Indian Institute of Technology Kharagpur

Department of Mechanical Engineering

**Session:** Spring, 2023-24 **Mid-Semester Examination**

**Subject:** Vibration of Structures (ME60428) **Number of students:** 46

**Time:** 2 hours **Total Marks:** 60

1. Using **Hamilton's Principle**, derive the equation of motion and boundary conditions of an uniform fixed-fixed string undergoing transverse free vibration. Take $T$ as constant tension, $\rho$ as volume density, and $A$ as the uniform cross-sectional area of the string. The string vibrates in $xz$-plane. State all the assumptions used during the derivation. Express the final equation in terms of the wave speed $c = \sqrt{T/\rho A}$. (15)

2. Consider an uniform homogeneous shaft undergoing torsional vibration as shown in the **Fig. 1**. Two massive discs having mass moment of inertia $I_{D_1}$ and $I_{D_2}$, respectively, are mounted at two different locations of the shaft. Write the **equation of motion** for this shaft. All symbols have their usual meaning. (5)

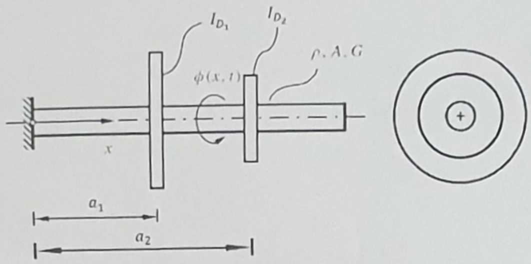



Fig. 1

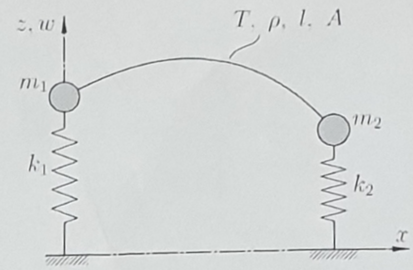



Fig. 2

3. A uniform taut string is undergoing transverse vibrations as shown in **Fig. 2**. Parameter definitions are as follows: field variable $w(x,t)$, volume density ($\rho$), length ($l$), cross-sectional area ($A$), string tension ($T$), point masses ($m_1$, $m_2$) and spring stiffness ($k_1$, $k_2$).
(a) Determine the **boundary conditions** of the given system using force balance. (6)
(b) Assuming harmonic solution of the form $w(x,t) = W(x)e^{i\omega t}$, where $W(x)$ denotes the eigenfunction and $\omega$ denotes the eigenfrequency, show that the **characteristic equation** for this system is given by: $\tan\left(\frac{\omega l}{c}\right) = \frac{-T\omega c\left[(k_1+k_2)-(m_1+m_2)\omega^2\right]}{c^2(k_1-m_1\omega^2)(k_2-m_2\omega^2)-T^2\omega^2}$, where $c = \sqrt{T/\rho A}$. (14)

4. A fixed-fixed string is given an initial pull of height $h$ at $x = al$ and released at $t = 0$. The initial shape of the string is shown in **Fig. 3**. Take $l$ as the length of the string and parameter $0 < a < 1$. Write the **initial conditions** for this string. (5)

5. **Figure 4** shows a homogeneous tapered bar having circular cross-section. The governing equation of this bar under free vibration is given by $\rho A(x)u_{,tt} - [EA(x)u_{,x}]_{,x} = 0$, where $u(x,t)$ is the field variable, $\rho$ the volume density, $A(x)$ is the cross-sectional variation, and $E$ is the elastic modulus. Using **Galerkin's method**, discretize the equation of motion using the comparison functions as $P_i(x) = \left(1 - \frac{x}{l}\right)^{k+1} - 1$ ($k = 1, 2, ..., N$). For $N = 2$ determine the first two natural frequencies from the discretized system. Take $R = 0.02$ m, $l = 1$ m, $E = 200$ GPa, $\rho = 7800$ kg/m$^3$. (15)

