TALLER DE TRANSFORMADA DE LAPLACE

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1. Calcular las siguientes transformadas de laplace

a) $\mathcal{L}\{(t^2+3)^2\} = \frac{24}{s^5} + \frac{12}{s^3} + \frac{9}{s}$

$$\mathcal{L}\{(t^2+3)^2\} = \mathcal{L}\{t^4+6t^2+9\}$$
$$= \mathcal{L}\{t^4\} + 6\{t^2\} + \mathcal{L}\{9\}$$
$$= \frac{4!}{s^5} + 6\left[\frac{2!}{s^3}\right] + \frac{9}{s}$$
$$= \frac{24}{s^5} + \frac{12}{s^3} + \frac{9}{s}$$

b) $\mathcal{L}\{e^{3t}\cos(5t)\} = \mathcal{L}\{e^{3t}\cos(5t)\} = \frac{s-3}{(s-3)^2+25}$

c) $\mathcal{L}\{e^{6t}t^4 + (t-8)^4 u(t-8)\} = \frac{24}{(s-6)^5} + e^{-8s}\frac{24}{s^5}$

$$\mathcal{L}\{e^{6t}t^4\} + \mathcal{L}\{(t-8)^4 u(t-8)\} = \frac{24}{(s-6)^5} + e^{-8s}\mathcal{L}\{(t^4)\}$$
$$= \frac{24}{(s-6)^5} + e^{-8s}\frac{24}{s^5}$$

d) $\mathcal{L}\{\cos(t-4)\,u(t-4)\} = e^{-4s}\mathcal{L}\{\cos(t)\} = e^{-4s}\frac{s}{s^2+1}$

2) Calcular las siguientes tranformadas inversas

a) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+7s+10}\right\} = 4/3\, e^{-5t} - 1/3\, e^{-2t}$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+5)(s+2)}\right\} = \mathcal{L}\left\{\frac{4/3}{s+5}\right\} + \mathcal{L}\left\{\frac{-1/3}{s+2}\right\}$$

$$\frac{s+1}{(s+5)(s-2)} = \frac{A}{s+5} + \frac{B}{s+2} = \frac{4/3}{s+5} + \frac{-1/3}{s+2}$$

$$s+1 = A(s+2) + B(s+5)$$

$\star$ Si $s=-2 \rightarrow B = -1/3$ $\quad\star$ Si $s=-5 \rightarrow A = 4/3$

b) $\mathcal{L}^{-1}\left\{\frac{s^2+4}{s^3+9s}\right\} = \frac{4}{9} + \frac{5}{9}\cos(3t)$

$$\frac{s^2+4}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9} = \frac{4}{9s} + \frac{5s}{9(s^2+9)}$$

$$s^2+4 = A(s^2+9) + Bs^2 + Cs$$
$$s^2+4 = (A+B)s^2 + 9A + Cs$$

$\star$ Si $s=0 \Rightarrow A = 4/9$ $\quad\star$ $A+B=1 \Rightarrow B = \frac{5}{9}$ $\quad\star$ $C=0$

c) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+6s+13}\right\} = e^{-3t}\cos(2t) - e^{-3t}\text{sen}(2t)$

$$\mathcal{L}^{-1}\left\{\frac{(s+3)-2}{(s+3)^2+2^2}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+3)}{(s+3)^2+2^2} - \frac{2}{(s+3)^2+2^2}\right\}$$

$$= e^{-3t}\cos(2t) - e^{-3t}\text{sen}(2t)$$

d) $\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2(s-2)}\right\} = \frac{-3}{4} + \frac{1}{2}t + \frac{3}{4}e^{2t}$

$$\frac{2s-1}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} = \frac{-3}{4s} + \frac{1}{2s^2} + \frac{3}{4(s-2)}$$

$$2s-1 = As(s-2) + B(s-2) + Cs^2$$
$$2s-1 = As^2 - 2As + Bs - 2B + Cs^2$$
$$2s-1 = s^2(A+C) + s(B-2A) - 2B$$
$$2 = B - 2A \qquad 2 - \frac{1}{2} = -2A$$
$$2 = \frac{1}{2} - 2A$$

