

Indian Institute of Technology Kharagpur
Department of Mechanical Engineering

Session: Spring, 2022-23

Subject: Vibration of Structures (ME60428)

Time: 2 hours

Mid-Semester Examination

Number of students: 34

Total Marks: 60

Instructions: Answer all questions showing all derivation steps. No marks without steps.

- Using **Hamilton's Principle**, derive the governing differential equation of a homogeneous bar (modeled as a 1-D elastic continua) undergoing axial vibration. State all the assumptions used during the derivation. (12)
- Consider a homogeneous uniform bar connected to a string which is under tension T , as shown in the Fig. 1. What are the **initial conditions** for the given system, when the string suddenly snaps at $t = 0$. All symbols have their usual meaning. (8)

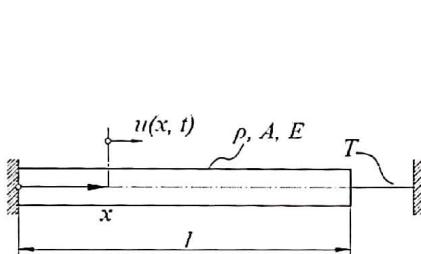


Fig. 1

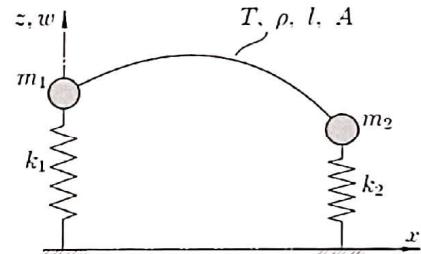


Fig. 2

- State the **boundary conditions** of a uniform taut string undergoing transverse vibrations as shown in Fig. 2, in terms of the field variable $w(x, t)$. Parameter definitions are as follows: Volume density (ρ), length (l), cross-sectional area (A), string tension (T), point masses (m_1, m_2) and spring stiffness (k_1, k_2). (8)
- Consider a uniform taut string with an interacting mass element placed at $x = al$ ($0 < a < 1$) undergoing transverse vibrations as shown in the Fig. 3. We use the **Ritz method** to approximately determine its first two natural frequencies. For this purpose, we assume the solution form as $w(x, t) = b_1(t)x(l - x) + b_2(t)x^2(l - x)$ which discretizes the system as $M\ddot{b} + Kb = 0$. Determine the elements of the first row in the stiffness matrix K . (8)
- The governing equation of a uniform hanging string is given by $w_{tt} - [g(l-x)w_x]_x = 0$, where $w(x, t)$ is the field variable, g is the acceleration due to gravity, and l is the length of the beam. Using **Galerkin's method**, discretize the equation of motion using the comparison functions as $P_i(x) = x^i$ ($i = 1, 2, \dots, N$). For $N = 2$ determine the eigenfrequencies from the discretized system. Take $g = 10 \text{ m/s}^2$, and $l = 1 \text{ m}$. (14)
- A homogeneous tapered bar of circular cross-section undergoing axial vibrations is shown in Fig. 4. Assume an admissible function for the field variable $u(x, t)$ of the form $H_k(x) = (x/l)^k$, where k is an integer. Using **Rayleigh's quotient**, determine the value of k that yields the lowest value of the fundamental natural frequency of the system. (10)

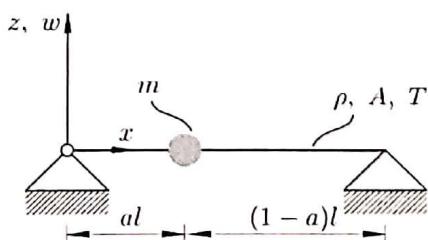


Fig. 3

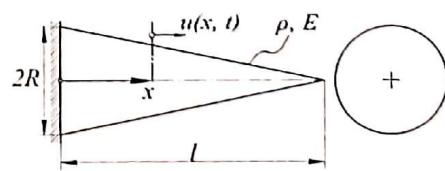
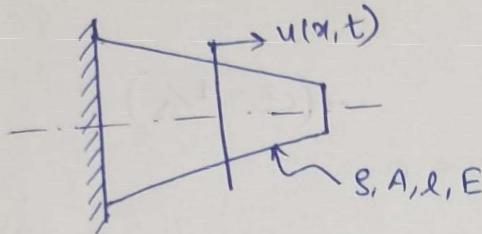


