



GRUNDLAGEN DER ELEKTROTECHNIK II

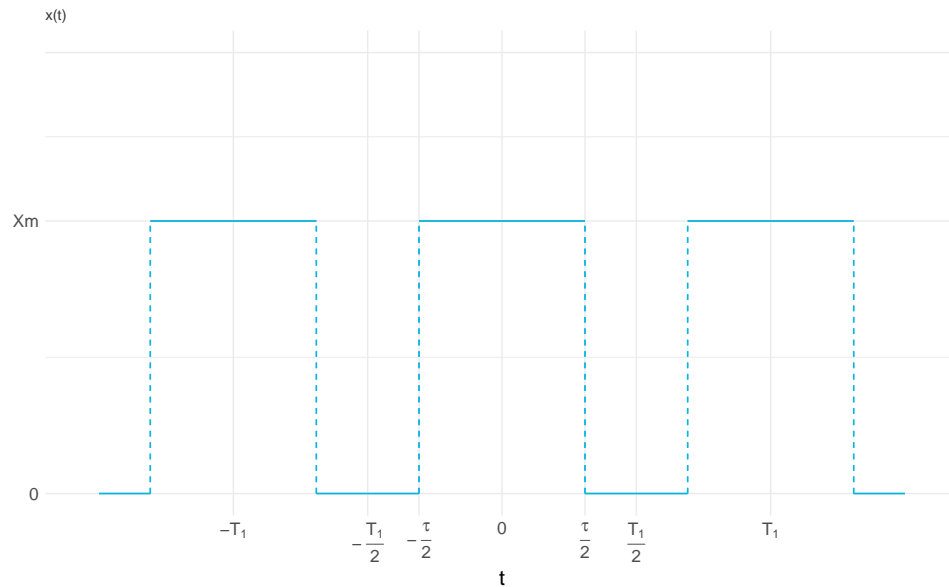
# Frequenzanalyse periodischer Signale

Studien- und Versuchsaufgaben

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23.5.2019

# 1 Vorbereitungsaufgaben

## 1.1



$$\begin{aligned}
 \underline{X}_\nu &= \frac{1}{T_1} \cdot \int_{T_1} x(t) \cdot e^{-j\nu \cdot \omega_1 t} dt \\
 &= \frac{1}{T_1} \cdot \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} X_m \cdot e^{-j\nu \cdot \omega_1 t} dt \\
 &= -\frac{X_m}{T_1 \cdot j\nu \omega_1} \cdot \left[ e^{-j\nu \cdot \omega_1 t} \right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\
 &= -\frac{X_m}{T_1 \cdot j\nu \omega_1} \cdot \left( e^{-j\nu \cdot \omega_1 \frac{\tau}{2}} - e^{j\nu \cdot \omega_1 \frac{\tau}{2}} \right)
 \end{aligned}$$

$\omega_1 = \frac{2\pi}{T_1}$  und Erweiterung mit  $\frac{-1}{-1}$ :

$$\begin{aligned}
 \underline{X}_\nu &= \frac{X_m}{2j\pi\nu} \cdot \left( e^{j\nu \cdot \pi \frac{\tau}{T_1}} - e^{-j\nu \cdot \pi \frac{\tau}{T_1}} \right) \\
 &= \frac{X_m}{\pi\nu} \cdot \frac{\left( e^{j\nu \cdot \pi \frac{\tau}{T_1}} - e^{-j\nu \cdot \pi \frac{\tau}{T_1}} \right)}{2j}
 \end{aligned}$$

mit  $\frac{(e^{jx} - e^{-jx})}{2j} = \sin(x)$  und  $\frac{\tau}{T_1} = D$ :

$$\underline{X}_\nu = \frac{X_m}{\pi\nu} \cdot \sin(\pi\nu D)$$

Erweitert man wieder mit  $\frac{D}{D}$  erhält man das Bild einer Spaltfunktion  $\text{si}(x) = \frac{\sin x}{x}$ :

$$\underline{X}_\nu = D \cdot X_m \cdot \frac{\sin(\pi\nu D)}{\pi\nu D} = D \cdot X_m \cdot \text{si}(\pi\nu D)$$

Als reelle Reihe:

$$x(t) = X_0 + \sum_{\nu=1}^{\infty} \hat{X}_\nu \cos(\nu \cdot \omega_1 t + \phi_\nu)$$

$$X_0 = \frac{1}{T_1} \cdot \int_{T_1} x(t) dt = \frac{X_m}{2}$$

Aus der komplexen Reihendarstellung folgt

$$b_\nu = -2 \cdot \text{Im}(\underline{X}_\nu) = 0$$

$$\hat{X}_\nu = \sqrt{a_\nu^2 + b_\nu^2} = 2 \cdot |\underline{X}_\nu| \implies a_\nu = 2 \cdot |D \cdot X_m \cdot \text{si}(\nu\pi D)|$$

