

PROBLEM #1

Use the Maclaurin series to determine if $f(x) = \tan^{-1}(x)$ is even, odd or neither.

Suppose $\tan^{-1}(x)$ can be modelled with the Maclaurin series,

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f^{(1)}(x) = \frac{d}{dx} (\tan^{-1} x)$$

Proof of $\frac{d}{dx} \arctan(x)$,

$$y = \arctan x$$

$$\tan y = x$$

$$\frac{d}{dx} \tan y = \frac{d}{dx} x$$

$$\sec^2 y \frac{dy}{dx} = 1 \Leftrightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

Since $\sec^2 y = 1 + \tan^2 y$, since $y = \arctan x$,
 $\tan^2 y = x^2$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sec^2 y} \\ &= \frac{1}{1 + \tan^2 y} \\ &= \frac{1}{1 + x^2}. \end{aligned}$$

$$g_1(x) = \frac{f'(0)}{1!} x^1 + g_0(x)$$

$$= \frac{x}{(1)} + \arctan(0)$$

$$g_2(x) = -2x^3 + x + 0$$

Based on $f^{(2)}(0)x^3$ the series is odd.

$$g(x) = \sum_{n=0}^{\infty} \frac{(n-2)!(-1)^n x^{(n+2)}}{(n+2)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{(n \cdot 2 + 1)}}{n \cdot 2 + 1}, \text{ powers always odd!}$$

PROBLEM #2.

Given $f(x) = 2/(x^2 - x)$, which ranges can a power series be formed for?

There are asymptotes at the roots of $x^2 - x$.

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(0)}}{2(1)}$$

$$= \frac{1 \pm 1}{2}, \text{ roots at } 0, 1.$$

Taylor series can be found between $[-\infty, 0)$, $(0, 1)$, $(1, \infty]$.

PROBLEM #3

For $f(x) = 2/(x^2 - x)$ which statements are true about its power series representation.

- ☐ approx. properly captures asymptotes
- ☐ this is a well behaved function
- ☒ approx ignores segments of function
- ☐ approx converges quickly \rightarrow sign flip in approximation
- ☒ approx ignores asymptotes. \rightarrow not quick conv.

PROBLEM #4

Given $f(x) = \frac{1}{(1+x^2)}$, performing Taylor expansions

for the first 3 terms around point $x=2$ are,

$$f(x) = \frac{1}{5} - \frac{4(x-2)}{25} + \frac{22(x-2)^2}{125} + \dots$$

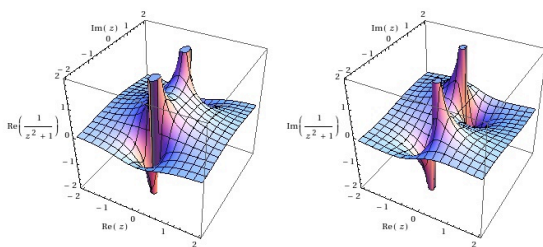
Why does the expansion perform poorly @ $x=0$.

- ☒ asymptotes are in the complex plane.
- ☐ function does not differentiate well.
- ☐ the function has no real roots
- ☐ none of these options.
- ☒ it is a discontinuous function in the complex plane.

$$\text{quadratic form.} = \frac{- (0) \pm \sqrt{(0)^2 - 4(1)(1)}}{2(1)}$$

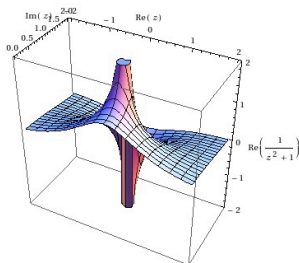
$$= \frac{0 \pm \sqrt{-4}}{2}, \text{ roots @ } \frac{\sqrt{-4}}{2} = \pm \frac{4i}{2}$$

Runge phenomenon - any attempt to approx. with a polynomial fails bc. poles in complex plane.



These are plots of the real and imaginary parts of the function $\frac{1}{z^2 + 1}$, where $z = x + iy$. The poles of the function at $z = \pm i$ are easily seen in the plots. Since we're limited to seeing (a two-dimensional projection of) three dimensions, we are forced here to illustrate the poles by plotting the real and imaginary parts of the function separately.

Now, look at a "slice" of the real part:



PROBLEM #5.

Given $f(x) = 1/x$, calculate the Taylor expansions at $x=4$ in the order of 2 degrees of accuracy.

$$f'(x) = -1/x^2$$

$$f''(x) = 2/x^3$$

$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(4) (-1)^n x^{-n+1}}{n!} (x-4)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (4)^{-n+1} (x-4)^n$$

$$g_1(x) = \frac{1}{4} - \frac{(x-4)}{4^2(1)!} + O(\Delta x^2)$$

$$= \frac{1}{4} - \frac{x-4}{16} + O(\Delta x^2)$$