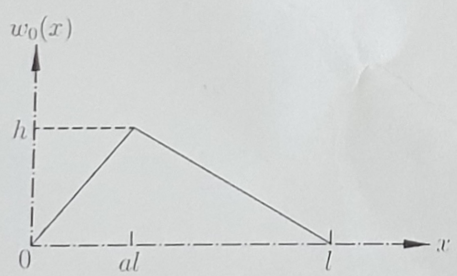



Fig. 3

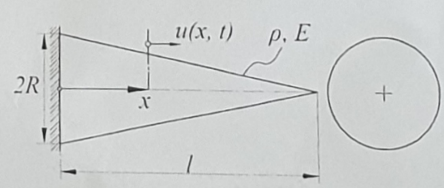
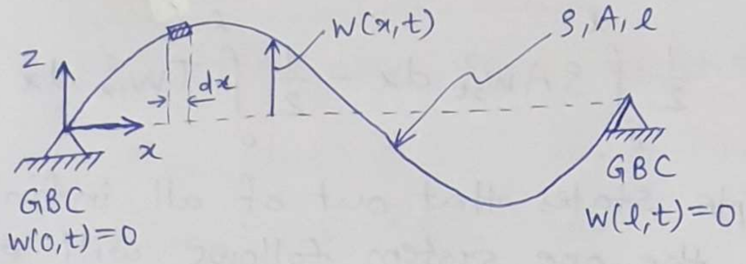



Fig. 4

# Vibration of Structures (Mid Sem 2024)

https://github.com/LastElectron

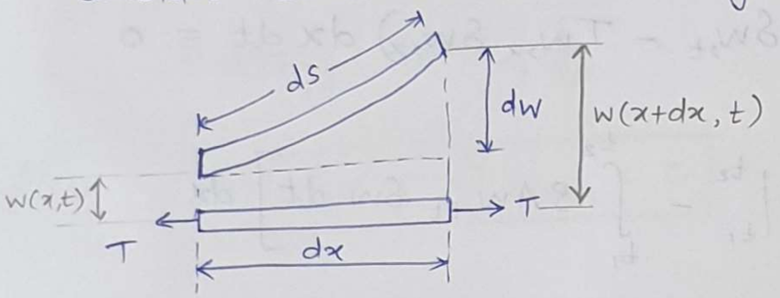
Q.1 Ans:- $\boxed{\text{Hamilton's Principle}}$




Model Assumptions

1) Motion is planar, so displacement of string perpendicular to x-z plane i.e. in y-direction is neglected.
2) Slope is small i.e. transverse displacements are assumed to be small to ensure linear behaviour.
3) Tension does not change with displacement.
4) No damping or external force acting on system.

Consider small element of length 'd' of string.




$$ds^2 = dx^2 + dw^2$$
$$ds^2 = dx^2(1 + w_{,x}^2)$$
$$ds = dx\sqrt{1 + w_{,x}^2}$$

- Kinetic Energy $T$ of string, $T = \int_0^l \frac{1}{2} \rho A\, dx\, w_{,t}^2$

$$T = \frac{1}{2}\int_0^l \rho A w_{,t}^2\, dx$$

- Potential Energy $V$ of string, $V = \int_0^l T(ds - dx)$

$$= \int_0^l T\, dx\left(\sqrt{1 + w_{,x}^2} - 1\right)$$

$$= \int_0^l T\, dx\left(\cancel{1} + \frac{1}{2}w_{,x}^2 - \cancel{1}\right)$$

... Using Binomial expansion & ignoring higher order terms as $\frac{dw}{dx}$ is very small)

$$\mathcal{V} = \frac{1}{2}\int_0^l T w_{,x}^2 \, dx$$

Lagrangian, $\mathcal{L} = \mathcal{T} - \mathcal{V}$

$$= \frac{1}{2}\int_0^l \rho A w_{,t}^2 \, dx - \frac{1}{2}\int_0^l T w_{,x}^2 \, dx$$

Hamilton's principle states that out of all infinitely available paths, the one system follows will extremize the action.

$$\delta \int_{t_1}^{t_2} \mathcal{L} \, dt = 0$$

$$\Rightarrow \quad \delta \int_{t_1}^{t_2} \frac{1}{2}\int_0^l [\rho A w_{,t}^2 - T w_{,x}^2] \, dx \, dt = 0$$

$$\Rightarrow \quad \int_{t_1}^{t_2}\int_0^l (\rho A w_{,t}\, \delta w_{,t} - T w_{,x}\, \delta w_{,x}) \, dx \, dt = 0$$

$$\Rightarrow \quad \int_0^l \left[ \rho A w_{,t}\, \delta w \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \rho A w_{,tt}\, \delta w \, dt \right] dx$$

$$- \int_{t_1}^{t_2} \left[ T w_{,x}\, \delta w \Big|_0^l - \int_0^l (T w_{,x})_{,x}\, \delta w \, dx \right] dt = 0$$

$$\Rightarrow \quad \int_0^l \rho A w_{,t}\, \delta w \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} T w_{,x}\, \delta w \Big|_0^l \, dt$$

$$- \int_{t_1}^{t_2}\int_0^l [\rho A w_{,tt} - T w_{,xx}]\, \delta w \, dx \, dt = 0$$

... since $T$ is constant.

