Diferencia de medias

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$H_0: \mu_1 = \mu_2$ $H_a: \mu_1 \neq \mu_2$

Comparación de 2 medias

T_1	8	6	3	4	5
T_2	7	5	4	3	4
dif	1	1	-1	1	1

$H_0: \mu_{dif} = 0$ $H_a: \mu_{dif} \neq 0$

Si hay p-variables

X_{1j1}: Variab 1 con trat 1 X_{2j1}: var 1 trat 2

X_{1j2} " 2 " " 1 X_{2j2}: var 2 trat 2

$\vdots$ $\vdots$

$D_{j1} = X_{1j1} - X_{2j1}$

$\vdots$

$D_{jp} = X_{1jp} - X_{2jp}$

$D_j = \begin{pmatrix} D_{j1} \\ \vdots \\ D_{jp} \end{pmatrix}$ $j = 1 \ldots n$

Supongamos:

$E(D_j) = \delta = \begin{pmatrix} \delta_1 \\ \vdots \end{pmatrix}$ $cov(D_j) = \Sigma_d$

$$E(D_j) = \delta = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_p \end{pmatrix} \qquad cov(D_j) = \Sigma_d$$

Teorema:

Sean $D_1 \dots D_n$ independientes, asumimos que son $N_p(\delta, \Sigma_d)$

Entonces: $T^2 = n(\bar{D} - \delta)' S_d$

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

tiene distribución $\frac{(n-1)p}{(n-p)} F_{p,n-p}$

Si $n-p$ grande, $T^2 \overset{aprox}{\sim} \chi^2_p$

Dados los valores observados $d_j = \begin{pmatrix} d_{j1} \\ \vdots \\ d_{jp} \end{pmatrix}$

Podemos evaluar

$$H_0 : \delta = 0 \quad H_a : \delta \neq 0 \quad \text{con nivel } \alpha$$

Usando: $T^2 = n \bar{d}' S_d^{-1} \bar{d} \sim \frac{n-1}{n-p} p F_{p,n-p}$

Si H_0 es cierta

rechazamos H_0 si $T^2 > \frac{n-1}{n-p} p F_{p,n-p}(\alpha)$

Obs:

Una región de confianza $1-\alpha$ de δ son todos los δ t.q.

$$n(\bar{d}-\delta)' S_d^{-1} (\bar{d}-\delta) \leq \frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)$$

Los I.C. simultáneos con conf. $1-\alpha$ están dados por:

$$\bar{d}_i \pm \sqrt{\underbrace{\frac{(n-1)p}{n-p} F_{p,n-p}(\alpha)}_{\triangle}} \sqrt{\frac{S_{d_i}^2}{n}}$$

Si $n-p$ es grande $\triangle \approx \chi_p^2(\alpha)$

IC de Bonferroni

$$\bar{d}_i \pm t_{n-1}\left(\frac{\alpha}{2m}\right) \sqrt{\frac{S_{d_i}^2}{n}}$$

m normalmente $m = p$

Example 1: Effluent Data

$2p$

Sample j	Commercial lab		State lab of hygiene		dif V1	dif V2
	x_{1j1} (BOD)	x_{1j2} (SS)	x_{2j1} (BOD)	x_{2j2} (SS)		
1	6	27	25	15		
2	6	23	28	13		
3	18	64	36	22		
4	8	44	35	29		
5	11	30	15	31		
6	34	75	44	64		
7	28	26	42	30		
8	71	124	54	64		
9	43	54	34	56		
10	33	30	29	20		
11	20	14	39	21		
n					med	med

Example 1: Checking for a mean difference with paired observations

The T^2-statistic for testing $H_0 : \boldsymbol{\delta}' = [\delta_1, \delta_2] = [0, 0]$ is constructed from the differences of paired observations:

$d_{j1} = x_{1j1} - x_{2j1}$	-19	-22	-18	-27	-4	-10	-14	17	9	4	-19
:	12	10	42	15	-1	11	-4	60	-2	10	-7

$$\bar{\mathbf{d}} = \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \end{bmatrix} = \begin{bmatrix} -9.36 \\ 13.27 \end{bmatrix}, \quad \mathbf{S}_d = \begin{bmatrix} 199.26 & 88.38 \\ 88.38 & 418.61 \end{bmatrix}$$

$$T^2 = 11 \begin{bmatrix} -9.36, & 13.27 \end{bmatrix} \begin{bmatrix} .0055 & -.0012 \\ -.0012 & .0026 \end{bmatrix} \begin{bmatrix} -9.36 \\ 13.27 \end{bmatrix} = 13.6$$

For $\alpha = 0.05$: $[p(n-1)/(n-p)]F_{p,n-p}(.05) = [2(10)/9]F_{2,9}(.05) = 9.74$

Conclusion

Since $T^2 = 13.6 > 9.47$, we reject H_0 and conclude that there is a nonzero mean difference between the measurements of the two laboratories.

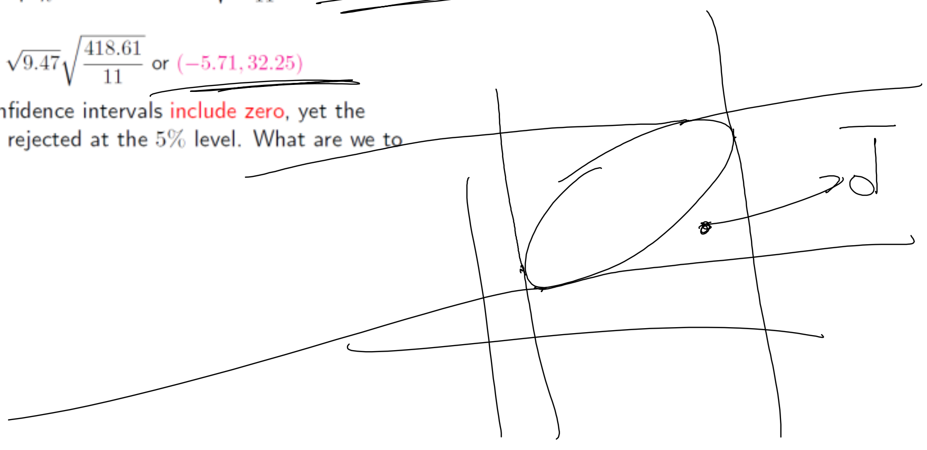
Example 1: Checking for a mean difference with paired observations

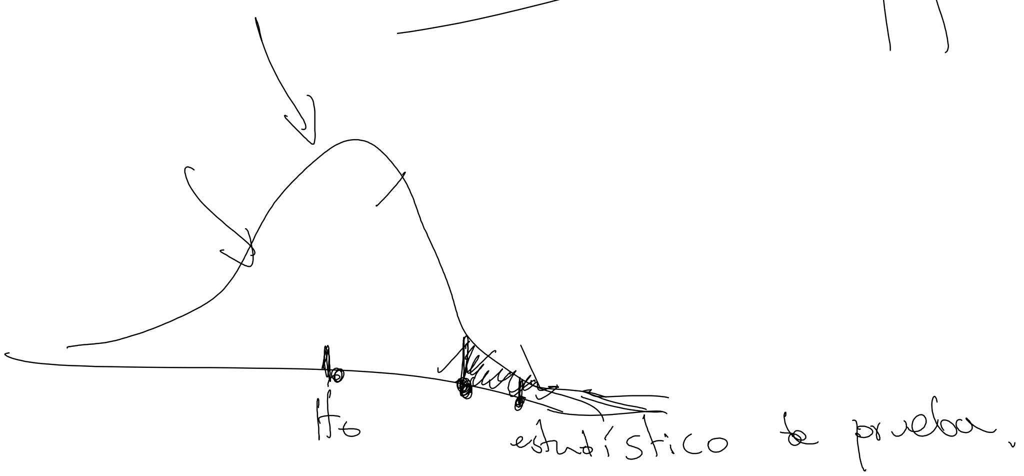
- It appears, that the commercial lab tends to produce lower BOD measurements and higher SS measurements than the State Lab of Hygiene.
- The 95% simultaneous confidence intervals for the mean differences δ_1 and δ_2 can be computed using $(6-10)$. These intervals are

