Modal Analysis of Beam and Plate: Analytical Method with Validation Using ANSYS and/or Mathematica

Submitted in partial fulfillment of the requirements
for the Vibrations of Structures (ME60428)
of

Master of Technology

by

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Under the guidance of

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CERTIFICATE

This is to certify that the term project report entitled **Modal Analysis of Beam and Plate: Analytical Method with Validation Using ANSYS and/or Mathematica**, submitted by **Mungekar Gaurav Bholanath Madhavi (23ME63R11)** to the Indian Institute of Technology Kharagpur, in partial fulfilment of the requirements for the completion of the course of **Vibrations of Structures (ME60428)** is a record of bona fide research work under my supervision. No part of this report has been submitted elsewhere for award of any other degree.

Prof. Korak Sarkar

Date:

DECLARATION

I certify that

a. The work contained in the thesis is original and has been done by myself under the general supervision of my supervisor.
b. The work has not been submitted to any other Institute for any degree or diploma.
c. I have followed the guidelines provided by the Institute in writing the thesis.
d. I have conformed to the norms and guidelines in the Ethical Code of Conduct of the Institute.
e. Whenever I have used materials (data, theoretical analysis, and text) from other sources, I have given due credit to them by citing them in the text of the thesis and giving their details in the references.
f. Whenever I have quoted written materials from other sources, I have put them under quotation marks and given due credit to the sources by citing them and giving required details in the references.

(Signature)

Mungekar Gaurav Bholanath Madhavi (23ME63R11)

Contents

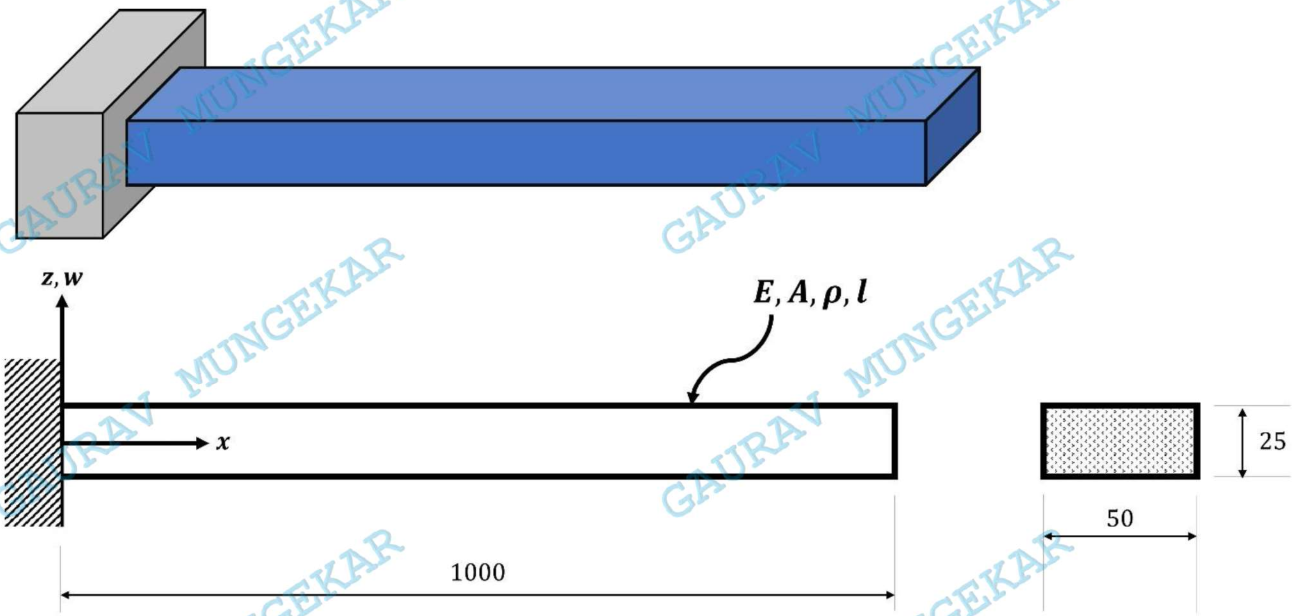
List of Figures

List of Tables

1. Vibration of Beam

Problem Statement:

1. Analytically derive the first four natural frequencies and mode shapes of a uniform Euler-Bernoulli cantilever beam undergoing transverse vibration. (Dimensions given below)
2. Determine the first four natural frequencies and mode shapes of the same beam using Ritz method with a 10-term approximation. Plot the mode shapes and compare them with the exact mode shapes in four separate plots.
3. Determine the first four natural frequencies and transverse bending modes of the same beam using Ansys.



All Dimensions are in mm

Figure 1 Cantilever Beam with Fixed Support

Given Data:

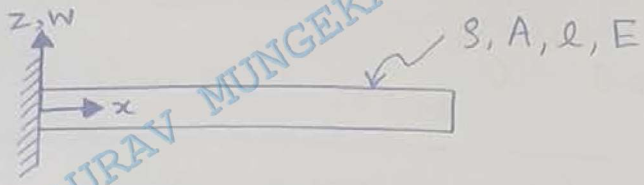
Young's Modulus of Beam material, E = 140 GPa

Density of beam material, ρ = 3700 kg/m^3

Poisson's ratio of beam, υ = 0.33

Beam Dimensions: 1m x 50 mm x 25 mm (length (l), breadth, height)

Analytical Solution of uniform Euler-Bernoulli cantilever Beam undergoing transverse vibration



Cantilever Beam

Step 1:- Derivation of Governing Differential Equation.

$$T = \frac{1}{2}\int_0^l (\rho A\, dx).\, w_{,t}^2$$

$$V = \frac{1}{2}\int_{\forall} \sigma\, \varepsilon\, d\forall$$

$$= \frac{1}{2}\int_0^l \int_A (-E z w_{,xx})(-z w_{,xx})\, dA\, dx$$

$$= \frac{1}{2}\int_0^l E w_{,xx}^2 \int_A z^2\, dA\, dx$$

$$= \frac{1}{2}\int_0^l E I w_{,xx}^2\, dx$$

By Hamilton's principle, $\delta \int_{t_1}^{t_2} (T-V)\, dt = 0$

$$\Rightarrow \delta \int_{t_1}^{t_2}\int_0^l \left(\frac{1}{2}\rho A w_{,t}^2 - \frac{1}{2} E I w_{,xx}^2\right) dx\, dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\int_0^l \rho A w_{,t}\, \delta w_{,t} - E I w_{,xx}\, \delta w_{,xx}\big)\, dx\, dt = 0$$

$$\Rightarrow \underbrace{\cancel{\int_0^l \rho A w_{,t}\, \delta w \Big|_{t_1}^{t_2} dx}}_{0\ \text{(Boundary Term)}} - \int_{t_1}^{t_2} E I w_{,xx} (\delta w)_{,x} \Big|_0^l dt$$

$$- \int_{t_1}^{t_2}\int_0^l \left[\rho A w_{,tt}\, \delta w - (E I w_{,xx})_{,x} (\delta w)_{,x}\right] dx\, dt = 0$$

$$\Rightarrow - \int_{t_1}^{t_2} E I w_{,xx} (\delta w)_{,x} \Big|_0^l dt + \int_{t_1}^{t_2} (E I w_{,xx})_{,x}\, \delta w \Big|_0^l dt$$

$$- \int_{t_1}^{t_2}\int_0^l \left[\rho A w_{,tt} + (E I w_{,xx})_{,xx}\right] \delta w\, dx\, dt = 0$$

Equation of Motion :- $\boxed{\rho A w_{,tt} + EI w_{,xxxx} = 0}$ ... (#1)

... $E, I = \text{constant} \neq f(x)$

Boundary Conditions :-

$w(0,t) = 0$ ... Deflection	$EI w_{,xx}(l,t) = 0$ ... B.M.
$w_{,x}(0,t) = 0$ ... Slope	$EI w_{,xxx}(l,t) = 0$ ... S.F.

... (#2)

Step 2 :- Find Eigenfrequency from assumed solution.

Let Assumed solution, $w(x,t) = W(x)\, e^{i\omega t}$ ... (#3)

... Variable separable type solution.

So,
$$w_{,t} = W(x)\,(i\omega)\, e^{i\omega t} = W(i\omega)\, e^{i\omega t}$$
$$w_{,tt} = W\,(i\omega)^2 e^{i\omega t} = -\omega^2 W e^{i\omega t}$$
$$w_{,x} = W'(x)\, e^{i\omega t} \qquad \dots\ (\ )' = \frac{d}{dx}(\ )$$
$$w_{,xx} = W''(x)\, e^{i\omega t}$$
$$w_{,xxx} = W'''(x)\cdot e^{i\omega t}$$
$$w_{,xxxx} = W''''(x)\, e^{i\omega t} = W''''\, e^{i\omega t}$$

Substitute these values in EOM,

$$\Rightarrow \quad -\omega^2 \rho A W \cancel{e^{i\omega t}} + EI W'''' \cancel{e^{i\omega t}} = 0$$
$$\Rightarrow \quad EI W'''' - \omega^2 \rho A W = 0$$

Divide both sides by EI,

$$\Rightarrow \quad W'''' - \omega^2 \frac{\rho A}{EI} W = 0$$

Assume $c = \sqrt{\dfrac{EI}{\rho A}}$

$$\boxed{\Rightarrow \quad W'''' - \frac{\omega^2}{c^2} W = 0}$$ ... Eigen Value Problem

New Boundary Conditions :-

① $w(0,t) = W(0)\, e^{i\omega t} = 0 \quad \Rightarrow \quad W(0) = 0$

② $w_{,x}(0,t) = W'(0)\, e^{i\omega t} = 0 \quad \Rightarrow \quad W'(0) = 0$

③ $EI w_{,xx}(l,t) = EI W''(l)\, e^{i\omega t} = 0 \quad \Rightarrow \quad W''(l) = 0$

④ $EI w_{,xxx}(l,t) = EI W'''(l)\, e^{i\omega t} = 0 \quad \Rightarrow \quad W'''(l) = 0$

... (#4)

Let $\frac{\omega^2}{c^2} = \beta^4$ i.e. $\beta = \sqrt{\frac{\omega}{c}}$

$$\therefore \quad W'''' - \beta^4 W = 0 \quad \cdots \#5$$

characteristic equation of above differential equation is,

$$s^4 - \beta^4 = 0$$

$$(s^2 - \beta^2)(s^2 + \beta^2) = 0$$

$$s^2 = \beta^2 \quad \text{or} \quad s^2 = -\beta^2$$

$$s = \pm\beta \quad \text{or} \quad s = \pm i\beta$$

Solution of EVP is given by,

$$W(x) = c_1 e^{\beta x} + c_2 e^{-\beta x} + c_3 e^{i\beta x} + c_4 e^{-i\beta x}$$

Above solution can be written in another form as trigonometric & hyperbolic functions as,

$$\boxed{W(x) = B_1 \cosh\beta x + B_2 \sinh\beta x + B_3 \cos\beta x + B_4 \sin\beta x} \quad \cdots \#6$$

Using New Boundary condition 1,

$$\Rightarrow W(0) = B_1(1) + B_2(0) + B_3(1) + B_4(0) = 0$$

$$\Rightarrow \quad B_1 + B_3 = 0 \quad \cdots 5$$

Using New Boundary Condition 2,

$$W'(x) = B_1\beta \sinh\beta x + B_2\beta\cosh\beta x - B_3\beta\sin\beta x + B_4\beta\cos\beta x$$

$$\Rightarrow W'(0) = B_1\beta(0) + B_2\beta(1) - B_3\beta(0) + B_4\beta(1) = 0$$

$$\Rightarrow \quad B_2\beta + B_4\beta = 0$$

$$\Rightarrow \quad B_2 + B_4 = 0 \quad \cdots 6$$

Using New Boundary Condition 3,

$$W''(x) = B_1\beta^2\cosh\beta x + B_2\beta^2\sinh\beta x - B_3\beta^2\cos\beta x - B_4\beta^2\sin\beta x$$

$$\cancel{\Rightarrow W''(0) = B_1\beta^2(1) + B_2\beta^2(0) - B_3\beta^2(1) - B_4\beta^2(0) = 0}$$

$$\Rightarrow W''(l) = B_1\beta^2\cosh\beta l + B_2\beta^2\sinh\beta l - B_3\beta^2\cos\beta l - B_4\beta^2\sin\beta l = 0$$

$$\Rightarrow \quad B_1\cosh\beta l + B_2\sinh\beta l - B_3\cos\beta l - B_4\sin\beta l = 0 \quad \cdots 7$$

Using New Boundary condition (4),

$$W'''(x) = B_1 \beta^3 \sinh \beta x + B_2 \beta^3 \cosh \beta x + B_3 \beta^3 \sin \beta x - B_4 \beta^3 \cos \beta x$$

$$\Rightarrow W'''(l) = B_1 \beta^3 \sinh \beta l + B_2 \beta^3 \cosh \beta l + B_3 \beta^3 \sin \beta l - B_4 \beta^3 \cos \beta l = 0$$

$$\Rightarrow \quad B_1 \sinh \beta l + B_2 \cosh \beta l + B_3 \sin \beta l - B_4 \cos \beta l = 0 \quad \ldots (8)$$

From (5), (6), (7), (8), we get

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \cosh \beta l & \sinh \beta l & -\cos \beta l & -\sin \beta l \\ \sinh \beta l & \cosh \beta l & \sin \beta l & -\cos \beta l \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From (5), $B_1 + B_3 = 0 \Rightarrow B_1 = -B_3$ ... (9)

From (6), $B_2 + B_4 = 0 \Rightarrow B_2 = -B_4$ ... (10)

From (7), $B_1 \cosh \beta l + B_2 \sinh \beta l + B_1 \cos \beta l + B_2 \sin \beta l = 0$

$$(\cosh \beta l + \cos \beta l) B_1 + (\sinh \beta l + \sin \beta l) B_2 = 0 \quad \ldots (11)$$

From (8), $B_1 \sinh \beta l + B_2 \cosh \beta l - B_1 \sin \beta l + B_2 \cos \beta l = 0$

$$(\sinh \beta l - \sin \beta l) B_1 + (\cosh \beta l + \cos \beta l) B_2 = 0 \quad \ldots (12)$$

Also, for non-trivial solution,

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \cosh \beta l & \sinh \beta l & -\cos \beta l & -\sin \beta l \\ \sinh \beta l & \cosh \beta l & \sin \beta l & -\cos \beta l \end{vmatrix} = 0$$

$$\Rightarrow 1 \times [1 \times (\cos^2 \beta l + \sin^2 \beta l) + 1 \times (\sinh \beta l \cdot \sin \beta l + \cosh \beta l \cdot \cos \beta l)]$$
$$+ 1 \times [-1 \times (-\cosh \beta l \cdot \cos \beta l + \sin \beta l \cdot \sinh \beta l) + 1 \times (\cosh^2 \beta l - \sinh^2 \beta l)] = 0$$

$$\Rightarrow 1 \, [1 + \sinh \beta l \cdot \sin \beta l + \cosh \beta l \cdot \cos \beta l] + \cosh \beta l \cos \beta l$$
$$- \sin \beta l \cdot \sinh \beta l + 1 = 0$$

$$\Rightarrow \cancel{\sinh \beta l \cdot \sin \beta l} + 2 \cosh \beta l \cos \beta l - \cancel{\sinh \beta l \cdot \sin \beta l} + 2 = 0$$

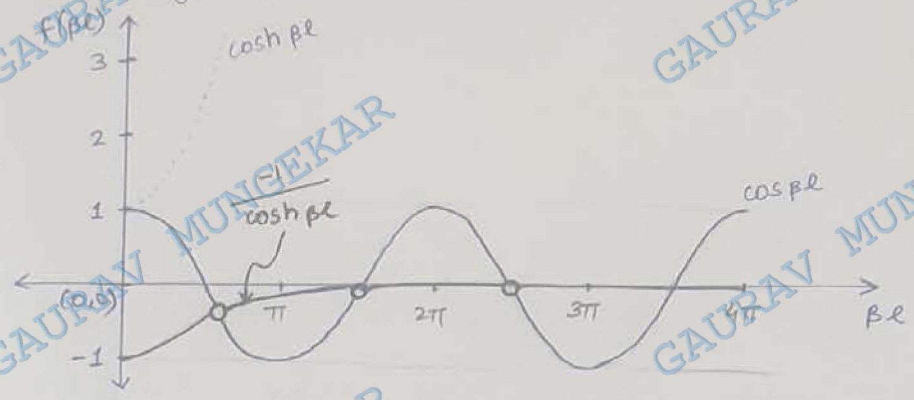
$$\Rightarrow \boxed{\cos \beta l \cdot \cosh \beta l + 1 = 0}$$

... Characteristic Eq^n of cantilever E-B Beam.

