Contents

two-sided Z-Transform

Definition

The ***two-sided Z-Transform*** of a general sequence $x[n]$ is defined as

$$X(z) := \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{1}$$

Notice that

- z is a complex variable, commonly written in exponential form

$$z = \Sigma + j\Phi = |z| e^{j\Omega}$$

- The Z-Transform of a DT sequence is continuous complex-valued function. Moreover, $X(z)$ is a polynomial comprising positive and negative powers of z (i.e., $X(z)$ is ***always*** rational [Oppenheim and Willsky, 1998, Alkin, 2014])

$$\underset{\textbf{Discrete}}{x[n]} \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad \underset{\textbf{Continuous}}{X(z)}$$

Region of Convergence (ROC)

Definition

The ***Region of Convergence (ROC)*** of a Z-Transform is the interval of values of z on which the infinite series

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

converges [Oppenheim and Willsky, 1998].

In other words, it consists of the values of $z = |z|e^{j\Omega}$ on which the DT Fourier Transform of $x[n]|z|^{-n}$ converges

Note that the ROC of a Z-Transform only depends on the magnitude of z, not on the angular frequency parameter Ω. Therefore, the shape of the ROC will be circular

Relationship between Z and DTFT

There is a close relationship between both mappings. We may establish a dual relationship for both transforms

$$\mathcal{F} \rightarrow \mathcal{Z} \quad \mathcal{Z} \rightarrow \mathcal{F}$$

Relationship $\mathcal{Z} \rightarrow$ DTFT

DTFT as a Z-Transform

If the Z-Transform of $\mathcal{Z}\left\{x\left[n\right]\right\} = X\left(z\right)$ exists and its ROC contains the unit circle where $z = e^{j\Omega}$, then the DT Fourier Transform of $x\left[n\right]$ corresponds to the Z-Transform $X\left(z\right)$ evaluated at $z = e^{j\Omega}$

$$\mathcal{F}\left\{x\left[n\right]\right\} = X\left(e^{j\Omega}\right) = X\left(z\right)\big|_{z=e^{j\Omega}} \tag{2}$$

Relationship DTFT→ $\mathcal{Z}$

Z-Transform as a DTFT

The Z-Transform of $x[n]$ can be interpreted as the DT Fourier Transform of $x[n]$ after premultiplying it by a real exponential signal $|z|^{-n}$

$$\mathcal{Z}\{x[n]\} = \mathcal{F}\left\{x[n]\,|z|^{-n}\right\} = \sum_{n=-\infty}^{\infty}\left[x[n]\,|z|^{-n}\right]e^{-j\Omega n} \tag{3}$$

The values of z for which the DTFT $\mathcal{F}\{x[n]\,|z|^{-n}\}$ exists correspond also to the region on the z-plane in which the Z-Transform $\mathcal{Z}\{x[n]\}$ exists

Properties of the ROC

Property 1: the ROC of $X(z)$ consists of a ring in the z-plane centered about the origin

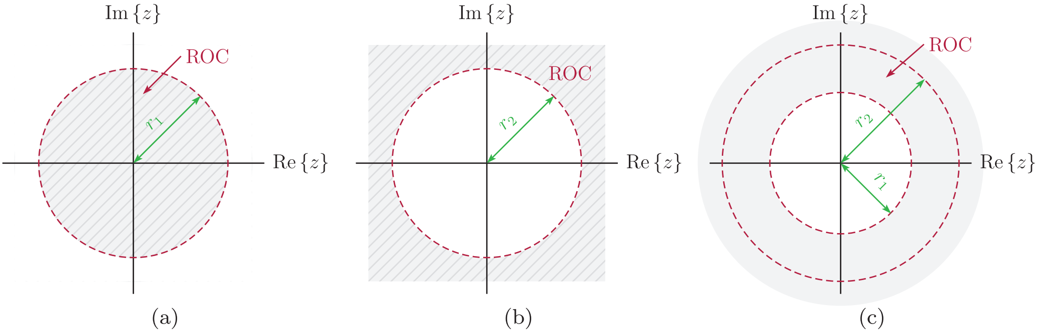

FIGURE: Shape of the ROC of a Z-Transform [Alkin, 2014]. Another option (not shown) is the union (disjoint) of the interior of a circle and the exterior of a circle.

