NAC-COLORINGS SEARCH: COMPLEXITY AND ALGORITHMS

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Goals

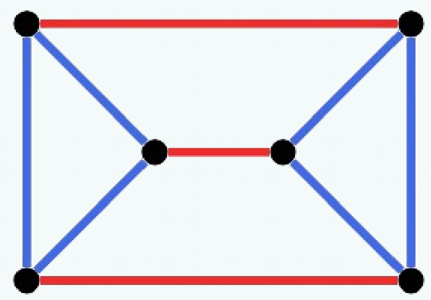
- Study basics of Rigidity theory & flexible realizations.
- Show that NAC-coloring existence is NP-complete on graphs with maximum degree five.
- Design, implement and evaluate an algorithm and heuristics for NAC-coloring search.

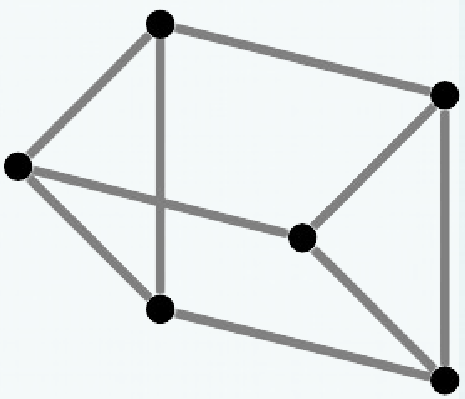
NAC-coloring

- *NAC-coloring* is an edge coloring by red and blue such that it is surjective and there are no cycles with exactly one red or blue edge.
- It is NP-complete to decide if a graph has a NAC-coloring.
- Certifiable by connected components search.

Flexible realizations

- Realization of a graph into the space is *flexible* if it can be transformed while preserving edge lengths, otherwise it is *rigid*.
- Generically rigid graphs may still have some flexible realizations.
- A graph has a flexible realization iff. there exist a NAC-coloring of the graph.