Since characteristic equation is transcendental equation. We will solve it graphically.

$$\cos \beta l = \frac{-1}{\cosh \beta l}$$

For very high value of $\cosh \beta l$, $\cos \beta l$ tends to zero.



Graphical solution of characteristic Equation of cantiler beam

Since function $\frac{-1}{\cosh \beta l}$ converges rapidly to zero, we can say

$\boxed{\cos \beta_n l = 0}$ ... for higher modes. ($\beta_n \equiv$ Higher Modes here)

$$\beta_n l = \left(\frac{2n-1}{2}\right)\pi \quad \text{where } n = 1, 2, 3, \ldots, \infty$$

$$\boxed{\beta_n = \left(\frac{2n-1}{2}\right)\frac{\pi}{l}} \quad \text{... For Higher Modes}$$

We know, $\beta^2 = \frac{\omega}{c}$ i.e. $\omega_n = \beta_n^2 c$

$$\therefore \quad \omega_n = \left[\left(\frac{2n-1}{2}\right)\frac{\pi}{l}\right]^2 c$$

$$\boxed{\omega_n = \left[\left(\frac{2n-1}{2}\right)\frac{\pi}{l}\right]^2 \sqrt{\frac{EI}{\rho A}}}$$

... Circular Natural Frequency

... For Higher Modes

Please note that above expressions only valid for higher modes. In case of lower modes, small correction factor e_n added such that,

$$\boxed{\begin{aligned} \beta_n &= \left(\frac{2n-1}{2}\pi + e_n\right)\frac{1}{l} \\ \omega_n &= \left[\frac{2n-1}{2}\pi + e_n\right]^2 \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} \end{aligned}}$$

... For Lower Modes

$$\begin{bmatrix} e_1 = 0.3042 \\ e_2 = -0.018 \\ e_3 = 0.001 \\ \vdots \\ e \approx 0 \end{bmatrix}$$

Step 3 :- Calculation of Eigenfunction

From (11), $B_1 = \left[\dfrac{-(\sinh \beta l + \sin \beta l)}{(\cosh \beta l + \cos \beta l)}\right] B_2 = \alpha B_2$ (say)

Put, $B_2 = 1$ to get one possible solution,

$B_1 = \alpha_n (1) = \alpha_n$

$B_2 = 1$

$B_3 = -B_1 = -\alpha_n$

$B_4 = -B_2 = -1$

Put these values in (#6) to get n^{th} eigenfunction,

$W_n(x) = \alpha_n \cosh \beta_n x + \sinh \beta_n x - \alpha_n \cos \beta_n x - \sin \beta_n x$

Resubstitute $\alpha_n = -\dfrac{(\sinh \beta_n l + \sin \beta_n l)}{(\cosh \beta_n l + \cos \beta_n l)}$

$$\boxed{W_n(x) = \sinh \beta_n x - \sin \beta_n x - \left[\frac{\sinh \beta_n l + \sin \beta_n l}{\cosh \beta_n l + \cos \beta_n l}\right](\cosh \beta_n x - \cos \beta_n x)}$$

... Eigenfunction

Step 4 :- General Solution

We know, $w(x,t) = W(x)\, e^{i\omega t}$

Summing all solutions,

$$\boxed{w(x,t) = \sum_{n=1}^{\infty} [C_n \cos \omega_n t + S_n \sin \omega_n t]\, W_n(x)}$$

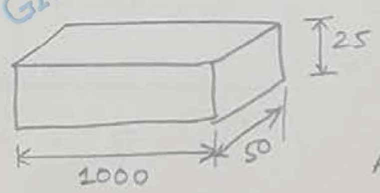
where $W_n(x) = \sinh \beta_n x - \sin \beta_n x - \left[\dfrac{\sinh \beta_n l + \sin \beta_n l}{\cosh \beta_n l + \cos \beta_n l}\right](\cosh \beta_n x - \cos \beta_n x)$

Step 5 :- Calculation of first four natural frequencies

$E = 140$ GPa $= 140 \times 10^9$ Pa

$I = \dfrac{50 \times 25^3}{12}$ mm^4

$= 65104.1667$ mm^4



All dimensions in mm

$\rho = 3700$ kg/m^3

$A = 50 \times 25$ mm^2

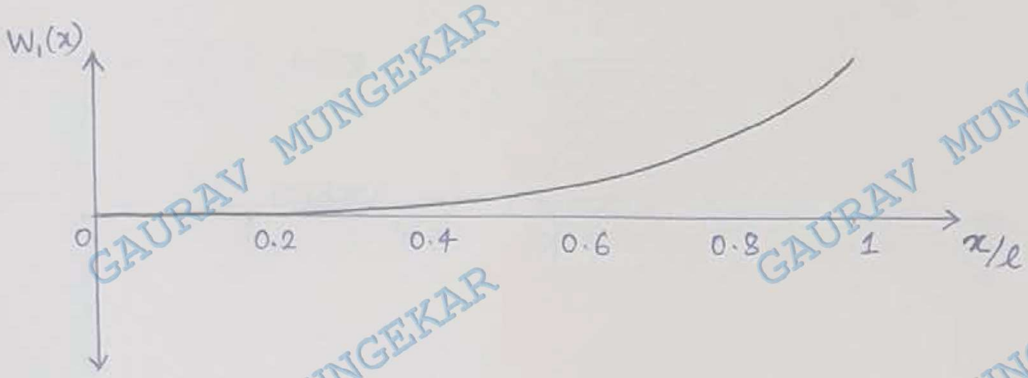
$= 1250$ mm^2

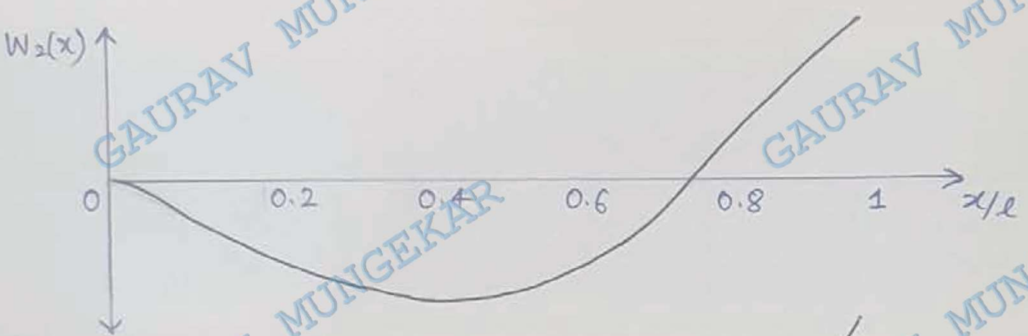
$= 0.00125$ m^2 & $I = 65104.1667 \times 10^{-12}$ m^4

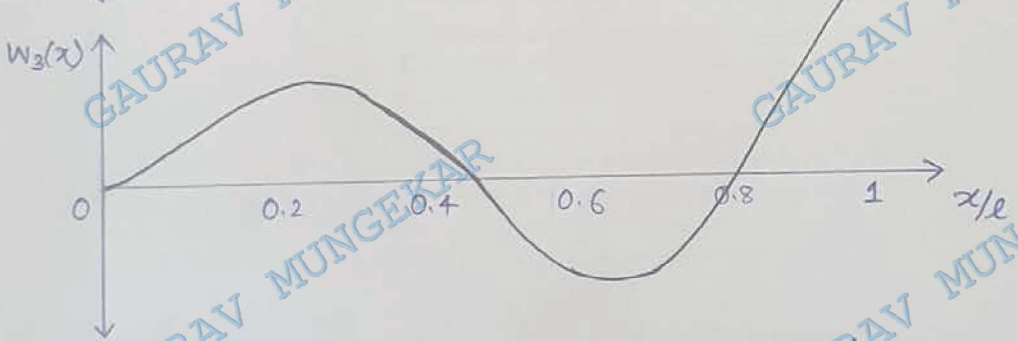
Without Considering Error Correction Factor

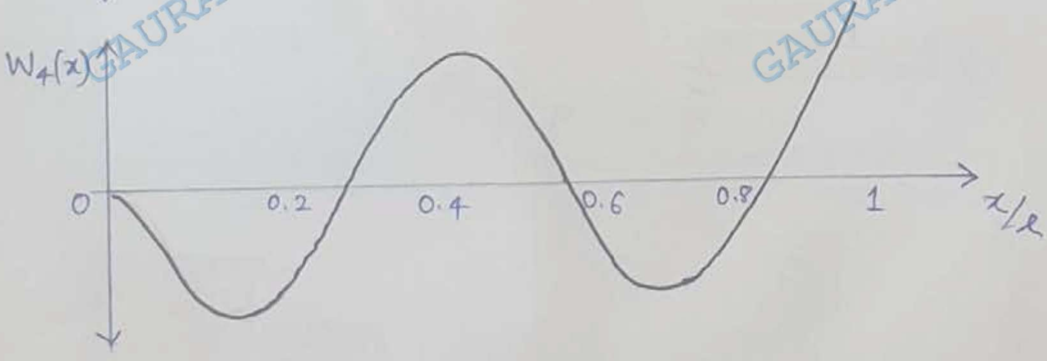
$$\omega_n = \left[\left(\frac{2n-1}{2}\right)\frac{\pi}{\ell}\right]^2 \sqrt{\frac{EI}{\rho A}} \quad \text{in rad/s} \quad \text{and} \quad f_n = \frac{\omega_n}{2\pi} \quad \text{in Hz}$$

Sr.	n	Value of ω_n (rad/s)	Value of f_n (Hz)
1	1	109.5348	17.433
2	2	985.8136	156.8971
3	3	2738.3711	435.8253
4	4	5367.2073	854.2176









Considering Error Correction Factor (e_n)

$$\omega_n = \left[\frac{2n-1}{2}\pi + e_n\right]^2 \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} \quad \ldots (rad/s) \quad \text{and} \quad f_n = \frac{\omega_n}{2\pi} \ldots (Hz)$$

Here $e_1 = 0.3042$

$e_2 = -0.018$

$e_3 = 0.001$

Sr.	n	e_n	ω_n (rad/s)	f_n (Hz)
1	1	0.3042	156.0678	24.8389
2	2	-0.018	978.2969	155.7007
3	3	0.001	2739.0684	435.9363
4	4	0	5367.2074	854.2176

Similar plots with small difference in values of $W_n(x)$.

1.2 Ritz Method (Numerical Method)

```
In[1]:= Clear["Global`*"]

In[2]:= num = 10; (* Number of terms in Ritz expansion *)
        numFreq = 4 ; (* Number of frequencies you want *)

In[4]:= h = Table[(x / l)^(i+1), {i, num}]; (* Defining the admissible functions *)
        h // MatrixForm;

In[6]:= m = Table[0, {i, num}, {j, num}]; (* Mass and stiffness matrix *)
        k = Table[0, {i, num}, {j, num}];

In[8]:= For[i = 1, i ≤ num, i++,
         For[j = 1, j ≤ num, j++,
          m⟦i, j⟧ = Integrate[ρa * h⟦i⟧ * h⟦j⟧, {x, 0, l}];
          k⟦i, j⟧ = Integrate[ei * D[h⟦i⟧, {x, 2}] * D[h⟦j⟧, {x, 2}], {x, 0, l}]
         ]
        ]

In[9]:= FullSimplify[m] // MatrixForm;

In[10]:= FullSimplify[k] // MatrixForm;

In[11]:= ei = (140 * 10^9 * 0.05 * 0.025^3) / 12;
         l = 1;
         ρa = 3700 * (0.05 * 0.025); (* Values required for numerical solution *)

In[12]:= ω = Sqrt[Eigenvalues[N@{k, m}]]
          (* Exact frequencies : 156.0678, 978.2969, 2739.0684, 5367.2073 *)

Out[12]=
         {246 282., 48 228.7, 34 354.5, 19 096.9,
          13 458.2, 8873.52, 5367.27, 2738.91, 978.173, 156.086}

In[13]:= (*Sort ω values along with their indices*)
         sortedOmegaValues = SortBy[Transpose[{ω, Range[Length[ω]]}], First];

         (*Display sorted ω values with corresponding indices*)
         For[i = 1, i ≤ numFreq , i++,
          Print["ω", Subscript[Length[sortedOmegaValues] + 1 - sortedOmegaValues⟦i, 2⟧, ""],
           " = ", sortedOmegaValues⟦i, 1⟧, " rad/s"]]

         ω1 = 156.086 rad/s

         ω2 = 978.173 rad/s

         ω3 = 2738.91 rad/s

         ω4 = 5367.27 rad/s

In[15]:= For[i = 1, i ≤ numFreq , i++,
          Print["f", Subscript[Length[sortedOmegaValues] + 1 - sortedOmegaValues⟦i, 2⟧, ""],
           " = ", sortedOmegaValues⟦i, 1⟧ / (2 * Pi), " Hz"]]

         f1 = 24.8418 Hz

         f2 = 155.681 Hz

         f3 = 435.911 Hz

         f4 = 854.228 Hz

In[16]:= Transpose[Eigensystem[N@{k, m}]] // MatrixForm;
```

