Numerical Analysis MATH50003 (2024–25) Problem Sheet 2

Problem 1 Using dual number arithmetic, compute the following polynomials evaluated at the dual number $2 + \epsilon$ and use this to deduce their derivative at 2:

$$2x^2 + 3x + 4, (x + 1)(x + 2)(x + 3), (2x + 1)x^3.$$

Problem 2 What should the following functions applied to dual numbers return for $x = a+b\epsilon$:

$$f(x) = x^{100} + 1, g(x) = 1/x, h(x) = \tan x.$$

Problem 3(a) What is the correct definition of division on dual numbers, i.e., for what choice of s and t does the following hold:

$$(a + b\epsilon)/(c + d\epsilon) = s + t\epsilon.$$

Problem 3(b) A *field* is a commutative ring such that $0 \neq 1$ and all nonzero elements have a multiplicative inverse, i.e., there exists a^{-1} such that $aa^{-1} = 1$. Can we use the previous part to define $a^{-1} := 1/a$ to make $\mathbb{D}$ a field? Why or why not?

Problem 4 Use dual numbers to compute the derivative of the following functions at $x = 0.1$:

$$\exp(\exp x \cos x + \sin x), \prod_{k=1}^{3} \left(\frac{x}{k} - 1\right), \text{ and } f_2^{\rm s}(x) = 1 + \frac{x-1}{2 + \frac{x-1}{2}}$$

Consider a 2D analogue of dual numbers $a + b\epsilon_x + c\epsilon_y$ defined by the relationship $\epsilon_x\epsilon_y = \epsilon_x^2 = \epsilon_y^2 = 0$.

Problem 5(a) Derive the formula for writing the product of two 2D dual numbers $(a + a_x\epsilon_x + a_y\epsilon_y)(b + b_x\epsilon_x + b_y\epsilon_y)$ where $a, a_x, a_y, b, b_x, b_y \in \mathbb{R}$ as a 2D dual number.

Problem 5(b) Show for all 2D polynomials

$$p(x, y) = \sum_{k=0}^{n} \sum_{j=0}^{m} c_{kj} x^k y^j$$

that

$$p(x + a\epsilon_x, y + b\epsilon_y) = p(x, y) + a\frac{\partial p}{\partial x}\epsilon_x + b\frac{\partial p}{\partial y}\epsilon_y.$$

Problem 5(c) Use 2D dual numbers to compute the gradient of $p(x, y) = (1+x+3xy)(1+y)$ at $x = 1$ and $y = 2$.

Problem 6 Suppose f is twice-differentiable in a neighbourhood of B of r such that $f(r) = f'(r) = 0$, where f'' does not vanish in B. Show that the error of the k-th Newton iteration $\varepsilon_k := r - x_k$ satisfies

$$|\varepsilon_{k+1}| \leq \tilde{M}|\varepsilon_k|$$

where

$$\tilde{M} = \frac{1}{2} \sup_{y \in B} |f''(y)| \sup_{y \in B} \frac{1}{|f''(y)|}.$$