Grundlagen der Elektrotechnik II

Frequenzanalyse periodischer Signale

Studien- und Versuchsaufgaben

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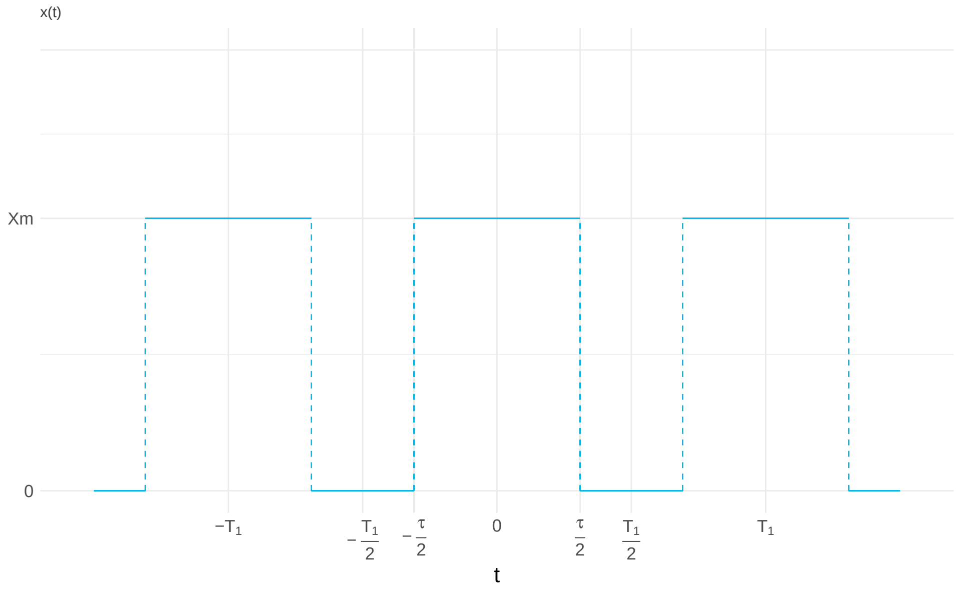
1 Vorbereitungsaufgaben

1.1



$$\underline{X}_\nu = \frac{1}{T_1} \cdot \int_{T_1} x(t) \cdot e^{-(j\nu \cdot \omega_1 t)}\,\mathrm{d}t$$

$$= \frac{1}{T_1} \cdot \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} X_m \cdot e^{-(j\nu \cdot \omega_1 t)}\,\mathrm{d}t$$

$$= -\frac{X_m}{T_1 \cdot j\nu\omega_1} \cdot \left[e^{-(j\nu \cdot \omega_1 t)}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

$$= -\frac{X_m}{T_1 \cdot j\nu\omega_1} \cdot \left(e^{-j\nu \cdot \omega_1 \frac{\tau}{2}} - e^{j\nu \cdot \omega_1 \frac{\tau}{2}}\right)$$

$\omega_1 = \frac{2\pi}{T_1}$ und Erweiterung mit $\frac{-1}{-1}$:

$$\underline{X}_\nu = \frac{X_m}{2j\pi\nu} \cdot \left(e^{j\nu \cdot \pi \frac{\tau}{T_1}} - e^{-j\nu \cdot \pi \frac{\tau}{T_1}}\right)$$

$$= \frac{X_m}{\pi\nu} \cdot \frac{\left(e^{j\nu \cdot \pi \frac{\tau}{T_1}} - e^{-j\nu \cdot \pi \frac{\tau}{T_1}}\right)}{2j}$$

mit $\dfrac{\left(e^{jx} - e^{-jx}\right)}{2j} = \sin(x)$ und $\dfrac{\tau}{T_1} = D$:

$$\underline{X}_\nu = \frac{X_m}{\pi\nu} \cdot \sin(\pi\nu D)$$

Erweitert man wieder mit $\dfrac{D}{D}$ erhält man das Bild einer Spaltfunktion $\mathrm{si}(x) = \dfrac{\sin x}{x}$:

$$\underline{X}_\nu = D \cdot X_m \cdot \frac{\sin(\pi\nu D)}{\pi\nu D} = D \cdot X_m \cdot \mathrm{si}(\pi\nu D)$$

