H 335: Computer Structures

Spring 2016 – Homework 2

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1. **Prove, using induction, that for radix-r, the largest number that can be represented with N digits is $r^N - 1$:**

   $(d_{N-1}, ..., d_0)_r = \sum_{i=0}^{N-1} d_i \times r^i, d_i \in \{0..r-1\}$
   Base case: $N = 1$

   $\sum_{i=0}^{N-1} d_i \times r^i = r^N - 1$

   $\sum_{i=0}^{0} 0 \times r^0 = r^0 - 1$

   $0 = 0$

   Inductive hypothesis: $N = k$

   $\sum_{i=0}^{k-1} d_i \times r^i = r^k - 1$

   Inductive step: $N = k + 1$

   $\sum_{i=0}^{k} d_i \times r^i = r^{k+1} - 1$

   $(\sum_{i=0}^{1} d_i \times r^i) + (\sum_{i=0}^{k-1} d_i \times r^i) = r^{k+1} - 1$

   $(\sum_{i=0}^{1} d_i \times r^i) + (r^k - 1) = r^{k+1} - 1$

   $(\sum_{i=0}^{1} d_i \times r^i) + r^k = r^{k+1}$

   $(\sum_{i=0}^{1} d_i \times r^i) = r^{k+1} - r^k$

   $(\sum_{i=0}^{1} d_i \times r^i) = (r-1)r^k$

2. **Prove that for radix-r addition, the carry bits are always 0 or 1:**
   The carry bit for any size radix will always be either 0 or 1. This is because when adding radix number we simply add the two first digits and move on. A single digit number when aded to another single digit number will never give a number greater than or equal to 20. Therefore, every carry can only be 1 or zero to represent carrying a ten or a radix-r where r is the given radix system.

$$9 + 9 = 18$$
$$18 < 20 \therefore \text{carry would only be a 1.}$$

3. **Given the formal definition, derive the minimum and maximum two's complement numbers that can be represented in N bits:**

$(b_N - 1, ..., b_0)_{\bar{2}} = -b_{N-1} \times 2^{N-1} + \sum_{i=0}^{N-2} b_i \times 2^i$

$$\text{When finding the largest two's complement number given N bits: } \boxed{2^{N-1} - 1}$$
$$2^{8-1} - 1 = 2^7 - 1 = 128 - 1 = +127$$
$$\boxed{\text{largest} = +127 = 0111\ 1111}$$
$$\text{When finding the smallest two's complement number given N bits: } \boxed{-2^{N-1}}$$
$$-2^{8-1} = -2^7 = -128$$
$$\boxed{\text{smallest} = -128 = 1000\ 0000}$$

4. **For a number B with magnitute less than 2^{N-2}, show that if B is represented by 2's complement number with N bits $b_{N-1}..b_0$ then $-(b_N - 1, ..., b_0)_{\bar{2}} = (\overline{b_N - 1, ..., b_0})_{\bar{2}} + 1$**

$-(b_N - 1, ..., b_0)_{\bar{2}} = (\overline{b_N - 1, ..., b_0})_{\bar{2}} + 1$

$-(0010)_{\bar{2}} = (\overline{0010})_{\bar{2}} + 1$

$lhs = -(0111)_{\bar{2}} = (1111)_{\bar{2}}$

$rhs = (\overline{0111})_{\bar{2}} + 1 = (1000)_{\bar{2}} + 1 = (1001)_{\bar{2}}$

$lhs = rhs$

5. **Prove the "sign-extension" is value preserving:**

Sign-extension is value preserving because when using the two's complement, the most significant bit is the value needed to show if a value is positive or negative. That is why the greatest value cannot be used with an N-bit number. When using two's complement, the gratest value in an N-bit number must be $(r^N - 1)$ because if we used the most significant bit as a vlue instead of a marker for positive or negative, we would not be able to know if the value was positive or negative. This was shown in question 3.