Properties of the ROC

Property 2: the ROC does not contain any poles

Property 3: if $x[n]$ is of finite duration, then the ROC is the complete z-plane except possibly $z = 0$ and/or $z = \infty$

Properties of the ROC

The third property will be distilled through an example

Example 11.1

Compute the Z-Transform and define the ROC of the following sequences:

$x[n] = \delta[n-1]$

$$\mathcal{Z}\{\delta[n-1]\} = \sum_{n=-\infty}^{\infty} \delta[n-1] z^{-n}$$

$$= z^{-n}\Big|_{n=1} = z^{-1} = \frac{1}{z}$$

The ROC does not include $z = 0$ since it is a pole of $X(z)$

$x[n] = \delta[n+1]$

$$\mathcal{Z}\{\delta[n+1]\} = \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n}$$

$$= z^{-n}\Big|_{n=-1} = z^{1} = z$$

The ROC consists of the entire finite z-plane but does not include $z = \infty$ since it is a pole of $X(z)$

Properties of the ROC

Property 4: **if $x[n]$ is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC. In other words, the ROC of a right-sided sequence is the exterior of a circle centered at the origin**

Property 5: **if $x[n]$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $0 < |z| < r_0$ will also be in the ROC. In other words, the ROC of a left-sided sequence is the interior of a circle centered at the origin**

Property 6: **if $x[n]$ is two-sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$**

Properties of the ROC

Property 7: the ROC of a Z-Transform $X(z)$ is bounded by its poles or extends to infinity

Property 8: if $x[n]$ is right-sided, then the ROC of $X(z)$ is the region in the z-plane outside the outermost pole. In addition, if $x[n]$ is positive-time, the ROC includes $z = \infty$

Property 9: if $x[n]$ is left-sided, then the ROC of $X(z)$ is the region in the z-plane inside the innermost pole. In addition, if $x[n]$ is negative-time, the ROC includes $z = 0$

Properties of the two-sided Z-Transform

Property	$x[n]$	$\mathcal{Z}\{x[n]\}$	ROC
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(z) + a_2 X_2(z)$	At least $R_1 \cap R_2$
Time shifting	$x[n-n_0]$	$z^{-n_0} X(z)$	R except for the possible addition or deletion of the origin
Scaling in the z-domain	$e^{j\Omega_0 n} x[n]$	$X\left(e^{-j\Omega_0} z\right)$	R
	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
	$a^n x[n]$	$X\left(a^{-1} z\right)$	$\|a\| R$
Time reversal	$x[-n]$	$X\left(z^{-1}\right)$	Inverted R (R^{-1} : the set of the points z^{-1} where $z \in R$)

Properties of the two-sided Z-Transform

Property	$x[n]$	$\mathcal{Z}\{x[n]\}$	**ROC**
Time expansion	$x_k[n] = \begin{cases} x[r] & n = rk \\ 0 & n \neq rk \end{cases} \; r \in \mathbb{Z}$	$X(z^k)$	$R^{1/k} = \{z^{1/k} \mid z \in R\}$
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z) X_2(z)$	At least $R_1 \cap R_2$
First difference	$x[n] - x[n-1]$	$(1 - z^{-1}) X(z) = \frac{z-1}{z} X(z)$	At least $R \cap \lvert z\rvert > 0$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-z^{-1}} X(z) = \frac{z}{z-1} X(z)$	At least $R \cap \lvert z\rvert > 1$
Differentiation in the z-domain	$nx[n]$	$-z \frac{d}{dz}[X(z)]$	R

Applications of the two-sided Z-Transform

Let $h[n]$ be the impulse response of an LTI system whose Z-Transform is $H(z)$

$$H(z) := \mathcal{Z}\{h[n]\}$$

Causality

(Time-domain) An LTI system is ***causal*** if $h[n]$ is a positive-time signal

$$h[n] = 0 \quad n < 0 \tag{4}$$

Applications of the two-sided Z-Transform

Let $h[n]$ be the impulse response of an LTI system whose Z-Transform is $H(z)$

$$H(z) := \mathcal{Z}\{h[n]\}$$

Causality

(z-domain) An LTI system is ***causal*** if the ROC of $H(z)$ is the exterior of a circle, including infinity

$$\begin{array}{ccc} \text{Causal} & \Rightarrow & \text{ROC of } H(z) \\ \text{system} & \Leftarrow & \text{exterior of a circle} \\ & & \text{including } \infty \end{array} \tag{5}$$

