Direct sums express the nature of addition between abstract groups with more rigour than typical summations.

NOTATION

Typical sums are written:

$$C = A + B$$

Typical series of sums are written:

$$c = \sum^{i} a_i$$

A direct sum is written:

$$C = A \oplus B$$

A series of direct sums are written:

$$c = \bigoplus_{i \in A} a_i$$

THE SPECIFICITY OF DIRECT SUMS

Beyond expressing the additive result of summing n-groups, the direct sum specifies that every sum is unique.

$U = U_1 \oplus U_2 \Rightarrow$ $\forall u \in U$: $u = u_1 + u_2 \Rightarrow u_1$ and u_2 uniquely sum to u.

where $u_1 \in U_1$, $u_2 \in U_2$.

This can be expressed in another way: that the Cartesian Product has a total ordering.

That is there is a bijective mapping between

$f: \mathbb{N} \to X_0 \times \dots \times X_n$, where X_i are the groups of the direct sum.

Example #1 – $\mathbb{R}^3$

$U_1 = \{(x,y,0) \in \mathbb{R}^3 \mid x,y \in \mathbb{R}\}$

$U_2 = \{(0,0,z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\}$

Then $\mathbb{R}^3 = U_1 \oplus U_2$

<u>However</u>, if $U_2 = \{(0,w,z) \in \mathbb{R}^3 \mid w,z \in \mathbb{R}\}$ then while $\mathbb{R}^3 = U_1 + U_2$, $\mathbb{R}^3$ is not a direct sum!

Corollories

#1: If $U_1, U_2 \subset V$ are subspaces then $V = U_1 \oplus U_2$ iff:

I. $V = U_1 + U_2$

II. $0 = u_1 + u_2$, where $u_1 \in U_1$ and $u_2 \in U_2$, this implies $u_1 = u_2 = 0$.

#2: If $U_1, U_2 \subset V$ are subspaces then $V = U_1 \oplus U_2$ iff:

I. $V = U_1 + U_2$

II. $\{\emptyset\} = U_1 \cap U_2$.