Transportation Research Part A 126 (2019) 67–82

Contents lists available at ScienceDirect

Transportation Research Part A

journal homepage: www.elsevier.com/locate/tra

A link criticality index embedded in the convex combinations solution of user equilibrium traffic assignment

Amirmasoud Almotahari[a,*], M. Anil Yazici[b]

[a] Department of Civil Engineering, Stony Brook University, 1208 Civil Engineering, Stony Brook, NY, 11794, United States
[b] Department of Civil Engineering, Stony Brook University, 2425 Civil Engineering, Stony Brook, NY, 11794, United States

A R T I C L E I N F O

Keywords:
Vulnerability
Criticality
Traffic assignment
Convex combinations algorithm

A B S T R A C T

Identification of critical components in transportation networks is an essential part of designing robust and resilient systems. Topological criticality measures are based on graph theory and are applicable in multiple domains including communication and social networks. However, the non-linearity of link performance functions in transportation systems does not allow a perfect domain transfer of topological measures. Hence, transportation researchers take traffic flow characteristics into account while developing criticality measures. In such approaches, typically, a network performance measure is selected, then links are removed one-by-one, and traffic demand is re-assigned to the updated network to calculate the impacts of each link failure. This consecutive link removal procedure requires multiple assignments which create a computational burden, especially for large networks. Overall objectives of this paper are (1) to compare and contrast selected criticality measures, and (2) to develop a new measure to identify critical components of transportation network, considering both traffic characteristics and network topology. For this purpose, the user equilibrium traffic assignment formulation is utilized, and the convex combinations solution algorithm is exploited for identification of link criticality ranking within a single traffic assignment. The developed measure is named Link Criticality Index (LCI). The LCI is compared with the existing measures in the literature through three numerical examples. Pros and cons of the LCI and selected measures are discussed in detail. The results indicate the proposed link criticality measure provides a balanced ranking with respect to connectivity/redundancy as well as the traffic conditions in the network.

1. Introduction

Besides day-to-day movements of goods and people (which serves as an important factor from economic and welfare perspectives), transportation network plays a key role in complementing other systems in case of disruptions and hazards, e.g. emergency response and supply chain management (Mattsson and Jenelius, 2015). Depending on the disruption, the consequences can range from a limited and temporary change in one specific network to a severe failure of several networks (Murray, 2013). Identification of critical network components relates to robustness and resilience analysis which focus on system performance degradation and restoration after a disruption.

Existing criticality metrics utilize either graph theory and network science models (i.e. topological approaches), or take traffic flow characteristics into account. Topological approaches utilize the network graph structure and are generally applicable to

* Corresponding author.
E-mail addresses: Am.Almotahari@stonybrook.edu (A. Almotahari), Anil.Yazici@stonybrook.edu (M.A. Yazici).

https://doi.org/10.1016/j.tra.2019.06.005
Received 4 December 2018; Received in revised form 1 May 2019; Accepted 4 June 2019
Available online 11 June 2019
0965-8564/ Published by Elsevier Ltd.

networks in multiple domains including telecommunication networks, biological networks, social networks, energy networks, and transportation networks. For instance, *Betweenness-Centrality* (BC) (which considers the number of shortest paths passing through a node/link) is a criticality measure, which is mainly developed for social network analysis, but can also be utilized in transportation networks. However, the non-linear link performance functions (i.e. traffic congestion) in transportation networks do not allow a perfect domain transfer of topological measures. For example, assume a social network of three individuals (A, B, and C) connected to each other with same link costs, (i.e. a graph with nodes placed on the corners of an equilateral triangle). The failure of a direct link between nodes A and B in a social network may result in a "*re-route*" from node-C, causing one additional link cost for the path. In case of a roadway link failure of the same nature between three urban areas, the cost of re-route from node-C would be non-linear and depends on the link performance functions and the network flows. Cats and Jenelius (2014) argue that the nature of transportation services requires more refined model for supply and demand interactions in order to evaluate the impacts of disruptions. Accordingly, researchers (Gauthier et al., 2018; Jenelius et al., 2006; Nagurney and Qiang, 2007) suggest criticality measures which investigate traffic flow characteristics. In such measures, typically, a network performance measure (e.g. total travel time) is considered and a full-scan impact analysis of the network is performed by deleting each link in the network one-by-one (Cats et al., 2017). Then the selected measure is recalculated by performing traffic assignment on the adjusted network. The critical links are identified/ranked based on the change in the selected measure between the original and the adjusted networks. Such approaches are effective and true to the nature of the transportation network dynamics since the metrics are based on traffic assignment flows. However, performing multiple traffic assignments increases the computational burden and deleting links can cause disconnectivity in the network. In turn, this can create unsatisfied demand between origin-destination (OD) pairs and makes it problematic to perform traffic assignment (Jenelius et al., 2006; Rupi et al., 2015). In order to address this issue, either case-specific metrics are suggested for disconnectivity cases (Jenelius et al., 2006), or link performance functions are adjusted (e.g. infinite travel time (Nagurney and Qiang, 2007)) to mathematically circumvent disconnectivity issue. Rupi et al. (2015) consider unsatisfied demand by taking into account the value of a missing trip between the OD pairs due to closure of the link. Their method aims to provide criticality ranking for all links including both *"cut-links"* (which cause disconnectivity) and non-cut links. In the application of their method, they assume that all cut-links are more critical than non-cut links. In these respects, there is a need for a criticality measure that considers both traffic characteristics and connectivity/redundancy in the network.

Policy makers can utilize criticality rankings for resource prioritization before disruptive events to design/maintain robust transportation networks, i.e. small loss of network performance/functionality. They also can use the rankings to provide recovery plans after disruptions to enhance network resilience, i.e. restoration of sufficient network functionality/performance in a timely manner. Based on the literature, the practitioners have multiple measures at their toolbox for decision and policy making. However, unless the existing measures consistently point out similar set of links as critical in a particular network, the use of available metrics for policy making is problematic. Hence, it is necessary to compare and contrast the existing criticality measures, understand pros and cons, and inform practitioners about the caveats associated with the metrics. Such comparison offers insights also for network science and transportation researchers in terms of formulating new metrics and advancing the knowledge on analysis of transportation networks under disruptions. This paper undertakes the tasks of both the comparison of metrics and suggestion of a new metric, namely Link Criticality Index (LCI). Since there is no *"ground truth"* or *"absolute"* criticality ranking for a network for benchmarking, the comparison of measures is mainly based on special links that can provide meaningful comparisons, e.g. a link whose failure can create disconnectivity. The selected measures and LCI are also discussed in terms of computational burden and complexity.

2. Literature review

Several studies develop criticality measures for road networks (Berdica, 2002; Demirel et al., 2015; Khademi et al., 2015; Taylor and Susilawati, 2012; Watling and Balijepalli, 2012; Yu et al., 2012), air transport systems (Mokhtarimousavi et al., 2018; Voltes-Dorta et al., 2017) and transit and railway networks (Bababeik et al., 2017; Cats et al., 2016; Cats and Jenelius, 2018, 2015). Some studies develop models for a single-link failure mechanism (Jenelius et al., 2006; Knoop et al., 2012; Nagurney and Qiang, 2007) while others consider multi-link failure mechanisms (Bababeik et al., 2017; Jenelius and Mattsson, 2012). Despite considering different modes or failure mechanisms, these criticality measures do not necessarily imply different metrics in terms of mathematical modeling. For example, Betweenness-Centrality can be used for analyzing both road (Gauthier et al., 2018) and transit (Cats and Jenelius, 2015) networks. Different categorizations from other perspectives can be found in the studies authored by Faturechi and Miller-Hooks (2014) and Reggiani et al. (2015).

Mattsson and Jenelius (2015) divide metrics into two distinct traditions and characterize them as topological or system-based. Topological approaches are based on the graph representation of the transportation network and involves graph theory-based properties, such as connectivity, centrality, etc. Topological measures fail to capture the non-linearity that characterizes congestion in transport systems (Cats et al., 2017). In contrast, system-based approach investigates the changes in supply-demand and the resulting traffic flow variations. Similar to Mattsson and Jenelius (2015), Gauthier et al. (2018) label such measures as demand-sensitive metrics. Some studies (Cats et al., 2017; Cats and Jenelius, 2015; Gauthier et al., 2018; Jenelius et al., 2006; Nagurney and Qiang, 2007) incorporate network flows into existing topological measures (e.g. Betweenness-Centrality) and develop hybrid metrics for link criticality ranking identification in transportation network. In this paper, following the categorization in previous studies (Faturechi and Miller-Hooks, 2014; Gauthier et al., 2018; Mattsson and Jenelius, 2015), criticality measures are categorized into topological and traffic assignment-based approaches.