Fig. 4

Q.1 Ans:-

Hamilton's Principle

Homogenous bar (1D elastic continuum) undergoing axial vibration



Assumptions

- 1) Bar is homogenous (S, E constant)
- 2) Bar modeled as 1D continuum, so any transverse vibrations neglected.
- 3) Bar subjected to axial load only, neglect any external forces or moments
- 4) Bar is linear elastic (Hooke's Law valid)
- 5) Displacement ' u ' is very small compared to length of bar ' l ', allows us to use linear elasticity theory.

$$\textcircled{1} \quad \text{Kinetic Energy} = T = \int_0^l \frac{1}{2} S A d x \underbrace{u_{,t}^2}_{m/v^2}$$

$$= \frac{1}{2} \int_0^l S A u_{,t}^2 d x$$

$$\textcircled{2} \quad \text{Potential Energy} = V = \frac{1}{2} \int_0^l \sigma \epsilon (A dx)$$

$$= \frac{1}{2} \int_0^l E A u_{,x}^2 d x \quad \dots \quad \begin{bmatrix} \epsilon = u_{,x} \\ \sigma = E \epsilon = E u_{,x} \end{bmatrix}$$

$$\textcircled{3} \quad \text{Lagrangian Density}, L = T - V$$

$$= \frac{1}{2} \left[\int_0^l S A u_{,t}^2 d x - \int_0^l E A u_{,x}^2 d x \right]$$

Using Hamilton Principle,

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\Rightarrow \delta \int_{t_1}^{t_2} \int_0^l \left[\frac{1}{2} S A u_{,t}^2 - \frac{1}{2} E A u_{,x}^2 \right] dx dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \int_0^l \left[S A u_{,t} (S u)_{,t} - E A u_{,x} (S u)_{,x} \right] dx dt = 0$$

$$\Rightarrow \int_0^l \left[S A u_{,t} \cancel{S u} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} S A u_{,tt} S u dt \right] dx$$

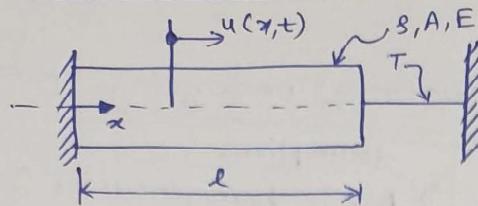
$$- \int_{t_1}^{t_2} \left[E A u_{,x} \cancel{S u} \Big|_0^l - \int_0^l (E A u_{,x})_{,x} S u dt \right] = 0$$

$$\Rightarrow \int_0^l \int_{t_1}^{t_2} \left[- S A u_{,tt} + (E A u_{,x})_{,x} \right] S u dx dt = 0$$

Governing Differential Equation

$$- S A u_{,tt} + (E A u_{,x})_{,x} = 0 \quad \text{OR} \quad S A u_{,tt} - (E A u_{,x})_{,x} = 0$$

Q.2 Ans:- Collapse of a stretched bar



EOM :- $u_{,tt} - c^2 u_{,xx} = 0 \quad \dots (c^2 = E/G)$

BCs :- $u(0, t) = 0, \quad u_{,x}(l, t) = 0$

\hookrightarrow No strain at end
 $\epsilon = \frac{\partial u}{\partial x} \Big|_{x=l} = 0$

Variational Formulation
 $\left[\frac{\partial}{\partial u_{,x}} \left(\frac{1}{2} u_{,x}^2 - \frac{1}{2} \frac{1}{c^2} u_{,xx}^2 \right) \right]_{x=l} = 0$
 $- c^2 u_{,xxl} \Big|_{x=l} = 0$

ICs :- $u(x, 0) = \frac{Tx}{EA}, \quad u_{,t}(x, 0) = 0 \quad \dots$

$T = EA u_{,x}$
 $\Rightarrow u_{,x} = \frac{T}{EA}$
 $\Rightarrow \frac{\partial u}{\partial x} = \frac{T}{EA}$
 $\Rightarrow u = \frac{Tx}{EA}$

