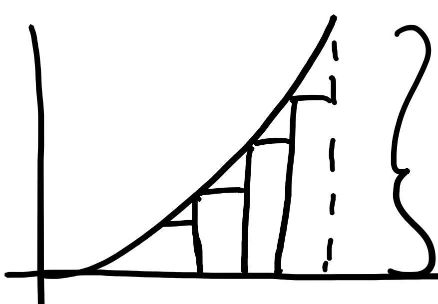
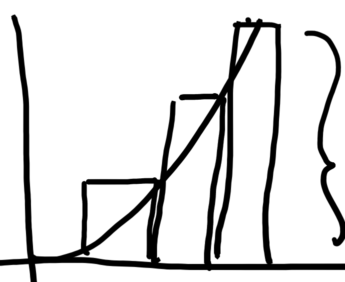
Integrals via Exhaustion

вторник, 26 февраля 2019 г. 23:30

Method for Parabolic Segment Integration





Construct sums for upper and lower bound approx.

LBound = $s_n = \left(\frac{b}{n}\right)\left(\frac{kb}{n}\right)^2 = \frac{b^3}{n^3}\cdot k^2$

$= \frac{b^3}{n^3}(1^2+2^2+\dots+(n-1)^2)$

UBound = $S_n = \frac{b^3}{n^3}(1^2+2^2+\dots+n^2)$

Equate $\sum_{k=1}^{n} k^2 = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$

Derive,

$1^2+2^2+\dots+(n-1)^2 < \frac{n^3}{3} < 1^2+2^2+\dots+n^2$

Then multiply by b^3/n^3,

$s_n < \frac{b^3}{3} < S_n$

Make assertion $A = b^3/3$.

$s_n < A < S_n$

Reframe as,

$\frac{b^3}{3}-\frac{b^3}{n} < A < \frac{b^3}{3}+\frac{b^3}{n}$

Replace s_n and S_n,

UBound

$1^2+2^2+\dots+n^2 < \frac{n^3}{3}+n^2$

$\Leftrightarrow S_n < \frac{b^3}{3}+\frac{b^3}{n}$

LBound

$\frac{n^3}{3} < 1^2+2^2+\dots+n^2$

$\Leftrightarrow \frac{n^3}{3}-n^2 < \sum_{k=1}^{n-1}k^2$

$\Leftrightarrow \frac{b^3}{3}-\frac{b^3}{n} < \frac{b^3}{n^3}\cdot\sum_{k=1}^{n-1}k^2$

$\Leftrightarrow \frac{b^3}{3}-\frac{b^3}{n} < S_n$

Cases:

1. $A > b^3/3$
2. $A < b^3/3$
3. $A = b^3/3$

Contradiction of $A > \frac{b^3}{3}$

$A > \frac{b^3}{3} \Rightarrow A < \frac{b^3}{3}+\frac{b^3}{n}$

$\Leftrightarrow A-\frac{b^3}{3} < \frac{b^3}{n}$

$\Leftrightarrow n < \frac{b^3}{(A-b^3/3)}$

This inequality is false when $n \geq b^3/(A-b^3/3)$, and is thus contradictory. A can not be $> b^3/3$.

Contradiction of $A < b^3/3$.

Given the inequality is true,

$\frac{b^3}{3}-\frac{b^3}{n} < A < \frac{b^3}{3}+\frac{b^3}{n}$

Then re-express in terms of n

$\frac{b^3}{3}-\frac{b^3}{n} < A \Leftrightarrow A-\frac{b^3}{3} > -\frac{b^3}{n}$

$\Leftrightarrow \frac{b^3}{n} > \frac{b^3}{3}-A$

$\Leftrightarrow \frac{b^3}{(\frac{b^3}{3}-A)} > n$

This inequality is contrad. when $n \geq \frac{b^3}{(\frac{b^3}{3}-A)}$; Therefor the Area can not be $A < b^3/3$.

Therefore, since

A is not $< b^3/3$

A is not $> b^3/3$

Thus A must $= b^3/3$

Theorem $\sum^n(k^2) = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$

Begin with the equality,

$(k+1)^3 = k^3+3k^2+3k+1$

$\Leftrightarrow (k+1)^3-k^3 = 3k^2+3k+1$

$\Leftrightarrow \forall k$ where $1\le k\le n-1$,

$\cancel{(2)^3}-(1)^3 = 3(1)^2+3(1)+1$

$\cancel{(3)^3}-\cancel{(2)^3} = 3(2)^2+3(2)+1$

$\cdots = \cdots$

$(n)^3-\cancel{(n-1)^3} = 3(n-1)^2+3(n-1)+1$

The summation of these equations,

$n^3-1 = 3\sum^{n-1}k^2+3(n-1)(n)\cdot\frac{1}{2}+(n-1)$

$\Leftrightarrow 3\sum^{n-1}k^2 = n^3-1-3(n-1)(n)\cdot\frac{1}{2}-(n-1)$

$\Leftrightarrow \quad = n^3-n-\left[\frac{3n^2-3n}{2}\right]$

$= n^3-n-\frac{3n^2}{2}+\frac{3n}{2}$

$= n^3-\frac{3n^2}{2}+\frac{n}{2}$

$\sum^{n-1}k^2 = \left[n^3-\frac{3n^2}{2}+\frac{n}{2}\right]/3$

$\sum^{n-1}k^2 = \frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}$

$\sum^{n-1}k^2+n^2 = \frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}+n^2$

$\boxed{\sum^{n}k^2 = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}}$ □