Contents

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Left-sided, Right-sided and Two-Sided Signals

Definition

A signal is said to be ***left-sided*** if there exists a t_ℓ such that

$$x(t) = 0 \;\; \forall \; t > t_\ell \tag{1}$$

Likewise, a signal is called ***right-sided*** if there exists a t_r such that

$$x(t) = 0 \;\; \forall \; t < t_r \tag{2}$$

A signal is said to be ***two-sided*** or ***bilateral*** if it is not left-sided nor right-sided

Examples

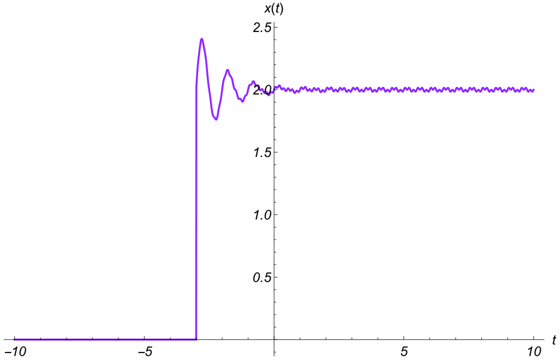

FIGURE: A right-sided signal

Examples

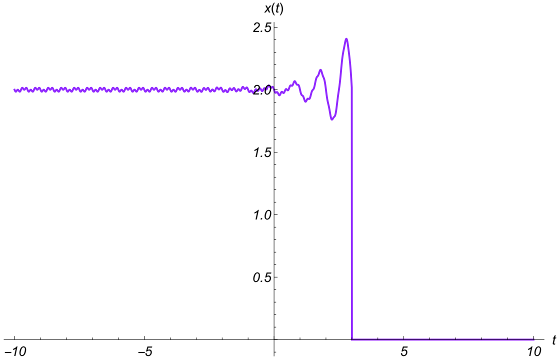

Figure: A left-sided signal

Examples

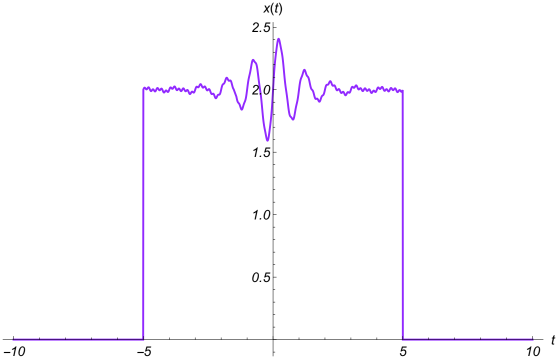

Figure: A two-sided or bilateral signal

Positive-time and Negative-time Signals

Definition

A signal is called ***positive-time*** if

$$x(t) = 0 \quad t < 0 \tag{3}$$

Likewise, a signal is called ***negative-time*** if

$$x(t) = 0 \quad t > 0 \tag{4}$$

All positive-time signals are right-sided signals and all negative-time signals are left-sided signals. However, *not all* right-sided signals are positive-time and *not all* left-sided signals are negative time

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Examples

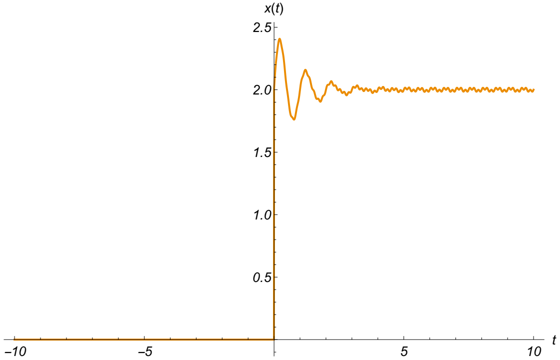

FIGURE: A positive-time signal (also right-sided)

