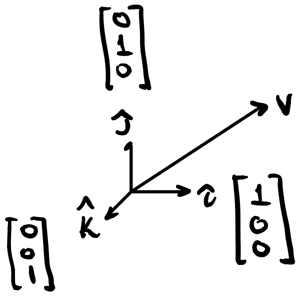
All vectors are located in a coordinate space.
When performing projection, the vector upon which you are projecting is a "new" coordinate space.

Standard Basis

where $i, j, k \ldots z$ are (can also use "e_i") unit vectors on their axes
→ corollorally orthonormal



The hat in vectors terminology represents a basis vector.

A vector v can be represented as a typical vector,

$$v = [\alpha e_1 \ \beta e_2 \ldots \zeta e_n]^T$$

or as a sum of basis vector,

$$v = \alpha e_1 + \beta e_2 + \gamma e_3 \ldots + \zeta e_n$$

By expressing a vector as a combination of other vectors — this expression can be re-expressed in the frame of a different set of vectors.

changing $E = \{e_i\} \rightarrow F = \{f_i\}$.

These sets of vectors are known as basis vectors.

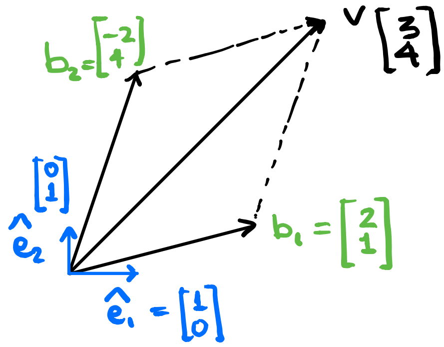
↳ The vector v, will take on different values depending on the coord-sys; the vector is independent of the the coord. sys..

Use of the Projectional Dot-Product

- can use the projectional dot-product to find the vector v in the new coord. sys.
  - provided you know how the new basis vectors relate to the old.

⇒ This is extremely easy when the new coord. sys. vectors are orthogonal to each other.

Translation of Vector v into second coord. sys.:



① Test basis of new coord sys are orthogonal (i.e $\cos\theta = 1$ for all basis vectors).

$$b_1 \cdot b_2 = |b_1||b_2|\cos\theta$$

$$\cos\theta = \frac{b_1 \cdot b_2}{|b_1||b_2|}$$

(*) using the example, $b_1 = [2\ 1]^T$
$b_2 = [-2\ 4]^T$

$$\cos(\theta) = \frac{2(-2) + 1(4)}{\sqrt{5}\sqrt{20}}$$

$$= 0.$$

② Perform vector projection of the vector v onto the basis vectors.

$$\underset{\text{of } b_1}{\text{Vector proj}} = \underbrace{\frac{b_1}{|b_1|}}_{\text{normalized projection in the direction of } b_1.} \times \underbrace{\frac{v \cdot b_1}{|b_1|}}_{\text{Scalar Proj.}}$$

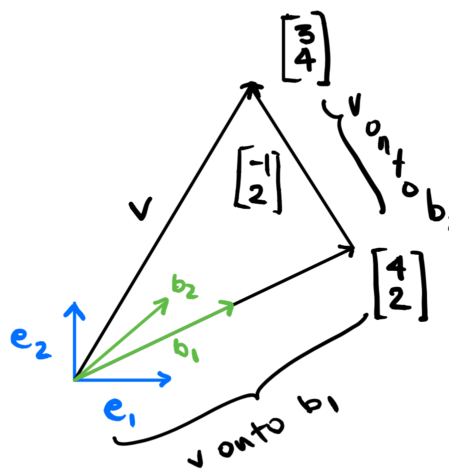
$$= \begin{bmatrix}2\\1\end{bmatrix} \frac{10}{(\sqrt{2^2+1^2})^2}$$

$$= \begin{bmatrix}4\\2\end{bmatrix}$$

$$\underset{\text{of } b_2}{\text{vector proj}} = \frac{b_2}{|b_2|} \times \frac{v \cdot b_2}{|b_2|}$$

$$= \begin{bmatrix}-2\\4\end{bmatrix} \times \frac{10}{20}$$

$$= \begin{bmatrix}-1\\2\end{bmatrix}$$



$$\longrightarrow \begin{bmatrix}-1\\2\end{bmatrix} + \begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$$

Summary,

All vectors aren't tied to their axises, they can be reexpressed by other bases. Those bases describe the space of those vectors; the selection of those bases is very important. This translation can be done very easily using the dot product when the bases are orthogonal to one another.