In[17]:=
```
v = Eigenvectors[N@{k, m}];
v // MatrixForm
```

Out[18]//MatrixForm=

$$\begin{pmatrix}
-0.0000440851 & 0.00117275 & -0.0123358 & 0.0689738 & -0.229786 & 0.478984 & -0.630715 & 0.509703 & -0.230769 & 0.0448155 \\
-0.0000795777 & 0.00185914 & -0.0174658 & 0.0883121 & -0.268567 & 0.514735 & -0.626732 & 0.470429 & -0.198535 & 0.0360437 \\
-0.000275355 & 0.00536167 & -0.0415833 & 0.171263 & -0.416283 & 0.6205 & -0.563732 & 0.294914 & -0.0761313 & 0.00596699 \\
0.0000202631 & 0.000347114 & -0.00866971 & 0.0628616 & -0.230732 & 0.492557 & -0.638404 & 0.495874 & -0.212545 & 0.0386908 \\
0.000126385 & 0.000856868 & -0.0235936 & 0.141161 & -0.404098 & 0.640359 & -0.574356 & 0.271494 & -0.0506603 & -0.00129048 \\
-0.00019553 & 0.0011629 & -0.00422874 & 0.0332643 & -0.171003 & 0.449141 & -0.65185 & 0.535217 & -0.233958 & 0.0424476 \\
0.00169198 & -0.00687672 & 0.00976599 & -0.0637306 & 0.304131 & -0.623191 & 0.63577 & -0.325424 & 0.0694539 & -0.00161851 \\
-0.023751 & 0.0619857 & 0.00252838 & -0.0203133 & -0.154547 & -0.00933189 & 0.569458 & -0.717782 & 0.356883 & -0.0658996 \\
0.411557 & -0.655784 & -0.00100343 & 0.00693055 & 0.526955 & -0.307752 & -0.116072 & 0.119498 & -0.0191764 & -0.00250906 \\
0.908433 & -0.416821 & -1.36303\times10^{-7} & 8.84102\times10^{-7} & 0.0311921 & -0.0061263 & -0.0000123763 & 0.0000118864 & 0.0000698691 & -7.87071\times10^{-6}
\end{pmatrix}$$

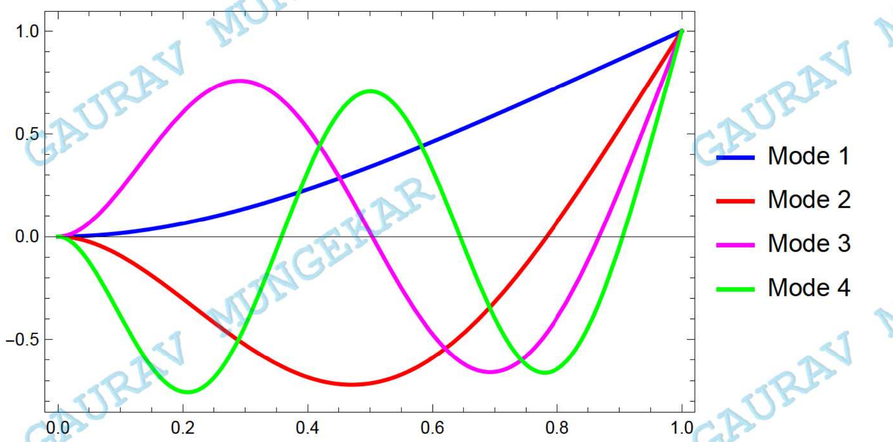
In[19]:=
```
mode = Table[0, {i, num}]; newmode = Table[0, {i, num}];
For[i = num;
 p = 1, i > 0, i--;
 p++, mode[[p]] = Sum[(h[[j]] * v[[i, j]]), {j, 1, num}];
 newmode[[p]] = mode[[p]] / (mode[[p]] /. x → 1)]
Simplify[newmode] // MatrixForm
```

Out[21]//MatrixForm=

$$\begin{pmatrix}
x^2 \left(1.75801 - 0.806636\,x - 2.63774\times10^{-7}\,x^2 + 1.71092\times10^{-6}\,x^3 + 0.0603633\,x^4 - 0.0118557\,x^5 - 0.0000239507\,x^6 + 0.0000230027\,x^7 + 0.000135211\,x^8 - 0.0000152315\,x^9\right) \\
x^2 \left(-11.0172 + 17.555\,x + 0.0268613\,x^2 - 0.185528\,x^3 - 14.1063\,x^4 + 8.23838\,x^5 + 3.1072\,x^6 - 3.19891\,x^7 + 0.513344\,x^8 + 0.0671664\,x^9\right) \\
x^2 \left(30.8436 - 80.4959\,x - 3.28341\,x^2 + 26.3793\,x^3 + 200.699\,x^4 + 12.1186\,x^5 - 739.511\,x^6 + 932.127\,x^7 - 463.456\,x^8 + 85.5787\,x^9\right) \\
x^2 \left(-61.2139 + 248.792\,x - 353.323\,x^2 + 2305.7\,x^3 - 11003.1\,x^4 + 22546.4\,x^5 - 23001.4\,x^6 + 11773.5\,x^7 - 2512.77\,x^8 + 58.5558\,x^9\right) \\
x^2 \left(103.277 - 614.231\,x + 2233.57\,x^2 - 17569.8\,x^3 + 90321.8\,x^4 - 237232.\,x^5 + 344300.\,x^6 - 282696.\,x^7 + 123574.\,x^8 - 22420.4\,x^9\right) \\
x^2 \left(-105.604 - 715.979\,x + 19714.2\,x^2 - 117950.\,x^3 + 337654.\,x^4 - 535069.\,x^5 + 479918.\,x^6 - 226854.\,x^7 + 42330.6\,x^8 + 1078.29\,x^9\right) \\
x^2 \left(127.496 + 2184.06\,x - 54550.2\,x^2 + 395528.\,x^3 - 1.45178\times10^6\,x^4 + 3.09919\times10^6\,x^5 - 4.01687\times10^6\,x^6 + 3.12006\times10^6\,x^7 - 1.33734\times10^6\,x^8 + 243444.\,x^9\right) \\
x^2 \left(-699.333 + 13617.3\,x - 105611.\,x^2 + 434965.\,x^3 - 1.05725\times10^6\,x^4 + 1.57591\times10^6\,x^5 - 1.43174\times10^6\,x^6 + 749008.\,x^7 - 193354.\,x^8 + 15154.7\,x^9\right) \\
x^2 \left(893.028 - 20863.5\,x + 196003.\,x^2 - 991046.\,x^3 + 3.01388\times10^6\,x^4 - 5.77641\times10^6\,x^5 + 7.03325\times10^6\,x^6 - 5.2792\times10^6\,x^7 + 2.22798\times10^6\,x^8 - 404486.\,x^9\right) \\
x^2 \left(-187.021 + 4975.14\,x - 52332.\,x^2 + 292607.\,x^3 - 974816.\,x^4 + 2.03199\times10^6\,x^5 - 2.67567\times10^6\,x^6 + 2.16231\times10^6\,x^7 - 978987.\,x^8 + 190120.\,x^9\right)
\end{pmatrix}$$

In[22]:=
```
Plot[{newmode[[1]], newmode[[2]], newmode[[3]], newmode[[4]]}, {x, 0, 1},
 PlotStyle → {{Blue, Thick}, {Red, Thick}, {Magenta, Thick}, {Green, Thick}},
 Frame → True, PlotLegends → {"Mode 1", "Mode 2", "Mode 3", "Mode 4"}]
```

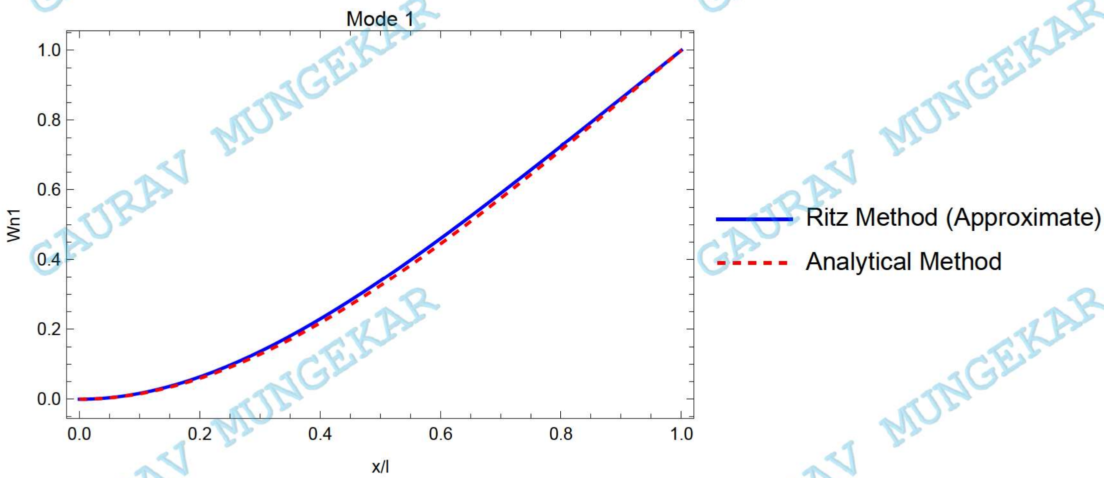
Out[22]=



In[23]:=
```
(*Given values*)l = 1;
beta[n_] := - (2 * n - 1) * Pi / 2;

(*Define the expression for Wn (x) *)
Wn[x_, n_] := Sinh[beta[n] * x] - Sin[beta[n] * x] -
   ((Sinh[beta[n] * l] + Sin[beta[n] * l]) / (Cosh[beta[n] * l] + Cos[beta[n] * l])) *
    (Cosh[beta[n] * x] - Cos[beta[n] * x]);

(*Normalize Wn (x) for n=1*)
maxAbsWn1 = MaxValue[{Abs[Wn[x, 1]], 0 ≤ x ≤ l}, x];
NormalizedWn1[x_] := Wn[x, 1] / maxAbsWn1;
Print["Wn1_(Normalized) = ", NormalizedWn1[x]]

(*Plot*)
Plot[{newmode[[1]], NormalizedWn1[x]}, {x, 0, l}, PlotStyle →
  {{Blue, Thick, Thickness[0.005]}, {Directive[Red, Dashed, Thickness[0.005]]}},
 Frame → True, FrameLabel → {"x/l", "Wn1"}, PlotLegends →
  {"Ritz Method (Approximate)", "Analytical Method"}, PlotLabel → "Mode 1"]
```

Wn1_(Normalized) = $\frac{1}{2}\left(\text{Sin}\left[\frac{\pi x}{2}\right] - \left(-\text{Cos}\left[\frac{\pi x}{2}\right] + \text{Cosh}\left[\frac{\pi x}{2}\right]\right)\text{Sech}\left[\frac{\pi}{2}\right]\left(-1 - \text{Sinh}\left[\frac{\pi}{2}\right]\right) - \text{Sinh}\left[\frac{\pi x}{2}\right]\right)$

Out[29]=

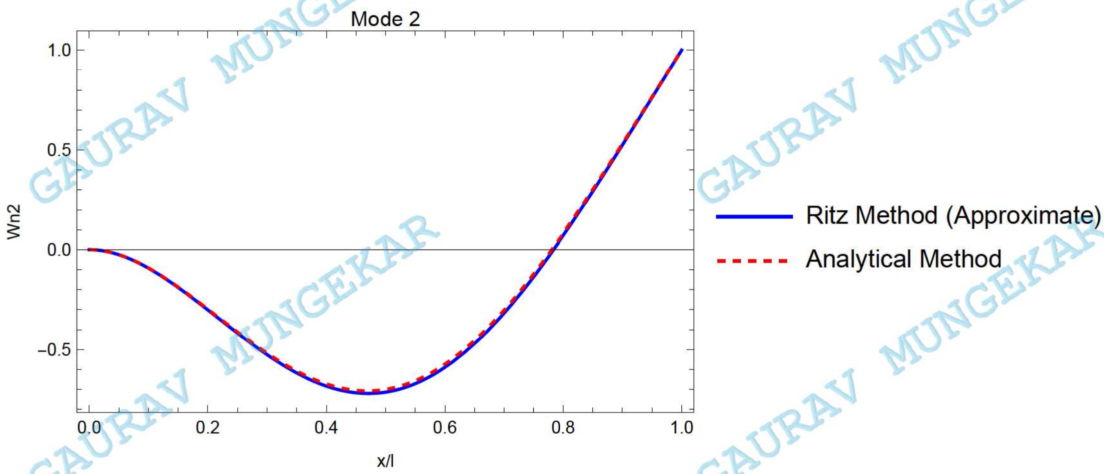


In[30]:=
```
(*Given values*)l = 1;
beta[n_] := (2 * n - 1) * Pi / 2;

(*Define the expression for Wn(x)*)
Wn[x_, n_] := Sinh[beta[n] * x] - Sin[beta[n] * x] -
   ((Sinh[beta[n] * l] + Sin[beta[n] * l]) / (Cosh[beta[n] * l] + Cos[beta[n] * l])) *
    (Cosh[beta[n] * x] - Cos[beta[n] * x]);

(*Normalize Wn(x) for n=2*)
maxAbsWn2 = MaxValue[{Abs[Wn[x, 2]], 0 ≤ x ≤ l}, x];
NormalizedWn2[x_] := Wn[x, 2] / maxAbsWn2;
Print["Wn2_(Normalized)= ", NormalizedWn2[x]]

(*Plot*)
Plot[{newmode[[2]], NormalizedWn2[x]}, {x, 0, l}, PlotStyle →
   {{Blue, Thick, Thickness[0.005]}, {Directive[Red, Dashed, Thickness[0.005]]}},
  Frame → True, FrameLabel → {"x/l", "Wn2"}, PlotLegends →
   {"Ritz Method (Approximate)", "Analytical Method"}, PlotLabel → "Mode 2"]
```

Wn2_(Normalized) =

$$\frac{1}{2}\left(-\mathrm{Sin}\left[\frac{3\pi x}{2}\right]-\left(-\mathrm{Cos}\left[\frac{3\pi x}{2}\right]+\mathrm{Cosh}\left[\frac{3\pi x}{2}\right]\right)\mathrm{Sech}\left[\frac{3\pi}{2}\right]\left(-1+\mathrm{Sinh}\left[\frac{3\pi}{2}\right]\right)+\mathrm{Sinh}\left[\frac{3\pi x}{2}\right]\right)$$