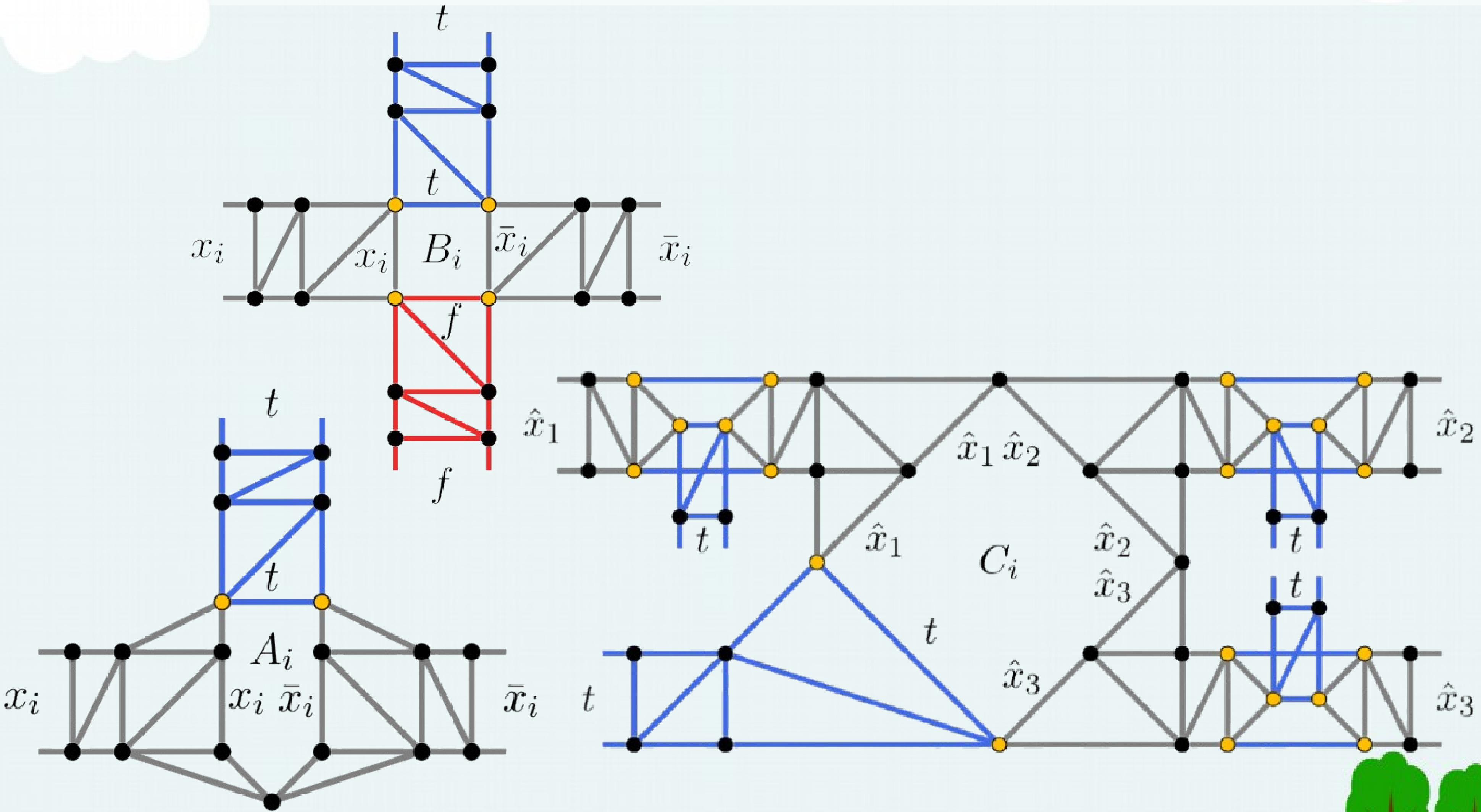




NP-comp. & degree five

- It has been already known that it is NP-complete to decide wether a graph has a NAC-coloring.
- We show that it is NP-complete to answer also for graphs with maximum degree five.
- Reduction from 3-SAT.

Reduction from 3-SAT



FPT algorithm

- Algorithms polynomial in graph size with a factor *f(k)*, where *k* is a graph parameter.
- Algorithm for NAC-coloring counting parametrized by treewidth *k*.
- Treewidth represents kind of a similarity of a graph with trees.
- Decomposition tree where nodes are bags of vertices (vertex cuts) of the original graph.
- Dynamic programming alg. run on the decomp.

FPT algorithm

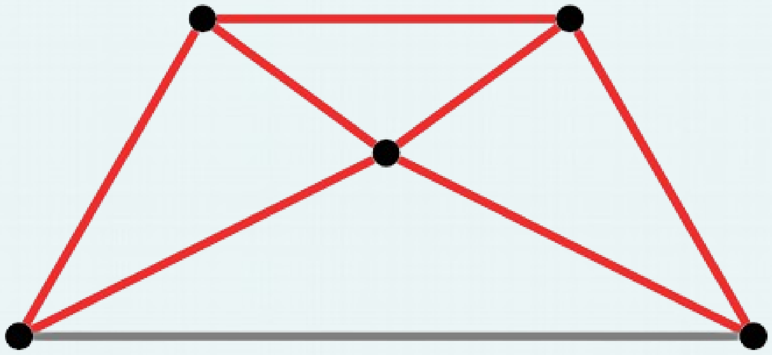
- Information about connectivity in bags needs to be preserved, hence superexponential complexity in k, linear in graph size.
- Recursive definition of the cache function.
- Additional optimizations proposed.
- Not implemented, only proved.

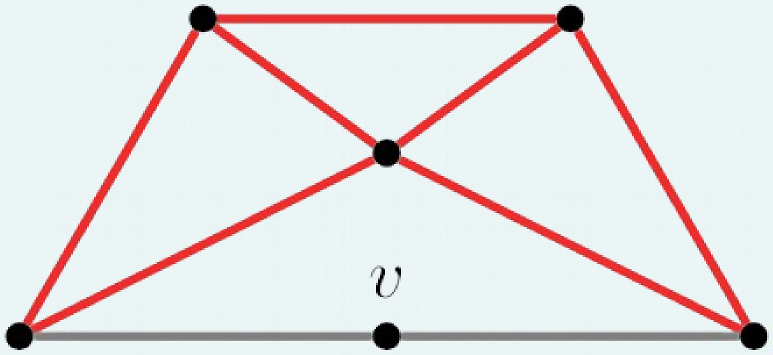
Stable cuts

- A *stable cut* is a vertex cut that is also an independent set.
- If a stable cut exists, a NAC-coloring also trivially exists.
- Algorithm for stable cut search in flexible graphs implemented.

