MATH50003 Numerical Analysis

I.4 Newton's Method

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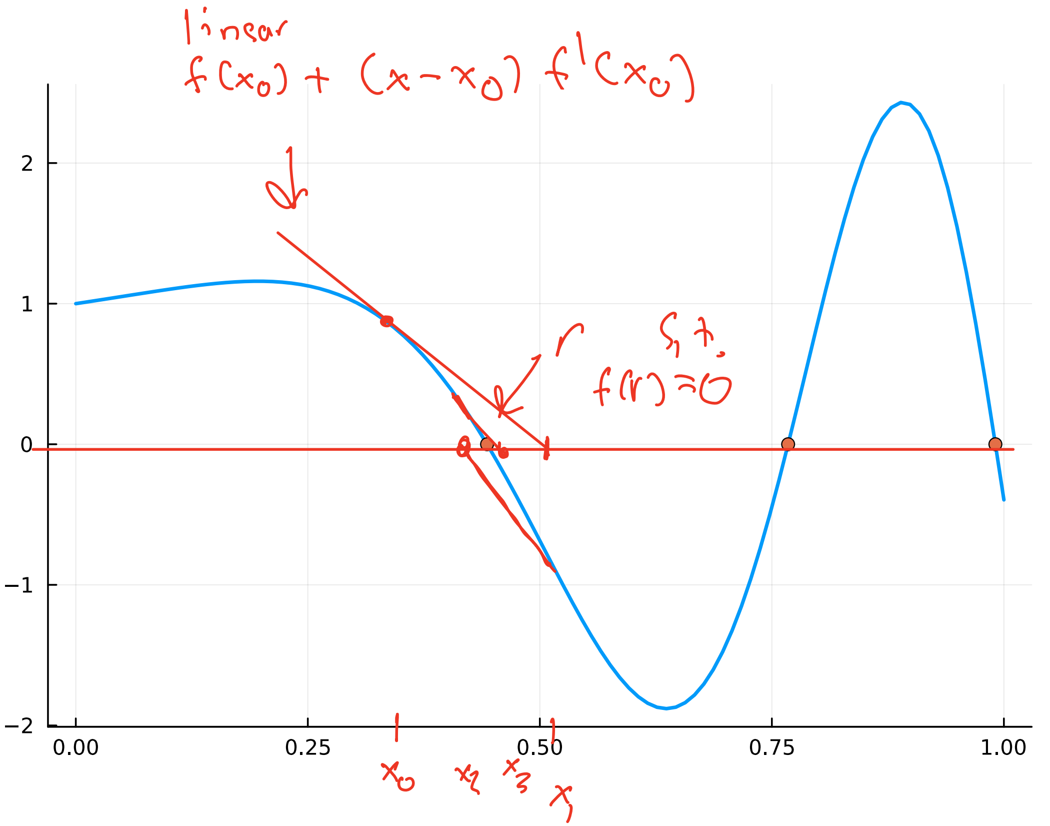
Part I

Calculus on a Computer

1. Rectangular rules for integration
2. Divided differences for differentiation
3. Dual numbers for differentiation
4. Newton's method for root finding

Given a function, how can we find a *single* root/zero?



Newton's method

Find roots of affine functions

Given initial guess x_0:

$$f(x) \approx \underbrace{f(x_0) + (x - x_0)f'(x)}_{\text{linear}}$$

Root of right-hand side:

$$f(x_0) + (x - x_0)f'(x_0) = 0 \quad \Leftrightarrow \quad x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

root of linear funct.

Algorithm

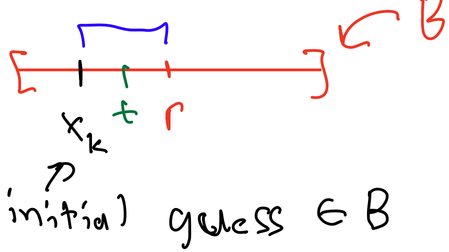
x_0 = initial guess

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Does it converge?

Yes if x_0 is close to r.

ϵ_k



$[r-\delta, r+\delta]$

Theorem 3 (Newton error). *Suppose f is twice-differentiable in a neighbourhood B of r such that $f(r)=0$, and f' does not vanish in B. Denote the error of the k-th Newton iteration as $\varepsilon_k := r - x_k$. If $x_k \in B$ then*

error

$$\underbrace{|\varepsilon_{k+1}|}_{\text{small}^2 = \text{very small}} \le M\underbrace{|\varepsilon_k|}_{\text{small}}^2$$

where

$$M := \frac{1}{2}\sup_{x\in B}|f''(x)|\sup_{x\in B}\left|\frac{1}{f'(x)}\right|.$$

Proof Use Taylor's theorem:

$$0 = f(r) = f(x_k + \epsilon_k) = f(x_k) + f'(x_k)\,\epsilon_k + \frac{f''(t)}{2}\,\epsilon_k^2$$

where $t \in [x_k, r]$ or $[r, x_k] \subset B$.

Rearrange

$$f(x_k) = -f'(x_k)\,\epsilon_k - \frac{f''(t)}{2}\,\epsilon_k^2$$

$\Rightarrow$

$$\epsilon_{k+1} := r - \underbrace{x_{k+1}}_{= x_k - \frac{f(x_k)}{f'(x_k)}}$$

$$= \underbrace{r - x_k}_{\epsilon_k} + \frac{f(x_k)}{f'(x_k)}$$

$$= \epsilon_k - \frac{\cancel{f'(x_k)\,\epsilon_k} + f''(t)/2\ \epsilon_k^2}{f'(x_k)}$$

$$= -\frac{f''(t)}{2 f'(x_k)}\ \epsilon_k^2$$

$$\Rightarrow\ |\epsilon_{k+1}| \leq \frac{|\epsilon_k|^2}{2}\, |f''(t)|\, \left|\frac{1}{f'(x_k)}\right| \leq M|\epsilon_k|^2$$

$\in B$

Corollary 1 (Newton convergence). *If $x_0 \in B$ is sufficiently close to r then $x_k \to r$.*

Proof

Need to show $x_k \in B \quad \forall k$.

Suppose $x_k \in B$ s.t. $|\epsilon_k| = |r - x_k| \leq M^{-1}$.

Then

$$|\epsilon_{k+1}| \leq M |\epsilon_k|^2 \leq |\epsilon_k|$$

$$\Rightarrow \quad x_{k+1} \in B$$

By induction if $|\epsilon_0| < M^{-1}$ then $x_k \in B \quad \forall k$.

Then for large k

$$|\epsilon_k| \le M|\epsilon_{k-1}|^2 \le \underbrace{M\left(M|\epsilon_{k-2}|^2\right)^2}_{M^3 |\epsilon_{k-2}|^4}$$

$$\le \underbrace{M^3\left(M|\epsilon_{k-3}|^2\right)^4}_{M^7 |\epsilon_{k-3}|^8} \le \text{———}$$

$$\le M^{2^k-1}|\epsilon_0|^{2^k} = \frac{(\overbrace{M|\epsilon_0|}^{<1})^{2^k}}{M}$$

$$\xrightarrow[k\to\infty]{} 0 \qquad \text{(Very fast!)}$$

Let’s see how it works in practice