Als reele Reihe:

$$x(t) = X_0 + \sum_{\nu=1}^{\infty} \hat{X}_\nu \cos(\nu \cdot \omega_1 t + \phi_\nu)$$

$$X_0 = \frac{1}{T_1} \cdot \int_{T_1} x(t)\,\mathrm{d}t = \frac{X_m}{2}$$

Aus der komplexen Reihendarstellung folgt

$$b_\nu = -2 \cdot \mathrm{Im}(\underline{X}_\nu) = 0$$

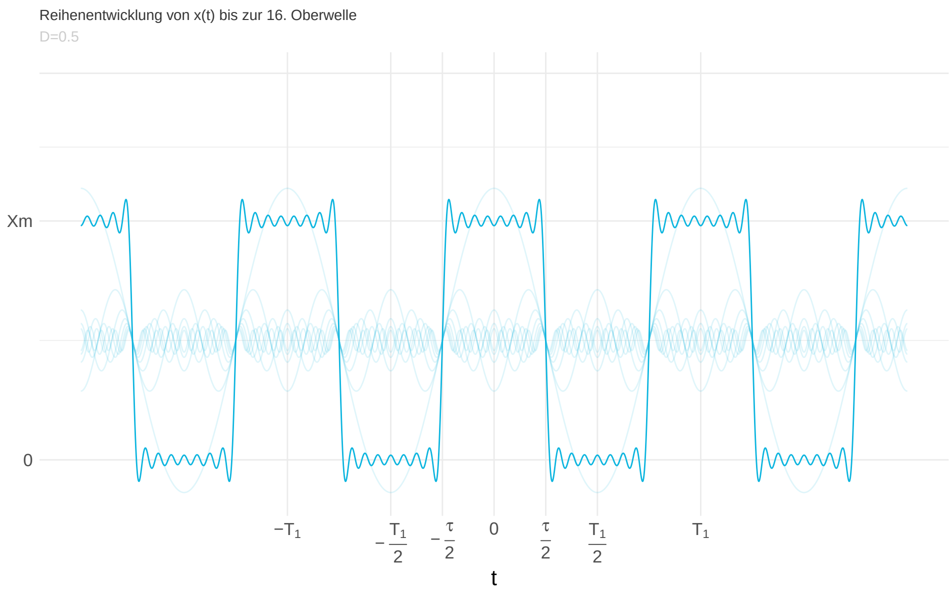
$$\hat{X}_\nu = \sqrt{a_\nu^2 + b_\nu^2} = 2 \cdot \mid \underline{X}_\nu \mid \Longrightarrow \; a_\nu = 2 \cdot \mid D \cdot X_m \cdot \mathrm{si}(\nu\pi D) \mid$$

ϕ_ν hängt nur vom Wert von $\mathrm{si}(\nu\pi D)$ ab, da $\underline{X}_\nu$ rein reell ist:

$$\phi_\nu = \begin{cases} 0 & ; \nu = \dfrac{4k+1}{2D} \\ \pi & ; \nu = \dfrac{4k-1}{2D} \\ \text{n.d.} & ; \text{sonst} \end{cases}$$

Somit ist

$$x(t) = \frac{X_m}{2} + \sum_{\nu=1}^{\infty} 2DX_m \cdot \mid \mathrm{si}(\pi\nu D) \mid \cdot \cos(\nu \cdot \frac{2\pi}{T_1} \cdot t + \phi_\nu)$$



Effektivwert:

$$X_{\text{eff}} = \sqrt{\frac{1}{T_1} \cdot \int_{T_1} x^2(t)\,\mathrm{d}t} = \sqrt{\frac{X_m^2}{T_1} \cdot \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} 1\,\mathrm{d}t}$$

