"Last time I asked you: "What does mathematics mean?", Some of you answered "The manipulation of structures of numbers". And if I had asked you what music means to you, would you have answered: "The manipulation of notes!""

— Serge Lang —

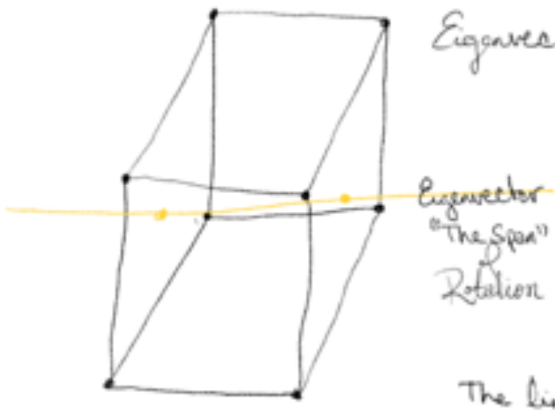
Original Vector Space & Span

Vector Space After Transformation

Some vectors after their transformation will continue in the same span. These vectors are eigenvectors; the degree to which they change in magnitude is their eigenvalue.

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

Can think of a Linear Transformation as Eigenvectors and Eigenvalues.



$$\underbrace{A\vec{v}}_{} = \underbrace{\lambda\vec{v}}_{}$$

The linear transformation of an Eigenvector is the same as multiplying the eigenvector by its Eigenvalue.

Matrix vector multiplication differs from scalar vector multiplication.

To bring about a more useful formula - extract lambda and express the operation as matrix multiplication.

$$A\vec{v} = (\lambda I)\vec{v} \iff A\vec{v} - (\lambda I)\vec{v} = \vec{0}$$
$$\iff \underline{(A - \lambda I)}\vec{v} = \vec{0}$$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 1 & 5-\lambda & 9 \\ 2 & 6 & 5-\lambda \end{bmatrix}$$

Matrix of subtractions look something like this.

The only way possible for matrix multiplication to become zero (*when multiplied by a non-zero vector) is when the determinant is Zero. Meaning dimensionality has collapsed.

This means $\det(A - I\lambda) = 0$.

If $(A - I\lambda) = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} \Leftrightarrow \det(A) = (1-\lambda)^2 - (1)(0)$
$= 0$

So implying $\lambda = 1$.

A Eigenvalue can have more than one Eigenvector.

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, scales all possible spans by the singular eigenvalue. (2).

Eigenbasis

A diagnol matrix has only one row with a non-zero value in all rows of a matrix's column.

Diagnol matrices have easily computable squares.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ These values are eigenvalues, their corresponding eigenvectors are the basis vectors which the matrices columns represent.

$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \ldots \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \vec{v} = \begin{bmatrix} 3^n & 0 \\ 0 & 2^n \end{bmatrix} \vec{v}$

If you have enough eigenvectors to span the subspace you can exchange the coordinate system (e.g $\hat{i}, \hat{j}, \hat{k}$) for the coordinate system based on the eigenvectors as basis vectors.

To the [illegible]

In the case you wanted to perform a complex transformation, that would be computed easier (e.g. in a eigen-based coordinate system), you can translate the transformation in terms of the eigenspace — perform the computation and then translate the result back to the original coordinate system.

If eigenvectors were $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$,

and the transformation to compute was $\begin{bmatrix}3&1\\0&2\end{bmatrix}^n$. The one could,

$$\underbrace{\begin{bmatrix}1&-1\\0&1\end{bmatrix}^{-1}\begin{bmatrix}3&1\\0&2\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}}_{\text{Translate to eigenspace.}} = \begin{bmatrix}3&0\\0&2\end{bmatrix}$$

then apply

$$\begin{bmatrix}3&0\\0&2\end{bmatrix}^n \vec{v} = \vec{w} \text{ (in eigenspace)}$$

Then translate back to the original coordinate system.

① Determine if the transformation vector is diagonizable.
$\Rightarrow$ does that mean $\det A = 0$?

② Compute $P^{-1}AP = D$

P columns are

[illegible] are made from eigenvectors,
D diagonal from Eigenvalues.

(3) Compute power.

$$A = P D P^{-1}$$
$$A^2 = P D P^{-1} P D P^{-1}$$
$$= \underbrace{P}\,\underbrace{D^2}\,\underbrace{P^{-1}}$$

Translate original vector into eigenspace representation.
Do transformation
Translate back to standard representation.

The reason why you can utilize eigenspaces to simplify matrix operations is because those matrix operations are simple matrix transformations abstracted