Assumed solution, $u(x, t) = \sum_{k=1}^{\infty} (c_k \cos \omega_k t + s_k \sin \omega_k t) V_k(x)$

where $\omega_k = \frac{(2k-1)\pi c}{2l}$

$V_k(x) = \sin \left[\frac{(2k-1)\pi x}{2l} \right]$

Using 1st initial condition, Put $t=0$ in assumed solution,

$$u(x, 0) = \sum_{k=1}^{\infty} [c_k(1) + 0] V_k(x) = \frac{Tx}{EA}$$

$$\Rightarrow \sum_{k=1}^{\infty} c_k V_k(x) = \frac{Tx}{EA}$$

Do inner product w.r.t. V_j ,

$$\Rightarrow c_j \langle V_j, V_j \rangle = \langle \frac{Tx}{EA}, V_j \rangle$$

$$\Rightarrow c_j = \frac{\langle \frac{Tx}{EA}, V_j \rangle}{\langle V_j, V_j \rangle}$$

$$\Rightarrow c_j = \frac{\int_0^l \frac{Tx}{EA} \sin \left[\frac{(2j-1)\pi x}{2l} \right] dx}{\int_0^l \sin^2 \left[\frac{(2j-1)\pi x}{2l} \right] dx}$$

$$\Rightarrow c_j = \frac{\left[\frac{Tx}{EA} \left[-\frac{\cos \left[\frac{(2j-1)\pi x}{2l} \right]}{\frac{(2j-1)\pi}{2l}} \right] \right]_0^l - \int_0^l \frac{T}{EA} \cdot \left[\frac{-\cos \left[\frac{(2j-1)\pi x}{2l} \right]}{\frac{(2j-1)\pi}{2l}} \right] dx}{\int_0^l \frac{1 - \cos \left[\frac{(2j-1)\pi x}{2l} \right]}{2} dx}$$

$\dots \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$

$$\Rightarrow c_j = \frac{\left\{ \frac{T\ell}{EA} \left[-\frac{\cos(\frac{(2j-1)\pi}{2})}{\frac{(2j-1)\pi}{2\ell}} \right] - 0 \right\} + \frac{T}{EA} \left[\frac{\sin(\frac{(2j-1)\pi x}{2\ell})}{\left(\frac{(2j-1)\pi}{2\ell} \right)^2} \right]^{\ell}}{\left[\frac{1}{2}\ell - \frac{\sin(\frac{(2j-1)\pi x}{2\ell})}{\frac{2x(2j-1)\pi}{\ell}} \right]^{\ell}}.$$

$$\Rightarrow c_j = \frac{\left(-\frac{2T\ell^2}{EA} \frac{\cos(\frac{(2j-1)\pi}{2})}{\frac{(2j-1)\pi}{2\ell}} \right) + \frac{T}{EA} \left[\frac{\sin(\frac{(2j-1)\pi}{2})}{\left(\frac{(2j-1)\pi}{2\ell} \right)^2} - 0 \right]}{\left(\frac{1}{2}\ell - \frac{\sin(\frac{(2j-1)\pi}{2\ell})}{\frac{2(2j-1)\pi}{\ell}} \right) - (0 - 0)}$$

Here $\cos\left[\frac{(2j-1)\pi}{2}\right] = \cos\left[m\frac{\pi}{2}\right]$ where $m = 2j-1$
 $j = 1, 2, 3, \dots, \infty$
 $m \rightarrow \text{Always odd number}$

$$\Rightarrow c_j = \frac{0 + \frac{T}{EA} \frac{4\ell^2}{(2j-1)^2\pi^2} \sin\left(\frac{(2j-1)\pi}{2}\right)}{\frac{1}{2}\ell - \frac{\ell}{2} \frac{\sin(2j-1)\pi}{(2j-1)\pi}}$$

Here, $\sin[(2j-1)\pi] = \sin[m\pi]$ where $m \rightarrow \text{always odd number}$
 $= 0$

$$\Rightarrow c_j = \frac{\frac{T}{EA} \frac{4\ell^2}{(2j-1)^2\pi^2} \sin\left(\frac{(2j-1)\pi}{2}\right)}{\frac{1}{2}\ell - 0}$$

$$\Rightarrow c_j = \boxed{\frac{8T\ell}{(2j-1)^2\pi^2 EA} \sin\left(\frac{(2j-1)\pi}{2}\right)}$$

\therefore Assumed solution's "c_k" value now known.

