

Calculate how a vector v will be reflected in a plane.

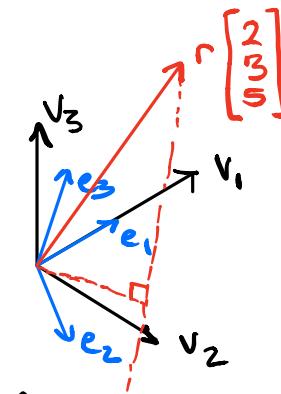
$$V = \{v_1, v_2, v_3\}$$

$$v_1 = [1 \ 1 \ 1]^T$$

$$v_2 = [2 \ 0 \ 1]^T$$

$$v_3 = [3 \ 1 \ -1]^T$$

These vectors are in the plane.



$$\frac{1}{3} \begin{bmatrix} 11 \\ 14 \\ 5 \end{bmatrix}$$

- ① Find orthonormal vectors describing the plane and v_3 . Do this with Gram Schmidt.

$$u_1 = v_1, \quad e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = v_2 - \frac{(v_2 \cdot e_1)e_1}{e_1 \cdot e_1} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \left[\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right] \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad e_2 = \frac{u_2}{\|u_2\|}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$u_3 = v_3 - \frac{(v_3 \cdot e_1)e_1}{e_1 \cdot e_1} - \frac{(v_3 \cdot e_2)e_2}{e_2 \cdot e_2}$$

$$= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \left[\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right] \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left[\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right] \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \quad e_3 = \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

- ② Construct transformation matrix from the orthonormal basis vectors (back to V 's basis).

$$E = [e_1 | e_2 | e_3] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix}$$

Plane vectors.

- ⑤ Construct a transformation to do the reflection in the orthonormal basis.

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$e_1 \quad e_2 \quad e_3$

Flip the e_3 component
which is not part of the plane.

This is a transformation matrix defined
in the basis of the plane

- ④ Get the translation matrix to convert from v 's basis to ℓ .

$E^{-1} = E^T$, because matrix is orthogonal and
vectors have norm of 1.
e.g $A^T A = I$.

- ⑤ Translate \underline{c} into the e -basis, transform r_e using the
reflection matrix T_e , convert the reflected vector back into
the basis of v .

$$r' = ET_eE^{-1}r.$$

$$T_e E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & (1 & 1 & 1) \\ \frac{1}{\sqrt{2}} & (1 & -1 & 0) \\ \frac{1}{\sqrt{6}} & (-1 & 1 & -2) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & (1 & 1 & 1) \\ \frac{1}{\sqrt{2}} & (1 & -1 & 0) \\ \frac{1}{\sqrt{6}} & (-1 & 1 & -2) \end{bmatrix}$$

$$\begin{aligned} ET_eE^{-1} &= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & (1 & 1 & 1) \\ \frac{1}{\sqrt{2}} & (1 & -1 & 0) \\ \frac{1}{\sqrt{6}} & (-1 & 1 & -2) \end{bmatrix} \\ &= \frac{1}{\sqrt{3}} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix} \end{aligned}$$

$$r' = E T_e E^{-1} r = \frac{1}{3} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$r' = \frac{1}{3} \begin{bmatrix} 11 \\ 14 \\ 5 \end{bmatrix}$$