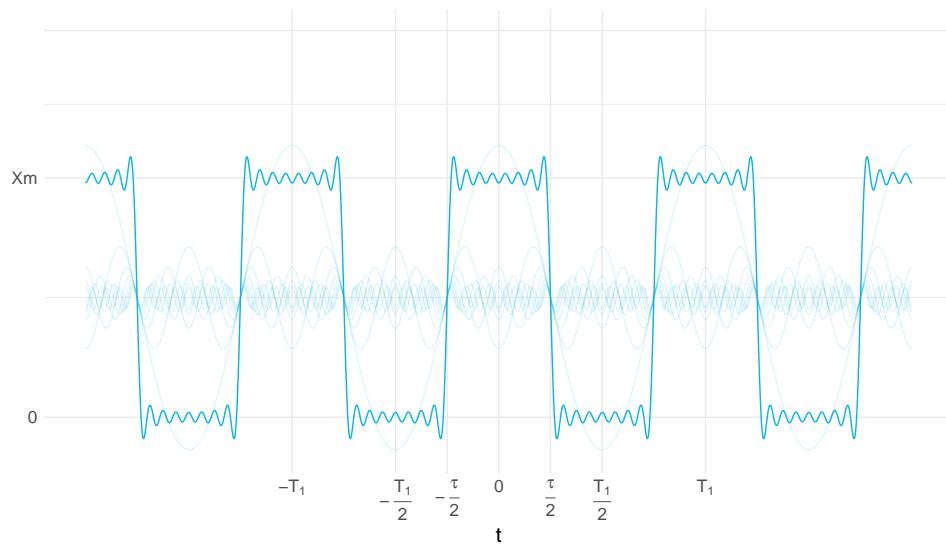
$\phi_\nu$  hängt nur vom Wert von  $\text{si}(\nu\pi D)$  ab, da  $\underline{X}_\nu$  rein reell ist:

$$\phi_\nu = \begin{cases} 0 & ; \nu = \frac{4k+1}{2D} \\ \pi & ; \nu = \frac{4k-1}{2D} \\ \text{n.d.} & ; \text{sonst} \end{cases}$$

Somit ist

$$x(t) = \frac{X_m}{2} + \sum_{\nu=1}^{\infty} 2DX_m \cdot |\operatorname{si}(\pi\nu D)| \cdot \cos\left(\nu \cdot \frac{2\pi}{T_1} \cdot t + \phi_\nu\right)$$

Reihenentwicklung von  $x(t)$  bis zur 16. Oberwelle  
 $D=0.5$



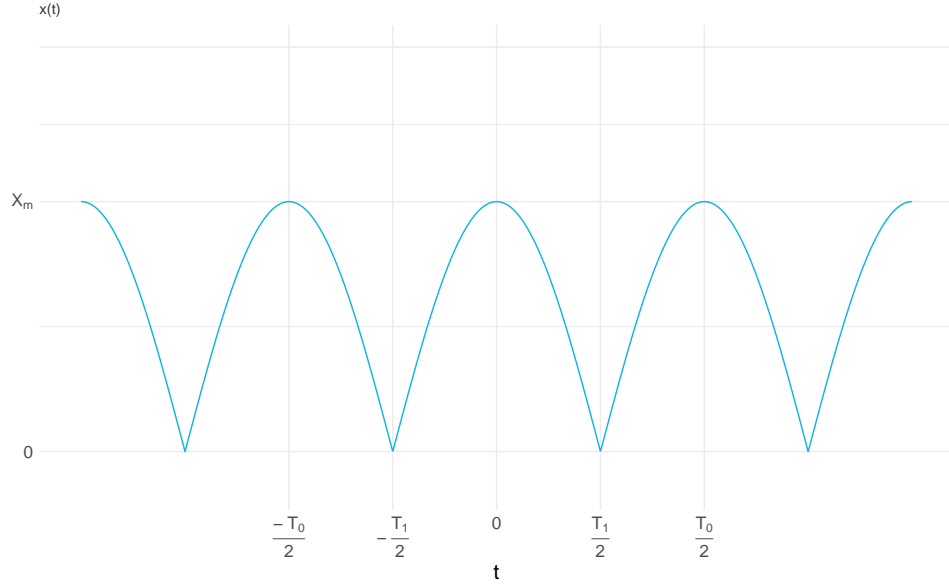
Effektivwert:

$$X_{\text{eff}} = \sqrt{\frac{1}{T_1} \cdot \int_{T_1} x^2(t) dt} = \sqrt{\frac{X_m^2}{T_1} \cdot \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} 1 dt}$$

