THEOREM 1.39. *For arbitrary real numbers x and y, we have*

$$|x + y| \le |x| + |y| .$$

Note: This property is called the *triangle inequality,* because when it is generalized to vectors it states that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides.

Proof. Adding the inequalities $-|x| \le x \le |x|$ and $-|y| \le y \le |y|$, we obtain

$$-(|x| + |y|) \le x + Y \le |x| + |y| ,$$

and hence, by Theorem 1.38, we conclude that $|x + y| \le |x| + |y|$.

If we take $x = a - c$ and $y = c - b$, then $x + y = a - b$ and the triangle inequality becomes

$$|a - b| \le |a - c| + |b - c| .$$

This form of the triangle inequality is often used in practice.

Using mathematical induction, we may extend the triangle inequality as follows:

THEOREM 1.40. *For arbitrary real numbers $a_„$ $a_2, \ldots, a_„$ we have*

$$\left|\sum_{k=1}^{n} a_k\right| \le \sum_{k=1}^{n} |a_k| .$$

Proof. When $n = 1$ the inequality is trivial, and when $n = 2$ it is the triangle inequality. Assume, then, that it is true for n real numbers. Then for $n + 1$ real numbers $a, , a_2, \ldots, a_{n+1}$, we have

$$\left|\sum_{k=1}^{n+1} a_k\right| = \left|\sum_{k=1}^{n} a_k + a_{n+1}\right| \le \left|\sum_{k=1}^{n} a_k\right| + |a_{n+1}| \le \sum_{k=1}^{n} |a_k| + |a_{n+1}| = \sum_{k=1}^{n+1} |a_k| .$$

Hence the theorem is true for $n + 1$ numbers if it is true for n. By induction, it is true for every positive integer n.

The next theorem describes an important inequality that we shall use later in connection with our study of vector algebra.

THEOREM 1.41. THE CAUCHY-SCHWARZ INEQUALITY. *If $a_1, \ldots$ ..a. and$b_{,,} \ldots, b_n$ are arbitrary real numbers, we have*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) . \tag{1.23}$$

The equality sign holds if and only if there is a real number x such that $a_k x + b_k = 0$ for each $k = 1, 2, \ldots, n$.

Proof. We have $\sum_{k=1}^{n} (a_k x + b_k)^2 \ge 0$ for every real x because a sum of squares can never be negative. This may be written in the form

$$Ax^2 + 2Bx + C \ge 0 , \tag{1.24}$$

where

$$A = \sum_{k=1}^{n} a_k^2 , \qquad B = \sum_{k=1}^{n} a_k b_k , \qquad C = \sum_{k=1}^{n} b_k^2 .$$

We wish to prove that $B^2 \le AC$. If $A = 0$, then each $a_k = 0$, so $B = 0$ and the result is trivial. If $A \ne 0$, we may complete the square and write

$$Ax^2 + 2Bx + C = A\left(x + \frac{B}{A}\right)^2 + \frac{AC - B^2}{A} .$$

The right side has its smallest value when $x = -B/A$. Putting $x = -B/A$ in (1.24), we obtain $B^2 \le AC$. This proves (1.23). The reader should verify that the equality sign holds if and only if there is an x such that $a_k x + b_k = 0$ for each k.