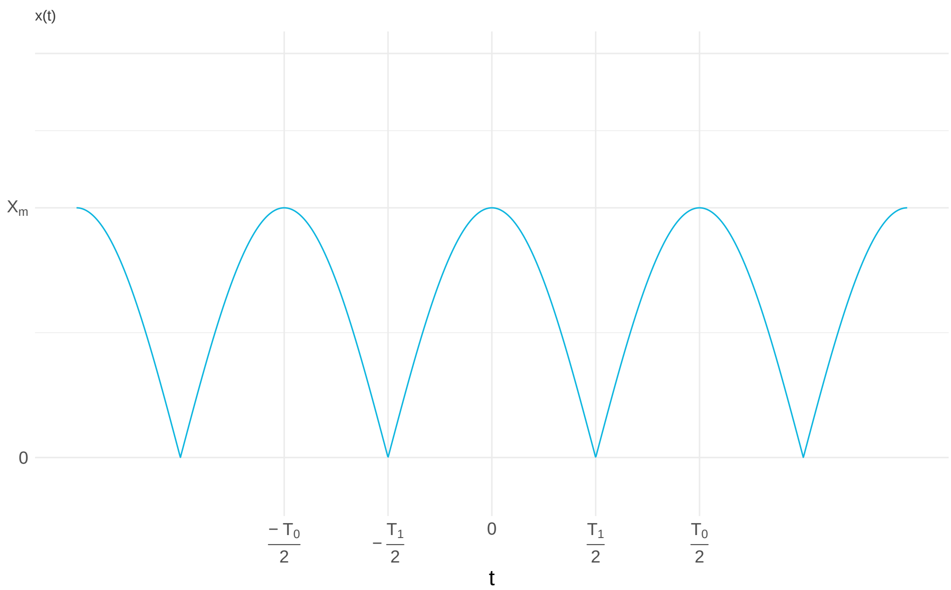
$$X_{\text{eff}} = X_m \cdot \sqrt{\frac{\tau}{T_1}} = X_m \cdot \sqrt{D}$$

$D = \frac{1}{2}, X_{\text{eff}} = \frac{X_m}{\sqrt{2}}$			
ν	$\underline{X}_\nu$	$\hat{X}_\nu$	ϕ_ν
1	$\frac{1}{\pi}X_m$	$\frac{2}{\pi}X_m$	0
2	–	–	–
3	$-\frac{1}{3\pi}X_m$	$\frac{2}{3\pi}X_m$	π
4	–	–	–
5	$\frac{1}{5\pi}X_m$	$\frac{2}{5\pi}X_m$	0
6	–	–	–
7	$-\frac{1}{7\pi}X_m$	$\frac{2}{7\pi}X_m$	π
8	–	–	–
9	$\frac{1}{9\pi}X_m$	$\frac{2}{9\pi}X_m$	0
10	–	–	–
11	$-\frac{1}{11\pi}X_m$	$\frac{2}{11\pi}X_m$	π
12	–	–	–
13	$\frac{1}{13\pi}X_m$	$\frac{2}{13\pi}X_m$	0
14	–	–	–
15	$-\frac{1}{15\pi}X_m$	$\frac{2}{15\pi}X_m$	π
16	–	–	–

$D = \frac{1}{4}, X_{\text{eff}} = \frac{X_m}{2}$		
ν	$\hat{X}_\nu$	ϕ_ν
1	$\frac{\sqrt{2}}{\pi}X_m$	0
2	$\frac{1}{\pi}X_m$	0
3	$\frac{\sqrt{2}}{3\pi}X_m$	0
4	–	–
5	$\frac{\sqrt{2}}{5\pi}X_m$	π
6	$\frac{1}{3\pi}X_m$	π
7	$\frac{\sqrt{2}}{7\pi}X_m$	π
8	–	–
9	$\frac{\sqrt{2}}{9\pi}X_m$	0
10	$\frac{1}{5\pi}X_m$	0
11	$\frac{\sqrt{2}}{11\pi}X_m$	0
12	–	–
13	$\frac{\sqrt{2}}{13\pi}X_m$	π
14	$\frac{1}{7\pi}X_m$	π
15	$\frac{\sqrt{2}}{15\pi}X_m$	π
16	–	–