$\star$ Si $s=2 \Rightarrow C = 3/4$

$\star$ Si $s=0 \Rightarrow B = 1/2$

$\star$ $-\frac{3}{4} = A$

e) $\mathcal{L}^{-1}\left\{\dfrac{e^{2s}}{s^2-4}\right\} = \dfrac{1}{2}\cdot \operatorname{senh}(2(t+2))\, u(t+2)$

$$\mathcal{L}^{-1}\left\{\dfrac{e^{-(-2s)}}{s^2-2^2}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2-2^2}\right\}\Bigg|_{t\to t-(-2)} u(t-(-2))$$

$$= \mathcal{L}^{-1}\left\{\dfrac{2}{2}\cdot\dfrac{1}{(s^2-2^2)}\right\}\Bigg|_{t\to t+2} u(t+2)$$

$$= \dfrac{1}{2}\cdot \operatorname{senh}(2t+4)\, u(t+2)$$

f) $\mathcal{L}^{-1}\left\{\dfrac{s}{(3-s)^3}\right\} = -te^{3t} - 3/2\, t^2 e^{3t}$

$$-\dfrac{s}{(s-3)^3} = \dfrac{A}{s-3} + \dfrac{B}{(s-3)^2} + \dfrac{C}{(s-3)^3} = \dfrac{-1}{(s-3)^2} - \dfrac{3}{(s-3)^3}$$

$$-s = A(s-3)^2 + B(s-3) + C$$
$$-s = As^2 - 6As + 9A + Bs - 3B + C$$

* Si $s=3 \Rightarrow C=-3$ * $0=A$ * $-1 = 3A+B \rightarrow B=-1$

$$\mathcal{L}^{-1}\left\{\dfrac{-1}{(s-3)^2} - \dfrac{3}{(s-3)^3}\right\} = -\mathcal{L}^{-1}\left\{\dfrac{1}{(s-3)^2}\right\} - 3\mathcal{L}^{-1}\left\{\dfrac{1}{(s-3)^3}\right\}$$

$$= \dfrac{-1}{(2-1)!}\, t^{2-1}e^{3t} - 3\left[\dfrac{1}{(3-1)!}\, t^{3-1} e^{3t}\right]$$

$$= -te^{3t} - 3/2\, t^2 e^{3t}$$

3. Use la transformada de Laplace para encontrar la solución del problema del valor inicial.

a) $\begin{cases} y'' - 4y = e^t \\ y(0) = y'(0) = 0 \end{cases}$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y\} = \frac{1}{s-1}$$

$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) - 4\mathcal{L}\{y\} = \frac{1}{s-1}$$

$$s^2 Y - 4Y = \frac{1}{s-1}$$

$$Y(s^2-4) = \frac{1}{s-1}$$

$$Y = \frac{1}{(s^2-4)(s-1)} = \frac{1}{(s-2)(s+2)(s-1)}$$

$$= \frac{A}{(s-2)} + \frac{B}{(s+2)} + \frac{C}{(s-1)}$$

$$1 = A(s+2)(s-1) + B(s-2)(s-1) + C(s+2)(s-2)$$

⋆ Si $s=-2 \Rightarrow 1/12 = B$ ⋆ Si $s=1 \rightarrow 1/-3 = C$

⋆ Si $s=2 \Rightarrow 1/4 = A$

$$Y = \frac{1}{4(s-2)} + \frac{1}{12(s+2)} - \frac{1}{3(s-1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{4(s-2)} + \frac{1}{12(s+2)} - \frac{1}{3(s-1)}\right\}$$

$$\boxed{y(t) = \frac{1}{4}e^{2t} + \frac{1}{12}e^{-2t} - \frac{1}{3}e^{t}}$$

b) $\begin{cases} y'' + 9y = 1 \\ y(0)=0 \quad y'(0)=1 \end{cases}$

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = 1/s$$
$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) + 9\mathcal{L}\{y\} = 1/s$$
$$s^2 Y - 1 + 9Y = 1/s$$
$$Y(s^2+9) = \frac{1}{s} + \frac{s}{s} = \frac{1+s}{s}$$
$$Y = \frac{1+s}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

