Indian Institute of Technology Kharagpur
Department of Mechanical Engineering

Session: Spring, 2022-23 — **Mid-Semester Examination**
Subject: Vibration of Structures (ME60428) — **Number of students:** 34
Time: 2 hours — **Total Marks:** 60

Instructions: Answer all questions showing all derivation steps. No marks without steps.

1. Using **Hamilton's Principle**, derive the governing differential equation of a homogeneous bar (modeled as a 1-D elastic continua) undergoing axial vibration. State all the assumptions used during the derivation. (12)

2. Consider a homogeneous uniform bar connected to a string which is under tension T, as shown in the **Fig. 1**. What are the **initial conditions** for the given system, when the string suddenly snaps at $t = 0$. All symbols have their usual meaning. (8)

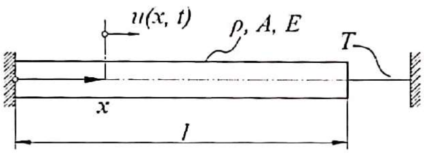


Fig. 1

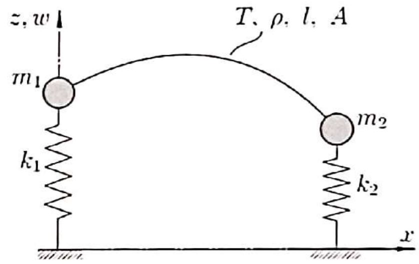


Fig. 2

3. State the **boundary conditions** of a uniform taut string undergoing transverse vibrations as shown in **Fig. 2**, in terms of the field variable $w(x, t)$. Parameter definitions are as follows: Volume density (ρ), length (l), cross-sectional area (A), string tension (T), point masses (m_1, m_2) and spring stiffness (k_1, k_2). (8)

4. Consider a uniform taut string with an interacting mass element placed at $x = al$ $(0 < a < 1)$ undergoing transverse vibrations as shown in the **Fig. 3**. We use the **Ritz method** to approximately determine its first two natural frequencies. For this purpose, we assume the solution form as $w(x,t) = b_1(t)x(l-x) + b_2(t)x^2(l-x)$ which discretizes the system as $\mathbf{M}\ddot{\mathbf{b}} + \mathbf{K}\mathbf{b} = \mathbf{0}$. Determine the elements of the first row in the stiffness matrix $\mathbf{K}$. (8)

5. The governing equation of a uniform hanging string is given by $w_{,tt} - [g(l-x)w_{,x}]_x = 0$, where $w(x,t)$ is the field variable, g is the acceleration due to gravity, and l is the length of the beam. Using **Galerkin's method**, discretize the equation of motion using the comparison functions as $P_i(x) = x^i$ $(i = 1, 2, ..., N)$. For $N = 2$ determine the eigenfrequencies from the discretized system. Take $g = 10$ m/s^2, and $l = 1$ m. (14)

6. A homogeneous tapered bar of circular cross-section undergoing axial vibrations is shown in **Fig. 4**. Assume an admissible function for the field variable $u(x,t)$ of the form $H_k(x) = (x/l)^k$, where k is an ~~integer~~ Real +ve. Using **Rayleigh's quotient**, determine the value of k that yields the lowest value of the fundamental natural frequency of the system. (10)

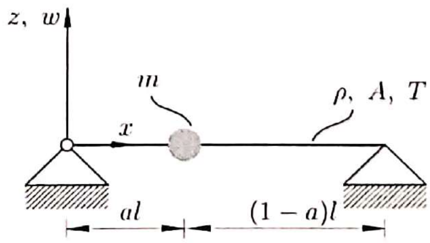


Fig. 3

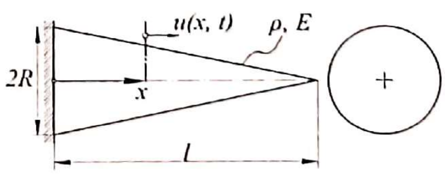
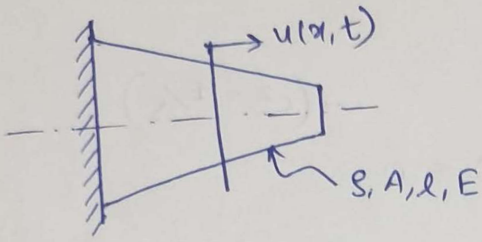


Fig. 4

Q.1 Ans:- $\boxed{\text{Hamilton's Principle}}$

Homogenous bar (1D elastic continua) undergoing axial vibration



Assumptions

1) Bar is homogenous (ρ, E constant)
2) Bar modeled as 1D continuum, so any transverse vibrations neglected.
3) Bar subjected to axial load only, neglect any external forces or moments
4) Bar is linear elastic (Hooke's Law valid)
5) Displacement 'u' is very small compared to length of bar 'l', allows us to use linear elasticity theory.

