

1. Using **Hamilton's Principle**, derive the equation of motion and boundary conditions of an uniform fixed-fixed string undergoing transverse free vibration. Take  $T$  as constant tension,  $\rho$  as volume density, and  $A$  as the uniform cross-sectional area of the string. The string vibrates in  $xz$ -plane. State all the assumptions used during the derivation. Express the final equation in terms of the wave speed  $c = \sqrt{T/\rho A}$ . (15)

2. Consider an uniform homogeneous shaft undergoing torsional vibration as shown in the **Fig. 1**. Two massive discs having mass moment of inertia  $I_{D_1}$  and  $I_{D_2}$ , respectively, are mounted at two different locations of the shaft. Write the **equation of motion** for this shaft. All symbols have their usual meaning. (5)

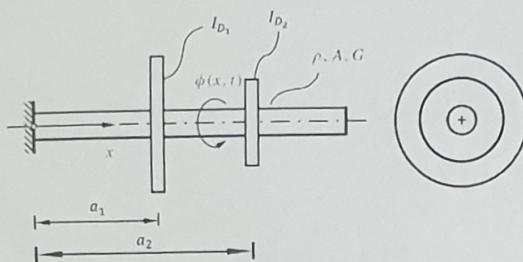


Fig. 1

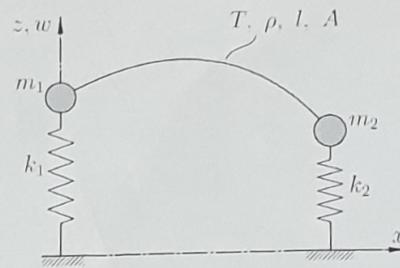


Fig. 2

3. A uniform taut string is undergoing transverse vibrations as shown in **Fig. 2**. Parameter definitions are as follows: field variable  $w(x, t)$ , volume density ( $\rho$ ), length ( $l$ ), cross-sectional area ( $A$ ), string tension ( $T$ ), point masses ( $m_1, m_2$ ) and spring stiffness ( $k_1, k_2$ ). (6)

(a) Determine the **boundary conditions** of the given system using force balance. (6)

(b) Assuming harmonic solution of the form  $w(x, t) = W(x)e^{i\omega t}$ , where  $W(x)$  denotes the eigenfunction and  $\omega$  denotes the eigenfrequency, show that the **characteristic equation** for this system is given by:  $\tan\left(\frac{\omega l}{c}\right) = \frac{-T\omega c[(k_1+k_2)-(m_1+m_2)\omega^2]}{c^2(k_1-m_1\omega^2)(k_2-m_2\omega^2)-T^2\omega^2}$ , where  $c = \sqrt{T/\rho A}$ . (14)

4. A fixed-fixed string is given an initial pull of height  $h$  at  $x = al$  and released at  $t = 0$ . The initial shape of the string is shown in **Fig. 3**. Take  $l$  as the length of the string and parameter  $0 < a < 1$ . Write the **initial conditions** for this string. (5)

5. **Figure 4** shows a homogeneous tapered bar having circular cross-section. The governing equation of this bar under free vibration is given by  $\rho A(x)u_{tt} - [EA(x)u_{xx}]_{,x} = 0$ , where  $u(x, t)$  is the field variable,  $\rho$  the volume density,  $A(x)$  is the cross-sectional variation, and  $E$  is the elastic modulus. Using **Galerkin's method**, discretize the equation of motion using the comparison functions as  $P_i(x) = (1 - \frac{x}{l})^{k+1} - 1$  ( $k = 1, 2, \dots, N$ ). For  $N = 2$  determine the first two natural frequencies from the discretized system. Take  $R = 0.02$  m,  $l = 1$  m,  $E = 200$  GPa,  $\rho = 7800$  kg/m<sup>3</sup>. (15)

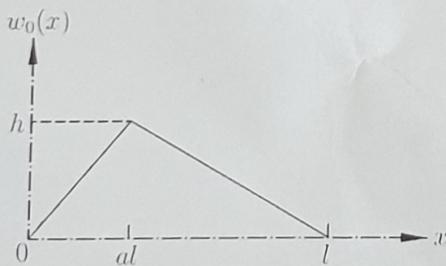


Fig. 3

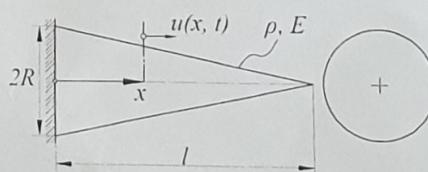
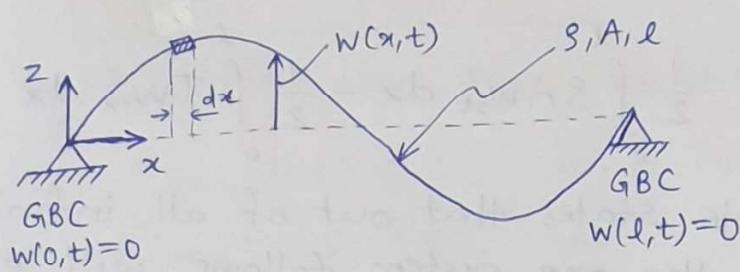


