On the analytical and numerical solutions of Laplace's equation for different electrostatic configurations

S. Brown, F. Hayes, L. Heikkilä, W. Liu, D. Richardson
School of Physics and Astronomy,
University of Glasgow,
Glasgow, United Kingdom

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Abstract

The behaviour of the electric field under the presence and influence of solid conducting objects is modelled and studied. Finite difference methods, namely the relaxation method, are employed to solve the Laplace equation numerically, to find the approximate form of the electrostatic potential, and hence the electric field, present in various electrostatic systems. We explicitly focus on two such systems: the case of a long perfectly conducting cylinder centred between two infinite planes at potential $+V$ and $-V$; and the case of a silicon detector—a system composed of two silicon wafers with segmented doped implants at ground potential on one side and a uniform doped implant, referred to as the backplane, on the other side, held at a potential $+V$. We compare the analytical solution of the first system to its approximate numerical solution, and study the convergence of the numerical method.

1 Introduction

1.1 Derivation of Laplace's Equation

Gauss' Law may used to express the relation between the *divergence* of the electric field, represented by $\mathbf{E}$, and the *charge density* ρ, where ϵ_0 is the *permittivity of free space*:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

Alternatively (c.f. the Divergence Theorem), it can be expressed in its integral form, as:

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q}{\epsilon_0} \tag{2}$$

where the integral is taken over a closed surface, S, and Q is the enclosed charge.

In the absence of charge, as is the case outside a conductor, this reduces to the statement the electric field is *solenoidal.*

One can also relate the *curl* of the electric field to the time derivative of the magnetic field, denoted **B**:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{3}$$

If the magentic field is constant, in other words unchanging with time, this equation reduces to the statement that the electric field is *irrotational.* A consequence of irrotationality is that the electric field may then be written as negative the gradient of some scalar potential, ϕ say. Mathematically:

$$\boldsymbol{E} = -\nabla\phi \tag{4}$$

This scalar potential is called the *electrostatic potential.* One then has *Laplace's equation*:

$$\nabla^2\phi = 0 \tag{5}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the *Laplacian operator*[i].

This is a second-order partial differential equation that can be solved, given some well-posed boundary conditions, to find the electrostatic potential and electric field for some physical system.

1.2 Finite Difference Schemes

It is often the case that differential equations, such as the Laplace equation, cannot be solved analytically, and their solution must be numerically approximated.

To numerically approximate the solution to a differential equation it is necessary to approximate the derivative of a function. The usual method employed to do this is finite differencing. This approximates the derivative by using explicit differencing to step the function from one value to the next in small increments of some variable. The smaller the increment used, the more accurate the approximation, but the longer and more computationally intensive the process.

Suppose one wishes to approximate the first derivative of a function $f(t)$, say, with respect to the variable t. The simplest way to do this is to discretise the derivative and write

$$\frac{df}{dt} \approx \frac{f(t+\Delta t) - f(t)}{\Delta t} \tag{6}$$

where $f(t+\Delta t)$ and $f(t)$ are two values of f, evaluated at two points a distance Δt apart.

To approximate the second derivative of a function, one writes

$$\frac{d^2 f}{dt^2} \approx \frac{f'(t) - f'(t-h)}{\Delta t} = \frac{\frac{f(t+\Delta t)-f(t)}{\Delta t} - \frac{f(t)-f(t-\Delta t)}{\Delta t}}{\Delta t} = \frac{f(t+\Delta t) - 2f(t) + f(t-\Delta t)}{\Delta t^2} \tag{7}$$

1.2.1 Derivation of the Relaxation Method

For our purposes, we wish to discretise the following equation:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{8}$$

[i]Specifically in Cartesian co-ordinates

One can approximate this partial differential equation as:

$$\frac{\phi_{j+1,k} - 2\phi_{j,k} + \phi_{j-1,k}}{\Delta x^2} = -\frac{\phi_{j,k+1} - 2\phi_{j,k} + \phi_{j,k-1}}{\Delta y^2} \tag{9}$$

where j and k index the x and y directions respectively.

If we use the same step-size $\Delta = \Delta x = \Delta y$ in both x and y directions this reduces to:

$$\phi_{j,k} = \frac{1}{4}(\phi_{j-1,k} + \phi_{j+1,k} + \phi_{j,k-1} + \phi_{j,k+1}) \tag{10}$$

In other words, the value of the potential at a specific point is the average of the four surrounding points. This is the relaxation method.

