$$\operatorname{Var} X = \frac{1}{N}\sum_{n=0}^{N}(x_n - \overline{x}_n)^2$$

Let $S_n = N \cdot \operatorname{Var} X$. S_n can be expressed,

$$S_n = S_{n-1} + (x_n - \mu_{n-1})(x_n - \mu_n).$$

This equality can be proven by,

$$S_n = N \cdot \operatorname{Var} X$$

$$= N \cdot \frac{1}{N}\sum_{i=0}^{N}(x_i - \overline{x}_n)^2$$

$$= \sum^{n}(x_i^2 - 2\overline{x}_n x_i + \overline{x}_n^2)$$

$$= \sum^{n} x_i^2 - 2\overline{x}_n \sum^{n} x_i + \overline{x}_n^2 \sum^{n} 1$$

$$= \sum x_i^2 - 2\overline{x}_n n \frac{\sum^{n} x_i}{n} + n\overline{x}_n^2$$

$$= \sum x_i^2 - 2\overline{x}_n^2 n + n\overline{x}_n^2$$

$$= \sum x_i^2 + n\overline{x}_n^2(-2+1)$$

$$= \sum x_i^2 - n\overline{x}_n^2$$

This expression can be used to express S_{n-1}.

$$S_{n-1} = \sum^{n-1} x_i^2 - (n-1)\overline{x}_{n-1}^2$$

The incremental difference between S_n and S_{n-1} can be found by converting the closed form of the series into a recurrence relation,

$$S_n - S_{n-1} = \sum x_i^2 - n\overline{x}_n^2 - \left[\sum^{n-1} x_i^2 - (n-1)\overline{x}_{n-1}^2\right]$$

$$= \sum^{n} x_i^2 - \sum^{n-1} x_i^2 \quad - n\overline{x}_n^2 + (n-1)\overline{x}_{n-1}^2$$

$$= x_n^2 - n\overline{x}_n^2 + n\overline{x}_{n-1}^2 - \overline{x}_{n-1}^2$$

$$= x_n^2 + n(\overline{x}_{n-1}^2 - \overline{x}_n^2) - \overline{x}_{n-1}^2$$

$$= x_n^2 + n(\overline{x}_{n-1} + \overline{x}_n)(\overline{x}_{n-1} - \overline{x}_n) - \overline{x}_{n-1}^2$$

Given the identity,

$$x_n = n\overline{x}_n - (n-1)\overline{x}_{n-1}$$

$$= n(\overline{x}_n - \overline{x}_{n-1}) + \overline{x}_{n-1}$$

$$\overline{x}_{n-1} - x_n = n(\overline{x}_n - \overline{x}_{n-1})(-1)$$

$$\overline{x}_{n-1} - x_n = n(\overline{x}_{n-1} - \overline{x}_n)$$, producing the factor $\overline{x}_{n-1} - \overline{x}_n$.

$$\overline{x}_{n-1} - \overline{x}_n = \frac{\overline{x}_{n-1} - x_n}{n}$$, isolating $\overline{x}_{n-1} - \overline{x}_n$.

Sub in the equivalent expression and reduce the expression.

$$= x_n^2 + n(\overline{x}_{n-1} + \overline{x}_n)(\overline{x}_{n-1} - \overline{x}_n) - \overline{x}_{n-1}^2$$

$$= x_n^2 + n(\overline{x}_{n-1} + \overline{x}_n)\frac{(\overline{x}_{n-1} - x_n)}{n} - \overline{x}_{n-1}^2$$

$$= x_n^2 + (\overline{x}_{n-1} + \overline{x}_n)(\overline{x}_{n-1} - x_n) - \overline{x}_{n-1}^2$$

$$= x_n^2 + \overline{x}_{n-1}^2 - x_n\overline{x}_{n-1} + \overline{x}_n\overline{x}_{n-1} - \overline{x}_n x_n - \overline{x}_{n-1}^2$$

$$= x_n^2 - x_n\overline{x}_{n-1} + \overline{x}_n\overline{x}_{n-1} - \overline{x}_n x_n$$

$$= (x_n - \overline{x}_{n-1})(x_n - \overline{x}_n)$$

$$= S_n - S_{n-1}$$

Thus,

$$S_n = S_{n-1} + (x_n - \overline{x}_{n-1})(x_n - \overline{x}_n)$$

This implies,

$$\mathrm{Var}_n(x) = \frac{n-1}{n}\mathrm{Var}_{n-1}X + \frac{1}{n}(x_n - \overline{x}_{n-1})(x_n - \overline{x}_n)$$