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а

$$f(x) = e^{|x|}, \quad [-\pi, \pi]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} e^{|x|}\, dx = \frac{2}{\pi} \int\limits_{0}^{\pi} e^{x}\, dx = \frac{2}{\pi} \left(e^x\right) \Big|_0^{\pi} = \frac{2(e^{\pi} - 1)}{\pi}$$

$$a_n = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} e^{|x|} \cos nx\, dx = \frac{2}{\pi} \int\limits_{0}^{\pi} e^{x} \cos nx\, dx$$

отдельно посчитаем первообразную

$$\int e^x \cos nx\, dx = [u = \cos nx, du = -n \sin nx, dv = e^x\, dx, v = e^x] = e^x \cos nx + n \int e^x \sin nx\, dx =$$

$$= [u = \sin nx, du = n \cos nx\, dx, dv = e^x\, dx, v = e^x] = e^x \cos nx + n \left(e^x \sin nx - n \int e^x \cos nx\, dx \right) =$$

$$= e^x \cos nx + n e^x \sin nx - n^2 \int e^x \cos nx\, dx$$

$$\int e^x \cos nx\, dx = e^x \cos nx + n e^x \sin nx - n^2 \int e^x \cos nx\, dx$$

$$\int e^x \cos nx\, dx = \frac{1}{1+n^2} e^x \cos nx + \frac{n}{1+n^2} e^x \sin nx$$

$$\frac{2}{\pi} \int\limits_{0}^{\pi} e^x \cos nx\, dx = \frac{2}{\pi} \left(\frac{1}{1+n^2} e^x \cos nx + \frac{n}{1+n^2} e^x \sin nx \right) \Bigg|_0^{\pi} =$$

$$= \frac{2}{\pi} \left(\frac{e^{\pi} \cos \pi n}{1+n^2} - \frac{1}{1+n^2} \right) = \frac{2((-1)^n e^{\pi} - 1)}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} e^{|x|} \sin nx\, dx = 0,$$

потому что под интегралом нечётная функция и симметричный промежуток

$$f(x) = \frac{e^{\pi} - 1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n e^{\pi} - 1}{(1+n^2)} \cos nx$$

б

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int\limits_0^{\pi} e^{|x|} \sin nx \, dx = \frac{2}{\pi} \int\limits_0^{\pi} e^{x} \sin nx \, dx$$

отдельно посчитаем первообразную

$$\int e^x \sin nx \, dx = [u = \sin nx, du = n \cos nx, dv = e^x \, dx, v = e^x] = e^x \sin nx - n \int e^x \cos nx \, dx =$$

$$= [u = \cos nx, du = -n \sin nx \, dx, dv = e^x \, dx, v = e^x] = e^x \sin nx - n \left(e^x \cos nx + n \int e^x \sin nx \, dx \right) =$$

$$= e^x \sin nx - n e^x \cos nx - n^2 \int e^x \sin nx \, dx$$

$$\int e^x \sin nx \, dx = e^x \sin nx - n e^x \cos nx - n^2 \int e^x \sin nx \, dx$$

$$\int e^x \cos nx \, dx = \frac{1}{1+n^2} e^x \sin nx - \frac{n}{1+n^2} e^x \cos nx$$

$$b_n = \frac{2}{\pi} \int\limits_0^{\pi} e^x \sin nx \, dx = \frac{2}{\pi} \left(\frac{1}{1+n^2} e^x \sin nx - \frac{n}{1+n^2} e^x \cos nx \right) \Bigg|_0^{\pi} =$$

$$= \frac{2}{\pi} \left(-\frac{n}{1+n^2} e^{\pi} \cos \pi n + \frac{n}{1+n^2} \right) = \frac{2n(1 - e^{\pi}(-1)^n)}{\pi(1+n^2)}$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n(1 - e^{\pi}(-1)^n)}{1+n^2} \sin nx$$

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$$\iint\limits_D \frac{\ln(x^2+y^2)}{x^2+y^2} \, dxdy, \quad D : e \leqslant x^2 + y^2 \leqslant e^4, x > 0$$

Перейдём к полярным координатам

$$x = r\cos\varphi, y = r\sin\varphi, |J| = r, x^2 + y^2 = (r\cos\varphi)^2 + (r\sin\varphi)^2 = r^2$$

$$e \leqslant x^2 + y^2 \leqslant e^4 \Longrightarrow e \leqslant r^2 \leqslant e^4$$

$$x > 0 \Longrightarrow r\cos\varphi > 0 \Longrightarrow \cos\varphi > 0 \Longrightarrow -\frac{\pi}{2} < \varphi < \frac{\pi}{2}$$

$$\iint\limits_D \frac{\ln(x^2+y^2)}{x^2+y^2} \, dxdy = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_e^{e^4} \frac{\ln r^2}{r^2} \cdot r \, dr = 2\pi \int\limits_e^{e^4} \frac{\ln r}{r} \, dr = 2\pi \left(\frac{\ln^2 r}{2} \right) \Bigg|_e^{e^4} =$$

$$= 2\pi \cdot \left(\frac{4^2}{2} - \frac{1^2}{2} \right) = 2\pi \cdot \frac{15}{2} = 15\pi$$