$$1+s = A(s^2+9) + (Bs+C)s$$
$$1+s = As^2 + 9A + Bs^2 + Cs \qquad \star\, C=1 \quad \star\, 1/9 = A \quad \star\, B = -1/9$$

$$Y = \frac{1}{9s} + \frac{\frac{-s}{9}+1}{s^2+9} = \frac{1}{9s} + \frac{-s+9}{9(s^2+9)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{9s} + \frac{-s+9}{9(s^2+9)}\right\} = \frac{1}{9} - \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{s-9}{s^2+9}\right\}$$

$$\boxed{y(t) = \frac{1}{9}\left(1 - \cos(3t) + 3\,\mathrm{sen}(3t)\right)}$$

c) $\begin{cases} y'' - 5y' + 6y = u(t-1) \\ y(0)=0 \quad y'(0)=1 \end{cases}$

$$\mathcal{L}\{y''\} - 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{u(t-1)\}$$
$$s^2\mathcal{L}\{y\} - 1 - 5s\mathcal{L}\{y\} + 6\mathcal{L}\{y\} = e^{-s}/s$$
$$s^2 Y - 1 - 5sY + 6Y = e^{-s}/s$$
$$Y(s^2 - 5s + 6) = \frac{e^{-s}+s}{s}$$
$$Y = \frac{e^{-s}}{s(s-2)(s-3)} + \frac{1}{(s-2)(s-3)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s-2)(s-3)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\}$$

$$1 = A(s-3) + B(s-2) \qquad \star\, \text{Si } s=3 \Rightarrow B=1 \qquad \star\, \text{Si } s=2 \Rightarrow A=-1$$

i) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-2)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{-1}{s-2} + \dfrac{1}{s-3}\right\} = -e^{2t} + e^{3t}$

ii) $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s(s-2)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s(s-2)(s-3)}\right\}\Bigg|_{t\to t-1} u(t-1)$

$1 = A(s-2)(s-3) + Bs(s-3) + Cs(s-2)$

* Si $s=2 \Rightarrow -1/2 = B$ * Si $s=0 \Rightarrow 1/6 = A$
* Si $s=3 \Rightarrow 1/3 = C$

$\mathcal{L}^{-1}\left\{\dfrac{1}{2s} - \dfrac{1}{2(s-2)} + \dfrac{1}{3(s-3)}\right\}\Bigg|_{t\to t-1} = \dfrac{1}{6} - \dfrac{1}{2}e^{2t-2} + \dfrac{1}{3}e^{3t-3}$

$$\boxed{y(t) = -e^{2t} + e^{3t} + \left[\frac{1}{6} - \frac{1}{2}e^{2t-2} + \frac{1}{3}e^{3t-3}\right]u(t-1)}$$

d) $\begin{cases} y'' - y' = e^{t}\cos t \\ y(0) = y'(0) = 0 \end{cases}$

$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} = \mathcal{L}\{e^{t}\cos t\}$

$s^2\mathcal{L}\{y\} - s\mathcal{L}\{y\} = \dfrac{s-1}{(s-1)^2+1}$

$Y(s^2 - s) = \dfrac{s-1}{(s-1)^2+1}$

$Y = \dfrac{s-1}{[(s-1)^2+1](s^2-s)}$

$Y = \dfrac{\cancel{s-1}}{(s^2-2s+2)s\cancel{(s-1)}} = \dfrac{1}{(s^2-2s+2)s} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2-2s+2}$

$1 = A(s^2-2s+2) + Bs^2 + Cs$

$1 = As^2 - 2As + 2A + Bs^2 + Cs$ $\quad A = 1/2 \quad B = -1/2 \quad C = 1$

$Y = \dfrac{1}{2s} + \dfrac{-\frac{s}{2}+1}{(s^2-2s+2)} = \dfrac{1}{2s} + \dfrac{-\frac{1}{2}(s-1) + \frac{1}{2}}{(s-1)^2+1}$