First term is always zero since variation of field variable '$w$' at initial and final time is zero.

$$\therefore \quad \delta w(x, t_1) \equiv 0 \quad \text{and} \quad \delta w(x, t_2) \equiv 0$$

Integrand of third term has to be zero, as it gives equation of transverse vibration of string.

$$\boxed{\rho A w_{,tt} - T w_{,xx} = 0}$$

So, second term also zero which gives

$$T w_{,x}(0,t) \equiv 0 \quad \text{or} \quad w(0,t) \equiv 0$$

$$T w_{,x}(l,t) \equiv 0 \quad \text{or} \quad w(l,t) \equiv 0$$

These are possible boundary conditions.

For fixed-fixed string, boundary conditions are

$$w(0,t) \equiv 0$$

$$w(l,t) \equiv 0$$

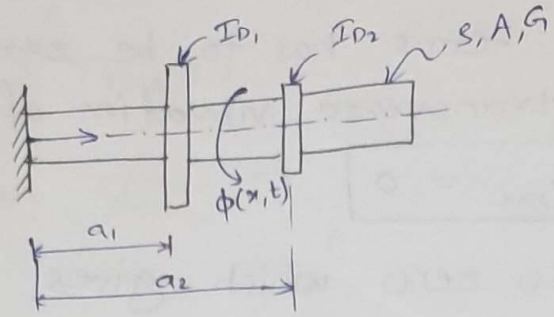
We can modify EOM as,

$$w_{,tt} - \frac{T}{\rho A} w_{,xx} = 0$$

Putting $c = \sqrt{\frac{T}{\rho A}}$,

$$\boxed{w_{,tt} - c^2 w_{,xx} = 0}$$

Q.2 Ans:-




- Kinetic Energy of Shaft : $T = \frac{1}{2}\int_0^l \rho I_p \phi_{,t}^2 \, dx$

- Potential Energy of Shaft : $V = \frac{1}{2}\int_0^l G I_p \phi_{,x}^2 \, dx$

- Lagrangian Density, $\mathcal{L}$ :-
$$\mathcal{L} = \frac{1}{2}\left(\rho I_p \phi_{,t}^2 - G I_p \phi_{,x}^2\right)$$

EOM by generalised formulation :-

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \phi_{,t}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi_{,x}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = \tau_{ext} \quad \text{——} (\#)$$

- $\frac{\partial \mathcal{L}}{\partial \phi_{,t}} = \rho I_p \phi_{,t}$ $\qquad \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \phi_{,t}}\right) = \rho I_p \phi_{,tt}$

- $\frac{\partial \mathcal{L}}{\partial \phi_{,x}} = -G I_p \phi_{,x}$ $\qquad \boxed{\frac{\partial}{\partial}\left(\frac{\partial \mathcal{L}}{\partial \phi_{,x}}\right) = -G I_p \phi_{,xx}}$

- $\frac{\partial \mathcal{L}}{\partial \phi} = 0$

- $\tau_{ext} = \tau_{ext_1} + \tau_{ext_2}$

$\tau_{ext_1}$: External Torque per unit length due to disc 1

$\tau_{ext_2}$: External Torque per unit length due to disc 2.

$\because$ $\tau_{ext_1} = n_{E_1}(x,t) = -I_{D_1}\phi_{,tt}\,\delta(x-a_1)$

$\tau_{ext_2} = n_{E_2}(x,t) = -I_{D_2}\phi_{,tt}\,\delta(x-a_2)$

Negative sign indicate disc opposing motion of shaft.

$$= -I_{D_1}\phi_{,tt}\,\delta(x-a_1) - I_{D_2}\phi_{,tt}\,\delta(x-a_2)$$

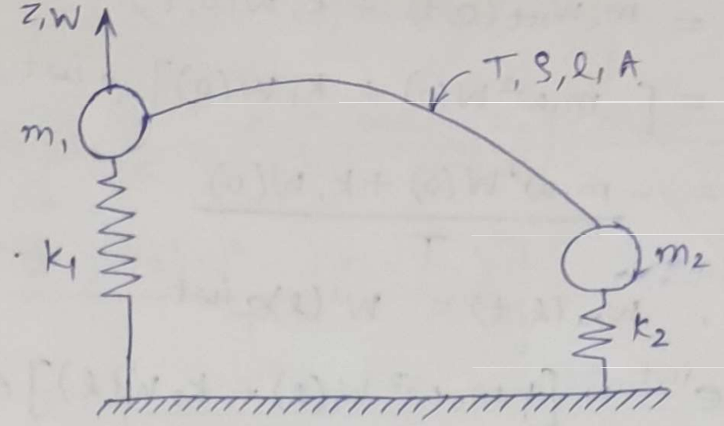
Combining all these terms, & put in (#),

$$\boxed{\rho I_p \phi_{,tt} - G I_p \phi_{,xx} = \left[-I_{D_1}\delta(x-a_1) - I_{D_2}\delta(x-a_2)\right]\phi_{,tt}}$$