14.9 Exercises

1. Prove each of the following properties of absolute values.
   (a) $|x| = 0$ if and only if $x = 0$.
   (b) $|-x| = |x|$.
   (c) $|x - y| = |y - x|$.
   (d) $|x|^2 = x^2$.
   (e) $|x| = \sqrt{x^2}$.
   (f) $|xy| = |x|\,|y|$.
   (g) $|x/y| = |x|/|y|$ if $y \ne 0$.
   (h) $|x - y| \le |x| + |y|$.
   (i) $|x| - |y| \le |x - y|$.
   (j) $||x| - |y|| \le |x - y|$.
2. Each inequality (a_i), listed below, is equivalent to exactly one inequality (b_j). For example, $|x| < 3$ if and only if $-3 < x < 3$, and hence (a_1) is equivalent to (b_2). Determine all equivalent pairs.
   (a_1) $|x| < 3$.
   (a_2) $|x - 1| < 3$.
   (a_3) $|3 - 2x| < 1$.
   (a_4) $|1 + 2x| \le 1$.
   (a_5) $|x - 1| > 2$.
   (a_6) $|x + 2| \ge 5$.
   (a_7) $|5 - x^{-1}| < 1$.
   (a_8) $|x - 5| < |x + 1|$.
   (a_9) $|x^2 - 2| \le 1$.
   (a_{10}) $x < x^2 - 12 < 4x$.
   (b_1) $4 < x < 6$.
   (b_2) $-3 < x < 3$.
   (b_3) $x > 3$ or $x < -1$.
   (b_4) $x > 2$.
   (b_5) $-2 < x < 4$.
   (b_6) $-\sqrt{3} \le x \le -1$ or $1 \le x \le \sqrt{3}$.
   (b_7) $1 < x < 2$.
   (b_8) $x \le -7$ or $x \ge 3$.
   (b_9) $\frac{1}{6} < x < \frac{1}{4}$.
   (b_{10}) $-1 \le x \le 0$.
3. Determine whether each of the following is true or false. In each case give a reason for your decision.
   (a) $x < 5$ implies $|x| < 5$.
   (b) $|x - 5| < 2$ implies $3 < x < 7$.
   (c) $|1 + 3x| \le 1$ implies $x \ge -\frac{2}{3}$.
   (d) There is no real x for which $|x - 1| = |x - 2|$.
   (e) For every $x > 0$ there is a $y > 0$ such that $|2x + y| = 5$.
4. Show that the equality sign holds in the Cauchy-Schwarz inequality if and only if there is a real number x such that $a_k x + b_k = 0$ for every $k = 1, 2, \ldots, n$.

$$(A \cdot B^T)^2 \le (A \circ A)(B \circ B)$$

$$\begin{bmatrix} a_1b_1 & & \\ & a_2b_2 & \\ & \ddots & \\ & & a_nb_n \end{bmatrix}$$

sum and square

$$\begin{bmatrix} a_1^2b_1^2 & a_1^2b_2^2 & a_1^2b_3^2 & \cdots & a_1^2b_m^2 \\ \vdots & & & & \\ a_n^2b_1^2 & a_n^2b_2^2 & \cdots & & a_n^2b_m^2 \end{bmatrix} = \sum_i^n \left[a_i^2 \sum_j^n b_j^2\right]$$

sum of all elements.

let $c_{ij} = a_ib_j$.

~~c_nc_n c_nc_n~~

let $c_i = c_{ii}$.

$a_1b_1a_2b_2$

$$\begin{bmatrix} c_1c_1 & c_1c_2 & \cdots & c_1c_n \\ c_2c_1 & c_2c_2 & \cdots & c_2c_n \\ \vdots & & & \vdots \\ c_nc_1 & c_nc_2 & \cdots & c_nc_n \end{bmatrix}$$, where $c_i = a_ib_i$.

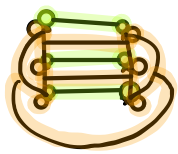
uses $a_i, a_j \in A$:

Diagonal is always the same.

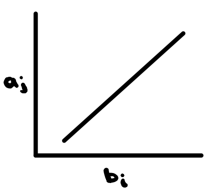
$$\begin{bmatrix} a_1b_1a_1b_1 & a_1b_1a_2b_2 & \cdots & a_1b_1a_nb_n \\ a_2b_2a_1b_1 & a_2b_2a_2b_2 & & a_2b_2a_nb_n \\ \vdots & & & \\ a_nb_na_1b_1 & a_nb_na_2b_2 & \cdots & a_nb_na_nb_n \end{bmatrix}$$

$$\sum_i^n c_i \sum_j^n c_j = \sum_i^n \left[a_ib_i \cdot C\right], \text{ where } C = \sum_i^n a_ib_i$$

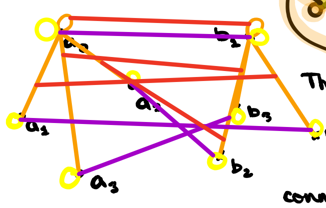
$$= \sum_i^n \left[a_ib_i \sum_j^n a_jb_j\right]$$



Correlation



Must have linear correlation between each $\langle a_i, b_i \rangle$ pair.



There are many more edges, pairwise connections produced by the cartesian product:

$$\binom{n}{2}^2 \xleftarrow{\text{Card}} (A \times A) \times (B \times B), \text{ when } \left(\sum^k a_kb_k\right)^2$$

double check

$$n^2 \xleftarrow{\text{Card}} \{\forall a_i \in A : (a_i, a_i)\} \times \{\forall b_i \in B : (b_i, b_i)\},$$

when

$$\sum^k a_k^2 \cdot \sum b_k^2$$