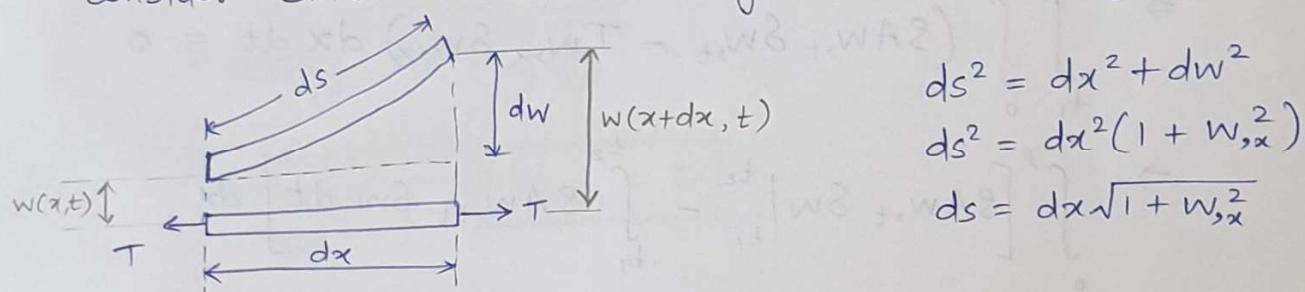
Fig. 4

Q.1 Ans:-

Hamilton's PrincipleModel Assumptions

- 1) Motion is planar, so displacement of string perpendicular to  $x-z$  plane i.e. in  $y$ -direction is neglected.
- 2) Slope is small i.e. transverse displacements are assumed to be small to ensure linear behaviour.
- 3) Tension does not change with displacement.
- 4) No damping or external force acting on system.

Consider small element of length ' $ds$ ' of string.



$$ds^2 = dx^2 + dw^2$$

$$ds^2 = dx^2(1 + w_x^2)$$

$$ds = dx\sqrt{1 + w_x^2}$$

- Kinetic Energy  $T$  of string,  $T = \int_0^l \frac{1}{2} s A dx w_t^2$

$$T = \frac{1}{2} \int_0^l s A w_t^2 dx$$

- Potential Energy  $V$  of string,  $V = \int_0^l T(ds - dx)$

$$= \int_0^l T dx (\sqrt{1 + w_x^2} - 1)$$

$$= \int_0^l T dx \left( 1 + \frac{1}{2} w_x^2 - 1 \right)$$

... Using Binomial expansion & ignoring higher order terms as  $\frac{dw}{dx}$  is very small

$$V = \frac{1}{2} \int_0^l T W_{,x}^2 dx$$

Lagrangian,  $\mathcal{L} = T - V$

$$= \frac{1}{2} \int_0^l S A W_{,t}^2 dx - \frac{1}{2} \int_0^l T W_{,x}^2 dx$$

Hamilton's principle states that out of all infinitely available paths, the one system follows will extremize the action.

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\Rightarrow \delta \int_{t_1}^{t_2} \frac{1}{2} \int_0^l [S A W_{,t}^2 - T W_{,x}^2] dx dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \int_0^l (S A W_{,t} S W_{,t} - T W_{,x} S W_{,x}) dx dt = 0$$

$$\Rightarrow \int_0^l [S A W_{,t} S W \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} S A W_{,tt} S W dt] dx$$

$$- \int_{t_1}^{t_2} [T W_{,x} S W \Big|_0^l - \int_0^l (T W_{,xx})_x S W dx] dt = 0$$

$$\Rightarrow \int_0^l S A W_{,t} S W \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} T W_{,x} S W \Big|_0^l dt$$

$$- \int_{t_1}^{t_2} \int_0^l [S A W_{,tt} - T W_{,xx}] S W dx dt = 0$$

... since  $T$  is constant.

first term is always zero since variation of field variable 'w' at initial and final time is zero.

$$\therefore S W(x, t_1) \equiv 0 \quad \text{and} \quad S W(x, t_2) \equiv 0$$

Integrand of third term has to be zero, as it gives equation of transverse vibration of string.

$$SAw_{,tt} - Tw_{,xx} = 0$$

So, second term also zero which gives

$$Tw_{,x}(0, t) \equiv 0 : \text{ or } w(0, t) \equiv 0$$

$$Tw_{,x}(l, t) \equiv 0 \quad \text{or} \quad w(l, t) \equiv 0$$

These are possible boundary conditions.

