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Vector Space of Signals

The signals of practical interest in engineering can be classified in the following categories:

- Energy signals;
- Power signals;
- Polynomials;
- Exponential functions;
- Finite and infinite linear combinations of Dirac-delta impulses;
- Well-behaved complex-valued functions;
- (Piecewise) differentiable real-valued functions that do not grow faster than a polynomial;
- (Piecewise) continuous real-valued functions that do not grow faster than a polynomial.

All signals of practical interest belong to an infinite-dimensional vector space called the *signal space*

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Definition of Norm

Definition (Norm)

A ***norm*** is a real-valued function that assigns a positive *length* or *size* to a vector in a vector space

$$||\cdot|| \mapsto \mathbb{R} \tag{1}$$

Note that a zero length is just assigned to a zero vector

Properties of a Norm

Let $\mathbf{x}$ be an element of a finite or infinite-dimensional vector space[a]

[a] If $\mathbf{x} \in \mathbb{R}^{n\times 1}$, $\mathbf{x}$ is a vector, If $\mathbf{x}$ has infinite dimensions, then it is a function. For simplicity, we will only consider scalar one-dimensional signals. Thus, if $\mathbf{x}$ has infinite dimensions, we write $\mathbf{x} = x(t)$

Properties of a Norm

- $\|\mathbf{x}\| \geq 0$
- $\|\mathbf{x}\| = 0 \Leftrightarrow x = 0$
- $\|\bar{a}\,\mathbf{x}\| = |\bar{a}|\,\|\mathbf{x}\| \quad \forall\, \bar{a} \in \mathbb{C}$
- (Triangle Inequality) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

$\mathcal{L}$1-Norm

Let $x(t)$ be a real-valued signal (i.e., $x(t): (-\infty, \infty) \mapsto \mathbb{R}$)

Definition ($\mathcal{L}$1-Norm)

The ***$\mathcal{L}$1-Norm***[a] (or simply 1-Norm) of a signal $x(t)$ is *the integral of its absolute value*

$$\|x(t)\|_1 := \int_{-\infty}^{\infty} |x(t)|\, dt \tag{2}$$

[a] **(Nice to know :])** The $\mathcal{L}$ in the $\mathcal{L}$1-Norm refers to the fact that the signal must be *integrable in the sense of Lebesgue* for the integral to converge

$x(t)$ has a finite 1-Norm $\Leftrightarrow$ **$x(t)$ is absolutely integrable**

$\mathcal{L}$2-Norm

Definition ($\mathcal{L}$2-Norm)

The ***$\mathcal{L}$2-Norm*** (or simply 2-Norm) of a signal $x(t)$ is defined as

$$\|x(t)\|_2 := \sqrt{\left(\int_{-\infty}^{\infty} x^2(t)\, dt\right)} \tag{3}$$

Notice that the square of the 2-Norm of a signal corresponds to its energy

$$\|x(t)\|_2^2 = E_x \tag{4}$$

$x(t)$ has a finite 2-Norm $\Leftrightarrow$ **$x(t)$ is an energy signal**

$\mathcal{L}_p$ or p-Norm

Definition ($\mathcal{L}_p$ or p-Norm)

The ***$\mathcal{L}_p$ or p-Norm*** of a signal $x(t)$ is defined as

$$\|x(t)\|_p := \left(\int_{-\infty}^{\infty} x^p(t)\, dt \right)^{1/p} \tag{5}$$

Signals as Infinite Sets

Consider a real-valued signal $x(t)$. $x(t)$ can be seen as a ***mapping***, a ***transformation*** or a ***functional relation*** between two sets

$$x(t) \; : \; \underbrace{\overset{\overset{t}{\downarrow}}{\mathbb{R}}}_{\mathcal{D}[x(t)]} \mapsto \underbrace{\overset{\overset{x(t)}{\downarrow}}{\mathbb{R}}}_{\mathcal{R}[x(t)]} \tag{6}$$

The set $\mathcal{D}[x(t)]$ is known as the ***domain*** of $x(t)$, and $\mathcal{R}[x(t)]$ is the ***range*** of $x(t)$

For a CT function $x(t)$, the sets $\mathcal{D}[x(t)]$ and $\mathcal{R}[x(t)]$ are ***infinite sets*** (i.e., they contain an infinite amount of elements)

Maximum and Minimum of a Signal

Definition (Global Maximum)

A real-valued signal $x(t)$ has a ***global*** (or ***absolute***) ***maximum point at*** t^* if

$$x(t^*) \geq x(t) \quad \forall\, t \in \mathcal{D}[x(t)] \tag{7}$$

If $x(t^*) > x(t)\ \forall\, t \neq t^*$, then $x(t^*)$ is called a ***strict global maximum***

Definition (Global Minimum)

A real-valued signal $x(t)$ has a ***global*** (or ***absolute***) ***minimum point at*** t^* if

$$x(t^*) \leq x(t) \quad \forall\, t \in \mathcal{D}[x(t)] \tag{8}$$

If $x(t^*) < x(t)\ \forall\, t \neq t^*$, then $x(t^*)$ is called a ***strict global minimum***

Maximum and Minimum of a Signal

Definition (Local Maximum)

A real-valued signal x has a ***local*** (or ***relative***) ***maximum point at*** t^* if

$$\exists\, \varepsilon > 0 \mid x\left(t^*\right) \geq x\left(t\right)\ \forall\, t \in \left[t^* - \varepsilon,\ t^* + \varepsilon\right] \tag{9}$$

If $x\left(t^*\right) > x\left(t\right)\ \forall\, t \in \left[t^* - \varepsilon, t^* + \varepsilon\right]$ and $t \neq t^*$, then $x\left(t^*\right)$ is called a ***strict local maximum***

Definition (Local Minimum)

A real-valued signal x has a ***global*** (or ***absolute***) ***minimum point at*** t^* if

$$\exists\, \varepsilon > 0 \mid x\left(t^*\right) \leq x\left(t\right)\ \forall\, t \in \left[t^* - \varepsilon,\ t^* + \varepsilon\right] \tag{10}$$

If $x\left(t^*\right) < x\left(t\right)\ \forall\, t \in \left[t^* - \varepsilon, t^* + \varepsilon\right]$ and $t \neq t^*$, then $x\left(t^*\right)$ is called a ***strict local minimum***

Maximum and Minimum of a Signal

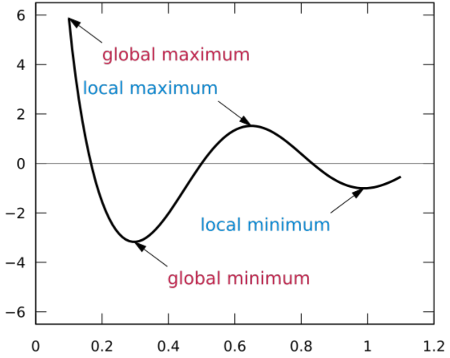


FIGURE: Global and local maxima and minima of a signal with domain $0.1 \leq t \leq 1.1$ (taken from https://goo.gl/5789rr)

