There is a generalization of the dot product which can be used to compute the angles length and distances between vectors.

It is in some cases useful to measure geometric properties unconventionally.

The Inner Product is defined,

$x, y \in V$

$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$, having the properties,

- symmetric
- positive-definite
- bilinear

<u>Bilinear</u>

$x, y, z \in V$, $\lambda \in \mathbb{R}$

$$\left.\begin{aligned} \langle \lambda x + z, y \rangle &= \lambda \langle x, y \rangle + \langle z, y \rangle \\ \langle x, \lambda y + z \rangle &= \lambda \langle x, y \rangle + \langle x, z \rangle \end{aligned}\right\}$$

Bi-linearity both arguments behave linearly.

<u>Positive Definite</u>

$\langle x, x \rangle \geq 0$, $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

<u>Symmetry</u>

$\langle x, y \rangle = \langle y, x \rangle$

Manipulations of the Inner Product

$\langle x, y \rangle = x^T I y$, i.e the dot product

Suppose,

$$\langle x, y \rangle = x^T A y, \text{ where } A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2y_1 & y_2 \\ y_1 & 2y_2 \end{bmatrix}$$

$$= 2y_1x_1 + x_2y_1 + x_1y_2 + 2y_2x_2$$

, this is different than the dot product because of the transformation matrix.

(*) Any symmetric, positive definite matrix can be used to form a valid inner product of the form,

$$\langle x, y \rangle = x^T A y$$