$D = \frac{1}{8}, X_{\text{eff}} = \frac{X_m}{\sqrt{8}}$		
ν	$\hat{X}_\nu$	ϕ_ν
1	$\frac{\sqrt{2-\sqrt{2}}}{\pi}X_m$	0
2	$\frac{\sqrt{2}}{2\pi}X_m$	0
3	$\frac{\sqrt{2+\sqrt{2}}}{3\pi}X_m$	0
4	$\frac{1}{2}X_m$	0
5	$\frac{\sqrt{2+\sqrt{2}}}{5\pi}X_m$	0
6	$\frac{\sqrt{2}}{16\pi}X_m$	0
7	$\frac{\sqrt{2-\sqrt{2}}}{7\pi}X_m$	0
8	–	–
9	$\frac{\sqrt{2-\sqrt{2}}}{9\pi}X_m$	π
10	$\frac{\sqrt{2}}{5\pi}X_m$	π
11	$\frac{\sqrt{2+\sqrt{2}}}{11\pi}X_m$	π
12	$\frac{1}{6\pi}X_m$	π
13	$\frac{\sqrt{2+\sqrt{2}}}{13\pi}X_m$	π
14	$\frac{\sqrt{2}}{7\pi}X_m$	π
15	$\frac{\sqrt{2-\sqrt{2}}}{15\pi}X_m$	π
16	–	–

1.2



$$\underline{X}_\nu = \frac{1}{T_1} \cdot \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} X_m \cos\left(\frac{\pi}{T_1}t\right) \cdot e^{-(j\nu\frac{\pi}{T_1}t)}\,\mathrm{d}t$$

$$= \frac{X_m}{T_1} \cdot \left[\frac{e^{-(j\nu\frac{\pi}{T_1}t)}}{(-j\nu\frac{\pi}{T_1})^2 + (\frac{\pi}{T_1})^2} \cdot \left((-j\nu\frac{\pi}{T_1}) \cdot \cos\left(\frac{\pi}{T_1}t\right) + \frac{\pi}{T_1} \cdot \sin\left(\frac{\pi}{T_1}t\right)\right)\right]_{-\frac{T_1}{2}}^{\frac{T_1}{2}}$$

$$= \frac{X_m}{\pi(1-4\nu^2)} \cdot \left[e^{-j\nu\pi} \cdot (\underbrace{-j2\nu \cdot \cos\left(\frac{\pi}{2}\right)}_{0} + 1) - e^{j\nu\pi} \cdot (\underbrace{-j2\nu \cdot \cos\left(-\frac{\pi}{2}\right)}_{0} - 1)\right]$$

$$= \frac{X_m}{\pi(1-4\nu^2)} \cdot (e^{-j\nu\pi} + e^{j\nu\pi}) = 2\frac{X_m}{\pi(1-4\nu^2)} \cdot \left(\frac{e^{j\nu\pi} + e^{-j\nu\pi}}{2}\right)$$

$$\underline{X}_\nu = \frac{2X_m}{\pi(1-4\nu^2)} \cdot \cos(\nu\pi)$$

$$\hat{X}_\nu = 2 \cdot \mid \underline{X}_\nu \mid = \frac{4 \cdot X_m}{\pi(1-4\nu^2)} \cdot \cos(\nu\pi)$$

Mittelwert:

$$X_0 = \frac{1}{T_1} \cdot \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} X_m \cdot \cos\left(\frac{\pi}{T_1}t\right) \mathrm{d}t = \frac{X_m}{\pi} \cdot \left[\sin\left(\frac{\pi}{T_1}t\right)\right]_{-\frac{T_1}{2}}^{\frac{T_1}{2}}$$

$$X_0 = \frac{2X_m}{\pi}$$

Effektivwert:

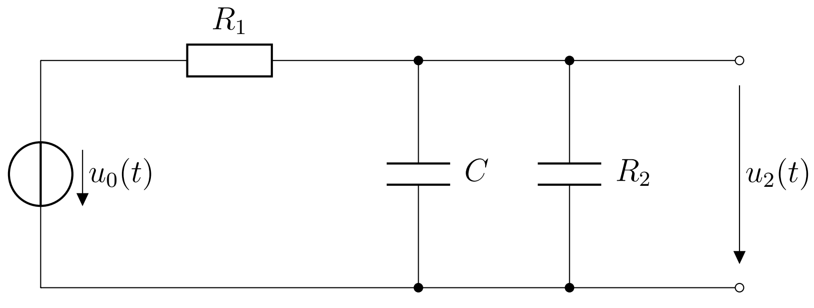
$$X_{\text{eff}} = \sqrt{\frac{X_m^2}{T_1} \cdot \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} \left(\cos\left(\frac{\pi}{T_1}t\right)\right)^2 \mathrm{d}t}$$