Out[36]=

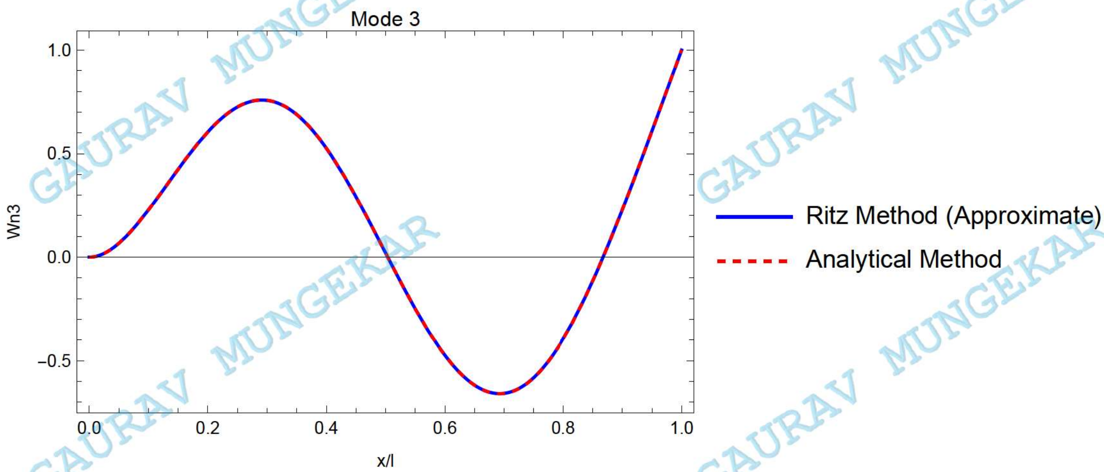


In[37]:=
```
(*Given values*)l = 1;
beta[n_] := - (2 * n - 1) * Pi / 2;

(*Define the expression for Wn(x)*)
Wn[x_, n_] := Sinh[beta[n] * x] - Sin[beta[n] * x] -
   ((Sinh[beta[n] * l] + Sin[beta[n] * l]) / (Cosh[beta[n] * l] + Cos[beta[n] * l])) *
    (Cosh[beta[n] * x] - Cos[beta[n] * x]);

(*Normalize Wn(x) for n=3*)
maxAbsWn3 = MaxValue[{Abs[Wn[x, 3]], 0 ≤ x ≤ l}, x];
NormalizedWn3[x_] := Wn[x, 3] / maxAbsWn3;
Print["Wn3_(Normalized) = ", NormalizedWn3[x]]

(*Plot*)
Plot[{newmode[[3]], NormalizedWn3[x]}, {x, 0, l}, PlotStyle →
   {{Blue, Thick, Thickness[0.005]}, {Directive[Red, Dashed, Thickness[0.005]]}},
  Frame → True, FrameLabel → {"x/l", "Wn3"}, PlotLegends →
   {"Ritz Method (Approximate)", "Analytical Method"}, PlotLabel → "Mode 3"]
```

Wn3_(Normalized) =

$$\frac{1}{2}\left(\text{Sin}\left[\frac{5\pi x}{2}\right] - \left(-\text{Cos}\left[\frac{5\pi x}{2}\right] + \text{Cosh}\left[\frac{5\pi x}{2}\right]\right)\text{Sech}\left[\frac{5\pi}{2}\right]\left(-1 - \text{Sinh}\left[\frac{5\pi}{2}\right]\right) - \text{Sinh}\left[\frac{5\pi x}{2}\right]\right)$$

Out[43]=

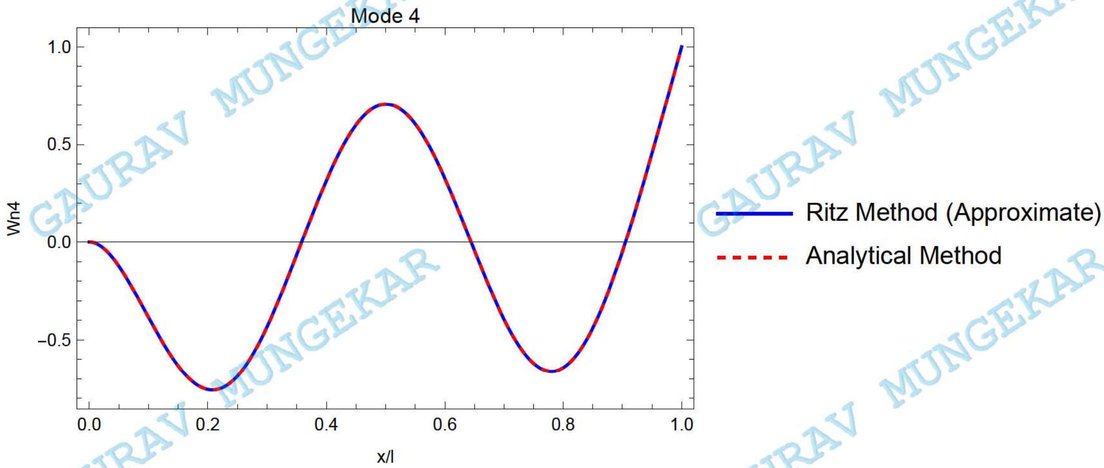


```
In[44]:= (*Given values*)l = 1;
beta[n_] := (2 * n - 1) * Pi / 2;

(*Define the expression for Wn(x)*)
Wn[x_, n_] := Sinh[beta[n] * x] - Sin[beta[n] * x] -
   ((Sinh[beta[n] * l] + Sin[beta[n] * l]) / (Cosh[beta[n] * l] + Cos[beta[n] * l])) *
    (Cosh[beta[n] * x] - Cos[beta[n] * x]);

(*Normalize Wn(x) for n=4*)
maxAbsWn4 = MaxValue[{Abs[Wn[x, 4]], 0 ≤ x ≤ l}, x];
NormalizedWn4[x_] := Wn[x, 4] / maxAbsWn4;
Print["Wn4_(Normalized)= ", NormalizedWn4[x]]

(*Plot*)
Plot[{newmode[[4]], NormalizedWn4[x]}, {x, 0, l}, PlotStyle →
  {{Blue, Thick, Thickness[0.005]}, {Directive[Red, Dashed, Thickness[0.005]]}},
 Frame → True, FrameLabel → {"x/l", "Wn4"}, PlotLegends →
  {"Ritz Method (Approximate)", "Analytical Method"}, PlotLabel → "Mode 4"]
```

Wn4_(Normalized)=

$$\frac{1}{2}\left(-\text{Sin}\left[\frac{7\pi x}{2}\right]-\left(-\text{Cos}\left[\frac{7\pi x}{2}\right]+\text{Cosh}\left[\frac{7\pi x}{2}\right]\right)\text{Sech}\left[\frac{7\pi}{2}\right]\left(-1+\text{Sinh}\left[\frac{7\pi}{2}\right]\right)+\text{Sinh}\left[\frac{7\pi x}{2}\right]\right)$$

Out[50]=



1.3 ANSYS Results

My Uploaded YouTube video Details:

Video Title – Modal Analysis of Cantilever Beam | ANSYS Workbench 2023

Video Link - https://youtu.be/f_rqLNfbnDM

Frequencies from ANSYS Results:

Table 1 Natural Frequencies of Cantilever beam from ANSYS Results

Sr	Mode	Natural Frequency Notation	ANSYS Results (In Hz)
1	1st Mode	f_1	24.899
2	2nd Mode	f_2	155.58
3	3rd Mode	f_3	433.63
4	4th Mode	f_4	844.18

Mode Shapes from ANSYS Results:

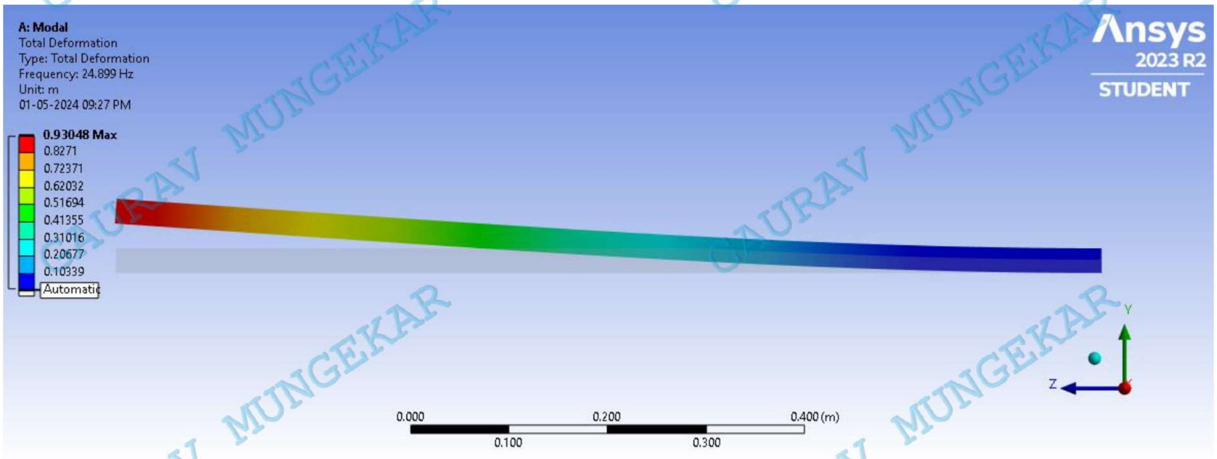


Figure 2 First Mode of Cantilever Beam (Frequency: 24.899 Hz)

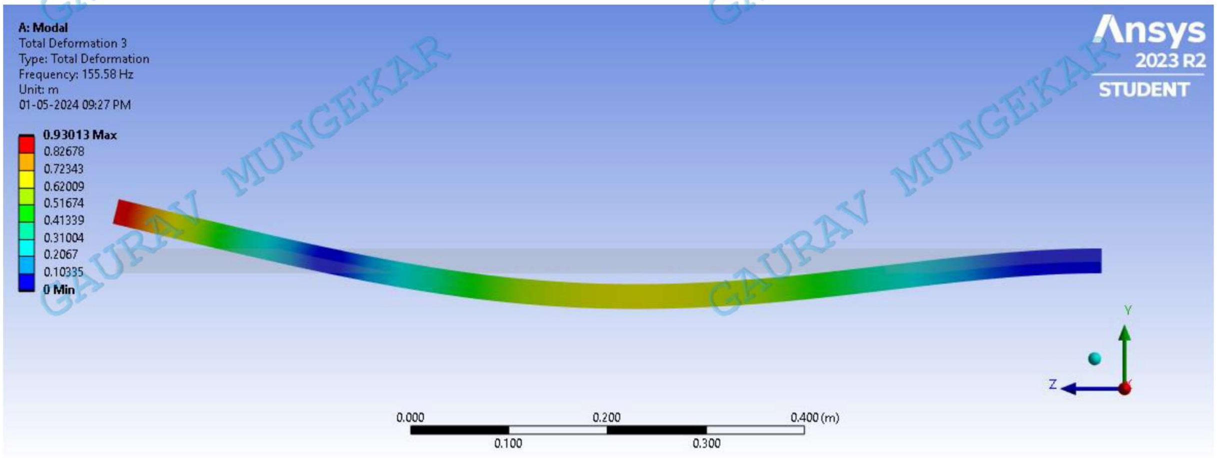


Figure 3 Second Mode of Cantilever Beam (Frequency: 155.58 Hz)

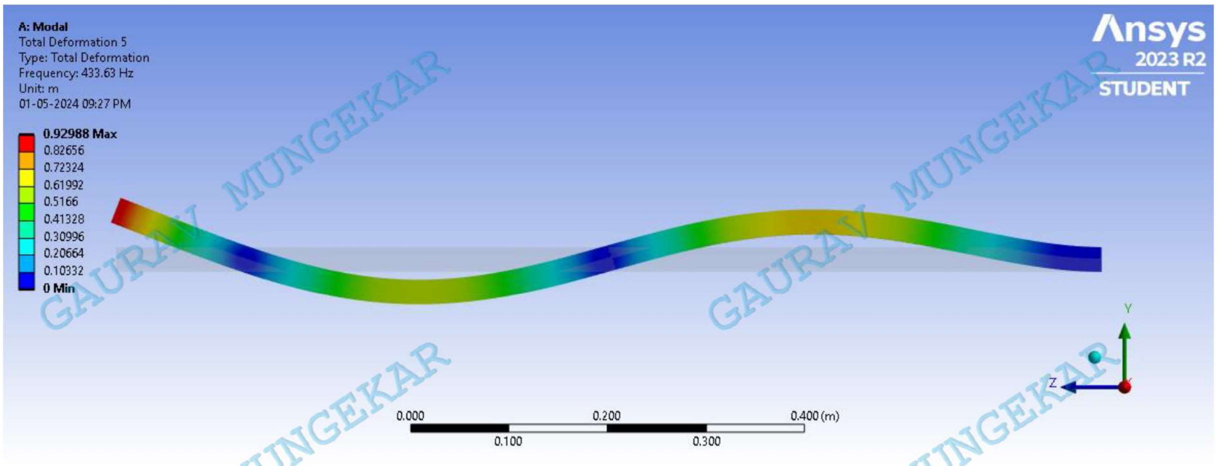


Figure 4 Third Mode of Cantilever Beam (Frequency: 433.63 Hz)

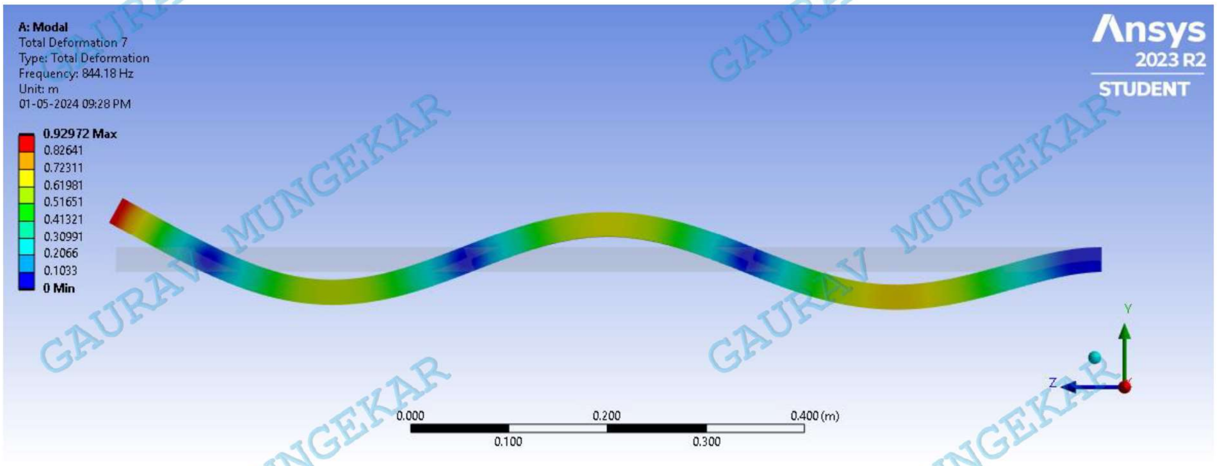


Figure 5 Fourth Mode of Cantilever Beam (Frequency: 844.18 Hz)

1.4 Observations

Comparison of results of first four natural frequencies (in Hz):

1) With Correction factor e_n

Table 2 Comparison of Natural Frequencies from Analytical Method (with e_n), Ritz Method, and ANSYS Results

Mode	Frequency (In Hz)	Analytical Method (With Correction factor e_n)	Ritz Method (Approximate)	ANSYS Results	Error %
1st	f_1	24.8389	24.8418	24.899	0.241959
2nd	f_2	155.7007	155.681	155.58	0.077521
3rd	f_3	435.9362	435.911	433.63	0.529022
4th	f_4	854.2176	854.228	844.18	1.176281

2) Without Correction factor e_n

Table 3 Comparison of Natural Frequencies from Analytical Method (without e_n), Ritz Method, and ANSYS Results

Mode	Frequency (In Hz)	Analytical Method (Without Correction factor e_n)	Ritz Method (Approximate)	ANSYS Results	Error %
1st	f_1	17.433	24.8418	24.899	42.8268
2nd	f_2	156.8971	155.681	155.58	0.8394
3rd	f_3	435.8253	435.911	433.63	0.5233
4th	f_4	854.2176	854.228	844.18	1.1762

1.5 Conclusions

1) Besides in-plane (transverse) bending modes, other modes such as out-of-plane (transverse) bending and twisting modes are observed. **Filtering out the required bending modes** from ANSYS results is essential.
2) In Mathematica, the initial 5 modes closely match the analytical results using a 10-term approximation by the Ritz method. However, discrepancies emerge in the later modes.
3) **Impact of Correction Factor e_n** : When utilizing the analytical method with the correction factor e_n, the results closely align with the Ritz method and ANSYS results, with minimal error percentages ranging from 0.0775% to 1.1763%. In contrast, when e_n is omitted, higher errors are encountered, particularly noticeable at the initial eigenfrequencies.