Topological measures are basically developed by network scientists and graph theorists to study the behavior of networks in general. In terms of mathematical network modeling, Fulkerson and Harding (1977) are the first that approached the shortest path problem from the network vulnerability perspective. Also, Corley and Sha (1982) propose a specialized algorithm to identify the component whose removal causes the greatest increase in the shortest distance between two nodes. Murray (2013) provides a general review on well-recognized system performance measures such as maximal flow, connectivity, access fortification, component attributes, and system flow which are related only to the abstract graph represented a real network. Some topological measures are network measures (e.g. transitivity (Newmann, 2003), modularity (Newman, 2006), and, assortativity (Newman, 2002)), and some others are node/link-specific measures (e.g. closeness-centrality (Freeman, 1977), and participation coefficient (Joyce et al., 2010)). In general, the topological models do not conform to traffic flow dynamics (e.g. non-linear link performance functions) on transportation systems and they do not necessarily produce realistic results for transportation planners. In order to overcome this issue, researchers take traffic characteristics (travel time, flow, etc.) into account while developing criticality metrics. Unlike topological models, traffic assignment-based measures consider the traffic supply/demand dynamics. For this purpose, researchers approach the problem by utilizing traffic assignment to modify existing topological measures (e.g. efficiency, BC, etc.), or by introducing simulation-based models according to the changes in traffic conditions (e.g. total travel time, path travel time, etc.) in case of a link failure in the network. Table 1 provides a non-exhaustive summary of criticality measures in the literature, including 4 measures (Efficiency Index – EI, Network Robustness Index – NRI, Travel Time Weighted Betweenness-Centrality – TTWBC, Importance Score – IS) which are further analyzed and compared with the metric proposed in this paper. A short conceptual description for each measure is also provided in Table 1. The interested reader is encouraged to read the original sources for details.

Network Robustness Index (NRI) (Sullivan et al., 2010) measures the link criticality based on the change in total network travel time when a link is removed. However, when a link failure results in disconnected parts, NRI cannot evaluate the impact of failure, because the travel time between disconnected OD pairs cannot be calculated. When a link failure causes disconnectivity in the network, Jenelius et al. (2006) introduce a case-specific criticality measure for the links that cause disconnectivity. Such case-specific criticality assessment can rank disconnectivity links among each other but does not allow comparison with other links. For addressing this issue, Nagurney and Qiang (2007) incorporate traffic demands and travel times into the efficiency measure proposed by Latora and Marchiori (2001) and develop a criticality measure for transportation network. Since their metric can handle the disconnectivity without alteration, Nagurney and Qiang (2007) argue that their model provides a uniform and unified metric for the criticality of transportation network components.

On one hand, network-wide measures (Efficiency, total travel time, etc.) require multiple traffic assignments (as many as the number of network links) to calculate the link criticality. In contrast, link/node-specific measures give the link criticality ranking with single traffic assignment. Gauthier et al. (2018) modify the original Betweenness-Centrality (BC) measure and propose a travel time weighted Betweenness-Centrality for OD pairs (TTWBC). TTWBC is calculated by considering travel times as weights for the roadway links. TTWBC requires path enumeration however avoids multiple UE assignment for link removals. To increase the computational efficiency, Furno et al. (2018) develop a framework for fast computation of BC for real-time vulnerability assessment of transportation networks. Accordingly, Gauthier et al. (2018) also develop a simulation-based criticality measure called Stress Test Criticality (STC). STC captures the impact of day-to-day disruptive events by investigating the evolution of the network performance loss for different capacity-disruption levels. They utilize the importance measure proposed by Jenelius et al. (2006) to evaluate the network performance. Besides ignoring the impact of disconnectivity and resulting unsatisfied demand, their method requires multiple traffic assignments (i.e. number of link failures multiplied by number of different capacity-disruption levels) which create computational burden.

In general, graph theory-based topological measures are relatively easier to calculate and interpret, and computationally more efficient to identify critical links on a given network. In contrast, traffic assignment-based measures capture the nature of the traffic flow dynamics. However, traffic assignment can be costly to perform for large networks, especially when performed multiple times for each link. Regarding this trade-off, this paper suggests an approach that utilizes traffic assignment without link removal while considering network characteristics, e.g. alternative paths between ODs. Accordingly, a link criticality index (LCI) embedded in the iterations of convex combinations solution algorithm for the UE assignment is introduced. Also, four measures are further studied and compared with the proposed LCI; namely Efficiency Index (EI), Network Robustness Index (NRI), Travel Time Weighted Betweenness-Centrality (TTWBC), and Importance Score (IS).

3. Methodology

The proposed link criticality index (LCI) is calculated within the iterations of the convex combinations (Frank and Wolfe, 1956; Sheffi, 1985) algorithm for user equilibrium (UE) traffic assignment for fixed OD demands and without link interactions. LCI provides link criticality rankings within a single UE assignment without recalculation of a performance function for each link removal. UE assignment formulation is well known and omitted in the text for space considerations. Interested reader can refer to (Sheffi, 1985) for detailed discussion of UE assigment. Frank-Wolfe (FW) algorithm is one of the traditional approaches for solving the UE program. The algorithm utilizes the feasible descent to approximate the objective function iteratively. At each iteration, two main steps are required: Direction finding and step-size determination. To find the descent direction, for each OD pair, demand is assigned to the shortest path between that OD. This process gives the auxiliary flows in that iteration. After finding the direction, the step-size in the iteration is calculated by minimizing the objective function along the obtained direction. This process gives link flows (hence link travel times) in that specific iteration. Path travel times can then be calculated by summing up the link travel times on each path. The link/path travel times are updated with the new flow on links at the subsequent iterations, until the convergence threshold is reached. Overall, a link flow increases only when the link is on the shortest path(s) during the iteration. The underlying idea for the LCI is embedded in the flow fluctuations of individual links under iterations: if a certain link is consistently assigned additional flows (i.e.

Table 1
Selected criticality measures in the literature.

Study	Name and Measure Type	Formulation	Variables	Brief Description
Hansen (1959)	Accessibility (Topological)	$A_i = \frac{\sum_j B_j f(c_{ij})}{\sum_j B_j}$	A_i: Accessibility of region B_j: attractiveness of region j $f(c_{ij})$: Impedance function representing the separation between two regions	● More applicable for evaluating cities and towns, particularly those are well separated from each other across a network. ● Less suitable for rural areas.
Sullivan et al. (2010)	Network Robustness Index – ***NRI*** (Traffic Assignment Based)	$NRI(a) = TC_a - TC$	$NRI(a)$: Network robustness index of link TC: System travel time for the base case (complete network) TC_a: System travel time when link a is failed and traffic is re-routed	● NRI evaluates link criticality in transportation networks by using the change in total travel time of the network. ● If a failure causes disconnectivity, it cannot evaluate the impacts.
Jenelius et al. (2006)	Link importance – ***IS*** (Traffic Assignment Based)	$I_1(a) = \frac{TC_a - TC}{Q}$ $I_2(a) = \frac{\sum_{w \in W} u_w(a)}{Q}$	Q: Total demand for all OD pairs TC: System travel time for the base case TC_a: System travel time when link a is failed $u_w(a)$: Unsatisfied demand for OD pair w after link a is removed	● Two different case-specific criticality measures are defined for failures with and without disconnectivity. ● In case of disconnectivity, the measure ranks links based on the amount of unsatisfied demand. ● These separate metrics does not allow a direct comparison with the scores calculated
Latora and Marchiori (2001)	Efficiency (Topological)	$E(G) = \frac{1}{N(N-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}}$	$E(G)$: Efficiency of network N: Number of nodes in G d_{ij}: Shortest path length between node i and node j	● Efficiency allows a quantitative analysis of unweighted and weighted networks. ● A multi-discipline network-specific measure which does not directly evaluate link criticality. Changes in efficiency (e.g. relative change) can be considered as a criticality measure.
Nagurney and Qiang (2007)	Transportation Network Efficiency – ***EI*** (Traffic Assignment Based)	$E(G) = \frac{\sum_{w \in W} \frac{q_w}{\lambda_w}}{n_w}$	λ_w: Shortest path travel time of OD pair w q_w: Demand of OD pair w	● It incorporates traffic demands and travel times into the original Efficiency measure. ● This metric circumvents the disconnectivity issue mathematically and provides a uniform/unified criticality ranking.
Holme et al. (2002)	Betweenness-Centrality (Topological)	$BC(a) = \sum_{w \in W} \frac{d_w(a)}{d_w}$	$BC(a)$: Betweenness-Centrality of link a d_w: Total number of shortest paths between the OD w $d_w(a)$: Number of shortest paths between the OD w traversing link a	● BC score inherently includes information on existence of alternative paths and redundancy of links. ● Unlike Efficiency, BC can directly calculate a link/node criticality.
Gauthier et al. (2018)	Travel time weighted Betweenness-Centrality – ***TTWBC*** (Mixed)	$TTWBC(a) = w_a BC(a)$	w_a: Weight of link a based on link travel times (higher weights are assigned to links with higher travel times)	● TTWBC is calculated by considering travel times as weights for the roadway links. ● This link/node-specific measure merges travel times with a topological measure (BC).

frequently on the shortest path(s) during multiple iterations) despite the increasing link saturation (hence higher link marginal costs), the link is more critical. In order to capture this persistent flow assignment despite increasing congestion, link marginal cost (MC) is utilized. Marginal cost of link a in iteration n can be calculated as:

$$mc_a(x_a^n) = \frac{d(x_a^n \cdot t_a(x_a^n))}{dx_a^n} = t_a(x_a^n) + x_a^n \cdot \frac{dt_a(x_a^n)}{dx_a^n} \tag{1}$$

where x_a^n and $t_a(x_a^n)$ are flow and travel time for the link a in iteration n. By definition, MC provides the increase in travel time on a link with additional *unit flow*. Hence, if a saturated link is assigned additional flow of $\Delta x = x_a^{n+1} - x_a^n$ between iterations $n$and $n+1$, the travel time increase can be estimated as $\Delta x \times mc_a(x_a^n)$. Wardrop principle for UE ensures that travelers cannot switch route to reduce their travel times at the equilibrium. Hence, the links with high MC are less likely to be assigned additional flow (i.e. be on the shortest paths) during the iterations, unless the alternative paths are not viable alternatives, i.e. travelers cannot reduce their travel time. If a high level flow is assigned to an already saturated link (i.e. high MC), the travel time increases considerably. This can lead to the corresponding path not to be on the shortest path in the subsequent iteration(s). So, if the flow increases on an already saturated link, the link at hand is relatively more critical compared to the links at alternative paths in the network.

