

## Numerical Analysis MATH50003 (2024–25) Problem Sheet 5

**Problem 1** Compute the LU factorisation (if possible) and the PLU factorisation, where the entry of largest magnitude is always permuted to the diagonal, of the following matrices:

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 5 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 5 & 5 & 5 \\ 1 & 2 & 0 & 0 \\ 3 & 3 & 3 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

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**Problem 2** By computing the Cholesky factorisation, determine which of the following matrices are symmetric positive definite:

$$\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix}$$

**Problem 3** Show that a matrix  $A \in \mathbb{R}^{n \times n}$  is symmetric positive definite if and only if it has a *reverse* Cholesky factorisation of the form

$$A = UU^\top$$

where  $U$  is upper triangular with positive entries on the diagonal.

**Problem 4(a)** Use the Cholesky factorisation to prove that the following  $n \times n$  matrix is symmetric positive definite for any  $n$ :

$$\Delta_n := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Hint: consider a matrix  $K_n^{(\alpha)}$  that equals  $\Delta_n$  apart from the top left entry which is  $\alpha > 1$  and use a proof by induction.

**Problem 4(b)** Deduce its Cholesky factorisations:  $\Delta_n = L_n L_n^\top$  where  $L_n$  is lower triangular.