$= \dfrac{1}{2s} - \dfrac{s-1}{2[(s-1)^2+1]} + \dfrac{1}{2[(s-1)^2+1]}$

$$\mathcal{L}^{-1}\left\{ \frac{1}{2s} - \frac{s-1}{2[(s-1)^2+1]} + \frac{1}{2[(s-1)^2+1]} \right\} = \frac{1}{2}$$

$$y(t) = \frac{1}{2} - \frac{1}{2}\cdot e^{t}\cos(t) + \frac{1}{2}e^{t}\operatorname{sen}(t)$$

$$\boxed{y(t) = \frac{1}{2}\left(1 - e^{t}\cos t + e^{t}\operatorname{sen} t\right)}$$

e) $\begin{cases} y'' + 2y' + y = 2(t-3)u(t-3) \\ y(0) = 2 \qquad y'(0) = 1 \end{cases}$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 2\mathcal{L}\{(t-3)u(t-3)\}$$

$$s^2Y - s2 - 1 + 2sY - 4 + Y = \frac{2e^{-3s}}{s^2}$$

$$Y(s^2+2s+1) = \frac{2e^{-3s}}{s^2} + 2s + 5$$

$$Y = \frac{2e^{-3s}}{s^2(s+1)^2} + \frac{5}{(s+1)^2} + \frac{2s}{(s+1)^2}$$

$$\mathcal{L}^{-1}\left\{ \frac{2e^{-3s}}{s^2(s+1)^2} + \frac{1}{(s+1)^2} + \frac{2s}{(s+1)^2} \right\}$$

i) $\mathcal{L}^{-1}\left\{ \frac{5}{(s+1)^2} \right\} = \boxed{5te^{-t}}$

ii) $\mathcal{L}^{-1}\left\{ \frac{2e^{-3s}}{s^2(s+1)^2} \right\} = 2\,\mathcal{L}^{-1}\left\{ \frac{e^{-3s}}{s^2(s+1)^2} \right\}$

$$= 2\,\mathcal{L}^{-1}\left\{ \frac{1}{s^2(s+1)^2} \right\}\Bigg|_{t\to t-3} u(t-3)$$

$$\frac{1}{s^2(s+1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+1)} + \frac{D}{(s+1)^2}$$

$$1 = As(s+1)^2 + B(s+1)^2 + Cs^2(s+1) + Ds^2$$

$$1 = As^3 + 2As^2 + As + Bs^2 + 2Bs + B + Cs^3 + Cs^2 + Ds^2$$

$B = 1 \quad 0 = A + 2B \quad 0 = A + C \quad 0 = 2A + B + C + D$

$A = -2 \quad C = 2 \quad 0 = 2(-2) + 1 + 2 + D \Rightarrow D = 1$

$$\mathcal{L}^{-1}\left\{\frac{-2}{s} + \frac{1}{s^2} + \frac{2}{(s+1)} + \frac{1}{(s+1)^2}\right\}\Bigg|_{t\to t-3} = -2 + t + 2e^{-t} + e^{-t}t\ \Bigg|_{t\to t-3}$$

$$\boxed{= \left[-2 + t - 3 + 2e^{-t+3} + e^{-t+3}(t-3)\right]u(t-3)}$$

iii) $\mathcal{L}^{-1}\left\{\frac{2s}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s+1)} - \frac{2}{(s+1)^2}\right\}$

$$\boxed{= 2e^{-t} - 2te^{-t} = 2(e^{-t} - te^{t})}$$

$\frac{A}{s+1} + \frac{B}{(s+1)^2} \qquad 2s = A(s-1) + B \qquad A = 2$

$2s = As + A + B \qquad 0 = 2 + B \Rightarrow B = -2$

$$\boxed{y(t) = 5te^{-t} + \left[-2 + t - 3 + 2e^{-t+3} + e^{-t+3}(t-3)\right]u(t-3) + 2(e^{-t} - te^{t})}$$

f) $\begin{cases} y'' - 5y' + 6y = u(t-1) \\ y(0) = 0 \quad y'(0) = 1 \end{cases}$ Es el mismo que el c.