In addition, since $H(z)$ is rational, we can determine if the system is causal if the ROC is the exterior of a circle outside the outermost pole, or equivalently, if $H(z)$ is ***proper*** when written in positive powers of z

$$\begin{array}{ccccc} \text{Causal} & \Rightarrow & H(z) \text{ is proper} & \Rightarrow & \text{ROC of } H(z) \\ \text{system} & \Leftarrow & \text{(positive powers)} & \Leftarrow & \text{exterior of a circle} \\ & & & & \text{outside the outermost pole} \end{array}$$

UNIVERSIDAD NACIONAL DE COLOMBIA

Applications of the two-sided Z-Transform

Let $h[n]$ be the impulse response of an LTI system whose Z-Transform $H(z)$ is rational

$$H(z) := \mathcal{Z}\{h[z]\} = \frac{N(z)}{D(z)}$$

The roots of $N(z)$ are known as *zeros* of $H(z)$ whereas those of $D(z)$ correspond to the *poles* of $H(z)$

Stability

(Time-domain) An LTI system with impulse response $h[n]$ is ***BIBO stable*** if $h[n]$ is *absolutely summable*

$$\sum_{n=-\infty}^{\infty} |h[n]| \leq M < \infty$$

(z-domain) An LTI system will be BIBO stable if the ROC of $X(z) = \mathcal{Z}\{x[n]\}$ includes the unit circle where $z = e^{j\Omega}$. In other words, the DT Fourier transform of $h[n]$ must exist

Initial Value Theorem (IVT)

Initial Value Theorem (IVT)

For any Z-Transform pair $X(z) = \mathcal{Z}\{x[n]\}$

$$x[0] = \lim_{z \to \infty} X(z) \tag{6}$$

Final Value Theorem (FVT)

Final Value Theorem (FVT)

If all the poles of $(z-1)\,X(z)$ are inside the unit circle on the z-plane, then

$$x_{ss} := \lim_{n\to\infty} x[n] = \lim_{z\to 1}\left(1 - z^{-1}\right) X(z) = \lim_{z\to 1}\left(\frac{z-1}{z}\right) X(z) \tag{7}$$

Notice that this theorem requires $X(z)$ to be stable, or having at most one pole at $z=1$ (i.e., type I system). If the system has more than one pole at $z=1$ or on the unit circle, or is unstable, the Final value theorem will yield an erroneous result

One-sided Z-Transform

One-sided or Unilateral Z-Transform

The ***one-sided Z-Transform*** of $x[n]$ is the function of the complex variable $z = |z|e^{j\Omega}$ given by:

$$X(z) := Z\{x[n]\} = \sum_{n=0}^{\infty} x[n]\, z^{-n} \tag{8}$$

The unilateral Z-Transform can handle only positive-time signals and treat causal systems

Properties of the one-sided Z-Transform

Property	$x[n]$	$Z\{x[n]\}$
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(z) + a_2 X_2(z)$
Frequency scaling	$a^n x[n]$	$X\left(\frac{z}{a}\right)$
Frequency differentiation	$nx[n]$	$-z\frac{d}{dz}[X(z)]$
	$n^2 x[n]$	$z\frac{d}{dz}[X(z)] + z^2\frac{d^2}{dz^2}[X(z)]$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\left(\frac{1}{1-z^{-1}}\right) X(z) = \left(\frac{z}{z-1}\right) X(z)$
Convolution	$x_1[n] * x_2[n]$	$X_1(z) X_2(z)$
Modulation by $\cos[\Omega_0 n]$	$x[n]\cos(\Omega_0 n)$	$\frac{1}{2}\left[X\left(e^{j\Omega_0}z\right) + X\left(e^{-j\Omega_0}z\right)\right]$
Modulation by $\sin[\Omega_0 n]$	$x[n]\sin[\Omega_0 n]$	$\frac{j}{2}\left[X\left(e^{j\Omega_0}z\right) - X\left(e^{-j\Omega_0}z\right)\right]$