$$X_{\text{eff}} = X_m \cdot \sqrt{\frac{\tau}{T_1}} = X_m \cdot \sqrt{D}$$

$D = \frac{1}{2}, X_{\text{eff}} = \frac{X_m}{\sqrt{2}}$				$D = \frac{1}{4}, X_{\text{eff}} = \frac{X_m}{2}$			$D = \frac{1}{8}, X_{\text{eff}} = \frac{X_m}{\sqrt{8}}$		
$\nu$	$\underline{X}_\nu$	$\hat{X}_\nu$	$\phi_\nu$	$\nu$	$\hat{X}_\nu$	$\phi_\nu$	$\nu$	$\hat{X}_\nu$	$\phi_\nu$
1	$\frac{1}{\pi}X_m$	$\frac{2}{\pi}X_m$	0	1	$\frac{\sqrt{2}}{\pi}X_m$	0	1	$\frac{\sqrt{2-\sqrt{2}}}{\pi}X_m$	0
2	—	—	—	2	$\frac{1}{\pi}X_m$	0	2	$\frac{\sqrt{2}}{2\pi}X_m$	0
3	$-\frac{1}{3\pi}X_m$	$\frac{2}{3\pi}X_m$	$\pi$	3	$\frac{\sqrt{2}}{3\pi}X_m$	0	3	$\frac{\sqrt{2+\sqrt{2}}}{3\pi}X_m$	0
4	—	—	—	4	—	—	4	$\frac{1}{2}X_m$	0
5	$\frac{1}{5\pi}X_m$	$\frac{2}{5\pi}X_m$	0	5	$\frac{\sqrt{2}}{5\pi}X_m$	$\pi$	5	$\frac{\sqrt{2+\sqrt{2}}}{5\pi}X_m$	0
6	—	—	—	6	$\frac{1}{3\pi}X_m$	$\pi$	6	$\frac{\sqrt{2}}{16\pi}X_m$	0
7	$-\frac{1}{7\pi}X_m$	$\frac{2}{7\pi}X_m$	$\pi$	7	$\frac{\sqrt{2}}{7\pi}X_m$	$\pi$	7	$\frac{\sqrt{2-\sqrt{2}}}{7\pi}X_m$	0
8	—	—	—	8	—	—	8	—	—
9	$\frac{1}{9\pi}X_m$	$\frac{2}{9\pi}X_m$	0	9	$\frac{\sqrt{2}}{9\pi}X_m$	0	9	$\frac{\sqrt{2-\sqrt{2}}}{9\pi}X_m$	$\pi$
10	—	—	—	10	$\frac{1}{5\pi}X_m$	0	10	$\frac{\sqrt{2}}{5\pi}X_m$	$\pi$
11	$-\frac{1}{11\pi}X_m$	$\frac{2}{11\pi}X_m$	$\pi$	11	$\frac{\sqrt{2}}{11\pi}X_m$	0	11	$\frac{\sqrt{2+\sqrt{2}}}{11\pi}X_m$	$\pi$
12	—	—	—	12	—	—	12	$\frac{1}{6\pi}X_m$	$\pi$
13	$\frac{1}{13\pi}X_m$	$\frac{2}{13\pi}X_m$	0	13	$\frac{\sqrt{2}}{13\pi}X_m$	$\pi$	13	$\frac{\sqrt{2+\sqrt{2}}}{13\pi}X_m$	$\pi$
14	—	—	—	14	$\frac{1}{7\pi}X_m$	$\pi$	14	$\frac{\sqrt{2}}{7\pi}X_m$	$\pi$
15	$-\frac{1}{15\pi}X_m$	$\frac{2}{15\pi}X_m$	$\pi$	15	$\frac{\sqrt{2}}{15\pi}X_m$	$\pi$	15	$\frac{\sqrt{2-\sqrt{2}}}{15\pi}X_m$	$\pi$
16	—	—	—	16	—	—	16	—	—

## 1.2



$$\begin{aligned}
 \underline{X}_\nu &= \frac{1}{T_1} \cdot \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} X_m \cos\left(\frac{\pi}{T_1}t\right) \cdot e^{-j\nu\frac{\pi}{T_1}t} dt \\
 &= \frac{X_m}{T_1} \cdot \left[ \frac{e^{-j\nu\frac{\pi}{T_1}t}}{(-j\nu\frac{\pi}{T_1})^2 + (\frac{\pi}{T_1})^2} \cdot \left( (-j\nu\frac{\pi}{T_1}) \cdot \cos\left(\frac{\pi}{T_1}t\right) + \frac{\pi}{T_1} \cdot \sin\left(\frac{\pi}{T_1}t\right) \right) \right]_{-\frac{T_1}{2}}^{\frac{T_1}{2}} \\
 &= \frac{X_m}{\pi(1-4\nu^2)} \cdot \left[ e^{-j\nu\pi} \cdot \underbrace{(-j2\nu \cdot \cos(\frac{\pi}{2}) + 1)}_0 - e^{j\nu\pi} \cdot \underbrace{(-j2\nu \cdot \cos(-\frac{\pi}{2}) - 1)}_0 \right] \\
 &= \frac{X_m}{\pi(1-4\nu^2)} \cdot (e^{-j\nu\pi} + e^{j\nu\pi}) = 2 \frac{X_m}{\pi(1-4\nu^2)} \cdot \left( \frac{e^{j\nu\pi} + e^{-j\nu\pi}}{2} \right)
 \end{aligned}$$

$$\underline{X}_\nu = \frac{2X_m}{\pi(1-4\nu^2)} \cdot \cos(\nu\pi)$$

$$\hat{X}_\nu = 2 \cdot |\underline{X}_\nu| = \frac{4 \cdot X_m}{\pi(1-4\nu^2)} \cdot \cos(\nu\pi)$$