Examples

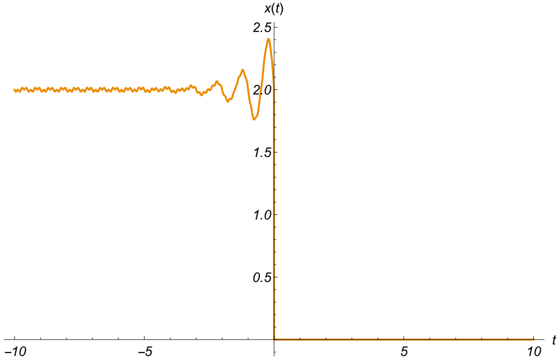

FIGURE: A negative-time signal (also left-sided)

Two-sided Laplace Transform

Definition

The ***two-sided Laplace Transform*** of a general signal $x(t)$ is defined as:

$$X(s) := \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \qquad (5)$$

The advantage of the bilateral Laplace transform is that it can handle both right-sided and left-sided signals [Sadiku, 2015]. Other than that, there is almost *no practical application* of this transform [Chen, 2009]

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Notation and Complex Frequency

Notation and Complex Frequency

When the Laplace transform of $x(t)$ exists, we write

$$\begin{aligned} X(s) &= \mathcal{L}\{x(t)\} \\ x(t) &\xleftrightarrow{\mathcal{L}} X(s) \end{aligned} \tag{6}$$

The independent, complex variable s is referred to as ***complex frequency*** and will be defined as

$$s = \sigma + j\omega \tag{7}$$

where

- $\mathrm{Re}\{s\} = \sigma \ \left[\frac{\mathrm{Np}}{\mathrm{s}}\right]$ is known as the ***neperian frequency*** (real part of s)
- $\mathrm{Im}\{s\} = \omega \ \left[\frac{\mathrm{rad}}{\mathrm{s}}\right]$ is known as the ***angular frequency*** (imaginary part of s)

Region of Convergence (ROC)

Definition

The ***Region of Convergence (ROC)*** of a Laplace transform is the interval of values of s on which the integral

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

converges [Oppenheim and Willsky, 1998].

In other words, it consists of the values of $s = \sigma + j\omega$ on which the Fourier transform of $x(t)\, e^{-\sigma t}$ converges

The convergence of $\mathcal{F}\left\{x(t)\, e^{-\sigma t}\right\}$ depends only on $\sigma = \mathrm{Re}\{s\}$. Therefore, the ROC of a Laplace Transform has the shape of a strip

Relationship between $\mathcal{F}$ and $\mathcal{L}$

Many texts on dynamic systems claim that the Laplace transform is a general version of the Fourier transform. Although there is a close relationship between both integral mappings, the affirmation is not completely true.

We may establish a dual relationship between both transforms

$$\mathcal{F} \rightarrow \mathcal{L} \quad \mathcal{L} \rightarrow \mathcal{F}$$

Relationship between $\mathcal{L}$ and $\mathcal{F}$

Relationship $\mathcal{L} \rightarrow \mathcal{F}$

If the Laplace Transform of $\mathcal{L}\left\{x\left(t\right)\right\} = X\left(s\right)$ exists and its ROC contains the imaginary axis $s = j\omega$, then the Fourier Transform of $x\left(t\right)$ corresponds to the Laplace transform $X\left(s\right)$ evaluated at $s = j\omega$

$$\mathcal{F}\left\{x\left(t\right)\right\} = X\left(j\omega\right) = X\left(s\right)\big|_{s=j\omega} \tag{8}$$

Relationship between $\mathcal{F}$ and $\mathcal{L}$

Relationship $\mathcal{F} \rightarrow \mathcal{L}$

The Laplace Transform of $x(t)$ can be interpreted as the CT Fourier Transform of $x(t)$ after premultiplying it by a real exponential signal $e^{-\sigma t}$

$$\mathcal{L}\{x(t)\} = \mathcal{F}\left\{x(t)\, e^{-\sigma t}\right\} = \int_{-\infty}^{\infty} \left[x(t)\, e^{-\sigma t}\right] e^{-j\omega t} \qquad (9)$$

