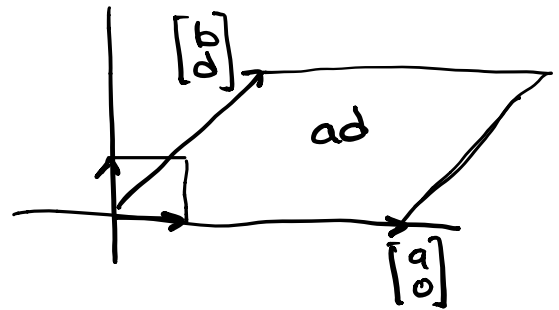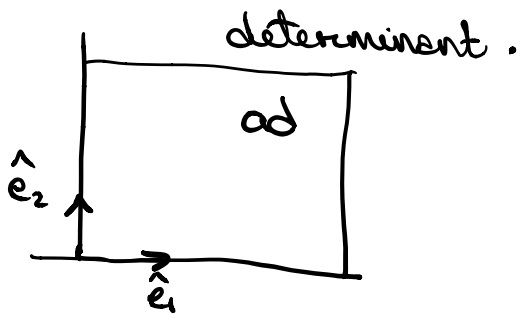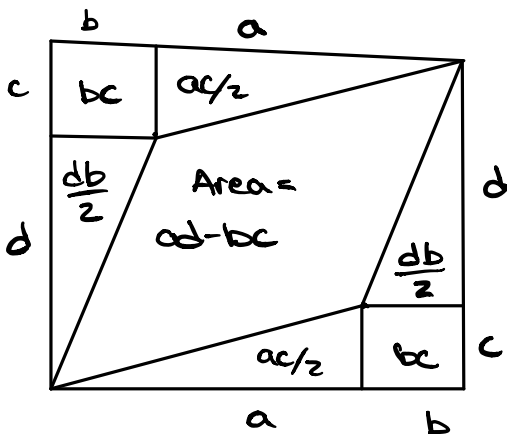⇒ about the determinant when there is and isn't linear independent basis vectors.

The matrix $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ scales space.

determinant.

ad

$\hat{e}_2$

$\hat{e}_1$

$\begin{bmatrix} b \\ d \end{bmatrix}$

ad

$\begin{bmatrix} a \\ 0 \end{bmatrix}$

The matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ produces a parallelogram.

This matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ produces another parallelogram,

b   a

c | bc | ac/2

db/2 | Area = ad-bc | d

d | | db/2

ac/2 | bc | c

a   b

$A_{Area} = (a+b)(c+d) - ac - bd - 2bc$

$= \not{ac} + ad + bc + \not{bd} - \not{ac} - \not{bd} - 2bc$

$= ad - bc$

$= |A|$

Finding the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, flip diagonal negate accross 2nd diag.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$

$\frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = I$

$A^{-1} = \frac{1}{ad-b} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$

The determinant is the ratio of space before and after transformation (1x1: ad-bc).

The inverse needs to shrink the transformation by $\frac{1}{ad-bc}$.

QR decomp to calc det(A) generally.

If you collapse the dimensions the space is also collapsed.

A 2×2 space will reduce to 0. This happens when the cols are linearly dependent.

There is no inverse for rebuilding the original space because that dimensions information is lost on encoding it into a lesser dimension.

↳ That is why when $\det(A) = 0$, there is no inverse.

A Generalized Method for Computing the Determinant

- LU Factorization is faster than QR.- by a fractional constant.

An orthogonal matrix is a matrix whose columns and rows (same operation) are orthogonal unit vectors.

The following are proofs and consequences of a matrix's orthogonality,

$$A^T A = A A^T = I$$

DETERMINANT OF ORTH MATRIX

The determinant of an orthogonal matrix is either negative or positive 1; because the vectors are all of unit length and they are all orthogonal → all inputs will not be scaled and all dimensions will be preserved that is inversion is always possible.

INTUITION BEHIND ORTHOGANAL UNITARY MATRICES.

just a compressed way of expressing dot prod between columns of A. multiplying a column/row of A/AT to the other row will be 0 because columns are orthogonal to one another; the row of AT is just a column of original matrix A. when not zero the col and row will be equal (i.e $i = j$) and their $\langle A_i, A^T_j \rangle$ will $= 1$ because vectors in an orthogonal matrix have a modulus of 1.