$\therefore$ EOM of shaft

OR $$\boxed{\left[\rho I_p + I_{D_1}\delta(x-a_1) + I_{D_2}\delta(x-a_2)\right]\phi_{,tt} - G I_p \phi_{,xx} = 0}$$

Q.3 Ans :-

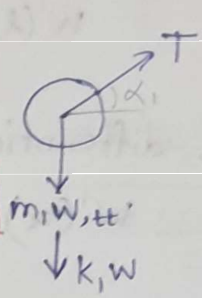



[a] Boundary conditions

① Force balance at $m_1$,

$$\Sigma F_y = m a_y$$

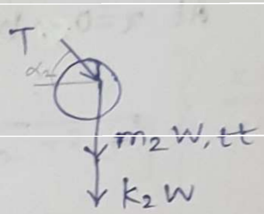
$$\boxed{T w_{,x}(0,t) = m_1 w_{,tt}(0,t) + k_1 w(0,t)}$$




$T\sin\alpha \approx T\tan\alpha \approx T\frac{dw}{dx} = T w_{,x}$

② Force balance at $m_2$,

$$\boxed{T w_{,x}(l,t) = -m_2 w_{,tt}(l,t) - k_2 w(l,t)}$$




[b] Assumed solution $\quad w(x,t) = W(x) e^{i\omega t}$

So, EOM is $\quad w_{,tt} - \frac{T}{\rho A} w_{,xx} = 0$

$$\Rightarrow \quad w_{,tt} - c^2 w_{,xx} = 0 \qquad \ldots\; c = \sqrt{\frac{T}{\rho A}}$$

Substituting assumed solution,

$$\boxed{\begin{aligned} w_{,tt} &= W(x)(i\omega)^2 e^{i\omega t} = -\omega^2 W e^{i\omega t} \\ w_{,xx} &= W'' e^{i\omega t} \end{aligned}}$$

$$\Rightarrow \quad -\omega^2 W e^{i\omega t} - c^2 W'' e^{i\omega t} = 0$$

$$\Rightarrow \quad (c^2 W'' + \omega^2 W) e^{i\omega t} = 0$$

$$\Rightarrow \quad \boxed{W'' + \frac{\omega^2}{c^2} W = 0}$$

Solution of this differential equation is, ——— Ⓐ

$$W = D\cos\left(\frac{\omega}{c}x\right) + H\sin\left(\frac{\omega}{c}x\right)$$

For Boundary conditions, using $w_{,x}(x,t) = W' e^{i\omega t}$

For BC 1, $\quad w_{,x}(0,t) = W'(0) e^{i\omega t}$

Putting above value in BC ①,

$$T W'(0) e^{i\omega t} = m_1 w_{,tt}(0,t) + k_1 w(0,t)$$

$$T W'(0) e^{i\omega t} = [-m_1 \omega^2 W(0) + k_1 W(0)] e^{i\omega t}$$

$$W'(0) = \frac{-m_1 \omega^2 W(0) + k_1 W(0)}{T} \quad \text{——— (B)}$$

Similarly for BC 2, $w_{,x}(l,t) = W'(l) e^{i\omega t}$

$$T W'(l) e^{i\omega t} = [+m_2 \omega^2 W(l) - k_2 W(l)] e^{i\omega t}$$

$$W'(l) = \frac{m_2 \omega^2 W(l) - k_2 W(l)}{T} \quad \text{——— (C)}$$

Using (A), differentiate (A) wrt $x$,

$$W'(x) = D \frac{\omega}{c}\left(-\sin\left(\frac{\omega}{c}x\right)\right) + H \frac{\omega}{c} \cos\left(\frac{\omega}{c}x\right) \quad \text{——— (D)}$$

At $x=0$, $W'(0) = 0 + \frac{H\omega}{c}$ ——— (From (D))

$$\frac{-m_1 \omega^2 W(0) + k_1 W(0)}{T} = \frac{H\omega}{c} \quad \text{——— (From (B))}$$

$$H = \frac{c}{\omega}\left(\frac{-m_1\omega^2 + k_1}{T}\right) W(0) \quad \text{——— (E)}$$