Maximum and Minimum of a Signal

Theorem (Extreme Value Theorem)

If a real-valued signal $x(t)$ is continuous and with a closed domain $\mathcal{D}[x(t)] = [t_0, t_f]$, then $x(t)$ must reach a maximum and a minimum at least once

Nevertheless, the Extreme Value Theorem does not guarantee the existence of global maxima and minima for signals such as $x(t) = \sin(t)$ with domain $\mathcal{D}[x(t)] = (-\infty, \infty)$

Minima and maxima may exist (or not) for signals defined on open intervals such as $[0, \infty)$ or $(-\infty, \infty)$

Maximum and Minimum of a Signal

Maxima and minima can also be thought in terms of sets

- The ***maximum of a function***, if it exists, corresponds to the ***maximum element in the set*** $\mathcal{R}\left[x\left(t\right)\right]$
- Likewise, the ***minimum of a function***, if it exists, corresponds to the ***minimum value in the set*** $\mathcal{R}\left[x\left(t\right)\right]$

Upper and Lower Bound

Definition (Upper Bound)

$m \in \mathbb{R}$ is an ***upper bound*** of the set $\mathcal{R}[x(t)]$ if every element of $\mathcal{R}$ is less than m

$$m > \mathcal{R}[x(t)] \;\forall\; t \tag{11}$$

If m exists, $\mathcal{R}[x(t)]$ is said to be ***bounded above***

Definition (Lower Bound)

$l \in \mathbb{R}$ is a ***lower bound*** of the set $\mathcal{R}[x(t)]$ if every element of $\mathcal{R}$ is greater than l

$$l < \mathcal{R}[x(t)] \;\forall\; t \tag{12}$$

If l exists, $\mathcal{R}[x(t)]$ is said to be ***bounded below***

Notice that:

- A function $x(t)$ can have an endless quantity of upper and lower bounds
- The bounds do not have to belong to the range of the signal $x(t)$

Supremum and Infimum

Definition (Supremum)

If $\mathcal{R}\left[x\left(t\right)\right]$ is bounded above, then the ***supremum*** of $x\left(t\right)$ is the ***least upper bound*** of the signal

$$\sup_{t\,\in\,\mathcal{D}(t)} x\left(t\right) := \min\ M \text{ where } M = \left\{m_i \geq x\left(t\right)\ \forall\ t\right\}\ \ i = 1, 2, \ldots \tag{13}$$

Definition (Infimum)

If $\mathcal{R}\left[x\left(t\right)\right]$ is bounded below, then the ***infimum*** of $x\left(t\right)$ is the ***greatest lower bound*** of the signal

$$\inf_{t\,\in\,\mathcal{D}(t)} x\left(t\right) := \max\ L \text{ where } L = \left\{l_i \leq x\left(t\right)\ \forall\ t\right\}\ \ i = 1, 2, \ldots \tag{14}$$

The supremum and the infimum, if they exist, do not necessarily belong to the range of the signal $\mathcal{R}\left[x\left(t\right)\right]$

Supremum

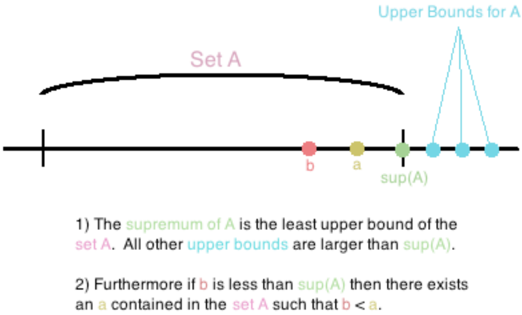


Figure: Supremum (taken from https://goo.gl/4bPaJL)

Infimum

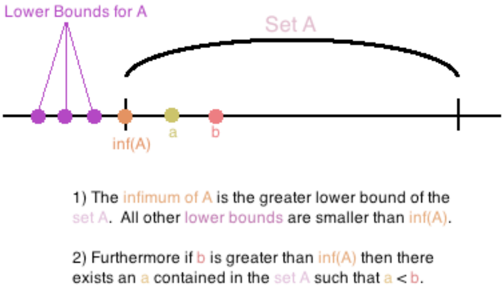


Figure: Infimum (taken from https://goo.gl/4bPaJL)

max AND sup; min AND inf

Lemma (Supremum and Global Maximum)

If the function $x(t)$ has a global maximum, then it corresponds to the supremum

$$\max_{t\,\in\,\mathcal{D}[x(t)]} x(t) = \sup_{t\,\in\,\mathcal{D}[x(t)]} x(t) \tag{15}$$

Lemma (Infimum and Global Minimum)

If the function $x(t)$ has a global minimum, then it corresponds to the infimum

$$\min_{t\,\in\,\mathcal{D}[x(t)]} x(t) = \inf_{t\,\in\,\mathcal{D}[x(t)]} x(t) \tag{16}$$

Example

Example 5.1

Find the supremum, infimum, maximum and minimum (if they exist) of the following signals

$$x_1(t) = 3e^{-t}u(t)$$

- **Supremum:** the least upper bound is 3. Therefore,

$$\sup_{t\in(-\infty,\infty)} x_1(t) = 3$$

- **Maximum:** the maximum exists (no formal proof will be given) and occurs at $t = 0$. Therefore,

$$\max\ x_1(t) = \sup_{t\in(-\infty,\infty)} x_1(t) = 3$$

Example

Example 5.1

Find the supremum, infimum, maximum and minimum (if they exist) of the following signals

$$x_1(t) = 3e^{-t}u(t)$$

- **Infimum:** the greatest lower bound is 0. Therefore,

$$\inf_{t \in (-\infty,\infty)} x_1(t) = 0$$

- **Minimum:** the minimum exists (no formal proof will be given) and occurs when $t < 0$. Thus,

$$\min x_1(t) = \inf_{t \in (-\infty,\infty)} x_1(t) = 0$$

Example

Example 5.2

Find the supremum, infimum, maximum and minimum (if they exist) of the following signals

$$x_2(t) = 0.5u(-t) + 2e^{-2t}u(t)$$

- **Supremum:** the least upper bound is 2.5. Therefore,

$$\sup_{t\,\in\,(-\infty,\infty)} x_2(t) = 2.5$$

- **Maximum:** since $u(-t) = 1$ and $u(t) = 1$ for $t = 0$, the maximum exists (no formal proof will be given) and occurs at $t = 0$. Therefore

$$\max x_2(t) = \sup_{t\,\in\,(-\infty,\infty)} x_2(t) = 2.5$$

Example

Example 5.2

Find the supremum, infimum, maximum and minimum (if they exist) of the following signals

$$x_2(t) = 0.5u(-t) + 2e^{-2t}u(t)$$

- **Infimum:** the greatest lower bound is 0 (as $t \to \infty$). Therefore,

$$\inf_{t \in (-\infty, \infty)} x_2(t) = 0$$

- **Minimum:** the minimum does not exist since the function $x_2(t)$ never attains the value of 0 (it approaches it asymptotically)