① Kinetic Energy $= T = \int_0^l \frac{1}{2} \underbrace{\rho A dx}_{m} \underbrace{u_{,t}^2}_{V^2}$

$$= \frac{1}{2}\int_0^l \rho A u_{,t}^2 \, dx$$

② Potential Energy $= V = \frac{1}{2}\int_0^l \underbrace{\sigma \varepsilon}_{J/m^3} \underbrace{(A\,dx)}_{\text{Volume}}$

$$= \frac{1}{2}\int_0^l EA u_{,x}^2 \, dx \quad \cdots \begin{bmatrix} \varepsilon = u_{,x} \\ \sigma = E\varepsilon = Eu_{,x} \end{bmatrix}$$

③ Lagrangian Density, $\mathcal{L} = T - V$

$$= \frac{1}{2}\left[\int_0^l \rho A u_{,t}^2 \, dx - \int_0^l EA u_{,x}^2 \, dx\right]$$

Using Hamilton Principle,

$$\delta \int_{t_1}^{t_2} \mathcal{L} \, dt = 0$$

$$\Rightarrow \delta \int_{t_1}^{t_2} \int_0^l \left[\frac{1}{2}\rho A u_{,t}^2 - \frac{1}{2} EA u_{,x}^2\right] dx\, dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \int_0^l \left[\rho A u_{,t} (\delta u)_{,t} - EA u_{,x} (\delta u)_{,x}\right] dx\, dt = 0$$

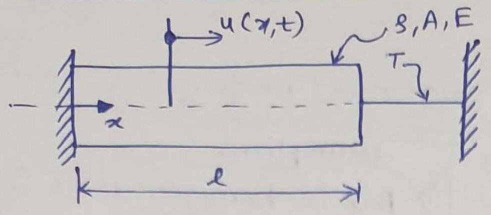
$$\Rightarrow \int_0^l \left[\underbrace{\cancel{\rho A u_{,t} \delta u \Big|_{t_1}^{t_2}}}_{0 \text{ (Boundary Term)}} - \int_{t_1}^{t_2} \rho A u_{,tt} \, \delta u \, dt\right] dx$$

$$- \int_{t_1}^{t_2} \left[\underbrace{\cancel{EA u_{,x} \delta u \Big|_0^l}}_{0 \text{ Boundary Conditions}} - \int_0^l (EA u_{,x})_{,x} \, \delta u \, dx\right] dt = 0$$

$$\Rightarrow \int_0^l \int_{t_1}^{t_2} \left[-\rho A u_{,tt} + (EA u_{,x})_{,x}\right] \delta u \, dx\, dt = 0$$

Governing Differential Equation $\boxed{-\rho A u_{,tt} + (EA u_{,x})_{,x} = 0}$ OR $\boxed{\rho A u_{,tt} - (EA u_{,x})_{,x} = 0}$

Q.2 Ans :– Collapse of a stretched bar



EOM :- $u_{,tt} - c^2 u_{,xx} = 0$ $\quad \ldots (c^2 = E/\rho)$

BCs :- $u(0,t) = 0$, $\underbrace{u_{,x}(l,t) = 0}$

↳ No strain at end $\varepsilon = \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0$ } Variational Formulation $\left[\frac{\partial \mathcal{L}}{\partial u_{,x}}\left(\frac{1}{2}u_{,t}^2 - c^2 \frac{1}{2} u_{,x}^2\right)\right]\Big|_{x=l} = 0$