$$\delta_1 : \bar{d}_1 \pm \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{s^2_{d}}{n}} = -9.36 \pm \sqrt{9.47} \sqrt{\frac{199.26}{11}} \text{ or } (-22.46, 3.74)$$

$$\delta_2 : 13.27 \pm \sqrt{9.47} \sqrt{\frac{418.61}{11}} \text{ or } (-5.71, 32.25)$$

- The 95% simultaneous confidence intervals include zero, yet the hypothesis $H_0 : \delta = 0$ was rejected at the 5% level. What are we to conclude?





Manera Alterna

$\bar{d}$ Sd y T^2 se pueden calcular a partir de $\bar{\mathbb{X}}$ y de S.

$$\underset{n\times 2p}{\mathbb{X}} = \left[\begin{array}{ccc|ccc} x_{111} & \cdots & x_{11p} & x_{121} & \cdots & x_{12p} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n11} & \cdots & x_{n1p} & x_{n21} & \cdots & x_{n2p} \end{array}\right]$$

$$\underset{2p\times 2p}{S} = \left(\begin{array}{c|c} \underset{p\times p}{S_{11}} & \underset{p\times p}{S_{12}} \\ \hline \underset{p\times p}{S_{21}} & \underset{p\times p}{S_{22}} \end{array}\right)$$

Si yo defino

$$\underset{p\times 2p}{C} = \left(\begin{array}{ccc|ccc} 1 & & 0 & -1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 1 & 0 & & -1 \end{array}\right)$$

$$\mathbb{d}_j = C\,\mathbb{X}_j$$

$$\bar{\mathbb{d}} = C\,\bar{\mathbb{X}} \qquad S_d = CSC'$$

$$T^2 = n\,\bar{\mathbb{X}}'C'(CSC')^{-1}C\bar{\mathbb{X}}$$

Los vectores fila de C se conocen como vectores de contraste.

Comparación de varios tratamientos:

Supongamos que se quieren comparar q tratamientos con una variable respuesta.

Cada item recibe cada tratamiento una

vez. La j-ésima observación es:

$$\mathbb{X}_j = \begin{pmatrix} X_{j1} \\ \vdots \\ X_{jq} \end{pmatrix} \qquad j=1,\dots,n$$

donde X_{ji} es la respuesta al i-ésimo tratamiento en el j-ésimo ítem.

Para comparar los q tratamientos podemos usar matrices de contraste sobre las componentes de $\mu = E(\mathbb{X}_j)$

Ejemplo

$$\begin{pmatrix} \mu_1-\mu_2 \\ \vdots \\ \mu_1-\mu_q \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \cdots & & & & \\ 1 & 0 & \cdots & 0 & -1 \end{pmatrix}}^{C} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_q \end{pmatrix}$$

Se puede estudiar "igualdad" de tratamientos evaluando $C\mu = 0$

En este caso, las medias muestrales son $C\bar{\mathbb{X}}$ y la matriz de cov muestrales CSC'

Y el estadístico de prueba

$$T^2 = n(C\bar{\mathbb{X}})'(CSC')^{-1}(C\bar{\mathbb{X}})$$

Concretamente.

Suponga una población $N_q(\mu, \Sigma)$

Sea C una matriz de contraste.

