Index notation is better for differentiation with respect to spatial coordinates; it also simplifies the presentation and manipulation of differential geometry.

Symbolic notation is good when the matrix calculus is simple, while the matrix algebra and matrix arithmetic is messy.

Nomenclature

Definition. Real Matrix of dimension $M \times N$, is when $a_{ij} \in \mathbb{R}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and the ordered rectangular matrix is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{M1} & \cdots & & a_{MN} \end{bmatrix}$$

A short formed version of A can be written

$$A = [a_{ij}], \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n$$

where the typical element is a_{ij}.

Matrix Multiplication

1 Definition. Matrix Product. The matrix product AB can be represented by the matrix C, where C:

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & \cdot & & \cdot \\ \vdots & & \cdot & \cdot \\ & & c_{ij} & \cdot \\ c_{m1} & \cdot & \cdot & c_{mp} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & \{i\text{-th row.}\} & & \cdot \\ \cdot & & & \\ a_{m1} & \cdot & \cdot & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & \{j\text{-th col}\} & b_{1p} \\ b_{21} & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & & \cdot \\ b_{n1} & & \cdot & \cdot & b_{np} \end{bmatrix}$$

this holds for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots p$. It can be written algebraically:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

(2) Proposition. Matrix-Vector Product. Let A be $m \times n$ and x be be $n \times 1$. the typical element of the product:

$$z = Ax$$

implies,

$$\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ \cdot \\ c_{m1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & \{i\text{-th row.}\} & & \cdot \\ \cdot & & & \\ a_{m1} & \cdot & \cdot & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}$$

This can be written formally:

$$z_i = \sum_{k=1}^{n} a_{ik} x_k, \quad i \in \{1, \ldots, m\}.$$

(3) Similarily,

$$z^T = y^T A$$

and can be formalized arithmetically as,

$$z_i = \sum_{k=1}^{n} a_{ki} y_k$$

This is visualized as,

$$\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix}^T = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} .$$

(4) $\alpha = y^T A x$

$$= \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} y_j x_k$$

$$= \sum_{j=1}^{m} y_j \left(\sum_{k=1}^{n} a_{jk} x_k \right)$$

$$\alpha = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \sum_{k=1}^{m} v_{1k}, \quad v = \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix}^T = y^T I A x$$

(5) $C = AB \Rightarrow c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

(6) $C^T = B^T A^T \Rightarrow c_{ij} = \sum_{k=1}^{n} a_{jk} b_{ki}$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & \ddots & & \vdots \\ \vdots & & c_{ij} & \vdots \\ c_{m1} & \cdots & \cdots & c_{mp} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & \cdots & \cdots & b_{np} \end{bmatrix}$$

(*) transpose is same multiplication but interchanges rows and columns.

(7) $C^{-1} = B^{-1}A^{-1} \iff C = AB \vee CB^{-1}A^{-1} = ABB^{-1}A^{-1}$
$= I.$

Partioned Matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$, where A_{ij} is a Matrix.

$$A^{-1} = \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix}$$

Matrix Defferentiation

Let $y = \psi(x)$, $y \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$,

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdot & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

This is called the Jacobian matrix of the transformation $\psi(x)$ - and represents the first order partial derivatives of the transformation from x to Y.