Using 2nd initial condition,

Differentiate assumed soln w.r.t. t,

$$u_{,t} = \sum_{k=1}^{\infty} [c_k (\sin \omega_k t) \cdot \omega_k + s_k (\cos \omega_k t) \cdot \omega_k] u_k(x)$$

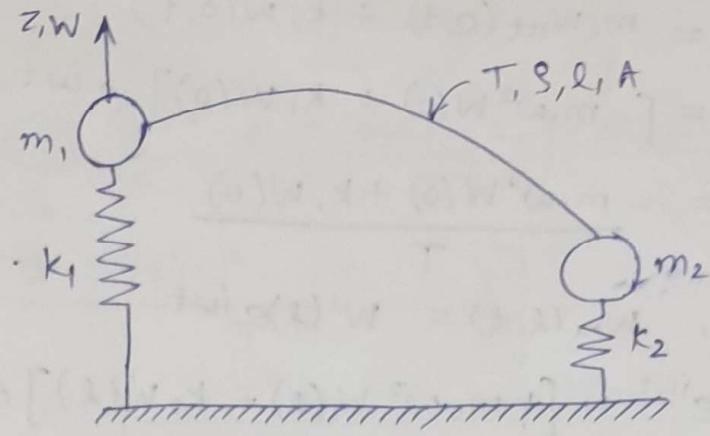
Put t = 0,

$$u_{,t}(x, 0) = \sum_{k=1}^{\infty} (0 + s_k(1) \cdot \omega_k) u_k(x) = 0$$

As $u_k(x) \neq 0$, $\boxed{s_k = 0}$

\therefore Final solution is, $u(x, t) = \sum_{k=1}^{\infty} \frac{8T\ell}{(2k-1)^2\pi^2 EA} (-1)^{k-1} \cos \omega_k t \cdot \sin\left[\frac{(2k-1)\pi x}{2\ell}\right]$ To get abs displacement (+ve) all time

Q.3 Ans:-

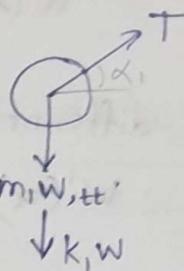


[a] Boundary Conditions

① Force balance at m_1 ,

$$\sum F_y = m_1 a_y$$

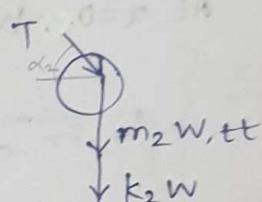
$$T_{W,x}(0, t) = m_1 w_{tt}(0, t) + k_1 w(0, t)$$



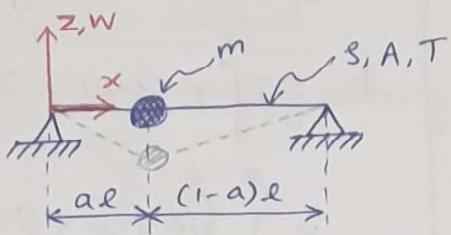
$$\begin{aligned} T \sin \alpha \\ \approx T \tan \alpha \\ \approx T \frac{dw}{dx} \\ = T_{W,x} \end{aligned}$$

② Force balance at m_2 ,

$$T_{W,x}(l, t) = -m_2 w_{tt}(l, t) - k_2 w(l, t)$$



Q. 4 Ans:-



Ritz Method

$$\delta \int_{t_1}^{t_2} (SAw_{,tt} - Tw_{,xx}) dx dt = 0$$

By considering mass,

$$SAw_{,tt} - Tw_{,xx} = p(t)$$

$$SAw_{,tt} - Tw_{,xx} = -mw_{,tt}^2$$

$$[SA + m\delta(x-al)]w_{,tt} - Tw_{,xx} = 0$$

In variational form,

$$\delta \int_{t_1}^{t_2} \int_0^l \{ [SA + m\delta(x-al)]w_{,tt}^2 - Tw_{,xx}^2 \} dx dt = 0$$

