Indian Institute of Technology Kharagpur
Department of Mechanical Engineering

Session: Spring, 2023-24 **End-Semester Examination**
Subject: Vibration of Structures (ME60428) **Number of students:** 46
Time: 3 hours **Total Marks:** 100

Instructions: Answer all questions showing all derivation steps. No marks without steps.

1. (a) Using **Hamilton's Principle**, derive the equation of motion of an Euler-Bernoulli beam (modeled as a 1-D elastic continua) undergoing transverse vibration. (10)
   (b) Also, state the possible combinations of boundary conditions at both ends of the beam and mark them as geometric or natural boundary conditions. (8)
   (c) State all assumptions used during the derivation. (2)

2. (a) The **comparison function** of an uniform **fixed-pinned** Euler-Bernoulli beam of length 1 m, undergoing transverse vibrations, is given by $f(x) = a_0 + a_1 x + 100x^2 + a_3 x^3 + a_4 x^4$. Determine the coefficients a_0, a_1, a_3 & a_4. (12)
   (b) Determine the values of this comparison function at intervals of 0.1 m along the length of the beam and draw a rough graph of the derived function. (6)
   (c) Does this function resemble any particular mode shape of this beam? (2)

3. The governing equation for the transverse free vibration of thin uniform **rectangular membranes** is given by $\mu w_{,tt} - T(w_{,xx} + w_{,yy}) = 0$, where $w(x,y,t)$ is the displacement field variable, μ is the mass per unit area, and T is the uniform tension per unit length.
   (a) Using modal analysis, determine the expressions for the mode-shape $W_{(m,n)}$ and circular natural frequencies $\omega_{(m,n)}$ of a thin rectangular membrane clamped on all sides. (10)
   (b) A thin sheet of steel of thickness 0.01 mm is stretched over a rectangular framework of size 50 mm $\times$ 25 mm under a tension of 2 kN per unit length of periphery. Determine the first six circular natural frequencies (rad/s) of the sheet. The density of steel is given by 7860 kg/m^3. (6)
   (c) Figure 1 shows four different mode-shapes of the given thin steel sheet. Determine their natural frequencies and identify the two degenerate modes. (4)

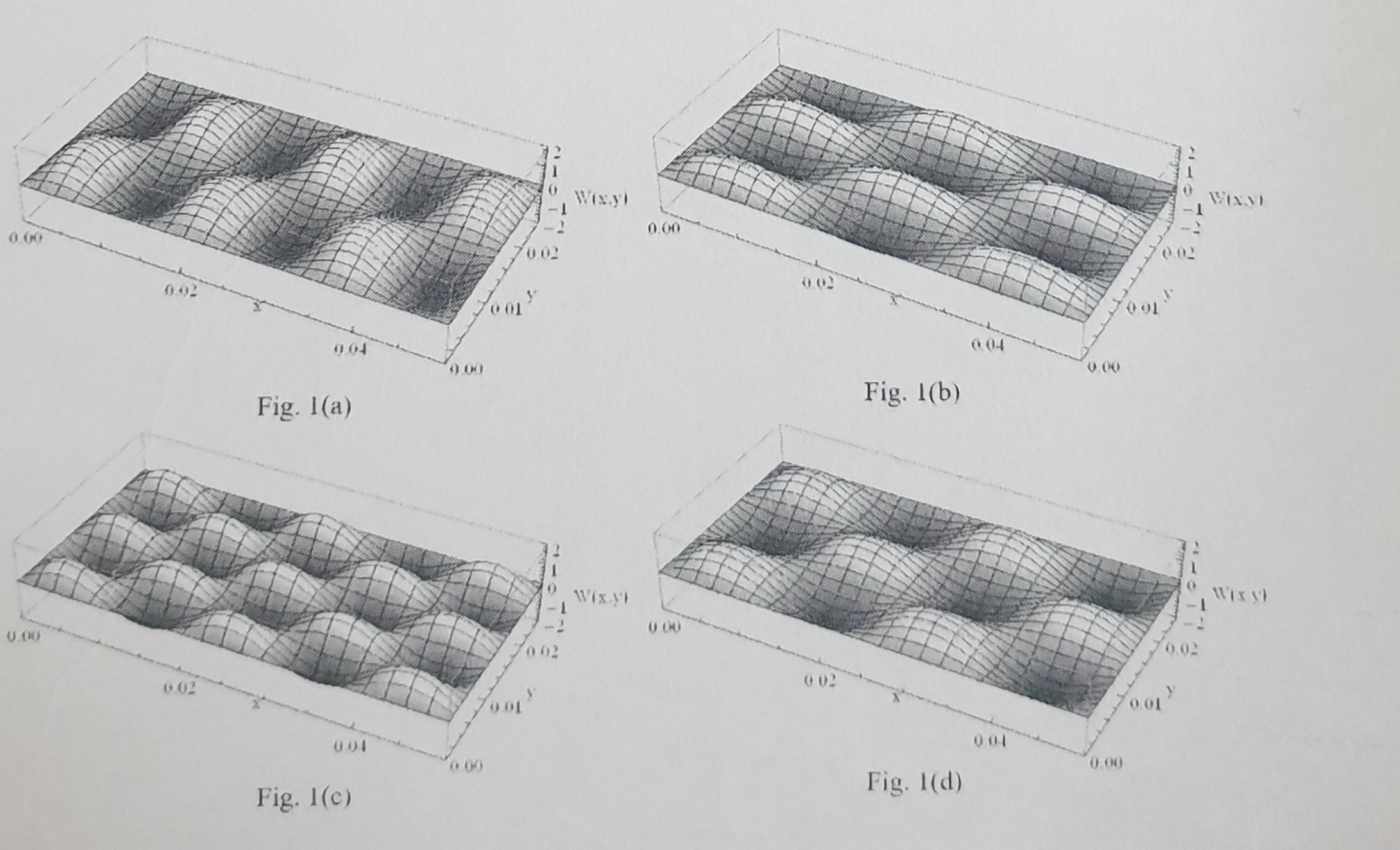


Figure 1: Different mode-shapes of the thin sheet in Problem 3(b)

4. (a) The equation of motion of an Euler-Bernoulli beam, having length l, with uniformly distributed **external damping** is given by $\rho A w_{,tt} + EI w_{,xxxx} + d_E w_{,t} = 0$, where d_E is the coefficient of external damping. Assuming simply-supported boundary conditions with nth modal solution as $w_n(x,t) = P_n(t)\sin\left(\frac{n\pi x}{l}\right)$, we obtain the discretized equation $\ddot{P}_n + 2\xi_n\omega_n\dot{P}_n + {\omega_n}^2 P_n = 0$. Determine the **damping factor** ξ_n in terms of the given parameters. (14)

(b) Figure 2 shows four different mode-shapes of a thin clamped **circular plate** undergoing transverse vibration. Identify each mode (e.g. W_{35}, W_{14}, etc.), based on the number of nodal lines and nodal circles in each. Also, identify the degenerate modes. (6)

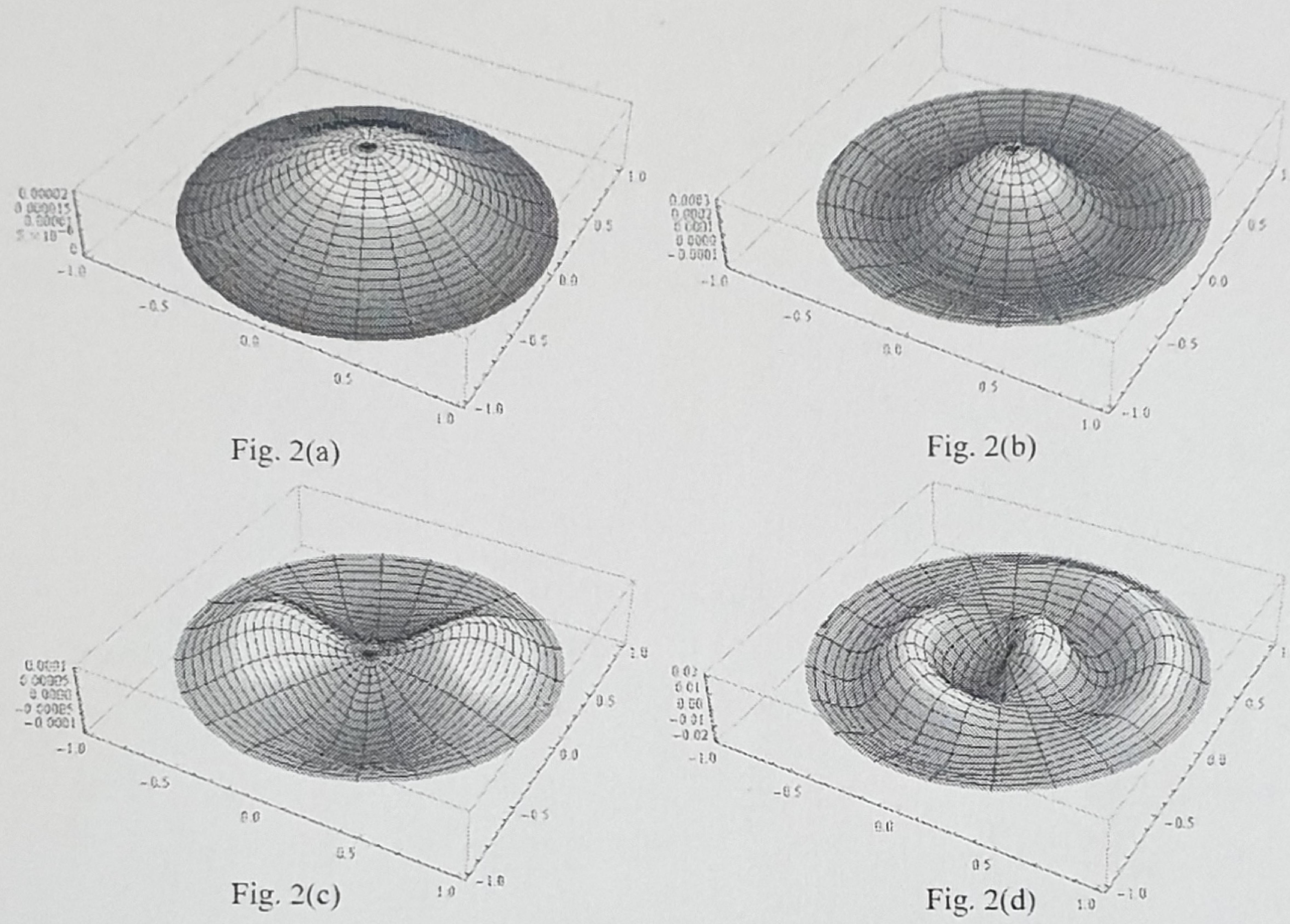


Figure 2: Mode-shapes of a thin circular plate

5. (a) Derive the equations of motion of a single DOF spring-mass-damper oscillator excited by a **non-ideal** DC motor with eccentricity considering motor system interaction for the system shown in Figure 3. (10)

