MATH50003 Numerical Analysis

I.3 Dual Numbers

Dr Sheehan Olver

Part I

Calculus on a Computer

1. Rectangular rules for integration
2. Divided differences for differentiation
3. Dual numbers for differentiation
4. Newton's method for root finding

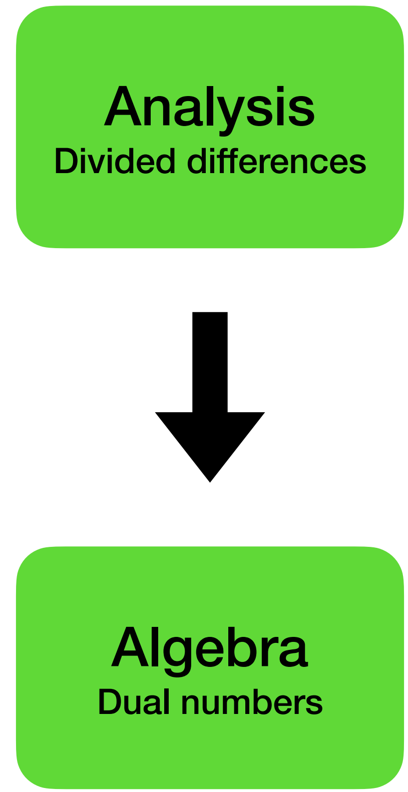
Divided differences can have large errors.

$\gtrsim 10^{-8}$

Is it even possible to algorithmically calculate derivatives to high accuracy?

$\lesssim 10^{-15}$

Yes: if we have access to the code.



has +, *

Dual numbers are a commutative ring that allow us to differentiate

Definition 1 (dual numbers)

$$\mathbb{D} := \{a + b\epsilon \quad : \quad a, b \in \mathbb{R}, \quad \epsilon^2 = 0\}$$

generated by 1 & ϵ

Compare with complex numbers:

$$\mathbb{C} := \{a + bi \quad : \quad a, b \in \mathbb{R}, \quad i^2 = -1\}$$

generated by 1 & i

Addition/multiplication

Follows from simple algebra

Complex

$(a + bi) + (c + di) = (a + c) + (b + d)i$

Eg

$(1+2i) + (3+4i) =$

$4 + 6i$

Dual

$(a + b\epsilon) + (c + d\epsilon) = (a + c) + (b + d)\epsilon$

$(1+2\epsilon) + (2+4\epsilon) =$

$3 + 6\epsilon$

$(a + bi)(c + di) = (ac - bd) + (bc + ad)i$

Since

$(a+bi)(c+di) =$

$ac + (bc+ad)i + bd\underbrace{i^2}_{=-1}$

Eg

$(1+2i)(3+4i) =$

$-3 + 10i$

$(a + b\epsilon)(c + d\epsilon) = ac + (bc + ad)\epsilon$

Since

$(a+b\epsilon)(c+d\epsilon) =$

$ac + (bc+ad)\epsilon + \cancel{bd\epsilon^2}$

$(1+2\epsilon)(3+4\epsilon)$

$3 + 10\epsilon$

$*$ & $+$ give us polynomials.

Eg for

$$p(x) = (2-x)(x+x^2)$$

we have

$$p(1+\epsilon) = (1-\epsilon)\underbrace{\left(1+\epsilon+\underbrace{(1+\epsilon)^2}_{1+2\epsilon}\right)}_{2+3\epsilon}$$

$$= \underbrace{2}_{p(1)} + \underbrace{\epsilon}_{p'(1)}$$

I.3.1 Differentiating polynomials

Addition/multiplication ⇒ dual numbers compute derivatives

Theorem 2 (polynomials on dual numbers). *Suppose p is a polynomial. Then*

$$p(a + b\epsilon) = p(a) + bp'(a)\epsilon$$

Eg $\quad p(a+\epsilon) = \underbrace{p(a)}_{\text{value}} + \underbrace{p'(a)}_{\text{derivative}}\epsilon$

Proof

consider x^n by induction. Base:

$\underbrace{n=0}_{p(x)=1}$: $\quad p(a+b\epsilon) = \underbrace{1}_{p(a)} + \underbrace{0}_{bp'(a)}\epsilon$

$n=1$: $\underbrace{}_{p(x)=x}$ $\quad p(a+b\epsilon) = \underbrace{a}_{p(a)} + b\epsilon$

$\overset{\wedge}{p'(a)}$

Assume true $\leq n$. Ie $(a+b\epsilon)^n = a^n + bna^{n-1}\epsilon$.