Assumed solution, $w(x,t) = b_1(t) \cdot x(l-x) + b_2(t) \cdot x^2(l-x)$

$$= \sum_{k=1}^2 b_k(t) H_k(x)$$

$$= \begin{bmatrix} x(l-x) & x^2(l-x) \end{bmatrix} \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$$

$$= \vec{H}_k^T \vec{b}_k$$

$$\Rightarrow \delta \int_{t_1}^{t_2} (\vec{b}_k^T M \vec{b}_k - \vec{b}_k^T K \vec{b}_k) dt = 0 \quad \dots \left[\begin{array}{l} M = \int_0^l SA + m\delta(x-al) \vec{H}_k \vec{H}_k^T dx \\ K = \int_0^l T \vec{H}_k \vec{H}_k^T dx \end{array} \right]$$

$$\Rightarrow M \ddot{\vec{b}}_k + K \vec{b}_k = 0$$

$$\text{Here, } K = \int_0^l T \frac{d}{dx} \begin{bmatrix} x(l-x) \\ x^2(l-x) \end{bmatrix} \cdot \frac{d}{dx} \begin{bmatrix} x(l-x) & x^2(l-x) \end{bmatrix} dx$$

$$= \int_0^l T \begin{bmatrix} l-2x \\ 2xl-3x^2 \end{bmatrix} \begin{bmatrix} l-2x & 2xl-3x^2 \end{bmatrix} dx$$

$$= \int_0^l T \begin{bmatrix} (l-2x)^2 & (l-2x)(2xl-3x^2) \\ (2xl-3x^2)(l-2x) & (2xl-3x^2)^2 \end{bmatrix} dx$$

$$= \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$\begin{aligned}
 K_{11} &= \int_0^l T(l-2x)^2 dx \\
 &= T \int_0^l (l^2 + 4x^2 - 4lx) dx \\
 &= T \left[l^2 x + \frac{4x^3}{3} - \frac{4lx^2}{2} \right]_0^l \\
 &= T \left[l^3 + \frac{4l^3}{3} - 2l^3 - 0 \right] \\
 &= \frac{Tl^3}{3}
 \end{aligned}$$

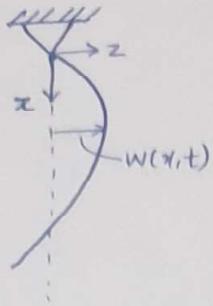
$$\begin{aligned}
 K_{12} &= \int_0^l T(l-2x)(2xl-3x^2) dx \\
 &= T \int_0^l (2xl^2 - 7x^2l + 6x^3) dx \\
 &= T \left[\frac{2x^2l^2}{2} - \frac{7x^3l}{3} + \frac{6x^4}{4} \right]_0^l \\
 &= T \left[l^4 - \frac{7}{3}l^4 + \frac{3}{2}l^4 - 0 \right] \\
 &= \frac{Tl^4}{6}
 \end{aligned}$$

$$\begin{aligned}
 k_{21} = k_{12} &= \frac{Tl^4}{6} \\
 k_{22} &= \int_0^l T(2xl-3x^2)^2 dx \\
 &= T \int_0^l (4x^2l^2 + 9x^4 - 12x^3l) dx \\
 &= T \left[\frac{4x^2l^3}{3} + \frac{9x^5}{5} - \frac{12lx^4}{4} \right]_0^l \\
 &= T \left[\frac{4l^2l^3}{3} + \frac{9l^5}{5} - 3l^5 \right] \\
 &= Tl^5 \left[\frac{4}{3} + \frac{9}{5} - 3 \right] \\
 &= \left[\frac{20+27-45}{15} \right] Tl^5 \\
 &= \frac{2}{15} Tl^5
 \end{aligned}$$

$$\boxed{[K] = \begin{bmatrix} \frac{I l^3}{3} & \frac{T l^4}{6} \\ \frac{T l^4}{6} & \frac{2}{15} T l^5 \end{bmatrix}}$$