$-c^2 u_{,x}\big|_{x=l} = 0$

$\therefore u_{,x}\big|_{x=l} = 0$

ICs :- $u(x,0) = \frac{Tx}{EA}$, $u_{,t}(x,0) = 0$ $\quad \ldots \left[\begin{array}{l} T = EA\, u_{,x} \\ \Rightarrow u_{,x} = \frac{T}{EA} \\ \Rightarrow \frac{\partial u}{\partial x} = \frac{T}{EA} \\ \Rightarrow u = \frac{Tx}{EA} \end{array}\right]$

$\frac{N}{m^2} \times m^2 \times \frac{m}{m}$

Assumed solution, $u(x,t) = \sum_{k=1}^{\infty} (C_k \cos\omega_k t + S_k \sin\omega_k t)\, U_k(x)$

where $\omega_k = \frac{(2k-1)\pi c}{2l}$

$U_k(x) = \sin\left[\frac{(2k-1)\pi x}{2l}\right]$

Using 1st initial condition, Put $t=0$ in assummed solution,

$$u(x,0) = \sum_{k=1}^{\infty} [C_k(1) + 0]\, U_k(x) = \frac{Tx}{EA}$$

$$\Rightarrow \sum_{k=1}^{\infty} C_k U_k(x) = \frac{Tx}{EA}$$

Do inner product w.r.t. U_j,

$$\Rightarrow C_j \langle U_j, U_j \rangle = \left\langle \frac{Tx}{EA}, U_j \right\rangle$$

$$\Rightarrow C_j = \frac{\left\langle \frac{Tx}{EA}, U_j \right\rangle}{\langle U_j, U_j \rangle}$$

$$\Rightarrow C_j = \frac{\int_0^l \frac{Tx}{EA} \sin\left[\frac{(2j-1)\pi x}{2l}\right] dx}{\int_0^l \sin^2\left[\frac{(2j-1)\pi x}{2l}\right] dx}$$

$$\Rightarrow C_j = \frac{\left\{\frac{Tx}{EA}\left[\frac{-\cos\left[\frac{(2j-1)\pi x}{2l}\right]}{\frac{(2j-1)\pi}{2l}}\right]\right\}_0^l - \int_0^l \frac{T}{EA}\cdot\left[\frac{-\cos\frac{(2j-1)\pi x}{2l}}{\frac{(2j-1)\pi}{2l}}\right] dx}{\int_0^l \frac{1-\cos\left[\frac{(2j-1)\pi x}{l}\right]}{2}\, dx}$$

$\ldots \left[\sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$

$$\Rightarrow C_j = \frac{\left\{ \frac{Tl}{EA}\left[-\frac{\cos\frac{(2j-1)\pi}{2}}{\frac{(2j-1)\pi}{2l}} \right] - 0 \right\} + \frac{T}{EA}\left[\frac{\sin\frac{(2j-1)\pi x}{2l}}{\left(\frac{(2j-1)\pi}{2l}\right)^2} \right]_0^l}{\left[\frac{1}{2}x - \frac{\sin\frac{(2j-1)\pi x}{l}}{2\times\frac{(2j-1)\pi}{l}} \right]_0^l}$$

$$\Rightarrow C_j = \frac{\left(-\frac{2Tl^2}{EA}\frac{\cos\frac{(2j-1)\pi}{2}}{(2j-1)\pi} \right) + \frac{T}{EA}\left[\frac{\sin\frac{(2j-1)\pi}{2}}{\left(\frac{(2j-1)\pi}{2l}\right)^2} - 0 \right]}{\left(\frac{1}{2}l - \frac{\sin(2j-1)\pi}{\left(\frac{2(2j-1)\pi}{l}\right)} \right) - (0-0)}$$

Here $\cos\left[\frac{(2j-1)\pi}{2}\right] = \cos\left[m\frac{\pi}{2}\right]$ where $m = 2j-1$

$= 0$

$j = 1, 2, 3, \ldots \infty$

$m \rightarrow$ Always odd number

$$\Rightarrow C_j = \frac{0 + \frac{T}{EA}\frac{4l^2}{(2j-1)^2\pi^2}\sin\left(\frac{(2j-1)\pi}{2}\right)}{\frac{1}{2}l - \frac{l}{2}\frac{\sin(2j-1)\pi}{(2j-1)\pi}}$$

Here, $\sin[(2j-1)\pi] = \sin[m\pi]$ where $m \rightarrow$ always odd number

$= 0$

$$\Rightarrow C_j = \frac{\frac{T}{EA}\frac{4l^2}{(2j-1)^2\pi^2}\sin\left(\frac{(2j-1)\pi}{2}\right)}{\frac{1}{2}l - 0}$$

$$\Rightarrow \boxed{C_j = \frac{8Tl}{(2j-1)^2\pi^2 EA}\sin\left(\frac{(2j-1)\pi}{2}\right)}$$

∴ Assummed solution's "C_K" value now known.