The values of σ for which the Fourier transform $\mathcal{F}\left\{x(t)\, e^{-\alpha t}\right\}$ exists correspond also to the region on the s-plane in which the Laplace transform $\mathcal{L}\{x(t)\}$ exists (i.e., the ROC of $X(s)$)

Rational Laplace Transform

Before analyzing the properties of the ROC, we will define a special kind of Laplace transform that arises frequently in practice

Definition

A ***rational Laplace Transform*** is a complex function of s expressed as a ratio of two polynomials. We may write a rational transfer function $X(s)$ as:

$$X(s) = \frac{N(s)}{D(s)} \tag{10}$$

where $N(s)$ and $D(s)$ are the polynomials of numerator and denominator respectively.

$\boxed{\textbf{$X(s)$ will be rational if $x(t)$ is a linear combination of complex (or real) exponentials}}$

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Properties of the ROC

Property 1: the ROC of $X(s)$ consists of parallel strips to the $j\omega$-axis in the s plane

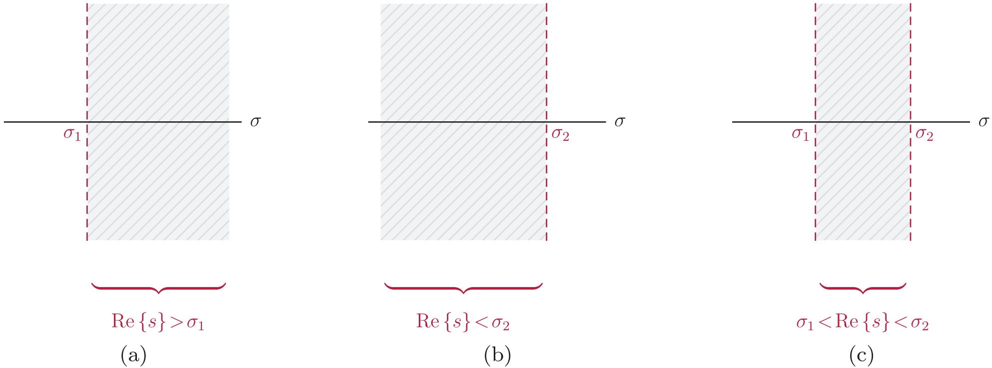

FIGURE: Form of ROC [Alkin, 2014]. Another option (not shown) is a union of two disjoint half-planes.

Properties of the ROC

Property 2: the ROC does not contain any pole

Property 3: if $x(t)$ is of finite duration and absolutely integrable, then the ROC of its Laplace transform is the complete s-plane. Extreme points such as $\mathrm{Re}\{s\} \to \pm\infty$ need to be analyzed separately

Properties of the ROC

Property 4: **if $x(t)$ is right-sided and the line $\text{Re}\{s\} = \sigma_0$ lies within the ROC, then all values of s for which $\text{Re}\{s\} > \sigma_0$ will also be on the ROC. In other words, the ROC of a right-sided signal is a right half-plane (RHP)**

Property 5: **if $x(t)$ is left-sided and the line $\text{Re}\{s\} = \sigma_0$ lies within the ROC, then all values of s for which $\text{Re}\{s\} < \sigma_0$ will also be on the ROC. In other words, the ROC of a left-sided signal is a left half-plane (LHP)**

Property 6: **if $x(t)$ is a bilateral or two-sided signal and the line $\text{Re}\{s\} = \sigma_0$ lies within the ROC of its Laplace transform, then the ROC will consists of a strip on the s-plane including the line $\text{Re}\{s\} = \sigma_0$**

Properties of the ROC

Property 7: **if $x(t)$ has a rational Laplace transform $X(s)$, then the ROC of its Laplace transform is bounded by its poles or extends to the infinite. Moreover, no poles of $X(s)$ are contained in the ROC**