Mittelwert:

$$X_0 = \frac{1}{T_1} \cdot \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} X_m \cdot \cos\left(\frac{\pi}{T_1}t\right) dt = \frac{X_m}{\pi} \cdot \left[ \sin\left(\frac{\pi}{T_1}t\right) \right]_{-\frac{T_1}{2}}^{\frac{T_1}{2}}$$

$$X_0 = \frac{2X_m}{\pi}$$

Effektivwert:

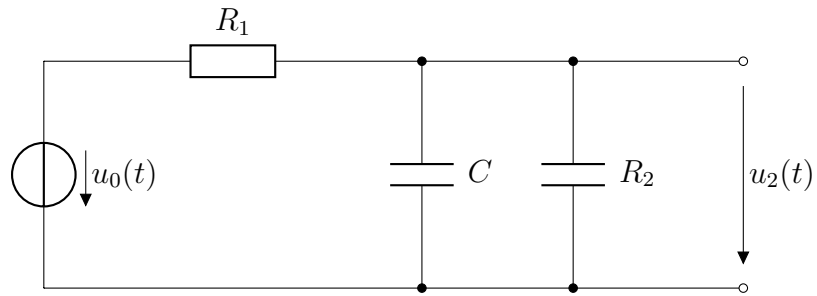
$$X_{\text{eff}} = \sqrt{\frac{X_m^2}{T_1} \cdot \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} \left(\cos\left(\frac{\pi}{T_1}t\right)\right)^2 dt}$$

$$X_{\text{eff}} = \sqrt{\frac{X_m^2}{T_1} \cdot \left[ \frac{t}{2} + \frac{\sin\left(\frac{2\pi}{T_1}t\right)}{4 \cdot \frac{\pi}{T_1}} \right]_{-\frac{T_1}{2}}^{\frac{T_1}{2}}}$$

$$X_{\text{eff}} = \frac{X_m}{\sqrt{2}}$$

$D = \frac{1}{4}, X_{\text{eff}} = \frac{X_m}{2}$		
$\nu$	$\hat{X}_\nu$	$\phi_\nu$
1	$\frac{4}{3\pi}X_m$	0
2	$\frac{4}{15\pi}X_m$	$\pi$
3	$\frac{4}{35\pi}X_m$	0
4	$\frac{4}{63\pi}X_m$	$\pi$
5	$\frac{4}{99\pi}X_m$	0
6	$\frac{4}{143\pi}X_m$	$\pi$
7	$\frac{4}{195\pi}X_m$	0
8	$\frac{4}{255\pi}X_m$	$\pi$
9	$\frac{4}{323\pi}X_m$	0
10	$\frac{4}{399\pi}X_m$	$\pi$
11	$\frac{4}{483\pi}X_m$	0
12	$\frac{4}{575\pi}X_m$	$\pi$
13	$\frac{4}{675\pi}X_m$	0
14	$\frac{4}{783\pi}X_m$	$\pi$
15	$\frac{4}{899\pi}X_m$	0

### 1.3





$$\underline{U}_2 = \underline{U}_0 \cdot \frac{\frac{1}{j\omega C + \frac{1}{R_2}}}{R_1 + \frac{1}{j\omega C + \frac{1}{R_2}}} = \frac{\underline{U}_0 \cdot R_2}{R_1 + R_2 + j\omega C R_1 R_2}$$

Betrag:

$$\hat{U}_2 = \frac{\hat{U}_0 \cdot R_2}{\sqrt{(R_1 + R_2)^2 + (\omega C R_1 R_2)^2}}$$

Phase:

$$\phi_{\underline{U}_2} = \phi_{\underline{U}_0} - \arctan \frac{\omega C R_1 R_2}{R_1 + R_2}$$

