There are also Jacobian Matrices.
They represent the combinations of Jacobian vectors, and decompose multivariate functions into Jacobian vector components.
These components when combined into a Jacobian matrix act as a change of basis from the original functions parameters $x_1, ..., x_n$ to the algebraic coordinate expressions of the Jacobian coordinate component vectors.

It is important to note that the algebraic expressions which are the partial derivatives of the components which construct the original function are often <u>highly</u> non-linear.
These functions <u>can</u> still be smooth. Meaning on small enough increments the transformations produced by the component jacobian vectors is approximately linear.
It would in that case be possible to add the cumulative determinants at each point in space to calculate the degree to which there was a change in space borne from the transformation.
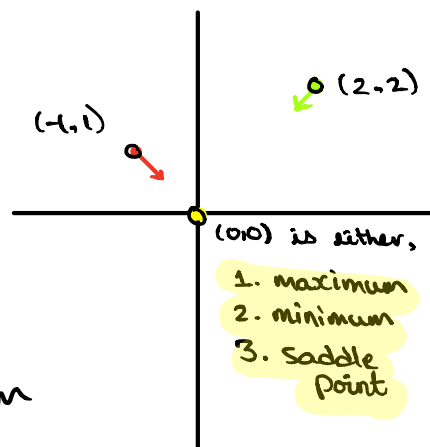
Example:
$$f(x, y) = e^{-(x^2 + y^2)}$$
$$J = [-2xe^{-(x^2 + y^2)}, -2ye^{(-x^2 + y^2)}]$$
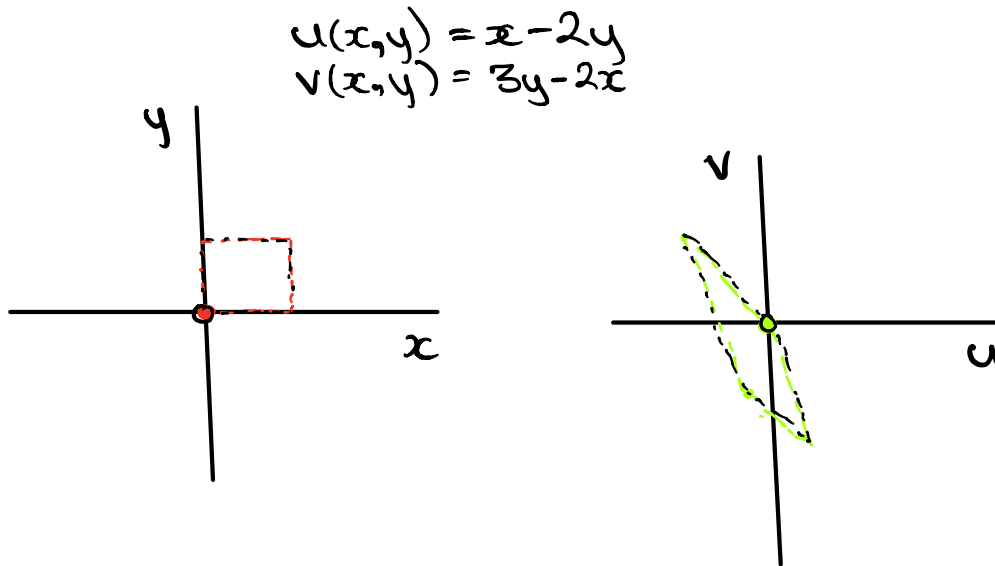$$J(1,1) = [2e^{-2}, -2e^{-2}] = [0.27, -0.27]$$
$$J(2,2) = [-0.001, -0.001]$$
$$J(0,0) = [0, 0] \text{ , direction of vector from original point.}$$
i.e the gradient!

(-1,1)

(2,2)

(0,0) is either,
1. maximum
2. minimum
3. saddle point

Jacobian matrices map vectors as inputs to algebraic functions. The matrices then output vectors instead of scalars or other algebraic equations.

Example of Jacobian component vectors,

$$u(x,y) = x - 2y$$
$$v(x,y) = 3y - 2x$$

$$J_u = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \end{bmatrix}$$

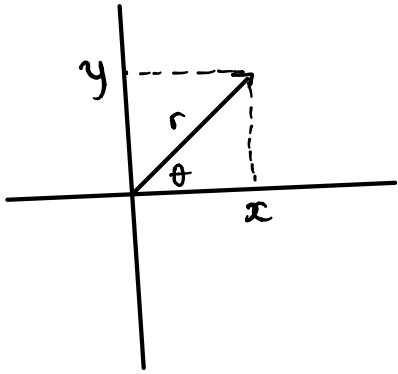$$J_v = \begin{bmatrix} \partial v/\partial x & \partial v/\partial y \end{bmatrix}$$

$$J = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial x \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$$

$$J \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

Example: using the Jacobian to determine the parameters which contribute to the transformation size of the conversion from Polar to Cartesian Coordinates.

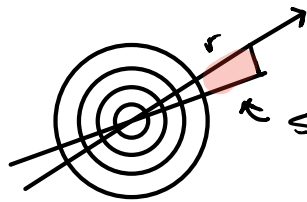$$x(r, \theta) = r\cos\theta$$
$$y(r, \theta) = r\sin\theta$$

$$J = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2mm] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix}$$

← multivariate!!!
produces x & y.
very cool ☺.

$$= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The conversion to cartesian expands space proportionate to r not θ!

$$\det(J) = r(\cos^2\theta + \sin^2\theta)$$
$$= r$$

↖ space (i.e x, y) get bigger with r.