(b) Apply steady state power/energy balance for the above motor-system to obtain the relationship between voltage and excitation frequency. The motor torque constant is given by μ. (10)

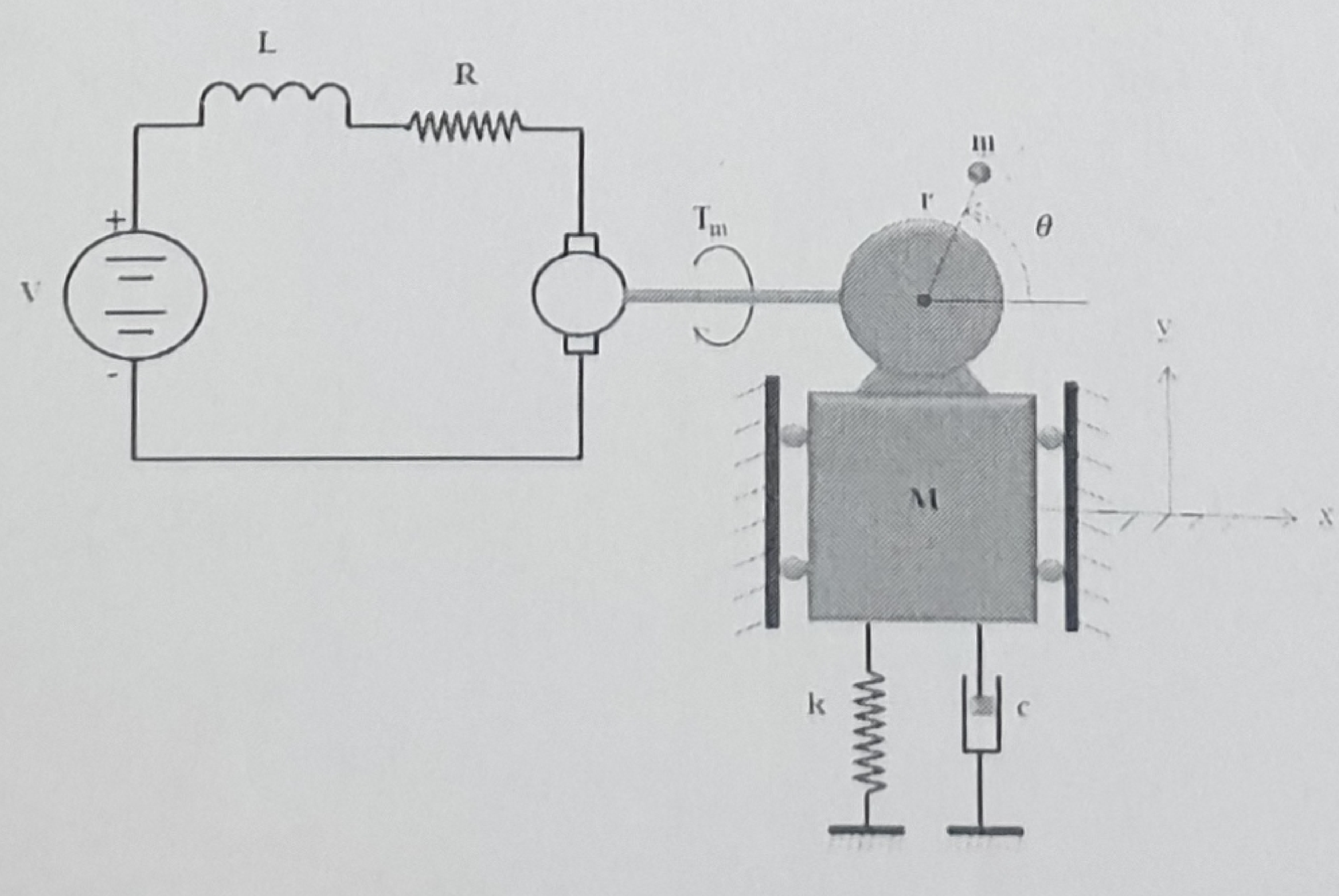


Figure 3: Single DOF spring-mass-damper oscillator excited by a non-ideal DC motor

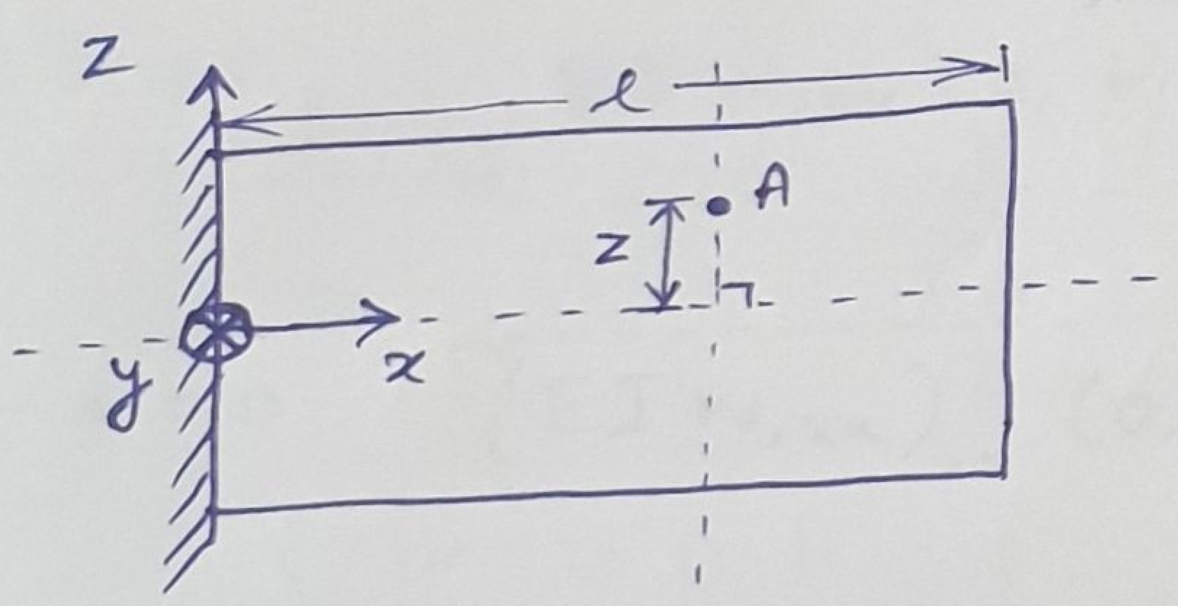
Q.1 Ans:- $\boxed{\text{Euler – Bernoulli Beam}}$

Beam is 1-D elastic continua which resist bending.

Assumptions:

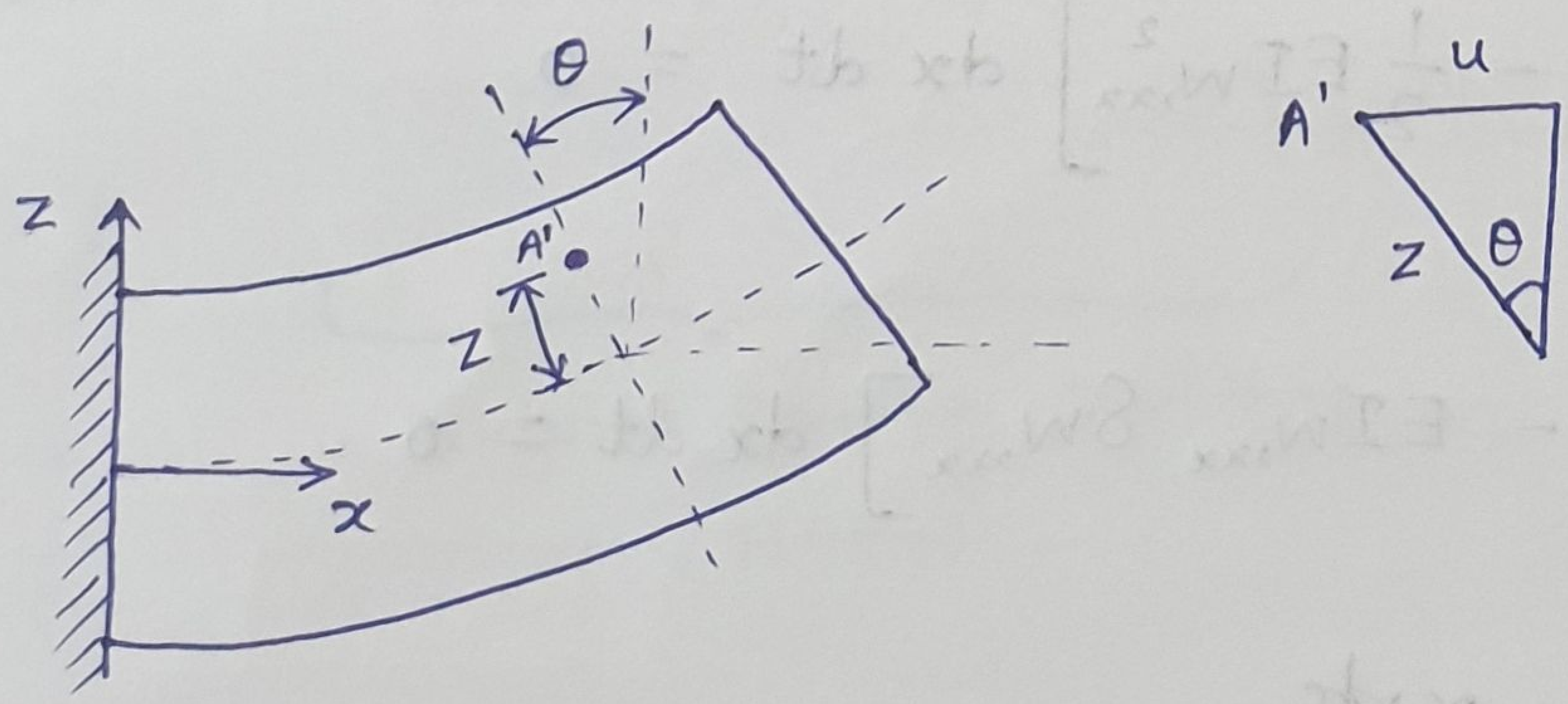
1) Material of beam is linear, homogenous and isotropic.
2) Cross section of beam remains plane before and after deformation.
3) Beam is assumed slender ($l \geq 10h$)
4) Cross sections that were orthogonal to neutral fiber before deformation remain orthogonal after deformation
5) Shear deformation is negligible (or beam is infinitely stiff in shear)
6) Beam is assumed to deflect in a single plane.
7) Slope is small.