g) $\begin{cases} y'' - 4y' + 4y = t^2e^t \\ y(0) = 0 \quad y'(0) = 1 \end{cases}$

$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{t^2e^t\}$

$s^2Y - 1 - 4sY + 4Y = \frac{2}{(s-1)^3}$

$$Y = \frac{2}{(s-1)^3(s-2)^2} + \frac{1}{(s-2)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3(s-2)^2} + \frac{1}{(s-2)^2}\right\}$$

i) $\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3(s-2)^2}\right\}$

$$\frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{(s-2)} + \frac{E}{(s-2)^2}$$

$$2 = A(s-1)^2(s-2)^2 + B(s-1)(s-2)^2 + C(s-2)^2 + D(s-2)(s-1)^3 + E(s-1)^3$$

$$2 = (As^2 - 2As + A)(s^2 - 4s + 4) + Bs - B(s^2 - 4s + 4) + Cs^2 - 4Cs + 4C + Ds - 2D(s-1)^3 + Es^3 + 3Es^2 + 3Es + E$$

$$2 = As^4 - 4As^3 + 4As^2 - 2As^3 + 8As^2 - 8As + As^2 - 4As + 4A + Bs^3 - 4Bs^2 + 4Bs - Bs^2 + 4Bs - 4B + Cs^2 - 4Cs + 4C$$
$$Ds^4 - Ds^3 - 3Ds^2 - 5Ds - 2D + Es^3 + 3Es^2 + 3Es + E$$

$2 = 4A - 4B + 4C - 2D + E$

$0 = -8A - 4A + 8B - 4C - 5D + 3E$

$0 = 13A - 5B + C - 3D + 3E$

$0 = -6A + B - D + E$

$0 = A + D \Rightarrow A = D$

$$2\,\mathcal{L}^{-1}\left\{ \frac{3}{(s-1)} + \frac{2}{(s-1)^2} + \frac{1}{(s-1)^3} - \frac{3}{(s-2)} + \frac{1}{(s-2)^2} \right\}$$

$$\boxed{= 2\left[3e^t + 2te^t + \frac{1}{2}t^2e^t - 3e^{2t} + te^{2t}\right]}$$

ii) $$\mathcal{L}^{-1}\left\{ \frac{1}{(s-2)^2} \right\} = te^{2t}$$

$$\boxed{y(t) = 2\left[3e^t + 2te^t + \frac{1}{2}t^2e^t - 3e^{2t} + te^{2t}\right] + te^{2t}}$$

h) $\dfrac{dx}{dt} + 3y = t$; $\dfrac{dy}{dt} + 3x = -1$; $x(0) = 0$, $y(0) = 1$

i) $\mathcal{L}\{x'\} + 3\mathcal{L}\{y\} = 1/s^2$

$sX + 3Y = 1/s^2$

ii) $\mathcal{L}\{y'\} + 3\mathcal{L}\{x\} = -1/s$

$3X + sY = \frac{s-1}{s}$

$$A = \begin{bmatrix} s & 3 \\ 3 & s \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1/s^2 & 3 \\ \frac{s-1}{s} & s \end{bmatrix} \qquad A_2 = \begin{bmatrix} s & 1/s^2 \\ 3 & \frac{s-1}{s} \end{bmatrix}$$

$$|A_1| = \frac{1}{s} - \frac{3(s-1)}{s} = \frac{4-3s}{s} \qquad |A_2| = s - 1 - \frac{3}{s^2} \qquad |A| = s^2 - 9$$

$$X = \frac{\frac{4-3s}{s}}{s^2-9} = \frac{4-3s}{s(s^2-9)} \qquad Y = \frac{s-1-\frac{3}{s^2}}{s^2-9}$$

$$X = \frac{4}{s(s^2-9)} - \frac{3}{s^2-9} \quad (1) \qquad Y = \frac{s}{s^2-9} - \frac{1}{s^2-9} - \frac{3}{s^2(s^2-9)} \quad (2)$$