For fixed-fixed string, boundary conditions are

$$w(0, t) \equiv 0$$

$$w(l, t) \equiv 0$$

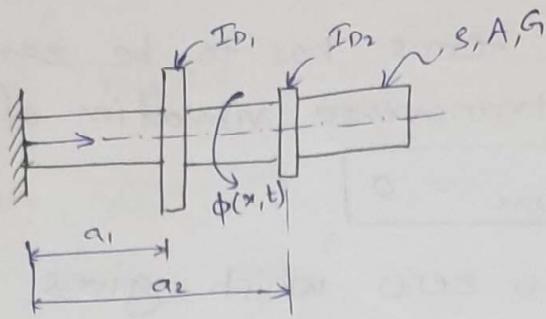
We can modify EOM as,

$$w_{,tt} - \frac{T}{SA} w_{,xx} = 0$$

$$\text{Putting } c = \sqrt{\frac{T}{SA}}$$

$$w_{,tt} - c^2 w_{,xx} = 0$$

Q.2 Ans:-



- Kinetic Energy of shaft :  $T = \frac{1}{2} \int_0^l S I_p \phi_{,t}^2 dx$
- Potential Energy of shaft :  $V = \frac{1}{2} \int_0^l G I_p \phi_{,x}^2 dx$
- Lagrangian density  $L = \frac{1}{2} (S I_p \phi_{,t}^2 - G I_p \phi_{,x}^2)$

EOM by generalised formulation :-

$$\Rightarrow \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{\phi}_{,t}} \right) + \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial \dot{\phi}_{,x}} \right) - \frac{\partial L}{\partial \phi} = \tau_{ext} \quad \#$$

$$\cdot \frac{\partial L}{\partial \dot{\phi}_{,t}} = S I_p \phi_{,t} \quad \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{\phi}_{,t}} \right) = S I_p \phi_{,tt}$$

$$\cdot \frac{\partial L}{\partial \dot{\phi}_{,x}} = -G I_p \phi_{,x} \quad \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial \dot{\phi}_{,x}} \right) = -G I_p \phi_{,xx}$$

$$\cdot \frac{\partial L}{\partial \phi} = 0$$

$$\cdot \tau_{ext} = \tau_{ext,1} + \tau_{ext,2}$$

$\downarrow$   
External Torque per unit length due to disc 1

$\downarrow$   
External Torque per unit length due to disc 2.

$$\begin{aligned} \tau_{ext,1} &= n_{E_1}(x, t) \\ &= -I_{D_1} \phi_{,tt} S(x-a_1) \\ \tau_{ext,2} &= n_{E_2}(x, t) \\ &= -I_{D_2} \phi_{,tt} S(x-a_2) \end{aligned}$$

Negative sign indicate disc opposing motion of shaft.

$$= -I_{D_1} \phi_{,tt} S(x-a_1) - I_{D_2} \phi_{,tt} S(x-a_2)$$

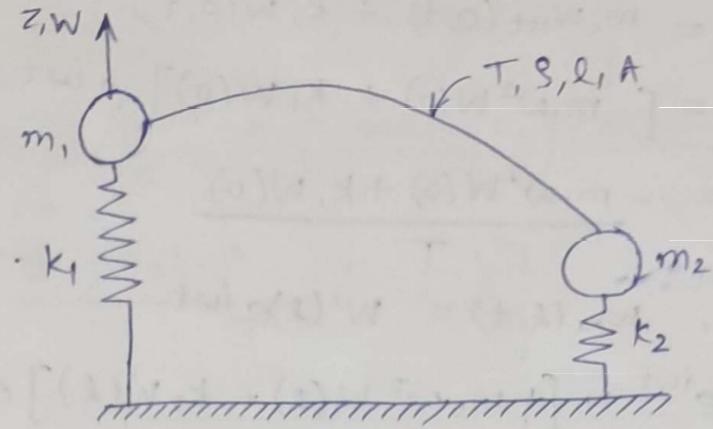
Combining all these terms, & put in #,

$$S I_p \phi_{,tt} - G I_p \phi_{,xx} = [-I_{D_1} S(x-a_1) - I_{D_2} S(x-a_2)] \phi_{,tt}$$