∞-Norm

Definition (∞-Norm)

The ***∞-Norm*** of a signal $x(t)$ is the supremum of its absolute value (i.e., the least upper bound)

$$\left\|x(t)\right\|_{\infty} := \sup_{t \in \mathcal{D}[x(t)]} \left|x(t)\right| \tag{17}$$

For example, the ∞-Norm of

$$x(t) = 2e^{-3t}u(t)$$

equals $\left\|2e^{-3t}u(t)\right\|_{\infty} = 2$

Bounded Signals and ∞-Norm

Bounded Signals Revisited

A signal $x(t)$ is ***bounded*** if and only if its ∞-Norm is finite

$$|x(t)| \leq B < \infty \Leftrightarrow \|x(t)\|_{\infty} < \infty \tag{18}$$

Relationship between Norms

1-, ∞- and 2-Norms

If the signal $x(t)$ has finite 1- and ∞-Norms

$$\begin{aligned} \|x(t)\|_1 &< \infty \\ \|x(t)\|_\infty &< \infty \end{aligned}$$

then $x(t)$ has a finite 2-Norm

$$\|x(t)\|_2 \leq \sqrt{\|x(t)\|_1 \, \|x(t)\|_\infty} \tag{19}$$

If the signal $x(t)$ is absolutely integrable (i.e., finite 1-Norm) and bounded above (i.e., finite ∞-Norm), then it has finite energy

Dot Product and Inner Product: Finite Vectors

Recall that if $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^{n\times 1}$, their ***dot product*** is given by

$$\mathbf{u}\cdot\mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

If we take the ***dot product of a vector with itself***, we obtain its ***squared length***

$$\mathbf{u}\cdot\mathbf{u} = ||\mathbf{u}||_2^2$$

Dot Product and Inner Product: Infinite Vectors

Now we will consider *vectors* with an infinite number of components, the so-called ***infinite vectors***. Let $\mathbf{u}$ and $\mathbf{w}$ be infinite vectors

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \dots & u_n & u_{n+1} & \dots \end{bmatrix}$$
$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \dots & v_n & v_{n+1} & \dots \end{bmatrix}$$

In this point, we will generalize the notion of dot product and define it as the ***inner product*** between vectors $\mathbf{u}$ and $\mathbf{v}$. In other words, **for finite vectors the dot and the inner product are the same**

$$\underbrace{\mathbf{u}\cdot\mathbf{v}}_{\text{Dot product for finite vectors}} = \underbrace{\langle \mathbf{u}, \mathbf{v}\rangle}_{\text{Inner product for finite vectors}}$$

Inner Product

(Nice to know :]) The inner product is a mapping from a vector space to a scalar field (usually $\mathbb{C}$) that satisfies three axioms.
Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ belong to the vector field, and $\bar{\alpha} \in \mathbb{C}$. The axioms are:

- **Conjugate symmetry**

$$< \mathbf{x}, \mathbf{y} > = (< \mathbf{y}, \mathbf{x} >)^*$$

- **Linearity in the first argument**

$$< \bar{a}\mathbf{x}, \mathbf{y} > = \bar{\alpha} < \mathbf{x}, \mathbf{y} >$$
$$< \mathbf{x} + \mathbf{y}, \mathbf{z} > = \ < \mathbf{x}, \mathbf{z} > + < \mathbf{y}, \mathbf{z} >$$

- **Positive-definiteness**

$$< \mathbf{x}, \mathbf{x} > \geq 0$$
$$< \mathbf{x}, \mathbf{x} > = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$$

Dot and Inner Product: Infinite Vectors

At this point we can observe that the inner product of two infinite vectors becomes an infinite series

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n + \cdots = \sum_{i=1}^{\infty} u_i v_i$$

What happens with the inner product of an infinite vector with itself?

If the dot product of a finite vector with itself yields the squared length of the vector, we can expect the same for infinite vectors

Inner Product of an Infinite Vector

Length Squared of an Infinite Vector

The ***length squared*** of an infinite vector is the inner product with itself

$$||\mathbf{u}||^2 := \langle \mathbf{u}, \mathbf{u} \rangle = \|\mathbf{u}\|_2^2 \tag{20}$$

If an infinite vector has finite length, we say that it belongs to a vector space known as *Hilbert space*

A one-dimensional vector with infinite components is equivalent to a function (signal)

Inner Product of Functions

Consider two functions $x_1(t)$ and $x_2(t)$ defined on an interval $[a, b]$. Since an integral fulfills all the properties of an inner product as long as the integral exists, we can make the following definition

Inner Product of Functions

The ***inner product*** of two functions $x_1(t)$ and $x_2(t)$ on an interval $[a, b]$ is the number

$$\langle x_1, x_2 \rangle := \int_a^b x_1(t)\, x_2(t)\, dt \tag{21}$$

Inner Product: Norm or Length of a Function

The definition of the inner product of two functions induces the idea of the norm or length of a function

2-Norm of a Function

The ***2-norm or length of a function*** on an interval $[a, b]$ is given by

$$\|x_1(t)\|_2 := \sqrt{\langle x_1, x_1 \rangle} = \sqrt{\int_a^b x_1^2(t)\, dt} \tag{22}$$

Orthogonal Functions and Sets

Motivated by the fact that two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal whenever their inner product is zero, we define orthogonal functions in a similar manner

Orthogonal Functions

Two functions $x_1(t)$ and $x_2(t)$ are ***orthogonal*** on an interval $[a, b]$ if

$$\langle x_1, x_2 \rangle = \int_a^b x_1(t)\, x_2(t)\, dt = 0 \qquad (23)$$

Orthogonal Sets

A set of real-valued functions $\{\phi_0(t), \phi_1(t), \phi_2(t), \ldots\}$ is said to be ***orthogonal*** on an interval $[a, b]$ if

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(t)\, \phi_n(t)\, dt = 0 \quad m \neq n \qquad (24)$$

Orthogonal Series Expansion

Suppose $\{\phi_n(t)\}$ is an infinite orthogonal set of functions on an interval $[a, b]$. Our concern is as follows:

If $x(t)$ is a function defined on the interval $[a, b]$, we have to determine a set of coefficients $c_n, \; n = 0, 1, 2, \ldots$ for which

$$x(t) = c_0\phi_0(t) + c_1\phi_1(t) + \cdots + c_n\phi_n(t) + \ldots$$

In other words

We want to write $x(t)$ as an infinite linear combination of the set $\{\phi_n(t)\}$