At $x=l$, $W'(l) = D\frac{\omega}{c}\left(-\sin\frac{\omega l}{c}\right) + \frac{H\omega}{c}\left(\cos\frac{\omega l}{c}\right)$ ——— (From (D))

$$\Rightarrow \frac{m_2\omega^2 W(l) - k_2 W(l)}{T} = -\frac{D\omega}{c}\left(\sin\frac{\omega l}{c}\right) + \frac{H\omega}{c}\left(\cos\frac{\omega l}{c}\right)$$

——— From (C)

Substitute value of $H$ from (E),

$$\Rightarrow \left(\frac{m_2\omega^2 - k_2}{T}\right) W(l) = -\frac{D\omega}{c}\left(\sin\frac{\omega l}{c}\right) + \frac{c}{\omega}\frac{\omega}{c}\left(\frac{-m_1\omega^2 + k_1}{T}\right) W(0)\left(\cos\frac{\omega l}{c}\right)$$

We know, $W(0) = D(1) + H(0) = D$

$W(l) = D\left(\cos\frac{\omega l}{c}\right) + H\left(\sin\frac{\omega l}{c}\right)$ ... from (A)

Put above values, we get

$$\Rightarrow \left(\frac{m_2\omega^2 - k_2}{T}\right)\left[\cancel{D}\left(\cos\frac{\omega l}{c}\right) + \underbrace{\frac{c}{\omega}\left(\frac{-m_1\omega^2 + k_1}{T}\right)\cancel{D}}_{H}\left(\sin\frac{\omega l}{c}\right)\right]$$

$$= -\frac{\cancel{D}\omega}{c}\left(\sin\frac{\omega l}{c}\right) + \left(\frac{-m_1\omega^2 + k_1}{T}\right)\cancel{D}\left(\cos\frac{\omega l}{c}\right)$$

$$\Rightarrow \cos\frac{\omega l}{c}\left[\left(\frac{m_2\omega^2-k_2}{T}\right)-\left(\frac{-m_1\omega^2+k_1}{T}\right)\right]$$
$$=\sin\frac{\omega l}{c}\left[-\frac{\omega}{c}-\frac{c}{\omega}\left(\frac{-m_1\omega^2+k_1}{T}\right)\left(\frac{m_2\omega^2-k_2}{T}\right)\right]$$

$$\Rightarrow \cos\frac{\omega l}{c}\left[\frac{(m_1+m_2)\omega^2-(k_1+k_2)}{T}\right]$$
$$=\sin\frac{\omega l}{c}\left[(-1)\left\{\frac{\omega^2T^2+c^2(k_1-m_1\omega^2)(m_2\omega^2-k_2)}{c\omega T^2}\right.\right]$$

$$\Rightarrow \frac{\left[\dfrac{(m_1+m_2)\omega^2-(k_1+k_2)}{T}\right]}{(-1)\left[\dfrac{\omega^2T^2-c^2(k_1-m_1\omega^2)(k_2-m_2\omega^2)}{c\omega T^2}\right]}=\frac{\sin\frac{\omega l}{c}}{\cos\frac{\omega l}{c}}$$

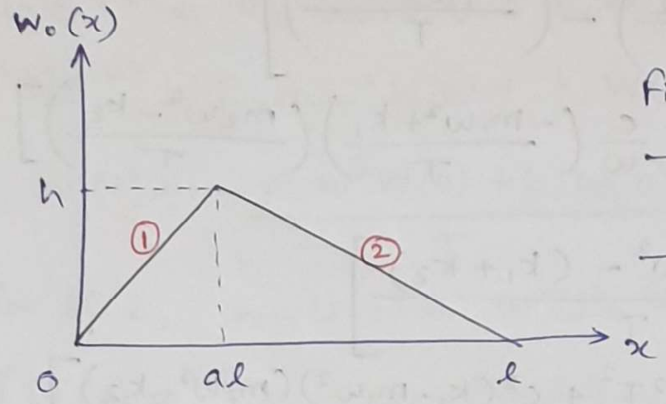
$$\Rightarrow \tan\left(\frac{\omega l}{c}\right)=\left\{\frac{(k_1+k_2)-(m_1+m_2)\omega^2}{(-1)\left[c^2(k_1-m_1\omega^2)(k_2-m_2\omega^2)-T^2\omega^2\right]}\right\}\frac{c\omega T^2}{T}$$

$$\Rightarrow \boxed{\tan\left(\frac{\omega l}{c}\right)=\frac{-T\omega c\left[(k_1+k_2)-(m_1+m_2)\omega^2\right]}{c^2(k_1-m_1\omega^2)(k_2-m_2\omega^2)-T^2\omega^2}}$$

where $c=\sqrt{\frac{T}{\rho A}}$

Hence Proved.

