

## Linear Algebra Background

CONJUGATE TRANSPOSE  
(AKA HERMITIAN)

or  
Bedeagored Matrix.

$A^*$   
 $A^H$   
 $A^T$

Take the transpose of  $A_{m \times n}$  with complex entries, then take complex conjugate of each entry.

$a+ib \rightarrow a-ib$ , where  $a, b \in \mathbb{R}$ .

Formal definition, for  $A_{m \times n}$

$(A^H)_{ij} = \overline{A}_{ji}$ , notice i and j are transposed

can also be written,

$$A^H = (\bar{A})^T = \bar{A}^T$$

NOT  $\text{adj}(A)$ , the ADJUGATE.

When matrix is square

Hermitian or self-adjoint

$$A = A^H \rightarrow a_{ij} = \overline{a_{ji}}$$

Skew Hermitian / Anti hermitian

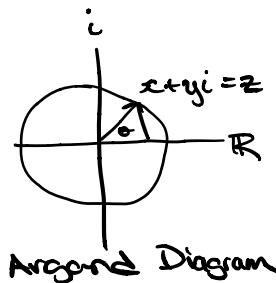
$$A = -A^H \rightarrow a_{ij} = -\overline{a_{ji}}$$

Normal (Hermitian w-commutativity)

$$A^H A = A A^H$$

Unitary

$$A^H = A^{-1}$$



Complex Numbers as  $2 \times 2$  matrix, works for multiplication & addition,

$$a+ib = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad z = a+ib$$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \underbrace{a^2 - b^2}_{\text{new } a} & \underbrace{-2(a+b)}_{\text{new } b+1} \\ \underbrace{2(a+b)}_{\text{new } b} & \underbrace{-b^2 + a^2}_{\text{new } a} \end{bmatrix},$$

$a$  - is  $x$  of  
an Argand D.  
 $b$  - is  $y$  of the  
Argand (imag-axis)

Motivation.

Picture an  $A_{m \times n}$  matrix of  $\mathbb{C}$  numbers,  $\Rightarrow A_{2m \times 2n}$  of  $\mathbb{R}$ ,

$$\begin{bmatrix} C_1 & \dots \\ C_2 & \dots \end{bmatrix} \Rightarrow \begin{bmatrix} R_1 & R_1 & \dots & \dots \\ R_1 & R_1 & \dots & \dots \\ R_2 & R_2 & \dots & \dots \\ R_2 & R_2 & \dots & \dots \end{bmatrix}$$

$$z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

The transpose of  $2 \times 2$  represented complex numbers switch it's sign to maintain mult/adj. Thus reducing  $\mathbb{R}_{2m \times 2n}$  to  $\mathbb{C}_{m \times n}$  and then performing a transpose must account for the  $\mathbb{R}$ -based transpose mechanics  $\rightarrow$  yielding the reason for performing the complex conjugate after transposing a  $\mathbb{C}$  matrix.

## Unitary Matrix

A complex square matrix  $U$  is unitary if its conjugate transpose  $U^*$  is its inverse.

$$\left. \begin{array}{l} U^*U = UU^* = I \\ \text{or} \\ U^H = U^{-1} \end{array} \right\} \begin{array}{l} \text{If } U \in \mathbb{R}^N \\ \rightarrow U^* = U^T \end{array}$$

In physics

$$U^T U = U U^T = I$$

## Diagonalizable Matrix

A matrix can be called similar to a diagonal matrix if there exists an invertible matrix  $P$ , such that  $P^{-1}AP$  is a diagonal matrix.

also like a change of basis.

## Diagonal Matrix

Entries outside the main diagonal are zero.

$$\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} = A.$$

Usually refers to square matrix.

Q: Eigenvalues + Eigenvectors of a diagonalizable matrix?

Q<sub>2</sub>: What are normal matrices and normal equations?

Q<sub>3</sub>. Gram Schmidt Process?

How linear independence transforms to orthonormal?

Q. QR Factorization  
orthogonal vs. unitary matrix

## Norms

Most commonly the Euclidean norm,

$$\|\vec{a}\| = \sqrt{x^2 + y^2 + z^2}$$

in a 3-dimensional space.

## Hermitian Matrix

A hermitian  $\Leftrightarrow a_{ij} = \overline{a_{ji}}$ ,  
 $A = A^H$

## Orthonormal Vectors

A collection of real  $m$ -vectors  $a_1, a_2, a_3, \dots, a_n$  is orthonormal if,

- vectors have unit norm:

$$\|a_i\| = 1$$

- mutually orthogonal:

$$a_i^T a_j = 0, \text{ when } i \neq j.$$

Examples,

$$\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

## Norms with Scripts

Superscript is a power.

Subscript is dimension.

$$\|\vec{v}\|_y^x \rightarrow \left[ \sum_{i=1}^d |v_i|^y \right]^{\frac{1}{y}}$$

## Matrix with Orthonormal Vectors

$$A^T A = [a_1, a_2, \dots, a_n]^T [a_1, a_2, \dots, a_n]$$

$$= \begin{bmatrix} a_1^T a_1 & a_1^T a_2 \dots a_1^T a_n \\ a_2^T a_1 & a_2^T a_2 \dots a_2^T a_n \\ \vdots & \vdots \ddots \vdots \\ a_n^T a_1 & a_n^T a_2 \dots a_n^T a_n \end{bmatrix}$$

Gram Matrix

$$\left\{ = \begin{bmatrix} a_1^0, a_1^1, \dots, a_1^n \\ a_2^0, a_2^1, \dots, a_2^n \\ \vdots \\ a_n^0, a_n^1, \dots, a_n^n \end{bmatrix} \begin{bmatrix} a_1^0, a_2^0, \dots \\ a_1^1, a_2^1, \dots \\ \vdots \\ a_1^n, a_2^n, \dots \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

$$= I \Leftrightarrow \forall x \in A \rightarrow \| \vec{x} \|$$