2 System A

Suppose one has a perfectly uniform electric field between two infinite planes at potentials V and $-V$, and they are a distance $2d$ apart. Suppose one then places an infinitely long perfectly conducting cylinder, of radius R, into the centre of the field, at ground potential. We wish to find the resulting form of the electrostatic potential, and hence the electric field, surrounding the cylinder.

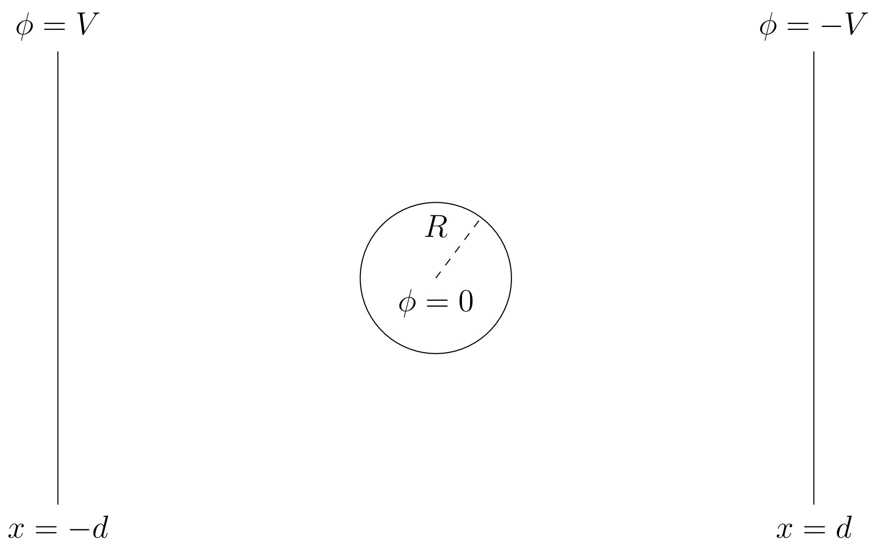

Figure 1: Cross-sectional diagram of System A

2.1 Analytic Solution

First, one realises that the three-dimensional problem can be reduced entirely to two dimensions due to the translation symmetry of the system along the length of the cylinder. So, we consider a cross-section of the system and introduce a polar co-ordinate system, with origin centred on the centre of the cylinder.

Laplace's equation in polar co-ordinates is:

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{11}$$

We employ separation of variables and posit a solution of the form $\phi = f(r)g(\theta)$ for two unknown functions f and g. Upon substitution, one finds that:

$$\frac{r}{f(r)}\frac{d}{dr}(r\frac{df(r)}{dr}) = -\frac{1}{g(\theta)}\frac{d^2g(\theta)}{d\theta^2} \tag{12}$$

Since this is true for arbitrary values of r and θ, it is constant, and we—with some foreknowledge—set this equal to k^2, for some constant $k \in \mathbb{R}$. This gives two second order ordinary differential equations:

$$r\frac{d}{dr}(r\frac{df(r)}{dr}) = k^2 f(r) \qquad \frac{d^2g(\theta)}{d\theta^2} = -k^2 g(\theta) \tag{13}$$

For the case $k = 0$, these equations have solutions $f(r) = \alpha\ln(r)+\beta$ and $g(\theta) = \gamma\theta+\delta$. For non-zero k, they have solutions

$$f(r) = \alpha_k r^k + \beta_k r^{-k} \qquad g(\theta) = \gamma_k \sin(k\theta) + \delta_k\cos(k\theta) \tag{14}$$

With the physically reasonable requirement that $g(\theta) = g(\theta + 2\pi)$, we have that k is an integer. Hence, by the principle of superposition and the linearity of differentiation, the general solution to the Laplace equation in polar co-ordinates is a sum of these terms:

$$\phi(r,\theta) = f(r)g(\theta) = (\alpha\ln(r) + \beta)(\gamma\theta + \delta) + \sum_{n=1}^{\infty}(\alpha_n r^n + \beta_n r^{-n})(\gamma_n\sin(n\theta) + \delta_n\cos(n\theta)) \tag{15}$$

For a particular solution to the system considered here one must impose boundary conditions.