Diagram :-



Let l = Length of Beam
E = Young's modulus of beam
I = Area moment of inertia

Before Deformation



$$AA' = z\sin\theta \approx z\theta \approx z\frac{\partial w}{\partial x} \approx z\, w_{,x}$$

$$\therefore\; u = -z\, w_{,x}$$

∴ Negative sign due to displacement along negative x-direction

Formulation :

- Kinetic Energy of system,

$$T = \int_0^l \frac{1}{2} \rho A w_{,t}^2\, dx + \cancel{\int_0^l \frac{1}{2}\rho I \theta_{,t}^2\, dx} \rightarrow 0$$

(Rotary inertia term is zero in E-B beam wrt Rayleigh beam)

$$T = \int_0^l \frac{1}{2} \rho A w_{,t}^2 \, dx$$

- Potential Energy of system

$$V = \frac{1}{2} \int_V \sigma \varepsilon \, dV \quad \ldots \left[\varepsilon = \frac{\partial u}{\partial x} = -z w_{,xx} \ldots (\text{as } u = -z w_{,x}) \right]$$

$$\left[\sigma = E\varepsilon = -zEw_{,xx} \right]$$

$$= \frac{1}{2} \int_0^l \int_0^A (-Ezw_{,xx})(-zw_{,xx}) \, dA \, dx$$

$$= \frac{1}{2} \int_0^l E w_{,xx}^2 \left[\int_0^A z^2 \, dA \right] dx$$

$$= \frac{1}{2} \int_0^l EI w_{,xx}^2 \, dx \quad \ldots \left[I = \int_0^A z^2 \, dA \right]$$

- Lagrangian

$$\mathcal{L} = T - V$$

$$= \int_0^l \left(\frac{1}{2} \rho A w_{,t}^2 - \frac{1}{2} EI w_{,xx}^2 \right) dx$$

- Using Hamilton's principle,

$$\delta \int_{t_1}^{t_2} \mathcal{L} \, dt = 0$$

$$\Rightarrow \delta \int_{t_1}^{t_2} \int_0^l \left[\frac{1}{2} \rho A w_{,t}^2 - \frac{1}{2} EI w_{,xx}^2 \right] dx \, dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \int_0^l \left[\rho A w_{,t} \, \delta w_{,t} - EI w_{,xx} \, \delta w_{,xx} \right] dx \, dt = 0$$

Integrating by parts,

$$\Rightarrow \int_0^l \rho A w_{,t} \, \cancel{\delta w} \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} EI w_{,xx} \, \delta w_{,x} \Big|_0^l dt$$

No change in w

$$- \int_{t_1}^{t_2} \int_0^l \left[\rho A w_{,tt} \, \delta w - (EI w_{,xx})_{,x} \, \delta w_{,x} \right] dx \, dt = 0$$

$$\Rightarrow \quad 0 - \int_{t_1}^{t_2} EI w_{,xx}\, \delta w_{,x} \Big|_0^{l} dt + \int_{t_1}^{t_2} (EI w_{,xx})_{,x}\, \delta w \Big|_0^{l} dt$$

$$- \int_{t_1}^{t_2}\int_0^{l} \left[\rho A w_{,tt} + (EI w_{,xx})_{,xx} \right] \delta w \, dx\, dt = 0$$

$\therefore$ Governing Differential Equation for Euler-Bernoulli beam model,

$$\rho A w_{,tt} + (EI w_{,xx})_{,xx} = 0$$

Boundary Conditions :-

At $x = 0$, $\quad EI w_{,xx}(0,t) = 0 \quad$ OR $\quad w_{,x}(0,t) = 0$

AND

At $x = l$, $\quad EI w_{,xx}(l,t) = 0 \quad$ OR $\quad w_{,x}(l,t) = 0$

↳ Moment

At $x = 0$, $\quad (EI w_{,xx})_{,x}(0,t) = 0 \quad$ OR $\quad w(0,t) = 0$

AND

At $x = l$, $\quad (EI w_{,xx})_{,x}(l,t) = 0 \quad$ OR $\quad w(l,t) = 0$

↳ Shear Force

Natural Boundary Conditions (NBCs)

Geometric Boundary Conditions (GBCs)

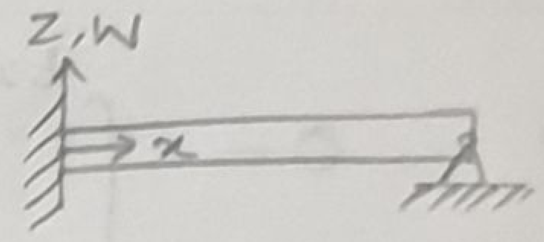
https://github.com/LastElectron

Q.2 Ans:- $\boxed{\text{Euler - Bernoulli Beam}}$

(a) GDE :- $\rho A W_{,tt} + (EI W_{,xx})_{,xx} = 0$

consider E, I as constant.

$$\rho A W_{,tt} + EI W_{,xxxx} = 0 \quad \text{——— (1)}$$



Boundary Conditions : (Fixed- Pinned)

At fixed end, deflection & slope is zero.

At pinned end, deflection & moment is zero.

$$\left.\begin{aligned} \text{At } x=0, \quad & w(0,t) = 0 \\ & W_{,x}(0,t) = 0 \\ \text{At } x=l, \quad & w(l,t) = 0 \\ & W_{,xx}(l,t) = 0 \end{aligned}\right\} \text{——— (2)}$$

... $\left[\begin{array}{l} EI\, W_{,xx}(l,t) = 0 \\ \text{since } E,I \text{ is constant} \\ W_{,xx}(l,t) = 0 \end{array}\right]$

... $(l = 1m)$

Comparison Function :-

$$f(x) = a_0 + a_1 x + 100 x^2 + a_3 x^3 + a_4 x^4 \equiv w(x) \quad \text{——— (3)}$$

It should satisfy all boundary conditions.