... EOM of shaft

OR

$$[S I_p + I_{D_1} S(x-a_1) + I_{D_2} S(x-a_2)] \phi_{,tt} - G I_p \phi_{,xx} = 0$$

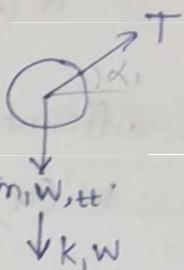


[a] Boundary conditions

① Force balance at  $m_1$ ,

$$\sum F_y = m_1 g$$

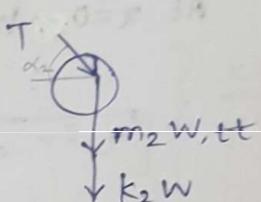
$$T_{W,x}(0, t) = m_1 w_{tt}(0, t) + k_1 w(0, t)$$



$$\begin{aligned} T \sin \alpha &\approx T \text{tan} \alpha \\ &\approx T \frac{dw}{dx} \\ &= T w_{xx} \end{aligned}$$

② Force balance at  $m_2$ ,

$$T_{W,x}(l, t) = -m_2 w_{tt}(l, t) - k_2 w(l, t)$$



[b] Assumed solution  $w(x, t) = W(x) e^{i\omega t}$

$$\text{So, EOM is } w_{tt} - \frac{T}{SA} w_{xx} = 0$$

$$\Rightarrow w_{tt} - c^2 w_{xx} = 0 \quad \dots c = \sqrt{\frac{T}{SA}}$$

Substituting assumed solution,

$$\begin{aligned} w_{tt} &= W(x) (i\omega)^2 e^{i\omega t} = -\omega^2 W e^{i\omega t} \\ w_{xx} &= W'' e^{i\omega t} \end{aligned}$$

$$\Rightarrow -\omega^2 W e^{i\omega t} - c^2 W'' e^{i\omega t} = 0$$

$$\Rightarrow (c^2 W'' + \omega^2 W) e^{i\omega t} = 0$$

$$\Rightarrow W'' + \frac{\omega^2}{c^2} W = 0$$

Solution of this differential equation is,

$$W = D \cos\left(\frac{\omega}{c} x\right) + H \sin\left(\frac{\omega}{c} x\right) \quad \text{A}$$

For Boundary conditions, using  $w_{xx}(x, t) = W'' e^{i\omega t}$   
for BC 1,  $w_{xx}(0, t) = W'(0) e^{i\omega t}$

Putting above value in BC ①,

$$TW'(0)e^{i\omega t} = m_1 w_{stt}(0, t) + k_1 w(0, t)$$

$$TW'(0)e^{i\omega t} = [-m_1 \omega^2 W(0) + k_1 W(0)] e^{i\omega t}$$

$$W'(0) = -\frac{m_1 \omega^2 W(0) + k_1 W(0)}{T} \quad \text{--- (B)}$$

Similarly for BC 2,  $w_{xx}(l, t) = W'(l)e^{i\omega t}$

$$TW'(l)e^{i\omega t} = [+m_2 \omega^2 W(l) - k_2 W(l)] e^{i\omega t}$$

$$W'(l) = \frac{m_2 \omega^2 W(l) - k_2 W(l)}{T} \quad \text{--- (C)}$$

Using (A), differentiate (A) wrt  $x$ ,

$$W'(x) = D \frac{\omega}{c} \left( -\sin \left( \frac{\omega}{c} x \right) \right) + H \frac{\omega}{c} \cos \left( \frac{\omega}{c} x \right) \quad \text{--- (D)}$$

$$\text{At } x=0, W'(0) = 0 + \frac{H\omega}{c} \quad \text{--- (From (D))}$$

$$\frac{-m_1 \omega^2 W(0) + k_1 W(0)}{T} = \frac{H\omega}{c} \quad \text{--- (From (B))}$$

$$H = \frac{c}{\omega} \left( \frac{-m_1 \omega^2 + k_1}{T} \right) W(0) \quad \text{--- (E)}$$

$$\text{At } x=l, W'(l) = D \frac{\omega}{c} \left( -\sin \frac{\omega l}{c} \right) + H \frac{\omega}{c} \left( \cos \frac{\omega l}{c} \right) \quad \text{--- (From (D))}$$

$$\Rightarrow \frac{m_2 \omega^2 W(l) - k_2 W(l)}{T} = -\frac{D\omega}{c} \left( \sin \frac{\omega l}{c} \right) + \frac{H\omega}{c} \left( \cos \frac{\omega l}{c} \right)$$