Then for $p(x) = x^{n+1}$

$$(a+b\epsilon)^{n+1} = (a+b\epsilon)(a+b\epsilon)^n$$

assumption

$$= (a+b\epsilon)(a^n + nba^{n-1}\epsilon)$$

dual *

$$= \underbrace{a^{n+1}}_{p(a)} + \underbrace{(ba^n + nba^n)}_{b(n+1)a^n = b\,p'(a).}\epsilon$$

In general, if

$$p(x) = \sum_{k=0}^{n} c_k x^k$$

then

$$p(a+b\epsilon) = \sum_{k=0}^{n} c_k (a+b\epsilon)^k = c_0 + \sum_{k=1}^{n} c_k \left(a^k + b k a^{k-1} \epsilon\right)$$

$$= \underbrace{c_0 + \sum_{k=1}^{n} c_k a^k}_{p(a)} + b \underbrace{\left(\sum_{k=1}^{n} c_k k a^{k-1} \right)}_{p'(a)} \epsilon$$

Example 1 (differentiating polynomial)**.** Consider computing $p'(2)$ where

$$p(x) = (x-1)(x-2) + x^2.$$

Since

$$p(2+\epsilon) = \underbrace{(1+\epsilon)\,\epsilon}_{\epsilon} + \underbrace{(1+\epsilon)^2}_{1+2\epsilon}$$

$$= \underbrace{1}_{p(2)} + \underbrace{3}_{p'(2)}\,\epsilon$$

I.3.2 Differentiating other functions

Theorem 1 gives us a rule to extend differentiation via duals

Motivation: consider a Taylor series

$$f(x) = \sum_{k=0}^{\infty} f_k x^k$$

And assume a is in radius of convergence.

What will $f(a + b\epsilon)$ return?

