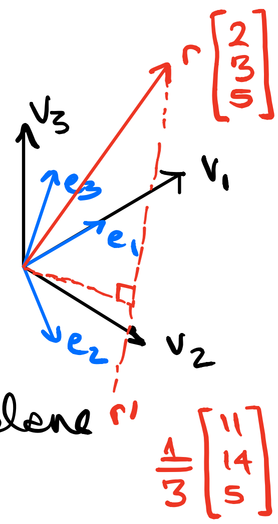
Calculate how a vector $u$ will be reflected in a plane.

$$V = \{v_1, v_2, v_3\}$$

$$v_1 = [1\ \ 1\ \ 1]^T$$
$$v_2 = [2\ \ 0\ \ 1]^T$$
$$v_3 = [3\ \ 1\ \ -1]^T$$

These vectors are in the plane.




① Find orthonormal vectors describing the plane and $v_3$. Do this with Gram Schmidt.

$$u_1 = v_1 \qquad e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$u_2 = v_2 - \frac{(v_2\cdot e_1)e_1}{e\cdot e} = \begin{bmatrix}2\\0\\1\end{bmatrix} - \left[\begin{bmatrix}2\\0\\1\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\right]\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad e_2 = \frac{u_2}{\|u_2\|}$$

$$= \begin{bmatrix}1\\-1\\0\end{bmatrix} \qquad = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$u_3 = v_3 - \frac{(v_3\cdot e_1)e_1}{e_1\cdot e_1} - \frac{(v_3\cdot e_2)e_2}{e_2\cdot e_2}$$

$$= \begin{bmatrix}3\\1\\-1\end{bmatrix} - \left[\begin{bmatrix}3\\1\\-1\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\right]\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} - \left[\begin{bmatrix}3\\1\\-1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}\right]\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$$e_3 = \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

② Construct transformation matrix from the orthonormal basis vectors (back to v's basis).

$$E = [e_1 | e_2 | e_3] = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}}\end{bmatrix}$$

Plane vectors.

③ Construct a transformation to do the reflection in the orthonormal bases.

$$T = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ \begin{matrix} e_1 & e_2 & e_3 \end{matrix} \end{array}$$

Flip the $e_3$ component which is not part of the plane.

This is a transformation matrix defined in the basis of the plane

④ Get the translation matrix to convert from v's basis to e.

$E^{-1} = E^T$, because matrix is orthogonal and vectors have norm of 1.

e.g $A^T A = 1$.

⑤ Translate r into the e-basis, transform $r_e$ using the reflection matrix $T_e$, convert the reflected vector back into the basis of v.

$$r' = E T_e E^{-1} r.$$

$$T_e E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3}\,(1 \ 1 \ 1) \\ 1/\sqrt{2}\,(1 \ -1 \ 0) \\ 1/\sqrt{6}\,(1 \ 1 \ -2) \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{3}\,(1 \ 1 \ 1) \\ 1/\sqrt{2}\,(1 \ -1 \ 0) \\ 1/\sqrt{6}\,(-1 \ -1 \ 2) \end{bmatrix}$$

$$E T_e E^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3}\,(1 \ 1 \ 1) \\ 1/\sqrt{2}\,(1 \ -1 \ 0) \\ 1/\sqrt{6}\,(-1 \ -1 \ 2) \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$r' = E T_e E^{-1} r = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$r' = \frac{1}{3}\begin{bmatrix} 11 \\ 14 \\ 5 \end{bmatrix}$$