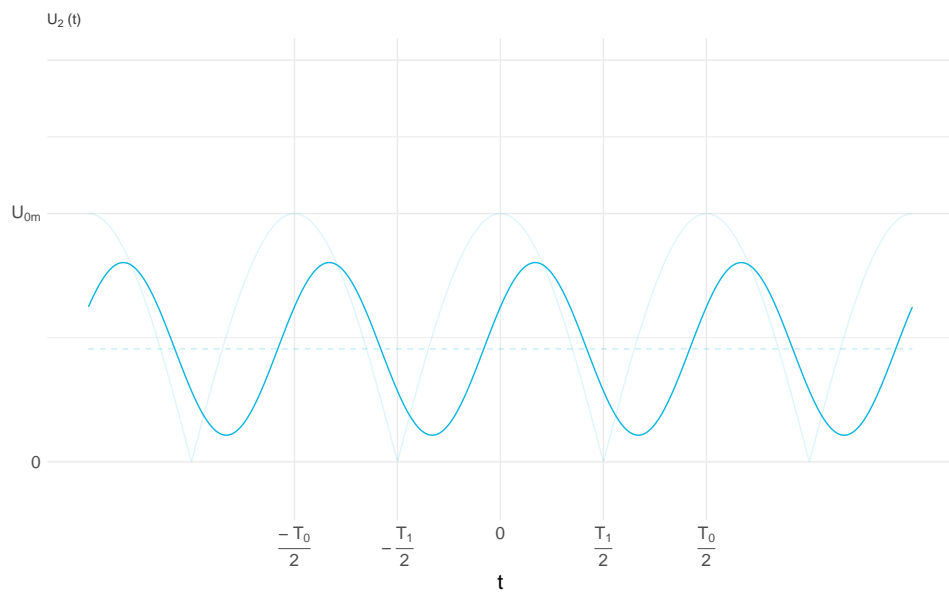
Mittelwert:

$$\underline{U}_{0_\nu}(\nu = 0) = \hat{U}_{0_\nu}(\nu = 0) \cdot \frac{R_2}{R_1 + R_2} = \frac{2 \cdot 1\text{V} \cdot 1\text{k}\Omega}{\pi \cdot (1\text{k}\Omega + 400\text{k}\Omega)} \approx 0.455 \text{ V}$$

$\nu$	$\underline{U}_{0\nu}/V$	$\hat{U}_{0\nu}/V$	$\phi$	$\hat{U}_{2\nu}/V$	$\phi_{U_{2\nu}}/^\circ$	$U_{2\text{eff}}/V$
1	0.2122	-0.4244	0		-60.88	
2	-0.4244	0.08488	$\pi$		105.56	
3	0.0182	0.03638	0		-79.48	
4	-0.0101	0.02021	$\pi$		97.93	
5	0.0064	0.01286	0		-83.64	
6	-0.0045	0.00890	$\pi$		95.3	
7	0.0033	0.00652	0		-85.45	
8	-0.0025	0.00499	$\pi$		93.19	
9	0.0020	0.00394	0		-86.45	
10	-0.0016	0.00319	$\pi$		93.19	
11	0.0013	0.00264	0		-87.1	
12	-0.0011	0.00221	$\pi$		92.28	
13	-0.0009	0.00189	0		-87.55	
14	-0.0008	0.00163	$\pi$		92.28	
15	0.0007	0.00142	0		-87.87	
16	-0.0006	0.0012	$\pi$		91.99	

Effektivwert aus Summe der einzelnen Effektivwerte:

$$U_{2\text{eff}} = \sqrt{U_{2\nu}^2 + \sum_{\nu=1}^{\infty} U_{2\text{eff}\nu}^2} \approx 0.4667 \text{ V}$$



## 2 Versuchsaufgaben

### 2.1

Messwerte der Aufgabe 3.1 für  $T = 10 \text{ } \mu\text{s}$ :

$D = \frac{1}{2}, X_{\text{eff}} = ?$			
$\nu$	$\hat{X}_\nu/\text{mV}$	$X_{\nu_{\text{eff}}}/\text{mV}$	$f_\nu/\text{kHz}$
1	608.112	430	100
2	-	-	-
3	208.045	147.11	300
4	-	-	-
5	123.546	87.36	500
6	-	-	-
7	91.040	64.375	700
8	-	-	-
9	68.059	48.125	900
10	-	-	-
11	57.544	40.69	1100
12	-	-	-
13	48.691	34.43	1300
14	-	-	-
15	41.606	29.42	1500

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$$D = \frac{1}{4}, X_{\text{eff}} = 500 \text{ mV}$$


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$\nu$	$\hat{X}_\nu/\text{mV}$	$X_{\nu_{\text{eff}}}/\text{mV}$	$f_\nu/\text{kHz}$
1	445.477	315	100
2	311.070	219.96	200
3	143.401	101.4	300
4	-	-	-
5	88.388	62.5	500
6	105.217	74.4	600
7	64.488	45.6	700
8	-	-	-
9	48.225	34.1	900
10	60.670	42.9	1000
11	38.891	27.5	1100
12	-	-	-
13	34.083	24.1	1300
14	45.255	32	1400
15	30.123	21.3	1500

