{\Omega_{m0}}(1+z')^{3/2}} \\
&= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{m0}}}\int_z^\infty \frac{dz'}{(1+z')^{3/2}} \\
&= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{m0}}}\left[-2\frac{1}{\sqrt{1+z'}}\right]_z^\infty \\
&= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{m0}}}\frac{2}{\sqrt{1+z}} \\
&= 2c\frac{1}{H_0\sqrt{\Omega_{m0}}(1+z)^{3/2}} \\
&= \frac{2c}{H_0} \\
&\sim \frac{c}{H_0}
\end{aligned}$$

For a radiation-dominated universe the Hubble parameter can instead be expressed as

$$H(z) = H_0\sqrt{\Omega_{r0}}(1+z)^2$$

We do the same calculation:

$$\begin{aligned}
d_{P,PH}(z) &= \frac{c}{1+z}\int_z^\infty \frac{dz'}{H(z')} \\
&= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{r0}}}\int_z^\infty \frac{dz'}{(1+z')^2} \\
&= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{r0}}}\left[-\frac{1}{1+z'}\right]_z^\infty \\
&= \frac{c}{1+z}\frac{1}{H_0\sqrt{\Omega_{r0}}}\frac{1}{1+z} \\
&= c\frac{1}{H_0\sqrt{\Omega_{r0}}}\frac{1}{(1+z)^2} \\
&= \frac{c}{H_0}
\end{aligned}$$

Which is also what we were to show.

c)

We want to find the redshift that corresponds to the radius of the observable universe (proper distance to the particle horizon) to be equal that of a typical neutron star with a radius of around 10 km and a mass around 1.5 solar masses. This will be in the incredibly early universe, when the universe should have been radiation-dominated. We then can put these values into the expression for $d_{P,PH}$ which was shown above, and solve for the redshift z

$$\begin{aligned}
d_{P,PH} &= \frac{c}{H(z)} \\
&= \frac{c}{H_0\sqrt{\Omega_{r0}}(1+z)^2} \\
\Rightarrow (1+z)^2 &= \frac{c}{H_0\sqrt{\Omega_{r0}}d_{P,PH}} \\
z &= \sqrt{\frac{c}{H_0\sqrt{\Omega_{r0}}d_{P,PH}}} - 1 \\
&= \sqrt{\frac{2.998\times10^8\ m/s}{2.269\times10^{-18}\ s^{-1}\ \sqrt{10^{-4}}10\times10^3\ m}} - 1 \\
&= \underline{1.150\times10^{12}}
\end{aligned}$$

where I used the values $h = 7$ for the Hubble constant, and $\Omega_{r0} = 10^{-4}$.

If one instead calculated for a matter-dominated universe at this point one would instead get $z = 8.3498\times10^{14}$, but the universe should not be matter dominated at that point in time.

The mass density at this redshift can be calculated from

$$\begin{aligned}
\rho_m &= \rho_{m0}(1+z)^3 \\
&= \rho_{c0}\Omega_{m0}(1+z)^3 \\
&= \underline{4.1948\times10^9\ kg/m^3} \\
&= \underline{5.891\times10^{-9}\rho_{\text{neutron}*}}
\end{aligned}$$

Where $\rho_{\text{neutron}*} = 7.120\times10^{17}\ kg/m^3$ is the average mass density of a typical neutron star, calculated from the typical radius and typical mass. And where $\rho_{c0} = \frac{3H_0^2}{8\pi G}$ is the critical density.

The radiation density can be calculated in the same manner

$$\begin{aligned}
\rho_r &= \rho_{r0}(1+z)^4 \\
&= \rho_{c0}\Omega_{r0}(1+z)^4 \\
&= \underline{1.607\times10^{18}\ kg/m^3} \\
&= \underline{2.267\rho_{\text{neutron}*}}
\end{aligned}$$

Which is **significantly** greater than the mass density, at this redshift/point in time.

d)

We can use the same approach as previously, the photon temperature (CMB temperature) can be calculated from

$$\begin{aligned}
T &= T_0(1+z) \\
&= 2.725\ K\ (1+1.150\times10^{12}) \\
&= \underline{3.133\times10^{12}\ K}
\end{aligned}$$

Meaning the CMB would have had the above temperature when the radius of the observable universe was equal to the typical neutron star radius.