6. **Convert the following decimal numbers to binary:**

(a) **107 = 110 1011**

Repeated Division (mod 2)							
Quotients	107	53	26	13	6	3	1
Remainders	1	1	0	1	0	1	1

$$(107)_{10} = (1101011)_2$$

(b) **2312 = 1001 0000 1000**

Repeated Division (mod 2)												
Quotients	2312	1156	578	289	144	72	36	18	9	4	2	1
Remainders	0	0	0	1	0	0	0	0	1	0	0	1

$(2312)_{10} = (100100001000)_2$

(c) **31333 = 111 1010 0110 0101**

Repeated Division (mod 2)															
Quotients	31333	15666	7833	3916	1958	979	489	244	122	61	30	15	7	3	1
Remainders	1	0	1	0	0	1	1	0	0	1	0	1	1	1	1

$(31333)_{10} = (111101001100101)_2$

(d) **97 = 110 0001**

Repeated Division (mod 2)							
Quotients	97	48	24	12	6	3	1
Remainders	1	0	0	0	0	1	1

$(97)_{10} = (1100001)_2$

7. **Perform the following subtraction operations using complements:**

(a) **103 - 92 = 11**

$$\boxed{\begin{array}{l} 0: (10-1)-0 = 9-0 = 9 \\ 9: (10-1)-9 = 9-9 = 0 \\ 2: (10-1)-2 = 9-2 = 7 \end{array}}$$

$$\begin{array}{cccc} \boxed{\cancel{1}} & \boxed{0} & \boxed{1} & \\ & 1 & 0 & 3 \\ & 9 & 0 & 7 \\ + & & & 1 \\ \hline & & 1 & 1 \end{array}$$

(b) **1027 - 11 = 1016**

$$\boxed{\begin{array}{l} 0: (10-1)-0 = 9-0 = 9 \\ 0: (10-1)-0 = 9-0 = 9 \\ 1: (10-1)-1 = 9-1 = 8 \\ 1: (10-1)-1 = 9-1 = 8 \end{array}}$$

$$\begin{array}{ccccc} \boxed{\cancel{1}} & \boxed{1} & \boxed{1} & \boxed{1} & \\ & 1 & 0 & 2 & 7 \\ & 9 & 9 & 8 & 8 \\ + & & & & 1 \\ \hline & 1 & 0 & 1 & 6 \end{array}$$

(c) **129 - 33 = 96**

$$\boxed{\begin{array}{l} 0: (10-1)-0 = 9-0 = 9 \\ 3: (10-1)-3 = 9-3 = 6 \\ 3: (10-1)-3 = 9-3 = 6 \end{array}}$$

$$\begin{array}{cccc} \boxed{\cancel{1}} & \boxed{0} & \boxed{1} & \\ & 1 & 2 & 9 \\ & 9 & 6 & 6 \\ + & & & 1 \\ \hline & & 9 & 6 \end{array}$$

(d) **2222 - 222 = 2000**

$$\boxed{\begin{array}{l} 0: (10-1)-0 = 9-0 = 9 \\ 2: (10-1)-2 = 9-2 = 7 \\ 2: (10-1)-2 = 9-2 = 7 \\ 2: (10-1)-2 = 9-2 = 7 \end{array}}
\qquad
\begin{array}{ccccc}
\boxed{\cancel{1}} & \boxed{1} & \boxed{1} & \boxed{1} & \\
 & 2 & 2 & 2 & 2 \\
 & 9 & 7 & 7 & 7 \\
+ & & & & 1 \\
\hline
 & 2 & 0 & 0 & 0
\end{array}$$

8. **Convert the following decimal numbers to 8-bit two's complement (show your work)**

(a) **-91 = 1101 1011**

Repeated Division (mod 2)							
Quotients	91	45	22	11	5	2	1
Remainders	1	1	0	1	1	0	1

$(-91)_{10} = (1101\ 1011)_2$

The most significant bit will be the bit to show if the number is positive or negative. Since the decimal number will be an 8-bit two's complement, the most significant bit will be in the last position on the left and for -91 the most significant bit will be 1 to represent a negative.

(b) **-96 = 1110 0000**

Repeated Division (mod 2)							
Quotients	96	48	24	12	6	3	1
Remainders	0	0	0	0	0	1	1

$(-96)_{10} = (1110\ 0000)_2$

(c) **-126 = 1111 1110**

Repeated Division (mod 2)							
Quotients	126	63	31	15	7	3	1
Remainders	0	1	1	1	1	1	1

$(-126)_{10} = (1111\ 1110)_2$

(d) **101 = 0110 0101**

Repeated Division (mod 2)							
Quotients	101	50	25	12	6	3	1
Remainders	1	0	1	0	0	1	1

$(101)_{10} = (0110\ 0101)_2$

Now that the decimal is positive, the most significant bit will be a 0. 1 is for negative and 0 is for positive.

(e) **78 = 0100 1110**

Repeated Division (mod 2)							
Quotients	78	39	19	9	4	2	1
Remainders	0	1	1	1	0	0	1

$(78)_{10} = (0100\ 1110)_2$