NAC-coloring search

- Naive approach tries all the colorings.
- Triangle connected components evolved into monochromatic classes.
- Quick check for small cycles using bit masks.





NAC-coloring search

- Strategy:
  - Decompose into smaller subgraphs.
  - Find all the NAC-colorings of the subgraphs.
  - Choose colorings merge order for the subgraphs.
- Multiple heuristics for each stage.

Benchmarks

- Flexible vs. minimally rigid vs. globally rigid.
- Any NAC-coloring / all NAC-colorings / the number of NAC-colorings.

Graphs with many NAC-colorings

- Improved naive approach is the fastest.
- Fast for finding any NAC-coloring on over ~100 vertex graphs.
- Listing all NAC-colorings runs fast enough for ~30 vertex graphs.

Graphs with no/few NAC-colorings

- Naive search in not feasible.
- Monochromatic classes reduce search space significantly (hard to find random hard cases).
- Tens of vertices / monochromatic classess run in few seconds.

NAC-colorings search

- Extension of our paper from VýLeT.
- Code contributed to PyRigi.
- I am ready for your questions and discussion.

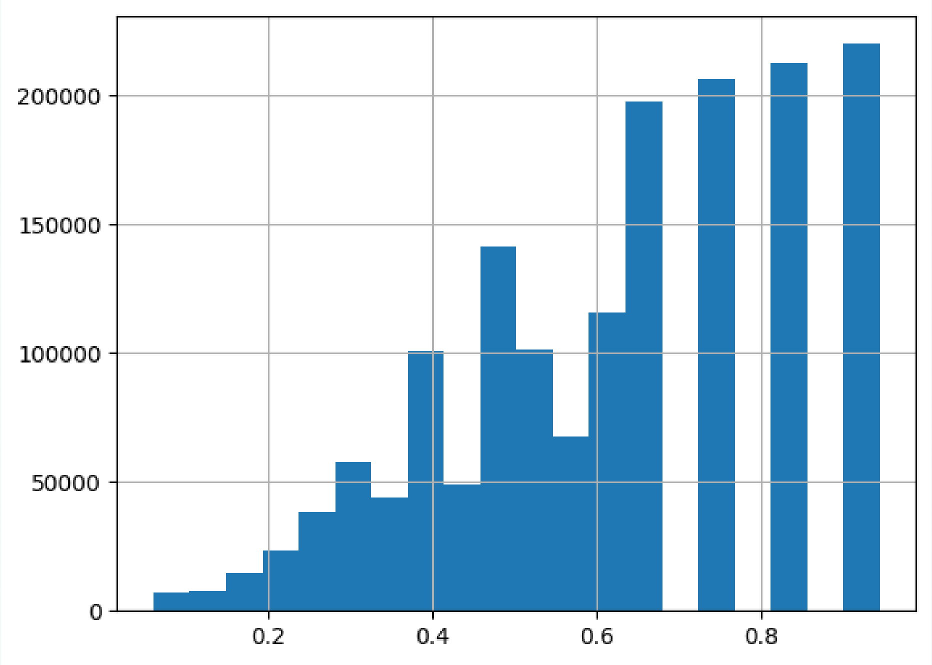
Reviewer’s question

Is it typically the case that the color classes in a NAC-coloring are balanced with respect to the two colors (especially when the graph has only few NAC-colorings)?

Or does it often happen (in practice) that there exists some NAC-coloring that is largely biased towards one of the two colors?