$$X_{\text{eff}} = \sqrt{\frac{X_m^2}{T_1} \cdot \left[\frac{t}{2} + \frac{\sin\left(\frac{2\pi}{T_1}\right)t}{4 \cdot \frac{\pi}{T_1}}\right]_{-\frac{T_1}{2}}^{\frac{T_1}{2}}}$$

$$X_{\text{eff}} = \frac{X_m}{\sqrt{2}}$$

$D = \frac{1}{4}, X_{\text{eff}} = \frac{X_m}{2}$		
ν	$\hat{X}_\nu$	ϕ_ν
1	$\frac{4}{3\pi} X_m$	0
2	$\frac{4}{15\pi} X_m$	π
3	$\frac{4}{35\pi} X_m$	0
4	$\frac{4}{63\pi} X_m$	π
5	$\frac{4}{99\pi} X_m$	0
6	$\frac{4}{143\pi} X_m$	π
7	$\frac{4}{195\pi} X_m$	0
8	$\frac{4}{255\pi} X_m$	π
9	$\frac{4}{323\pi} X_m$	0
10	$\frac{4}{399\pi} X_m$	π
11	$\frac{4}{483\pi} X_m$	0
12	$\frac{4}{575\pi} X_m$	π
13	$\frac{4}{675\pi} X_m$	0
14	$\frac{4}{783\pi} X_m$	π
15	$\frac{4}{899\pi} X_m$	0

1.3



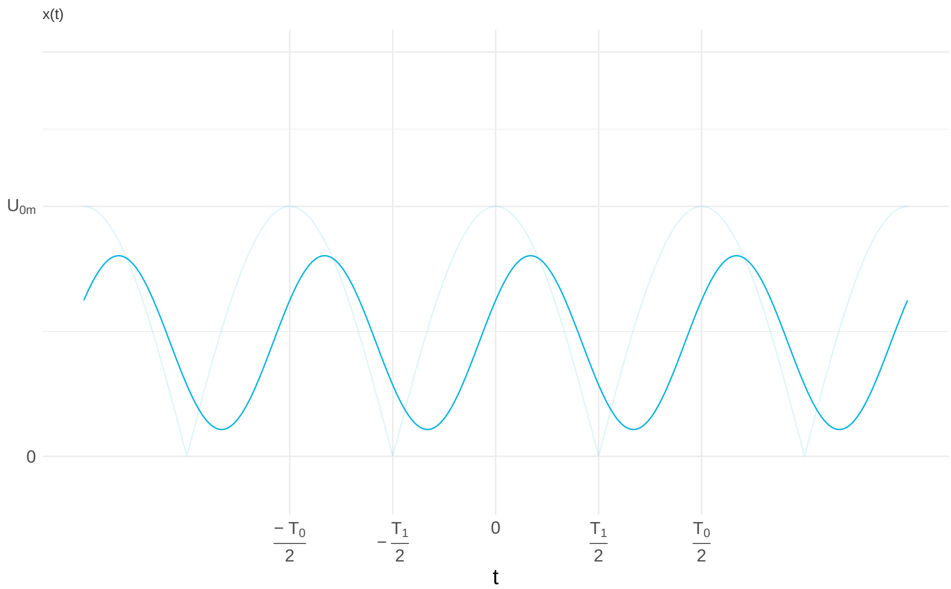
$$\underline{U}_2 = \underline{U}_0 \cdot \frac{\frac{1}{j\omega C + \frac{1}{R_2}}}{R_1 + \frac{1}{j\omega C + \frac{1}{R_2}}} = \frac{\underline{U}_0 \cdot R_2}{R_1 + R_2 + j\omega C R_1 R_2}$$

Betrag:

$$\hat{U}_2 = \frac{\hat{U}_0 \cdot R_2}{\sqrt{(R_1 + R_2)^2 + (\omega C R_1 R_2)^2}}$$

Phase:

$$\phi_{\underline{U}_2} = \phi_{\underline{U}_0} - \arctan \frac{\omega C R_1 R_2}{R_1 + R_2}$$



