"Using the chain rule is like pealing an onion: you have to deal with each layer at a time, and if it's to big you will start crying"

— Anonymous.

Complex functions can be of three forms.

① Addition

$$\frac{d}{dx}(\sin(x) + x^2)$$

② Multiplication

$$\frac{d}{dx}(\sin(x)x^2)$$

③ Composition

$$\frac{d}{dx}(\sin(x^2))$$

Sum Rule

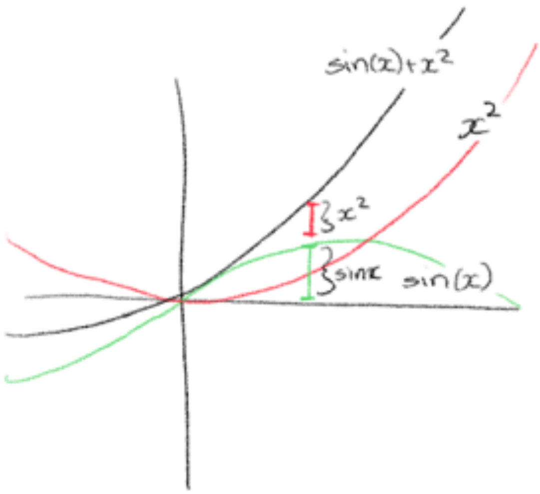
The derivative of the sum of two functions is the sum of their derivatives.

$$\frac{d}{dx}(g(x) + f(x)) = \frac{dg}{dx} + \frac{df}{dg}$$

↳ This has

... linearity, like linear algebra.
& only have to check scaling

$$\frac{d}{dx} f(c \cdot x) = c \cdot \frac{df}{dx}$$



$$f \circ g = f(x) + g(x)$$

$$f(x) = f \circ g - g(x)$$
$$g(x) = f \circ g - f(x)$$

$$\frac{d(f \circ g - g(x))}{dx} = \frac{df}{dx}$$

$$\frac{d(f \circ g - f(x))}{dx} = \frac{dg}{dx}$$

derivative of parts is the same as derivative of whole. therfor derivative of individual pieces is the same as their sum.

$$\frac{d(f \circ g)}{dx} = \frac{d(f \circ g - g(x))}{dx} + \frac{d(f \circ g - f(x))}{dx}$$

$$= \frac{d(f(x))}{dx} + \frac{d(g(x))}{dx}$$

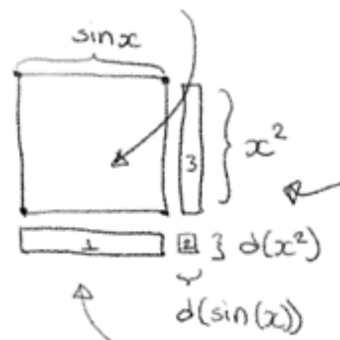
$$= \frac{df}{dx} + \frac{dg}{dx}$$

□

Product Rule.

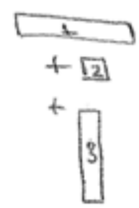
$f(x) = \sin(x)\, x^2$

Analogous to Area.



$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}$$

df =



$$= d(\sin(x))\cdot d(x^2) + d(\sin(x))\,x^2 + d(x^2)\sin(x)$$

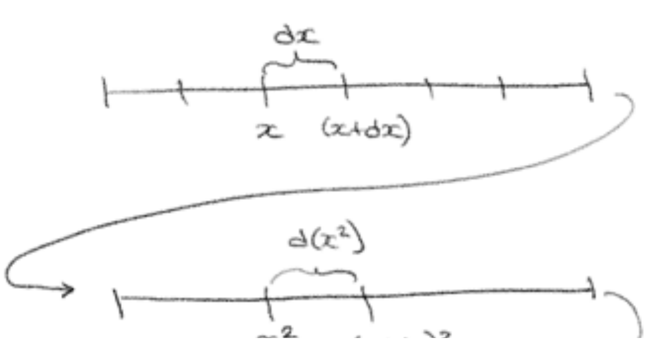
$$df = 0 + \cos(x)dx\cdot x^2 + 2(x)dx\cdot\sin(x)$$

$$\boxed{\frac{df}{dx} = \cos(x)\cdot x^2 + 2(x)\cdot\sin(x)}$$

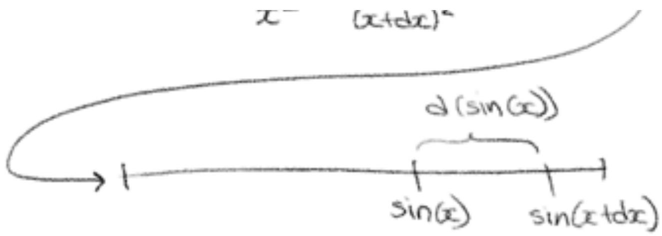
Function Composition – Chain Rule.

$$f\circ g = \sin(x^2) \qquad \frac{d}{dx}g(h(x)) = \frac{dg}{dh}(h(x))\frac{dh}{dx}(x)$$

Imagine the number lines,



LAYER #2 –



Change with respect to $\underline{x}$
Derivative is $2(x)dx$.

LAYER #3 –

Changes respective to it's input.
Therefor it's derivative is $\cos(in)d(in)$.
With respect to the small change of $in \to 0$.

$$\therefore \quad \cos(x^2)\cdot 2(x) = \frac{d}{dx} g(h(x)) \Longleftarrow$$

$$= \frac{dg}{\cancel{dh}} \frac{\cancel{dh}}{dx}$$

$$\frac{d}{dx} g(h(x)) \quad = \frac{dg}{dx}$$