Property 8: **if $x(t)$ is a right-sided signal with a rational Laplace transform $X(s)$, then its ROC is the region in the s-plane to the right of the rightmost pole. If $x(t)$ is a left-sided signal, the ROC of its Laplace transform is the region in the s-plane to the left of the leftmost pole**

Properties of the Two-sided Laplace Transform

Property	$x(t)$	$\mathcal{L}\{x(t)\}$	**ROC**
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(s) + a_2 X_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0} X(s)$	R
Complex frequency shifting	$e^{s_0 t} x(t)$	$X(s - s_0)$	$R + \mathrm{Re}\{s_0\}$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{s}{a}\right)$	$\frac{R}{a}$
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Time convolution	$x_1(t) * x_2(t)$	$X_1(s) X_2(s)$	At least $R_1 \cap R_2$
Time differentiation	$\frac{d}{dt}[x(t)]$	$sX(s)$	R
Frequency differentiation	$-tx(t)$	$\frac{d}{ds}[X(s)]$	R
Time integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{1}{s} X(s)$	At least $R \cap \mathrm{Re}\{s\} > 0$

Initial Value Theorem (IVT)

Initial Value Theorem (IVT)

For any Laplace transform pair $X(s) = \mathcal{L}\{x(t)\}$

$$x\left(0^{+}\right) = \lim_{s \to \infty} sX(s) \tag{11}$$

Final Value Theorem (FVT)

Final Value Theorem (FVT)

If all the poles of $sX(s)$ are in the left half of the s-plane, then

$$x_{ss} := \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{12}$$

Notice that this theorem requires $X(s)$ to be stable, or having at most one pole at $s = 0$ (i.e., type I system). If the system has more than one pole at the origin or is unstable, the Final value theorem will yield an erroneous result.

Applications of the Two-sided Laplace Transform

Let $h(t)$ be the impulse response of an LTI system whose Laplace transform is $H(s)$

$$H(s) := \mathcal{L}\{h(t)\}$$

Causality

(Time-domain) An LTI system with impulse response $h(t)$ is ***causal*** if and only if its impulse response $h(t)$ is a positive-time signal

$$h(t) = 0 \quad t < 0 \tag{13}$$

Applications of the Two-sided Laplace Transform

Let $h(t)$ be the impulse response of an LTI system whose Laplace transform is $H(s)$

$$H(s) := \mathcal{L}\{h(t)\}$$

Causality

(s-domain) An LTI system with impulse response $h(t)$ is ***causal*** if the ROC of its Laplace transform $H(s)$ is a right-half plane:

$$\begin{array}{ccc} \text{Causal} & \Rightarrow & \text{ROC of } H(s) \\ \text{system} & \not\Leftarrow & \text{right-half plane} \end{array} \tag{14}$$

Note that a right-half plane as ROC does not guarantee that the system will be causal. There is, however, a special case in which the implication goes both ways:

If $H(s)$ is a rational Laplace transform, then:

$$\begin{array}{ccc} \text{Causal} & \Rightarrow & \text{ROC of } H(s) \\ \text{system} & \Leftarrow & \text{right-half plane} \end{array}$$

Applications of the Two-sided Laplace Transform

Let $h(t)$ be the impulse response of an LTI system whose Laplace transform is $H(s)$

$$H(s) := \mathcal{L}\{h(t)\} = \frac{N(s)}{D(s)}$$

Stability

(Time-domain) An LTI system with impulse response $h(t)$ is BIBO stable if and only if $h(t)$ is absolutely integrable

Applications of the Two-sided Laplace Transform

Let $h(t)$ be the impulse response of an LTI system whose Laplace transform is $H(s)$

$$H(s) := \mathcal{L}\{h(t)\} = \frac{N(s)}{D(s)}$$

The roots of $N(s)$ are known as *zeros* of $H(s)$ whereas those of $D(s)$ correspond to the *poles* of $H(s)$.