QR Decomposition/Factorization produces,

$A = QR$, a combo of an

    I. orthogonal matrix (Q)

    II. a triangular matrix. (R)
        "upper"

One method for QR decomp is Gram-Schmidt process.

Begin with the matrix you want to factorize,

    A

Observe A is square and can be expressed by its columns,

$$A = [a_1 | a_2 | a_3 | \ldots | a_n].$$

Factorize A's columns,

$$u_1 = a_1, \quad e_1 = \frac{u_1}{\|u_1\|}$$

$$u_2 = a_2 - (a_2 \cdot e_1) e_1, \quad e_2 = \frac{u_2}{\|u_2\|}.$$

$$u_{k+1} = a_{k+1} - (a_{k+1} \cdot e_1) e_1 - \ldots - (a_{k+1} \cdot e_k) k, \quad e_{k+1} = \frac{u_{k+1}}{\|u_{k+1}\|}.$$

* $\|\cdot\|$ is L_2 norm — root of sum of squares.

Express the factorization,

$$A = [a_1 | a_2 | \ldots | a_n] = [e_1 | e_2 | \ldots | e_n] \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & \ldots & a_n \cdot e_1 \\ 0 & a_2 \cdot e_2 & \ldots & a_n \cdot e_2 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \vdots & \ddots & a_n \cdot e_n \end{bmatrix}$$

Example of Gram-Schmidt

Given the matrix A - perform QR factorization.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

R-diagonally factorized vectors   Q-orthogonal factor vectors

1. $u_1 = a_1$

$= [1\ 1\ 0]^T$

$e_1 = u_1 / \|u_1\|$

$= \dfrac{1}{\sqrt{2}} [1\ 1\ 0]^T$

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2. $u_2 = a_2 - (a_2 \cdot e_1) e_1$

$= [1\ 0\ 1]^T - \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{\sqrt{2}} [1\ 1\ 0]^T$

$= [\tfrac{1}{2}\ -\tfrac{1}{2}\ 1]^T$

$e_2 = \dfrac{u_2}{\|u_2\|}$

$= \dfrac{1}{\sqrt{3/2}} [\tfrac{1}{2}\ -\tfrac{1}{2}\ 1]^T$

$= [\tfrac{1}{\sqrt{6}}\ -\tfrac{1}{\sqrt{6}}\ \tfrac{2}{\sqrt{6}}]$

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3. $u_3 = a_3 - (a_3 \cdot e_1) e_1 - (a_3 \cdot e_2) e_2$

$= [0\ 1\ 1]^T - \dfrac{1}{\sqrt{2}} [\tfrac{1}{\sqrt{2}}\ \tfrac{1}{\sqrt{2}}\ 0]^T$

$\quad - \dfrac{1}{\sqrt{6}} [\tfrac{1}{\sqrt{6}}\ -\tfrac{1}{\sqrt{6}}\ \tfrac{2}{\sqrt{6}}]^T$

$= [-\tfrac{1}{3}\ \tfrac{1}{\sqrt{3}}\ \tfrac{1}{\sqrt{3}}]^T$

$e_3 = \dfrac{u_3}{\|u_3\|}$

$= [\tfrac{-1}{\sqrt{3}}\ \tfrac{1}{\sqrt{3}}\ \tfrac{1}{\sqrt{3}}]^T$

$Q = [e_1 | e_2 | \dots | e_n]$

$= \begin{bmatrix} \tfrac{1}{\sqrt{2}} & \tfrac{1}{\sqrt{6}} & -\tfrac{1}{\sqrt{3}} \\ \tfrac{1}{\sqrt{2}} & -\tfrac{1}{\sqrt{6}} & \tfrac{1}{\sqrt{3}} \\ 0 & \tfrac{2}{\sqrt{6}} & \tfrac{1}{\sqrt{3}} \end{bmatrix}$

$R = \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & a_3 \cdot e_1 \\ 0 & a_2 \cdot e_2 & a_3 \cdot e_2 \\ 0 & 0 & a_3 \cdot e_3 \end{bmatrix}$

$= \begin{bmatrix} \tfrac{2}{\sqrt{2}} & \tfrac{1}{\sqrt{2}} & \tfrac{1}{\sqrt{2}} \\ 0 & \tfrac{3}{\sqrt{6}} & \tfrac{1}{\sqrt{6}} \\ 0 & 0 & \tfrac{2}{\sqrt{3}} \end{bmatrix}$