MEASURE SPACES

A measure on a set is, the assignment a size to each of the possible subsets. Measures are monotone

– a proper subset of another not necessarily proper subset must always have a measure less than or equal to the encompassing subset.

This is a generalization of the concepts of length, area and volume.

In particular length, area and volume in Euclidean Geometry is the appointment of Lebesgue Measures to the Euclidean Space (for n-dimensional subsets $\mathbb{R}^n$).

The Lebesgue measure of $[0,1] \in \mathbb{R}$ is a length of 1.

A measure space is a triple, $(X, \mathcal{S}, \mu)$, where
- X is nonempty
- $\mathcal{S}$ is σ-algebra on set X
- μ is a measure on $(X, \mathcal{S})$.

- measure spaces are monotone mappings
- mapping must be to a value ≥ 0
- probability spaces are measure spaces, which space is equal to 1.

PROBABILITY SPACES

Probability space is a measure space formed by the triple (Ω, F, P).

- Ω is the sample space - an arbitrary non empty set
- F is the σ-algebra, where $\underbrace{F \subseteq 2^{\Omega}}_{\sigma\text{-field}}$, a set of subsets Ω called events.
  - F contains the sample space: $\Omega \in F$
  - F is closed under complements. if $A \in F \rightarrow (\Omega \setminus A) \in F$
  - F is closed under countable unions: $A_i \in F$ $i = 1, 2 \ldots$ then $(\bigcup_{i=1}^{\infty} A_i) \in F$

    by DeMorgan's then $(\bigcap_{i=1}^{\infty} A_i) \in F$

(*) the idea is "can we measure things about the sample space. Is probability tractable?

- the probability measure

  $P: F \rightarrow [0, 1]$, a function which,
  - p is countably additive (called σ-additive), if $\{A_i\}_{i=1}^{\infty} \subseteq F$ is a countable collection of pairwise disjoint sets, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$
  - the measure of the entire sample space is equal to one i.e $P(\Omega) = 1$.

(*) sometimes this probability measure is called a probability distribution!

The Probability Space of One Flip of a Fair Coin

Let the experiment be the flip of a fair coin.
The probability space can be expressed:

Ω (Sample Space - the possible outcomes):

The outcome can be head or tails.

$\Omega = \{H, T\}$

F (σ-algebra field - the possible events):

$F \subseteq 2^{\Omega}$, more precisely $F = 2^{\Omega}$, $F = 2^2$

1. $\{H\}$, heads
2. $\{T\}$, tails
3. $\{\}$, neither head or tail
4. $\{H, T\}$, head or tail

$F = \{\{\}, \{H\}, \{T\}, \{H, T\}\}$

P ($P: F \to [0, 1]$ - the probability measure):

1. $P(\{H\}) = 1/2$
2. $P(\{T\}) = 1/2$
3. $P(\{\}) = 0$
4. $P(\{H, T\}) = 1$.

A MEASURABLE SPACE - BOREL SPACE

Given a non-empty set X and a σ-algebra on set X called A, then the Measurable space can be formed by the tuple (X, A).

$$X = \{1, 2, 3\}$$

$$A_1 = \{X, \emptyset\}$$

Then (X, A_1) is a measurable space.

If the set X is finite or countably infinite, then it is common the σ-algebra is the Power Set on X, that is $A = P(X)$.