In order to illustrate the logic, assume that there are two bridges between two islands (with multiple ODs in each island). The bridges will appear in the shortest paths at most iterations (if not in all of them) and will have high criticality scores. If there is only one bridge between the two islands, then the link will definitely appear in the shortest path at all iterations, resulting in a higher score even if the link has a very high cost. In other words, if the demand is continuously assigned to a congested link, it shows the link plays a critical role in the network. In addition to the above considerations, the impact of redundancy on the link criticality is also reflected in the proposed metric by incorporating alternative path(s) between ODs into the calculations. For this purpose, path enumeration is performed for the link and all alternative paths between each OD are identified. The mathematical form of the proposed metric and additional discussion are presented below.

3.1. Link Criticality Index – LCI

In this paper, MC is selected to show the level of saturation on each link. When the MCs for two links are equal, their lengths and travel times are not necessarily equal. To make MCs comparable for all links in the whole network, link marginal costs are normalized by link travel times. Also, for links which do not experience any flow increase from one iteration to the next, the MC is assigned as the link score. If the flow increases from iteration n to $n+1$ on link a, the excess flow (i.e. $\mathrm{x}_\mathrm{a}^{n+1} - \mathrm{x}_\mathrm{a}^{n}$) is multiplied by $mc_a(x_a^n)$ to consider the impact of the additional flow to the link a on the total travel time on this link at iteration n. Accordingly, the link score can be calculated with respect to updated flows, travel times, and marginal costs at each iteration:

$$Score_a = \sum_{n=0}^{N-1} \max([x_a^{n+1} - x_a^n], 1) \cdot \frac{mc_a(x_a^n)}{t_a(x_a^n)} \tag{2}$$

The score, as presented above, does not fully reflect a link's criticality in terms of the OD pairs served and redundancy. For instance, if a link connects ODs with higher demand, then the link has more contribution to the functionality of the system. Meanwhile, if the OD flows can follow alternative paths (i.e. alternative link sequences), then the link's criticality should be reduced. In order to account for the importance of the links serving multiple ODs and alternative paths between each OD, the link scores are calculated for all the involved links on all alternative paths (based on path enumeration). During this calculation, two weighing coefficients are introduced.

The first coefficient (γ_w) weighs the criticality of link a based on the OD pairs (w) that it serves:

$$\gamma_\mathrm{w} = \frac{q_w}{Q} \tag{3}$$

where q_w is the demand for the OD pair$w$ and Q is the total demand for all OD pairs. The coefficient increases as the level of OD demand increases, i.e. links serving ODs with higher demands are more critical than links serving ODs with lower demand levels.

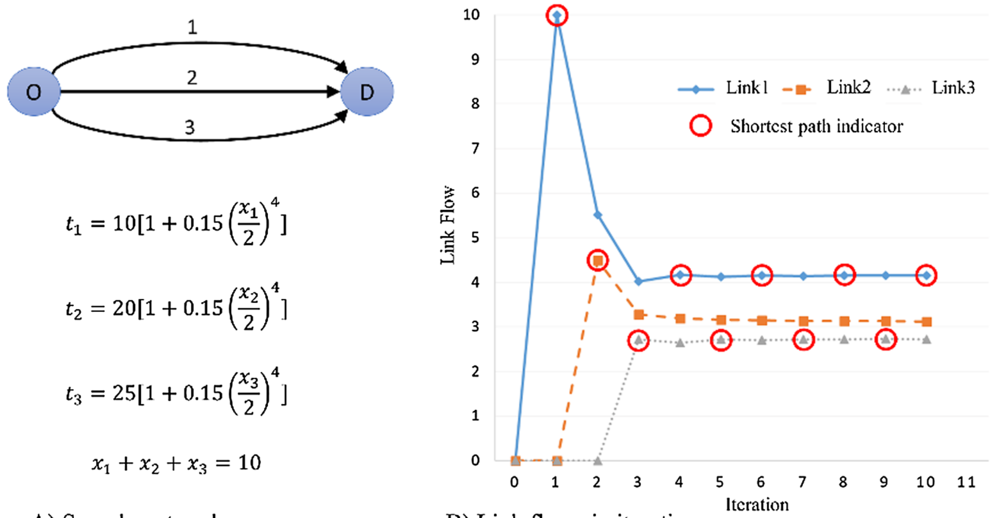
The second coefficient ($\mu_{p,w}^n$) weighs the criticality of a link based on path travel time, e.g. an alternative path with twice travel time of the shortest path should not be treated equally with an alternative path that is slightly longer than the shortest path. The coefficient is calculated based on path travel times at each iteration in convex combinations process as:

$$\mu_{p,w}^n = \frac{\frac{1}{c_{p,w}^n}}{\sum_{p' \in P_w} \frac{1}{c_{p',w}^n}} \tag{4}$$

where $c_{p,w}^n$ is the path travel time for path p, between the OD pair w, in iteration n. This coefficient guarantees that in case there are multiple alternative routes serving for an OD pair, higher weights are assigned to the faster routes. These weights are used for calculation of the LCI as:

$$LCI_\mathrm{a} = \sum_{\mathrm{n}=0}^{\mathrm{N}-1} \sum_{\mathrm{w} \in \mathrm{W}} \sum_{\mathrm{p} \in \mathrm{P_w}} \max([\mathrm{x}_\mathrm{a}^{\mathrm{n}+1} - \mathrm{x}_\mathrm{a}^\mathrm{n}], 1) \cdot \frac{\mathrm{mc_a}(\mathrm{x}_\mathrm{a}^\mathrm{n})}{\mathrm{t_a}(\mathrm{x}_\mathrm{a}^\mathrm{n})} \cdot \delta_{\mathrm{a,p}}^\mathrm{w} \cdot \mu_{\mathrm{p,w}}^\mathrm{n} \cdot \gamma_\mathrm{w} \tag{5}$$

where LCI_a is the link criticality index for link a, $\delta_{a,p}^w$ is a dummy variable showing whether path p between the OD pair w passes



A) Sample network B) Link flows in iterations

Fig. 1. A sample network: (A) Network topology and link performance functions. (B) Link flows in each iteration using convex combinations (FW) algorithm.

through link a (if so, $\delta^w_{a,p} = 1$; otherwise, $\delta^w_{a,p} = 0$). Briefly, the level of OD demands and corresponding functionality of the links are taken into account through γ_w. Similarly, redundancy of a link in the network is considered using path weights ($\mu^n_{p,w}$). Accordingly, the proposed procedure and calculations can be embedded in the convex combinations algorithm as follows (the additional calculations are marked with **bold**):

Step 0: Perform all-or-nothing assignment based on free flow travel times and set the iteration counter, n, equals to 1. (Calculate $t_a(0)$ and x_a^1)
+ **{enumerate all paths between each OD}**
Step 1: Update travel times based on new flows. (calculate $t_a(x_a^n)$)
+ **{calculate $mc_a(x_a^n)$, and $(x_a^{n+1} - x_a^n)$}**
Step 2: Direction finding. Perform all-or-nothing assignment based on new travel times.
Step 3: Step-size determination by solving a single variable optimization problem.
Step 4: Move to new flows using the direction and step-size calculated in steps 2 and 3.
+ **{calculate $c^n_{p,w}$ and, $\mu^n_{p,w}$}**
Step 5: Check convergence; if met, stop; otherwise set the iteration counter to $n + 1$ and go to step 1.
+ **{calculate LCI_a}**

The stopping criterion used in this paper for evaluating the convergence is selected as:

$$\sum_{w \in W} \frac{|u_w^n - u_w^{n-1}|}{u_w^n} \leq 0.01 \tag{6}$$

where u_w^n denotes the shortest path travel time between the OD pair w at the nth iteration (Sheffi, 1985).

