An isomorphism is a morphism which is inverse to another morphism. Again a morphism is nothing more than a map; it relates one algebraic structure to another in an arrangement that preserves the underlying operations that constitute the algebraic structures.

The definition of a <u>map</u> – which preserves an underlying operation of it's algebraic structures (that is a homomorphism) is:

A map $f: A \to B$ which preserves an operation μ, of arity k, where the operation is defined on A and B,

$$f(\mu_A(a_1, \dots, a_k)) = \mu_B(f(a_1), \dots, f(a_k)), \text{ for all } a_1, \dots, a_k \in A.$$

(*) <u>NOTE</u>: the notation for the operations does not need to be the same in the source and target of the homomorphism.

For Example: the exponential function is a homomorphism – but the notation differs.

Let $x \to e^x$ be the exponential map, it satisfies the operation μ:

$$e^{x+y} = e^x e^y, \text{ however the } \mu = \begin{cases} \overset{\mu_A}{+}, & x, y \in \mathbb{R} \\ \underset{\mu_B}{\times}, & x, y \in \mathbb{R}^+ \end{cases}$$

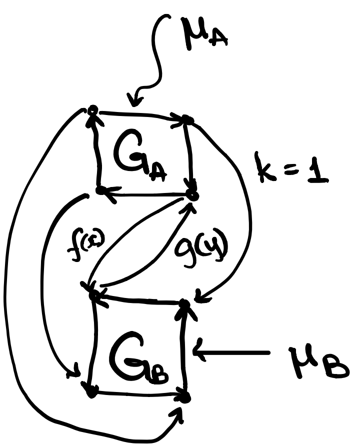

DIAGRAM 1: Depicts an isomorphism between two isomorphic Graphs: $G = \{G_A, G_B\}$, where $k=1$ for a unary and only operation μ defined on G. $f: G_A \to G_B$ and $g: G_B \to G_A$ are inverse homomorphisms.

Two objects are isomorphic if an isomorphism exists between them. Again an isomorphism is qualified by inverse homomorphisms.

Let A, B be arbitrary algebraic structures.
Let $f: A \to B$, be a morphism which satisfies:

$$f(\mu_A(x_1, \dots, x_n)) = \mu_B(f(x_1), \dots, f(x_n)), \ \forall_{x_1, \dots, x_n} \in A$$

Let $g: B \to A$, be a morphism which satisfies:

$$g(\mu_B(x_1, \dots, x_n)) = \mu_A(g(x_1), \dots, g(x_n)), \ \forall_{x_1, \dots, x_n} \in B$$

Let $f(g(x_B)) = x_B$ and $g(f(x_A)) = x_A$, ergo $f = g^{-1}(x_B)$
and $g = f^{-1}(x_A)$!!

Since there is a map - that is a homomorphism, from $A \to B$ and a second homomorphism $B \to A$, and it so happens they are mutually inverse of one another, and both those homomorphisms preserve <u>all</u> of the underlying operations belonging to the algebraic structures A and B, then the mathematical objects A and B are isomorphic.

$\square$

A real world example of an isomorphism is:

When a group of people agree on the same courses of action across the entire ~~a~~ range of topics for differing reasons.
Their <u>shared sense of policy</u> is the isomorphism betwixt them.