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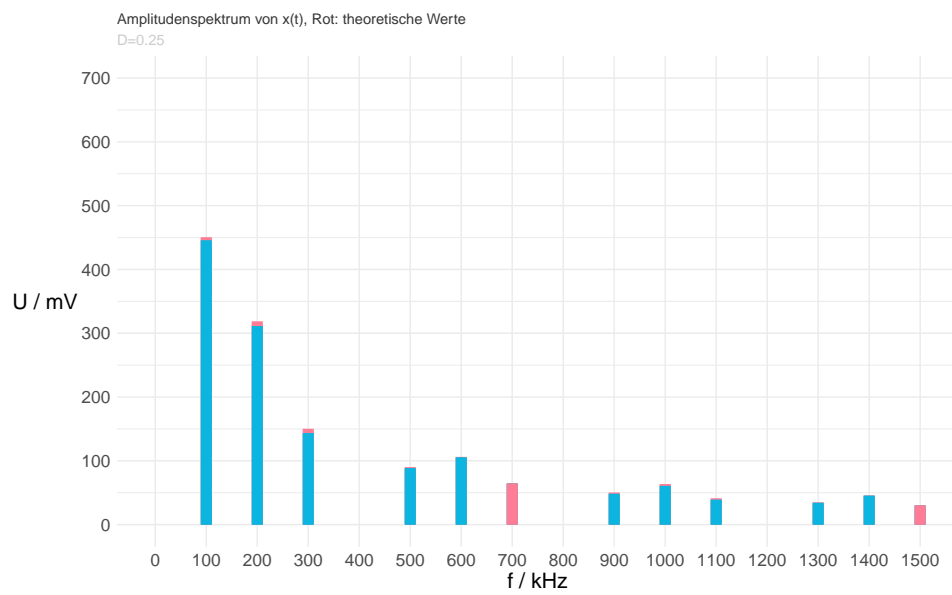
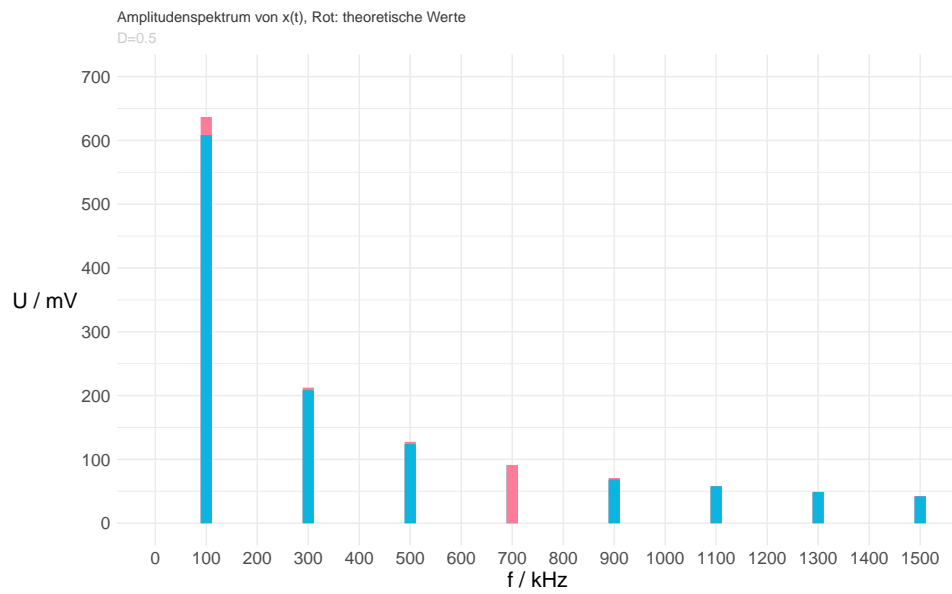
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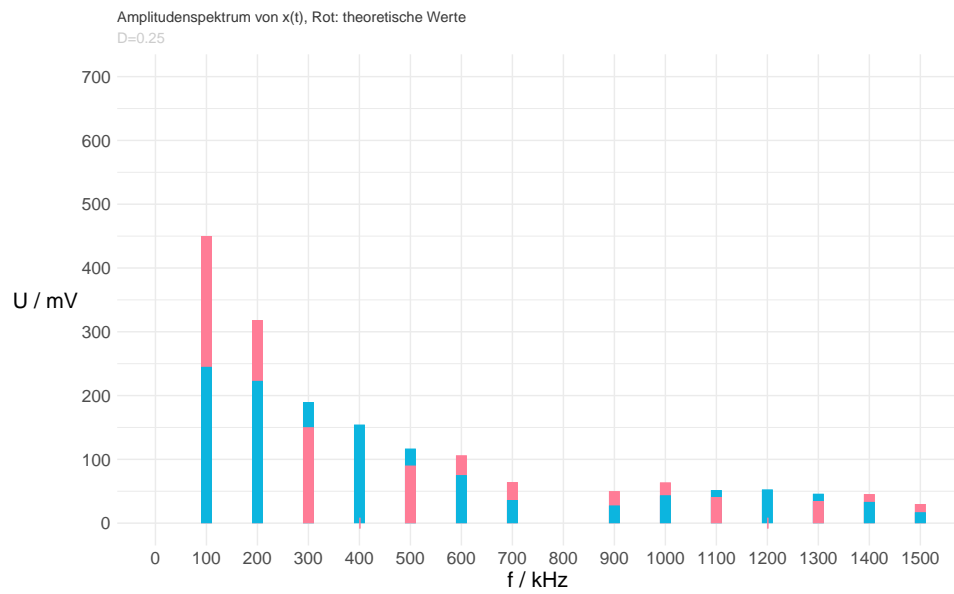

