

PROBLEM #4.

Use the MacLaurin series to formulate approximations of the first three non-zero terms for the function,

$$f(x) = e^{x^2}$$

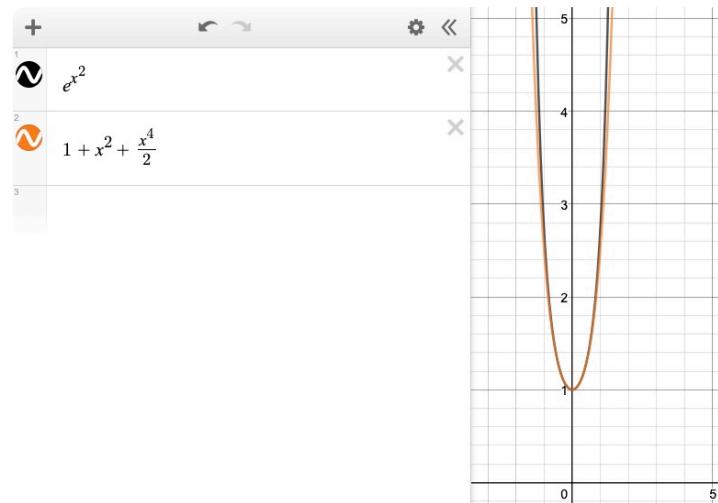
The MacLaurin series is

$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\begin{aligned} g_0(0) &= \frac{f^{(0)}(0)}{0!} x^{(0)} \\ &= \frac{1}{1}(1) \end{aligned}$$

$$g_1(x) = 1 + \frac{f^{(1)}(0)}{1!} x^1$$

$$\begin{aligned} g_1(x) &= 1 + e^{x^2} 2x \cdot x^1 \\ &= 1 + e^0 2(0) \cdot x \\ &= 1 + 0 \end{aligned}$$



$$g_2(x) = 1 + 0 + \frac{f^{(2)}(0)}{2!} x^2$$

$$\begin{aligned} &= 1 + [e^{x^2} 4x^2 + 2e^{x^2}] x^2 \\ &= 1 + 2x^2/2 \end{aligned}$$

$$g_3(x) = 1 + x^2 + \frac{f^{(3)}(0)}{3!} x^3$$

$$\begin{aligned} &= 1 + x^2 + [e^{x^2} 8x^3 + 8x e^{x^2} + 2e^{x^2} 2x] x^3 \\ &= 1 + x^2 \end{aligned}$$

$$\begin{aligned} g_4(x) &= 1 + x^2 + [e^{x^2} 16x^4 + 24x^2 e^{x^2} + 8e^{x^2} + e^{x^2} 16x^2 \\ &\quad + 4e^{x^2} + 8x^2 e^{x^2}] x^4 \\ &= 1 + x^2 + 12(0)x^4/4! \end{aligned}$$

PROBLEM #2.

Use Taylor Series to approximate the first three terms of $f(x) = \frac{1}{x}$ at $x=p=4$.

$$\text{Taylor Series: } \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (x-p)^n$$

