

# Linear Algebra Background

**CONJUGATE TRANSPOSE**  
(AKA HERMITIAN) or **Bedaggered** Matrix.  $A^\dagger$   
 $A^H$   
 $A^T$

Take the transpose of  $A_{m \times n}$  with complex entries, then take complex conjugate of each entry.

$a + ib \rightarrow a - ib$ , where  $a, b \in \mathbb{R}$ .

Formal definition, for  $A_{m \times n}$

$$(A^H)_{ij} = \overline{A_{ji}}, \text{ notice } i \text{ and } j \text{ are transposed}$$

can also be written,

$$A^H = (\overline{A})^T = \overline{A^T}$$

NOT  $\text{adj}(A)$ , <sup>not</sup> the ADJUGATE.

When matrix is square.

Hermitian or self-adjoint

$$A = A^H \rightarrow a_{ij} = \overline{a_{ji}}$$

Skew Hermitian / Anti hermitian

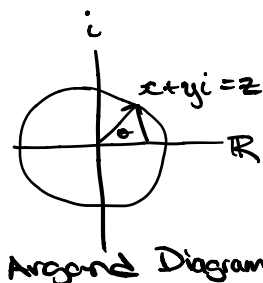
$$A = -A^H \rightarrow a_{ij} = -\overline{a_{ji}}$$

Normal (Hermitian w/ commutativity)

$$A^H A = A A^H$$

Unitary

$$A^H = A^{-1}$$



Complex Numbers as  $2 \times 2$  matrix, works for multiplication & addition,

$$a+ib \equiv \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad \} \quad z = a+ib$$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \overbrace{a^2 - b^2}^{\text{new } a} & \overbrace{-2(a+b)}^{\text{new } b \times -1} \\ \underbrace{2(a+b)}_{\text{new } b} & \underbrace{-b^2 + a^2}_{\text{new } a} \end{bmatrix},$$

$a$  - is  $x$  of an Argand D.  
 $b$  - is  $y$  of the Argand (imag-axis)

Motivation.

Picture an  $n \times n$  matrix of  $\mathbb{C}$  numbers,  $\Rightarrow A_{2m \times 2n}$  of  $\mathbb{R}$ ,

$$\begin{bmatrix} \mathbb{C}_1 & \dots \\ \mathbb{C}_2 & \dots \end{bmatrix} \Rightarrow \begin{bmatrix} \overbrace{\mathbb{R}_1 \mathbb{R}_1 \dots}^{\mathbb{C}_1} \\ \mathbb{R}_1 \mathbb{R}_1 \dots \\ \underbrace{\left\{ \begin{matrix} \mathbb{R}_2 \mathbb{R}_2 \\ \mathbb{R}_2 \mathbb{R}_2 \end{matrix} \right\}}_{\mathbb{C}_2} \dots \\ \dots \end{bmatrix}$$

$$z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

The transpose of  $2 \times 2$  represented complex numbers switch it's sign to maintain mult/add  
 Thus reducing  $\mathbb{R}_{2m \times 2n}$  to  $\mathbb{C}_{m \times n}$  and then performing a transpose must account for the  $\mathbb{R}$ -based transpose mechanics  $\rightarrow$  yielding the reason for performing the complex conjugate after transposing a  $\mathbb{C}$  matrix.

## Unitary Matrix

A complex square matrix  $U$  is unitary if its conjugate transpose  $U^*$  is its inverse.

$$\left. \begin{array}{l} U^*U = UU^* = I \\ \text{OR} \\ U^H = U^{-1} \end{array} \right\} \begin{array}{l} \text{If } U \in \mathbb{R}^N \\ \rightarrow U^* = U^T \\ \text{In physics} \\ U^+U = UU^+ = I \end{array}$$

## Diagonalizable Matrix

A matrix can be called similar to a diagonal matrix if there exists an invertible matrix  $P$ , such that  $P^{-1}AP$  is a diagonal matrix.  
↖ also like a change of basis.

## Schur Lemma

If  $A$  is a square complex matrix there exists a  $U$  upper triangular matrix and  $S$  a unitary matrix such that,  
 $A = SUS^*$   
 $= SUS^{-1}$

## Hermitian Matrix

A hermitian  $\Leftrightarrow a_{ij} = \overline{a_{ji}}$ ,  
 $A = A^H$

## Diagonal Matrix

Entries outside the main diagonal are zero.

$$\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} = A.$$

Usually refers to square matrix.

Q: Eigenvalues & Eigenvectors of a diagonalizable matrix?

Q2: What are normal matrices and normal equations?

Q3. Gram Schmidt Process?  
How linear independence transforms to orthonormal?

Q. QR Factorization  
orthogonal vs. unitary matrix

## Norms

Most commonly the euclidean norm,

$$\|\vec{a}\| = \sqrt{x^2 + y^2 + z^2}$$

in a 3 dimensional space.

## Orthonormal Vectors

A collection of real  $n$ -vectors  $a_1, a_2, a_3 \dots a_n$  is orthonormal if,

- vectors have unit norm:

$$\|a_i\| = 1$$

- mutually orthogonal:

$$a_i^T a_j = 0, \text{ when } i \neq j.$$

Examples,

$$\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

## Norms with Scripts

Superscript is a power.

Subscript is dimension.

$$\|\vec{v}\|_y^x \rightarrow \left[ \sum_{i=1}^d |v_i|^y \right]^{\frac{1}{y}}$$

## Matrix with Orthonormal Vectors

$$A^T A = [a_1 \ a_2 \ \dots \ a_n]^T [a_1 \ a_2 \ \dots \ a_n]$$

$$= \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \dots & a_1^T a_n \\ a_2^T a_1 & a_2^T a_2 & \dots & a_2^T a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^T a_1 & a_n^T a_2 & \dots & a_n^T a_n \end{bmatrix}$$

Gram Matrix

$$= \begin{bmatrix} a_1^0 & a_1^1 & \dots & a_1^n \\ a_2^0 & a_2^1 & \dots & a_2^n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^0 & a_n^1 & \dots & a_n^n \end{bmatrix} \begin{bmatrix} a_1^0 & a_2^0 & \dots & a_n^0 \\ a_1^1 & a_2^1 & \dots & a_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^n & a_2^n & \dots & a_n^n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 \end{bmatrix}$$

$$= I \Leftrightarrow \forall x \in A \rightarrow 1 = \|x\|$$