(A) Using BC 1, $w(0) = 0$ in (3),

$$f(0) = a_0 + 0 + 0 + 0 + 0 \equiv w(0)$$

$$\boxed{a_0 = 0}$$

(B) Using BC 2, $W_{,x}(0) = 0$ in (3),

$$f'(x) = a_1 + 200x + 3a_3 x^2 + 4a_4 x^3 \equiv W_{,x}(x)$$

$$f'(0) = a_1 + 0 + 0 + 0 = W_{,x}(0) = 0.$$

$$\boxed{a_1 = 0}$$

$$\therefore f(x) = 100x^2 + a_3 x^3 + a_4 x^4 \quad \text{——— (4)}$$

(C) Using BC 3, $w(l) = 0$ in (4),

$$f(l) = 100\, l^2 + a_3 l^3 + a_4 l^4 \equiv w(l) = 0$$

Putting $l = 1$,

$$100 + a_3 + a_4 = 0$$

$$a_3 + a_4 = -100 \quad \text{——— (5)}$$

(D) Using BC4, $w_{,xx}(l) = 0$ in (4),

$$f'(x) = 200x + 3a_3 x^2 + 4a_4 x^3 \equiv w_{,x}(x)$$

$$f''(x) = 200 + 6a_3 x + 12a_4 x^2 \equiv w_{,xx}(x)$$

$$\therefore f''(l) = 200 + 6a_3 l + 12a_4 l^2 \equiv w_{,xx}(l) = 0$$

Put $l = 1$,

$$200 + 6a_3 + 12a_4 = 0$$

$$3a_3 + 6a_4 = -100 \quad \text{——— (6)}$$

Using (5) & (6),

$$3a_3 + 6a_4 = -100 \quad \text{... by (6)}$$

$$- \quad 3a_3 + 3a_4 = -300 \quad \text{... by multiplying (5) by 3}$$

$$\ominus \quad \ominus \quad \oplus$$

$$3a_4 = 200$$

$$\boxed{a_4 = \frac{200}{3}}$$

Put this in (5),

$$a_3 = -100 - \frac{200}{3}$$

$$\boxed{a_3 = -\frac{500}{3}}$$

Values of Coefficients :- $a_0 = 0$

$a_1 = 0$

$a_3 = \frac{-500}{3} = -166.667$

$a_4 = \frac{200}{3} = 66.667$

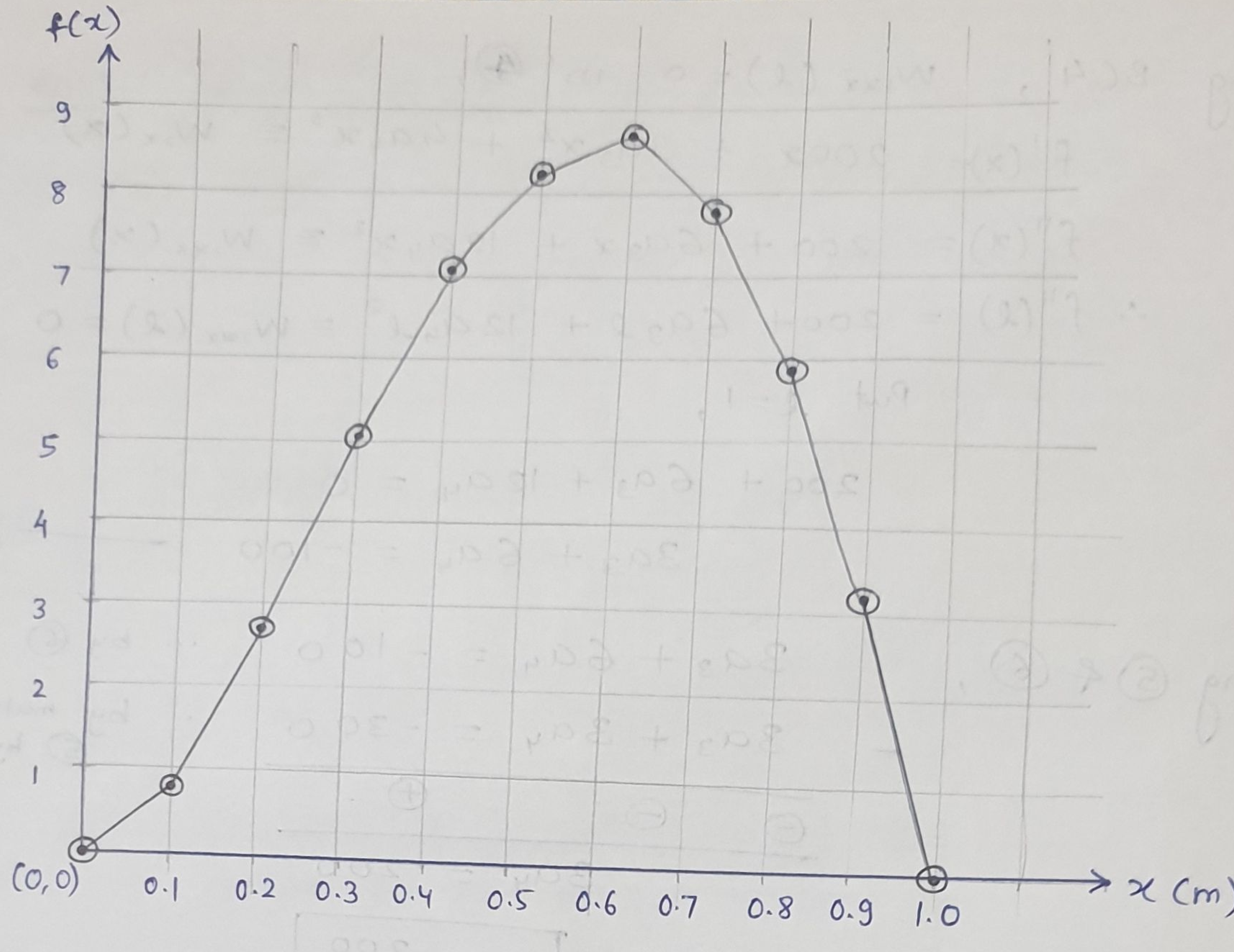
(b) Final Comparison function,

$$f(x) = 100x^2 - \frac{500}{3}x^3 + \frac{200}{3}x^4$$

Values of Comparison Function at interval of 0.1 m :-

x	$f(x)$
0	0
0.1	0.84
0.2	2.773
0.3	5.04
0.4	7.04
0.5	8.333

x	$f(x)$
0.6	8.64
0.7	7.84
0.8	5.973
0.9	3.24
1.0	0



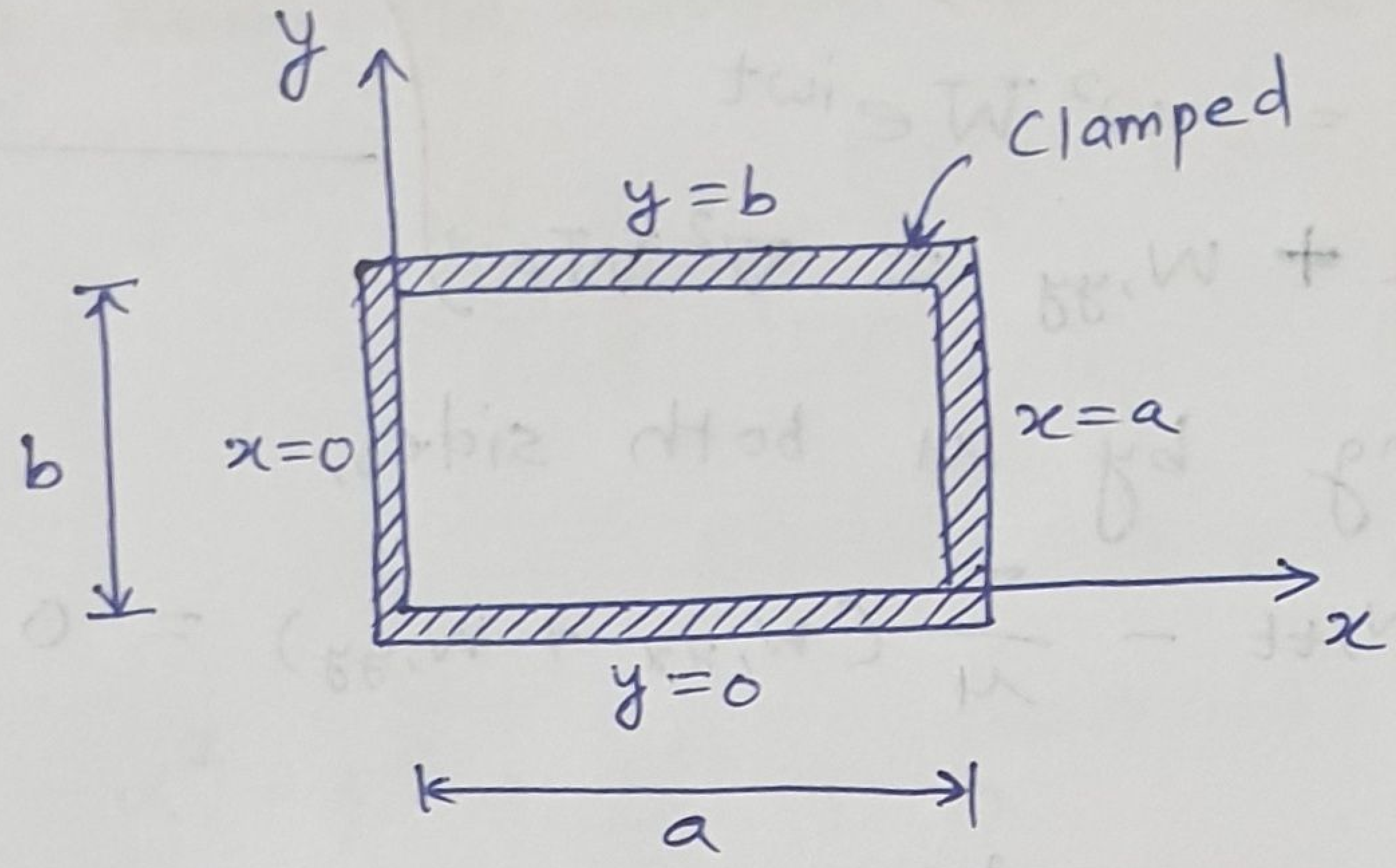
(c) This function resemble mode shape of beam corresponding to first natural frequency i.e. representing first mode.

https://github.com/LastElectron

Q.3 Ans :- $\boxed{\text{Rectangular Membrane}}$

GDE :- $\mu w_{,tt} - T(w_{,xx} + w_{,yy}) = 0$ —— (#1)

(a)



https://github.com/LastElectron

Rectangular Membrane clamped on all sides

Boundary conditions :

$$
\left.
\begin{array}{ll}
\text{Along edge } x = 0, & w(0, y, t) = 0 \\
\text{Along edge } x = a, & w(a, y, t) = 0 \\
\text{Along edge } y = 0, & w(x, 0, t) = 0 \\
\text{Along edge } y = b, & w(x, b, t) = 0
\end{array}
\right\} \text{—— (\#2)}
$$

Assumed solution :- $w(x, y, t) = W(x, y)\, e^{i\omega t}$ —— (#3)

Assume eigenfunction $W(x,y)$ as variable separable type and ω is circular eigenfrequency.

$\therefore \; w(x, y, t) = X(x) \cdot Y(y)\, e^{i\omega t}$ —— (1)

From (#3) , $w \equiv w(x,y,t) = W e^{i\omega t}$ ... $W(x,y) \equiv W$

$$w_{,tt} = W(i\omega)^2 e^{i\omega t}$$

$$= -\omega^2 W e^{i\omega t}$$ ——— (2)

$$w_{,xx} + w_{,yy} = \nabla^2 W e^{i\omega t}.$$

From (#1), dividing by μ both sides,

$$\Rightarrow \quad w_{,tt} - \frac{T}{\mu}(w_{,xx} + w_{,yy}) = 0$$

$$\Rightarrow \quad w_{,tt} - c^2(w_{,xx} + w_{,yy}) = 0 \quad \ldots \left(c^2 = \frac{T}{\mu}\right)$$

$$\Rightarrow \quad -\omega^2 W e^{i\omega t} - c^2 \nabla^2 W e^{i\omega t} = 0$$

$$\Rightarrow \quad \nabla^2 W + \frac{\omega^2}{c^2} W = 0$$

EOM :- $\nabla^2 W + \frac{\omega^2}{c^2} W = 0$

EBCs :- $W(0,y) = 0$ ... $\left(w(0,y,t) = W(0,y) e^{i\omega t} = 0 \right.$ As $\left. e^{i\omega t} \neq 0, \ W(0,y) = 0\right)$

$W(a,y) = 0$

$W(x,0) = 0$

$W(x,b) = 0$

... EVP — (3)

EOM can be written as,

$$W_{,xx} + W_{,yy} + \frac{\omega^2}{c^2} W = 0$$ ——— (4)

Rewriting assumed solution, $W(x,y) = X(x).Y(y)$

$$= e^{i(\alpha x + \beta y)}$$ — (5)

$W_{,xx} = X'' Y$ ... $(\ )'' = \frac{\partial^2}{\partial x^2}(\ )$

$W_{,yy} = X \ddot{Y}$ ... $(\ddot{\ }) = \frac{\partial^2}{\partial y^2}(\ )$

$W = XY$

Put above values in 4,

$$X'' Y + X \ddot{Y} + \frac{\omega^2}{c^2} XY = 0$$

Divide both sides by XY;

$$\Rightarrow \quad \frac{X''}{X} + \frac{\ddot{Y}}{Y} + \frac{\omega^2}{c^2} = 0$$

Using Reassumed solution from (5), $W(x,y) = e^{i\alpha x} \cdot e^{i\beta y}$
$= X(x) \cdot Y(y)$

$$\therefore \quad \frac{X''}{X} = \frac{(i\alpha)^2 e^{i\alpha x}}{e^{i\alpha x}} = -\alpha^2$$

$$\frac{\ddot{Y}}{Y} = \frac{(i\beta)^2 e^{i\beta y}}{e^{i\beta y}} = -\beta^2$$

$$\Rightarrow \quad -\alpha^2 - \beta^2 + \frac{\omega^2}{c^2} = 0$$

$$\Rightarrow \quad \alpha^2 + \beta^2 = \frac{\omega^2}{c^2} \qquad \text{——— (6)}$$

We can write reassumed solution in form of,

$$W(x,y) = e^{i\alpha x} \cdot e^{i\beta y}$$
$$= (B_1 \cos \alpha x + B_2 \sin \alpha x)(C_1 \cos \beta y + C_2 \sin \beta y)$$
$$= (B_1 C_1) \cos \alpha x \cos \beta y + (B_1 C_2) \cos \alpha x \sin \beta y$$
$$+ (B_2 C_1) \sin \alpha x \cos \beta y + (B_2 C_2) \sin \alpha x \sin \beta y$$

$$\Rightarrow \quad W(x,y) = A_1 \cos \alpha x \cos \beta y + A_2 \cos \alpha x \sin \beta y$$
$$+ A_3 \sin \alpha x \cos \beta y + A_4 \sin \alpha x \sin \beta y \qquad \text{——— (7)}$$

Using Boundary conditions from EVP to find constants.