4. Calculation of LCI

The simple network shown in Fig. 1 is used to illustrate step-by-step calculation of LCI. In this example, there is only one OD pair in the network, i.e. w = {1}. Hence the value of γ_1 equals to 1. Table 2 shows calculation of the LCI for three links in the network within the iterative solution of UE traffic assignment. As the iterations progress, the shortest path alternates. Consequently, the flow on a certain link increases when it is on the shortest path and decreases when it is not on the shortest path. The FW algorithm for this network converges after seven iterations. The flows, travel times, marginal costs, and the coefficient $\mu^n_{p,1}$ are calculated for each iteration, and eventually LCI is obtained according to Eq. (5).

Results show that among the three links, link #1 is the most critical link, followed by link #2 and link #3. All the paths in the network have only one link with identical performance functions except free flow travel time. As expected, the LCI ranking is opposite of the free flow travel time ranking. The following analysis and examples apply LCI in larger networks to compare with other measures and investigate the impacts of redundancy and connectivity on link criticality ranking.

Table 2
Calculation of LCI for the network depicted in Fig. 1.

Iter	Link	x_a^n	$t_a(x_a^n)$	$mc_a(x_a^n)$	$\max([x_a^{n+1}-x_a^n],1)\cdot\frac{mc_a(x_a^n)}{t_a(x_a^n)}$	$\mu_{p,1}^n$	$\frac{\lvert u_1^n-u_1^{n-1}\rvert}{u_1^n}$	step-size
0	1	0	10	10	10	0.526	–	–
	2	0	20	20	1	0.263		
	3	0	25	25	1	0.211		
1	1	10	947.5	4697.5	4.958	0.012	0.5	
	2	0	20	20	4.49	0.549		0.449
	3	0	25	25	1	0.439		
2	1	5.51	96.413	442.064	4.585	0.171	0.2	
	2	4.49	96.206	401.028	4.168	0.171		0.271
	3	0	25	25	2.71	0.658		
3	1	4.017	34.406	132.028	3.837	0.365	0.273	
	2	3.273	41.523	127.613	3.073	0.302		0.024
	3	2.71	37.641	88.206	2.343	0.333		
4	1	4.16	38.087	150.435	3.95	0.332	0.057	
	2	3.195	39.530	117.649	2.976	0.32		0.01
	3	2.645	36.471	82.353	2.258	0.347		
5	1	4.119	36.980	144.902	3.918	0.341	0.014	
	2	3.163	38.76	113.801	2.936	0.325		0.006
	3	2.719	37.801	89.004	2.355	0.334		
6	1	4.154	37.917	149.584	3.945	0.333	0.014	
	2	3.144	38.314	111.57	2.912	0.33		0.003
	3	2.702	37.496	87.481	2.333	0.337		
7	1	4.142	–	–	–	–	0.002	
	2	3.134	–	–	–	–		–
	3	2.724	–	–	–	–		
$LCI_1 = 11.466$				$LCI_2 = 7.239$		$LCI_3 = 5.57$		

5. Analysis and results

From a practitioner perspective, it is crucial to know how criticality measures compare so that they can be utilized for planning and decision making purposes. In order to address this issue, the LCI is evaluated for three different numerical examples and the results are compared with selected existing criticality metrics, i.e. Efficiency Index – EI (Nagurney and Qiang, 2007), Travel Time Weighted Betweenness-Centrality – TTWBC (Gauthier et al., 2018), Importance Score – IS (Jenelius et al., 2006), and, Network Robustness Index – NRI (Sullivan et al., 2010). The first example is a network used previously in the literature, the second one is a hypothetical network introduced to evaluate the LCI for networks with disconnectivity and redundancy. Last, Sioux Falls network is analyzed to illustrate the implementation of LCI on a larger network. The networks and rankings based on different approaches are investigated in detail, and advantages and limitations of those metrics are analyzed. The main handicap during comparison is the fact that there is no absolute, ground truth for link criticality. Hence, there is no basis to declare one measure being superior to the other. Hence, the comparison is carried out by focusing on network and traffic flow characteristics, e.g. disconnectivity links and final equilibrium flows.

5.1. Example 1: Preliminary comparison

The first example (Fig. 2) is adopted from Nagurney and Qiang (2007) and consists of 20 nodes, 28 links, and 2 OD pairs. The set of OD pairs, W, is {(1, 19), (1, 20)} and the demand for each OD pair is 100 vehicles. The link functions (which are different for each link) are given in Appendix A for the interested reader. The links are ranked with respect to the selected approaches and presented along with the rankings based on the proposed metric in Table 3.

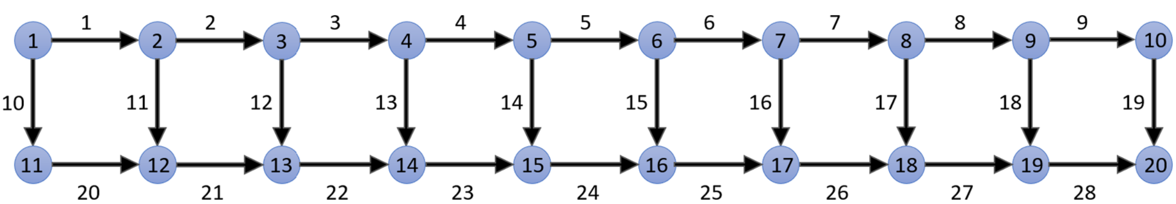


Fig. 2. Network for example 1 (Nagurney and Qiang, 2007).

Table 3
Top-10 Critical Links Based on Different Methods for Example 1.

	Criticality Ranking Method											
	LCI		TTWBC		EI		NRI		IS		UE Flows	
	link	Score	link	Score	Link	Score	link	Score	link	Score	Link	Flow
Link Criticality Ranking	1	1666.3	1	0.1883	27	0.9295	27	103,566,172	27	518,200	27	103.85
	27	1393.3	27	0.1782	26	0.9210	26	91,649,135	26	458,573	26	103.85
	2	1347.3	2	0.1663	1	0.9095	1	78,862,023	1	394,590	25	103.06
	26	1183.3	26	0.1555	2	0.8994	2	70,223,621	2	351,351	24	102.63
	3	1100.9	3	0.1435	25	0.8951	25	66,998,360	25	335,204	23	102.63
	25	984.2	25	0.1328	24	0.8947	24	66,885,331	24	334,642	22	102.63
	4	865.4	4	0.1217	3	0.8802	3	57,713,231	3	288,756	21	100.67
	5	707.9	24	0.1118	4	0.8679	4	52,343,393	4	259,196	20	100.67
	24	331.8	5	0.0993	23	0.8614	23	48,907,823	23	243,636	10	100.67
	28	294.7	28	0.0899	5	0.8449	5	43,507,655	5	213,115	1	99.33

The ranking results in Table 3 for all methods identify similar sets of links as critical, i.e. four of top-five and nine out of top-ten links are the same for all approaches. However, the individual rankings among the top links vary between approaches. As one of the discrepancies, link #1 is ranked at the top by TTWBC (Gauthier et al., 2018) whereas EI, IC and NRI ranks link #1 as the third critical link. For both OD pairs, all paths except one pass through the link #1, which increases the Betweenness-Centrality score, hence TTWBC score. On the other hand, one path between the OD pair (1, 19), and two paths between the OD pair (1, 20) do not pass through link #27. This reduces TTWBC score of link #27, which is ranked at the top by EI, IC and NRI. LCI provides top-10 rankings close to TTWBC and reasonably similar to other selected approaches. In the network shown in Fig. 2, none of the link removals causes disconnectivity in the network and unsatisfied demand for any OD pair. In order to compare the methods within a network with potential disconnectivity, the network with a bridge link is analyzed in example 2.

5.2. *Example 2: Connectivity and redundancy*

In the example network-2 (Fig. 3), removal of link #7 causes unsatisfied demand between OD pair (3,8). This network consists of 10 nodes, 15 links, and 4 OD pairs. The rankings based on the selected methods are given in Table 4. Note that, when link failure causes unsatisfied demand, Jenelius et al. (2006) propose a separate importance measure. Similarly, the measure proposed by Sullivan et al. (2010) is infinite for link #7 because the new travel time between the OD pair (3, 8) is infinite. In these respects, the score for link #7 cannot be directly compared and ranked among other links. The EI method (Nagurney and Qiang, 2007) does not need to be modified in case of disconnectivity.

According to ranking results (Table 4), links #3, #9, #8, and #13 are the most critical links based on IS, NRI and EI (excluding link #7 from NRI and IS since disconnectivity link scores are not calculated with the same equation). TTWBC ranks the same set of links at the bottom-4, and ranks links #7, #5, and #11 at the top. Meanwhile, links #5 and #11 are low-ranked by IS, NRI and EI. This is because IS, NRI and EI are sensitive to the link flows (i.e. higher flows on links #3, #9, #8, and #13), whereas TTWBC is sensitive to the BC score of links (i.e. links #5 and #11 serve the demand between all four OD pairs).