Q. 5 Ans:- Galerkin Method

Uniform Hanging String



$$N = 2 \text{ (No. of terms)}$$

$$g = 10 \text{ m/s}^2$$

$$l = 1 \text{ m}$$

$$\text{EOM} : -w_{,tt} - [g(l-x)w_{,x}]_{,x} = 0 \quad \text{--- } ①$$

$$\text{Assumed Solution} : -w(x,t) = \sum_{k=1}^N p_k(t) P_k(x) = \bar{P}^T \bar{P} \quad \text{--- } ②$$

Substituting in EOM we get,

$$\bar{P}^T \ddot{\bar{P}} + K[\bar{P}^T] \bar{P} = e(x,t) \quad \text{--- } ③$$

$$\langle e(x,t), H_j(x) \rangle = 0$$

$$\text{In Galerkin method, } H_j(x) = P_j(x)$$

$$\therefore \langle e(x,t), P_j(x) \rangle = 0$$

$$\therefore \int_0^l e(x,t) P_j(x) dx = 0$$

Multiply ③ by $P_j(x)$ and integrate wrt x ,

$$\therefore M(x) \bar{P}^T \ddot{\bar{P}} \bar{P} + K[\bar{P}^T] \bar{P} \bar{P} = 0$$

$$M \ddot{\bar{P}} + K \bar{P} = 0$$

$$\text{Here } [M] = \int_0^l M(x) \bar{P} \bar{P}^T dx$$

$$[K] = \int_0^l \bar{P} K [\bar{P}^T] dx$$

$$\text{Here } M(x) = 1$$

$$K[\bar{P}] = -[g(l-x)(\dots)_{,x}]_{,x}$$

$$P(x) = P_i(x) = x^i \quad \dots (i=1, 2, \dots, N) \dots \text{given}$$

$$\therefore [M] = \int_0^l M(x) \begin{bmatrix} P_1(x) \\ P_2(x) \end{bmatrix}_{2 \times 1} \begin{bmatrix} P_1(x) & P_2(x) \end{bmatrix}_{1 \times 2} dx$$

$$= \int_0^1 1 \begin{bmatrix} x \\ x^2 \end{bmatrix} [x \quad x^2] dx$$

$$[M] = \int_0^1 \begin{bmatrix} x^2 & x^3 \\ x^3 & x^4 \end{bmatrix} dx$$

$$= \begin{bmatrix} x^3/3 & x^4/4 \\ x^4/4 & x^5/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/4 \\ 1/4 & 1/5 \end{bmatrix}$$

$$[K] = \int_0^1 \begin{bmatrix} P_1(x) \\ P_2(x) \end{bmatrix} Eg(l-x) [P_1(M) \quad P_2(M)]' J' dx$$

$$= \int_0^1 \begin{bmatrix} x \\ x^2 \end{bmatrix} [10(1-x) \quad 1 \quad 2x]' dx$$

$$= \int_0^1 \begin{bmatrix} x \\ x^2 \end{bmatrix} x(-1) \left[(-1)[1 \quad 2x] + (1-x)[0 \quad 2] \right] dx$$

$$= \int_0^1 \begin{bmatrix} x \\ x^2 \end{bmatrix} x(-1) \left[[-1 \quad -2x] + [0 \quad 2-2x] \right] dx$$

$$= \int_0^1 \begin{bmatrix} x \\ x^2 \end{bmatrix} x(-1) \left[-1 \quad +2-4x \right] dx$$

$$= (-10) \int_0^1 \begin{bmatrix} -x & 2x-4x^2 \\ -x^2 & 2x^2-4x^3 \end{bmatrix} dx$$

$$= (-10) \begin{bmatrix} -x^2/2 & 2x^2/2 - 4x^3/3 \\ -x^3/3 & 2x^3/3 - 4x^4/4 \end{bmatrix} \Big|_0^1$$

$$= (-10) \begin{bmatrix} -1/2 & 1-4/3 \\ -1/3 & 2/3-1 \end{bmatrix}$$

$$= \begin{bmatrix} +5 & +10/3 \\ +10/3 & +10/3 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 1/4 \\ 1/4 & 1/5 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} + \begin{bmatrix} +5 & +10/3 \\ -10/3 & +10/3 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Put } \bar{p} = \bar{v} e^{i\omega t} \Rightarrow -\ddot{\bar{p}} = -\omega^2 \bar{v} e^{i\omega t}$$

