

Numerical Analysis MATH50003 (2024–25) Problem Sheet 8

Problem 1 Give explicit formulae for \hat{f}_k and \hat{f}_k^n for the following functions:

$$\cos \theta, \cos 4\theta, \sin^4 \theta, \frac{3}{3 - e^{i\theta}}, \frac{1}{1 - 2e^{i\theta}}$$

Problem 2 Prove that if the first $\lambda - 1$ derivatives $f(\theta), f'(\theta), \dots, f^{(\lambda-1)}(\theta)$ are 2π -periodic and $f^{(\lambda)}$ is uniformly bounded that

$$|\hat{f}_k| = O(|k|^{-\lambda}) \quad \text{as } |k| \rightarrow \infty$$

Use this to show for the Taylor case ($0 = \hat{f}_{-1} = \hat{f}_{-2} = \dots$) that

$$|f(\theta) - \sum_{k=0}^{n-1} \hat{f}_k e^{ik\theta}| = O(n^{1-\lambda}) \quad \text{as } n \rightarrow \infty$$

Problem 3(a) If f is a trigonometric polynomial ($\hat{f}_k = 0$ for $|k| > m$) show for $n \geq 2m + 1$ that we can exactly recover f :

$$f(\theta) = \sum_{k=-m}^m \hat{f}_k^n e^{ik\theta}$$

Problem 3(b) For the general (non-Taylor) case and $n = 2m + 1$, prove convergence for

$$f_{-m:m}(\theta) := \sum_{k=-m}^m \hat{f}_k^n e^{ik\theta}$$

to $f(\theta)$ as $n \rightarrow \infty$. What is the rate of convergence if we know that the first $\lambda - 1$ derivatives $f(\theta), f'(\theta), \dots, f^{(\lambda-1)}(\theta)$ are 2π -periodic and $f^{(\lambda)}$ is uniformly bounded?

Problem 3(c) Show that $f_{-m:m}(\theta)$ interpolates f at $\theta_j = 2\pi j/n$ for $n = 2m + 1$.

Problem 4(a) Show for $0 \leq k, \ell \leq n - 1$

$$\frac{1}{n} \sum_{j=1}^n \cos k\theta_j \cos \ell\theta_j = \begin{cases} 1 & k = \ell = 0 \\ 1/2 & k = \ell \\ 0 & \text{otherwise} \end{cases}$$

for $\theta_j = \pi(j - 1/2)/n$. Hint: Be careful as the θ_j differ from before, and only cover half the period, $[0, \pi]$. Using symmetry may help. You may also consider replacing \cos with complex exponentials:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Problem 4(b) Consider the Discrete Cosine Transform (DCT)

$$C_n := \begin{bmatrix} \sqrt{1/n} & & & \\ & \sqrt{2/n} & & \\ & & \ddots & \\ & & & \sqrt{2/n} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \cos \theta_1 & \cdots & \cos \theta_n \\ \vdots & \ddots & \vdots \\ \cos(n-1)\theta_1 & \cdots & \cos(n-1)\theta_n \end{bmatrix}$$

for $\theta_j = \pi(j - 1/2)/n$. Prove that C_n is orthogonal: $C_n^\top C_n = C_n C_n^\top = I$.

Problem 5 What polynomial interpolates $\cos z$ at $1, \exp(2\pi i/3)$ and $\exp(-2\pi i/3)$?