$$D = \frac{1}{8}, X_{\text{eff}} = 343 \text{ mV}$$


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$\nu$	$\hat{X}_\nu/\text{mV}$	$X_{\nu_{\text{eff}}}/\text{mV}$	$f_\nu/\text{kHz}$
1	244.942	173.2	100
2	222.965	157.66	200
3	189.787	134.2	300
4	154.291	109.1	400
5	116.673	82.5	500
6	75.095	53.1	600
7	36.204	25.6	700
8	-	-	-
9	27.436	19.4	900
10	43.416	30.7	1000
11	51.760	36.6	1100
12	52.609	37.2	1200
13	45.962	32.5	1300
14	33.234	23.5	1400
15	16.829	11.9	1500

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Messwerte der Aufgabe 3.1 für  $T = 1 \text{ } \mu\text{s}$ :



$D = \frac{1}{2}, X_{\text{eff}} = 692 \text{ mV}$				$D = \frac{1}{4}, X_{\text{eff}} = 489 \text{ mV}$			
$\nu$	$\hat{X}_\nu/\text{mV}$	$X_{\nu_{\text{eff}}}/\text{mV}$	$f_\nu/\text{MHz}$	$\nu$	$\hat{X}_\nu/\text{mV}$	$X_{\nu_{\text{eff}}}/\text{mV}$	$f_\nu/\text{MHz}$
1	627.911	444	1	1	445.477	315	1
2	-	-	-	2	308.864	218.4	2
3	203.647	144	3	3	143.401	101.4	3
4	-	-	-	4	-	-	-
5	119.077	84.2	5	5	87.540	61.9	5
6	-	-	-	6	103.379	73.1	6
7	85.984	60.8	7	7	106.915	75.6	7
8	-	-	-	8	-	-	-
9	63.922	45.2	9	9	47.800	33.8	9
10	-	-	-	10	59.256	41.9	10
11	53.599	37.9	11	11	38.467	27.2	11
12	-	-	-	12	-	-	-
13	46.528	32.9	13	13	32.244	22.8	13
14	-	-	-	14	43.416	30.7	14
15	40.305	28.5	15	15	29.274	20.7	15