1) Using BC 1, $W(0,y) = 0$

Put $x = 0$, $W = 0$ in (7),

$$\Rightarrow \quad 0 = A_1 (1) \cos \beta y + A_2 \sin \beta y + A_3 (0) \cos \beta y$$
$$+ A_4 (0) \sin \beta y$$

$$\Rightarrow \quad A_1 \cos \beta y + A_2 \sin \beta y = 0 \qquad \text{——— (8)}$$

2) Using BC3, $W(x,0) = 0$

Put $y = 0$, $W = 0$ in (7),

$$\Rightarrow \quad 0 = A_1 \cos \alpha x (1) + A_2 \cos \alpha x (0) + A_3 \sin \alpha x (1)$$
$$+ A_4 \sin \alpha x (0)$$

$$\Rightarrow \quad A_1 \cos \alpha x + A_3 \sin \alpha x = 0 \qquad \text{——— (9)}$$

From ⑧, to satisfy all values of $y \in [0,b]$ and
From ⑨, to satisfy all values of $x \in [0,a]$, we must have

$$A_1 = A_2 = A_3 = 0$$

So, eqⁿ ⑦ reduces to

$$W(x,y) = A_4 \sin \alpha x \sin \beta y \qquad \text{--- (10)}$$

3) Using BC 2 in (10), i.e. $W(a,y) = 0$.

$$\Rightarrow W(a,y) = A_4 \sin \alpha a \sin \beta y = 0$$

$$\Rightarrow \sin \alpha a \cdot \sin \beta y = 0 \qquad \text{--- (11)}$$

4) Using BC4 in (10) i.e. $W(x,b) = 0$,

$$\Rightarrow W(x,b) = A_4 \sin \alpha x \sin \beta b = 0$$

$$\Rightarrow \sin \alpha x \cdot \sin \beta b = 0 \qquad \text{--- (12)}$$

From (11), as $\sin \beta y \neq 0$, $\sin \alpha a = 0$

$$\alpha a = m\pi$$

$$\boxed{\alpha = \frac{m\pi}{a}} \quad \text{where } m = 1, 2, \ldots, \infty \qquad \text{--- (13)}$$

From (12), as $\sin \alpha x \neq 0$, $\sin \beta b = 0$

$$\beta b = n\pi$$

$$\boxed{\beta = \frac{n\pi}{b}} \quad \ldots \text{where } n = 1, 2, \ldots, \infty \qquad \text{--- (14)}$$

Substitute (13) & (14) in (10),

$$W(x,y) = A_4 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

For simplest solution, take $A_4 = 1$

$$\boxed{W_{mn}(x,y) = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)} \qquad \text{--- (15)}$$

... Eigenfunction / Mode Shape

To find eigenfrequency put (13) & (14) in ⑥,

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \frac{\omega^2}{c^2}$$

$$\therefore \quad \boxed{\omega_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}} \qquad \text{--- (16)}$$

... circular natural frequencies

(b) Given :-

$t = 0.01 \text{ mm} = 10^{-5} \text{ m}$

$a = 50 \text{ mm} = 0.05 \text{ m}$

$b = 25 \text{ mm} = 0.025 \text{ m}$

$T = 2 \text{ kN} = 2000 \text{ N}$

$\rho = 7860 \text{ kg/m}^3$

To find :- First six natural frequencies = ? ... using (16)

Sol^n :-

- $\mu = \frac{\text{Mass}}{\text{Area}} = \frac{\text{Density} \times \text{Volume}}{\text{Area}} = \frac{\rho \times (abt)}{ab} = \rho t$

$\mu = 7860 \times 10^{-5} \text{ kg/m}^2$

- $c = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{2000}{7860 \times 10^{-5}}} = 159.5158 \text{ m/s}$

First six natural frequencies obtain by putting different values of m and n starting from 1.

(1) $\omega_{11} = \pi \times 159.5158 \sqrt{\frac{1^2}{0.05^2} + \frac{1^2}{0.025^2}} = 22411.38 \text{ rad/s}$

(2) $\omega_{12} = \pi \times 159.5158 \sqrt{\left(\frac{1}{0.05}\right)^2 + \left(\frac{2}{0.025}\right)^2} = 41324.542 \text{ rad/s}$

(3) $\omega_{21} = \pi \times 159.5158 \sqrt{\left(\frac{2}{0.05}\right)^2 + \left(\frac{1}{0.025}\right)^2} = 28348.402 \text{ rad/s}$

(4) $\omega_{22} = \pi \times 159.5158 \sqrt{\left(\frac{2}{0.05}\right)^2 + \left(\frac{2}{0.025}\right)^2} = 44822.76 \text{ rad/s}$

(5) $\omega_{13} = \pi \times 159.5158 \sqrt{\left(\frac{1}{0.05}\right)^2 + \left(\frac{3}{0.025}\right)^2} = 60965.54 \text{ rad/s}$

(6) $\omega_{31} = \pi \times 159.5158 \sqrt{\left(\frac{3}{0.05}\right)^2 + \left(\frac{1}{0.025}\right)^2} = 36137.264 \text{ rad/s}$

(7) $\omega_{23} = \pi \times 159.5158 \sqrt{\left(\frac{2}{0.05}\right)^2 + \left(\frac{3}{0.025}\right)^2} = 63388.95 \text{ rad/s}$

(8) $\omega_{32} = \pi \times 159.5158 \sqrt{\left(\frac{3}{0.05}\right)^2 + \left(\frac{2}{0.025}\right)^2} = 50113.368 \text{ rad/s}$

$\therefore$ First six natural frequencies in ascending order

$\omega_{11} = 22411.38$ rad/s

$\omega_{21} = 28348.402$ rad/s

$\omega_{31} = 36137.264$ rad/s

$\omega_{12} = 41324.542$ rad/s

$\omega_{22} = 44822.76$ rad/s

$\omega_{32} = 50113.368$ rad/s

(c)

Figure	ω_{mn}	Natural frequency (rad/s)	Degenerate Modes $\left(\frac{a}{b}=2\right)$
1(a)	ω_{62}	72274.525	✓
1(b)	ω_{34}	85633.758	×
1(c)	ω_{55}	112056.89	×
1(d)	ω_{43}	72274.525	✓

$\therefore$ figure 1(a) and 1(d) represents degenerate modes. because they have same natural frequency.

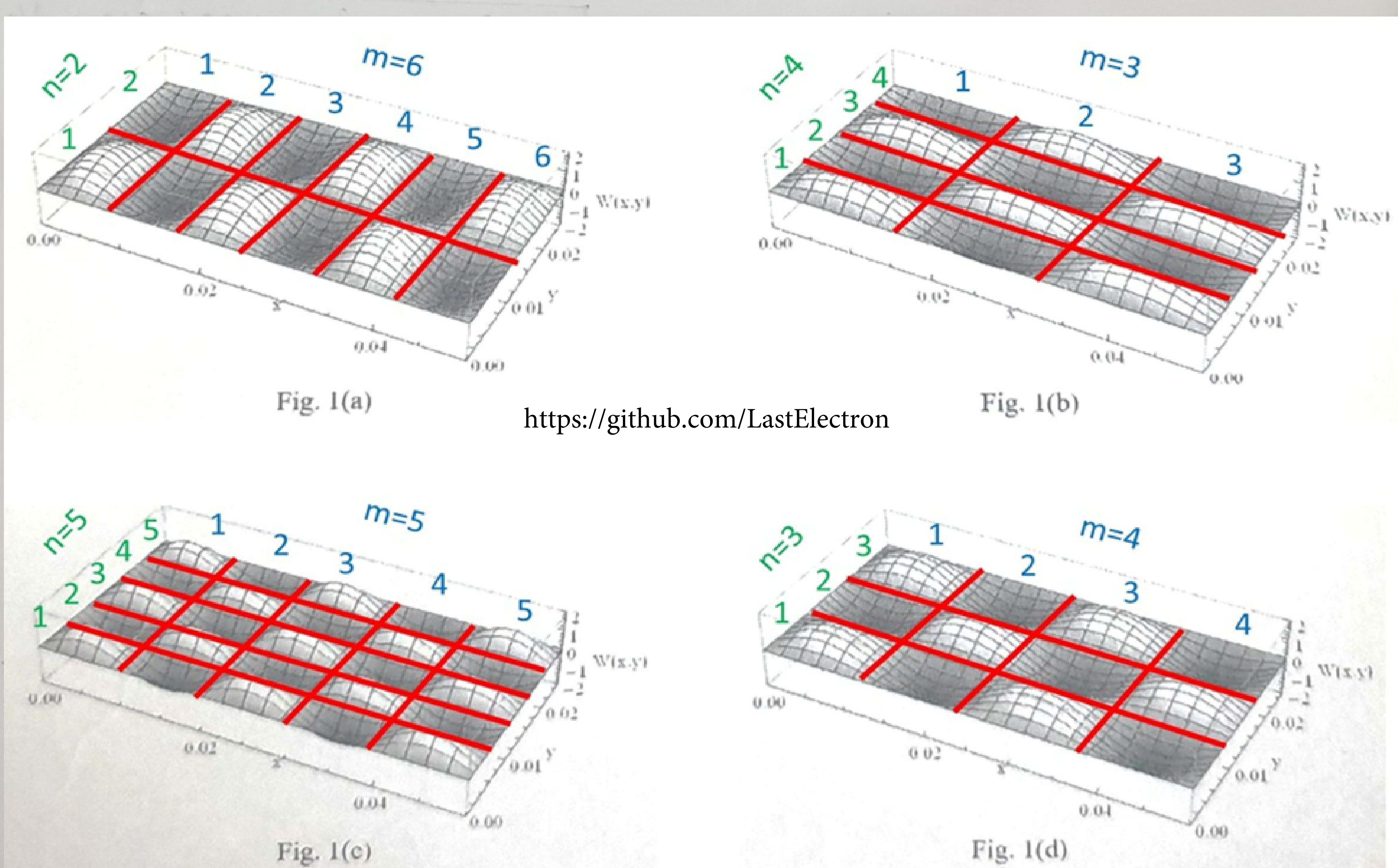


Fig. 1(a)

Fig. 1(b)

Fig. 1(c)

Fig. 1(d)

Q.4 Ans :- (a) $\boxed{\text{Euler - Bernoulli Beam with External Damping}}$

GDE :- $\rho A W_{,tt} + EI W_{,xxxx} + d_E W_{,t} = 0$ —— ①

Assuming simply supported boundary conditions with n^{th} modal solution, $W_n(x,t) = P_n(t) \sin\left(\frac{n\pi x}{l}\right)$

$$\therefore \quad W_{n,t} = \dot{P}_n \sin\left(\frac{n\pi x}{l}\right) \quad \ldots \; (\dot{\;}) = \frac{d}{dt}()$$

$$W_{n,tt} = \ddot{P}_n \sin\left(\frac{n\pi x}{l}\right)$$

$$W_{n,x} = P_n\left(\frac{n\pi}{l}\right) \cos\left(\frac{n\pi x}{l}\right)$$

$$W_{n,xx} = P_n\left(\frac{n\pi}{l}\right)^2 (-1) \sin\left(\frac{n\pi x}{l}\right)$$

$$W_{n,xxx} = P_n\left(\frac{n\pi}{l}\right)^3 (-1) \cos\left(\frac{n\pi x}{l}\right)$$

$$W_{n,xxxx} = P_n\left(\frac{n\pi}{l}\right)^4 \sin\left(\frac{n\pi x}{l}\right)$$

Substitute this in GDE,

$$\Rightarrow \rho A \ddot{P}_n \sin\left(\frac{n\pi x}{l}\right) + EI P_n\left(\frac{n\pi}{l}\right)^4 \sin\left(\frac{n\pi x}{l}\right) + d_E \cdot \dot{P}_n \sin\left(\frac{n\pi x}{l}\right) = 0$$

$$\Rightarrow \left[\rho A \ddot{P}_n + EI P_n\left(\frac{n\pi}{l}\right)^4 + d_E \dot{P}_n\right] \sin\left(\frac{n\pi x}{l}\right) = 0$$