Using 2nd initial condition,

Differentiate assumed sol^n w.r.t. t,

$$u_{,t} = \sum_{k=1}^{\infty} \left[C_K(\sin\omega_K t)\cdot\omega_K + S_K\cdot(\cos\omega_K t)\cdot\omega_K \right] U_K(x)$$

Put $t = 0$,

$$u_{,t}(x,0) = \sum_{k=1}^{\infty} (0 + S_K(1)\cdot\omega_K)\, U_K(x) = 0$$

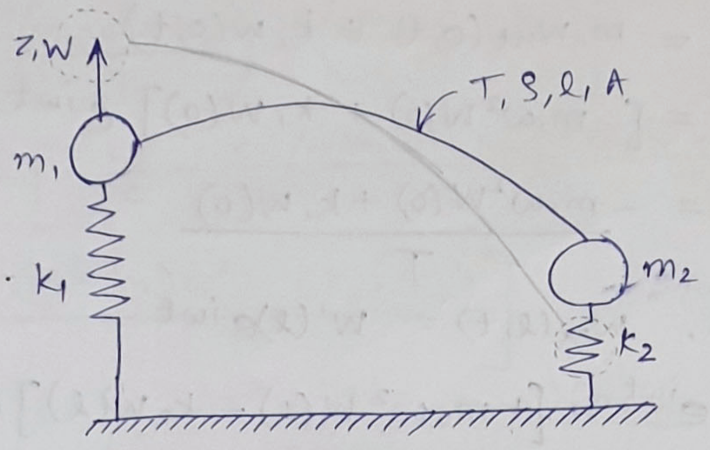
As $U_K(x) \neq 0$, $\boxed{S_K = 0}$

∴ Final solution is,

To get abs. displacement (+ve) all time

$$u(x,t) = \sum_{k=1}^{\infty} \frac{8Tl}{(2k-1)^2\pi^2 EA}(-1)^{k-1}\cdot\cos\omega_K t\cdot\sin\left[\frac{(2k-1)\pi x}{2l}\right]$$

Q.3 Ans :-

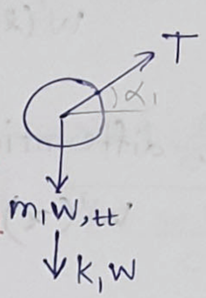


[a] Boundary conditions

① Force balance at m_1,

$$\Sigma F_y = m a_y$$

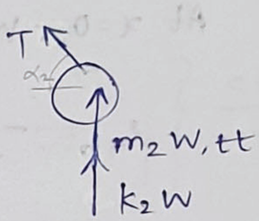
$$\boxed{T w_{,x}(0,t) = m_1 w_{,tt}(0,t) + k_1 w(0,t)}$$



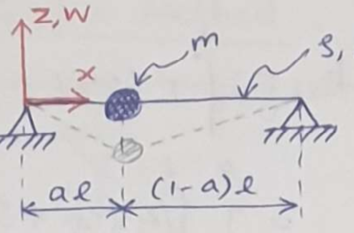
$T\sin\alpha$
$\approx T\tan\alpha$
$\approx T\frac{dw}{dx}$
$= T w_{,x}$

② Force balance at m_2,

$$\boxed{T w_{,x}(l,t) = -m_2 w_{,tt}(l,t) - k_2 w(l,t)}$$



Q.4 Ans:-



Ritz Method

$$\cancel{\delta \int_{t_1}^{t_2}\int_0^{\ell} (\rho A w_{,tt} - T w_{,xx})\, dx\, dt = 0}$$

By considering mass,

$$\rho A w_{,tt} - T w_{,xx} = p(t)$$

$$\rho A w_{,tt} - T w_{,xx} = -m w_{,tt}\, \delta(x - a\ell)$$

$$[\rho A + m\,\delta(x-a\ell)]\, w_{,tt} - T w_{,xx} = 0$$

In variational form,

$$\delta \int_{t_1}^{t_2} \int_0^{\ell} \{[\rho A + m\,\delta(x-a\ell)]\, w_{,t}^2 - T w_{,x}^2\}\, dx\, dt = 0$$