e)

We can derive the expression for the age of the universe for a given redshift z by first integrating the definition of the Hubble parameter:

$$\begin{aligned}
H &\equiv \frac{1}{a}\frac{da}{dt} \\
dt &= \frac{da}{aH(a)}
\end{aligned}$$

We integrate from the beginning of the universe at $t = 0$, where we image the scale factor will somehow be equal to zero $a = 0$:

$$\begin{aligned}
\int_0^t dt' &= \int_0^a \frac{da'}{a'H(a')} \\
t &= \int_0^a \frac{da'}{a'H(a')}
\end{aligned}$$

Now we substitute to the redshift z, again using that

$$a = \frac{a_0}{1+z}$$

here I choose to set $a_0 = 1$ such that

$$a = \frac{1}{1+z}$$

which gives

$$da = -\frac{dz}{(1+z)^2}$$

We now do the substitution

$$\begin{aligned} t(a) &= \int_0^a \frac{da'}{a'H(a')} \\ t(z) &= \int_\infty^z -\frac{\frac{dz'}{(1+z')^2}}{(1+z')H(z')} \\ &= \int_z^\infty \frac{dz'}{(1+z')H(z')} \end{aligned}$$

which is what we were to show.

For the radiation-dominated universe we can put in its expression for the Hubble parameter:

$$\begin{aligned} t(z) &= \int_z^\infty \frac{dz'}{(1+z')H(z')} \\ &= \int_z^\infty \frac{dz'}{(1+z')H_0\sqrt{\Omega_{r0}}(1+z')^2} \\ &= \frac{1}{H_0\sqrt{\Omega_{r0}}} \int_z^\infty \frac{dz'}{(1+z')^3} \\ &= \frac{1}{H_0\sqrt{\Omega_{r0}}} \left[-\frac{1}{2} \frac{1}{(1+z')^2} \right]_z^\infty \\ &= \frac{1}{2} \frac{1}{H_0\sqrt{\Omega_{r0}}} \frac{1}{(1+z)^2} \\ &= \frac{1}{2H(z)} \end{aligned}$$

which we were to show. Using the redshift found in c) $z = 1.150 \times 10^{12}$ we get

$$\begin{aligned} t(z = 1.150 \times 10^{12}) &= \frac{1}{2} \frac{1}{H_0\sqrt{\Omega_{r0}}} \frac{1}{(1+z)^2} \\ &= \underline{1.668 \times 10^{-5}\ s} \end{aligned}$$

Meaning that at the redshift from c), the age of the universe (calculated as radiation-dominated) was $0.167\ \mu s$.

II. On inflation

In this section we use $\hbar = c = 1$.

Problem 4

The horizon problem is the formulated problem to as how the entire universe looks basically homogeneous in every direction (isotropic) if the typical processes which can smooth out the temperatures after the last scattering only seem to be able to cover around 1° of angular size in the sky today. In other words, how could the temperatures look basically the same in every direction with angles up to 180° away from each other, if separated regions of space seemingly couldn't interact?

The flatness problem is refers to the problem of why the universe looks basically flat with very-close-to zero curvature, because of the fact that the curvature should increase with time. The density of matter and energy today is very close to the critical density ρ_{c0} which would imply zero curvature, and more so this means that the densities must have been even closer at the early universe. The problem asks why there seems to have been some sort of fine-tuning or even 'coincidence' for why the universe started out this way. Inflation can resolve this problem by explaining how the rapid expansion could drive the curvature parameter so close to zero, that even today it would looks almost flat.

Problem 5

We assume inflation is driven by a scalar field with the potential

$$V(\phi) = \lambda \phi^p$$

where λ is a positive constant, $p \geq 2$, and the field is only a function of time $\phi = \phi(t)$. We also assume that the slow-roll conditions are fulfilled, which are the following

$$\begin{aligned} \epsilon &\equiv \frac{E_p^2}{16\pi} \left(\frac{V'}{V} \right)^2 \ll 1 \\ |\eta| &\equiv \frac{E_p^2}{8\pi} \frac{|V''|}{V} \ll 1 \end{aligned}$$