Substitute value of  $H$  from (E), --- From (C)

$$\Rightarrow \left( \frac{m_2 \omega^2 - k_2}{T} \right) W(l) = -\frac{D\omega}{c} \left( \sin \frac{\omega l}{c} \right) + \frac{c}{\omega c} \left( \frac{-m_1 \omega^2 + k_1}{T} \right) W(0) \left( \cos \frac{\omega l}{c} \right)$$

$$\text{We know, } W(0) = D(1) + H(0) = D \quad \left. \right\}$$

$$W(l) = D \left( \cos \frac{\omega l}{c} \right) + H \left( \sin \frac{\omega l}{c} \right) \quad \left. \right\} \text{from (A)}$$

Put above values, we get

$$\Rightarrow \left( \frac{m_2 \omega^2 - k_2}{T} \right) \left[ D \left( \cos \frac{\omega l}{c} \right) + \frac{c}{\omega} \left( \frac{-m_1 \omega^2 + k_1}{T} \right) D \left( \sin \frac{\omega l}{c} \right) \right]$$

$$= -\frac{D\omega}{c} \left( \sin \frac{\omega l}{c} \right) + \left( \frac{-m_1 \omega^2 + k_1}{T} \right) D \left( \cos \frac{\omega l}{c} \right)$$

$$\Rightarrow \cos \frac{wl}{c} \left[ \left( \frac{m_2 w^2 - k_2}{T} \right) - \left( \frac{-m_1 w^2 + k_1}{T} \right) \right]$$

$$= \sin \frac{wl}{c} \left[ -\frac{w}{c} - \frac{c}{\omega} \left( \frac{-m_1 w^2 + k_1}{T} \right) \left( \frac{m_2 w^2 - k_2}{T} \right) \right]$$

$$\Rightarrow \cos \frac{wl}{c} \left[ \frac{(m_1 + m_2) w^2 - (k_1 + k_2)}{T} \right]$$

$$= \sin \frac{wl}{c} \left[ (-1) \left\{ \frac{\omega^2 T^2 + c^2 (k_1 - m_1 w^2)(m_2 w^2 - k_2)}{c \omega T^2} \right\} \right]$$

$$\Rightarrow \frac{\left[ \frac{(m_1 + m_2) w^2 - (k_1 + k_2)}{T} \right]}{(-1) \left[ \frac{\omega^2 T^2 + c^2 (k_1 - m_1 w^2)(k_2 - m_2 w^2)}{c \omega T^2} \right]} = \frac{\sin \frac{wl}{c}}{\cos \frac{wl}{c}}$$

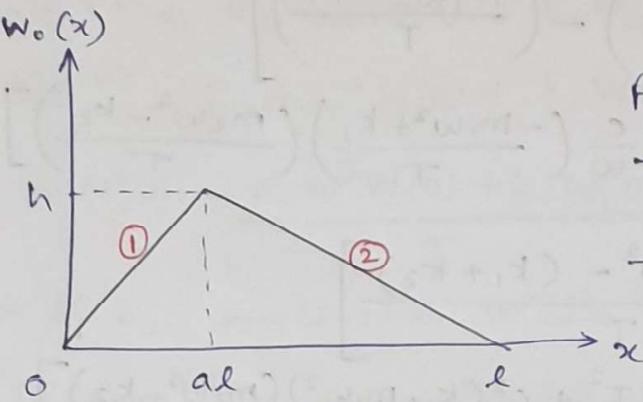
$$\Rightarrow \tan \left( \frac{wl}{c} \right) = \left\{ \frac{(k_1 + k_2) - (m_1 + m_2) w^2}{(-1) [c^2 (k_1 - m_1 w^2)(k_2 - m_2 w^2) - T^2 w^2]} \right\} \frac{c \omega T^2}{T}$$

$$\Rightarrow \boxed{\tan \left( \frac{wl}{c} \right) = \frac{-T w c [(k_1 + k_2) - (m_1 + m_2) w^2]}{c^2 (k_1 - m_1 w^2)(k_2 - m_2 w^2) - T^2 w^2}}$$

$$\text{where } c = \sqrt{\frac{T}{SA}}$$

Hence Proved

Q.4 Ans :-



fixed fixed string

→ Initial pull of height 'h'  
@  $x = al$   
→ Release at  $t = 0$

For Line ① bet<sup>n</sup>  $(0,0)$  to  $(al, h)$ ,

$$\frac{w_0(x) - 0}{h - 0} = \frac{x - 0}{al - 0} \quad \dots \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$w_0(x) = \frac{hx}{al} \quad \text{Also } v_0(x) = \frac{\partial w_0(x)}{\partial t} = 0$$