Stability

(s-domain) [General criterion] An LTI system with impulse response $h(t)$ is ***BIBO stable*** if its frequency response $H(j\omega)$ exists. In other words, the ROC of its Laplace transform $H(s)$ includes the $j\omega$ axis.

[Specific case] If in addition, the system is causal, then it is BIBO stable if $H(s)$ is rational and proper and its poles lie in the open left-half s-plane (LHP)

One-sided or Unilateral Laplace Transform $\mathcal{L}_-$

For most applications (e.g., where causal systems or positive-time signals are involved), it is useful to define a one-sided (or unilateral) Laplace transform, which uses 0^- (that is, a value just before $t = 0$) as the lower limit of integration in Equation (5).

Definition

The ***one-sided Laplace Transform*** of $x(t)$ is the function of the complex variable $s = \sigma + j\omega$ given by:

$$X(s) := \mathcal{L}_- \{x(t)\} = \int_{0^-}^{\infty} x(t)\, e^{-st} dt \tag{15}$$

The unilateral Laplace Transform can handle only positive-time or right-signals. However, these signals are in practice the most common. Hence, $\mathcal{L}_-$ is *widely used* in science and engineering

Properties of the One-sided Laplace Transform

Property	$x(t)$	$\mathcal{L}_-\{x(t)\}$
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(s) + a_2 X_2(s)$
Time shifting	$x(t - t_0)$	$e^{-st_0} X(s)$
Complex frequency shifting	$e^{s_0 t} x(t)$	$X(s - s_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(s) X_2(s)$
Frequency convolution	$x_1(t) x_2(t)$	$\frac{1}{2\pi j} X_1(s) * X_2(s)$
Modulation by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\left[X(s + j\omega_0) + X(s - j\omega_0)\right]$
Modulation by $\sin(\omega_0 t)$	$x(t)\sin(\omega_0 t)$	$\frac{j}{2}\left[X(s + j\omega_0) - X(s - j\omega_0)\right]$

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Properties of the One-sided Laplace Transform

Property	$x(t)$	$\mathcal{L}_-\{x(t)\}$
Time differentiation First derivative	$\frac{d}{dt}[x(t)]$	$sX(s) - x(0^-)$
Time differentiation Second derivative	$\frac{d^2}{dt^2}[x(t)]$	$s^2X(s) - sx(0^-) - \dot{x}(0^-)$
Time differentiation Third derivative	$\frac{d^3}{dt^3}[x(t)]$	$s^3X(s) - s^2x(0^-) - s\dot{x}(0^-) - \ddot{x}(0^-)$
Time differentiation nth derivative	$\frac{d^n}{dt^n}[x(t)]$	$s^nX(s) - s^{n-1}x(0^-) - s^{n-2}\dot{x}(0^-) - \ldots .x^{(n-1)}(0^-)$

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Properties of the One-sided Laplace transform

Property	$x(t)$	$\mathcal{L}_-\{x(t)\}$
Time integration	$\int_0^t x(\tau)\,d\tau$	$\frac{1}{s}X(s)$
Frequency integration	$\frac{x(t)}{t}$	$\int_s^\infty X(\lambda)\,d\lambda$
Frequency differentiation 1st order	$tx(t)$	$-\frac{d}{ds}[X(s)]$
Frequency differentiation Higher order	$t^n x(t)$	$(-1)^n \frac{d^n}{ds^n}[X(s)]$

Inverse Laplace Transform: Definition

Definition

The ***inverse Laplace Transform*** of the complex function $X(s)$ is defined as:

$$x(t) = \mathcal{L}^{-1}\left\{X(s)\right\} := \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)\, e^{st}\, ds \tag{16}$$

Notice that this is a line integral in the complex s-plane. Consequently, **we must use complex integration methods to solve it**.

Computing Inverse Laplace transforms by PFD

The line integral defined in (16) is rarely used in engineering to compute inverse Laplace transforms [Chen, 2004, Sadiku, 2015] either in the bilateral or one-sided case. It is much simpler to find the inverse of $X(s)$ by looking it up on a table.