Assumed Solution, $w(x,t) = b_1(t) \cdot x(\ell - x) + b_2(t) \cdot x^2(\ell - x)$

$$= \sum_{k=1}^{2} b_k(t)\, H_k(x)$$

$$= [x(\ell - x) \quad x^2(\ell-x)] \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$$

$$= \vec{H}_k^T \vec{b}_k$$

$$\Rightarrow \delta \int_{t_1}^{t_2} (\dot{\vec{b}}^T M \dot{\vec{b}} - \vec{b}^T K \vec{b})\, dt = 0 \quad \cdots \left[\begin{array}{l} M = \int_0^{\ell} (\rho A + m\,\delta(x - a\ell)) \vec{H}\vec{H}^T dx \\ K = \int_0^{\ell} T\, \vec{H}' \vec{H}'^T dx \end{array} \right]$$

$$\Rightarrow \quad \therefore M\ddot{\vec{b}} + K\vec{b} = 0$$

Here, $K = \int_0^{\ell} T \frac{d}{dx}\begin{bmatrix} x(\ell - x) \\ x^2(\ell - x) \end{bmatrix} \frac{d}{dx}[x(\ell-x) \quad x^2(\ell-x)]\, dx$

$$= \int_0^{\ell} T \begin{bmatrix} \ell - 2x \\ 2x\ell - 3x^2 \end{bmatrix} [\ell - 2x \quad 2x\ell - 3x^2]\, dx$$

$$= \int_0^{\ell} T \begin{bmatrix} (\ell-2x)^2 & (\ell-2x)(2x\ell - 3x^2) \\ (2x\ell - 3x^2)(\ell - 2x) & (2x\ell - 3x^2)^2 \end{bmatrix} dx$$

$$= \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$K_{11} = \int_0^l T(l-2x)^2\,dx$$

$$= T\int_0^l (l^2+4x^2-4lx)\,dx$$

$$= T\left[l^2x + \frac{4x^3}{3} - \frac{4lx^2}{2}\right]_0^l$$

$$= T\left[l^3 + \frac{4l^3}{3} - 2l^3 - 0\right]$$

$$= \frac{Tl^3}{3}$$

$$K_{12} = \int_0^l T(l-2x)(2xl-3x^2)\,dx$$

$$= T\int_0^l (2xl^2 - 7x^2l + 6x^3)\,dx$$

$$= T\left[\frac{2x^2l^2}{2} - \frac{7x^3l}{3} + \frac{6x^4}{4}\right]_0^l$$

$$= T\left[l^4 - \frac{7}{3}l^4 + \frac{3}{2}l^4 - 0\right]$$

$$= \frac{Tl^4}{6}$$

$$K_{21} = K_{12} = \frac{Tl^4}{6}$$

$$K_{22} = \int_0^l T(2xl-3x^2)^2\,dx$$

$$= T\int_0^l (4x^2l^2 + 9x^4 - 12x^3l)\,dx$$

$$= T\left[\frac{4l^2x^3}{3} + \frac{9x^5}{5} - \frac{12lx^4}{4}\right]_0^l$$

$$= T\left[\frac{4l^2l^3}{3} + \frac{9l^5}{5} - 3l^5\right]$$

$$= Tl^5\left[\frac{4}{3} + \frac{9}{5} - 3\right]$$

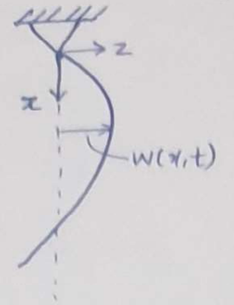
$$= \left[\frac{20+27-45}{15}\right]Tl^5$$

$$= \frac{2}{15}Tl^5$$

$$\therefore \boxed{[K] = \begin{bmatrix} \frac{Tl^3}{3} & \frac{Tl^4}{6} \\ \frac{Tl^4}{6} & \frac{2}{15}Tl^5 \end{bmatrix}}$$

Q.5 Ans:- Galerkin Method

Uniform Hanging String



$N = 2$ (No. of terms)