For Line ② bet<sup>n</sup>  $(al, h)$  to  $(l, 0)$

$$\frac{w_0(x) - h}{0 - h} = \frac{x - al}{l - al}$$

$$\begin{aligned} w_0(x) &= \frac{x - al}{l(1-a)}(c-h) + h \\ &= \frac{hal - hx + hl(1-a)}{l(1-a)} \\ &= \frac{hal - hx + hl - hal}{l(1-a)} \end{aligned}$$

$$w_0(x) = \frac{h(l-x)}{l(1-a)}$$

$$\text{Also } v_0(x) = \frac{\partial w_0(x)}{\partial t} = 0$$

Initial conditions of string

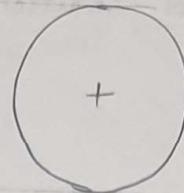
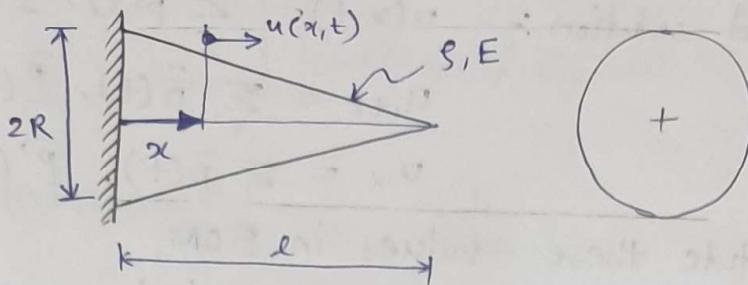
$$w_0(x) = \frac{hx}{al} \quad \dots 0 \leq x < al$$

$$= \frac{h(l-x)}{l(1-a)} \quad \dots al \leq x \leq l$$

$$v_0(x) = 0$$

Q.5 Ans:-

Galerkin Method



Given:-  $S A(x) u_{tt} - [E A(x) u_{xx}]_{xx} = 0$

Comparison function,  $P_i(x) = \left(1 - \frac{x}{l}\right)^{k+1} - 1$

$N = 2$  (Number of terms) ...  $k = 1, 2, \dots, N$

$R = 0.02 \text{ m}$

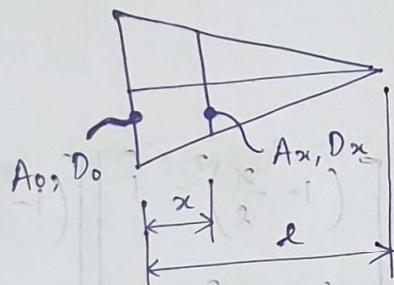
$l = 1 \text{ m}$

$E = 200 \text{ GPa}$

$S = 7800 \text{ kg/m}^3$

To find:- A) Discretize the system  
B) first two natural frequencies,  $\omega_1$  &  $\omega_2 = ?$

Soln:- A) Assume  $A_0$  &  $D_0$  be area & diameter at fixed end  
 $A_x$  &  $D_x$  be area & diameter at distance  $x$ .



$$1) \frac{D_x}{D_0} = \frac{l-x}{l}$$

$$2) \frac{A_x}{A_0} = \frac{\frac{\pi}{4} D_x^2}{\frac{\pi}{4} D_0^2} = \left(\frac{D_x}{D_0}\right)^2 = \left(\frac{l-x}{l}\right)^2 \quad \text{by } ①$$

$$A_x = A_0 \left(1 - \frac{x}{l}\right)^2$$

$$\text{we know, } A_0 = \frac{\pi}{4} (2R)^2 = \pi R^2$$

$$\therefore A_x = \pi R^2 \left(1 - \frac{x}{l}\right)^2$$

$$\text{EOM : } SA(x)u_{tt} - [EA(x)u_x]_{xx} = 0$$

Assumed solution :  $u(x, t) = \sum \bar{p}(t) \cdot \bar{P}(x)$

$$u_{tt} = \sum \ddot{\bar{p}}(t) \cdot \bar{P}(x)$$

$$u_{xx} = \sum \bar{p}(t) \cdot \bar{P}'(x) = \bar{P}^T \bar{p}$$

Substitute these values in EOM,

$$SA(x) \sum \ddot{\bar{p}}_i \bar{P}_i - [EA(x) \sum \bar{p}_i \bar{P}'_i]' = 0$$