Properties of the one-sided Z-Transform

Property	$x[n]$	$Z\{x[n]\}$
First difference*	$x[n-1]$	$z^{-1}X(z)$
Second difference *	$x[n-2]$	$z^{-2}X(z) - z^{-1}x[-1]$
mth difference *	$x[n-m]$	$z^{-m}X[z] + z^{-m+1}x[-1] + \cdots + z^{-1}x[-m+1] + x[-m]$
First advance	$x[n+1]$	$zX(z) - zx[0]$
Second difference	$x[n+2]$	$z^2X(z) - z^2x[0] - zx[1]$
mth advance	$x[n+m]$	$z^mX(z) - z^m\sum_{n=0}^{m-1} x[n]z^{-n}$

Inverse Z-Transform

Inverse Z-Transform: definition

The ***inverse Z-Transform*** of the complex function $X(z)$ is defined as:

$$x[n] = \mathcal{Z}^{-1}\{X(z)\} := \frac{1}{2\pi j}\oint X(z)\, z^{n-1} dz \tag{9}$$

Notice that this is a closed contour integral around the origin in the complex z-plane. The contour (counterclockwise) has radius r (r is any value for which $X(z)$ converges). Consequently, **we must use complex integration methods to solve it directly**.

Computing inverse Z-Transforms by PFD

The closed contour integral defined in (9) is rarely used in engineering to compute inverse Z-Transforms [Chen, 2004, Sadiku, 2015] either in the bilateral or one-sided case. It is much simpler to find the inverse of $X(z)$ by looking it up on a table.

However, we must first express $X(z)$ as a sum of terms available in the transform table. To do so, we will make use of partial fraction decomposition.

Inverse Z-Transform for the bilateral case

Since two different functions can have the same algebraic expression as their Z-Transform, the specification of the ROC enables one to determine the corresponding time signal for the given transform

Example 11.2

Consider the Z-Transform of the signal $x_1[n] = a^n u[n]$:

$$X_1(z) = \frac{1}{1 - az^{-1}} \quad |z| > |a|$$

and that of $x_2[n] = -a^n u[-n-1]$:

$$X_2(z) = \frac{1}{1 - az^{-1}} \quad |z| < |a|$$

Since the Z-Transforms $X_1(z)$ and $X_2(z)$ have the same algebraic form, it is the difference between the corresponding ROCs enables one to distinguish the signals in the complex frequency domain.

Inverse Z-Transform for the unilateral case

Since the one-sided Z-Transform can handle only right-sided signals, the ROC will be in most cases the exterior of a circle centered in the origin of the z-plane.

An interesting case happens when the transform is rational: the ROC begins from the outermost pole. If in addition the transform is proper, it extends to infinity

Rational Z-Transform	$\Rightarrow$	Partial fraction decomposition

Transfer Function

Definition

The **transfer function** $H(z)$ is the ratio of the Z-Transform of the output of an LTI system to that of its input *assuming all zero initial conditions*. In words, the ***transfer function is the Z-Transform of the unit impulse response*** $h[n]$

$$h[n] \xrightarrow{\mathcal{Z}} H(z) = \frac{Y(z)}{X(z)} \tag{10}$$

The transfer function characterizes the zero-state (forced) response of an LTI system

BIBO Stability of an LTI System

Theorem on BIBO Stability of an LTI System

A SISO LTI system with proper rational transfer function $H(z)$ is BIBO stable if and only if every pole of $H(z)$ lies inside the unit circle on the z-plane

References I

Alkin, O. (2014).
Signals and Systems - A MATLAB Integrated Approach.
CRC Press.

Chen, C.-T. (2004).
Signals and Systems.
3 edition.

Oppenheim, A. V. and Willsky, A. S. (1998).
Signals and systems.
Prentice Hall, 2 edition.

Sadiku, M. (2015).
Signals and Systems - A primer with MATLAB.
CRC Press.