(1) $$4\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-9)}\right\} = 4\mathcal{L}^{-1}\left\{\frac{-1}{9}\cdot\frac{1}{s} + \frac{1}{9}\cdot\frac{s}{(s^2-9)}\right\} = \frac{4}{9}[-1+\cosh(3t)]$$

$$\frac{A}{s} + \frac{Bs+C}{s^2-9} = As^2 - 9A + Bs^2 + Cs \qquad A = -1/9 \quad B = 1/9 \quad C = 0$$

$$x(t) = -\frac{4}{9} + \frac{4}{9}\cosh(3t) - \operatorname{senh}(3t) = -\frac{4}{9} + \frac{4}{9}\left[\frac{e^{3t}+e^{-3t}}{2}\right] - \frac{e^{3t}-e^{-3t}}{2}$$

$$= -\frac{4}{9} + \frac{2e^{3t}}{9} + \frac{2}{9}e^{-3t} - \frac{e^{3t}}{2} + \frac{e^{-3t}}{2}$$

$$\boxed{x(t) = -\frac{4}{9} - \frac{5e^{3t}}{18} + \frac{13e^{-3t}}{18}}$$

(2) $$-3\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-9)}\right\} = -3\mathcal{L}^{-1}\left\{-\frac{1}{9}\cdot\frac{1}{s^2} + \frac{1}{9}\cdot\frac{1}{s^2-9}\right\} = -\frac{1}{9}[-3t + \operatorname{senh}(3t)]$$

$$\frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2-9} = As^3 - 9As + Bs^2 - 9B + Cs^3 + Ds^2 \qquad B = -1/9 \quad C = 0 \quad A = 0 \quad D = 1/9$$

$$y(t) = \cosh(3t) - \frac{1}{3}\operatorname{senh}(3t) - \frac{1}{9}[-3t + \operatorname{senh}(3t)]$$

$$= \frac{9e^{3t}}{18} + \frac{9e^{-3t}}{18} - \frac{3e^{3t}}{18} + \frac{3e^{-3t}}{18} - \frac{e^{3t}}{18} + \frac{e^{-3t}}{18} + \frac{t}{3}$$

$$\boxed{y(t) = \frac{5e^{3t}}{18} + \frac{13e^{-3t}}{18} + \frac{t}{3}}$$

4. Use la transformada de laplace para resolver la ecuación integral.

4.a) $y' = 1 - \operatorname{sen}(t) - \int_0^t y(\tau)\,d\tau, \quad y(0)=0$

$$\mathcal{L}\{y'\} = \mathcal{L}\{1\} - \mathcal{L}\{\operatorname{sen} t\} - \mathcal{L}\left\{\int_0^t y(\tau)\,d\tau\right\}$$

$$sY = \frac{1}{s} - \frac{1}{s^2+1} - \mathcal{L}\left\{\int_0^t y(\tau)\,d\tau\right\}$$

$$\mathcal{L}\left\{\int_0^t y(\tau)\,d\tau\right\} = \frac{\mathcal{L}\{y(t)\}}{s} = \frac{Y}{s}$$

$$sY = \frac{1}{s} - \frac{1}{s^2+1} - \frac{Y}{s}$$

$$sY + \frac{Y}{s} = \frac{1}{s} - \frac{1}{s^2+1}$$

$$Y\left[\frac{s^2+1}{s}\right] = \frac{1}{s} - \frac{1}{s^2+1}$$

$$Y = \frac{s}{s^2+1}\left[\frac{1}{s} - \frac{1}{s^2+1}\right] = \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{(s^2+1)^2 - s(s^2+1)}{(s^2+1)^3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s^2+1) - s}{(s^2+1)^2}\right\}$$

$$\frac{s^2 - s + 1}{(s^2+1)^2} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s^2+1)^2}$$

$$(As+B)(s^2+1) + Cs + D = s^2 - s + 1$$

$$As^3 + As + Bs^2 + B + Cs + D = s^2 - s + 1$$

$$A = 0 \qquad B = 1 \qquad C = -1 \qquad D = 0$$

$$= \mathcal{L}^{-1}\left\{\frac{(s^2+1)-s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}\right\}$$