Multiply both sides by  $\bar{P}_j$  we get

$$\sum SA(x) \bar{P}_i^T \ddot{\bar{p}}_i \bar{P}_j - \sum [EA(x) \bar{P}_i^T \bar{P}_i \bar{P}_j]' = 0$$

$$M \ddot{\bar{p}} + K \bar{p} = 0$$

... Discretized form of EOM.

$$M = \int_0^L u(x) P P^T dx$$

$$K = - \int_0^L P \cdot K [P^T] dx$$

$$\text{Here } u(x) = SA(x)$$

$$K[u] = + [EA(x)u_x]_{xx}$$

B)  $M = \int_0^L S A(x) P P^T dx$

$$M = \int_0^L S \times \pi R^2 \left(1 - \frac{x}{L}\right)^2 \begin{bmatrix} \left(1 - \frac{x}{L}\right)^2 - 1 \\ \left(1 - \frac{x}{L}\right)^3 - 1 \end{bmatrix} \begin{bmatrix} \left(1 - \frac{x}{L}\right)^2 - 1 & \left(1 - \frac{x}{L}\right)^3 - 1 \end{bmatrix} dx$$

Putting values ;

$$M = \int_0^1 7800 \times \pi (0.02)^2 (1-x)^2 \begin{bmatrix} x - 2x + x^2 - x^3 \\ x - 3x + 3x^2 - x^3 - 1 \end{bmatrix} \begin{bmatrix} x - 2x + x^2 - x^3 & x - 3x + 3x^2 - x^3 - 1 \end{bmatrix}_{1 \times 2} dx$$

$$= 7800 \times \pi (0.02)^2 \int_0^1 (1-x)^2 \begin{bmatrix} (-2x + x^2)^2 & (-2x + x^2)(-3x + 3x^2 - x^3) \\ (-3x + 3x^2 - x^3)(-2x + x^2) & (-3x + 3x^2 - x^3)^2 \end{bmatrix} dx$$

$$M = 7800 \times \pi (0.02)^2 \int_0^1 \begin{bmatrix} (1-x)^2(-2x+x^2)^2 & (1-x)^2(-2x+x^2)(-3x+3x^2-x^3) \\ (1-x)^2(-3x+3x^2-x^3)(-2x+x^2) & (1-x)^2(-3x+3x^2-x^3)^2 \end{bmatrix} dx$$

$$= 7800 \times \pi (0.02)^2 \begin{bmatrix} \frac{8}{105} & \frac{11}{120} \\ \frac{11}{120} & \frac{1}{9} \end{bmatrix}$$

$$K = - \int_0^l P \cdot K[P^T] dx$$

$$= - \int_0^l P \times [EA(x) P^T]^T dx$$

$$= - \int_0^l \left[ \begin{bmatrix} (1-\frac{x}{l})^2 - 1 \\ (1-\frac{x}{l})^3 - 1 \end{bmatrix} \times \left[ E \times \pi R^2 \left( 1 - \frac{x}{l} \right)^2 \left[ \begin{bmatrix} (1-\frac{x}{l})^2 - 1 & (1-\frac{x}{l})^3 - 1 \end{bmatrix} \right]^T \right] \right] dx$$

$$= - \int_0^l \left[ \begin{bmatrix} (1-x)^2 - 1 \\ (1-x)^3 - 1 \end{bmatrix} \times \left[ 200 \times 10^9 \times \pi (0.02)^2 (1-x)^2 \left[ \begin{bmatrix} (1-x)^2 - 1 & (1-x^3) - 1 \end{bmatrix} \right]^T \right] \right] dx$$

$$= - \int_0^l \begin{bmatrix} -2x + x^2 \\ -3x + 3x^2 - x^3 \end{bmatrix} \times 200 \times 10^9 \times \pi (0.02)^2 \left\{ (1-x)^2 \left[ \begin{bmatrix} -2x + x^2 & -3x + 3x^2 - x^3 \end{bmatrix} \right]^T \right\} dx$$

$$= - 200 \times 10^9 \times \pi (0.02)^2 \int_0^l \begin{bmatrix} -2x + x^2 \\ -3x + 3x^2 - x^3 \end{bmatrix} \left\{ (1-x)^2 \left[ \begin{bmatrix} -2+2x & -3+6x-3x^2 \end{bmatrix} \right]^T \right\} dx$$

$$= - 200 \times 10^9 \times \pi (0.02)^2 \int_0^l \begin{bmatrix} -2x + x^2 \\ -3x + 3x^2 - x^3 \end{bmatrix} \cdot \left[ \begin{bmatrix} (1-2x+x^2)(-2+2x) & (1-2x+x^2)(-3+6x-3x^2) \end{bmatrix} \right] dx$$