$g = 10\ m/s^2$

$l = 1\ m$

EOM :- $w_{,tt} - [g(l-x)w_{,x}]_{,x} = 0$ —— ①

Assumed Solution :- $w(x,t) = \sum_{k=1}^{N} p_k(t)\, P(x) = P^T \bar{p}$ —— ②

Substituting in EOM we get,

$$P^T \ddot{\bar{p}} + K[P^T]\bar{p} = e(x,t)$$ —— ③

$$\langle e(x,t), H_j(x) \rangle = 0$$

In Galerkin method, $H_j(x) = P_j(x)$

$$\therefore \langle e(x,t), P_j(x) \rangle = 0$$

$$\therefore \int_0^l e(x,t)\, P_j(x)\, dx = 0$$

Multiply ③ by $P_j(x)$ and integrate wrt x,

$$\therefore \mu(x)\, \bar{P}^T \ddot{\bar{p}}\, P + K[P^T]\bar{p}\, P = 0$$

$$M\ddot{\bar{p}} + K\bar{p} = 0$$

Here $[M] = \int_0^l \mu(x)\, P P^T\, dx$

$[K] = \int_0^l \bar{P}\; K[P^T]\, dx$

Here $\mu(x) = 1$

$K[\;] = -[g(l-x)(\;)_{,x}]_{,x}$

$P(x) = P_i(x) = x^i$ ... $(i = 1, 2, \ldots, N)$ ... given

$$\therefore [M] = \int_0^{l=1} \mu(x) \begin{bmatrix} P_1(x) \\ P_2(x) \end{bmatrix}_{2\times1} [P_1(x) \;\; P_2(x)]_{1\times2}\, dx$$

$$= \int_0^1 1 \begin{bmatrix} x \\ x^2 \end{bmatrix} [x \;\; x^2]\, dx$$

$$[M] = \int_0^1 \begin{bmatrix} x^2 & x^3 \\ x^3 & x^4 \end{bmatrix} dx$$

$$= \begin{bmatrix} x^3/3 & x^4/4 \\ x^4/4 & x^5/5 \end{bmatrix}_0^1$$

$$= \begin{bmatrix} 1/3 & 1/4 \\ 1/4 & 1/5 \end{bmatrix}$$

$$[K] = \int_0^{l=1} \begin{bmatrix} p_1(x) \\ p_2(x) \end{bmatrix} \left[-g(l-x)[p_1(x) \quad p_2(x)]'\right]' dx$$

$$= \int_0^1 \begin{bmatrix} x \\ x^2 \end{bmatrix} \left[-10(1-x)[1 \quad 2x]\right]' dx$$

$$= \int_0^1 \begin{bmatrix} x \\ x^2 \end{bmatrix} \times(-10) \left[(-1)[1 \quad 2x] + (1-x)[0 \quad 2]\right] dx$$

$$= \int_0^1 \begin{bmatrix} x \\ x^2 \end{bmatrix} \times(-10) \left[[-1 \quad -2x] + [0 \quad 2-2x]\right] dx$$

$$= \int_0^1 \begin{bmatrix} x \\ x^2 \end{bmatrix} \times(-10)[-1 \quad +2-4x] \, dx$$

$$= (-10) \int_0^1 \begin{bmatrix} -x & 2x-4x^2 \\ -x^2 & 2x^2-4x^3 \end{bmatrix} dx$$

$$= (-10) \begin{bmatrix} -x^2/2 & 2x^2/2 - 4x^3/3 \\ -x^3/3 & 2x^3/3 - 4x^4/4 \end{bmatrix}_0^1$$

$$= (-10) \begin{bmatrix} -1/2 & 1 - 4/3 \\ -1/3 & 2/3 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} +5 & +10/3 \\ +10/3 & +10/3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1/3 & 1/4 \\ 1/4 & 1/5 \end{bmatrix} \begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix} + \begin{bmatrix} +5 & +10/3 \\ +10/3 & +10/3 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Put $\bar{p} = \bar{V} e^{i\omega t} \Rightarrow \ddot{\bar{p}} = -\omega^2 \bar{V} e^{i\omega t}$

$$-\omega^2 \begin{bmatrix} 1/3 & 1/4 \\ 1/4 & 1/5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 5 & 10/3 \\ 10/3 & 10/3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For nontrivial sol^n,