$$-\omega^2 \begin{bmatrix} 1/3 & 1/4 \\ 1/4 & 1/5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 5 & 10/3 \\ 10/3 & 10/3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for nontrivial soln,

$$\det[\omega^2 M + k] = 0$$

$$\det \left\{ \begin{bmatrix} -\omega^2/3 & -\omega^2/4 \\ -\omega^2/4 & -\omega^2/5 \end{bmatrix} + \begin{bmatrix} 5 & 10/3 \\ 10/3 & 10/3 \end{bmatrix} \right\} = 0$$
$$\begin{vmatrix} -\frac{\omega^2}{3} + 5 & -\frac{\omega^2}{4} + \frac{10}{3} \\ -\frac{\omega^2}{4} + \frac{10}{3} & -\frac{\omega^2}{5} + \frac{10}{3} \end{vmatrix} = 0$$
$$\left(-\frac{\omega^2}{2} + 5 \right) \left(-\frac{\omega^2}{5} + \frac{10}{3} \right) - \left(-\frac{\omega^2}{4} + \frac{10}{3} \right) \left(-\frac{\omega^2}{4} + \frac{10}{3} \right) = 0$$
$$\frac{\omega^4}{10} - \frac{10}{6}\omega^2 - \omega^2 + \frac{50}{3} - \frac{\omega^4}{16} + \frac{10}{12}\omega^2 + \frac{10}{12}\omega^2 - \frac{100}{9} = 0$$
$$\left(\frac{1}{10} - \frac{1}{16} \right)\omega^4 + \left(-\frac{10}{6} - 1 + \frac{10}{12} + \frac{10}{12} \right)\omega^2 + \left(\frac{50}{3} - \frac{100}{9} \right) = 0$$

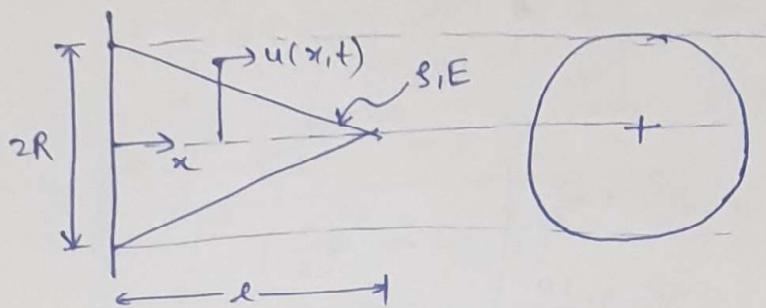
$$\omega^2 = 18.7766 \text{ or } 7.89$$

$$\boxed{\omega_1 = 4.333 \text{ rad/s}}$$

$$\boxed{\omega_2 = 2.8089 \text{ rad/s}}$$

<https://github.com/LastElectron>

Q.6 Ans:- [Rayleigh Method]

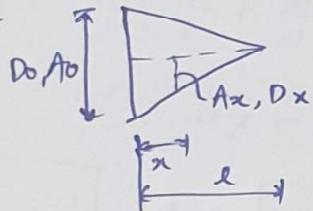


Conical bar (circular c/s)

Given :- $H_k(x) = \left(\frac{x}{l}\right)^k$... $k \rightarrow \text{Real +ve}$

To find :- $k = ?$

Soln :-



$$\frac{Dx}{D_o} = \frac{l-x}{l}$$

$$\frac{\frac{\pi}{4}(Dx)^2}{\frac{\pi}{4}(D_o)^2} = \frac{\frac{\pi}{4}(l-x)^2}{\frac{\pi}{4}l^2} = \frac{A_x}{A_o}$$

$$\therefore A_x = A_o \left(\frac{l-x}{l}\right)^2$$

$$A_x = A_o \left(1 - \frac{x}{l}\right)^2$$

$$T = KE = \frac{1}{2} \int_0^l S A U_{,t}^2 dx$$

$$V = PE = \frac{1}{2} \int_0^l E A U_{,x}^2 dx$$

$$\Sigma = KE + PE = 0$$

$$\frac{1}{2} \int_0^l (S A U_{,t}^2 + E A U_{,x}^2) dx = 0$$

Put $U(x,t) = H(x)e^{i\omega t}$
 $U_{,t} = H(i\omega)e^{i\omega t}$
 $U_{,x} = H'e^{i\omega t}$