Orthogonal Series Expansion

$$x(t) = c_0\phi_0(t) + c_1\phi_1(t) + \cdots + c_n\phi_n(t) + \dots$$

We can find the desired coefficients by using the inner product. Multiply the last expression by $\phi_n(t)$

$$x(t)\phi_n(t) = c_0\phi_0(t)\phi_n(t) + c_1\phi_1(t)\phi_n(t) + \cdots + c_n\phi_n(t)\phi_n(t) + \dots$$

Integrating over the interval $[a, b]$ gives

$$\int_a^b x(t)\phi_n(t)dt = c_0\int_a^b \phi_0(t)\phi_n(t)dt + c_1\int_a^b \phi_1(t)\phi_n(t)dt + \\ + \cdots + c_n\int_a^b \phi_n(t)\phi_n(t)dt + \dots$$

Each of these terms is an inner product with the function $\phi_m(t)$. Then,

$$\langle x, \phi_n\rangle = c_0\langle\phi_0, \phi_n\rangle + c_1\langle\phi_1, \phi_n\rangle + \cdots + c_n\langle\phi_n, \phi_n\rangle + \dots$$

Orthogonal Series Expansion

$$\langle x, \phi_n \rangle = c_0 \langle \phi_0, \phi_n \rangle + c_1 \langle \phi_1, \phi_n \rangle + \cdots + c_n \langle \phi_n, \phi_n \rangle + \ldots$$

Since the set is orthogonal, the inner products $\langle \phi_n, \phi_m \rangle$ will be zero for any $n \neq m$. Only $\langle \phi_n, \phi_n \rangle$ will not vanish. Thus, we have

$$\langle x, \phi_n \rangle = c_n \langle \phi_n, \phi_n \rangle$$

We can determine the desired coefficients as

$$c_n = \frac{\langle x, \phi_n (t) \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\int_a^b x (t) \phi_n (t) \, dt}{\int_a^b \phi_n^2 (t) \, dt}$$

Finally, the function $x (t)$ can be expressed as

$$x (t) = \sum_{n=0}^{\infty} c_n \phi_n (t)$$

which is known as an ***orthogonal series expansion*** or ***generalized Fourier series*** of x

Generalized Fourier Series

Generalized Fourier Series

The function $x(t)$ can be expressed as an infinite linear combination of the orthogonal set $\{\phi_n(t)\}$ on the interval $[a, b]$ as

$$x(t) = \sum_{n=0}^{\infty} c_n \phi_n(t) \tag{25}$$

This is known as a ***generalized Fourier series*** or an ***orthogonal series expansion***. The coefficients c_n are found by

$$c_n = \frac{\langle x, \phi_n(t) \rangle}{||\phi_n(t)||^2} = \frac{\int_a^b x(t)\,\phi_n(t)\,dt}{\int_a^b \phi_n^2(t)\,dt} \tag{26}$$

Important Results of Trigonometric Integrals

Let $m, n \in \mathbb{Z}$ such that $m \neq n$

Period 2π ($\omega_0 = 1$)	Angular frequency ($\omega_0 = \frac{2\pi}{T}$)
$\int_{-\pi}^{\pi} \cos(nt)\, dt = 0$	$\int_{<T>} \cos(n\omega_0 t)\, dt = 0$
$\int_{-\pi}^{\pi} \cos^2(nt)\, dt = \pi$	$\int_{<T>} \cos^2(n\omega_0 t)\, dt = \frac{T}{2}$
$\int_{-\pi}^{\pi} \sin(nt)\, dt = 0$	$\int_{<T>} \sin(n\omega_0 t)\, dt = 0$
$\int_{-\pi}^{\pi} \sin^2(nt)\, dt = \pi$	$\int_{<T>} \sin^2(n\omega_0 t)\, dt = \frac{T}{2}$
$\int_{-\pi}^{\pi} \cos(nt)\sin(nt)\, dt = 0$	$\int_{<T>} \cos(n\omega_0 t)\sin(n\omega_0 t)\, dt = 0$
$\int_{-\pi}^{\pi} \cos(nt)\sin(mt)\, dt = 0$	$\int_{<T>} \cos(n\omega_0 t)\sin(m\omega_0 t)\, dt = 0$

The functions sine and cosine are orthogonal over a period

CT Fourier Series: Definition

Fourier Series

The ***Fourier Series of a periodic signal*** $x(t)$ is a decomposition of $x(t)$ into a dc component and an ac component that consists of an infinite series of harmonically related sinusoids [Sadiku, 2015].

Harmonically Related Sinusoids

Sinusoids whose frequencies are multiples of a fundamental frequency ω_0 (or first harmonic) such as

$$\sin(\omega_0 t), \sin(2\omega_0 t), \sin(3\omega_0 t), \ldots$$
$$\cos(\omega_0 t), \cos(2\omega_0 t), \cos(3\omega_0 t), \ldots$$
$$e^{\pm j\omega_0 t}, e^{\pm j2\omega_0 t}, e^{\pm j3\omega_0 t}, \ldots$$

Essence of Frequency Domain Analysis

Signal Analysis: Change of Domain

A signal $x(t)$ will be represented as an infinite linear combination of single-frequency building blocks. Thus, a signal is seen as a superposition of ***single-frequency components.*** In other words, Fourier analysis helps see a signal in terms of frequency instead of time

LTI System Analysis

If a signal can be expressed as an infinite sum of single-frequency components, ***finding out how an LTI system responds to each term helps understand the overall or complete system behavior in response to the signal*** [Alkin, 2014]

CT Fourier Series: Different Forms

Depending on the complete set used to express the periodic function, the Fourier Series can have three different representations

- **Complex Exponential Fourier Series (EFS):** uses the complete set $\left\{e^{jk\omega_0 t}, e^{-jk\omega_0 t}\right\}$ as a basis to generate periodic functions
- **Trigonometric Fourier Series (TFS):** uses the complete set $\{1, \cos\left(k\omega_0 t\right), \sin\left(k\omega_0 t\right)\}$ as a basis to generate periodic functions
- **Amplitude-Phase Fourier Series (AFS):** uses the complete set $\{\cos\left(k\omega_0 t + \phi_k\right)\}$ as a basis to generate periodic functions

CT Fourier Series: Conditions of Convergence

Dirichlet Conditions

A periodic function $x(t)$ can be expanded as a Fourier series only if it fulfills the ***Dirichlet conditions*** given as:

- $x(t)$ should be absolutely integrable over any period

$$\int_{<T>} |x(t)|\,dt < \infty$$

- $x(t)$ has only a finite number of maxima and minima over any period;
- $x(t)$ has only a finite number of discontinuities over any period.