As $\sin\left(\frac{n\pi x}{l}\right) \neq 0$ for non-trivial solution,

$$\Rightarrow \rho A \ddot{P}_n + d_E \cdot \dot{P}_n + EI\left(\frac{n\pi}{l}\right)^4 P_n = 0$$

Divide by ρA throughout,

$$\Rightarrow \ddot{P}_n + \frac{d_E}{\rho A} \dot{P}_n + \frac{EI}{\rho A}\left(\frac{n\pi}{l}\right)^4 P_n = 0 \quad \text{—— ②}$$

Compare this equation with discrete form given i.e. $\ddot{P}_n + 2\xi_n \omega_n \dot{P}_n + \omega_n^2 P_n = 0$

By comparing,

$$2\xi_n \omega_n = \frac{d_E}{\rho A} \quad \text{—— ③} \quad \text{and} \quad \omega_n^2 = \frac{EI}{\rho A}\left(\frac{n\pi}{l}\right)^4 \quad \text{—— ④}$$

From (4), $\omega_n = \sqrt{\frac{EI}{\rho A}} \left(\frac{n\pi}{l}\right)^2$

OR

$$\omega_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}}$$

Put this in (3),

$$2\xi_n \frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}} = \frac{d_E}{\rho A}$$

$$\Rightarrow \quad \xi_n = \frac{l^2}{2n^2\pi^2}\sqrt{\frac{\rho A}{EI}}\ \frac{d_E}{\rho A}$$

$$\Rightarrow \quad \boxed{\xi_n = \frac{l^2 d_E}{2n^2\pi^2\sqrt{EI\rho A}}}$$

... damping factor in terms of given parameters

(b) Circular Plate (Thin Clamped)

- Fig. 2(a), Nodal lines $= 0 \Rightarrow m = 0$
  Nodal Circles $= 0 \Rightarrow n = 0+1 = 1$
  Mode $= W_{01}$

  Fig 2(b), Nodal lines $= 0 \Rightarrow m = 0$
  Nodal Circles $= 1 \Rightarrow n = 1+1 = 2$
  Mode $= W_{02}$

  Fig 2(c), Nodal lines $= 2 \Rightarrow m = 2$
  Nodal Circles $= 0 \Rightarrow n = 0+1 = 1$
  Mode $= W_{21}$

  Fig 2(d), Nodal lines $= 1 \Rightarrow m = 1$
  Nodal Circles $= 2 \Rightarrow n = 2+1 = 3$
  Mode $= W_{13}$

- For circular plate, if $m = 0 \rightarrow$ No degenerate modes
  if $m > 0 \rightarrow$ Two degenerate modes ($\sin m\theta$, $\cos m\theta$)

Clearly for 2(c) & 2(d), W_{21} and W_{13} are degenerate modes as $m>1$

1) $W_c^{21} = R_{21}\cos 2\theta$ 2) $W_c^{13} = R_{13}\cos\theta$ } cosine modes

$W_s^{21} = R_{21}\sin 2\theta$ $W_s^{13} = R_{13}\sin\theta$ } sine modes

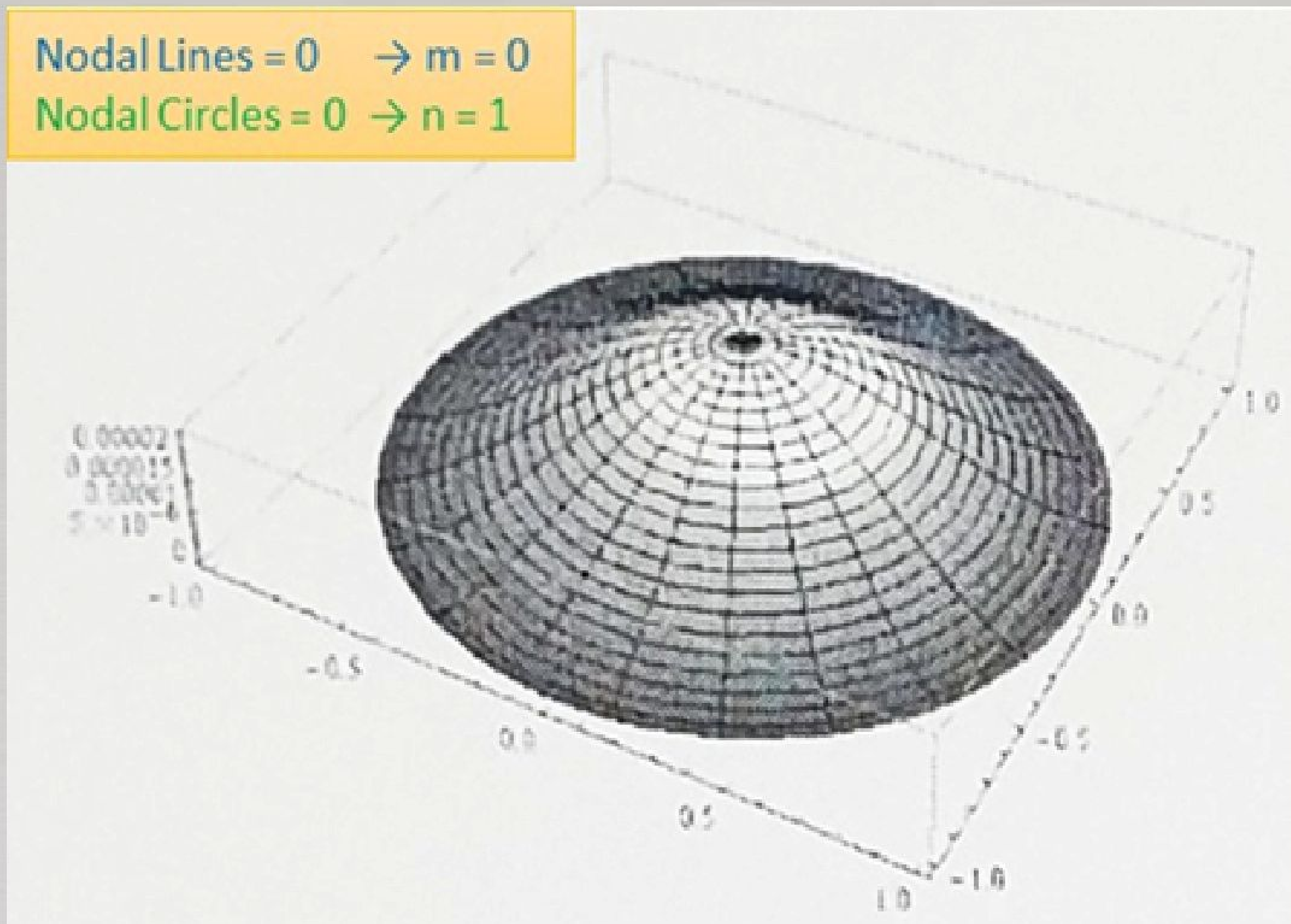


Fig. 2(a)

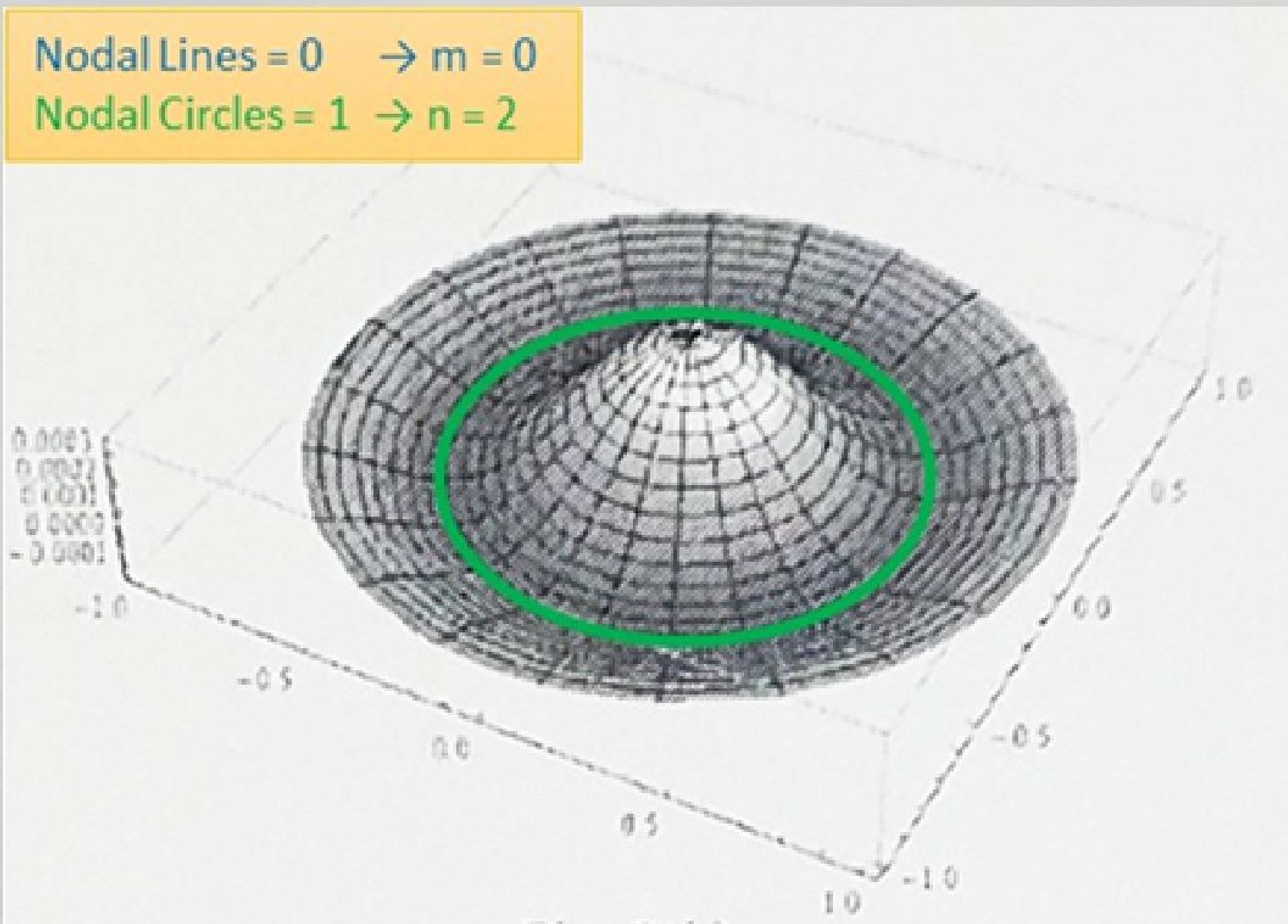


Fig. 2(b)