2. Vibration of Plate

Problem Statement

1. Analytically derive the first six natural frequencies and mode shapes of a uniform Kirchhoff rectangular plate with simply supported edges undergoing transverse vibration. (Dimensions given below)
2. Determine the first six natural frequencies and mode shapes of the same plate using Ansys.

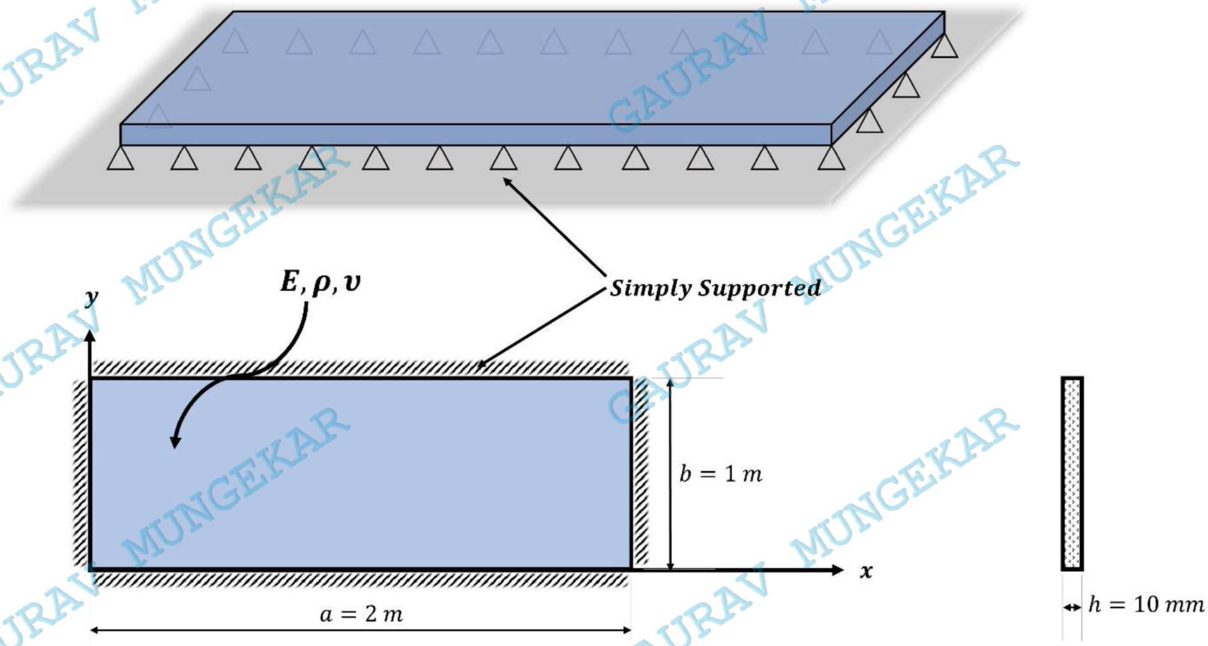


Figure 6 Simply-Supported Rectangular Plate

Given data:

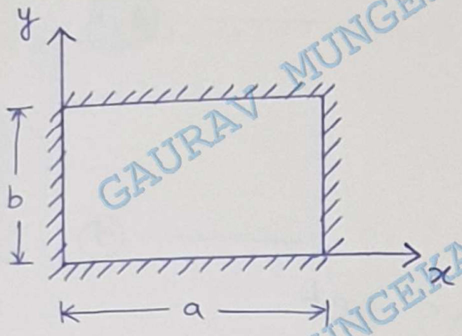
Young's Modulus of plate material, E = 140 GPa

Density of plate material, ρ = 3700 kg/m^3

Poisson's ratio of plate, υ = 0.33

Plate Dimensions: 2 m x 1 m x 10 mm (length (a), breadth (b), height (h))

Analytical Solution of uniform Kirchoff rectangular plate with simply supported edges undergoing transverse vibration



Step 1 :- Derivation of GDE solution

EOM : $\rho h w_{,tt} + D\nabla^4 w = 0$ —— ①

$$\ldots D = \frac{Eh^3}{12(1-\nu^2)}$$

Boundary Conditions :

$$\left.\begin{aligned} w|_{x=0,a} &= 0 \\ w|_{y=0,b} &= 0 \\ w_{,xx}|_{x=0,a} &= 0 \\ w_{,yy}|_{y=0,b} &= 0 \end{aligned}\right\} \quad \text{—— ②}$$

Assumed Solution : $w(x,y,t) = W(x,y)\, e^{i\omega t}$ —— ③

... Variable Separable Type
... $W(x,y)$ = Unknown Function
ω = Circular frequency

$$w_{,tt} = -\omega^2 W e^{i\omega t}$$

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}$$

$$= \left[\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4}\right] e^{i\omega t}$$

$$= (\nabla^4 W)\, e^{i\omega t}$$

Substituting these values in EOM we get,

$$-\rho h \omega^2 W \cancel{e^{i\omega t}} + D(\nabla^4 W)\cancel{e^{i\omega t}} = 0$$

$$\Rightarrow \quad -\rho h \omega^2 W + D\nabla^4 W = 0 \qquad \text{——— (4)}$$

$$\Rightarrow \quad \nabla^4 W - \frac{\rho h \omega^2}{D} W = 0$$

Assume $\gamma^4 = \dfrac{\rho h \omega^2}{D}$ ——— (4.1)

$$\Rightarrow \quad \nabla^4 W - \gamma^4 W = 0$$

$$\Rightarrow \quad (\nabla^4 - \gamma^4) W = 0 \qquad \text{——— (5)}$$

By factorizing the operator, we get

$$\Rightarrow \quad (\nabla^2 + \gamma^2)(\nabla^2 - \gamma^2) W = 0 \qquad \text{——— (6)}$$

Assume two functions $W_1(x,y)$ and $W_2(x,y)$ such that

$$(\nabla^2 + \gamma^2) W_1 = 0 \qquad \text{——— (7)}$$

$$(\nabla^2 - \gamma^2) W_2 = 0 \qquad \text{——— (8)}$$

Since operators $(\nabla^2 + \gamma^2)$ and $(\nabla^2 - \gamma^2)$ commute, so

$$W(x,y) = W_1(x,y) + W_2(x,y)$$ ... Valid for special cases ——— (9)

Note that above is the solution eqⁿ (5).

But converse is not true because there are many solutions of (5) which can't be written in form of (9)

Solution 1: $W_1(x,y)$:

EOM: $(\nabla^2 + \gamma^2) W_1 = 0$ → Helmholtz Equation

BCs:-
$$\left.\begin{aligned} W_1\big|_{x=0,a} &= 0 \\ W_1\big|_{y=0,b} &= 0 \\ W_{1,xx}\big|_{x=0,a} &= 0 \\ W_{1,yy}\big|_{y=0,b} &= 0 \end{aligned}\right\} \qquad \text{——— (10)}$$

Assumed Solution :- $W_1(x,y) = X_1(x) \cdot Y_1(y)$ ——— (11)

$$\nabla^2 W_1 = \frac{\partial^2 W_1}{\partial x^2} + \frac{\partial^2 W_1}{\partial y^2}$$

$$= W_{1,xx} + W_{1,yy}$$

$$= X_1'' Y_1 + X_1 \ddot{Y}_1 \qquad \cdots \quad (\;)'' = \frac{\partial^2(\;)}{\partial x^2}$$

$$\ddot{(\;)} = \frac{\partial^2(\;)}{\partial y^2}$$

EOM becomes,

$$\Rightarrow \quad X_1'' Y_1 + X_1 \ddot{Y}_1 + \gamma^2 X_1 Y_1 = 0$$

Divide both sides by $X_1 Y_1$,

$$\Rightarrow \quad \frac{X_1''}{X_1} + \frac{\ddot{Y}_1}{Y_1} + \gamma^2 = 0 \qquad \text{—— (12)}$$

Rewriting Assumed solution, $W_1(x,y) = e^{i(\alpha x + \beta y)}$

$$= X_1(x) \cdot Y_1(y)$$

So, $X_1 = e^{i\alpha x}$, $X_1'' = (i\alpha)^2 e^{i\alpha x} = -\alpha^2 e^{i\alpha x}$

$Y_1 = e^{i\beta y}$, $\ddot{Y}_1 = (i\beta)^2 e^{i\beta y} = -\beta^2 e^{i\beta y}$

$$\frac{X_1''}{X_1} = -\alpha^2 , \quad \frac{\ddot{Y}_1}{Y_1} = -\beta^2 \qquad \text{—— (13)}$$

Substituting these values,

$$\Rightarrow \quad -\alpha^2 - \beta^2 + \gamma^2 = 0$$

$$\Rightarrow \quad \alpha^2 + \beta^2 = \gamma^2 \qquad \text{—— (14)}$$

So, we can write final solution as

$$\begin{aligned} W_1(x,y) &= e^{i(\alpha x + \beta y)} \\ &= e^{i\alpha x} \cdot e^{i\beta y} \\ &= (B_1 \cos\alpha x + B_2 \sin\alpha x)(B_3 \cos\beta y + B_4 \sin\beta y) \\ &= A_4 \cos\alpha x \cdot \cos\beta y + A_3 \cos\alpha x \cdot \sin\beta y \\ &\quad + A_2 \sin\alpha x \cdot \cos\beta y + A_1 \sin\alpha x \cdot \sin\beta y \end{aligned} \qquad \text{—— (15)}$$

Solution 2 : $W_2(x,y)$

EOM : $(\nabla^2 - \gamma^2) W_2 = 0$

Assumed Solution :- $W_2(x,y) = X_2(x) \cdot Y_2(y)$ —— (16)

$$\nabla^2 W_2 = W_{2,xx} + W_{2,yy}$$

$$= X_2'' Y_2 + X_2 \ddot{Y}_2$$

So, EOM becomes,

$$W_{2,xx} + W_{2,yy} - \gamma^2 W_2 = 0$$

Substituting solution in above equation,

$$\Rightarrow \quad X_2'' Y_2 + X_2 \ddot{Y}_2 - \gamma^2 X_2 Y_2 = 0$$

Divide both sides by $X_2 Y_2$,

$$\Rightarrow \quad \frac{X_2''}{X_2} + \frac{\ddot{Y}_2}{Y_2} - \gamma^2 = 0 \qquad \text{——} (17)$$

$$\Rightarrow \quad \frac{1}{X_2}\frac{d^2X_2}{dx^2} + \frac{1}{Y_2}\frac{d^2Y_2}{dy^2} - \gamma^2 = 0$$

It is evident that a non-trivial solution exists if and only if,

$$\frac{1}{X_2}\frac{d^2X_2}{dx^2} = \bar{\alpha}^2 \quad \text{and} \quad \frac{1}{Y_2}\frac{d^2Y_2}{dy^2} = \bar{\beta}^2$$

$$\frac{d^2X_2}{dx^2} - \bar{\alpha}^2 X_2 = 0 \quad \text{and} \quad \frac{d^2Y_2}{dy^2} - \bar{\beta}^2 Y_2 = 0$$

Rewriting Assumed solution,
$W_2(x,y) = e^{\bar{\alpha}x + \bar{\beta}y}$
$= X_2(x) \cdot Y_2(y)$
$X_2 = e^{\bar{\alpha}x}$, $Y_2 = e^{\bar{\beta}y}$
$X_2'' = \bar{\alpha}^2 e^{\bar{\alpha}x}$, $\ddot{Y}_2 = \bar{\beta}^2 e^{\bar{\beta}y}$

where $\bar{\alpha}$ & $\bar{\beta}$ are constants.

$$\Rightarrow \quad \bar{\alpha}^2 + \bar{\beta}^2 - \gamma^2 = 0$$

$$\Rightarrow \quad \bar{\alpha}^2 + \bar{\beta}^2 = \gamma^2 \qquad \text{——} (18)$$

So, we can write final solution as

$$\begin{aligned} W_2(x,y) &= e^{\bar{\alpha}x + \bar{\beta}y} \\ &= e^{\bar{\alpha}x} \cdot e^{\bar{\beta}y} \\ &= (C_1 \sinh\bar{\alpha}x + C_2\cosh\bar{\alpha}x)(C_3\sinh\bar{\beta}y + C_4\cosh\bar{\beta}y) \\ &= A_5 \sinh\bar{\alpha}x \cdot \sinh\bar{\beta}y + A_6 \sinh\bar{\alpha}x \cdot \cosh\bar{\beta}y \\ &\quad + A_7\cosh\bar{\alpha}x \cdot \sinh\bar{\beta}y + A_8\cosh\bar{\alpha}x \cdot \cosh\bar{\beta}y \qquad \text{——} (19) \end{aligned}$$

Solution $W(x,y)$

$$\begin{aligned} W(x,y) &= W_1(x,y) + W_2(x,y) \\ &= A_1 \sin\alpha x \cdot \sin\beta y + A_2 \sin\alpha x \cdot \cos\beta y \\ &\quad + A_3\cos\alpha x \cdot \sin\beta y + A_4\cos\alpha x \cdot \cos\beta y \\ &\quad + A_5\sinh\bar{\alpha}x \cdot \sinh\bar{\beta}y + A_6\sinh\bar{\alpha}x \cdot \cosh\bar{\beta}y \\ &\quad + A_7\cosh\bar{\alpha}x \cdot \sinh\bar{\beta}y + A_8\cosh\bar{\alpha}x \cdot \cosh\bar{\beta}y \end{aligned}$$

$$\text{——} (20)$$

Step 2:- Use simply-supported plate boundary conditions

From (2) and (3),

$$W|_{x=0,a} = 0 \quad \text{——— (21)}$$

$$W|_{y=0,b} = 0 \quad \text{——— (22)}$$

$$W_{,xx}|_{x=0,a} = 0 \quad \text{——— (23)}$$

$$W_{,yy}|_{y=0,b} = 0 \quad \text{——— (24)}$$

- Using (21) in (20) at $x=0$,

$$W(0,y) = 0 + 0 + A_3 \sin\beta y + A_4 \cos\beta y + 0 + 0 + A_7 \sinh\bar{\beta}y + A_8 \cosh\bar{\beta}y = 0$$

$$A_3 \sin\beta y + A_4 \cos\beta y + A_7 \sinh\bar{\beta}y + A_8 \cosh\bar{\beta}y = 0 \quad \text{—(25)}$$

