The formalized algebraic expression of a matrix's eigenvalues is:

$Ax = \lambda x$, where $\lambda$ is a scaler value.

This formula expresses that $\underline{x}$ are eigenvectors and when multiplied on the matrix A, the end result will be a scaling of the original vector.

To determine the eigenvectors and values factorize the equation and solve for the characteristic polynomial.

$0 = (A - \lambda I)x$, setting to zero allows us to solve for when the matrix becomes zero.

$\Rightarrow$ a zero vector is trivial and not an eigenvector-by definition.

Then can use the determinant calculation knowing that the subtraction of the eigenvector-value combo from the vector by the matrice will negate each other,

$\det(A - \lambda I) = 0$, this approach is computationally expensive for a large number of dimensions.

If this example, A was a 2x2 matrix then,

$$
\begin{aligned}
|A - \lambda I| &= 0 \\
&= \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| \\
&= \left| \begin{bmatrix} (a-\lambda) & (b-0) \\ (c-0) & (d-\lambda) \end{bmatrix} \right| \\
&= (a-\lambda)(d-\lambda) - (b)(c) \\
&= \lambda^2 - (a+d)\lambda + ad - bc,
\end{aligned}
$$

the characteristic polynomial equation.

The eigenvalues are the solutions to the characteristic polynomial equation. They can be solved for and then substituted into the original equation to find their eigenvectors.

## Calculating Eigenvalues and Vectors Example.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, a vertical scaling transformation matrix.

① Express A as producer of eigenvalues,

$$Ax = \lambda x$$
$$0 = (A - \lambda I)x$$

② Use the determinant calculation to formulate the characteristic polynomial, and solve for the eigenvalues.

$$|A - \lambda I| = 0$$
$$= \left| \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$
$$= \left| \begin{bmatrix} (1-\lambda) & (0-0) \\ (0-0) & (2-\lambda) \end{bmatrix} \right|$$
$$= (1-\lambda)(2-\lambda) - (0)(0)$$
$$= (1-\lambda)(2-\lambda)$$

Therefor the roots of the characteristic polynomial are
$\lambda = 1, 2.$

③ Substitute the eigenvalues back into the original equation to solve for the eigenvectors,

@$\lambda = 1$: $\begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} x = 0$

$$= \begin{bmatrix} (1-1) & 0 \\ 0 & (2-1) \end{bmatrix} x$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$= \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$, meaning the $x_2$ value must be zero.
Can't conclude anything about $x_1$.

@ $\lambda = 2$: $\begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} x = 0$

$= \begin{bmatrix} (1-2) & 0 \\ 0 & 2-2 \end{bmatrix} x$

$= \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} x$

$= \begin{bmatrix} -x_1 \\ 0 \end{bmatrix}$, meaning the $x_1$ value must be zero.

Can't conclude anything about $x_2$.

Therefor the eigenvectors are,

@ $\lambda = 1$: $x = \begin{bmatrix} t \\ 0 \end{bmatrix}$, @ $\lambda = 2$: $x = \begin{bmatrix} 0 \\ t \end{bmatrix}$

Any vector complying with these definitions will only scale as they will be of a form where their transformations by A will only scale the original vector by it's appropriate eigenvalue.

# Calculating Eigenvectors & Values for non-geometric Eq.

A rotation does not have $\mathbb{R}$ (real) valued eigenvalues.

Ex.

$A' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad \left| \begin{bmatrix} (0-\lambda) & -1 \\ 1 & 0-\lambda \end{bmatrix} \right| = 0$

$= \lambda^2 + 1$, which is only solvable using $\mathbb{C}$ (complex) numbers.

(*) Using this approach of the determinant, the characteristic polynomial and solving for the roots of the polynomial is asympotically expensive – the more dimensions make this task computationally unfeasible.

$\Rightarrow$ This can become analytically untractable.

↳ where you are forced to use numerical methods instead.