$$\frac{1}{2} \int_0^l [-\omega^2 S A H^2 + E A H'^2] (e^{i\omega t})^2 dx = 0$$

$$\omega^2 = \frac{\int_0^l E A H'^2 dx}{\int_0^l S A H^2 dx} := R(x)$$

$$\therefore \omega^2 = \frac{E \int_0^l A \left(1 - \frac{x}{l}\right)^2 \cdot \left[k \left(\frac{x}{l}\right)^{k-1}\right]^2 dx}{S \int_0^l A_o \left(1 - \frac{x}{l}\right)^2 \left(\frac{x}{l}\right)^{2k} dx}$$

$$R[U(n)]^2 = \frac{E \int_0^l \left(1 - \frac{2x}{\ell} + \frac{x^2}{\ell^2}\right) K\left(\frac{x}{\ell}\right)^{2k-2} dx}{S \int_0^l \left(1 - \frac{2x}{\ell} + \frac{x^2}{\ell^2}\right) \left(\frac{x}{\ell}\right)^{2k} dx}$$

$$= \frac{EK}{S} \frac{\int_0^l \left(\frac{x}{\ell}\right)^{2k-2} - 2\left(\frac{x}{\ell}\right)^{2k-1} + \left(\frac{x}{\ell}\right)^{2k} dx}{\int_0^l \left(\frac{x}{\ell}\right)^{2k} - 2\left(\frac{x}{\ell}\right)^{2k+1} + \left(\frac{x}{\ell}\right)^{2k+2} dx}$$

$$= \frac{EK}{S} \frac{\left[\frac{x^{2k-1}}{(2k-1)\ell^{2k-2}} - \frac{2x^{2k}}{(2k)\ell^{2k-1}} + \frac{x^{2k+1}}{(2k+1)\ell^{2k}} \right]_0^l}{\left[\frac{x^{2k+1}}{(2k+1)\ell^{2k}} - \frac{2x^{2k+2}}{(2k+2)\ell^{2k+1}} + \frac{x^{2k+3}}{(2k+3)\ell^{2k+2}} \right]_0^l}$$

$$R[U(n)] = \frac{EK}{S} \left[\frac{\frac{l}{2k-1} - \frac{2l}{2k} + \frac{l}{2k+1}}{\frac{l}{2k+1} - \frac{2l}{2k+2} + \frac{l}{2k+3}} \right]$$

$$R = \frac{EK}{S} \frac{\cancel{l}(2k)(2k+1) - 2\cancel{l}(2k-1)(2k+1) + \cancel{l}(2k-1)(2k)}{\cancel{l}(2k+2)(2k+3) - 2\cancel{l}(2k+1)(2k+3) + \cancel{l}(2k+1)(2k+2)}$$

$$= \frac{EK}{S} \frac{(4k^2+2k-8k^2+2+4k^2-2k)/(2k-1)(2k)}{(4k^2+10k+6-8k^2-16k-6+4k^2+6k+2)/(2k+2)(2k+3)}$$

$$= \frac{EK}{S} \frac{2}{2} \frac{(2k+2)(2k+3)}{(2k-1)(2k)}$$

$$R = \frac{E}{S} \left(\frac{4k^2+10k+6}{4k-2} \right)$$

$$\text{To minimize } R, \quad \frac{\partial R}{\partial k} = 0$$

$$\Rightarrow \frac{E}{S} \left[\frac{(4k-2)(8k+10) - (4k^2+10k+6)(4)}{(4k-2)^2} \right] = 0$$

$$\Rightarrow (4k-2)(8k+10) - (4)(4k^2+10k+6) = 0$$

$$\Rightarrow (2k-1)(4k+5) - (4k^2+10k+6) = 0$$

$$\Rightarrow 8k^2+6k-5-4k^2-10k-6 = 0$$

$$\Rightarrow 4k^2-4k-11 = 0$$

$$k = \frac{-4 \pm \sqrt{16-4(-11)(4)}}{2 \times 4}$$

$$k = 2.232 \quad \text{or} \quad -1.232$$

Rejected
($k \rightarrow$ Real +ve)