$$y^0(\frac{1}{x}) = \frac{1}{x}$$

$$g_0(x) = \frac{1/x}{0!} (x-4)^0$$

$$\frac{d}{dx}(\frac{1}{x}) = -x^{-2}$$

$$= \frac{\frac{1}{x}}{1} (1)$$

$$y''(\frac{1}{x}) = 2x^{-3}$$

$$= \frac{1}{4}$$

$$y^{(3)}(\frac{1}{x}) = -6x^{-4}$$

$$g_1(x) = \frac{-(p)^{-2}}{1!} (x-4)^1 + g_0(x)$$

$$= -\frac{(x-4)}{4^2} + g_0(x)$$

$$= -\frac{(x-4)}{16} + \frac{1}{4}$$

$$g_2(x) = \frac{2(p)^{-3}}{2!} (x-p)^2 + g_1(x)$$

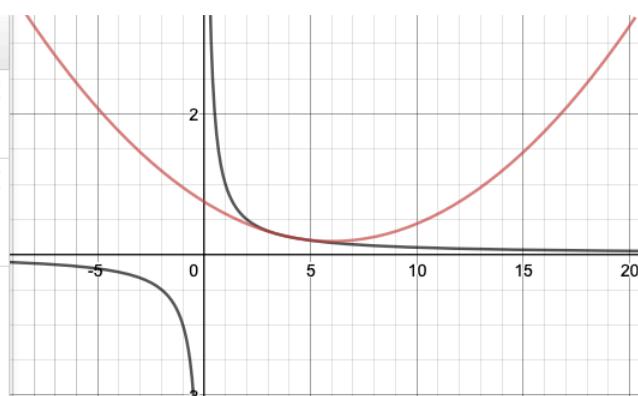
$$= 4^{-3} (x-4)^2 + g_1(x)$$

$$= \frac{(x-4)^2}{4^3} + g_1(x)$$

$$= \frac{(x-4)^2}{4^3} - \frac{(x-4)}{4^2} + \frac{1}{4}$$

$$1 \quad \frac{1}{x}$$

$$2 \quad \frac{(x-4)^2}{4^3} - \frac{(x-4)}{4^2} + \frac{1}{4}$$



PROBLEM #3

Determine the magnitude of difference, between the second order and first order expansion of the Taylor Series when approximating to find the value of $f(2)$, use $f(x) = \ln x$ at the point $x=10$.

$$f^{(0)}(x) = \ln x$$

$$f^{(1)}(x) = \frac{d}{dx} \ln x$$

The derivative of $\ln x$,

$$e^y = x \Leftrightarrow \frac{d}{dx} e^y = \frac{d}{dx} x$$

$$\Leftrightarrow e^y \frac{d}{dx}(y) = 1, \text{ provided } y \text{ has an object relative to } x.$$

$$\Leftrightarrow \frac{dy}{dx} = \frac{1}{e^y}$$

$$\Leftrightarrow \frac{dy}{dx} = \frac{1}{x}. \text{ given } e^y = x.$$

Therefore the derivative of $\ln x$ and the first order derivative of $f(x) = \ln x$ is,

$$= x^{-1}.$$

$$\begin{aligned} f^{(2)}(x) &= \frac{d}{dx} x^{-1} \\ &= -x^{-2} \end{aligned}$$

Taylor Series Approximation.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (x-p)^n,$$

$$\begin{aligned} g_0(x) &= \frac{\ln(p)}{1!} (x-10)^0 \\ &= \ln 10 \end{aligned}$$

$$\begin{aligned} g_1(x) &= \frac{1}{p \cdot 1!} (x-p)^1 + g_0(x) \\ &= \frac{(x-10)}{10} + \ln 10 \end{aligned}$$

$$\begin{aligned} g_2(x) &= \frac{f^{(2)}(p)}{n!} (x-p)^n + g_1(x) \\ &= \frac{-(x-10)^2}{(10)^2 2!} + \frac{x-10}{10} + \ln 10. \end{aligned}$$

The difference between the first order and second order approximation is $g_2(x) - g_1(x)$ at $x=2$.

$$\Delta f(2) = g_2(x) - g_1(x), \text{ where } x=2.$$

$$= -\frac{(x-10)^2}{2 \cdot 10^2} + \frac{x-10}{10} + \ln 10 - \left[\frac{x-10}{10} + \ln 10 \right]$$

$$= -\frac{(x-10)^2}{10^2 \cdot 2!}$$

$$= -\frac{(2-10)^2}{10^2 \cdot 2}$$

$$= -\frac{(8)^2}{10^2 \cdot 2}$$

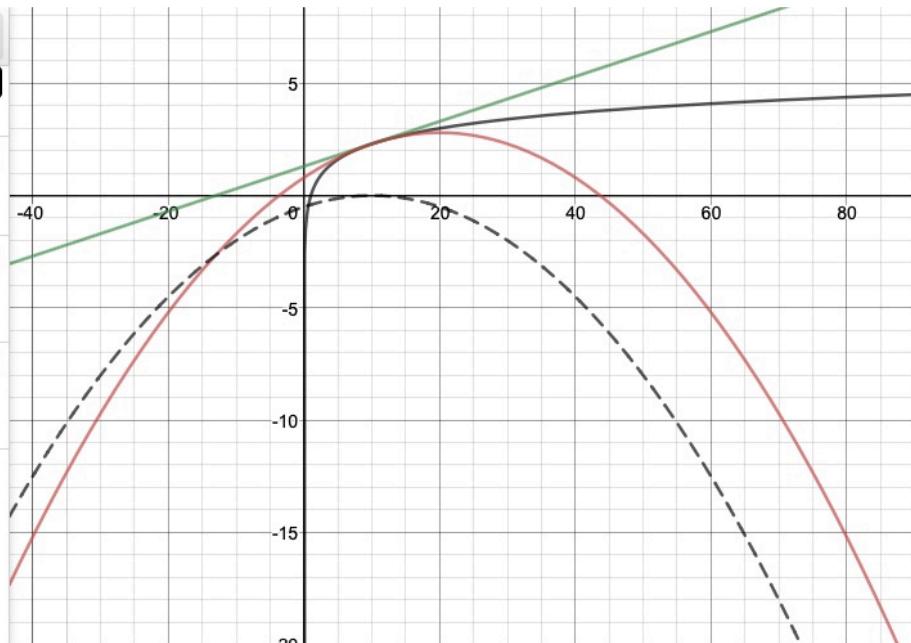
$$= -\frac{\left(\frac{8}{10}\right)^2}{2}$$