Q.4 Ans :-




Fixed fixed string
→ Initial pull of height 'h' @ $x = al$
→ Release at $t = 0$

For Line 1 betn $(0,0)$ to $(al, h)$,

$$\frac{w_0(x) - 0}{h - 0} = \frac{x - 0}{al - 0} \qquad \dots \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$w_0(x) = \frac{hx}{al} \qquad \text{Also } V_0(x) = \frac{\partial w_0(x)}{\partial t} = 0$$

For Line 2 betn $(al, h)$ to $(l, 0)$

$$\frac{w_0(x) - h}{0 - h} = \frac{x - al}{l - al}$$

$$w_0(x) = \frac{x - al}{l(1-a)}(-h) + h$$

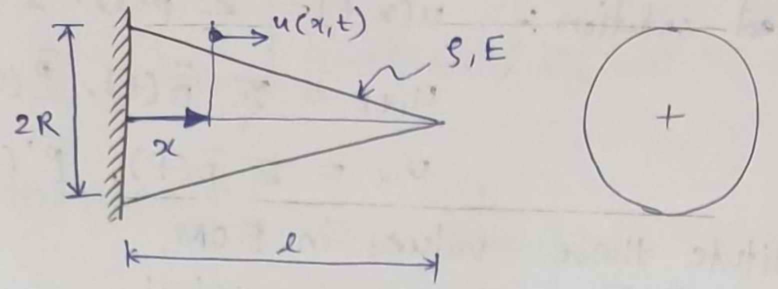
$$= \frac{hal - hx + hl(1-a)}{l(1-a)}$$

$$= \frac{\cancel{hal} - hx + hl - \cancel{hal}}{l(1-a)}$$

$$w_0(x) = \frac{h(l - x)}{l(1-a)} \qquad \text{Also } V_0(x) = \frac{\partial w_0(x)}{\partial t} = 0$$

$$\boxed{\begin{array}{l} \text{Initial conditions of string} \\ w_0(x) = \dfrac{hx}{al} \qquad \dots \quad 0 \leq x < al \\ \phantom{w_0(x)} = \dfrac{h(l-x)}{l(1-a)} \qquad \dots \quad al \leq x \leq l \\ V_0(x) = 0 \end{array}}$$

Q.5 Ans :- $\boxed{\text{Galerkin Method}}$




Given :- $\rho A(x) u_{,tt} - [EA(x) u_{,x}]_{,x} = 0$

Comparison function, $P_i(x) = \left(1 - \frac{x}{l}\right)^{k+1} - 1$ ... $k = 1, 2, \dots, N$

$N = 2$ (Number of terms)

$R = 0.02\ m$

$l = 1\ m$

$E = 200\ GPa$

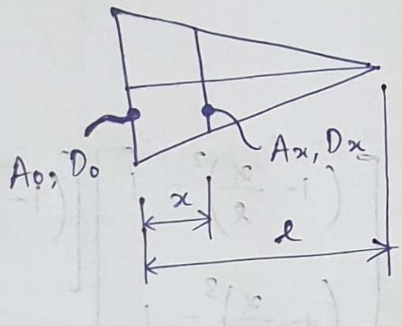
$\rho = 7800\ kg/m^3$

To find :- A) Discretize the system

B) First two natural frequencies, $\omega_1$ & $\omega_2 = ?$

Soln :- A) Assume $A_0$ & $D_0$ be area & diameter at fixed end

$A_x$ & $D_x$ be area & diameter at distance $x$.




1) $\frac{D_x}{D_0} = \frac{l - x}{l}$

2) $\frac{A_x}{A_0} = \frac{\frac{\pi}{4} D_x^2}{\frac{\pi}{4} D_0^2} = \left(\frac{D_x}{D_0}\right)^2 = \left(\frac{l-x}{l}\right)^2$ — by (1)

$A_x = A_0 \left(1 - \frac{x}{l}\right)^2$

we know, $A_0 = \frac{\pi}{4} (2R)^2 = \pi R^2$

$\therefore \boxed{A_x = \pi R^2 \left(1 - \frac{x}{l}\right)^2}$

EOM :- $\rho A(x) u_{,tt} - [EA(x) u_{,x}]_{,x} = 0$

Assumed solution :- $u(x,t) = \sum \bar{p}(t) \cdot \bar{\Phi}(x)$

$$u_{,tt} = \sum \ddot{\bar{p}}(t) \cdot \bar{\Phi}(x)$$

$$u_{,x} = \sum \bar{p}(t) \cdot \bar{\Phi}'(x) = \Phi_i'^T p_i$$

Substitute these values in EOM,

$$\rho A(x) \sum \ddot{\bar{p}}_i \bar{\Phi}_i - [EA(x) \sum \bar{p}_i \bar{\Phi}_i']' = 0$$