By considering the geometry of the system, we expect a solution that is symmetric about $\theta = 0$. This implies that $\gamma = 0$ and $\gamma_n = 0$, $\forall n$ as sin() is anti-symmetric about the origin (c.f. odd). Additionally, the potential is finite as $r \to \infty$, implying that α and α_n are both zero. We now have:

$$\phi(r,\theta) = \beta + \sum_{n=1}^{\infty}(\frac{\beta_n}{r^n}\cos(n\theta)) \tag{16}$$

where the β's have absorbed other constants. In particular, as $r \to \infty$, we require $\phi = -\frac{V}{d}x = -\frac{V}{d}r\cos(\theta)$. Since the infinite sum vanishes at infinity, $\beta = -\frac{V}{d}r\cos(\theta)$.

We now have that

$$\phi(r,\theta) = -\frac{V}{d}r\cos(\theta) + \sum_{n=1}^{\infty}(\frac{\beta_n}{r^n}\cos(n\theta)) = (\frac{\beta_1}{r} - \frac{V}{d}r)\cos(\theta) + \sum_{n=2}^{\infty}(\frac{\beta_n}{r^n}\cos(n\theta)) \tag{17}$$

We require that the potential is zero on the surface of the cylinder, $\phi(R,\theta) = 0$, so that $\beta_1 = \frac{VR^2}{d}$ and $\beta_{n\geq 2} = 0$

Thus, the final form for the electrostatic potential is

$$\phi(x,y) = \begin{cases} 0, & r \leq a \\ \frac{V}{d}(\frac{R^2}{r} - r)\cos(\theta), & r > a \end{cases} \tag{18}$$

a graph of which is shown in Figure 2.

/home/danmfr/p3t/laplace/pdfs/analytic.pdf

Figure 2: Analytic solution with $V = 1$, $R = 15$ on 100×100 grid

2.2 Numerical Solution

To numerically approximate the electrostatic potential for System A via the relaxation method the following rudimentary algorithm was developed.

Figure 3a shows the numerical approximation to the potential, and Figure 3b shows the electric field around the cylinder.

2.3 Comparison of Analytical and Numerical Solution

3 System B

The second system considered was a silicon detector—a system consisting of two silicon wafers, one segmented with doped implants at ground potential and the other, referred to as the backplane, uniformly doped, held at a potential $+V$.

3.1 Numerical Solution

4 Other Electrostatic Systems

We developed a general software package in C++ to solve and plot the potential and electric field for an arbitrary electrostatic system. It accepts input of a coloured bitmap, where the colours represent different potentials, and returns plots of the resultant electrostatic potential and electric field.

Here is a flowchart representing the algorithm

5 Conclusion

5.1 Further Work

A Appendices

```
procedure THE RELAXATION METHOD
    declare variables:
    V ← potential on plates
    δ ← position step-size
    d ← distance between plates
    h ← height of plates
    r ← radius of cylinder
    its ← number of iterations
    nx ← d/δ
    ny ← h/δ
    specify boundary potentials:
    for j = 1 to ny do
        u_{j,1} = +V
        u_{j,nx} = −V
    end for
    for k = 1 to nx do
        u_{1,k} ← V − 2Vj/nx
        u_{ny,k} ← V − 2Vj/nx
    end for
    find solution:
    for i = 1 to its do
        for j = 2 to ny − 1 do
            for k = 2 to nx − 1 do
                if (jδ − (1/2)d)^2 + (kδ − (1/2)h)^2 < r^2 then
                    u_{j,k} ← 0
                else
                    u_{j,k} ← (1/4)(u_{j−1,k} + u_{j+1,k} + u_{j,k−1} + u_{j,k+1})
                end if
            end for
        end for
    end for
    find electric field:
    for j = 1 to ny − 1 do
        for k = 1 to nx − 1 do
            (Ex)_{j,k} ← −(u_{j,k+1} − u_{j,k})/δ
            (Ey)_{j,k} ← −(u_{j+1,1} − u_{j,k})/δ
        end for
    end for
    plot potential and field
end procedure
```

/home/danmfr/p3t/laplace/pdfs/numerical.pdf

(a) electrostatic potential

/home/danmfr/p3t/laplace/pdfs/field.pdf

(b) electric field

Figure 3: Numerical approximations for electstatic potential and electric field with $V = 1$, $R = 15$ on 100×100 grid for 10000 iterations

