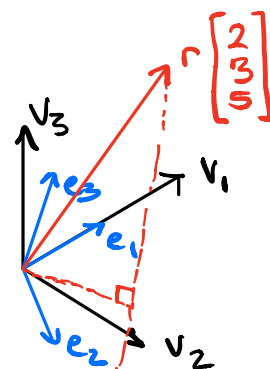


Calculate how a vector u will be reflected in a plane.

$$V = \{v_1, v_2, v_3\}$$

$$\begin{aligned} v_1 &= [1 \ 1 \ 1]^T \\ v_2 &= [2 \ 0 \ 1]^T \\ v_3 &= [3 \ 1 \ -1]^T \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \text{These} \\ \text{vectors} \\ \text{are in the} \\ \text{plane.} \end{array}$$



① Find orthonormal vectors describing the plane and v_3 . Do this with Gram Schmidt.

$$u_1 = v_1 \quad e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} u_2 &= v_2 - \frac{(v_2 \cdot e_1)}{e_1 \cdot e_1} e_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \left[\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right] \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \end{aligned} \quad \begin{aligned} e_2 &= \frac{u_2}{\|u_2\|} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} u_3 &= v_3 - \frac{(v_3 \cdot e_1)}{e_1 \cdot e_1} e_1 - \frac{(v_3 \cdot e_2)}{e_2 \cdot e_2} e_2 \\ &= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \left[\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right] \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left[\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right] \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \end{aligned} \quad e_3 = \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

② Construct transformation matrix from the orthonormal basis vectors (back to v 's basis).

$$E = [e_1 | e_2 | e_3] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

Plane vectors.

- ③ Construct a transformation to do the reflection in the orthonormal bases.

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\underbrace{\quad}_{e_1} \quad \underbrace{\quad}_{e_2} \quad \underbrace{\quad}_{e_3}$

Flip the e_3 component which is not part of the plane.
This is a transformation matrix defined in the basis of the plane

- ④ Get the translation matrix to convert from v 's basis to e .

$$E^{-1} = E^T, \text{ because matrix is orthogonal and vectors have norm of 1.}$$

e.g. $A^T A = 1$.

- ⑤ Translate r into the e -basis, transform r_e using the reflection matrix T_e , convert the reflected vector back into the basis of v .

$$r' = E T_e E^{-1} r.$$

$$T_e E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} (1 \ 1 \ 1) \\ \frac{1}{\sqrt{2}} (1 \ -1 \ 0) \\ \frac{1}{\sqrt{6}} (1 \ 1 \ -2) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} (1 \ 1 \ 1) \\ \frac{1}{\sqrt{2}} (1 \ -1 \ 0) \\ \frac{1}{\sqrt{6}} (-1 \ -1 \ 2) \end{bmatrix}$$

$$E T_e E^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} (1 \ 1 \ 1) \\ \frac{1}{\sqrt{2}} (1 \ -1 \ 0) \\ \frac{1}{\sqrt{6}} (-1 \ -1 \ 2) \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$r' = E^T e E^{-1} r = \frac{1}{3} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$r' = \frac{1}{3} \begin{bmatrix} 11 \\ 14 \\ 5 \end{bmatrix}$$