2 Versuchsaufgaben

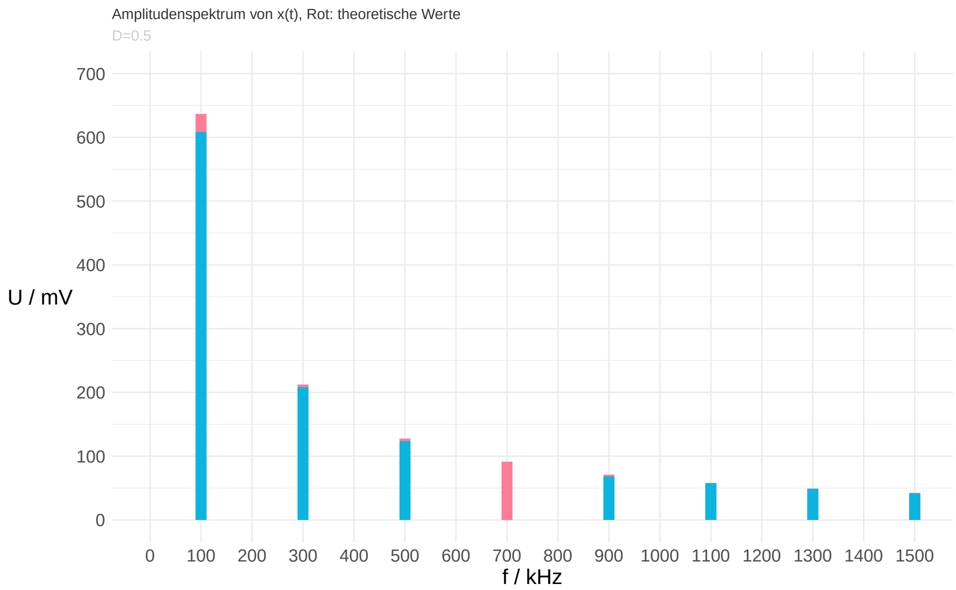
2.1

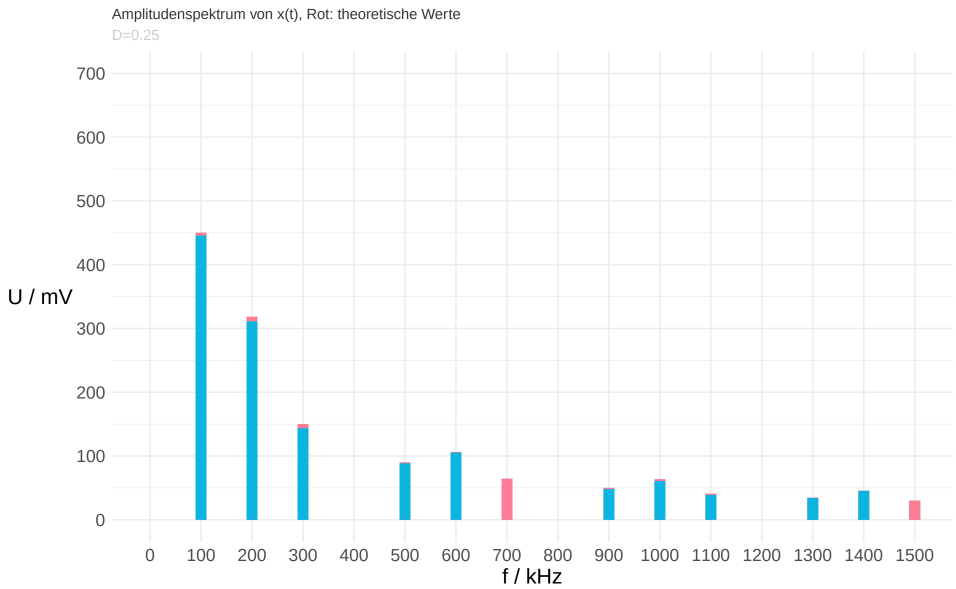
Messwerte der Aufgabe 3.1 für $T = 10$ µs:

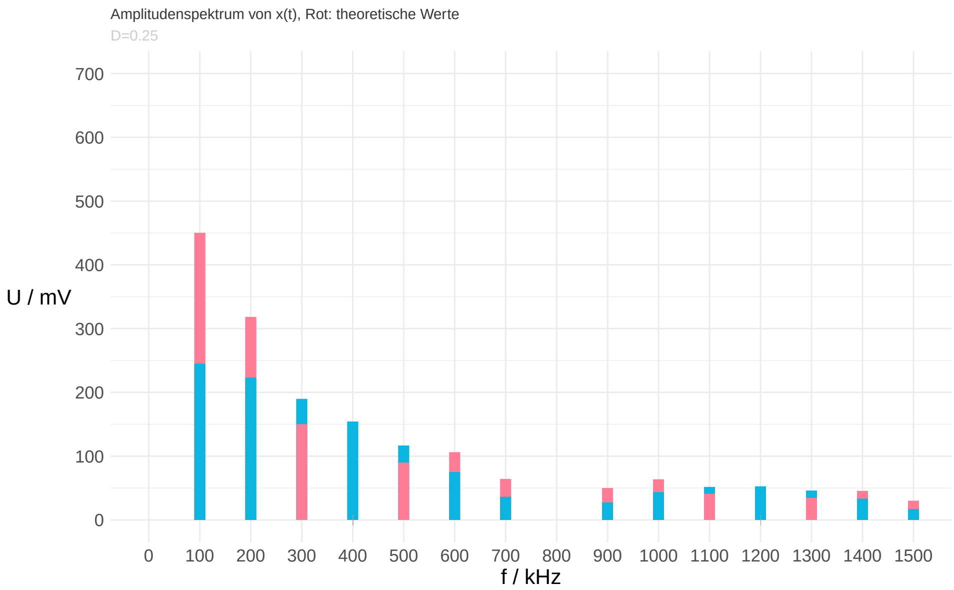
$D = \frac{1}{2}, X_{\text{eff}} =?$			
ν	$\hat{X}_\nu$/mV	$X_{\nu_{\text{eff}}}$/mV	f_ν/kHz
1	608.112	430	100
2	-	-	-
3	208.045	147.11	300
4	-	-	-
5	123.546	87.36	500
6	-	-	-
7	91.040	64.375	700
8	-	-	-
9	68.059	48.125	900
10	-	-	-
11	57.544	40.69	1100
12	-	-	-
13	48.691	34.43	1300
14	-	-	-
15	41.606	29.42	1500

$D = \frac{1}{4}, X_{\text{eff}} = 500 \text{ mV}$			
ν	$\hat{X}_\nu/\text{mV}$	$X_{\nu_{\text{eff}}}/\text{mV}$	f_ν/kHz
1	445.477	315	100
2	311.070	219.96	200
3	143.401	101.4	300
4	-	-	-
5	88.388	62.5	500
6	105.217	74.4	600
7	64.488	45.6	700
8	-	-	-
9	48.225	34.1	900
10	60.670	42.9	1000
11	38.891	27.5	1100
12	-	-	-
13	34.083	24.1	1300
14	45.255	32	1400
15	30.123	21.3	1500

$D = \frac{1}{8}, X_{\mathrm{eff}} = 343$ mV			
ν	$\hat{X}_\nu$/mV	$X_{\nu_{\mathrm{eff}}}$/mV	f_ν/kHz
1	244.942	173.2	100
2	222.965	157.66	200
3	189.787	134.2	300
4	154.291	109.1	400
5	116.673	82.5	500
6	75.095	53.1	600
7	36.204	25.6	700
8	-	-	-
9	27.436	19.4	900
10	43.416	30.7	1000
11	51.760	36.6	1100
12	52.609	37.2	1200
13	45.962	32.5	1300
14	33.234	23.5	1400
15	16.829	11.9	1500


Amplitudenspektrum von x(t), Rot: theoretische Werte
D=0.5
U / mV
700
600
500
400
300
200
100
0
0
100
200
300
400
500
600
700
800
900
1000
1100
1200
1300
1400
1500
f / kHz


Amplitudenspektrum von x(t), Rot: theoretische Werte
D=0.25
U / mV
700
600
500
400
300
200
100
0
0
100
200
300
400
500
600
700
800
900
1000
1100
1200
1300
1400
1500
f / kHz


Amplitudenspektrum von x(t), Rot: theoretische Werte
D=0.25
U / mV
700
600
500
400
300
200
100
0
0
100
200
300
400
500
600
700
800
900
1000
1100
1200
1300
1400
1500
f / kHz

2.2

2.3