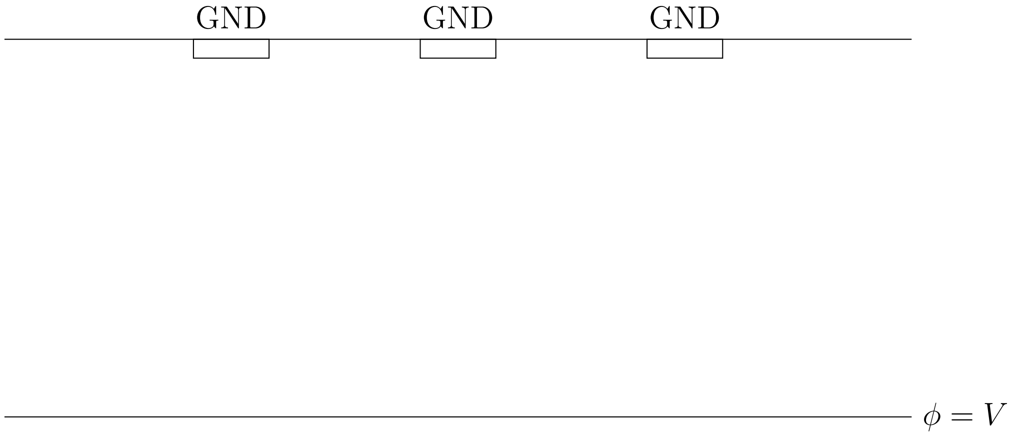

Figure 4: Cross-sectional diagram of System B

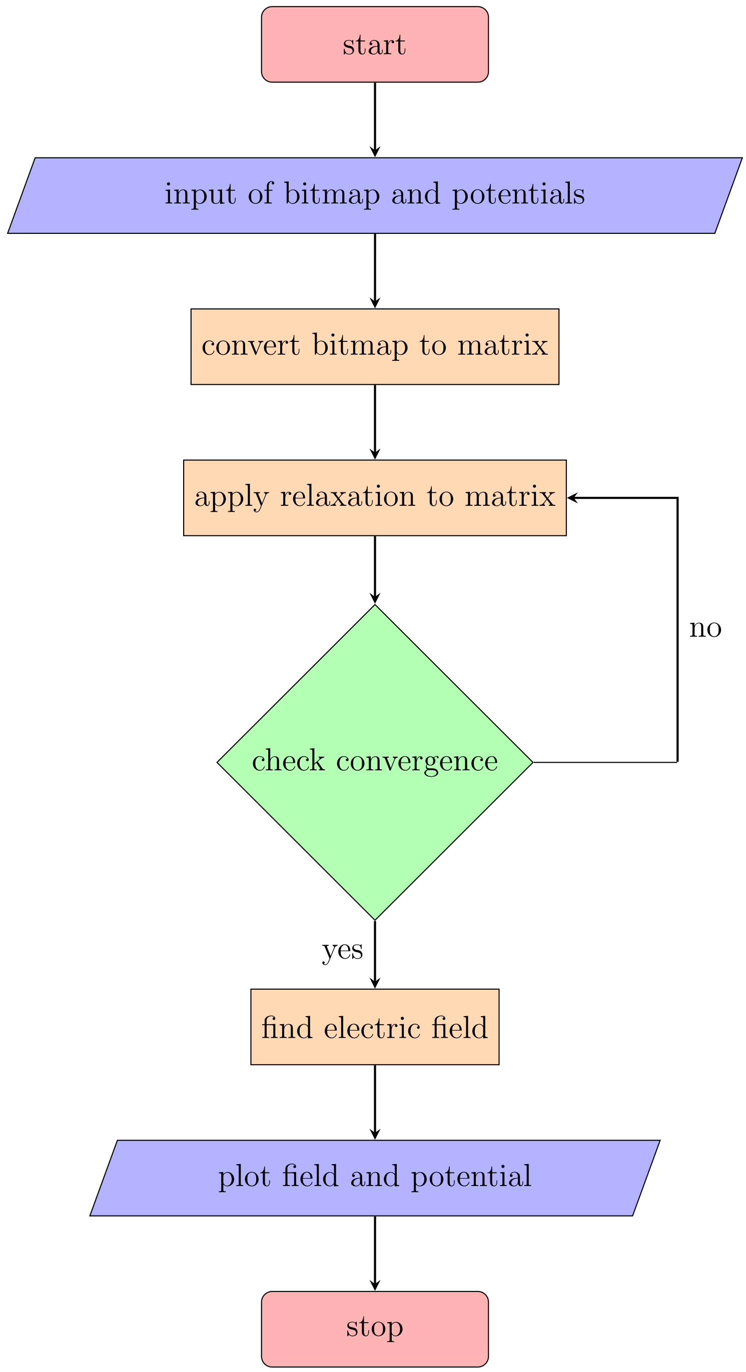

Figure 5: Flowchart of steps in software package