https://github.com/LastElectron

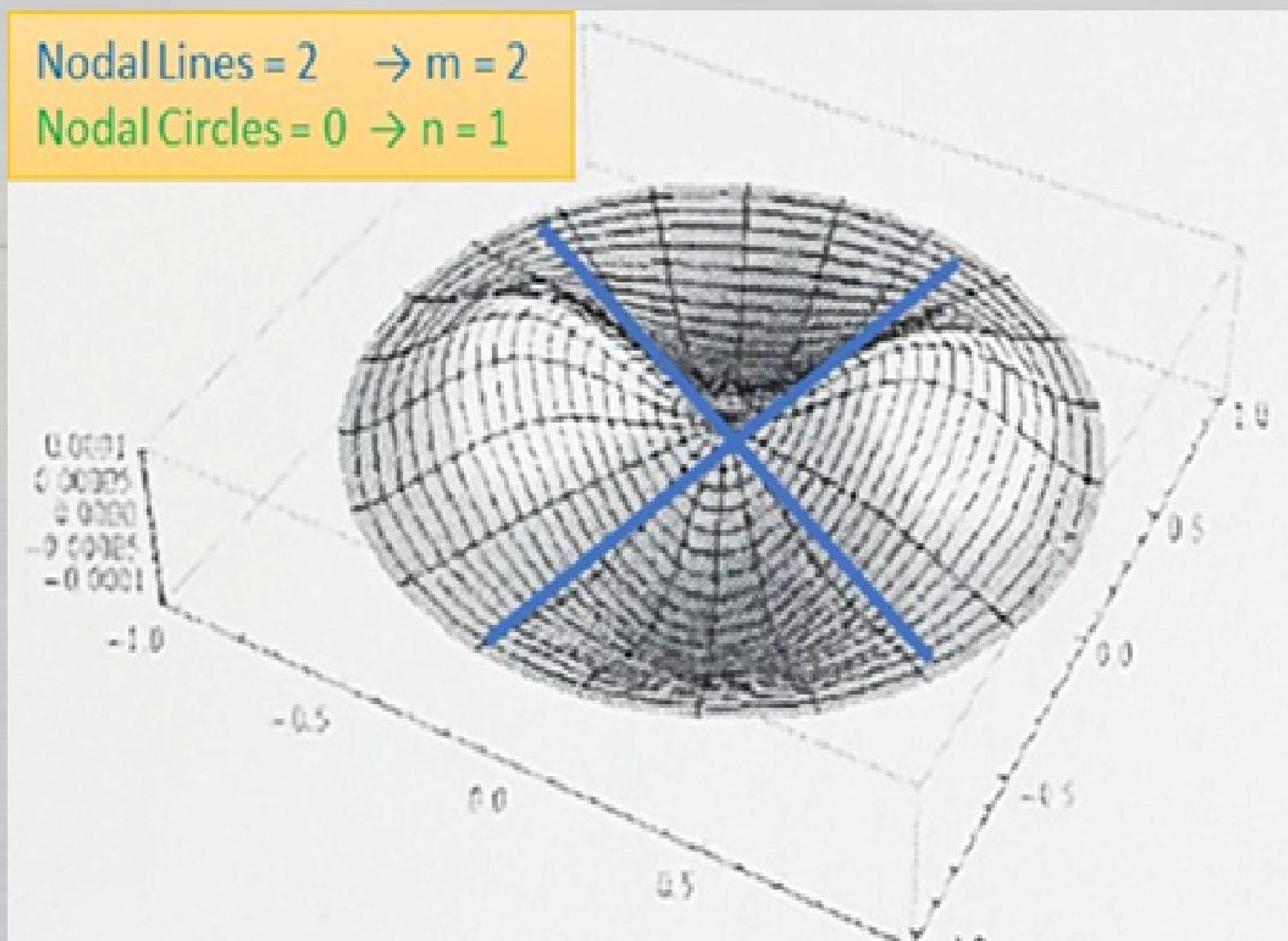


Fig. 2(c)

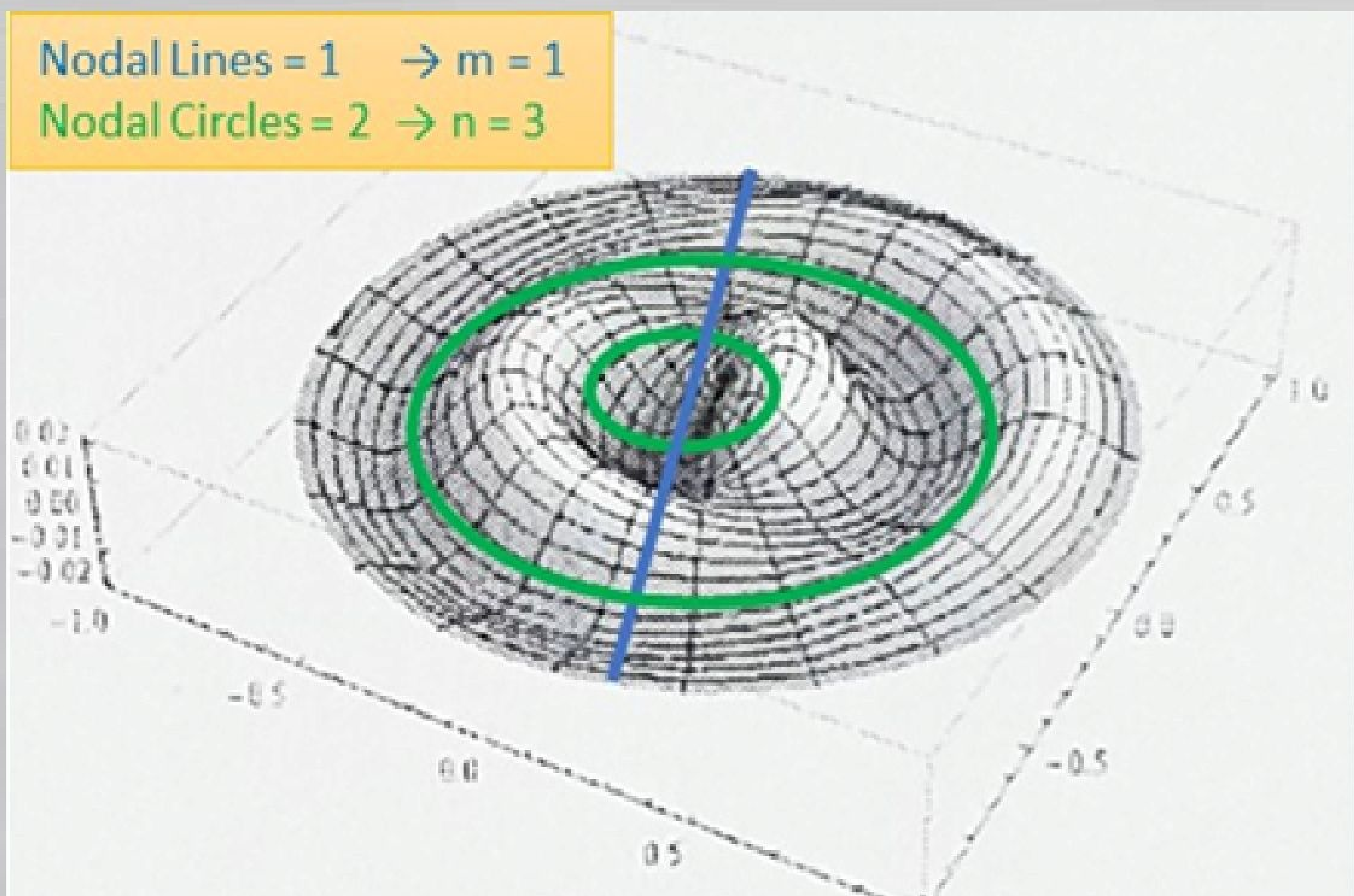
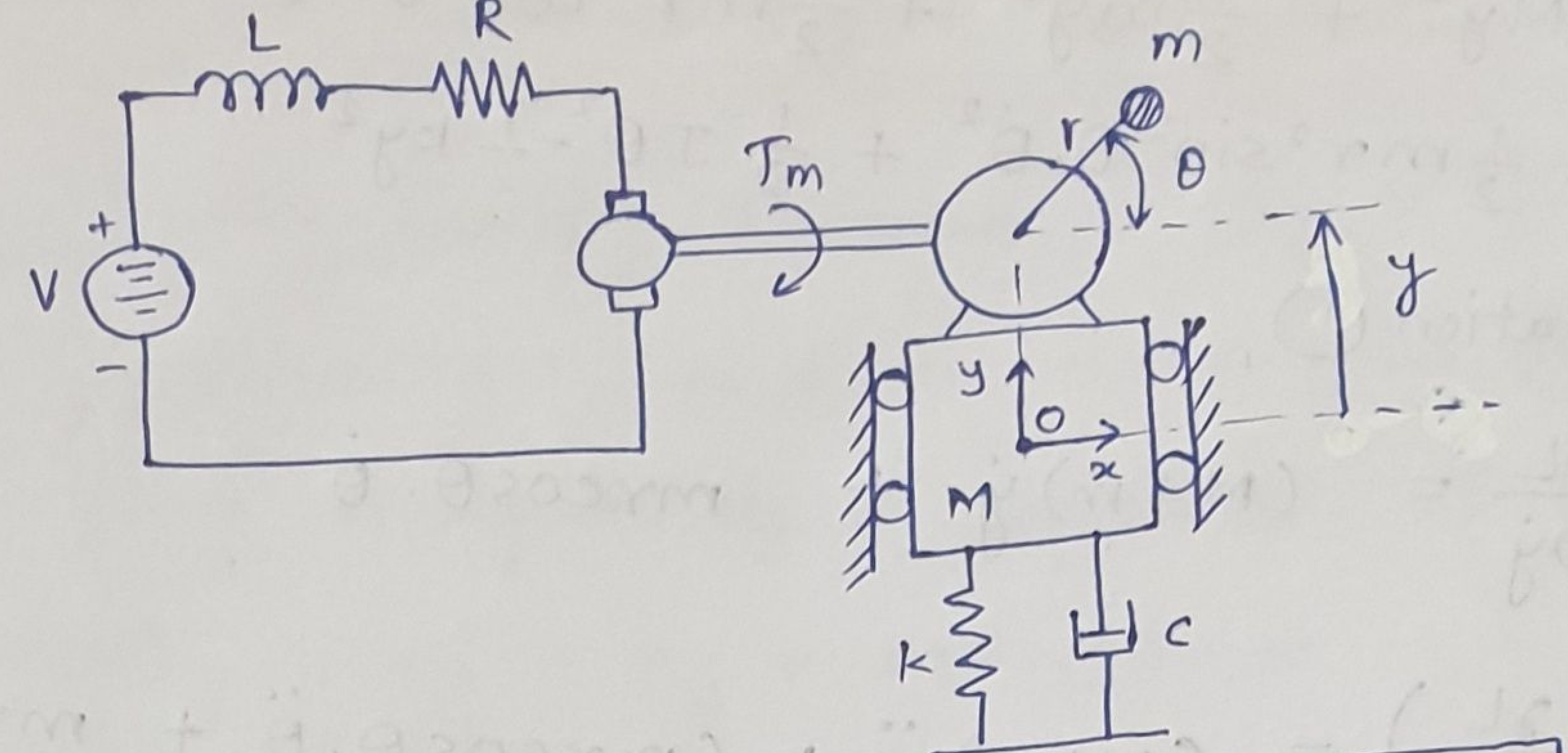