$$= -\frac{1}{2} \times \left(\frac{4}{5}\right)^2$$

$$= -\frac{(16/2)}{25}$$

$$= -\frac{8}{25}$$

$\ln x$	<input type="button" value="Hide Expression List"/>
$\frac{(x-10)}{10} + \ln 10$	
$-\frac{(x-10)^2}{200} + \frac{(x-10)}{10} + \ln 10$	
$-\frac{(x-10)^2}{200}$	$\Delta f = g_2(x) - g_1(x)$



PROBLEM #4.

Find the general Taylor series approximation for

$$f(x) = \frac{1}{(1-x)^2} \text{ where } x=0.$$

$$f^{(0)}(x) = \frac{1}{(1-x)^2}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (x-p)^n, \text{ where } p=0.$$

$$f^{(1)}(x) = -\frac{2(-1)}{(1-x)^3}$$

$$g_0(x) = \frac{1}{(1-0)^2} (x-0)^0 \\ = 1$$

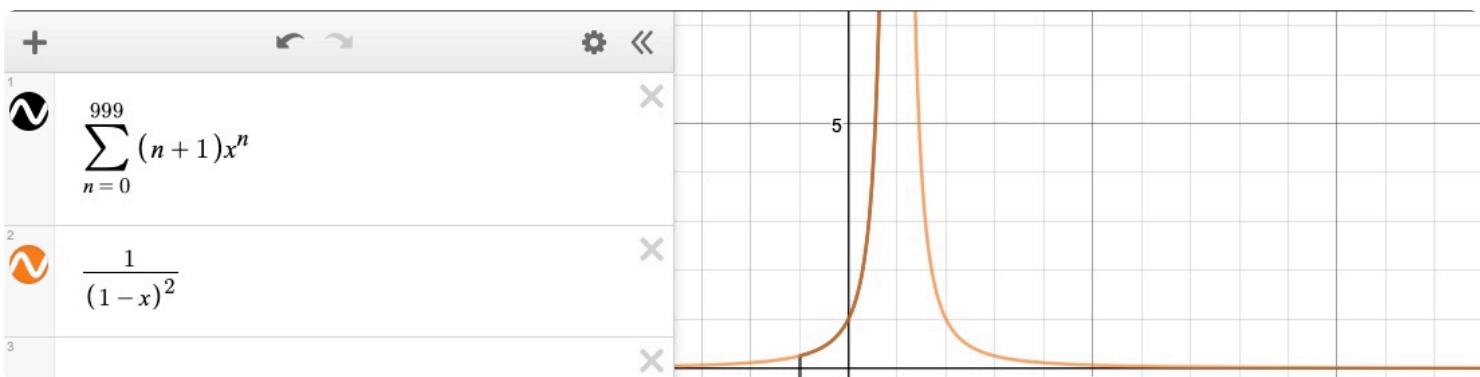
$$f^{(2)}(x) = \frac{3(2)}{(1-x)^4}$$

$$g_1(x) = \frac{2}{(1-0)^3} (x-0)^1 + g_0(x)$$

$$f^{(3)}(x) = \frac{4(3)(2)}{(1-x)^5}$$

$$f(x) \approx \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} x^n \\ \approx \sum_{n=0}^{\infty} (n+1)x^n$$

This can be generalized to,



PROBLEM #5.

Use the MacLaurin series to approximate $f(x) = \sqrt{4-x}$ as a quadratic equation.

$$f^{(0)}(x) = \sqrt{4-x}$$

$$M = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f^{(1)}(x) = \frac{1}{2} (4-x)^{-1/2} (-1)$$

$$g_0(x) = \frac{\sqrt{4-x}}{0!} x^0 \quad g_2(x) = -\frac{1}{4} \frac{(4-x)^{-3/2}}{2!} x^2$$

$$f^{(2)}(x) = +\frac{1}{4} (4-x)^{-3/2} (-1)$$

$$g_1(x) = \frac{1}{2} \frac{(4-x)^{-1/2}}{1!} x^1 + \sqrt{4-x} \quad -\frac{2!}{2} \frac{(4-x)^{1/2}}{2!} x^2 + \sqrt{4-x}$$

$$g_2(x) = -\frac{x^2}{64} - \frac{x}{4} + 2$$

