Linear Algebra Background

CONJUGATE TRANSPOSE (AKA HERMETIAN) or Bedaggered Matrix. A^* A^H $A^\dagger$

- Take the transpose of $A_{m \times n}$ with complex entries, then take complex conjugate of each entry.

$a + ib \rightarrow a - ib$, where $a, b \in \mathbb{R}$.

Formal definition, for $A_{m \times n}$

$$(A^H)_{ij} = \overline{A_{ji}},$$ notice i and j are transposed

can also be written,

$$A^H = (\overline{A})^T = \overline{A^T}$$

NOT $\text{adj}(A)$, not the <u>ADJUGATE</u>.

When matrix is square.

Hermition or self-adjoint

$$A = A^H \rightarrow a_{ij} = \overline{a_{ji}}$$

Skew Hermition / Anti hermition

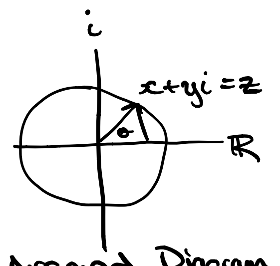
$$A = -A^H \rightarrow a_{ij} = -\overline{a_{ji}}$$

Normal (Hermition w commutativity) ;

$$A^H A = A A^H$$

Unitary

$$A^H = A^{-1}$$



Argand Diagram

Complex Numbers as 2x2 matrix, works for multiplication & addition,

$$a+ib \equiv \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad \Big\} \; z = a+ib$$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \overbrace{a^2-b^2}^{\text{new } a} & \overbrace{-2(a+b)}^{\text{new } b * -1.} \\ \underbrace{2(a+b)}_{\text{new } b} & \underbrace{-b^2+a^2}_{\text{new } a} \end{bmatrix},$$

a - is x of an Argand D.
b - is y of the argand (imag-axis)

Motivation.

Picture an $A_{m\times n}$ matrix of $\mathbb{C}$ numbers, $\Rightarrow A_{2m\times 2n}$ of $\mathbb{R}$,

$$\begin{bmatrix} \mathbb{C}_1 & \cdots \\ \mathbb{C}_2 & \cdots \end{bmatrix} \Longrightarrow \begin{array}{c c} & \overbrace{\qquad\qquad}^{\mathbb{C}_1} \\ & \begin{bmatrix} \mathbb{R}_1 & \mathbb{R}_1 & \cdots & \cdots \\ \mathbb{R}_1 & \mathbb{R}_1 & \cdots & \cdots \\ \mathbb{R}_2 & \mathbb{R}_2 & \cdots & \cdots \\ \mathbb{R}_2 & \mathbb{R}_2 & \cdots & \cdots \end{bmatrix} \\ \mathbb{C}_2 \Big\{ & \underbrace{\qquad\qquad}_{\mathbb{C}_2} \end{array}$$

$$z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

The transpose of 2x2 represented complex numbers switch it's sign to maintain mult/add

Thus reducing $\mathbb{R}_{2m\times 2n}$ to $\mathbb{C}_{m\times n}$ and then performing a transpose must account for the $\mathbb{R}$-based transpose mechanics $\rightarrow$ yielding the reason for performing the complex conjugate after transposing a $\mathbb{C}$ matrix.

Unitary Matrix

A complex square matrix U is unitary if its conjugate transpose U^* is it's inverse.

$U^*U = UU^* = I$

OR

$U^H = U^{-1}$

If $U \in \mathbb{R}^N$ $\rightarrow U^* = U^T$

In physics

$U^\dagger U = UU^\dagger = I$

Diagonalizable Matrix

A matrix can be called similar to a diagonal matrix if there exists an invertible matrix P, such that $P^{-1}AP$ is a diagonal matrix.

also like a change of basis.

Schur Lemma

If A is a square Complex matrix there exists a U upper triangular matrix and S a unitary matrix such that,

$A = SUS^*$
$= SUS^{-1}$

Hermitian Matrix

A hermetian $\Leftrightarrow a_{ij} = \overline{a_{ji}}$.

$A = A^H$

Diagonal Matrix

Entries outside the main diagonal are zero.

$$\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} = A.$$

Usually refers to square matrix.

Q: Eigenvalues & Eigenvectors of a diagonalizable matrix?

Q_2: What are normal matrices and normal equations?

Q_3. Grand Schmidt Process?
How linear independence transforms to orthonormal?

Q. QR Factorization
orthogonal vs. unitary matrix

Norms

Most commonly the euclidean norm,

$$\|\vec{a}\| = \sqrt{x^2 + y^2 + z^2}$$

in a 3 dimensional space.

Orthonormal Vectors

A collection of real m-vectors $a_1, a_2, a_3 \dots a_n$ is orthonormal if,

- vectors have unit norm:

  $\|a_i\| = 1$

- mutually orthogonal:

  $a_i^T a_j = 0$, when $i \neq j$.

Examples,

$$\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Norms with Scripts

Superscript is a power.
Subscript is dimension.

$$\|\vec{v}\|_y^x \rightarrow \left[\sum_{i=1}^{d} |v_i|^y \right]^{\frac{1}{y}}$$

Matrix with Orthonormal Vectors

$$A^T A = [a_1\ a_2 \dots a_n]^T [a_1\ a_2 \dots a_n]$$

$$= \begin{bmatrix} a_1^T a_1 & a_2^T a_2 & \dots & a_n^T a_n \\ a_2^T a_1 & a_2^T a_2 & \dots & a_2^T a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^T a_1 & a_n^T a_2 & \dots & a_n^T a_n \end{bmatrix}$$

Gram Matrix

$$= \begin{bmatrix} a_1^0 & a_1^1 & \dots & a_1^n \\ a_2^0 & a_2^1 & \dots & a_2^n \\ \vdots & & & \\ a_n^0 & a_n^1 & \dots & a_n^n \end{bmatrix} \begin{bmatrix} a_1^0 & a_2^0 & \dots \\ a_1^1 & a_2^1 & \dots \\ \vdots & & \\ a_1^n & a_2^n & \dots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots \\ \dots & & & \\ \dots & & 0 & 1 \end{bmatrix}$$

$$= I \Leftrightarrow \forall x \in A \rightarrow 1 = \|\vec{x}\|$$