For real-world signals, there is no need of evaluating these conditions because they will always be met

Every real-world signal is absolutely integrable and has finite total energy [Chen, 2009]

CT Trigonometric Fourier Series (TFS)

CT Trigonometric Fourier Series (TFS)

If $x\left(t\right)$ is a periodic signal with fundamental period $T_0 = T$ and fundamental angular frequency $\omega_0 = \frac{2\pi}{T_0}$, its ***Fourier Series expansion*** is given in ***trigonometric form (TFS)*** by

$$x\left(t\right) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos\left(k\omega_0 t\right) + b_k \sin\left(k\omega_0 t\right)\right] \tag{27}$$

where

$$\begin{aligned} a_0 &= \frac{1}{T}\int_{<T>} x\left(t\right) dt = \frac{\langle x(t), 1\rangle_{<T>}}{\|1\|^2_{<T>}} \\ a_k &= \frac{2}{T}\int_{<T>} x\left(t\right)\cos\left(k\omega_0 t\right) dt = \frac{\langle x\left(t\right), \cos\left(k\omega_0 t\right)\rangle_{<T>}}{\|\cos\left(k\omega_0 t\right)\|^2_{<T>}} \\ b_k &= \frac{2}{T}\int_{<T>} x\left(t\right)\sin\left(k\omega_0\right) tdt = \frac{\langle x\left(t\right), \sin\left(k\omega_0 t\right)\rangle_{<T>}}{\|\sin\left(k\omega_0 t\right)\|^2_{<T>}} \end{aligned} \tag{28}$$

CT Amplitude-Phase Fourier Series (AFS)

CT Amplitude-Phase Fourier Series (AFS)

If $x(t)$ is a periodic signal with fundamental period $T_0 = T$ and fundamental angular frequency $\omega_0 = \frac{2\pi}{T_0}$, its ***Fourier series expansion*** is given in ***Amplitude-Phase form (AFS)*** by

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[A_k \cos\left(k\omega_0 t + \phi_k\right)\right] \tag{29}$$

where

$$a_0 = \frac{1}{T}\int_{<T>} x(t)\, dt = \frac{\langle x(t), 1\rangle}{||1||^2_{<T>}}$$

$$a_k = \frac{2}{T}\int_{<T>} x(t)\cos\left(k\omega_0 t\right) dt = \frac{\langle x(t), \cos\left(k\omega_0 t\right)\rangle_{<T>}}{||\cos\left(k\omega_0 t\right)||^2_{<T>}}$$

$$b_k = \frac{2}{T}\int_{<T>} x(t)\sin\left(k\omega_0 t\right) dt = \frac{\langle x(t), \sin\left(k\omega_0 t\right)\rangle_{<T>}}{||\sin\left(k\omega_0 t\right)||^2_{<T>}} \tag{30}$$

$$A_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = -\tan^{-1}\left(\frac{b_k}{a_k}\right) \tag{31}$$

CT Complex Exponential Fourier Series (EFS)

CT Complex Exponential Fourier Series (EFS)

If $x(t)$ is a periodic signal with fundamental period $T_0 = T$ and fundamental angular frequency $\omega_0 = \frac{2\pi}{T_0}$, its ***Fourier Series expansion*** is given in ***Complex Exponential form (EFS)*** by:

$$x(t) = \sum_{k=-\infty}^{\infty} \bar{c}_k e^{jk\omega_0 t} \tag{32}$$

where

$$\bar{c}_k = \frac{1}{T} \int_{<T>} x(t)\, e^{-jk\omega_0 t} dt = \frac{\left\langle x(t), e^{-jk\omega_0 t} \right\rangle_{<T>}}{||1||^2_{<T>}} \tag{33}$$

with

$$\bar{c}_k := \frac{a_k - jb_k}{2} \tag{34}$$

Physical Definition of Frequency

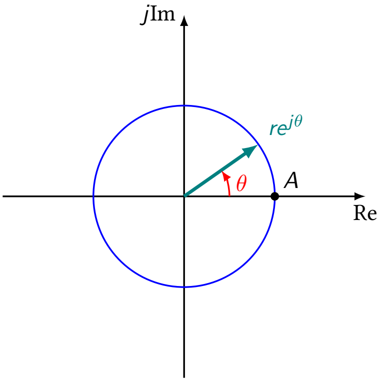


FIGURE: Spinning wheel.

Consider a spinning wheel with mark A as shown in Figure 4.

The following definitions hold only if the wheel rotates at a constant speed [Chen, 2009]

- **Period (T):** time (in seconds) for the mark A to complete one cycle
- **Frequency (f):** the number of cycles the mark A rotates in a second (in Hz - cycles per second)

Physical Definition of Frequency

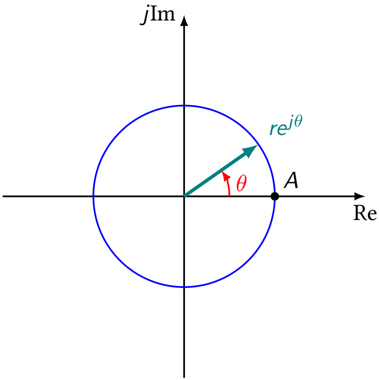


Figure: Spinning wheel.

If the wheel starts to rotate from $t = 0$ onward, then the angle or phase θ will increase with time. If the wheel rotates with a constant angular speed ω_0 [rad/s] in the counterclockwise direction, then we have:

$$\theta = \omega_0 t$$

Therefore, the rotation of mark A can be expressed as

$$x(t) = re^{j\omega_0 t}$$

Physical Definition of Frequency

Because one cycle has 2π radians, we have:

$$\frac{\overbrace{2\pi}^{\text{distance}}}{\underbrace{T}_{\text{time}}} = \underbrace{\omega_0}_{\substack{\text{angular}\\ \text{speed}}} \qquad \rightarrow T = \frac{2\pi}{\omega_0}$$

If we use f_0 to denote the frequency, then

$$f_0 = \frac{1}{T} \text{ [Hz]}$$

As one cycle has 2π radians, the angular frequency can also be defined as:

$$\omega_0 := 2\pi f_0 = \frac{2\pi}{T} \left[\frac{rad}{s}\right]$$

Physical Definition of Frequency

If $\omega_0 > 0$, then A or $e^{j\omega_0 t}$ rotates in the counterclockwise direction and $e^{-j\omega_0 t}$ rotates in the clockwise direction. Thus, we will encounter both positive and negative frequencies.

Physically there is no such thing as a negative frequency; the negative sign merely indicates its direction of rotation.