Minimally rigid graphs

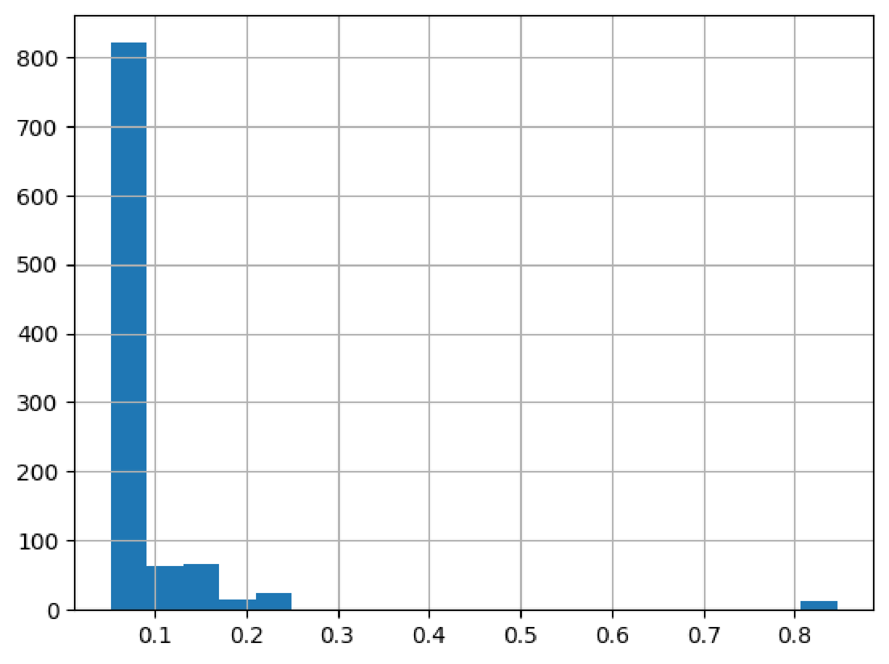
sample size	1000
NAC-colorings	1602001
weighted mean	0.60
mean	0.63
std	0.21
min	0.06
25	0.46
50	0.65
75	0.83
max	0.94



Minimally rigid graphs with 18 to 20 vertices.

Globally rigid graphs (single)

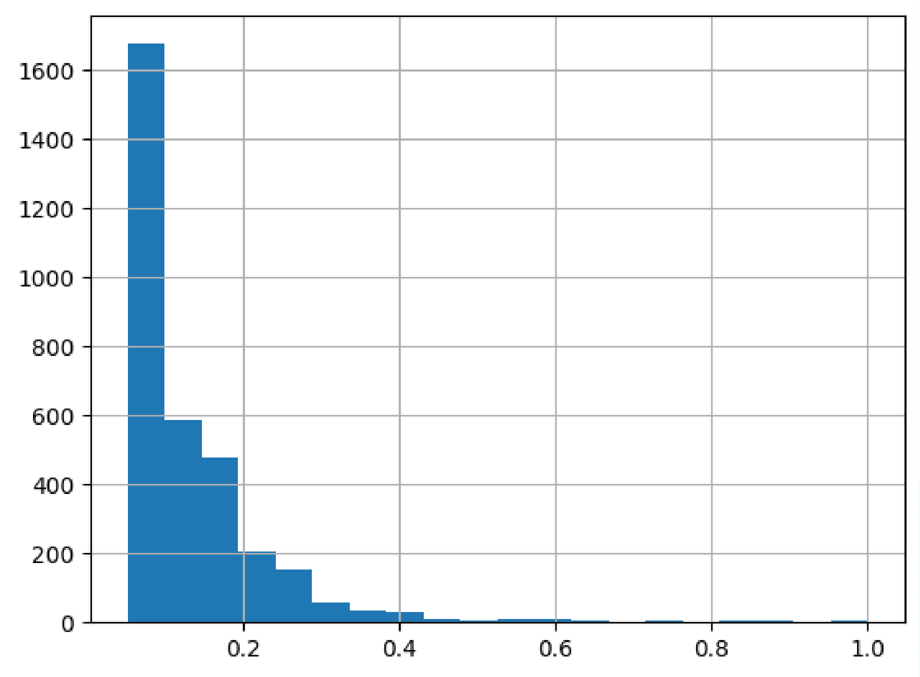
sample size	1000
NAC-colorings	1000
weighted mean	0.09
mean	0.09
std	0.09
min	0.05
25	0.06
50	0.07
75	0.09
max	0.85



Globally rigid graphs with 18+ vertices and one unique NAC-coloring.

Globally rigid graphs (few)

sample size	1000
NAC-colorings	3287
weighted mean	0.11
mean	0.13
std	0.11
min	0.05
25	0.07
50	0.10
75	0.16
max	1.00



Globally rigid graphs with 18+ vertices and
at most 10 unique NAC-colorings.

Reasoning

- Flexibility opportunities can be caused easily by small local defects while the rest of the graph can be far from having any NAC-coloring.