Numerical Analysis MATH50003 (2024–25) Problem Sheet 4

Problem 1 For intervals $X = [a, b]$ and $Y = [c, d]$ satisfying $0 < a < b$ and $0 < c < d$, and $n > 0$ prove that

$$\begin{aligned} X/n &= [a/n, b/n] \\ XY &= [ac, bd] \end{aligned}$$

Generalise (without proof) these formulæ to the case $n < 0$ and to where there are no restrictions on positivity of a, b, c, d. You may use the min or max functions.

Problem 2(a) Compute the following floating point interval arithmetic expression assuming half-precision F_{16} arithmetic:

$$[1, 1] \ominus ([1, 1] \oslash 6)$$

Hint: it might help to write $1 = (0.1111\ldots)_2$ when doing subtraction.

Problem 2(b) Writing

$$\sin\ x = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \delta_{x,2n+1}$$

Prove the bound $|\delta_{x,2n+1}| \leq 1/(2n+3)!$, assuming $x \in [0, 1]$.

Problem 2(c) Combine the previous parts to prove that:

$$\sin 1 \in [(0.11010011000)_2, (0.11010111101)_2] = [0.82421875, 0.84228515625]$$

You may use without proof that $1/120 = 2^{-7}(1.000100010001\ldots)_2$.

Problem 3 For $A \in F_{\infty,S}^{n\times n}$ and $\boldsymbol{x} \in F_{\infty,S}^{n}$ consider the error in approximating matrix multiplication with idealised floating point: for

$$A\boldsymbol{x} = \begin{pmatrix} \bigoplus_{j=1}^{n} A_{1,j} \otimes x_j \\ \vdots \\ \bigoplus_{j=1}^{n} A_{1,j} \otimes x_j \end{pmatrix} + \delta$$

use Problem 8 on PS3 to show that

$$\|\delta\|_\infty \leq 2\|A\|_\infty \|\boldsymbol{x}\|_\infty E_{n,\epsilon_{\rm m}/2}$$

for $E_{n,\epsilon} := \frac{n\epsilon}{1-n\epsilon}$, where $n\epsilon_{\rm m} < 2$ and the matrix norm is $\|A\|_\infty := \max_k \sum_{j=1}^{n} |a_{kj}|$.