In theory, the wheel can spin as fast as desired. Thus, we have

$$\text{Frequency range of } e^{j\omega_0 t} \rightarrow (-\infty, \infty)$$

To conclude this section, we mention that if we use $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$ to define the frequency, then the meaning of negative frequency is not clear because:

$$\sin(-\omega_0 t) = -\sin(\omega_0 t) = \sin(\omega_0 t + \pi)$$
$$\cos(-\omega_0 t) = \cos(\omega_0 t)$$

Synthesis and Analysis Equations

	Synthesis equations
Trigonometric form	$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)]$
Complex exponential form	$x(t) = \sum_{k=-\infty}^{\infty} \bar{c}_k e^{j\omega_0 kt}$
	Analysis equations
Trigonometric form	$a_0 = \frac{1}{T}\int_{<T>} x(t)\,dt = \frac{\langle x(t), 1\rangle_{<T>}}{\|\|1\|\|^2_{<T>}}$ $a_k = \frac{2}{T}\int_{<T>} x(t)\cos(k\omega_0 t)\,dt = \frac{\langle x(t), \cos(k\omega_0 t)\rangle_{<T>}}{\|\|\cos(k\omega_0 t)\|\|^2_{<T>}}$ $b_k = \frac{2}{T}\int_{<T>} x(t)\sin(k\omega_0 t)\,dt = \frac{\langle x(t), \sin(k\omega_0 t)\rangle_{<T>}}{\|\|\sin(k\omega_0 t)\|\|^2_{<T>}}$
Complex exponential form	$\bar{c}_k = \frac{1}{T}\int_{<T>} x(t)\,e^{-jk\omega_0 t}dt = \frac{\langle x(t), e^{-jk\omega_0 t}\rangle_{<T>}}{\|\|1\|\|^2_{<T>}}$

Relationships between Forms

TFS		AFS		EFS
$a_k - jb_k$	$=$	$\frac{\lvert \bar{c}_k \rvert}{2} \angle \tan^{1}\left(\frac{\mathrm{Im}\{\bar{c}_k\}}{\mathrm{Re}\{\bar{c}_k\}}\right)$	$=$	$2\bar{c}_k$
$\frac{\sqrt{a_k^2+b_k^2}}{2} \angle -\tan^{-1}\frac{b_k}{a_k}$	$=$	$\frac{A_k}{2}\angle\phi_k$	$=$	$\lvert \bar{c}_k \rvert \angle \theta_k$

Table: Equivalence between forms of the Fourier Series

Line Spectrum

Line Spectrum

The amplitude of each frequency component of a real-valued signal $x(t)$ is known as ***line spectrum or discrete spectrum***. It gives a great deal of information at a glance. In particular, we can see how many terms of the series are required to obtain a reasonable approximation of the original waveform. This is constructed from

$$\begin{aligned} A_k &= \sqrt{a_k^2 + b_k^2} && \text{(Trigonometric Form)} \\ 2|\bar{c}_k| &= \sqrt{a_k^2 + b_k^2} = |\bar{c}_k| + |\bar{c}_{-k}| && \text{(Complex Exponential Form)} \end{aligned} \tag{35}$$

Phase Spectrum

Phase Spectrum

The ***phase spectrum*** is the plot of frequency versus phase of

$$
\begin{aligned}
\phi_k &= -\tan^{-1}\left(\frac{b_k}{a_k}\right) && \text{(Trigonometric Form)} \\
\phi_k &= \theta_k = -\angle \bar{c}_k && \text{(Complex Exponential Form)}
\end{aligned}
\tag{36}
$$

Even Symmetry

Even Symmetry Revisited

A function possesses ***even symmetry*** if

$$x(t) = x(-t)$$

In other words, the replacement of t by $-t$ does not change the value of the function. Graphically, there exists mirror symmetry about the vertical axis

The Fourier Series of any even function $x(t)$ has $b_k = 0$. Conversely, if $b_k = 0$, then $x(t)$ must have even symmetry. Equivalently, all the coefficients $\bar{c}_k$ in the complex exponential form are real

Odd Symmetry

Odd Symmetry Revisited

A function has the property of ***odd symmetry*** if

$$x(t) = -x(-t)$$

In other words, if t is replaced by $-t$, then the negative of the given function is obtained. Graphically, we have symmetry about the origin, rather than about the vertical axis as we did for even signals.

The Fourier Series of any odd function $x(t)$ has $a_k = 0$. Conversely, if $a_k = 0$, then $x(t)$ must have even symmetry. Equivalently, all the coefficients $\bar{c}_k$ in the complex exponential form are imaginary

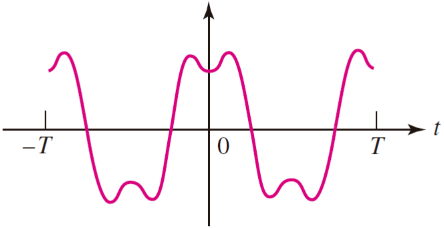
Half-Wave Symmetry

Half-Wave Symmetry

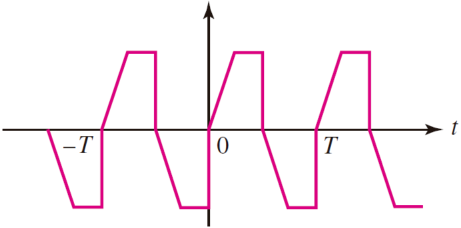
A signal is said to have ***half-wave symmetry*** if, except for a change of sign, each half cycle is like the adjacent half cycles. In other words, the shape of the negative half-cycle of the waveform is the same as that of the positive-half cycle, but inverted

$$-x\left(t+\frac{T}{2}\right)=x\left(t\right) \text{ or } -x\left(t-\frac{T}{2}\right)=x\left(t\right)$$

Half-Wave Symmetry



(a)



(b)

Figure: Examples of signals possessing half-wave symmetry [Hayt et al., 2008]

Half-Wave Symmetry

Testing Half-Wave Symmetry

- Choose any half-period $T/2$ length on the time axis
- Observe the values of $x(t)$ at the left and right points on the time axis
- If there is half-wave symmetry, these will always be equal but will have opposite signs as we slide the half-period $T/2$ length to the left or to the right on the time axis at non-zero values of $x(t)$

The Fourier Series of a signal $x(t)$ exhibiting half-wave symmetry has $a_k = b_k = 0$ for k even. In other words, it contains only odd harmonics. Conversely, if $a_k = b_k = 0$, then $x(t)$ must have half-wave symmetry. Equivalently, in the complex exponential form we have all the coefficients $\bar{c}_k = 0$ for k even

