The inner product can be used to compute lengths of vectors and distances between vectors.

The length of a vector is defined.

$\|x\| = \sqrt{\langle x,x\rangle}$, where the inner product is positive definite (≥ 0).

- The length of the inner product is entirely dependent on the inner product.
- Similar to how the inner product of the space affects the length of a vector it also affects the geometry.
- Length is also called the "norm".

Example: Normal Dot Product

Let the inner product be defined:

$\langle x,y\rangle = x^T y$

The length of the vector $v = [1\ 1]^T$.

$$\|x\| = \sqrt{\langle x,x\rangle}$$

$$= \left([1\ \ 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{1/2}$$

$$= \sqrt{2}$$

Example: Half the Dot Product

Let the inner product be defined:

$$\langle x,y\rangle = x^T \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} y$$

The length of a vector $v = [1\ 1]^T$

$$\|x\| = \sqrt{\langle x, x\rangle}$$

$$= \left([x_1\ \ x_2] \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right)^{1/2}$$

$$= \left[y_1\left(x_1 - \frac{x_2}{2}\right) + y_2\left(x_2 - \frac{x_1}{2}\right) \right]^{1/2}$$

$$= \left(y_1x_1 - \frac{y_1x_2}{2} + y_2x_2 - \frac{x_1y_2}{2} \right)^{1/2}$$

$$= \sqrt{y_1x_1 - \frac{1}{2}(y_1x_2 + y_2x_1) + y_2x_2}$$

@ $y = \vec{v}$, $x = \vec{v}$: $\|x\| = \sqrt{x_1^2 - x_1 x_2 + x_2^2}$

$$= \sqrt{(1)^2 - (1)(1) + (1)^2}$$

$$= 1.$$

Nice Properties of "The Half Dot Product"

① Scaled Norms are equivalent to their absolute scale.

$$\|\lambda x\| = |\lambda|\,\|x\|$$

② Triangle Inequality

$$\|x + y\| \leq \|x\| + \|y\|$$

③ Cauchy-Schwarz Inequality

$|\langle x, y\rangle| \leq \|x\|\,\|y\|$. or $|\langle u, v\rangle|^2 = \langle u, u\rangle \cdot \langle v, v\rangle$