Within these rankings, link #7 requires a particular attention since it is the only link that its removal creates unsatisfied demand for the OD pair (3,8). Based on the proposed LCI, links #7, #5, #8, and #13 are the most critical links in the network. These top links are a mixture of top-rated links based on other methods, because LCI takes into account both network connectivity and congestion.

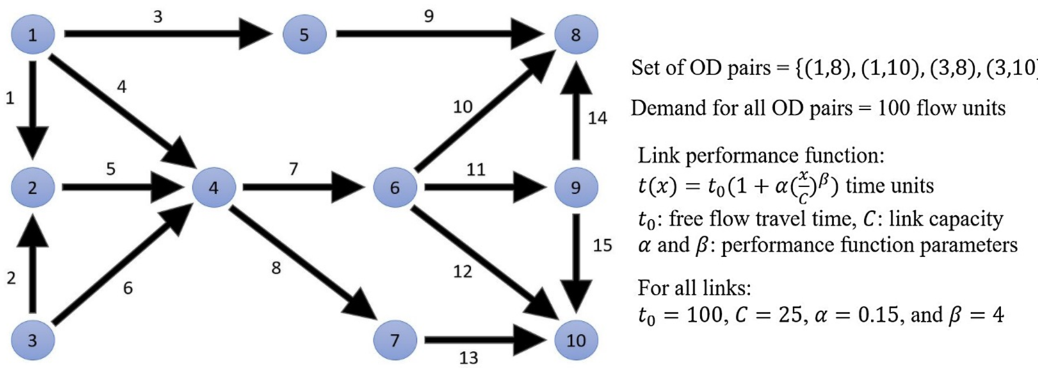


Fig. 3. Network for example 2.

Table 4
Link Criticality Rankings Based on Different Methods for Example 2.

	Criticality Ranking Method											
	LCI		TTWBC		EI		NRI		IS		UE Flows	
	link	Score	link	Score	link	Score	link	Score	link	Score	Link	Flow
Link Criticality Ranking	7	603.05	7	0.5741	3	0.8183	7	∞	7	0.25*	7	162.55
	5	340.23	5	0.3339	9	0.8183	8	86,378,709	8	216,433	8	137.45
	8	237.91	11	0.2871	8	0.3819	13	86,378,709	13	216,433	13	137.45
	13	237.91	6	0.2096	13	0.3819	3	29,652,491	3	73,993	6	114.26
	6	224.66	2	0.2071	6	0.2175	9	29,652,491	9	73,993	5	105.62
	2	166.96	14	0.1757	5	0.1753	6	23,315,460	6	58,445	3	100.00
	3	140.67	10	0.1756	4	0.1555	2	11,007,460	5	27,675	9	100.00
	9	140.67	4	0.1412	7	0.1403	5	10,642,442	2	26,773	2	85.74
	11	136.07	1	0.1268	2	0.1349	4	4,529,461	4	11,740	4	80.13
	10	118.63	12	0.1114	10	0.0244	10	755,195	10	1893	10	58.80
	4	77.00	15	0.1114	11	0.0130	12	600,925	12	1292	12	53.81
	14	66.26	8	0.1105	14	0.0112	11	361,195	11	908	11	49.94
	1	36.70	13	0.1105	15	0.0010	14	322,437	14	818	14	41.20
	12	35.42	3	0.0987	1	−0.0044	15	12,883	15	46	1	19.87
	15	9.21	9	0.0987	12	−0.0949	1	−250221	1	−442	15	8.74

* IS score for link #7 is calculated with a case-specific metric for disconnectivity links.

Links #7 and #5 appear at the top mainly due to their role in network connectivity, and links #8 and #13 are top-ranked mainly due to the high volume on those links. The LCI's balanced scoring is also reflected for links which are not ranked at the top. For example, link #3 (top-5 link for EI, NRI and IS) carries 25% of the total OD demand (100 units flow) at the equilibrium. However, it is ranked at the bottom-2 by TTWBC, because of its low BC score. In contrast, link #11 (ranked bottom-5 by EI, NRI and IS) is ranked at the top-3 by TTWBC due to its high BC score while it carries 50 flow units. LCI ranks link #3 and #11 as 7th and 9th, by not over-weighing their importance in network connectivity or utilization.

Briefly, the LCI provides a balanced score from multiple network-wide perspectives. Nevertheless, the varying rankings with different methods warrant further investigation to reinforce the discussion and better illustrate the strengths and limitations of criticality measures. For this purpose, a sensitivity analysis is performed by adjusting on OD demands and link performance functions in the example network-2. In the analysis, links #7, #3, and #11 are particularly investigated because link #7 complements link #11, i.e. the link #7 is the only link that connects the network flow to link #11. Link #3 serves as an alternate of link #7, i.e. it is on an alternative path between the OD pair (1, 8). Below, the ranking results are discussed in terms of disconnectivity, and alternate and complement links.

5.2.1. *Ranking of disconnectivity links*

Table 5 shows the changes in ranking of link #7 under two scenarios involving demand and link capacity changes. Fig. 4 illustrates the link #7's topological importance in the network, especially for the connectivity of the OD pair (3,8).

When there is no demand for the OD Pair (3,8) (scenario-1, $D^{3,8} = 0$), there is no unsatisfied demand. In both IS and NRI methods, link #7 is the most important link even when $D^{3,8} = 0$, because link #7 carries high flow between other OD pairs. In scenario-1, despite the fact that the demand for OD pair (3,8) increases and the link #7 is *still* an indispensable link that connects this OD pair, EI

Table 5
Changes in ranking of link #7 in different network conditions.

Scenario		Link #7 Ranking				
		LCI	TTWBC	EI	NRI	IS
Base Scenario (Table 4)	$C = 25$ $t_0 = 100$ $D = 100$	1	1	8	1	1
Scenario 1: Change Demand for OD Pair(3, 8)	$D^{3,8} = 0$	1(–)	1(–)	5(↑)	1(–)	1(–)
	$D^{3,8} = 200$	1(–)	1(–)	13(↓)	1(–)	1(–)
Scenario 2 (Extreme Case): Change Demand for OD Pair (3, 8) and Capacity of Link 7	$D^{3,8} = 5$ and $C^7 = 5$	6(↓)	1(–)	12(↓)	1(–)	1(–)

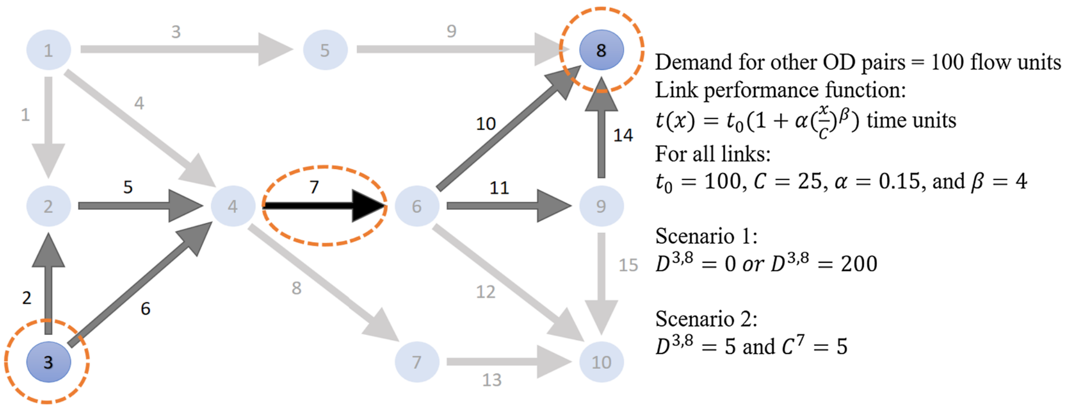


Fig. 4. Failure of link #7 causes unsatisfied demand for the OD (3,8).

yields a lower rank for link #7 (8th to 13th out of 15 links). In contrast, link #7 is ranked as the top critical link by TTWBC in both scenarios. This is because link #7 has the highest BC score in the network.

LCI ranks link#7 as the top critical link in scenario-1. Scenario-2 helps further illustrate LCI's balanced scoring of connectivity and network flows, i.e. a disconnectivity link is not always ranked at the top. In scenario-2, the link #7 capacity and demand of the OD pair (3,8) are reduced to 5 flow units simultaneously. This adjustment does not change the fact that removal of link #7 causes disconnectivity. However, it reduces the amount of UE flows on link #7 considerably. In this case, TTWBC still ranks link #7 as the 1st critical link despite the very low link flow. LCI, on the other hand, responds to the lower UE flow and places link #7 at 6th most critical link instead of the top critical link.

5.2.2. Ranking of complementary and alternate links

Fig. 5 illustrates how link #3 and link #11 function as alternate and complementary links for Link #7. Tables 6 and 7 show the changes in rankings of link #11 (complements link #7) and link #3 (alternate for link #7) for two scenarios.