- Using (22) in (20) at $y=0$,

$$W(x,0) = 0 + A_2 \sin\alpha x + 0 + A_4 \cos\alpha x + 0 + A_6 \sinh\bar{\alpha}x + 0 + A_8 \cosh\bar{\alpha}x = 0$$

$$A_2 \sin\alpha x + A_4 \cos\alpha x + A_6 \sinh\bar{\alpha}x + A_8 \cosh\bar{\alpha}x = 0 \quad \text{—(26)}$$

- Using (23) in (20) at $x=0$,

$$W_{,xx} = -A_1\alpha^2 \sin\alpha x \cdot \sin\beta y - A_2\alpha^2 \sin\alpha x \cdot \cos\beta y - A_3\alpha^2 \cos\alpha x \cdot \sin\beta y - A_4\alpha^2 \cos\alpha x \cdot \cos\beta y + A_5\bar{\alpha}^2 \sinh\bar{\alpha}x \cdot \sinh\bar{\beta}y + A_6\bar{\alpha}^2 \sinh\bar{\alpha}x \cdot \cosh\bar{\beta}y + A_7\bar{\alpha}^2 \cosh\bar{\alpha}x \cdot \sinh\bar{\beta}y + A_8\bar{\alpha}^2 \cosh\bar{\alpha}x \cdot \cosh\bar{\beta}y$$

At $x=0$, $W_{,xx}=0$,

$$0 = 0 + 0 - A_3\alpha^2 \sin\beta y - A_4\alpha^2 \cos\beta y + 0 + 0 + A_7\bar{\alpha}^2 \sinh\bar{\beta}y + A_8\bar{\alpha}^2 \cosh\bar{\beta}y$$

$$0 = -\alpha^2 \sin\beta y\, A_3 - \alpha^2 \cos\beta y\, A_4 + \bar{\alpha}^2 A_7 \sinh\bar{\beta}y + \bar{\alpha}^2 A_8 \cosh\bar{\beta}y \quad \text{—(27)}$$

- Using (24) in (20) at $y=0$

$$-\beta^2 \sin\alpha x\, A_2 - \beta^2 \cos\alpha x\, A_4 + \bar{\beta}^2 \sinh\bar{\alpha}x\, A_6 + \bar{\beta}^2 \cosh\bar{\alpha}x\, A_8 = 0 \quad \text{—(28)}$$

Using (21) in (20) at $x = a$,

$$A_1 \sin\alpha a \cdot \sin\beta y + A_2 \sin\alpha a \cdot \cos\beta y + A_3 \cos\alpha a \cdot \sin\beta y$$
$$+ A_4 \cos\alpha a \cdot \cos\beta y + A_5 \sinh\bar{\alpha} a \cdot \sinh\bar{\beta} y + A_6 \sinh\bar{\alpha} a \cdot \cosh\bar{\beta} y$$
$$+ A_7 \cosh\bar{\alpha} a \cdot \sinh\bar{\beta} y + A_8 \cosh\bar{\alpha} a \cdot \cosh\bar{\beta} y = 0 \quad \text{—— (29)}$$

Using (22) in (20) at $y = b$,

$$A_1 \sin\alpha x \cdot \sin\beta b + A_2 \sin\alpha x \cos\beta b + A_3 \cos\alpha x \cdot \sin\beta b$$
$$+ A_4 \cos\alpha x \cdot \cos\beta b + A_5 \sinh\bar{\alpha} x \cdot \sinh\bar{\beta} b + A_6 \sinh\bar{\alpha} x \cdot \cosh\bar{\beta} b$$
$$+ A_7 \cosh\bar{\alpha} x \cdot \sinh\bar{\beta} b + A_8 \cosh\bar{\alpha} x \cdot \cosh\bar{\beta} b = 0 \quad \text{—— (30)}$$

Using (23) in (20) at $x = a$,

$$-A_1 \alpha^2 \sin\alpha a \cdot \sin\beta y - A_2 \alpha^2 \sin\alpha a \cdot \cos\beta y - A_3 \alpha^2 \cos\alpha a \cdot \sin\beta y$$
$$- A_4 \alpha^2 \cos\alpha a \cdot \cos\beta y + A_5 \bar{\alpha}^2 \sinh\bar{\alpha} a \cdot \sinh\bar{\beta} y + A_6 \bar{\alpha}^2 \sinh\bar{\alpha} a \cdot \cosh\bar{\beta} y$$
$$+ A_7 \bar{\alpha}^2 \cosh\bar{\alpha} a \cdot \sinh\bar{\beta} y + A_8 \bar{\alpha}^2 \cosh\bar{\alpha} a \cdot \cosh\bar{\beta} y = 0 \quad \text{—— (31)}$$

Using (24) in (20) at $y = b$,

$$-A_1 \beta^2 \sin\alpha x \cdot \sin\beta b - A_2 \beta^2 \sin\alpha x \cdot \cos\beta b - A_3 \beta^2 \cos\alpha x \cdot \sin\beta b$$
$$- A_4 \beta^2 \cos\alpha x \cdot \cos\beta b + A_5 \bar{\beta}^2 \sinh\bar{\alpha} x \cdot \sinh\bar{\beta} b + A_6 \bar{\beta}^2 \sinh\bar{\alpha} x \cdot \cosh\bar{\beta} b$$
$$+ A_7 \bar{\beta}^2 \cosh\bar{\alpha} x \cdot \sinh\bar{\beta} b + A_8 \bar{\beta}^2 \cosh\bar{\alpha} x \cdot \cosh\bar{\beta} b = 0 \quad \text{—— (32)}$$

From (25) & (27),

$$A_3 \sin\beta y + A_4 \cos\beta y + A_7 \sinh\bar{\beta} y + A_8 \cosh\bar{\beta} y = 0 \quad \text{— by (25)}$$

$$-A_3 \sin\beta y - A_4 \cos\beta y + \frac{\bar{\alpha}^2}{\alpha^2} A_7 \sinh\bar{\beta} y + \frac{\bar{\alpha}^2}{\alpha^2} A_8 \cosh\bar{\beta} y = 0 \quad \text{— Divide (27) by } \alpha^2$$

Adding above two equations,

$$\cancel{\left(1 + \frac{\bar{\alpha}^2}{\alpha^2}\right)} \sinh\bar{\beta} y \, A_7 + \cancel{\left(1 + \frac{\bar{\alpha}^2}{\alpha^2}\right)} \cosh\bar{\beta} y \, A_8 = 0$$

$$(\sinh\bar{\beta} y) A_7 + (\cosh\bar{\beta} y) A_8 = 0 \quad \text{—— (33)}$$

Similarly, from (26) & (28),

$$A_2 \sin\alpha x + A_4 \cos\alpha x + A_6 \sinh\bar{\alpha} x + A_8 \cosh\bar{\alpha} x = 0 \quad \text{— by (26)}$$

$$-\sin\alpha x\, A_2 - \cos\alpha x\, A_4 + \frac{\bar\beta^2}{\beta^2}\sinh\bar\alpha x\, A_6 + \frac{\bar\beta^2}{\beta^2}\cosh\bar\alpha x\, A_8 = 0$$

~~Divide~~ (28) by β^2

Adding above two equations,

$$\cancel{\left(1+\frac{\bar\beta^2}{\beta^2}\right)}\sinh\bar\alpha x\, A_6 + \cancel{\left(1+\frac{\bar\beta^2}{\beta^2}\right)}\cosh\bar\alpha x\, A_8 = 0$$

$$(\sinh\bar\alpha x)\, A_6 + (\cosh\bar\alpha x)\, A_8 = 0 \quad \text{—— (34)}$$

Similarly, from (29) and (31),

$$(\sinh\bar\alpha a . \sinh\bar\beta y)\, A_5 + (\sinh\bar\alpha a . \cosh\bar\beta y)\, A_6 + (\cosh\bar\alpha a . \sinh\bar\beta y)\, A_7$$
$$+ (\cosh\bar\alpha a . \cosh\bar\beta y)\, A_8 = 0 \quad \text{—— (35)}$$

$$\Rightarrow (\sinh\bar\alpha a . \sinh\bar\beta y)\, A_5 + (\sinh\bar\alpha a . \cosh\bar\beta y)\, A_6$$
$$+ (\cosh\bar\alpha a . \cancel{\sinh\bar\beta y}) \times \left(\frac{\cosh\bar\beta y}{\cancel{\sinh\bar\beta y}}\right) A_8 + (\cosh\bar\alpha a . \cos\bar\beta y)\, A_8 = 0$$

... from (33)

$$\Rightarrow (\sinh\bar\alpha a . \sinh\bar\beta y)\, A_5 + (\sinh\bar\alpha a . \cosh\bar\beta y)\, A_6 = 0 \quad \text{—— (36)}$$

Similarly from (30) and (32),

$$(\sinh\bar\alpha x . \sinh\bar\beta b)\, A_5 + (\sinh\bar\alpha x . \cosh\bar\beta b)\, A_6 +$$
$$(\cosh\bar\alpha x . \sinh\bar\beta b)\, A_7 + (\cosh\bar\alpha x . \cosh\bar\beta b)\, A_8 = 0 \quad \text{—— (37)}$$

$$\Rightarrow (\sinh\bar\alpha x . \sinh\bar\beta b)\, A_5 + (\cancel{\sinh\bar\alpha x} . \cosh\bar\beta b)\left(\frac{-\cosh\bar\alpha x}{\cancel{\sinh\bar\alpha x}}\right) A_8$$
$$+ (\cosh\bar\alpha x . \sinh\bar\beta b)\, A_7 + (\cosh\bar\alpha x . \cosh\bar\beta b)\, A_8 = 0$$

$$\Rightarrow (\sinh\bar\alpha x . \sinh\bar\beta b)\, A_5 + (\cosh\bar\alpha x . \sinh\bar\beta b)\, A_7 = 0 \quad \text{—— (38)}$$

Equation (33), (34), (36) and (38) are four equations with four unknowns A_5, A_6, A_7, A_8.

So, clearly to satisfy all equations,

$$A_5 = A_6 = A_7 = A_8 = 0$$

Substitute A_7 & A_8 equal to zero in (25) & (26), we get two equations with two unknowns A_2, A_4.

$$\therefore A_2 = A_4 = 0$$

Put $A_2 = A_4 = A_6 = A_8 = 0$ in (28), it satisfies equation.

Similarly, put $A_4 = A_7 = A_8 = 0$ in (27), we get

$$A_3 \neq 0.$$

From all above,

$$A_2 = A_3 = A_4 = A_5 = A_6 = A_7 = A_8 = 0 \quad \text{——— (39)}$$

Note that putting all above values from (39) in (29),

$$A_1 \sin \alpha a \cdot \sin \beta y = 0$$

clearly $A_1 \neq 0$ & $\sin \beta y \neq 0$ (For non-trivial solution)

$$\therefore \sin \alpha a = 0$$

$$\therefore \sin \alpha_m a = m\pi \quad \text{where } m = 0, 1, 2, \dots, \infty$$

$$\boxed{\alpha_m = \frac{m\pi}{a}} \quad \text{——— (40)}$$

Put all values from (39) in (30),

$$A_1 \sin \alpha x \cdot \sin \beta b = 0$$

Clearly $A_1 \neq 0$ & $\sin \alpha x \neq 0$ (For non-trivial solution)

$$\therefore \sin \beta b = 0$$

$$\therefore \beta_n b = n\pi \quad \text{where } n = 0, 1, 2, \dots, \infty$$

$$\boxed{\beta_n = \frac{n\pi}{b}} \quad \text{——— (41)}$$

so, final solution becomes,

$$W(x, y) = A_1 \sin \alpha x \sin \beta y \quad \text{——— (42)}$$

$$W_{mn}(x, y) = A_1 \sin \alpha_m x \sin \beta_n y$$

$$= A_1 \sin\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi y}{b}\right)$$

For simplest solution, take $A_1 = 1$

$$\therefore \boxed{W_{mn} = \sin\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi y}{b}\right)} \quad \text{——— (43)}$$

... Mode shape / Eigenfunction

Step 3 :- Calculation of Natural frequencies

From eq^n (41) & (14),

$$\omega^2 = \gamma^4 \times \frac{D}{\rho h} \quad \text{where } \gamma^2 = \alpha^2 + \beta^2$$

$$\Rightarrow \omega_{mn} = \gamma_{mn}^2 \times \sqrt{\frac{D}{\rho h}}$$

$$\Rightarrow \omega_{mn} = \left(\alpha_m^2 + \beta_n^2\right)\sqrt{\frac{D}{\rho h}}$$

$$\boxed{\Rightarrow \omega_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\sqrt{\frac{D}{\rho h}}} \quad \text{— using (40), (41)} \quad \text{— (44)}$$

Given :-
$a = 2\ m$
$b = 1\ m$
$h = 10\ mm = 0.01\ m$
$E = 140\ GPa = 140 \times 10^9\ Pa$
$\nu = 0.33$
$\rho = 3700\ kg/m^3$

To Find :- First six natural frequencies,

Soln :- $D = \dfrac{Eh^3}{12(1-\nu^2)} = 13092.43257\ N \cdot m$

Sr.	m	n	ω_{mn}	ω_{mn} value (rad/s)	f_{mn} value (Hz)
1	1	1	ω_{11}	232.0700	36.9351
2	2	1	ω_{21}	371.3120	59.0961
3	3	1	ω_{31}	603.3820	96.0312
4	1	2	ω_{12}	789.0381	125.5793
5	2	2	ω_{22}	928.2801	147.7403
	4	1	ω_{41}	928.2801	147.7403
6	3	2	ω_{32}	1160.3501	184.6754

2.2 Mathematica Plots

```
In[1]:= Clear["Global`*"]

In[2]:= (*Parameters*)
        a = 2; b = 1; h = 0.01;
        Ep = 140 * 10^9; ν = 0.33; ρ = 3700;
        num = 7;

In[5]:= Dp = (Ep * (h^3)) / (12 * (1 - ν^2)); (*Material properties*)
        ω[m_, n_] = (π^2) * ((m / a)^2 + (n / b)^2) * Sqrt[Dp / (ρ * h)]; (* Frequency *)

In[7]:= (*Find Natural frequencies ω and sort them in Array*)
        omegaValues = {}; (*Initialize an empty list to store ω values*)
        For[i = 1, i ≤ num, i++,
         For[j = 1, j ≤ num, j++, AppendTo[omegaValues, {ω[i, j], i, j}];]]
        (*Iterate over m and n from 1 to 5*)
        sortedOmegaValues = SortBy[omegaValues, First];
        (*Sort omega values in ascending order*)
        sortedOmegaValues (*Display the sorted results*)

Out[10]=
        {{232.07, 1, 1}, {371.312, 2, 1}, {603.382, 3, 1}, {789.038, 1, 2},
         {928.28, 2, 2}, {928.28, 4, 1}, {1160.35, 3, 2}, {1346.01, 5, 1}, {1485.25, 4, 2},
         {1717.32, 1, 3}, {1856.56, 2, 3}, {1856.56, 6, 1}, {1902.97, 5, 2}, {2088.63, 3, 3},
         {2413.53, 4, 3}, {2413.53, 6, 2}, {2459.94, 7, 1}, {2831.25, 5, 3}, {3016.91, 1, 4},
         {3016.91, 7, 2}, {3156.15, 2, 4}, {3341.81, 6, 3}, {3388.22, 3, 4}, {3713.12, 4, 4},
         {3945.19, 7, 3}, {4130.85, 5, 4}, {4641.4, 6, 4}, {4687.81, 1, 5}, {4827.06, 2, 5},
         {5059.13, 3, 5}, {5244.78, 7, 4}, {5384.02, 4, 5}, {5801.75, 5, 5}, {6312.3, 6, 5},
         {6730.03, 1, 6}, {6869.27, 2, 6}, {6915.69, 7, 5}, {7101.34, 3, 6}, {7426.24, 4, 6},
         {7843.97, 5, 6}, {8354.52, 6, 6}, {8957.9, 7, 6}, {9143.56, 1, 7}, {9282.8, 2, 7},
         {9514.87, 3, 7}, {9839.77, 4, 7}, {10257.5, 5, 7}, {10768., 6, 7}, {11371.4, 7, 7}}

In[11]:= (*Display Natural frequencies*)
        For[i = 1, i ≤ num, i++,
         Print["ω", Subscript["(", sortedOmegaValues[[i, 2]], sortedOmegaValues[[i, 3]]],
          ") = ", sortedOmegaValues[[i, 1]], " rad/s"]]
        For[i = 1, i ≤ num, i++,
         Print["f", Subscript["(", sortedOmegaValues[[i, 2]], sortedOmegaValues[[i, 3]]],
          ") = ", sortedOmegaValues[[i, 1]] / (2 * π), " Hz"]]
```