where $E_p^2 = \frac{1}{G}$ is the Planck energy (again, this is with $\hbar = c = 1$). To calculate the number of e-foldings during inflation, we calculate the number of e-foldings at the start of the inflation $N(t_i)$, because we model that $N(t)$ decreases with time until the end of inflation where $N(t_{end}) = 0$ (Øystein Elgarøy, 2024, p. 114-115). We can use the lecture notes' equation (6.10) to calculate $N(t)$ and therefore $N_{tot} = N(t_i)$:

$$N_{tot} = N(t_i) = \frac{8\pi}{E_p^2} \int_{t_{end}}^{t_i} \frac{V}{V'} \dot{\phi} dt$$

which also can be written as

$$N_{tot} = N(\phi_i) = \frac{8\pi}{E_p^2} \int_{\phi_{end}}^{\phi_i} \frac{V}{V'} d\phi$$

where $\phi_{end} = \phi(t_{end})$ which is found from the criterion $\epsilon(\phi_{end}) = 1$.

First I find the potential differential

$$V' = p\lambda\phi^{p-1} = pV/\phi$$

Secondly I find the ϕ_{end} parameter

$$\epsilon = 1$$
$$\frac{E_p^2}{16\pi}\left(\frac{V'}{V}\right)^2 = 1$$
$$\frac{E_p^2}{16\pi}\left(\frac{pV/\phi_{end}}{V}\right)^2 = 1$$
$$\frac{E_p^2}{16\pi}\frac{p^2}{\phi_{end}^2} = 1$$
$$\phi_{end} = \frac{pE_p}{4\sqrt{\pi}}$$

We can also note here that the first slow-roll condition become

$$\frac{E_p^2}{16\pi}\left(\frac{V'}{V}\right)^2 \ll 1$$
$$\frac{E_p^2}{16\pi}\left(\frac{pV/\phi}{V}\right)^2 \ll 1$$
$$\frac{E_p^2}{16\pi}\frac{p^2}{\phi^2} \ll 1$$
$$\phi \gg \frac{p}{4\sqrt{\pi G}} = \phi_{end}$$

We can assume that at the start of inflation this condition will apply, when we approximate using slow-roll. This means that we can assume that the start of inflation ϕ_i, the following will be true:

$$\phi_i \gg \phi_{end}$$

Finally we can calculate the number of e-foldings

$$N_{tot} = N(\phi_i) = 8\pi G\int_{\phi_{end}}^{\phi_i}\frac{V}{V'}d\phi$$
$$= 8\pi G\int_{\phi_{end}}^{\phi_i}\frac{V}{pV/\phi}d\phi$$
$$= \frac{8\pi G}{p}\int_{\phi_{end}}^{\phi_i}\phi d\phi$$
$$= \frac{8\pi G}{p}\left(\phi_i^2 - \phi_{end}^2\right) \gg 0$$

Therefore, the number of e-foldings during inflation will be guaranteed to be large, since we have $\phi_i \gg \phi_{end}$ which naturally gives $\phi_i^2 - \phi_{end}^2 \gg 0$.

Problem 6

a)

Inflation is defined as the scale factor's rate of change accelerating

$$\ddot{a} > 0$$

We apply the second Friedmann equation (F2) to analyze this;

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3E_p^2}(\rho + 3p)$$

For scalar fields we have the following expressions for the density and pressure (project 1):

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \frac{1}{2}\dot{\phi}^2$$
$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) = \rho_\phi$$

when $V(\phi) = 0$. Assuming at inflation that the density and pressure is dominated by the scalar field ϕ we can write F2 as:

$$\frac{\ddot{a}}{a} \simeq -\frac{4\pi}{3E_p^2}(\rho_\phi + 3p_\phi)$$
$$= -\frac{4\pi}{3E_p^2}4\rho_\phi$$
$$= -\frac{4\pi}{3E_p^2}2\dot{\phi}^2$$
$$= -\frac{8\pi}{3E_p^2}\dot{\phi}^2$$

Since $a > 0$, thus $\ddot{a} > 0$ and we will have inflation when the right hand side is greater than zero. However, this is impossible since any value squared is positive, and we have negative constants in front. Meaning that the answer is no; inflation is *not* possible with the potential $V(\phi) = 0$.

b)

With the condition $\dot{\phi}^2 = 2V(\phi)$, the expressions for the density and pressure will then be

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) = 2V(\phi)$$
$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) = 0$$

Doing the same analysis for $\ddot{a} > 0$ as in problem a), F2 will in this case look like this

$$\frac{\ddot{a}}{a} \approx -\frac{4\pi}{3E_p^2}(\rho_\phi + 3p_\phi)$$
$$= -\frac{4\pi}{3E_p^2}2V(\phi)$$
$$= -\frac{8\pi}{3E_p^2}V(\phi)$$

Thus, we will have inflation ($\ddot{a} > 0$) if $V(\phi) > 0$, meaning the answer is yes: inflation is possible as long as the above condition is fulfilled.