$$\det[-\omega^2 M + k] = 0$$

$$\det\left\{\begin{bmatrix} -\omega^2/3 & -\omega^2/4 \\ -\omega^2/4 & -\omega^2/5 \end{bmatrix} + \begin{bmatrix} 5 & 10/3 \\ 10/3 & 10/3 \end{bmatrix}\right\} = 0$$

$$\begin{vmatrix} -\frac{\omega^2}{3}+5 & -\frac{\omega^2}{4}+\frac{10}{3} \\ -\frac{\omega^2}{4}+\frac{10}{3} & -\frac{\omega^2}{5}+\frac{10}{3} \end{vmatrix} = 0$$

$$\left(-\frac{\omega^2}{2}+5\right)\left(-\frac{\omega^2}{5}+\frac{10}{3}\right) - \left(-\frac{\omega^2}{4}+\frac{10}{3}\right)\left(-\frac{\omega^2}{4}+\frac{10}{3}\right) = 0$$

$$\frac{\omega^4}{10} - \frac{10}{6}\omega^2 - \omega^2 + \frac{50}{3} - \frac{\omega^4}{16} + \frac{10}{12}\omega^2 + \frac{10}{12}\omega^2 - \frac{100}{9} = 0$$

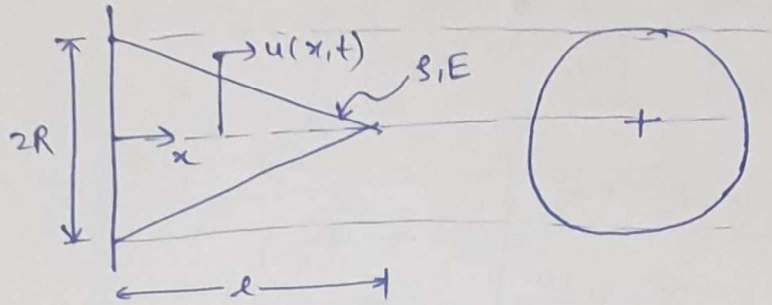
$$\left(\frac{1}{10}-\frac{1}{16}\right)\omega^4 + \left(-\frac{10}{6}-1+\frac{10}{12}+\frac{10}{12}\right)\omega^2 + \left(\frac{50}{3}-\frac{100}{9}\right) = 0$$

$$\omega^2 = 18.7766 \text{ or } 7.89$$

$$\boxed{\omega_1 = 4.333 \text{ rad/s}} \quad \text{and} \quad \boxed{\omega_2 = 2.8089 \text{ rad/s}}$$

https://github.com/LastElectron

Q.6 Ans:- $\boxed{\text{Rayleigh Method}}$

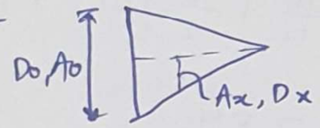


Conical bar (circular c/s)

Given :- $H_k(x) = \left(\frac{x}{l}\right)^k$ ... $k \rightarrow$ Real +ve

To find :- $k = ?$

Soln :-



$$\frac{D_x}{D_0} = \frac{l-x}{l}$$

$$\frac{\frac{\pi}{4}(D_x)^2}{\frac{\pi}{4}(D_0)^2} = \frac{\frac{\pi}{4}(l-x)^2}{\frac{\pi}{4}l^2} = \frac{A_x}{A_0}$$

$$\therefore A_x = A_0\left(\frac{l-x}{l}\right)^2$$

$$A_x = A_0\left(1-\frac{x}{l}\right)^2$$

$$T = KE = \frac{1}{2}\int_0^l \rho A u_{,t}^2 \, dx$$

$$V = PE = \frac{1}{2}\int_0^l E A u_{,x}^2 \, dx$$

$$\varepsilon = KE + PE = 0$$

$$\frac{1}{2}\int_0^l (\rho A u_{,t}^2 + E A u_{,x}^2)\, dx = 0$$

Put $u(x,t) = H(x) e^{i\omega t}$

$u_{,t} = H(i\omega) e^{i\omega t}$

$u_{,x} = H' e^{i\omega t}$

$$\frac{1}{2}\int_0^l [-\omega^2 \rho A H^2 + E A H'^2](e^{i\omega t})^2 \, dx = 0$$