Summary of Symmetry Results

	Simplification	Trigonometric	Complex exponential
Even	$b_k = 0$ $\bar{c}_k$ real	$a_k = \frac{4}{T}\int_{<\frac{T}{2}>} x(t)\cos(k\omega_0 t)\,dt$	$\bar{c}_k = \frac{2}{T}\int_{<\frac{T}{2}>} x(t)\cos(k\omega_0 t)\,dt$
Odd	$a_k = 0$ $\bar{c}_k$ imaginary	$b_k = \frac{4}{T}\int_{<\frac{T}{2}>} x(t)\sin(k\omega_0 t)\,dt$	$\bar{c}_k = \frac{-j2}{T}\int_{<\frac{T}{2}>} x(t)\sin(k\omega_0 t)\,dt$
Half-wave	$a_k = b_k = 0$ $\bar{c}_k = 0$ (for k even)	$a_k = \frac{4}{T}\int_{<\frac{T}{2}>} x(t)\cos(k\omega_0 t)\,dt$ $b_k = \frac{4}{T}\int_{<\frac{T}{2}>} x(t)\sin(k\omega_0 t)\,dt$	$\bar{c}_k = \begin{cases} \frac{2}{T}\int_{<\frac{T}{2}>} x(t)\,e^{-jk\omega_0 t}\,dt & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$
Half-wave and even	$b_k = 0$ $a_k = \bar{c}_k = 0$ (for k even)	$a_k = \begin{cases} \frac{8}{T}\int_{<\frac{T}{4}>} x(t)\cos(k\omega_0 t)\,dt & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$ $b_k = 0$ for all k	$\bar{c}_k = \begin{cases} \frac{4}{T}\int_{<\frac{T}{4}>} x(t)\cos(k\omega_0 t)\,dt & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$
Half-wave and odd	$a_k = 0$ $b_k = \bar{c}_k = 0$ (for k even)	$a_k = 0$ for all k $b_k = \begin{cases} \frac{8}{T}\int_{<\frac{T}{4}>} x(t)\sin(k\omega_0 t)\,dt & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$	$\bar{c}_k = \begin{cases} \frac{-j4}{T}\int_{<\frac{T}{4}>} x(t)\sin(k\omega_0 t)\,dt & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$

Linearity of CT Fourier Series

Linearity of FS

Let $x(t)$ and $y(t)$ be two periodic signals with the *same fundamental period* T. If they have the Fourier series expansion defined by the complex coefficients $\bar{\alpha}_k$ and $\bar{\beta}_k$

$$
\begin{aligned}
x(t) &\overset{\text{FS}}{\rightarrow} \bar{\alpha}_k \\
y(t) &\overset{\text{FS}}{\rightarrow} \bar{\beta}_k
\end{aligned}
$$

then

$$k_1 x(t) + k_2 y(t) \overset{\text{FS}}{\rightarrow} k_1\bar{\alpha}_k + k_2\bar{\beta}_k \tag{37}$$

Time Shifting of CT Fourier Series

Time Shifting of FS

If $x(t)$ is a periodic signal with period T such that

$$x(t) \quad \overset{\text{FS}}{\rightarrow} \quad \bar{\alpha}_k$$

The time-shifted version $x(t - t_0)$ will have the following Fourier representation

$$x(t - t_0) \quad \overset{\text{FS}}{\rightarrow} \quad e^{-jk\omega_0 t_0}\bar{\alpha}_k \tag{38}$$

When a time shift is applied to a periodic signal $x(t)$, the period T of the signal is preserved [Sadiku, 2015]

When a periodic signal is shifted in time, the magnitude of the Fourier coefficients remains unaltered [Sadiku, 2015]

Time Reversal of CT Fourier Series

Time Reversal

If $x(t)$ is a periodic signal with period T such that

$$x(t) \quad \overset{\text{FS}}{\rightarrow} \quad \bar{\alpha}_k$$

The time-inverted version $x(-t)$ will have the following Fourier representation

$$x(-t) \quad \overset{\text{FS}}{\rightarrow} \quad \bar{\alpha}_{-k} \tag{39}$$

When $x(t)$ is even, then its Fourier series coefficients are also even. When $x(t)$ is odd, then its Fourier series coefficients are also odd [Sadiku, 2015]

Time Scaling of CT Fourier Series

Time Scaling

If $x(t)$ is a periodic signal with period T such that

$$x(t) \overset{\text{FS}}{\rightarrow} \bar{\alpha}_k$$

The time-scaled version $x(at)$ will have the following Fourier representation

$$x(at) \overset{\text{FS}}{\rightarrow} \bar{\alpha}_k \tag{40}$$

The value of the harmonic contents does not change with time scaling. Nevertheless, the frequency representation is shifted depending on the transformation (the frequency ω_0 is moved to $a\omega_0$)

Derivative

FS of a Derivative

Let $x\left(t\right)$ is a periodic signal with period T such that

$$x\left(t\right) \quad \xrightarrow{\text{FS}} \quad \bar{\alpha}_k$$

Its derivative has the following Fourier Series representation

$$x'\left(t\right) \xrightarrow{\text{FS}} \left(jk\omega_0\right)\bar{\alpha}_k \tag{41}$$

Integral

FS of an Integral

Let $x(t)$ is a periodic signal with period T such that

$$x(t) \xrightarrow{\text{FS}} \bar{\alpha}_k$$

Its integral has the following Fourier Series representation

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xrightarrow{\text{FS}} \left(\frac{1}{jk\omega_0}\right) \bar{\alpha}_k \tag{42}$$

Parseval's Theorem

Parseval's Theorem

The average power of the signal $x(t)$ over one period equals the sum of the squared magnitudes of all the complex Fourier coefficients

$$P = \sum_{k=-\infty}^{\infty} ||\bar{c}_k||^2 \tag{43}$$

This results holds for the Trigonometric Fourier Series

$$P = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right) \tag{44}$$

and the Amplitude-Phase form

$$P = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty} A_k^2 \tag{45}$$

Gibbs Phenomemon

(Nice to know :]) Dirichlet conditions do not require a periodic signal to be continuous. Actually, we have computed FS representations for discontinuous signals such as pulse trains

However, this should be counter-intuitive

How is it possible to reconstruct a discontinuous signal using only continuous functions (such as sinusoids and complex exponentials)?

Gibbs Phenomenon

To gain insight into this idea, consider the following square wave

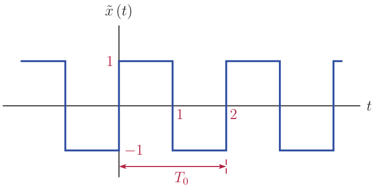


FIGURE: A square wave with fundamental period $T_0 = T = 2$ [Alkin, 2014]

This signal is *odd* ($a_k = 0$) and has *half-wave symmetry* ($b_k = 0 \; \forall \; k$ even). Therefore, its TFS is given by

$$x(t) = \sum_{k=1}^{\infty} b_k \sin\left[(2k-1)\,\omega_0 t\right] = \sum_{k=1}^{\infty} \left(\frac{4}{\pi k}\right) \sin\left[(2k-1)\,\pi t\right]$$

Gibbs Phenomenon

A finite harmonic approximation of $x\left(t\right)$ is

$$x^{(m)}\left(t\right) := \sum_{k=1}^{m} \left(\frac{4}{\pi k}\right) \sin\left[\left(2k-1\right)\pi t\right]$$

