
PROBLEM #1.

What is the characteristic polynomial and its solutions?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{aligned} Ax &= \lambda x \\ 0 &= Ax - \lambda x \\ &= (A - \lambda I)x \\ &= \begin{bmatrix} (1-\lambda) & 0 \\ 0 & (2-\lambda) \end{bmatrix}x \end{aligned}$$

Assume that matrix A has eigenvalues, use determinant as a structure with which to determine those eigenvalues,

$$\begin{aligned} |A - \lambda I| &= 0 \\ &= \left| \begin{bmatrix} (1-\lambda) & 0 \\ 0 & 2-\lambda \end{bmatrix} \right| \\ &= (1-\lambda)(2-\lambda) - (0)(0) \\ &= \lambda^2 - 3\lambda + 2 \end{aligned}$$

The eigenvalues are $\lambda = 1, 2$.

PROBLEM #2.

Find the eigenvectors of A, $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

From question #1, eigenvalues are

$$\lambda = 1, 2$$

Substitute λ s in to the eigen-matrix expression,

$$\begin{aligned} @\lambda=1: \begin{bmatrix} 1-1 & 0 \\ 0 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \\ &= \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \quad @\lambda=2: \begin{bmatrix} 1-2 & 0 \\ 0 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Therefore $x_2 = 0$.

Therefore $x_1 = 0$.

With the constraints found from the back substitution get the eigenvalues,

$$@\lambda=1: x = [x_1, 0]^T \quad @\lambda=2: x = [0, x_2]^T$$

PROBLEM #3.

What is the characteristic polynomial it's solution?

$$A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$$

Refer to Problem #1 for full solution steps.

$$|A - \lambda I| = 0$$

$$= (3-\lambda)(5-\lambda) - (4)(0)$$

$$= \lambda^2 - (3+5)\lambda + 15, \text{ characteristic polynomial}$$

Solutions

$$\lambda = 3, 5$$

PROBLEM #4.

Find eigenvectors for $A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$

$$0 = (A - \lambda I)x$$

$$= \begin{bmatrix} (3-\lambda) & 4 \\ 0 & (5-\lambda) \end{bmatrix} x, @\lambda=3.$$

$$= \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix} x$$

$$= \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix}$$

Therefore x_2 must be zero.

$$0 = (A - \lambda I)x$$

$$= \begin{bmatrix} (3-\lambda) & 4 \\ 0 & (5-\lambda) \end{bmatrix} x, @\lambda=5.$$

$$= \begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} x$$

$$= \begin{bmatrix} -2x_1 + 4x_2 \\ 0 \end{bmatrix}$$

Therefore $-2x_1$ must be equal to x_2 .

PROBLEM #5.

What is the characteristic polynomial its solution?

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}$$

$$C.P = \begin{vmatrix} (1-\lambda) & 0 \\ -1 & (4-\lambda) \end{vmatrix}$$

$$\begin{aligned} &= (1-\lambda)(4-\lambda) - (-1)(0), \text{ solutions are } \lambda = 1, 4. \\ &= \lambda^2 - 5\lambda + 4 \\ &= 0 \end{aligned}$$

PROBLEM #6.

Solve for eigenvectors of A

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}$$

Given the matrix lambda expression, back-substitute the lambda values which will negate the transformational rotations, shearing or mirroring.

$$Ax = \lambda x.$$

$$0 = (A - \lambda I)x$$

$$\begin{aligned} @\lambda = 1: & \begin{bmatrix} (1-1) & 0 \\ -1 & (4-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \\ &= \begin{bmatrix} 0 \\ -x_1 + 3x_2 \end{bmatrix} \end{aligned}$$

Therefore x_1 must be equal to $3x_2$.

$$\begin{aligned} @\lambda = 4: & \begin{bmatrix} 1-4 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \\ &= \begin{bmatrix} -3x_1 \\ -x_1 \end{bmatrix} \end{aligned}$$

Therefore if x_1 is zero x_2 can be anything.

PROBLEM #7.

What is the characteristic polynomial its solution?

$$A = \begin{bmatrix} -3 & 8 \\ 2 & 3 \end{bmatrix}$$

$$0 = \left| \begin{bmatrix} (-3-\lambda) & 8 \\ 2 & (3-\lambda) \end{bmatrix} \right|$$

$$= (-3-\lambda)(3-\lambda) - (8)(2)$$

$$= \lambda^2 - 25$$

$$= (\lambda - 5)(\lambda + 5)$$

Therefore the lambda solutions are $\lambda = 5, -5$.

PROBLEM #8

Solve for the eigenvectors of A.

$$A = \begin{bmatrix} -3 & 8 \\ 2 & 3 \end{bmatrix}$$

$$0 = (A - \lambda I)x$$

$$@\lambda=5: \begin{bmatrix} (-3-5) & 8 \\ 2 & (3-5) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -8x_1 + 8x_2 \\ 2x_1 - 2x_2 \end{bmatrix} = 0$$

Therefore x_1 must equal x_2 .

$$u_1 = \begin{bmatrix} t \\ t \end{bmatrix}$$

$$@\lambda=-5: \begin{bmatrix} (-3+5) & 8 \\ 2 & (3+5) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + 8x_2 \\ 2x_1 + 8x_2 \end{bmatrix}$$

Therefore x_2 must be equal to $-4x_1$.

$$u_2 = \begin{bmatrix} -4t \\ t \end{bmatrix}$$

PROBLEM #9.

What is the characteristic polynomial it's solution?

$$A = \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix}$$

$$0 = (A - \lambda I)x$$

$$\Leftrightarrow \det(A - \lambda I) = 0$$

$$= \det \begin{vmatrix} (5-\lambda) & 4 \\ -4 & (-3-\lambda) \end{vmatrix}$$

$$= (5-\lambda)(-3-\lambda) - (-4)(4)$$

$$= \lambda^2 - 2\lambda + 1$$

Solving for λ ,

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)} , \text{ find roots by quadratic eq.}$$

$$= \frac{2 \pm \sqrt{0}}{2} , \lambda = 1, 1$$

PROBLEM #10.

What is the characteristic polynomial it's solution?

$$A = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$$

The matrix equation which posits eigenvectors is,

$$Ax = \lambda x \Leftrightarrow 0 = (A - \lambda I)x$$

$$= \begin{bmatrix} (-2-\lambda) & -3 \\ 1 & (1-\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Use the determinant to create a polynomial which roots represent the lambda values which when coupled with their eigenvectors will make $(A - \lambda I)x = 0$.

Can do shortcut,

using intuition a quadratic $f(x)$,
if $Ax: \min f \leq f(x) \therefore$ No roots
that intersect x -axis.

$$\det(A - \lambda I) = 0$$

$$= (-2-\lambda)(1-\lambda) - (-3)(1)$$

$$= \lambda^2 + \lambda + 1 , \text{ the characteristic poly.}$$

Use the quadratic equation to solve for λ ,

$$\text{roots of } |A - \lambda I| = -(1) \pm \frac{\sqrt{(1)^2 - 4(1)(1)}}{2(1)} \\ = \frac{-1 \pm \sqrt{(-3)}}{2}$$

The roots of the characteristic polynomial equation are complex; $\lambda = \frac{-1 + \sqrt{-3}}{2}, \frac{-1 - \sqrt{-3}}{2}$.