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$$z = \sqrt{xy}, \quad y = 2x^2, y = x^3, z = 0$$

$$V = \iint\limits_D z\,dxdy = \iint\limits_D \sqrt{xy}\,dxdy = \int\limits_0^2 dx \int\limits_{x^3}^{2x^2} \sqrt{xy}\,dy = \int\limits_0^2 \sqrt{x}\,dx \cdot \left(\frac{2}{3}y\sqrt{y}\right)\Bigg|_{x^3}^{2x^2} =$$

$$= \int\limits_0^2 \sqrt{x} \cdot \left(\frac{2}{3} \cdot 2x^2 \cdot \sqrt{2x^2} - \frac{2}{3}x^3\sqrt{x^3}\right)\,dx = \int\limits_0^2 \left(\frac{4\sqrt{2}}{3}x^{\frac{7}{2}} - \frac{2}{3}x^5\right)\,dx =$$

$$= \left(\frac{4\sqrt{2}}{3} \cdot \frac{2}{9}x^{\frac{9}{2}} - \frac{2}{3} \cdot \frac{1}{6}x^6\right)\Bigg|_0^2 = \left(\frac{8\sqrt{2}}{27}x^{\frac{9}{2}} - \frac{1}{9}x^6\right)\Bigg|_0^2 = \frac{8\sqrt{2} \cdot 2^{\frac{9}{2}}}{27} - \frac{2^6}{9} = \frac{8\sqrt{2} \cdot 2^4\sqrt{2}}{27} - \frac{64}{9} =$$

$$= \frac{256}{27} - \frac{64}{9} = \frac{64}{27}$$

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a

$$f(x) = \begin{cases} \frac{\pi}{4}, & 0 \leqslant x < \pi \\ -\frac{\pi}{4}, & -\pi < x < 0 \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int\limits_{-\pi}^{0}\left(-\frac{\pi}{4}\right)\,dx + \frac{1}{\pi}\int\limits_0^{\pi}\frac{\pi}{4}\,dx = \frac{1}{\pi} \cdot \left(-\frac{\pi}{4} \cdot \pi\right) + \frac{1}{\pi} \cdot \left(\frac{\pi}{4} \cdot \pi\right) = 0$$

$$a_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int\limits_{-\pi}^{0}\left(-\frac{\pi}{4}\right)\cos nx\,dx + \frac{1}{\pi}\int\limits_0^{\pi}\frac{\pi}{4}\cos nx\,dx =$$

$$= -\frac{1}{4}\int\limits_{-\pi}^{0}\cos nx\,dx + \frac{1}{4}\int\limits_0^{\pi}\cos nx\,dx = -\frac{1}{4}\left(\frac{\sin nx}{n}\right)\Bigg|_{-\pi}^{0} + \frac{1}{4}\left(\frac{\sin nx}{n}\right)\Bigg|_0^{\pi} = 0 + 0 = 0$$

$$b_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int\limits_{-\pi}^{0}\left(-\frac{\pi}{4}\right)\sin nx\,dx + \frac{1}{\pi}\int\limits_0^{\pi}\frac{\pi}{4}\sin nx\,dx =$$

$$= -\frac{1}{4}\int\limits_{-\pi}^{0}\sin nx\,dx + \frac{1}{4}\int\limits_0^{\pi}\sin nx\,dx = -\frac{1}{4}\left(-\frac{\cos nx}{n}\right)\Bigg|_{-\pi}^{0} + \frac{1}{4}\left(-\frac{\cos nx}{n}\right)\Bigg|_0^{\pi} =$$

$$= -\frac{1}{4}\left(-\frac{1}{n} + \frac{\cos\pi n}{n}\right) + \frac{1}{4}\left(-\frac{\cos\pi n}{n} + \frac{1}{n}\right) = \frac{1}{4n} - \frac{(-1)^n}{4n} - \frac{(-1)^n}{4n} + \frac{1}{4n} = \frac{1-(-1)^n}{2n}$$

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n}\sin nx$$

б

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int\limits_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int\limits_0^{\pi} \frac{\pi}{4}\,dx = \frac{2}{\pi}\cdot\frac{\pi}{4}\cdot\pi = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int\limits_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int\limits_0^{\pi} \frac{\pi}{4}\cos nx\,dx = \frac{1}{2}\int\limits_0^{\pi}\cos nx\,dx = \frac{1}{2}\left(\frac{\sin nx}{n}\right)\Big|_0^{\pi} =$$

$$= \frac{1}{2}\left(\frac{\sin\pi n}{n} - 0\right) = 0$$

$$f(x) = \frac{\pi}{4}$$

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$$z = \sqrt{xy}, \quad z = 0, y = x^3, y = x^4$$

$$V = \iint\limits_D z\,dxdy = \iint\limits_D \sqrt{xy}\,dxdy = \int\limits_0^1 dx \int\limits_{x^4}^{x^3} \sqrt{xy}\,dy = \int\limits_0^1 \sqrt{x}\,dx \cdot \left(\frac{2}{3}y\sqrt{y}\right)_{x^4}^{x^3} =$$

$$= \int\limits_0^1 \sqrt{x}\left(\frac{2}{3}x^3\sqrt{x^3} - \frac{2}{3}x^4\sqrt{x^4}\right)\,dx = \int\limits_0^1 \left(\frac{2}{3}x^5 - \frac{2}{3}x^{\frac{13}{2}}\right)\,dx =$$

$$= \left(\frac{2}{3}\cdot\frac{1}{6}x^6 - \frac{2}{3}\cdot\frac{2}{15}x^{\frac{15}{2}}\right)\Big|_0^1 = \left(\frac{1}{9}x^6 - \frac{4}{45}x^{\frac{15}{2}}\right)\Big|_0^1 = \frac{1}{9} - \frac{4}{45} = \frac{1}{45}$$