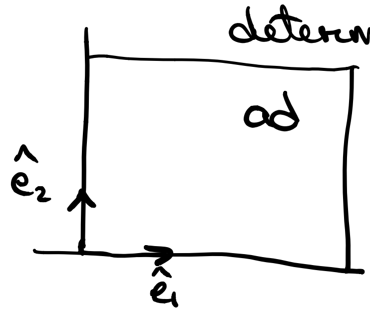
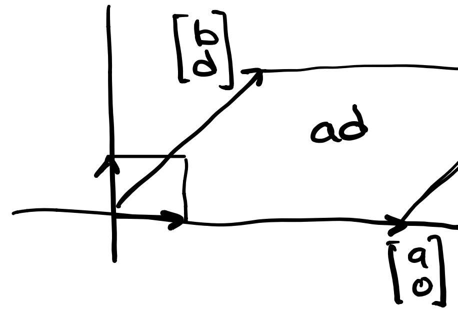
⇒ about the determinant when there is and isn't linear independent basis vectors.

The matrix $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ scales space.

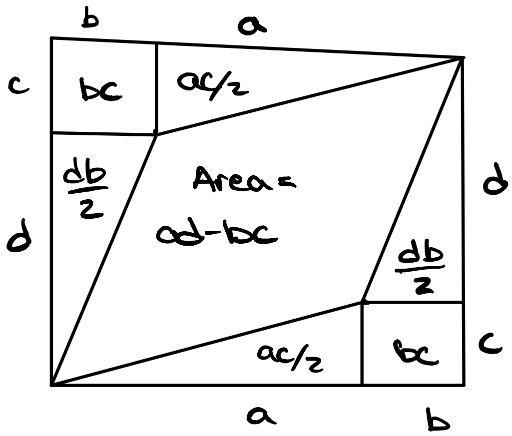
determinant.





The matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ produces a parallelogram.

This matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ produces another parallelogram,



$$A_{Area} = (a+b)(c+d) - ac - bd - 2bc$$
$$= \cancel{ac} + ad + bc + \cancel{bd} - \cancel{ac} - \cancel{bd} - 2bc$$
$$= ad - bc$$
$$= |A|$$

Finding the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. flip diagonal negate accross 2nd diag.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

The determinant is the ratio of space before and after transformation (1x1 : ad-bc).

$$\frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = I$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

The inverse needs to shrink the transformation by $\frac{1}{ad-bc}$.

QR decomp to calc det(A) generally.

If you collapse the dimensions the space is also collapsed.

A 2x2 space will reduce to $\underline{0}$. This happens when the cols are linearly dependent.

There is no inverse for rebuilding the original space because that dimensions information is lost on encoding it into a lesser dimension.

↳ That is why when $\det(A) = 0$, there is no inverse.

A Generalized Method for Computing the Determinant

- LU Factorization is faster than QR - by a fractional constant.

An orthogonal matrix is a matrix whose columns and rows (same operation) are orthogonal unit vectors.

The following are proofs and consequences of a matrix's orthogonality,

$$A^T A = A A^T = I$$

INTUITION BEHIND ORTHOGANAL UNITARY MATRICES.

just a compressed way of expressing dot prod between columns of A. multiplying a column/row of A/A^T to the other row will be 0 because columns are orthogonal to one another; the row of A^T is just a column of original matrix A. When not zero the col and row will be equal (i.e $i=j$) and their $\langle A_i, A^T_j \rangle$ will $= 1$ because vectors in an orthogonal matrix have a modulus of 1.

DETERMINANT OF ORTH MATRIX

The determinant of an orthogonal matrix is either negative or positive 1; because the vectors are all of unit length and they are all orthogonal → all inputs will not be scaled and all dimensions will be preserved that is inversion is always possible.

QR Decomposition/Factorization produces,

$A = QR$, a combo of an

I. orthogonal matrix (Q)

II. a triangular matrix. (R)
"upper"

One method for QR decomp is Gram-Schmidt process.

Begin with the matrix you want to factorize.

A

Observe A is square and can be expressed by its columns,

$$A = [a_1 | a_2 | a_3 | \dots | a_n].$$

Factorize A's columns,

$$u_1 = a_1, \quad e_1 = \frac{u_1}{\|u_1\|}$$

$$u_2 = a_2 - (a_2 \cdot e_1)e_1, \quad e_2 = \frac{u_2}{\|u_2\|}.$$

$$u_{k+1} = a_{k+1} - (a_{k+1} \cdot e_1)e_1 - \dots - (a_{k+1} \cdot e_k)_k, \quad e_{k+1} = \frac{u_{k+1}}{\|u_{k+1}\|}.$$

* $\|\cdot\|$ is L2 norm — root of sum of squares.

Express the factorization,

$$A = [a_1 | a_2 | \dots | a_n] = [e_1 | e_2 | \dots | e_n] \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & \dots & a_n \cdot e_1 \\ 0 & a_2 e_2 & \dots & a_n \cdot e_2 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \dots & a_n \cdot e_n \end{bmatrix}$$

Example of Gram-Schmidt

Given the matrix A - perform QR factorization.

$$A=\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$$

R-diagonally factorized vectors Q-orthogonal factor vectors

1. $u_1 = a_1$ $= [1\ 1\ 0]^T$	$e_1 = u_1/\|u_1\|$ $= \frac{1}{\sqrt{2}}[1\ 1\ 0]^T$
2. $u_2 = a_2 - (a_2 \cdot e_1)e_1$ $= [1\ 0\ 1]^T - \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}[1\ 1\ 0]^T$ $= [\frac{1}{2}\ -\frac{1}{2}\ 1]^T$	$e_2 = \frac{u_2}{\|u_2\|}$ $= \frac{1}{\sqrt{3/2}}[\frac{1}{2}\ -\frac{1}{2}\ 1]^T$ $= \begin{bmatrix}\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\end{bmatrix}$
3. $u_3 = a_3 - (a_3 \cdot e_1)e_1 - (a_3 \cdot e_2)e_2$ $= [0\ 1\ 1]^T - \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{bmatrix}^T$ $- \frac{1}{\sqrt{6}}\begin{bmatrix}\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\end{bmatrix}^T$ $= \begin{bmatrix}-\frac{1}{3} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}^T$	$e_3 = \frac{u_3}{\|u_3\|}$ $= \begin{bmatrix}\frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}^T$

$$Q = [e_1 | e_2 | \cdots | e_n] = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{bmatrix}$$

$$R = \begin{bmatrix}a_1\cdot e_1 & a_2\cdot e_1 & a_3\cdot e_1\\ 0 & a_2\cdot e_2 & a_3\cdot e_2\\ 0 & 0 & a_3\cdot e_3\end{bmatrix} = \begin{bmatrix}\frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 0 & \frac{3}{\sqrt{6}} & 1/\sqrt{6}\\ 0 & 0 & 2/\sqrt{3}\end{bmatrix}$$