Problem 7

a)

In the slow-roll approximations we have the two following equations (6.7) and (6.8) from the lecture notes (Øystein Elgarøy, 2024):

$$H^2 \approx \frac{1}{3M_p^2}V(\phi)$$

$$3H\dot{\phi} \approx -V'(\phi)$$

With the given potential

$$V(\phi) = V_0 e^{-\lambda\phi}$$

these become

$$H^2 \approx \frac{1}{3M_p^2} V_0 e^{-\lambda\phi}$$

$$3H\dot{\phi} \approx \lambda V(\phi) = \lambda V_0 e^{-\lambda\phi}$$

To solve for $\phi(t)$ we take the square root of the first expression and put it into the second one:

$$3\left(\frac{1}{3M_p^2} V_0 e^{-\lambda\phi}\right)^{1/2} \dot{\phi} \approx \lambda V_0 e^{-\lambda\phi}$$

$$\dot{\phi} = \frac{d\phi}{dt} \approx \lambda M_p \sqrt{\frac{V_0}{3}} e^{-\lambda\phi/2}$$

$$e^{\lambda\phi/2} d\phi \approx \lambda M_p \sqrt{\frac{V_0}{3}} dt$$

$$\int_{\phi}^{\phi_{end}} e^{\lambda\phi'/2} d\phi' \approx \lambda M_p \sqrt{\frac{V_0}{3}} \int_t^{t_{end}} dt'$$

$$\frac{2}{\lambda}\left(e^{\lambda\phi_{end}/2} - e^{\lambda\phi/2}\right) \approx \lambda M_p \sqrt{\frac{V_0}{3}} (t_{end} - t)$$

$$e^{\lambda\phi/2} \approx e^{\lambda\phi_{end}/2} - \frac{\lambda^2 M_p}{2} \sqrt{\frac{V_0}{3}} (t_{end} - t)$$

$$\frac{\lambda\phi}{2} \approx \ln\left[e^{\lambda\phi_{end}/2} - \frac{\lambda^2 M_p}{2} \sqrt{\frac{V_0}{3}} (t_{end} - t)\right]$$

$$\phi(t) \approx \frac{2}{\lambda} \ln\left[e^{\lambda\phi_{end}/2} - \frac{\lambda^2 M_p}{2} \sqrt{\frac{V_0}{3}} (t_{end} - t)\right]$$

where ϕ_i and t_i refers to the beginning of inflation.

To solve for $a(t)$ we integrate the first expression: [xxxxxxxxxxxxxxxxxxxxx]

$$\frac{1}{a}\frac{da}{dt} = H \approx \frac{1}{M_p} \sqrt{\frac{V_0}{3}} e^{-\lambda\phi/2}$$

$$\frac{da}{a} \approx \frac{1}{M_p} \sqrt{\frac{V_0}{3}} e^{-\lambda\phi/2} dt$$

$$\int_a^{a_{end}} \frac{da'}{a'} \approx \frac{1}{M_p} \sqrt{\frac{V_0}{3}} \dot{\phi}^{-1} \int_{\phi}^{\phi_{end}} e^{-\lambda\phi'/2} d\phi'$$

$$\ln\frac{a_{end}}{a} \approx \frac{1}{M_p} \sqrt{\frac{V_0}{3}} \frac{-2}{\lambda} \left(e^{-\lambda\phi_{end}/2} - e^{-\lambda\phi/2}\right)$$

$$\ldots$$

[xxxxxxxxxxxxxx, riktig fremgangsmåte?]

References

Øystein Elgarøy. (2024). *Ast3220 - cosmology i (lecture notes)*. https://www.uio.no/studier/emner/matnat/astro/AST3220/v24/undervisningsmateriale/lectures_ast3220.pdf.