Multiply both sides by $\bar{\Phi}_j$ we get

$$\therefore \sum \rho A(x) \bar{\Phi}_i^T \ddot{\bar{p}}_i \bar{\Phi}_j - \sum [EA(x) \bar{\Phi}_i'^T \bar{p}_i \bar{\Phi}_j]' = 0$$

$\boxed{M\ddot{\bar{p}} + K\bar{p} = 0}$ ... Discretized form of EOM.

$$M = \int_0^l \mu(x) \Phi \Phi^T dx$$

$$K = -\int_0^l \Phi \cdot K[\Phi^T] dx$$

Here $\mu(x) = \rho A(x)$

$K[u] = +[EA(x) u_{,x}]_{,x}$

B) $$M = \int_0^l \rho A(x) \Phi \Phi^T dx$$

$$M = \int_0^l \rho \times \pi R^2 \left(1 - \frac{x}{l}\right)^2 \begin{bmatrix} \left(1-\frac{x}{l}\right)^2 - 1 \\ \left(1-\frac{x}{l}\right)^3 - 1 \end{bmatrix} \begin{bmatrix} \left(1-\frac{x}{l}\right)^2 - 1 & \left(1-\frac{x}{l}\right)^3 - 1 \end{bmatrix} dx$$

Putting values;

$$M = \int_0^1 7800 \times \pi (0.02)^2 (1-x)^2 \begin{bmatrix} \cancel{1} - 2x + x^2 - \cancel{1} \\ \cancel{1} - 3x + 3x^2 - x^3 - \cancel{1} \end{bmatrix}_{2\times1} \begin{bmatrix} \cancel{1} - 2x + x^2 - \cancel{1} & \cancel{1} - 3x + 3x^2 - x^3 - \cancel{1} \end{bmatrix}_{1\times2} dx$$

$$= 7800 \times \pi (0.02)^2 \int_0^1 (1-x)^2 \begin{bmatrix} (-2x + x^2)^2 & (-2x+x^2)(-3x+3x^2-x^3) \\ (-3x+3x^2-x^3)(-2x+x^2) & (-3x+3x^2-x^3)^2 \end{bmatrix} dx$$

$$M = 7800 \times \pi (0.02)^2 \int_0^1 \begin{bmatrix} (1-x)^2(-2x+x^2)^2 & (1-x)^2(-2x+x^2)(-3x+3x^2-x^3) \\ (1-x)^2(-3x+3x^2-x^3)(-2x+x^2) & (1-x)^2(-3x+3x^2-x^3)^2 \end{bmatrix} dx$$

$$= 7800 \times \pi (0.02)^2 \begin{bmatrix} \frac{8}{105} & \frac{11}{120} \\ \frac{11}{120} & \frac{1}{9} \end{bmatrix}$$

$$K = -\int_0^l P \cdot K[P^T]\, dx$$

$$= -\int_0^l P \times [EA(x)\, P^{T'}]'\, dx$$

$$= -\int_0^l \begin{bmatrix} (1-\frac{x}{l})^2 - 1 \\ (1-\frac{x}{l})^3 - 1 \end{bmatrix} \times \left[ E \times \pi R^2 \left(1-\frac{x}{l}\right)^2 \left[ \left(1-\frac{x}{l}\right)^2 - 1 \quad \left(1-\frac{x}{l}\right)^3 - 1 \right]' \right]' dx$$

$$= -\int_0^1 \begin{bmatrix} (1-x)^2 - 1 \\ (1-x)^3 - 1 \end{bmatrix} \times \left[ 200 \times 10^9 \times \pi (0.02)^2 (1-x)^2 \left[ (1-x)^2 - 1 \quad (1-x^3) - 1 \right]' \right]' dx$$

$$= -\int_0^1 \begin{bmatrix} -2x + x^2 \\ -3x + 3x^2 - x^3 \end{bmatrix} \times 200 \times 10^9 \times \pi (0.02)^2 \left\{ (1-x)^2 \left[ -2x + x^2 \quad -3x + 3x^2 - x^3 \right]' \right\}' dx$$

$$= -200 \times 10^9 \times \pi (0.02)^2 \int_0^1 \begin{bmatrix} -2x + x^2 \\ -3x + 3x^2 - x^3 \end{bmatrix} \left\{ (1-x)^2 \left[ -2 + 2x \quad -3 + 6x - 3x^2 \right] \right\}' dx$$