Fig. 2(d)

Q.5 Ans :- (a)



SDOF spring mass damper oscillator excited by a non-ideal DC motor

Let J = MOI of motor spindle

The position of rotating mass at any instant t,

$$x_m = r\cos\theta$$
$$y_m = r\sin\theta + y$$

Differentiate w.r.t. time,

$$\dot{x}_m = -r\sin\theta.\dot{\theta}$$
$$\dot{y}_m = r\cos\theta.\dot{\theta} + \dot{y}$$

- Kinetic Energy of system,

$$T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\dot{y}_m^2 + \frac{1}{2}m\dot{x}_m^2 + \frac{1}{2}J\dot{\theta}^2$$
$$\Rightarrow T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m(r\cos\theta.\dot{\theta}+\dot{y})^2 + \frac{1}{2}m(-r\sin\theta.\dot{\theta})^2 + \frac{1}{2}J\dot{\theta}^2$$

- Potential Energy of system,

$$V = \frac{1}{2}ky^2$$

- Dissipating Energy of system (Rayleigh Potential),

$$R = \frac{1}{2}c\dot{y}^2 + \frac{1}{2}c_\theta\dot{\theta}^2 \quad \ldots\ c_\theta \equiv \text{Air / Bearing Resistance}$$

- Lagrange Equation for two variables y and θ,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} + \frac{\partial R}{\partial \dot{y}} = Q_{iy} \quad \text{——— (1)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial R}{\partial \dot{\theta}} = Q_{i\theta} \quad \text{——— (2)}$$

... Q = External Force/Moment

- Lagrangian ($L = T - V$),

$$L = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m(\dot{y}+r\cos\theta.\dot{\theta})^2 + \frac{1}{2}m(r\sin\theta.\dot{\theta})^2$$
$$+ \frac{1}{2}J\dot{\theta}^2 - \frac{1}{2}ky^2$$

$$\Rightarrow L = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} m \dot{y}^2 + \frac{1}{2} m r^2 \cos^2\theta \cdot \dot{\theta}^2 + m \dot{y} r \cos\theta \cdot \dot{\theta}$$
$$+ \frac{1}{2} m r^2 \sin^2\theta \cdot \dot{\theta}^2 + \frac{1}{2} J \dot{\theta}^2 - \frac{1}{2} k y^2$$

For Equation ①,

- $\frac{\partial L}{\partial \dot{y}} = (M+m)\dot{y} + m r \cos\theta \cdot \dot{\theta}$

- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = (M+m)\ddot{y} + (m r \cos\theta \cdot \ddot{\theta} + m r(-\sin\theta)\dot{\theta}\cdot\dot{\theta})$
  $= (m+M)\ddot{y} + m r \cos\theta \cdot \ddot{\theta} - m r \sin\theta \cdot \dot{\theta}^2$

- $\frac{\partial L}{\partial y} = -ky$

- $\frac{\partial R}{\partial \dot{y}} = c\dot{y}$

Put all terms in ①,

$$(m+M)\ddot{y} + m r \cos\theta \cdot \ddot{\theta} - m r \sin\theta \cdot \dot{\theta}^2 + ky + c\dot{y} = 0$$

$$\Rightarrow \boxed{(m+M)\ddot{y} + c\dot{y} + ky = m r(\sin\theta \cdot \dot{\theta}^2 - \cos\theta \cdot \ddot{\theta})} \quad —(\#)$$

For Equation ②,

- $\frac{\partial L}{\partial \dot{\theta}} = m r^2 \cos^2\theta \cdot \dot{\theta} + m \dot{y} r \cos\theta + m r^2 \sin^2\theta \cdot \dot{\theta} + J\dot{\theta}$
  $= m r^2 \dot{\theta} + m \dot{y} r \cos\theta + J\dot{\theta}$

- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m r^2 \ddot{\theta} + (m r \cos\theta\, \ddot{y} + m r(-\sin\theta)\cdot\dot{\theta}\,\dot{y}) + J\ddot{\theta}$
  $= m r^2 \ddot{\theta} + m r \cos\theta\, \ddot{y} - m r \sin\theta \cdot \dot{\theta} \cdot \dot{y} + J\ddot{\theta}$

- $\frac{\partial L}{\partial \theta} = m \dot{y} r(-\sin\theta)\dot{\theta}$ ... $\left[\frac{1}{2} m r^2 \dot{\theta}^2 (\cos^2\theta + \sin^2\theta) = \frac{1}{2} m r^2 \dot{\theta}^2 \neq f(\theta)\right]$

- $\frac{\partial R}{\partial \dot{\theta}} = C_\theta \cdot \dot{\theta}$

Putting all terms in ②,

$$\Rightarrow (J + m r^2)\ddot{\theta} + m r \cos\theta\, \ddot{y} - \cancel{m r \sin\theta \cdot \dot{\theta} \cdot \dot{y}} + \cancel{m r \sin\theta \cdot \dot{\theta} \cdot \dot{y}}$$
$$+ C_\theta \cdot \dot{\theta} = T_m \quad \ldots (T_m = \text{Motor Torque})$$

$$\Rightarrow \boxed{(J + mr^2)\ddot{\theta} + mr\cos\theta\,\ddot{y} + C_\theta \cdot \dot{\theta} = T_m} \quad \text{--- } \#_2$$

(b) For steady state energy balance,

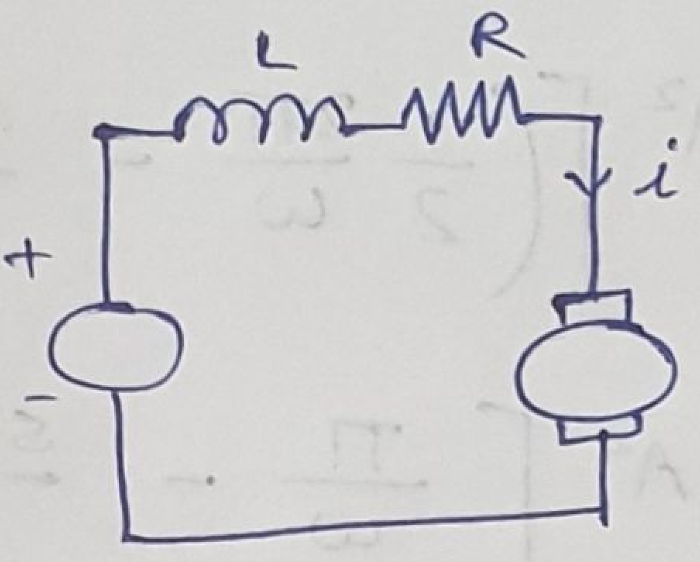
Put $\dot{\theta} = \omega =$ constant and so $\ddot{\theta} = 0$ in $\#_1$

$$(m+M)\ddot{y} + c\dot{y} + ky = mr(\sin\theta \cdot \omega^2 - 0)$$

$$(m+M)\ddot{y} + c\dot{y} + ky = mr\omega^2 \sin\theta \quad \text{--- (a)}$$

- Power delivered by permanent magnet DC motor (P_m)

$P_m = T.\omega$ ... $T =$ Torque $= \mu i$
$\mu =$ Torque constant
$i =$ Armature current



Using KVL,

$$i = \frac{V - \mu\omega}{R}$$

$$\therefore P_m = \mu i \omega$$

$$= \mu\left(\frac{V - \mu\omega}{R}\right)\omega$$

Energy delivered from motor (E_m)

$$E_m = \int_0^{2\pi/\omega} \frac{\mu(V - \mu\omega)}{R}\,\omega\, dt$$

$$= \frac{\mu(V - \mu\omega)\omega}{R}\,[t]_0^{2\pi/\omega}$$

$$= \frac{\mu(V - \mu\omega)\omega}{R} \times \frac{2\pi}{\omega}$$

$$= \frac{2\pi\mu(V - \mu\omega)}{R} \quad \text{--- (b)}$$

- Power dissipated by damper during vibration (P_d):

$$P_d = c\dot{y}^2 + C_\theta \cdot \omega^2$$

Let solution of (a) be, $y = A\cos(\omega t + \phi)$

$$\dot{y} = -A\omega \sin(\omega t + \phi)$$

$$\therefore P_d = c(-A\omega \cdot \sin(\omega t + \phi))^2 + C_\theta \cdot \omega^2$$

Dissipative energy per cycle (E_d) :

$$E_d = \int_0^{2\pi/\omega} P_d \, dt$$

$$= \int_0^{2\pi/\omega} c A^2 \omega^2 \sin^2(\omega t + \phi)\, dt + \int_0^{2\pi/\omega} C_\theta \omega^2 \, dt$$

$$= c\omega^2 A^2 \int_0^{2\pi/\omega} \left(\frac{1 - \cos 2(\omega t + \phi)}{2} \right) dt + C_\theta \omega^2 [t]_0^{2\pi/\omega}$$

$$= c\omega^2 A^2 \left[\frac{1}{2} t - \frac{\sin 2(\omega t + \phi)}{2(2\omega)} \right]_0^{2\pi/\omega} + C_\theta \omega^2 \left(\frac{2\pi}{\omega} \right)$$

$$= c\omega^2 A^2 \left[\left(\frac{1}{2} \frac{2\pi}{\omega} - \frac{\sin 2(2\pi + \phi)}{4\omega} \right) - \left(0 - \frac{\sin 2\phi}{4\omega} \right) \right] + 2\pi C_\theta \omega$$

$$= c\omega^2 A^2 \left[\frac{\pi}{\omega} - \cancel{\frac{\sin 2\phi}{4\omega}} + \cancel{\frac{\sin 2\phi}{4\omega}} \right] + 2\pi C_\theta \omega$$

$$= \frac{c\omega^2 A^2 \pi}{\omega} + 2\pi C_\theta \omega$$

$$= c A^2 \pi \omega + 2\pi C_\theta \omega \qquad \text{——— (c)}$$

By steady state energy balance,

$$E_m = E_d$$

$$\frac{2\pi \mu (V - \mu\omega)}{R} = \pi c \omega A^2 + 2\pi C_\theta \omega$$

$$\frac{2\pi \mu V}{R} = \left[\cancel{\pi} c A^2 + 2\cancel{\pi} C_\theta + \frac{2\cancel{\pi} \mu^2}{R} \right] \omega$$

Relation bet^n voltage and excitation frequency

$$\boxed{V = \frac{R}{2\mu} \left(c A^2 + 2 C_\theta + \frac{2\mu^2}{R} \right) \omega}$$