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### PROBLEM #1.

Calculate the expression  $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt}$  in matrix form.

$$\begin{aligned}f(x) &= f(x_1, x_2) \\&= x_1^2 x_2^2 + x_1 x_2\end{aligned}$$

$$\begin{aligned}x_1(t) &= 1 - t^2 \\x_2(t) &= 1 + t^2\end{aligned}$$

Calculate the vector derivative of  $x$ .

$$\frac{dx}{dt} = \begin{bmatrix} -2t \\ 2t \end{bmatrix}$$

Calculate the Jacobian row-vector of  $f$ ,

$$\begin{aligned}\bar{J}_f &= \frac{\partial f}{\partial x} \\&= \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right] \\&= [2x_1 x_2^2 + x_2, 2x_2 x_1^2 + x_1]\end{aligned}$$

The total derivative formed by the matrix representation chain rule,

$$\frac{df}{dt} = [2x_1 x_2^2 + x_2, 2x_2 x_1^2 + x_1] \begin{bmatrix} -2t \\ 2t \end{bmatrix}$$

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### PROBLEM #2.

Calc the total derivative in matrix form,

$$\begin{aligned}f(x) &= f(x_1, x_2, x_3) \\&= x_1^3 \cos(x_2) e^{x_3}\end{aligned}$$

$$\begin{aligned}x_1(t) &= 2t \\x_2(t) &= 1 - t^2\end{aligned}$$

$$x_3(t) = e^t$$

The total derivative is equal to,

$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt}$ , where  $\frac{\partial f}{\partial x}$  is the Jacobian row vector of  $f$  and  $\frac{dx}{dt}$  is the vector derivative of  $\underline{x}$  relative to  $t$ .

The Jacobian row vector is equal to,

$$\begin{aligned} J_f &= [3x_1^2 \cos(x_2) e^{x_3}, -\sin(x_2) x_1^3 e^{x_3}, e^{x_3} \cos x_2 x_1^3] \\ &= \frac{\partial f}{\partial x} \end{aligned}$$

The vector derivative of  $\underline{x}$ -to- $t$  is,

$$\frac{dx}{dt} = \begin{bmatrix} 2 \\ -2t \\ e^t \end{bmatrix}$$

The total derivative,

$$\frac{df}{dt} = [3x_1^2 \cos(x_2) e^{x_3}, -\sin(x_2) x_1^3 e^{x_3}, e^{x_3} \cos x_2 x_1^3] \begin{bmatrix} 2 \\ -2t \\ e^t \end{bmatrix}$$

### PROBLEM #3.

Calculate the total derivative in matrix form.

$$\begin{aligned} f(\underline{x}) &= f(x_1, x_2) \\ &= x_1^2 - x_2^2 \end{aligned}$$

$$\begin{aligned} x_1(u_1, u_2) &= 2u_1 + 3u_2 \\ x_2(u_1, u_2) &= 2u_1 - 3u_2 \end{aligned}$$

$$u_1(t) = \cos(t/2)$$

$$u_2(t) = \sin(2t)$$

The total derivative can be expressed.

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} \frac{du}{dt}$$

$J_f J_x \frac{du}{dt}$ , can be re-expressed in terms of Jacobian.

The Jacobian of  $f$  is,

$$J_f = [2x_1, -2x_2]$$

The Jacobian of  $x$  is,

$$J_x = \begin{bmatrix} Jx_1 \\ Jx_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix}$$

The vector derivative of  $u$  is,

$$\frac{du}{dt} = \begin{bmatrix} -\sin(t/2) \cdot \frac{1}{2} \\ \cos(2t) \cdot 2 \end{bmatrix}$$

The total derivative is,

$$\frac{df}{dt} = [2x_1, -2x_2] \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} -\sin(t/2) \cdot \frac{1}{2} \\ \cos(2t) \cdot 2 \end{bmatrix}$$

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#### PROBLEM # 4.

Calculate the total derivative in matrix form,

$$\begin{aligned} f(x) &= f(x_1, x_2) \\ &= \cos x_1, \sin x_2 \end{aligned}$$

$$x_1(u_1, u_2) = 2u_1^2 + 3u_2^2 - u_2$$

$$x_2(u_1, u_2) = 2u_1 - 5u_2^3$$

$$u_1(t) = e^{t/2}$$

$$u_2(t) = e^{-2t}$$

The total derivative can be expressed in terms of the Jacobian,

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} \frac{du}{dt}$$

$$= J_f J_x \frac{du}{dt}.$$

The Jacobian of  $f$ -relative-to- $x$ ,

$$J_f = [-\sin x_1, \sin x_2, \cos x_1, \cos x_2]$$

The Jacobian of  $x$ -relative-to- $u$ ,

$$J_x = \begin{bmatrix} J_{x_1} \\ J_{x_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 4u_1, 6u_2 - 1 \\ 2, -15u_1^2 \end{bmatrix}$$

The vector derivative of  $u$ ,

$$\frac{du}{dt} = \begin{bmatrix} e^{t/2} \cdot \frac{1}{2} \\ e^{-2t} \cdot (-2) \end{bmatrix}$$

The total derivative is,

$$\frac{df}{dt} = [-\sin x_1, \sin x_2, \cos x_1, \cos x_2] \begin{bmatrix} 4u_1, 6u_2 - 1 \\ 2, -15u_1^2 \end{bmatrix} \begin{bmatrix} e^{t/2} \cdot \frac{1}{2} \\ e^{-2t} \cdot (-2) \end{bmatrix}$$

PROBLEM # 5.

Calculate the total derivative in matrix form.

$$f(x) = f(x_1, x_2, x_3)$$

$$= \sin x_1 \cos x_2 e^{x_3}$$

$$x_1(u_1, u_2) = \sin(u_1) + \cos u_2$$

$$x_2(u_1, u_2) = \cos u_1 - \sin u_2$$

$$x_3(u_1, u_2) = e^{u_1+u_2}$$

$$u_1(t) = 1 + t/2$$

$$u_2(t) = 1 - t/2$$

The total derivative can be expressed as Jacobian products,

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} \frac{du}{dt}$$

$$= J_f J_x \frac{du}{dt}$$

The Jacobian of  $f$ ,

$$J_f = \begin{bmatrix} \cos x_1, \cos x_2 e^{x_3}, & -\sin x_1, \sin x_2 e^{x_3}, \sin x_1 \cos x_2 e^{x_3} \end{bmatrix}$$

The Jacobian of  $x$ ,

$$J_x = \begin{bmatrix} J_{x_1} \\ J_{x_2} \\ J_{x_3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} \end{bmatrix} = \begin{bmatrix} \cos u_1, & -\sin u_2 \\ -\sin u_1, & -\cos u_2 \\ e^{u_1+u_2}, & e^{u_1+u_2} \end{bmatrix}$$

The vector derivative of  $u$ ,

$$\frac{du}{dt} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

The total derivative in matrix form.

$$\frac{df}{dt} = \left[ \cos x, \cos x_2 e^{x_3}, -\sin x, \sin x_2 e^{x_3}, \sin x, \cos x_2 e^{x_3} \right]$$

$$\times \begin{bmatrix} \cos u_1, & -\sin u_2 \\ -\sin u_1, & -\cos u_2 \\ e^{u_1+u_2}, & e^{u_1+u_2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$