

The formalized algebraic expression of a matrix's eigenvalues is:

$$Ax = \lambda x, \text{ where } \lambda \text{ is a scalar value.}$$

This formula expresses that \underline{x} are eigenvectors and when multiplied on the matrix A, the end result will be a scaling of the original vector.

To determine the eigenvectors and values factorize the equation and solve for the characteristic polynomial.

$0 = (A - \lambda I)x$, setting to zero allows us to solve for when the matrix becomes zero.

\Rightarrow a zero vector is trivial and not an eigenvector-by definition.

Then can use the determinant calculation knowing that the subtraction of the eigenvector-value combo from the vector by the matrix will negate each other,

$\det(A - \lambda I) = 0$, this approach is computationally expensive for a large number of dimensions.

If this example, A was a 2×2 matrix then,

$$|A - \lambda I| = 0$$

$$= \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} (a-\lambda) & (b-0) \\ (c-0) & (d-\lambda) \end{bmatrix} \right|$$

$$= (a-\lambda)(d-\lambda) - (b)(c), \text{ the characteristic polynomial equation.}$$
$$= \lambda^2 - (a+d)\lambda + ad - bc,$$

The eigenvalues are the solutions to the characteristic polynomial equation. They can be solved for and then substituted into the original equation to find their eigenvectors.

Calculating Eigenvalues and Vectors Example.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, a vertical scaling transformation matrix.

- ① Express A as product of eigenvalues,

$$Ax = \lambda x$$

$$0 = (A - \lambda I)x$$

- ② Use the determinant calculation to formulate the characteristic polynomial, and solve for the eigenvalues.

$$\begin{aligned} |A - \lambda I| &= 0 \\ &= \left| \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| \\ &= \left| \begin{bmatrix} (1-\lambda) & (0-0) \\ (0-0) & (2-\lambda) \end{bmatrix} \right| \\ &= (1-\lambda)(2-\lambda) - (0)(0) \\ &= (1-\lambda)(2-\lambda) \end{aligned}$$

Therefore the roots of the characteristic polynomial are

$$\lambda = 1, 2.$$

- ③ Substitute the eigenvalues back into the original equation to solve for the eigenvectors,

$$\begin{aligned} @\lambda=1 : \quad & \begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} x = 0 \\ &= \begin{bmatrix} (1-1) & 0 \\ 0 & (2-1) \end{bmatrix} x \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 &= \begin{bmatrix} 0 \\ x_2 \end{bmatrix}, \text{ meaning the } x_2 \text{ value must be zero.} \\
 @\lambda=2: \quad &\begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}x = 0 & \text{Can't conclude anything about } x_1. \\
 &= \begin{bmatrix} (1-2) & 0 \\ 0 & 2-2 \end{bmatrix}x \\
 &= \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}x \\
 &= \begin{bmatrix} -x_1 \\ 0 \end{bmatrix}, \text{ meaning the } x_1 \text{ value must be zero.} \\
 && \text{Can't conclude anything about } x_2.
 \end{aligned}$$

Therefore the eigenvectors are,

$$@\lambda=1: x = \begin{bmatrix} t \\ 0 \end{bmatrix}, @\lambda=2: x = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Any vector complying with these definitions will only scale as they will be of a form where their transformations by A will only scale the original vector by it's appropriate eigenvalue.

Calculating Eigenvectors & Values for non-geometric Eq.

A rotation does not have FR (real) valued eigenvalues.

Ex.

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \left\{ \begin{bmatrix} (0-\lambda) & -1 \\ 1 & 0-\lambda \end{bmatrix} \right\} = 0 \\
 &= \lambda^2 + 1, \text{ which is only solvable using } \mathbb{C} \text{ (complex) numbers.}
 \end{aligned}$$

(*) Using this approach of the determinant, the characteristic polynomial and solving for the roots of the polynomial is asymptotically expensive — the more dimensions make this task computationally unfeasible.

⇒ This can become analytically untractable.

↳ where you are forced to use numerical methods instead.