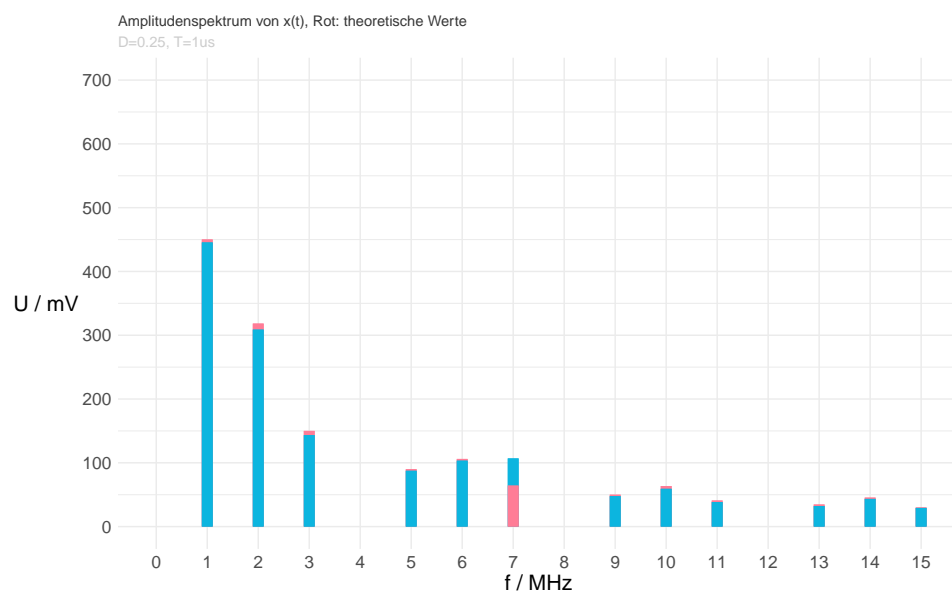
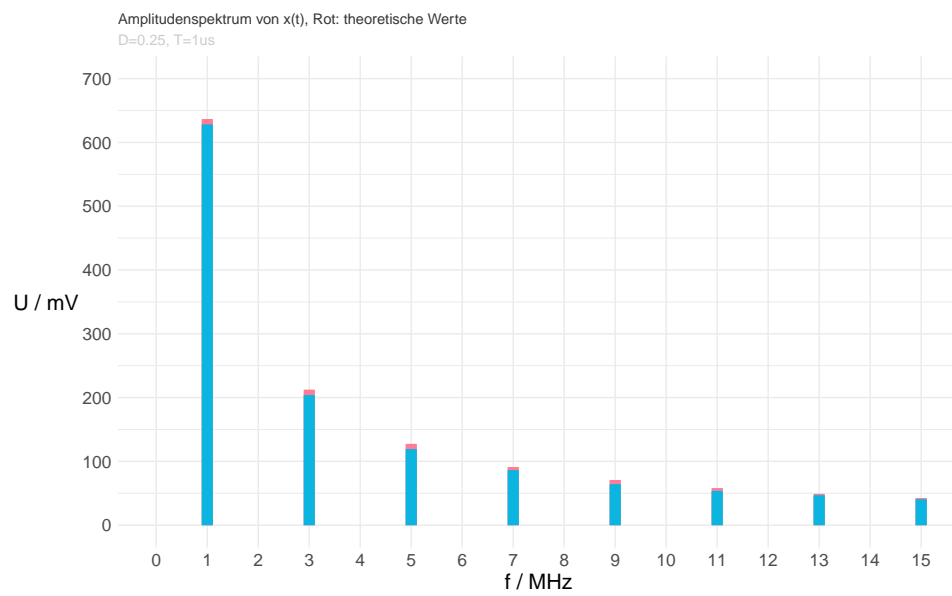
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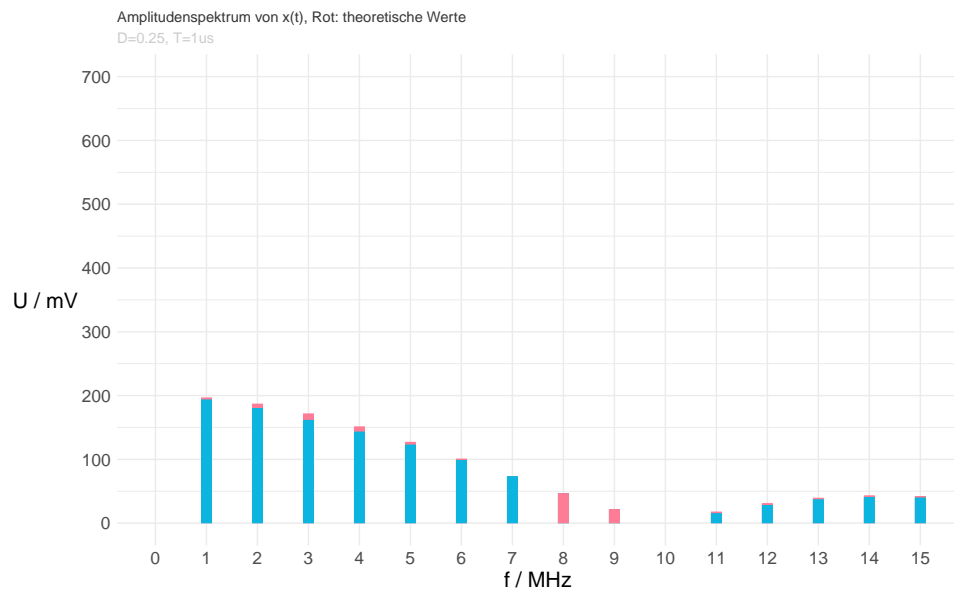

$$D = \frac{1}{10}, X_{\text{eff}} = 305 \text{ mV}$$


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$\nu$	$\hat{X}_\nu/\text{mV}$	$X_{\nu_{\text{eff}}}/\text{mV}$	$f_\nu/\text{MHz}$
1	193.606	136.9	1
2	180.312	127.5	2
3	161.786	114.4	3
4	143.260	101.3	4
5	122.895	86.9	5
6	98.995	70	6
7	73.398	51.9	7
8	46.810	33.1	8
9	22.062	15.6	9
10	-	-	-
11	15.415	10.9	11
12	28.426	20.1	12
13	37.052	26.2	13
14	40.871	28.9	14
15	39.881	28.2	15

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## 2.2

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$X_{\text{eff}} = 698 \text{ mV}, X_0 = 627 \text{ mV}$			
$\nu$	$\hat{X}_\nu/\text{mV}$	$X_{\nu\text{eff}}/\text{mV}$	$f_\nu/\text{kHz}$
1	406.021	287.1	1
2	85.843	60.7	2
3	39.881	28.2	3
4	23.617	16.7	4
5	17.395	12.3	5
6	13.011	9.2	6
7	9.475	6.7	7
8	8.202	5.8	8
9	6.930	4.9	9
10	5.940	4.2	10
11	4.808	3.4	11
12	4.384	3.1	12
13	4.243	3	13
14	3.536	2.5	14
15	3.253	2.3	15
16	2.828	2	16

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Messwerte der Aufgabe 3.2

## 2.3