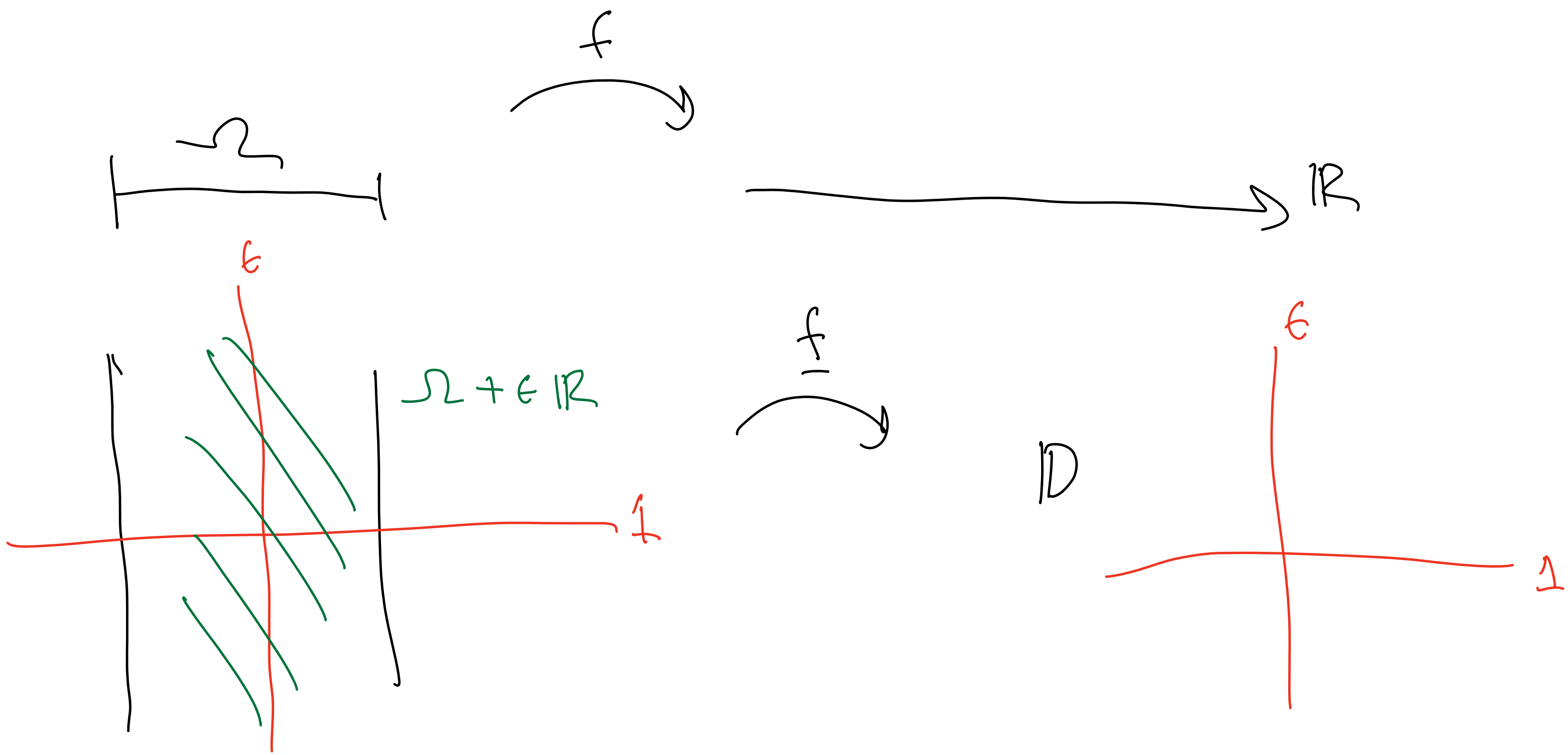
$$f(a+b\epsilon) = \sum_{k=0}^{\infty} f_k (a+b\epsilon)^k = f_0 + \sum_{k=1}^{\infty} f_k (a^k + b k a^{k-1} \epsilon)$$

$$= \underbrace{\sum_{k=0}^{\infty} f_k a^k}_{f(a)} + b \underbrace{\left(\sum_{k=1}^{\infty} f_k k a^{k-1} \right)}_{f'(a)} \epsilon$$

What if f is differentiable but not $\underbrace{\text{analytic}}_{\text{has Taylor}}$?

Definition 2 (dual extension)**.** Suppose a real-valued function $f : \Omega \to \mathbb{R}$ is differentiable in $\Omega \subset \mathbb{R}$. We can construct the *dual extension* $\underline{f} : \Omega + \epsilon\mathbb{R} \to \mathbb{D}$ by defining

$$\underline{f}(a + b\epsilon) := f(a) + bf'(a)\epsilon.$$



We can view $\mathbb{R} \simeq \{a + 0\epsilon : a \in \mathbb{R}\} \subset \mathbb{D}$ and write

$$f(a+b\epsilon) \equiv \underline{f}(a+b\epsilon).$$

Examples:

$$\exp(a+b\epsilon) := \exp(a) + b\exp(a)\epsilon \qquad a, b \in \mathbb{R}$$
$$\sin(a+b\epsilon) := \sin(a) + b\cos(a)\epsilon \qquad a, b \in \mathbb{R}$$
$$\cos(a+b\epsilon) := \cos(a) - b\sin(a)\epsilon \qquad a, b \in \mathbb{R}$$
$$\log(a+b\epsilon) := \log(a) + \frac{b}{a}\epsilon \qquad a > 0,\ b \in \mathbb{R}$$
$$\sqrt{a+b\epsilon} := \sqrt{a} + \frac{b}{2\sqrt{a}}\epsilon \qquad a > 0,\ b \in \mathbb{R}$$
$$|a+b\epsilon| := |a| + b\,\mathrm{sign}\,a\,\epsilon \qquad a \neq 0,\ b \in \mathbb{R}$$