$$= - 200 \times 10^9 \times \pi (0.02)^2 \int_0^l \begin{bmatrix} -2x + x^2 \\ -3x + 3x^2 - x^3 \end{bmatrix} \left[ \begin{bmatrix} (-2+2x)(-2+2x) & (-2+2x)(-3+6x-3x^2) \\ +(1-2x+x^2)(2) & +(1-2x+x^2)(6-6x) \end{bmatrix} \right] dx$$

$$= - 200 \times 10^9 \times \pi (0.02)^2 \int_0^l \begin{bmatrix} -2x + x^2 \\ -3x + 3x^2 - x^3 \end{bmatrix} \left[ \begin{bmatrix} 4-8x+4x^2+2-4x+2x^2 \\ 6-12x+6x^2-6x+12x^2-6x^3 \end{bmatrix} \right] dx$$

$$+ \begin{bmatrix} 6-6x-12x+12x^2+6x^2-6x^3 \end{bmatrix} dx$$

$$\begin{aligned}
 K &= -200 \times 10^9 \times \pi (0.02)^2 \int_0^1 \begin{bmatrix} -2x + x^2 \\ -3x + 3x^2 - x^3 \end{bmatrix} \begin{bmatrix} 6x^2 - 12x + 6 \\ -12x^3 + 36x^2 - 36x + 12 \end{bmatrix} dx \\
 &= -200 \times 10^9 \times \pi (0.02)^2 \int_0^1 \begin{bmatrix} (-2x + x^2)(6x^2 - 12x + 6) \\ (-3x + 3x^2 - x^3)(6x^2 - 12x + 6) \end{bmatrix} dx \\
 &= -200 \times 10^9 \times \pi (0.02)^2 \begin{bmatrix} -4/5 & -1 \\ -1 & -\frac{9}{7} \end{bmatrix} \\
 &= 200 \times 10^9 \times \pi (0.02)^2 \begin{bmatrix} \frac{4}{5} & 1 \\ 1 & \frac{9}{7} \end{bmatrix}
 \end{aligned}$$

Put values of M & K in discretized EOM,

$$7800 \times \pi (0.02)^2 \begin{bmatrix} \frac{8}{105} & \frac{11}{120} \\ \frac{11}{120} & \frac{1}{9} \end{bmatrix} \begin{Bmatrix} \ddot{P}_1 \\ \ddot{P}_2 \end{Bmatrix} + 200 \times 10^9 \times \pi (0.02)^2 \begin{bmatrix} \frac{4}{5} & 1 \\ 1 & \frac{9}{7} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$Put \quad \bar{P} = \bar{V} e^{i\omega t}$$

$$\ddot{\bar{P}} = -\omega^2 \bar{V} e^{i\omega t}$$

for non-trivial solution,

$$\det [-\omega^2 M + K] = 0$$

$$\Rightarrow \det \left\{ 7800 \times \pi (0.02)^2 \begin{bmatrix} -\frac{8}{105} \omega^2 & -\frac{11}{120} \omega^2 \\ -\frac{11}{120} \omega^2 & -\frac{1}{9} \omega^2 \end{bmatrix} + 200 \times 10^9 \times \pi (0.02)^2 \begin{bmatrix} \frac{4}{5} & 1 \\ 1 & \frac{9}{7} \end{bmatrix} \right\} = 0$$

$$\Rightarrow \det \begin{bmatrix} -\frac{4160}{7} \omega^2 + 1.6 \times 10^{11} & -715 \omega^2 + 200 \times 10^9 \\ -715 \omega^2 + 200 \times 10^9 & -\frac{2600}{3} \omega^2 + 2.57143 \times 10^{11} \end{bmatrix} = 0$$

$$\Rightarrow \left( -\frac{4160}{7} \omega^2 + 1.6 \times 10^{11} \right) \left( -\frac{2600}{3} \omega^2 + 2.57143 \times 10^{11} \right)$$

$$- \left( -715 \omega^2 + 200 \times 10^9 \right)^2 = 0$$

$$\Rightarrow (3822.619) \omega^4 - (5.483078 \times 10^{12}) \omega^2 + 1.14288 \times 10^{21} = 0$$

$$\omega^2 = 1181280579, 253096742$$

$$\omega = \pm 34369.76, \pm 15909.01$$

Ignoring negative values,

$$\omega_1 = 34369.76 \text{ rad/s}$$

$$\omega_2 = 15909.01 \text{ rad/s}$$

<https://github.com/LastElectron>