Let us define the approximation error as

$$\varepsilon^{(m)}\left(t\right) = x\left(t\right) - x^{(m)}\left(t\right)$$

Gibbs Phenomenon

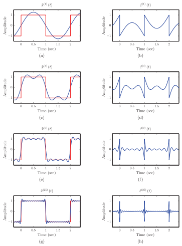


Figure: Several finite harmonic approximations for $x(t)$ [Alkin, 2014]

Gibbs Phenomenon

From this thought experiment, we conclude the following [Alkin, 2014]:

- The approximation error is discontinuous. It is always large at the discontinuities;
- However, $\varepsilon(t)$ gets smaller *away from discontinuities* and decreases as the number of harmonics in the approximation increases;
- The FS approximation *overshoots* the actual signal value at one side of the discontinuity, and *undershoots* it at the other side;
- At a discontinuity, the FS approximation of a discontinuous signal yields the *average value* of the signal amplitudes right before and after the discontinuity.

The amount of overshoot or undershoot is approximately 9% of the height of the discontinuity, and cannot be reduced by increasing m. This is called a particular case of *Gibbs Phenomenon*

Introduction - Discrete-Time Fourier Series (DTFS)

It is also possible to express a discrete-time periodic signal as a linear combination of discrete-time periodic basis functions (i.e., DT sinusoids and/or complex exponentials)

- **Continuous-Time:** a continuous-time periodic signal may have an infinite range of frequencies (i.e., an infinite number of harmonically related basis functions)
- **Discrete-Time:** DT periodic signals contain a ***finite*** range of angular frequencies. Therefore, they require a ***finite*** number of harmonically related basis functions

DTFS - Synthesis Equation

Let $x[n]$ be a DT *periodic* signal with a period of N samples:

$$x[n] = x[n+N]$$

We would like to write $x[n]$ as a linear combination of complex exponential basis functions of the form

$$\phi_k[n] = e^{j\Omega_k n}$$

so that the resulting expression of $x[n]$ will look like:

$$x[n] = \sum_k \bar{c}_k \phi_k[n] = \sum_k \bar{c}_k \phi_k[n]$$

DTFS - Synthesis Equation

$$x[n] = \sum_k \bar{c}_k \phi_k[n] = \sum_k \bar{c}_k \phi_k[n]$$

If $x[n]$ is periodic, the basis functions used in reconstructing the signal must also be periodic. That is,

$$\phi_k[n] = \phi_k[n+N]$$

Considering the form of $\phi_k[n]$, we have the following condition

$$\phi_k[n] = e^{j\Omega_k n} = e^{j\Omega_k(n+N)} = e^{j\Omega_k n} e^{j\Omega_k N}$$

Equivalently

$$e^{j\Omega_k n} = e^{j\Omega_k n} \underbrace{e^{j\Omega_k N}}_{=1}$$

So, we must have

$$\Omega_k N = 2\pi k \quad k \in \mathbb{Z}$$

DTFS - Synthesis Equation

The frequencies of the basis functions must follow

$$\Omega_k = \frac{2\pi k}{N} \quad k \in \mathbb{Z}$$

Under this condition, the form of the periodic basis functions is

$$\phi_k[n] = e^{j\frac{2\pi}{N}kn} = e^{j\Omega_0 kn}$$

where

$$\Omega_0 := \frac{2\pi}{N} \tag{46}$$

Harmonically Related DT Complex Exponentials

Harmonically Related DT Complex Exponentials

The set $\phi_k[n]$ has fundamental frequency parameters who are multiples of $\frac{2\pi}{N}$

$$\phi_k[n] = e^{jk\Omega_0 n} = e^{jk\frac{2\pi}{N}n} \; k = 0, \pm 1, \pm 2, \ldots \tag{47}$$

All of these signals are ***harmonically related***. Furthermore, there are only N distinct signals in this set because discrete-time complex exponentials which differ in frequency by a multiple of 2π are identical

$$\phi_k[n] = \phi_{k+rN}[n] \;\; k, r \in \mathbb{Z} \tag{48}$$

In other words, when k is changed by any integer multiple of N, the identical sequence is generated [Oppenheim and Willsky, 1998]. For any given periodic signal $x[n]$ with period N, there are only N different complex exponentials

DTFS - Analysis Equation

Since there are only N different harmonically related complex exponentials, we must add up to N terms. We may therefore add functions in the set $\phi_k[n]$ from $k = 0$ to $k = N-1$

$$x[n] = \sum_{k=0}^{N-1} \bar{c}_k e^{j\frac{2\pi}{N}kn}$$

If we multiply both sides of the equation by $e^{-j\frac{2\pi}{N}mn}$, we get

$$x[n]\, e^{-j\frac{2\pi}{N}mn} = \underbrace{e^{-j\frac{2\pi}{N}mn}}_{\text{Constant w.r.t. } k} \sum_{k=0}^{N-1} \bar{c}_k e^{j\frac{2\pi}{N}kn} = \sum_{k=0}^{N-1} \bar{c}_k e^{j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}mn}$$

Applying exponential properties and summing the terms on both sides for $n = 0, \ldots, N-1$ yields

$$\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}mn} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \bar{c}_k e^{j\frac{2\pi}{N}(k-m)n}$$

DTFS - Analysis Equation

$$\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}mn} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} \bar{c}_k e^{j\frac{2\pi}{N}(k-m)n}$$

We may interchange the order of the summations on the right-hand side of the last equation:

$$\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}mn} = \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} \bar{c}_k e^{j\frac{2\pi}{N}(k-m)n} = \sum_{k=0}^{N-1} \bar{c}_k \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n}$$

Recall now that the basis functions are orthogonal:

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n} = \begin{cases} N & (k-m) = rN \ r \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

As k ranges from 0 to $N-1$, the right-hand term is not equal to zero just when $k=m$. That is:

$$\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}mn} = N\bar{c}_m \overset{m=k}{\rightarrow} \bar{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

Discrete-Time Fourier Series

Discrete-Time Fourier Series

Let $x[n]$ be a periodic signal with fundamental period N where $\Omega_0 := \frac{2\pi}{N}$. Its Fourier Series expression is given by

$$\begin{aligned} x[n] &= \sum_{k=<N>} \bar{c}_k e^{jk\Omega_0 n} && \text{Synthesis equation} \\ \bar{c}_k &= \frac{1}{N} \sum_{n=<N>} x[n]\, e^{-jk\Omega_0 n} && \text{Analysis equation} \end{aligned} \tag{49}$$

where the Discrete-Time Fourier Series coefficients $\bar{c}_k$ are often referred to as the ***spectral coefficients*** of $x[n]$ and specify a decomposition of $x[n]$ into a sum of N harmonically related complex exponentials

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