Increasing the capacity of link #7 (scenario-3) increases the LCI rank for link #11 (9th to 3rd). Similarly, decreasing link #7's capacity decreases the link #11's rank from 9th to 12th. On the other hand, link #3 serves as an alternate of link #7 for carrying demand for the OD pair (1,8). Hence, a higher capacity for link #7 results in a lower rank for link #3 (7th to 11th), and vice versa (7th to 6th). These relationships are not reflected by other measures. For instance, decreasing the capacity of link #7 increases the rank of link #11 in the EI (11th to 6th), NRI (12th to 7th), and IS (12th to 7th) methods. Similarly in scenario-4, higher demand for the OD Pair (1,8) causes higher flow on link #3 at equilibrium. Hence, the link #3 is ranked higher in LCI (7th to 6th). Accordingly, link #11 which is on an alternative path for the OD (1,8) is ranked lower (9th to 10th). In contrast, when there is no demand for the OD (1,8), link #3 carries no flow and LCI ranks it at the bottom, while link #11 is ranked higher.

Overall, scenarios 1–4 help further illustrate LCI's balanced scoring of connectivity links as well as explain the ranking of complementary and alternate links. Example 2 (with identifiable disconnectivity links and alternate/complement links) illustrates the LCI's capability to reflect topological characteristics of the links in criticality rankings. Example 3 illustrates the implementation of the LCI on a larger Sioux Falls network, followed by a discussion on computational performance.

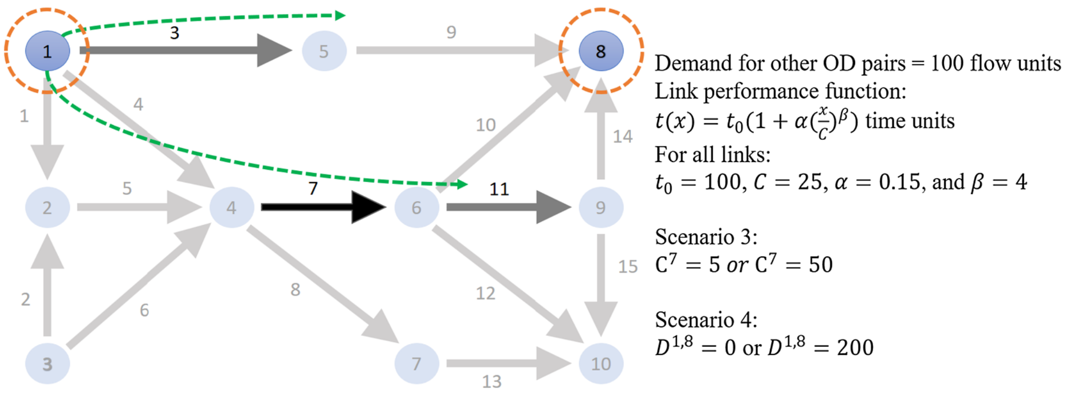


Fig. 5. Link #7 serves as a complement for link #11, and as an alternate for link #3.

Table 6
Changes in ranking of link #11 in different network conditions.

Scenario		Link #11 Ranking				
		LCI	TTWBC	EI	NRI	IS
Base Scenario (Table 4)	$C = 25$ $t_0 = 100$ $D = 100$	9	3	11	12	12
Scenario 3: Change the Capacity of Link 7	$C^7 = 5$ $C^7 = 50$	12(↓) 3(↑)	13(↓) 3(–)	6(↑) 11(–)	7(↑) 12(–)	7(↑) 12(–)
Scenario 4: Change Demand for OD Pair(1, 8)	$D^{1,8} = 0$ $D^{1,8} = 200$	7(↑) 10(↓)	3(–) 4(↓)	9(↑) 9(↑)	9(↑) 9(↑)	9(↑) 9(↑)

Table 7
Changes in ranking of links #3 in different network conditions.

Scenario		Link #3 Ranking				
		LCI	TTWBC	EI	NRI	IS
Base Scenario (Table 4)	$C = 25$ $t_0 = 100$ $D = 100$	7	14	1	4	4
Scenario 3: Change the Capacity of Link 7	$C^7 = 5$ $C^7 = 50$	6(↑) 11(↓)	6(↑) 14(–)	1(–) 1(–)	3(↑) 5(↓)	4(–) 5(↓)
Scenario 4: Change Demand for OD Pair(1, 8)	$D^{1,8} = 0$ $D^{1,8} = 200$	14(↓) 6(↑)	14(–) 13(↑)	13(↓) 14(↓)	13(↓) 14(↓)	13(↓) 14(↓)

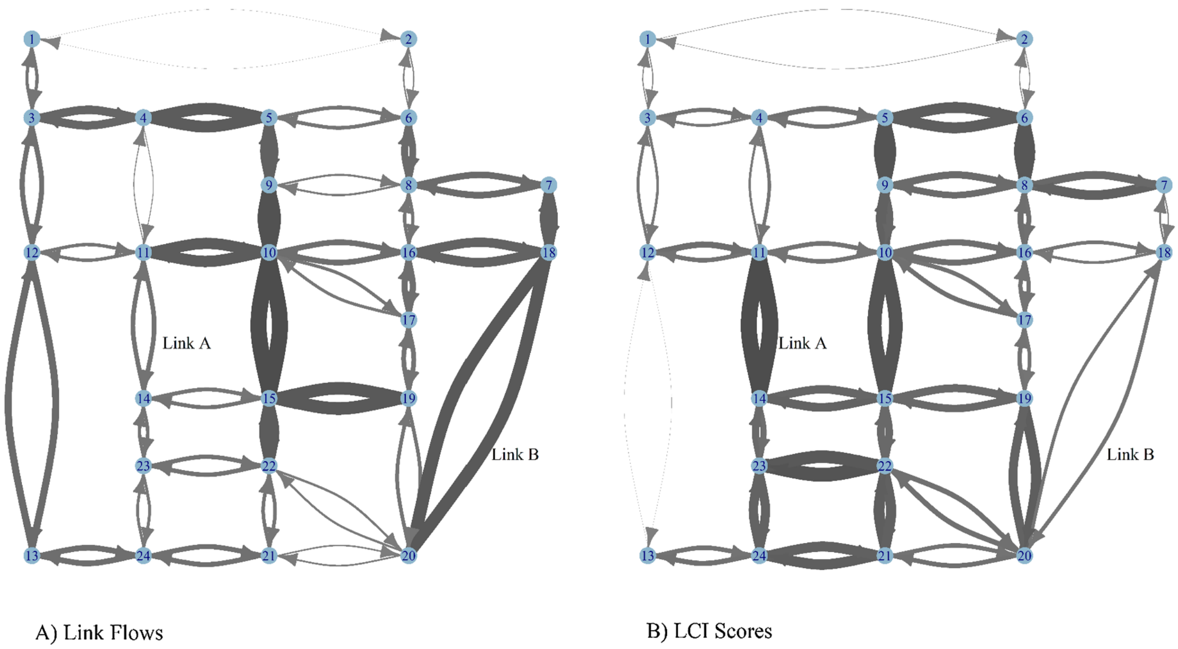


Fig. 6. Comparison of link criticality indices and link flows in Sioux Falls network: (A) Link flows and (B) LCI.

5.2.3. Example 3: Implementation on a large network

Sioux Falls is one of the benchmark networks used in network analysis and it consists of 76 links and 24 nodes (Fig. 6). The network data (e.g. OD demands, link performance functions, etc.) is obtained from (Stabler, n.d.). Sioux Falls network example is mainly used to illustrate the applicability of LCI on a large network and to compare the computational efficiency with other methods. The calculated scores for the proposed LCI and other criticality measures are given in Appendix B.