$$= -200 \times 10^9 \times \pi (0.02)^2 \int_0^1 \begin{bmatrix} -2x + x^2 \\ -3x + 3x^2 - x^3 \end{bmatrix} \left[ (1-2x+x^2)(-2+2x) \quad (1-2x+x^2)(-3+6x-3x^2) \right]' dx$$

$$= -200 \times 10^9 \times \pi \times (0.02)^2 \int_0^1 \begin{bmatrix} -2x + x^2 \\ -3x + 3x^2 - x^3 \end{bmatrix} \begin{bmatrix} (-2+2x)(-2+2x) + (1-2x+x^2)(2) & (-2+2x)(-3+6x-3x^2) + (1-2x+x^2)(6-6x) \end{bmatrix} dx$$

$$= -200 \times 10^9 \times \pi \times (0.02)^2 \int_0^1 \begin{bmatrix} -2x + x^2 \\ -3x + 3x^2 - x^3 \end{bmatrix} \begin{bmatrix} 4 - 8x + 4x^2 + 2 - 4x + 2x^2 & 6 - 12x + 6x^2 - 6x + 12x^2 - 6x^3 + 6 - 6x - 12x + 12x^2 + 6x^2 - 6x^3 \end{bmatrix} dx$$

$$K = -200\times10^9 \times \pi(0.02)^2 \int_0^1 \begin{bmatrix} -2x+x^2 \\ -3x+3x^2-x^3 \end{bmatrix} \begin{bmatrix} 6x^2-12x+6 & -12x^3+36x^2-36x+12 \end{bmatrix} dx$$

$$= -200\times10^9 \times \pi(0.02)^2 \int_0^1 \begin{bmatrix} (-2x+x^2)(6x^2-12x+6) & (-2x+x^2)(-12x^3+36x^2-36x+12) \\ (-3x+3x^2-x^3)(6x^2-12x+6) & (-3x+3x^2-x^3)(-12x^3+36x^2-36x+12) \end{bmatrix} dx$$

$$= -200\times10^9 \times \pi(0.02)^2 \begin{bmatrix} -4/5 & -1 \\ -1 & -\frac{9}{7} \end{bmatrix}$$

$$= 200\times10^9 \times \pi(0.02)^2 \begin{bmatrix} \frac{4}{5} & 1 \\ 1 & \frac{9}{7} \end{bmatrix}$$

Put values of M & K in discretized EOM,

$$7800\times\pi(0.02)^2 \begin{bmatrix} \frac{8}{105} & \frac{11}{120} \\ \frac{11}{120} & \frac{1}{9} \end{bmatrix} \begin{Bmatrix} \ddot{P}_1 \\ \ddot{P}_2 \end{Bmatrix} + 200\times10^9\times\pi(0.02)^2 \begin{bmatrix} \frac{4}{5} & 1 \\ 1 & \frac{9}{7} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Put $\bar{P} = \bar{V} e^{i\omega t}$

$\ddot{\bar{P}} = -\omega^2 \bar{V} e^{i\omega t}$

for non-trivial solution,

$\det[-\omega^2 M + K] = 0$

$$\Rightarrow \det\left\{ 7800\times\cancel{\pi(0.02)^2} \begin{bmatrix} -\frac{8}{105}\omega^2 & -\frac{11}{120}\omega^2 \\ -\frac{11}{120}\omega^2 & -\frac{1}{9}\omega^2 \end{bmatrix} + 200\times10^9\times\cancel{\pi(0.02)^2} \begin{bmatrix} \frac{4}{5} & 1 \\ 1 & \frac{9}{7} \end{bmatrix} \right\} = 0$$

$$\Rightarrow \det \begin{bmatrix} -\frac{4160}{7}\omega^2 + 1.6\times10^{11} & -715\omega^2 + 200\times10^9 \\ -715\omega^2 + 200\times10^9 & -\frac{2600}{3}\omega^2 + 2.57143\times10^{11} \end{bmatrix} = 0$$

$$\Rightarrow \left(-\frac{4160}{7}\omega^2 + 1.6\times10^{11}\right)\left(-\frac{2600}{3}\omega^2 + 2.57143\times10^{11}\right)$$

$$-\left(-715\omega^2 + 200\times10^9\right)^2 = 0$$

$$\Rightarrow (3822.619)\omega^4 - (5.483078\times10^{12})\omega^2 + 1.14288\times10^{21} = 0$$

$$\omega^2 = 1181280579 \;,\; 253096742$$

$$\omega = \pm 34369.76 \;,\; \pm 15909.01$$

Ignoring negative values,

$$\boxed{\begin{aligned}\omega_1 &= 34369.76 \text{ rad/s}\\ \omega_2 &= 15909.01 \text{ rad/s}\end{aligned}}$$