$$\boxed{y(t) = \operatorname{sen} t - \frac{1}{2} t \operatorname{sen} t}$$

b) $\frac{dy}{dt} + 6y(t) + 9\int_0^t y(\tau)\,d\tau = 1, \qquad y(0)=0$

$$\mathcal{L}\{y'\} + 6\,\mathcal{L}\{y\} + 9\,\mathcal{L}\left\{\int_0^t y(\tau)\,d\tau\right\} = \frac{1}{s}$$

$$sY + 6Y + 9\frac{Y}{s} = \frac{1}{s}$$

$$Y\left(s + 6 + \frac{9}{s}\right) = \frac{1}{s}$$

$$Y = \frac{1}{s^2+6s+9} = \frac{1}{(s+3)^2} \qquad \boxed{y(t) = e^{-3t}\,t}$$

c) $f(t) = \cos t + \int_0^t e^{-\tau} f(t-\tau)\,d\tau$

$$F = \mathcal{L}\{\cos t\} + \mathcal{L}\left\{\int_0^t e^{-\tau} f(t-\tau)\,d\tau\right\}$$

$$F = \frac{s}{s^2+1} + \mathcal{L}\{e^{-t}\}\,\mathcal{L}\{f(t)\}$$

$$F = \frac{s}{s^2+1} + \frac{F}{S+1}$$

$$F\left[1 - \frac{1}{s+1}\right] = \frac{s}{s^2+1}$$

$$F\cdot\frac{s}{s+1} = \frac{s}{s^2+1}$$

$$F = \frac{s+1}{s^2+1}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1} + \frac{1}{s^2+1}\right\}$$

$$\boxed{f(t) = \cos t + \operatorname{sen} t}$$

d) $y' + \text{sen}(t) = 1 - \int_0^t y(\tau)\,d\tau, \quad y(0)=0$

$$sY + \frac{1}{s^2+1} = \frac{1}{s} - \frac{Y}{s}$$

$$Y\left[s + \frac{1}{s}\right] = \frac{1}{s} - \frac{1}{s^2+1}$$

$$Y = \frac{\cancel{s}}{s^2+1}\left[\frac{s^2-s+1}{\cancel{s}(s^2+1)}\right]$$

$$Y = \frac{s^2}{(s^2+1)^2} - \frac{s}{(s^2+1)^2} + \frac{1}{(s^2+1)^2}$$

$$\boxed{y = \frac{1}{2}(\text{sen}\, t + t\cos t) - \frac{1}{2} t\,\text{sen}\, t + te^{-t}}$$

e) $f(t) = -1 + \int_0^t f(t-\tau)e^{-3\tau}\,d\tau$

$$F = \frac{-1}{s} + \mathcal{L}\left\{\int_0^t f(t-\tau)e^{-3\tau}\,d\tau\right\}$$

$$F = -\frac{1}{s} + \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{e^{-3t}\}$$

$$F = -\frac{1}{s} + \frac{F}{s+3}$$

$$F - \frac{F}{s+3} = -\frac{1}{s}$$

$$\frac{F(s+3)-F}{s+3} = -\frac{1}{s}$$

$$\frac{F(s+2)}{s+3} = -\frac{1}{s}$$

$$F = -\frac{s+3}{s(s+2)}$$

$$\mathcal{L}^{-1}\left\{\frac{-s}{s(s+2)} - \frac{3}{s(s+2)}\right\} = -\frac{3}{2} + \frac{1}{2}e^{-2t}$$

$$\frac{A}{s} + \frac{B}{s+2} = As + 2A + Bs \qquad \begin{matrix} A=0 \\ B=-1 \end{matrix} \qquad \mathcal{L}^{-1}\left\{\frac{-1}{s+2}\right\} = -e^{-2t}$$

$$\begin{matrix} A=-3/2 \\ B=3/2 \end{matrix} \qquad \mathcal{L}^{-1}\left\{-\frac{3}{s(s+2)}\right\} = \frac{-3}{2s} + \frac{3}{2(s+2)} = -\frac{3}{2} + \frac{3}{2}e^{-2t}$$

$$\boxed{f(t) = -\frac{3}{2} + \frac{1}{2}e^{-2t}}$$