Si se tiene $H_0: C\mu = 0$ $H_a: C\mu \neq 0$ con nivel α

Rechazamos H_0 si:

$$T^2 = n(C\bar{X})'(CSC')^{-1}(C\bar{X}) > \frac{(n-1)\overbrace{(q-1)}^{p}}{(n-q+1)} * F^{(\alpha)}_{\underbrace{q-1}_{p},\, n-q+1}$$

La región de confianza para $C\mu$ sería

$$n(C\bar{X}-C\mu)'(CSC')^{-1}(C\bar{X}-C\mu) \leq \boxed{\frac{(n-1)(q-1)}{(n-q+1)} * F^{(\alpha)}_{q-1,\, n-q+1}}$$

pepe $\longrightarrow$

Los intervalos de confianza simultáneos con conf $1-\alpha$ para $C\mu$ serían:

$$c'\bar{X} \pm \sqrt{\text{pepe}}\sqrt{\frac{CSC'}{n}}$$

Solo incluiría el contraste que nos interesa.

Example 2: Dataset

Dog	Treatment 1	2	3	4
1	426	609	556	600
2	253	236	392	395
3	359	433	349	357
4	432	431	522	600
5	405	426	513	513
6	324	438	507	539
7	310	312	410	456
8	326	326	350	504
9	375	447	547	548
10	286	286	403	422
11	349	382	473	497
12	429	410	488	547
13	348	377	447	514
14	412	473	472	446
15	347	326	455	468
16	434	458	637	524
17	364	367	432	469
18	420	395	508	531
19	397	556	645	625

Example 2:Testing for equal treatments in a repeated measures design

The data set contains the four measurements for each of the 19 dogs, where

$$\text{Treatment } 1 = \text{high } \mathrm{CO_2} \text{ pressure without } \mathrm{H}$$

$$\text{Treatment } 2 = \text{low } \mathrm{CO_2} \text{ pressure without } \mathrm{H}$$

$$\text{Treatment } 3 = \text{high } \mathrm{CO_2} \text{ pressure with } \mathrm{H}$$

$$\text{Treatment } 4 = \text{low } \mathrm{CO_2} \text{ pressure with } \mathrm{H}$$

We shall analyze the anesthetizing effects of $\mathrm{CO_2}$ pressure and halothane from this repeated-measures design.

Example 2:Testing for equal treatments in a repeated measures design

There are three treatment contrasts that might be of interest in the experiment. Let μ_1, μ_2, μ_3, and μ_4 correspond to the mean responses for treatments $1, 2, 3$, and 4, respectively. Then

$$(\mu_3 + \mu_4) - (\mu_1 + \mu_2) = \begin{pmatrix} \text{Halothane contrast representing the} \\ \text{difference between the presence and} \\ \text{absence of halothane} \end{pmatrix}$$

$$(\mu_1 + \mu_3) - (\mu_2 + \mu_4) = \begin{pmatrix} \mathrm{CO_2} \text{ contrast representing the difference} \\ \text{between high and low } \mathrm{CO_2} \text{ pressure} \end{pmatrix}$$

$$(\mu_1 + \mu_4) - (\mu_2 + \mu_3) = \begin{pmatrix} \text{Contrast representing the influence} \\ \text{of halothane on } \mathrm{CO_2} \text{ pressure differences} \\ (\mathrm{H} - \mathrm{CO_2} \text{ pressure "interaction"}) \end{pmatrix}$$

With $\boldsymbol{\mu}' = [\mu_1, \mu_2, \mu_3, \mu_4]$, the contrast matrix $\mathbf{C}$ is

$$\mathbf{C} = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

From the data,

$$\overline{\mathbf{x}} = \begin{bmatrix} 368.21 \\ 404.63 \\ 479.26 \\ 502.89 \end{bmatrix} \text{ and } \mathbf{S} = \begin{bmatrix} 2819.29 & & & \\ 3568.42 & 7963.14 & & \\ 2943.49 & 5303.98 & 6851.32 & \\ 2295.35 & 4065.44 & 4499.63 & 4878.99 \end{bmatrix}$$