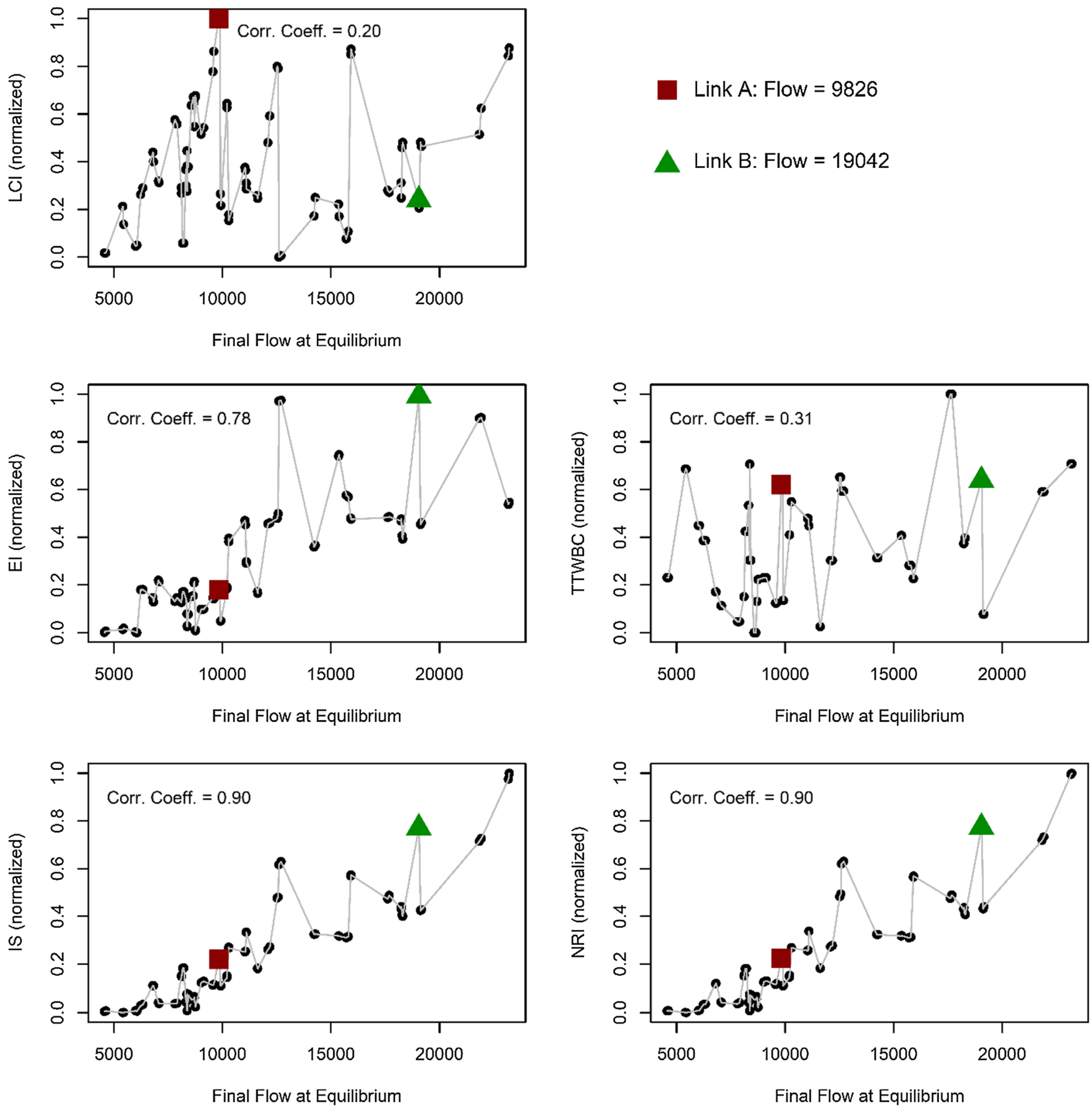


Fig. 7. Comparison of link scores with the final link flows for criticality measures.

5.2.4. Final equilibrium link flows vs. criticality ranking

Link criticality depends not only on the final equilibrium link flows, but also on the links' functionality (e.g. connectivity, redundancy) in the network. Otherwise, criticality ranking identification would not require any specific measure, and looking at the equilibrium link flows would suffice to identify the criticality rankings. The thickness of the links in Fig. 6A and B illustrate the UE link flows and LCI values in the Sioux Falls network, respectively. Fig. 7 shows the relationship between UE flows and link criticality rankings that are calculated with LCI, TTWBC, EI, NRI, and, IS. In order to provide comparable figure axes, the ranking scores for each metric are normalized between 0 and 1, i.e. $normalized\ \ score_i = \frac{score_i - score_{min}}{score_{max} - score_{min}}$, where $score_i$ represents the ranking score for link i, and $score_{max}$ and $score_{min}$ represent the highest and lowest score in the network for the selected measure, respectively. The results are represented in Fig. 7. The correlation coefficients between the UE flows and link criticality rankings are also provided, i.e. higher coefficient implies that a link's ranking score is more strongly correlated with the flow it carries at the equilibrium. To better illustrate the discussion, two links (Link A and Link B) are marked on both Figs. 6 and 7.

The correlation coefficients for IS, NRI and EI (0.90, 0.90, and 0.78, respectively) in Fig. 7 indicate that the link rankings highly correlate with the final flows. In other words, without calculating the link scores for EI, IS, and NRI, the criticality rankings can be

Table 8
Comparison of Running Time of Ranking Methods for Sioux Falls Network.

The Ranking Method	Path Enumeration		Traffic Assignment		Running Time for Other Calculations (min)	Total Running Time (min)
	Required? (Y/N)	Running Time (min)	#Assignments	Running Time (min)		
EI	N	0	77	11,079.25	5.21	11,084.46
NRI	N	0	77	11,079.25	4.32	11,083.57
IS	N	0	77	11,079.25	8.11	11,087.36
TTWBC	Y	3,671.42	1	139.93	2.67	3,814.02
LCI	Y	3,671.42	1	139.93	11.64	3,822.99

approximately identified for the Sioux Falls network. In contrast, the TTWBC and LCI do not indicate a strong correlation with the link flows, i.e. correlation coefficient is 0.31 for TTWBC and 0.2 for LCI. For example, Link A (connecting nodes 11 and 14) carries a substantially lower flow compared to Link B (which connects node 18 to node 20). However, LCI identifies Link-A as the most critical link while Link B is not among the top critical links in the network. Meanwhile, Link B is ranked higher than Link A in EI, IS, and NRI, mainly because of the higher flow it carries. TTWBC score for Link B is slightly higher that link A.

5.2.5. Computation time

In terms of computational cost, link criticality ranking methods vary considerably. On the one hand, EI, IS, and NRI methods need multiple traffic assignment (one assignment for each link failure + base assignment for complete network) while TTWBC and LCI require a single traffic assignment. On the other hand, TTWBC and LCI need one-time path enumeration which is computationally expensive.

In terms of complexity, finding the minimum paths is the most computation-intensive operations of most UE solution procedures (Sheffi, 1985). For finding the shortest path which is required in EI, IS, and NRI methods, Dijkstra's algorithm is a widely used method (Sheffi, 1985). Considering the worst-case scenario, Dijkstra's algorithm can find the shortest paths generating from a single origin in $O(n^2)$ time where n is the number of nodes in the network. Hence, for each link failure in EI, IS, and NRI methods, the running time associated with each assignment in these methods is proportional to (1) the product of the number of iterations; (2) the number of origins which determines the number of shortest paths to be calculated at each iteration; and (3) the number of nodes which determines the effort needed to calculate each shortest path. For path enumeration required in TTWBC and LCI methods, Binary Partition is a common algorithm (Marino, 2015). The worst-case running time for path enumeration between a single OD is $O((nm)^2)$ where n and m are the number of nodes and links, respectively. In TTWBC and LCI methods, there is no need to implement the Dijkstra's algorithm because all paths are already determined through path enumeration and the shortest paths can be obtained by simple sorting. Hence the running time associated with these TTWBC and LCI methods is proportional to the product of (1) the number of iterations in the UE assignment; (2) the number of ODs which determines the number of path enumerations; and (3) the number of nodes and links which determines the effort for path enumeration.

In order to compare the computational efficiency of ranking methods in practice, all algorithms are coded in R and run on a desktop computer with quad core 3.5 GHz CPU and 16 GB of memory. Accordingly, the running time of the ranking methods for this example are indicated in Table 8. The results show the total running time for EI, IS, and NRI methods are very close since the mechanism of removing links and recalculating the flows is similar in these methods. The total running time for the proposed LCI is better than EI, IS, and NRI methods, and very close to the TTWBC method. It should be noted that the most time consuming part for TTWBC and LCI methods is path enumeration which is required to evaluate the impact of connectivity/redundancy on link criticality ranking.

6. Conclusion

This paper suggests a new link criticality index (LCI) which is based on the link marginal costs under user equilibrium (UE) assignment iterations. LCI is embedded in the convex combinations solution of the UE problem. The links are assigned higher criticality score when they receive additional flow assignments at each iteration of convex combinations algorithm, despite the increasing saturation (thus the link marginal cost). The link score is calculated at each iteration and summed up for all iterations. Network-wide functionality (e.g. redundancy) of a link is taken into account by identifying alternative OD paths through a one-time path enumeration. Then the link scores are weighed with respect to OD demands and path travel times on alternative OD paths.

In order to inform policy makers about the advantages and limitations of available approaches, LCI is compared with selected measures from the literature, i.e. Efficiency Index – EI (Nagurney and Qiang, 2007), Travel Time Weighted Betweenness Centrality – TTWBC (Gauthier et al., 2018), Importance Score – IS (Jenelius et al., 2006), and, Network Robustness Index – NRI (Sullivan et al., 2010). The EI, IS, and NRI methods require removal of each link from the network and recalculation of the scores based on the UE flows on the adjusted network. In this procedure, they consider the changes in a specific network performance measure when network links are removed. LCI provides the criticality ranking without performing multiple traffic assignments, with the trade-off of requiring path enumeration (also the case for TTWBC). A sensitivity analysis of OD demands and link performance functions reveals that the existing measures put higher weight on either the final network flows (EI, IS, and NRI) or the connectivity role of the link (TTWBC). Consequently, certain set of links which are ranked at the top with one measure can be found to be ranked at the bottom with another measure. In these respects, the proposed LCI is shown to provide a balanced ranking with respect to connectivity/redundancy as well as the UE flow conditions in the network. In addition, LCI does not include any case-specific (i.e. disconnectivity link) scores and can rank all links consistently with the same mathematical formula. It is shown that the LCI calculates the criticality ranking faster for the Sioux Falls network.

