

The Taylor Series equations can be manipulated to express the expected error in a approximation.

The taylor series is,

$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (x-p)^n$$

The first order taylor series is,

$$g_1(x) = f(p) + f'(p)(x-p)$$

This can be reexpressed as a change in y ,

$$\begin{aligned} g_1(x) - f(p) &= (x-p) \cdot f'(p) \\ &= \Delta y \end{aligned}$$

The articulation of the approximation as a gradient of y at point p over some delta can be expressed,

$$g_1(x+\Delta x) = f(x) + f'(x)(\Delta x)$$

Given this expression the Taylor Series can be reexpressed in terms of x and Δx .

$$f(x+\Delta x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (\Delta x)^n$$

When forming a power series approximation and ignoring terms of degrees higher than Δx (e.g. $\Delta x^2, \Delta x^3, \dots$); especially when degrees of x produce smaller outcomes - is called linearisation.

The approximated input variable x , can be substituted as an expression of distance from p .

$$\begin{aligned} \Delta y &= g_1(x) - f(p) \\ &= g_1(p+\Delta p) - f(p) \\ &= (x-p) \cdot f'(p) \\ &= (\Delta p) \cdot f'(p) \end{aligned}$$

The error of a power series approximation can be expressed by the distance from the point where the approximation was derived, and the degree/order of the approximation's sophistication.

$$f(x+\Delta x) = f(x) + f'(x)(\Delta x) + O(\Delta x^2)$$

The $O(x)$ notation expresses an amount of error on the order of $\underline{\Delta x^2}$; can also be called - second order accurate.

The full Taylor series can be manipulated algebraically to express exactly the amount of error inherent in a subselection of the infinite number of terms.

$$f(x+\Delta x) = f(x) + f'(x)(\Delta x) + \frac{f''(x)(\Delta x)^2}{2} + \dots$$

$$f'(x)(\Delta x) = f(x+\Delta x) - f(x) - \frac{f''(x)(\Delta x)^2}{2} - \dots$$

$$f'(x) = \underbrace{\frac{f(x+\Delta x) - f(x)}{\Delta x}}_{\text{rise-over-run } a = mx + c} - \underbrace{\frac{f''(x)(\Delta x)^2}{2} - \frac{f'''(x)(\Delta x)^3}{6} - \dots}_{\text{higher order terms}}$$

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x), \text{ since } (\Delta x) \text{ dominates the higher order expressions when } \{ \Delta x \in \mathbb{R} | -1 < \Delta x < 1 \}.$$

In expressing a subselection of Taylor Series terms by the remainder of the rest of the series; that remainder can be simplified via expression in terms of the domination of Δx , as error. That error contextualizes using the non-error remainder as an approximation for the subselected terms.

In categorizing all Taylor series terms as a summation of the first order gradient; then expressing that summation as error in exclusion of $\frac{f(x+\Delta x) - f(x)}{\Delta x}$, the error can be found to be in the order of Δx . That is the first difference method is first order accurate.