However, we must first express $X(s)$ as a sum of terms available in the transform table. To do so, we will make use of Partial Fraction Decomposition (PFD)

Inverse Two-Sided Laplace Transform

Since two different functions can have the same algebraic expression as their Laplace transform, the specification of the ROC enables one to determine the corresponding time signal for the given transform.

Example 7.1

Consider the Laplace transform of the signal $x_1(t) = e^{-t}u(t)$:

$$X(s) = \frac{1}{s+1} \quad \text{Re}\{s\} > -1$$

and that of $x_2(t) = -e^{-t}u(-t)$:

$$X_2(s) = \frac{1}{s+1} \quad \text{Re}\{s\} < -1$$

Since the Laplace transforms $X_1(s)$ and $X_2(s)$ have the same algebraic form, it is the difference between the corresponding ROCs enables one to distinguish the signals in the complex frequency domain.

Inverse Unilateral Laplace Transform

Since the one-sided Laplace transform can handle only right-sided signals, the ROC will be in most cases a right half-plane in the s-domain

An interesting case happens when the transform is rational: the ROC begins from the rightmost pole and extends to infinity

$$\begin{array}{c} \text{Rational} \\ \text{Laplace} \\ \text{transform} \end{array} \Rightarrow \begin{array}{c} \text{Partial} \\ \text{fraction} \\ \text{decomposition} \end{array}$$

Response of LTI Systems: the Convolution Integral

Consider an LTI system. The response to the input $x(t)$ will be given by the convolution integral if the impulse response $h(t)$ is known:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

If the system is causal, then the upper limit of integration becomes:

$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

As the system is time-invariant, we may set the time $t = 0^-$ as the instant in which we start to observe the system. Thus:

$$y(t) = \int_{-\infty}^{0^-} x(\tau)\, h(t-\tau)\, d\tau + \int_{0^-}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

The first term of last equation summarizes the effects of the input from $-\infty$ to 0 on future outputs ranging from 0 to time t (see Tutorial 2). In other words, it corresponds to the initial state of the system (initial conditions in a differential equation model for LTI lumped systems).

Response of LTI Systems: the Convolution Integral

$$y(t) = \int_{-\infty}^{0^{-}} x(\tau)\, h(t-\tau)\, d\tau + \int_{0^{-}}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

The first term of last equation summarizes the effects of the input from $-\infty$ to 0 on future outputs ranging from 0 to time t (see Tutorial 2). In other words, it corresponds to the initial state of the system (initial conditions in a differential equation model for LTI lumped systems)

If we further assume the system to be initially relaxed, we have

$$y(t) = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

This convolution integral describes completely the zero-state (forced) response of the system (i.e., the response when the initial conditions are zero)

Transfer Function

$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

If we apply the one-sided Laplace transform $\mathcal{L}_-$, we have

$$Y(s) = \mathcal{L}_-\{y(t)\} = \mathcal{L}_-\left\{\int_{0^-}^t x(\tau)\, h(t-\tau)\, d\tau\right\} = X(s)\, H(s)$$

Thus,

$$Y(s) = X(s)\, H(s) \tag{17}$$

Transfer Function

Definition

The **transfer function** $H(s)$ is the ratio of the Laplace transform of the output of an LTI system to its input *assuming all zero initial conditions*. In words, the ***transfer function is the unilateral Laplace transform of the unit impulse response*** $h(t)$

$$h(t) \xrightarrow{\mathcal{L}_-} H(s) = \frac{Y(s)}{X(s)} \tag{18}$$

The transfer function characterizes the zero-state (forced) response of an LTI system

BIBO Stability of an LTI System

Theorem on BIBO Stability of an LTI System

A SISO LTI system with proper rational transfer function $H(s)$ is BIBO stable if and only if every pole of $H(s)$ has a negative real part or, equivalently, lies inside the left-half s-plane

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