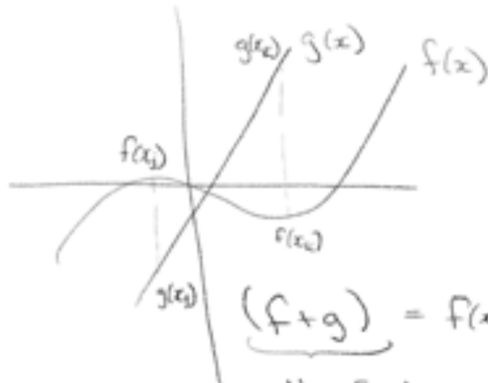


$$\underbrace{(f+g)}_{\text{New Function}} = f(x) + g(x)$$

Similar to coordinate (vector) addition.

Although functions have infinitely many values.

It is possible to apply concepts from linear algebra to functions.

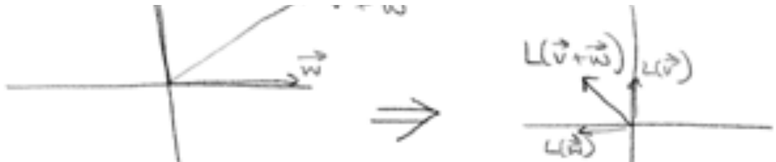
- Linear transformations
- Null Space
- Dot Products
- Eigen - everything.

$\Rightarrow$ For example, functional derivatives.

Formal Definition of Linearity

Additivity — $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$

Scaling — $L(c\vec{v}) = cL(\vec{v})$



The transformation of the sum of $\vec{v}$ and $\vec{w}$, is the same as adding the transformations $L(\vec{v})$ and $L(\vec{w})$.

Similarily, if you were to scale $\vec{v}$ and then transform that vector, it would be identical to transforming $L(\vec{v})$ then scaling it.

Often this (linear transformations) is described as preserving addition and scalar multiplication.

Calculus students subconciously know that the derivative $\frac{d}{dx}$ is linear.

For example,

$$\frac{d}{dx}(x^3 + x^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2)$$

$$\frac{d}{dx}(4x^3) = 4\frac{d}{dx}(x^3)$$

Writing a Function in Linear terms

$$\left.\begin{array}{l} x^{300} + 9x^2 \\ 1x^{1,000,000} + 1 \\ 3x^{(10^{100})} \end{array}\right\} \Rightarrow \text{Infinitely Many} \left\{\begin{array}{l} b_0 = 1 \\ b_1 = x \\ b_2 = x^2 \\ b_3 = x^3 \\ \vdots \end{array}\right.$$

The dimension is infinite.

$$1x^2 + 3x + 5\cdot 1 = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

A derivative would be described with an infinite matrix.

$$\frac{d}{dx} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 2 & \cdots & \\ \cdot & & & 3 & \cdots \\ \cdot & \cdots & & & 4 \;\cdot \end{bmatrix}}_{\text{This is a transformation from the original basis vectors } b_0, b_1, \ldots b_k \,(1, x, x^2, \ldots, x^k), \text{ that is linear (both additive and scaling).}} \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \Leftarrow (x^2 + 3x)$$

...

...

Parallels in functions

Linear Alg.	Functions
Linear Transformations	Linear Operators
Dot Product	Inner Products
Eigenvectors	Eigenfunctions

Axioms of Linear Systems

1. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
2. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
3. There is a vector $\vec{0}$ such that $\vec{0} + \vec{v} = \vec{v}$ for all $\vec{v}$
4. For every vector $\vec{v}$ there is a vector $-\vec{v}$ so
   $\vec{v} + (-\vec{v}) = \vec{0}$
5. $a(b\vec{v}) = (ab)\vec{v}$
6. $1\vec{v} = \vec{v}$
7. $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$
8. $(a+b)\vec{v} = a\vec{v} + b\vec{v}$

So what are vectors?

vectors?

$$\begin{bmatrix}4\\7\end{bmatrix} \begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{bmatrix}-9\\4\end{bmatrix} \begin{bmatrix}7\\9\end{bmatrix}$$

"Abstractness is the price of generality"

[They don't really matter]
as long as they are

— Linear —