$\omega_{(1,1)}$ = 232.07 rad/s

$\omega_{(2,1)}$ = 371.312 rad/s

$\omega_{(3,1)}$ = 603.382 rad/s

$\omega_{(1,2)}$ = 789.038 rad/s

$\omega_{(2,2)}$ = 928.28 rad/s

$\omega_{(4,1)}$ = 928.28 rad/s

$\omega_{(3,2)}$ = 1160.35 rad/s

$f_{(1,1)}$ = 36.9351 Hz

$f_{(2,1)}$ = 59.0961 Hz

$f_{(3,1)}$ = 96.0312 Hz

$f_{(1,2)}$ = 125.579 Hz

$f_{(2,2)}$ = 147.74 Hz

$f_{(4,1)}$ = 147.74 Hz

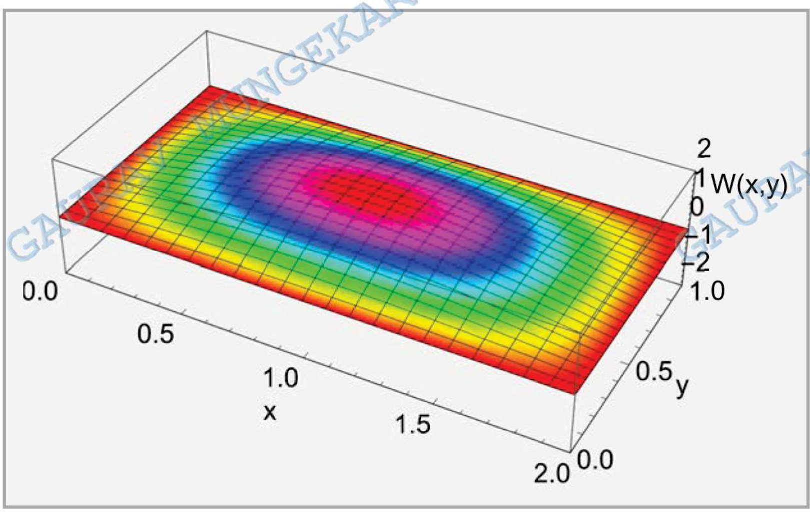
$f_{(3,2)}$ = 184.675 Hz

In[13]:=
```
(*Create Function to make plots*)
createModePlot[mv_, nv_] := Module[{modeshape, ω, mode, x, y},
  modeshape = Sin[(mv * π * x) / a] * Sin[(nv * π * y) / b]; (*Mode shape*)
  ω[m_, n_] := π^2 * ((m / a)^2 + (n / b)^2) * Sqrt[Dp / (ρ * h)];
   (*Frequency function*)
  Plot3D[modeshape, {x, 0, a}, {y, 0, b}, AxesLabel → {"x", "y", "W(x,y)"},
   PlotRange → {{0, a}, {0, b}, {-2, 2}}, BoxRatios → {a, b, 0.5}, Mesh → 21,
   AxesStyle → Larger, ImageSize → 400, ColorFunction → Hue]](*Return 3D plot*)
```

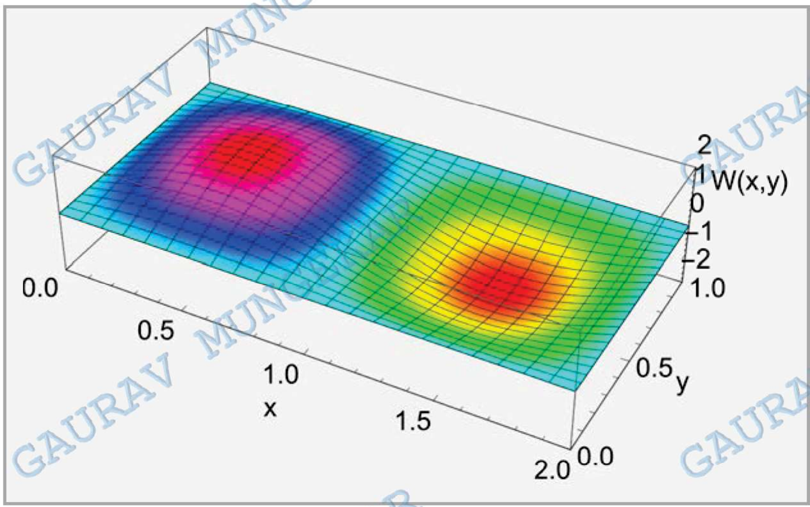
In[14]:=
```
(*Create Plots using for loop*)
plots = {};(*Initialize an empty list to store plots*)
For[i = 1, i ≤ num, i++, (*Loop over the first 6 sorted omega values*)AppendTo[
  plots, Panel[createModePlot[sortedOmegaValues⟦i, 2⟧, sortedOmegaValues⟦i, 3⟧],
   (*Wrap each plot in a panel*)Style[Row[{"Plot ", i, ": ",
      Subscript["ω", Row[{sortedOmegaValues⟦i, 2⟧, sortedOmegaValues⟦i, 3⟧}]],
      " = ", ToString[Round[sortedOmegaValues⟦i, 1⟧, 0.01]], " rad/s (",
      ToString[Round[sortedOmegaValues⟦i, 1⟧ / (2 π), 0.01]], " Hz)"}], Bold]]]]
(*Add a styled label with units for each plot*)
plots
```

Out[16]=

Plot 1: ω_{11} = 232.07 rad/s (36.94 Hz)

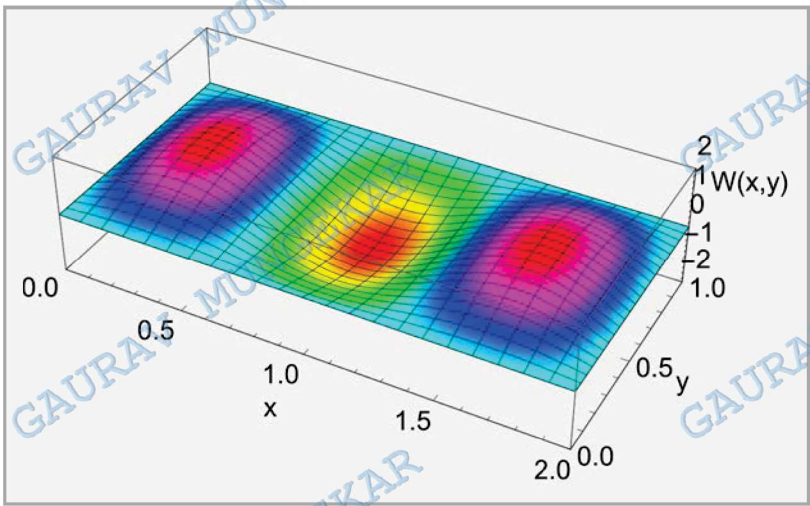
{  ,

Plot 2: ω_{21} = 371.31 rad/s (59.1 Hz)



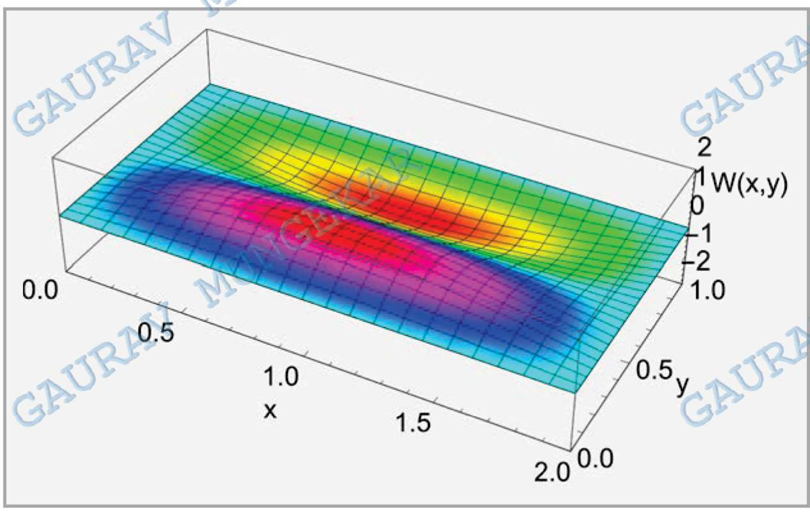
,

Plot 3: ω_{31} = 603.38 rad/s (96.03 Hz)



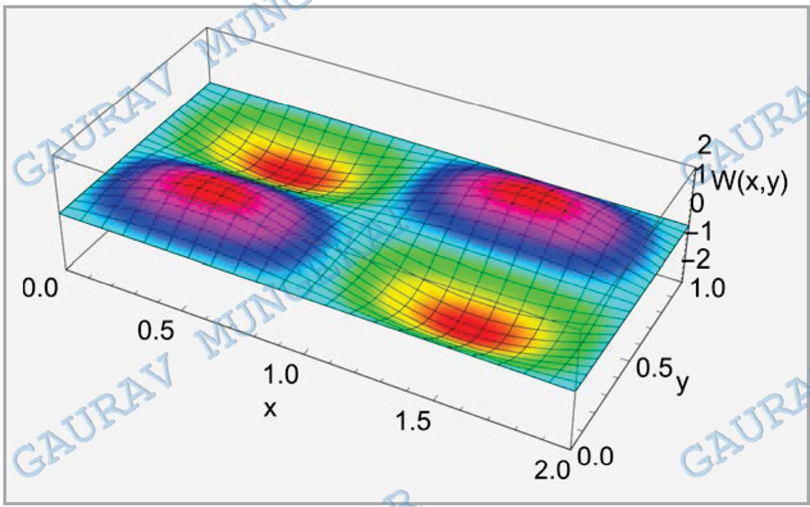
,

Plot 4: ω_{12} = 789.04 rad/s (125.58 Hz)



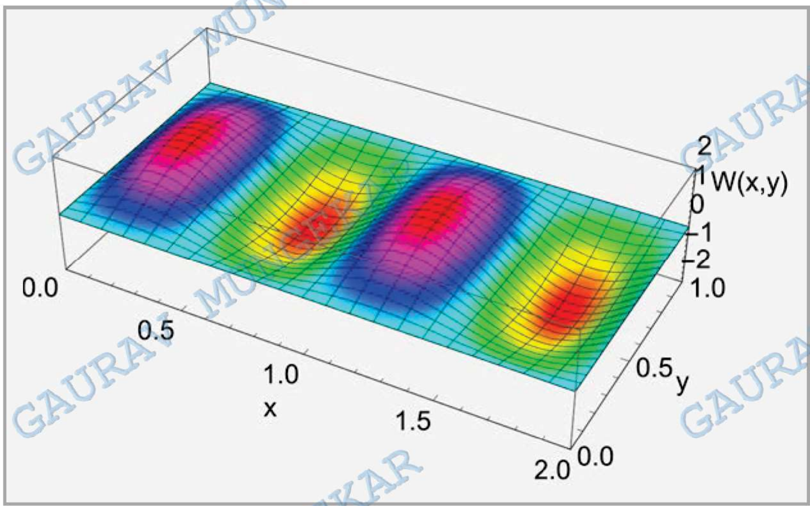
,

Plot 5: ω_{22} = 928.28 rad/s (147.74 Hz)



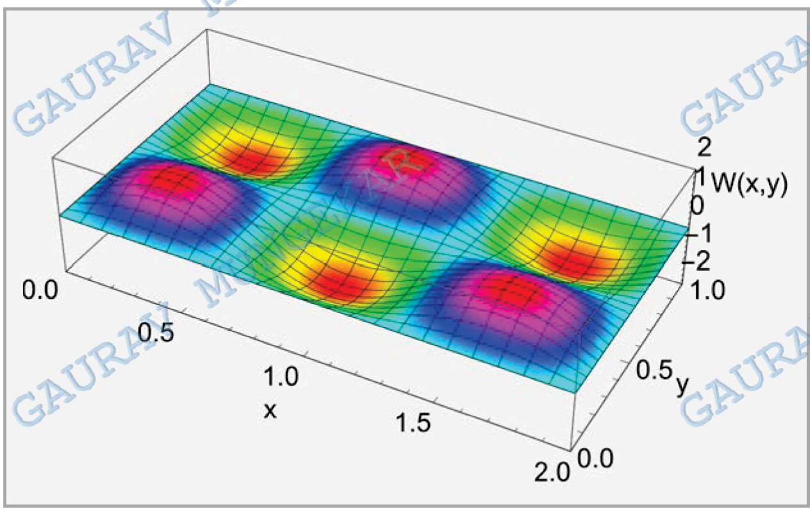
,

Plot 6: ω_{41} = 928.28 rad/s (147.74 Hz)



,

Plot 7: ω_{32} = 1160.35 rad/s (184.68 Hz)



}

2.3 ANSYS Results

<u>My Uploaded YouTube video Details:</u>

Video Title – Modal Analysis of Rectangular Plate | ANSYS Workbench 2023

Video Link - https://youtu.be/HzrzkBbOfjM

<u>Frequencies from ANSYS Results:</u>

Table 4 Natural Frequencies of Rectangular Plate from ANSYS Results

Sr	Mode	Frequency	ANSYS Results (In Hz)
1	1st Mode	f_{11}	37.002
2	2nd Mode	f_{21}	59.196
3	3rd Mode	f_{31}	96.426
4	4th Mode	f_{12}	126.92
5	5th Mode	f_{22}	149.02
	6th Mode	f_{41}	149.02
6	7th Mode	f_{32}	186.09