$\boxed{\text{Building Blocks}}$

Defining on $\mathbb{D}$ gives us duals on complicated functions

Lemma 2 (addition/multiplication)**.** *Suppose $f, g : \Omega \to \mathbb{R}$ are differentiable for $\Omega \subset \mathbb{R}$ and $c \in \mathbb{R}$. Then for $a \in \Omega$ and $b \in \mathbb{R}$ we have*

$$\begin{aligned}
\underline{f+g}(a+b\epsilon) &= \underline{f}(a+b\epsilon) + \underline{g}(a+b\epsilon) \\
\underline{cf}(a+b\epsilon) &= c\underline{f}(a+b\epsilon) \\
\underline{fg}(a+b\epsilon) &= \underline{f}(a+b\epsilon)\underline{g}(a+b\epsilon)
\end{aligned}$$

$$\begin{tikzcd}
f, g \arrow[r, "\text{addition}"] \arrow[d, "\text{extionsion}"'] & f+g \arrow[d, "\text{extension}"] \\
\underline{f}, \underline{g} \arrow[r, "\text{add}"'] & \underline{f+g} = \underline{f} + \underline{g}
\end{tikzcd}$$

Proof

add $$(f+g)(a+b\epsilon) = (f+g)(a) + b(f+g)'(a)\epsilon$$

$$= \underbrace{f(a) + bf'(a)\epsilon}_{f(a+b\epsilon)} + \underbrace{g(a) + bg'(a)\epsilon}_{g(a+b\epsilon)}$$

cf same.

mul $$fg(a+b\epsilon) = \overbrace{(fg)(a)}^{f(a)g(a)} + b\,\overbrace{(fg)'(a)}^{f(a)g'(a)+f'(a)g(a)}\,\epsilon$$

$$= \underbrace{\left(f(a) + bf'(a)\epsilon\right)}_{f(a+b\epsilon)}\underbrace{\left(g(a) + bg'(a)\epsilon\right)}_{g(a+b\epsilon)}$$

Lemma 3 (composition). *Suppose $f : \Gamma \to \mathbb{R}$ and $g : \Omega \to \Gamma$ are differentiable in $\Omega, \Gamma \subset \mathbb{R}$. Then*

$$\underline{(f \circ g)}(a + b\epsilon) = \underline{f}(\underline{g}(a + b\epsilon))$$

where $(f \circ g)(x) = f(g(x))$,

<u>Proof</u>

$$\underline{(f \circ g)}(a + b\epsilon) = f(\overbrace{g(a)}^{c}) + \overbrace{b\, g'(a)}^{d} f'(\overbrace{g(a)}^{c})\, \epsilon$$

$$= \underline{f}(\underbrace{g(a)}_{c} + \underbrace{b\, g'(a)}_{d}\, \epsilon)$$

$$= \underline{f}(\underline{g}(a + b\epsilon))$$

Example 2 (differentiating non-polynomial)**.** Consider differentiating $f(x) = \exp(x^2 + \cos x)$ at the point $a = 1$, where we automatically use the dual-extension of exp and cos.

$$f(1+\epsilon) = \exp\Big(\underbrace{\underbrace{(1+\epsilon)^2}_{1+2\epsilon} + \underbrace{\cos(1+\epsilon)}_{\cos 1 - \sin 1\,\epsilon}}_{1+\cos 1 + (2-\sin 1)\epsilon}\Big)$$

$$= \underbrace{\exp(1+\cos 1)}_{f(1)} + \underbrace{(2-\sin 1)\exp(1+\cos 1)}_{f'(1)}\,\epsilon$$

How do we implement this on a computer?