

— PROBLEM —

Prove by induction that for all $n \geq 0$,

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n.$$

In the inductive step use Pascal's Identity,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

— INDUCTIVE PROOF —

Basic Step,

$$\binom{0}{0} = \frac{0!}{0!0!} = 2^0$$

Inductive Step,

Let $k \in \mathbb{Z}_+$, where $k \neq 0$
Let $l \in [0, k]$,

$$\forall l \rightarrow \sum_{i=0}^l \binom{l}{i} = 2^l$$

— INDUCTIVE PROOF CONT. —

Using Pascal's Identity, two equivalences can easily be calculated,

$$\begin{aligned} \textcircled{1} \quad \binom{n+1}{n+1} &= \binom{n}{n} + \binom{n}{n+1} \\ &= \binom{n}{n} + 0 \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad \binom{k+1}{0} &= \binom{k}{-1} + \binom{k}{0} \\ &= 0 + \binom{k}{0} \end{aligned}$$

① Frame the summation in similar terms of Pascal's Identity.

$$\binom{k+1}{0} + \binom{k+1}{1} + \binom{k+1}{2} + \dots + \binom{k+1}{k} + \binom{k+1}{k+1}$$

② Replace intermediate terms with pascal equivalents,

$$\boxed{\binom{k+1}{0}} \underbrace{\binom{k}{0} + \binom{k}{1} + \dots + \binom{k}{k-1} + \binom{k}{k}}_{\text{TERM \#2}} + \boxed{\binom{k+1}{k+1}}_{\text{TERM \#(n-1)}}$$

③ Replace terminal terms,

$$\boxed{\binom{k}{0}} \xrightarrow{\text{SAME}} \binom{k}{0} + \binom{k}{0} + \underbrace{\binom{k+1}{1} + \binom{k+1}{1}}_{\text{TERM \#1}} + \binom{k+1}{2}$$

④ Remove grouped terms via factoring.

$$2 \cdot \left(\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{k} \right)$$

⑤ Re-express the summation as contiguous

$$2 \cdot \sum_{i=0}^k \binom{k}{i}$$

⑥ State the recurrence,

$$\sum_{i=0}^k \binom{k}{i} = 2 \cdot \sum_{i=0}^{k-1} \binom{k-1}{i}$$

⑦ Simplify the (geometric?) series

$$\sum_{i=0}^k \binom{k}{i} = 2 \cdot \underbrace{2 \cdot \dots \cdot 2}_{k-1 \text{ times}}$$

⑧ State the equivalence,

$$\sum_{i=0}^k \binom{k}{i} = 2^k \quad \text{QED}$$