6.1. Limitations and future work

The proposed LCI metric shows promising features yet also subject to some limitations. First, the method is developed based on the static UE assignment, which does not take into account link interactions, variable demand, and stochasticity. Also, path enumeration to calculate LCI is the most time-consuming module for identification of link criticality and may run into memory space considerations for larger networks with higher number of ODs. Another limitation is the use of the relatively less efficient link-based convex combinations method. An immediate future research direction is finding alternative ways to skip the path enumeration process while ensuring connectivity and redundancy are taken into account, e.g. using path-based traffic assignment methods. An interesting future work is implementation of LCI in UE assignment variants such as UE with variable demand and stochastic UE. Depending on the networks size, microsimulation can be an approach for evaluating dynamics of traffic flow and the changes in travelers' behavior when they encounter disruptions in the network. Despite these limitations, the LCI is shown to be an important initial step toward identification of transportation link criticality rankings within traffic flow dynamics and topological considerations.

Appendix A

See Table 9.

Table 9
Link performance functions for the network in example 1.

Link a	Link Function $t_a(x_a)$
1	$0.00005x^4 + 5x + 500$
2	$0.00003x^4 + 4x + 200$
3	$0.00005x^4 + 3x + 350$
4	$0.00003x^4 + 6x + 400$
5	$0.00006x^4 + 6x + 600$
6	$6x + 500$
7	$0.00008x^4 + 8x + 400$
8	$0.00004x^4 + 5x + 650$
9	$0.00001x^4 + 6x + 700$
10	$4x + 800$
11	$0.00007x^4 + 7x + 650$
12	$8x + 700$
13	$0.00001x^4 + 7x + 600$
14	$8x + 500$
15	$0.00003x^4 + 9x + 200$
16	$8x + 300$
17	$0.00003x^4 + 7x + 450$
18	$5x + 300$
19	$8x + 600$
20	$0.00003x^4 + 6x+300$
21	$0.00004x^4 + 4x+400$
22	$0.00002x^4 + 6x+500$
23	$0.00003x^4 + 9x+350$
24	$0.00002x^4 + 8x+400$
25	$0.00003x^4 + 9x+450$
26	$0.00006x^4 + 7x+300$
27	$0.00003x^4 + 8x+500$
28	$0.00003x^4 + 7x + 560$

Appendix B

See Table 10.

Table 10
Link Criticality Scores for Sioux Falls under Selected Methods.

Link	Flow	LCI	TTWBC	EI	NRI	IS	link	Flow	LCI	TTWBC	EI	NRI	IS
1 → 2	4567	1161.64	17.23	0.0197	0.0330	0.6717	13 → 24	11,095	4193.99	24.35	0.0549	0.1757	3.5943
1 → 3	8222	1617.57	23.61	0.0399	0.1081	2.2627	14 → 11	9886	12173.03	30.03	0.0403	0.1256	2.6182
2 → 1	4622	1162.69	17.23	0.0203	0.0333	0.6953	14 → 15	9018	6789.80	17.21	0.0312	0.0838	1.7142
2 → 6	6001	1480.03	24.39	0.0199	0.0336	0.7037	14 → 23	8425	5241.35	19.61	0.0287	0.0611	1.2819
3 → 1	8167	1622.01	23.61	0.0398	0.1078	2.2546	15 → 10	23,215	10876.47	32.91	0.0844	0.4609	9.5109
3 → 4	14,271	3786.26	19.93	0.0630	0.1697	3.5284	15 → 14	9138	7084.76	17.21	0.0313	0.0861	1.7692
3 → 12	10,294	2698.16	27.68	0.0648	0.1456	3.0030	15 → 19	19,128	6404.14	12.20	0.0734	0.2160	4.4083
4 → 3	14,217	2909.09	19.93	0.0622	0.1700	3.5159	15 → 22	18,271	6156.29	22.66	0.0677	0.2078	4.2368
4 → 5	18,230	4493.09	21.92	0.0751	0.2153	4.4686	16 → 8	8313	5110.10	27.18	0.0365	0.0486	1.0252
4 → 11	5409	3367.23	32.18	0.0212	0.0302	0.6330	16 → 10	11,063	5083.82	25.38	0.0732	0.1413	2.8870
5 → 4	18,240	3768.11	21.92	0.0761	0.2181	4.5453	16 → 17	11,624	3749.00	10.48	0.0392	0.1087	2.2396
5 → 6	8756	8556.60	16.97	0.0209	0.0397	0.8394	16 → 18	15,350	3473.55	23.02	0.1074	0.1670	3.4593
5 → 9	15,905	10570.77	17.08	0.0765	0.2734	5.6877	17 → 10	8106	4262.64	14.61	0.0346	0.0949	1.9708
6 → 2	6057	1530.97	24.39	0.0195	0.0343	0.6851	17 → 16	11,613	3887.69	10.48	0.0399	0.1083	2.2555
6 → 5	8750	8619.53	16.97	0.0205	0.0398	0.8489	17 → 19	9924	3413.75	14.09	0.0255	0.0780	1.5961
6 → 8	12,516	10003.06	31.06	0.0764	0.2382	4.8693	18 → 7	15,793	2183.53	18.91	0.0871	0.1650	3.4360
7 → 8	12,182	7656.09	19.60	0.0742	0.1497	3.0503	18 → 16	15,372	2890.71	23.02	0.1081	0.1677	3.4532
7 → 18	15,701	1832.21	18.91	0.0880	0.1638	3.3900	18 → 20**	19,042	3671.45	30.63	0.1373	0.3634	7.4765
8 → 6	12,565	9901.54	31.06	0.0787	0.2426	4.8999	19 → 15	19,162	6215.32	12.20	0.0742	0.2201	4.4332
8 → 7	12,091	6387.57	19.60	0.0737	0.1476	2.9492	19 → 17	9907	3958.10	14.09	0.0253	0.0766	1.6002
8 → 9	6826	5481.36	15.26	0.0348	0.0812	1.6034	19 → 20	8690	8279.62	13.95	0.0443	0.0563	1.1685
8 → 16	8327	4343.53	27.18	0.0369	0.0483	1.0264	20 → 18	19,056	3291.35	30.63	0.1382	0.3649	7.4822
9 → 5	15,921	10827.21	17.08	0.0759	0.2752	5.7297	20 → 19	8707	7151.94	13.95	0.0451	0.0574	1.1953
9 → 8	6797	5933.26	15.26	0.0370	0.0806	1.6144	20 → 21	6329	4243.37	22.33	0.0410	0.0453	0.9295
9 → 10	21,838	6781.86	29.05	0.1262	0.3398	6.9791	20 → 22	7052	4574.77	13.35	0.0457	0.0489	1.0038
10 → 9	21,924	8018.12	29.05	0.1266	0.3460	7.0867	21 → 20	6234	3937.48	22.33	0.0409	0.0444	0.8902
10 → 11	17,684	4031.63	42.48	0.0772	0.2411	4.9652	21 → 22	8656	8551.77	9.65	0.0379	0.0537	1.1173
10 → 15	23,171	10501.33	32.91	0.0834	0.4592	9.2937	21 → 24	10,203	8246.18	23.12	0.0422	0.0959	1.9686
10 → 16	11,039	5212.47	25.38	0.0753	0.1404	2.8604	22 → 15	18,300	6397.71	22.66	0.0661	0.2056	4.1854
10 → 17	8112	3989.72	14.61	0.0355	0.0961	1.9444	22 → 20	7078	4495.21	13.35	0.0448	0.0480	0.9689
11 → 4	5447	2518.24	32.18	0.0217	0.0301	0.6374	22 → 21	8562	8156.88	9.65	0.0375	0.0541	1.1279
11 → 10	17,609	4126.33	42.48	0.0769	0.2350	4.8371	22 → 23	9604	10707.30	13.71	0.0363	0.0808	1.6493
11 → 12	8373	4084.84	32.86	0.0229	0.0332	0.7030	23 → 14	8366	6000.54	19.61	0.0289	0.0618	1.3015
11 → 14*	9826	12265.65	30.03	0.0409	0.1267	2.5862	23 → 22	9564	9755.91	13.71	0.0369	0.0792	1.6221
12 → 3	10,292	2973.44	27.68	0.0666	0.1459	3.0358	23 → 24	7906	7272.77	11.17	0.0370	0.0481	0.9818
12 → 11	8375	5279.46	32.86	0.0226	0.0335	0.7087	24 → 13	11,095	4473.08	24.35	0.0540	0.1753	3.6029
12 → 13	12,598	965.89	29.15	0.1348	0.2976	6.1073	24 → 21	10,203	8053.23	23.12	0.0413	0.0926	1.9075
13 → 12	12,697	1030.93	29.15	0.1352	0.3017	6.2280	24 → 23	7806	7472.40	11.17	0.0351	0.0456	0.9595

* Link A in Figs. 6 and 7.
** Link B in Figs. 6 and 7.

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