<u>Mode Shapes from ANSYS Results:</u>

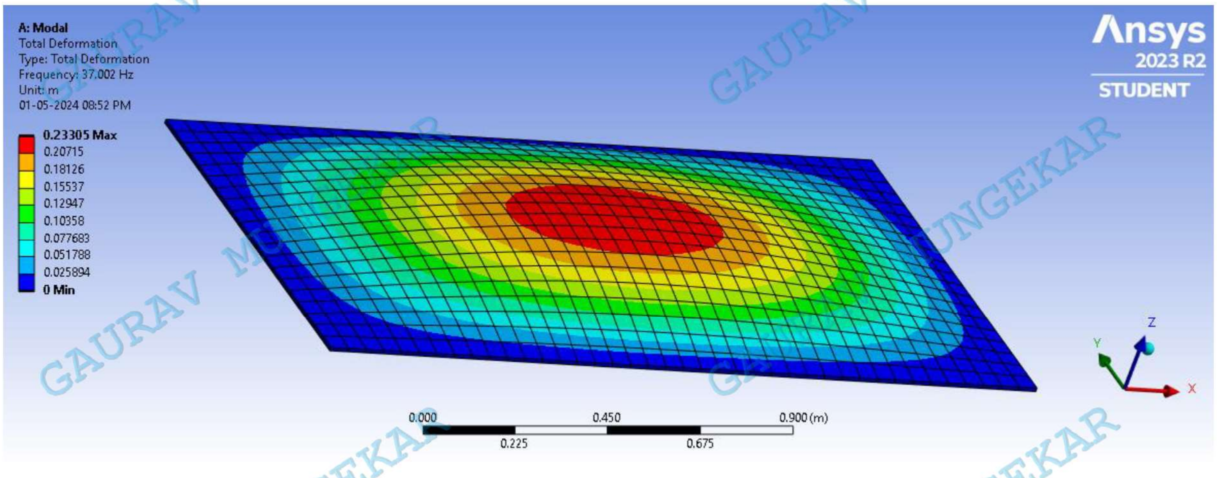


Figure 7 First Mode of Rectangular Plate (Frequency: 37.002 Hz)

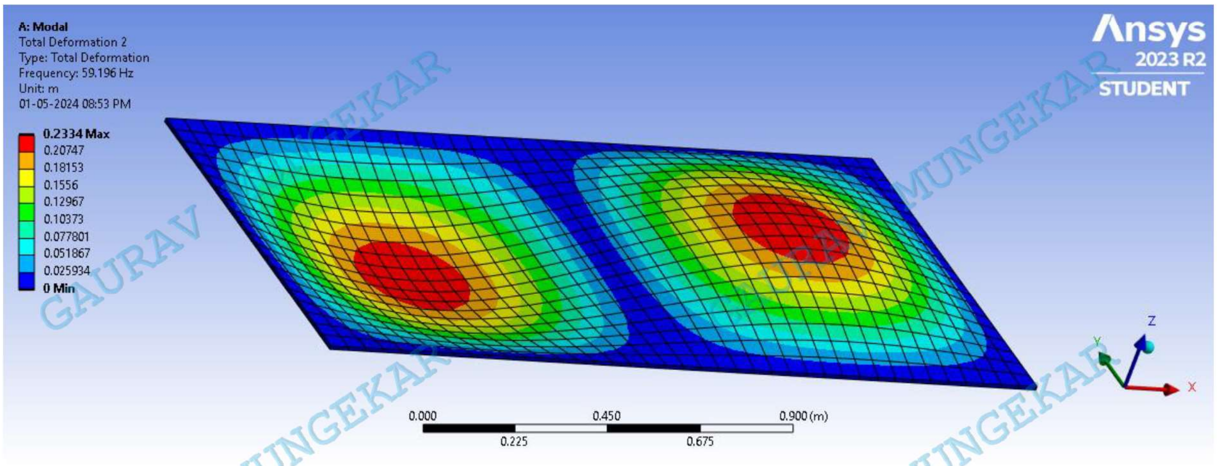


Figure 8 Second Mode of Rectangular Plate (Frequency: 59.196 Hz)

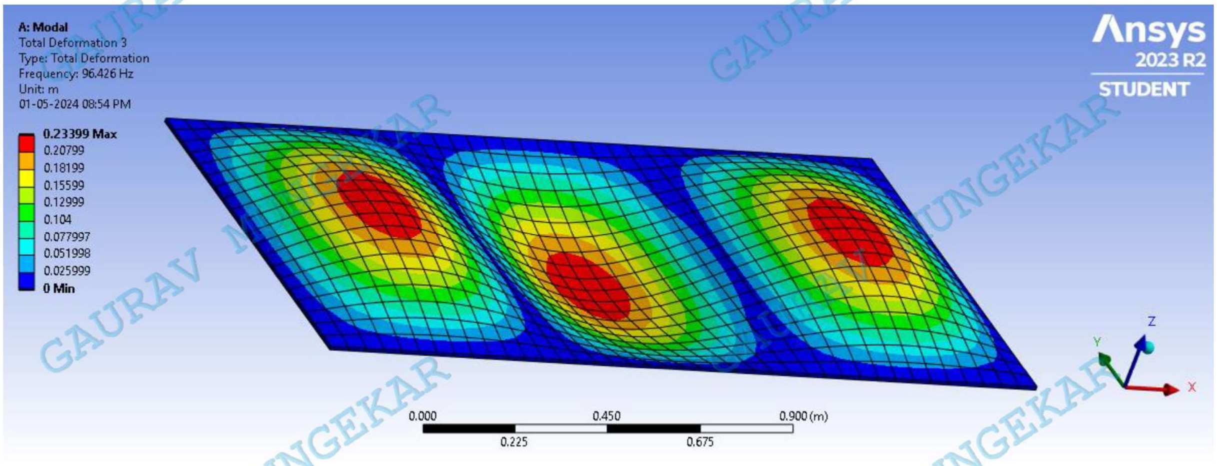


Figure 9 Third Mode of Rectangular Plate (Frequency: 96.426 Hz)

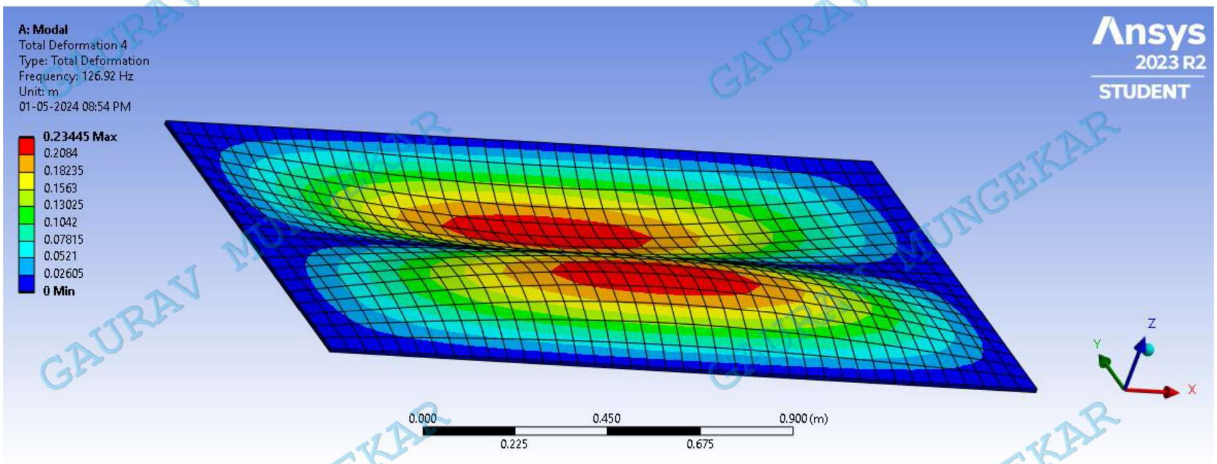


Figure 10 Fourth Mode of Rectangular Plate (Frequency: 126.92 Hz)

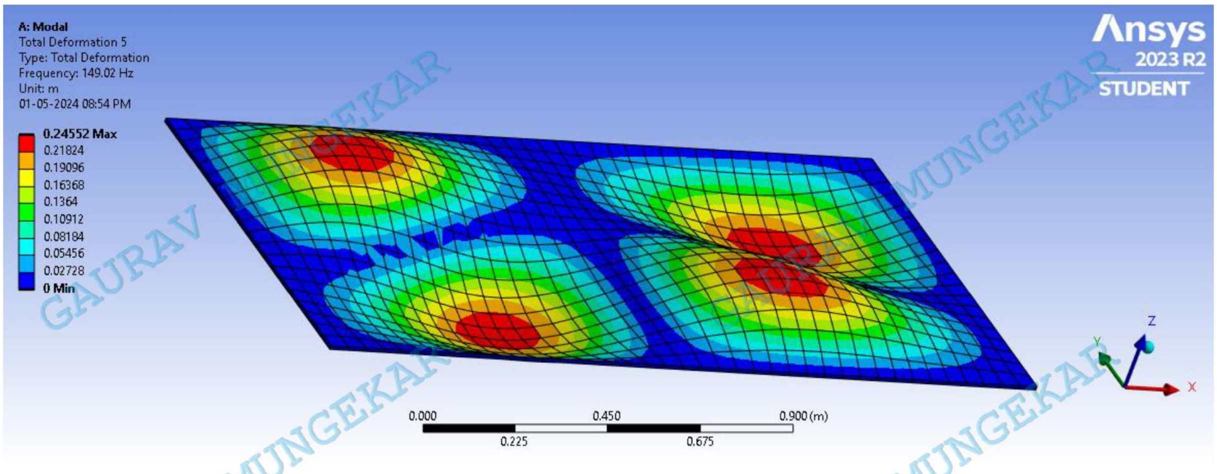


Figure 11 Fifth Mode of Rectangular Plate (Frequency: 149.02 Hz)

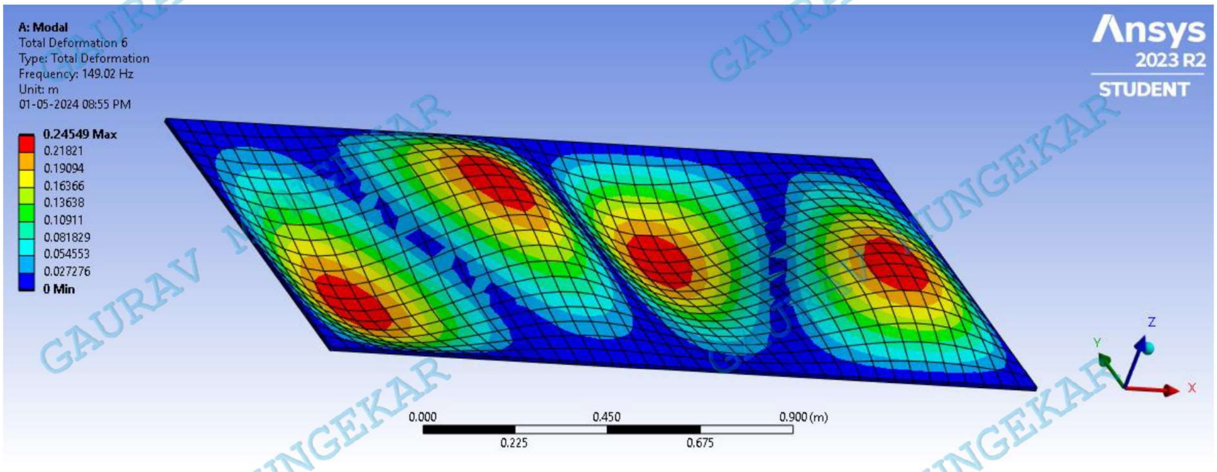


Figure 12 Sixth Mode of Rectangular Plate (Frequency: 149.02 Hz)

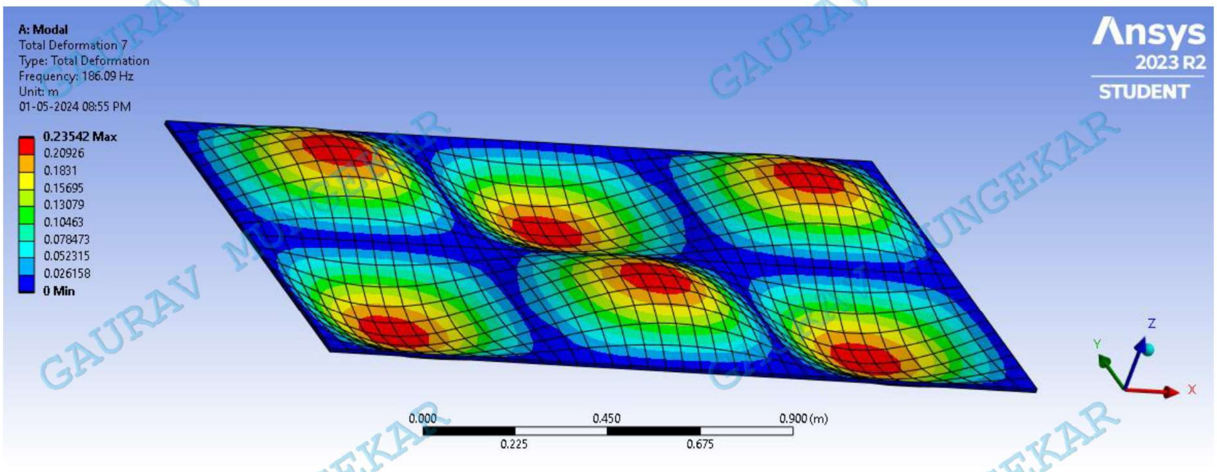


Figure 13 Seventh Mode of Rectangular Plate (Frequency: 186.09 Hz)

2.4 Observations

Comparison of results of first six natural frequencies (in Hz):

Table 5 Comparison of Natural Frequencies from Analytical Method and ANSYS Results

Sr	Mode	Frequency (In Hz)	Analytical Results	ANSYS Results	Error %
1	1st Mode	f_{11}	36.9351	37.002	0.1811
2	2nd Mode	f_{21}	59.0961	59.196	0.1690
3	3rd Mode	f_{31}	96.0312	96.426	0.4111
4	4th Mode	f_{12}	125.5793	126.92	1.0676
5	5th Mode	f_{22}	147.7403	149.02	0.8662
	6th Mode	f_{41}	147.7403	149.02	0.8662
6	7th Mode	f_{32}	184.6754	186.09	0.7660

2.5 Conclusions

1) Modes f_{22} and f_{41} share the exact same natural frequency of 147.7403 Hz, indicating degeneracy between mode pairs (2,2) and (4,1). Although they have the same eigenfrequency, they exhibit different eigenmodes based on mode shapes. So (2,2) and (4,1) are degenerate modes.
2) The comparison between analytical and ANSYS results reveals consistent trends, with error percentages ranging from 0.1690% to 1.0676%.
3) Both methods demonstrate accuracy in predicting frequencies, validating the reliability of the analytical approach.

References

[1] P. Hagedorn and A. DasGupta, Vibrations and Waves in Continuous Mechanical Systems. 2007. doi: https://doi.org/10.1002/9780470518434.

[2] “Modal Analysis of Cantilever Beam by ANSYS Workbench,” www.youtube.com. https://www.youtube.com/watch?v=NpVX-C2d2Ks (accessed May 01, 2024).

[3] “Modal Analysis of PLate in Ansys 19 (Natural Frequency, Resonance, Vibration),” www.youtube.com. https://youtu.be/4Ee6DNkHyc0?si=O9liGhboOmlQ9keP (accessed May 01, 2024).