$$\omega^2 = \frac{\int_0^l E A H'^2 \, dx}{\int_0^l \rho A H^2 \, dx} := R(x)$$

$$\therefore \omega^2 = E \frac{\int_0^l \cancel{A_0}\left(1-\frac{x}{l}\right)^2 \cdot \left[k\left(\frac{x}{l}\right)^{k-1}\right]^2 dx}{\rho \int_0^l \cancel{A_0}\left(1-\frac{x}{l}\right)^2 \left(\frac{x}{l}\right)^{2k} dx}$$

$$R[u(x)] = \omega^2 = \frac{E\int_0^l \left(1-\frac{2x}{l}+\frac{x^2}{l^2}\right) k\left(\frac{x}{l}\right)^{2k-2} dx}{\rho\int_0^l \left(1-\frac{2x}{l}+\frac{x^2}{l^2}\right)\left(\frac{x}{l}\right)^{2k} dx}$$

$$= \frac{Ek}{\rho}\,\frac{\int_0^l \left(\frac{x}{l}\right)^{2k-2} - 2\left(\frac{x}{l}\right)^{2k-1} + \left(\frac{x}{l}\right)^{2k} dx}{\int_0^l \left(\frac{x}{l}\right)^{2k} - 2\left(\frac{x}{l}\right)^{2k+1} + \left(\frac{x}{l}\right)^{2k+2} dx}$$

$$= \frac{Ek}{\rho}\,\frac{\left[\frac{x^{2k-1}}{(2k-1)l^{2k-2}} - \frac{2x^{2k}}{(2k)l^{2k-1}} + \frac{x^{2k+1}}{(2k+1)l^{2k}}\right]_0^l}{\left[\frac{x^{2k+1}}{(2k+1)l^{2k}} - \frac{2x^{2k+2}}{(2k+2)l^{2k+1}} + \frac{x^{2k+3}}{(2k+3)l^{2k+2}}\right]_0^l}$$

$$R[U(x)] = \frac{Ek}{\rho}\left[\frac{\frac{l}{2k-1} - \frac{2l}{2k} + \frac{l}{2k+1}}{\frac{l}{2k+1} - \frac{2l}{2k+2} + \frac{l}{2k+3}}\right]$$

$$R = \frac{Ek}{\rho}\,\frac{\cancel{l}(2k)(2k+1) - 2\cancel{l}(2k-1)(2k+1) + \cancel{l}(2k-1)(2k)\,/\,(2k-1)(2k)\cancel{(2k+1)}}{\cancel{l}(2k+2)(2k+3) - 2\cancel{l}(2k+1)(2k+3) + \cancel{l}(2k+1)(2k+2)\,/\,\cancel{(2k+1)}(2k+2)(2k+3)}$$

$$= \frac{Ek}{\rho}\,\frac{(\cancel{4k^2}+\cancel{2k}-\cancel{8k^2}+2+\cancel{4k^2}-\cancel{2k})/(2k-1)(2k)}{(\cancel{4k^2}+\cancel{10k}+\cancel{6}-\cancel{8k^2}-\cancel{16k}-\cancel{6}+\cancel{4k^2}+\cancel{6k}+2)/(2k+2)(2k+3)}$$

$$= \frac{Ek}{\rho}\,\frac{\cancel{2}}{\cancel{2}}\,\frac{(2k+2)(2k+3)}{(2k-1)(2\cancel{k})}$$

$$R = \frac{E}{\rho}\left(\frac{4k^2+10k+6}{4k-2}\right)$$

To minimize R, $\frac{\partial R}{\partial k} = 0$

$$\Rightarrow \frac{E}{\rho}\left[\frac{(4k-2)(8k+10) - (4k^2+10k+6)(4)}{(4k-2)^2}\right] = 0$$

$$\Rightarrow (4k-2)(8k+10) - (4)(4k^2+10k+6) = 0$$

$$\Rightarrow (2k-1)(4k+5) - (4k^2+10k+6) = 0$$

$$\Rightarrow 8k^2+6k-5-4k^2-10k-6 = 0$$

$$\Rightarrow 4k^2-4k-11 = 0$$

$$k = \frac{4 \pm \sqrt{16-4(-11)(4)}}{2\times 4}$$

$\boxed{k = 2.232}$ or $\cancel{-1.232}$ → Rejected ($k \to$ Real +ve)

https://github.com/LastElectron