Taller de Sistemas de Ecuaciones Diferenciales

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1) Determine la solución general del sistema dado.

a) $\frac{dx}{dt} = 2x + 2y \qquad \frac{dy}{dt} = x + 3y$

$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ → Los autovalores son $\lambda = 4$ y $\lambda = 1$

→ los autovectores son $\vec{K}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ y $\vec{K}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

La solución general es $X = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t}$

2) Resuelva el problema de valor inicial

a) $X' = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} X \quad , \quad X(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$A = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix}$ → Los autovalores son: $\lambda_1 = \frac{1}{2}, \ \lambda_2 = -\frac{1}{2}$

→ Los autovectores son: $K_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ K_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

La solución general es: $\vec{x} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2}$

Sea $X(0) = \begin{pmatrix} C_1 \\ C_1 \end{pmatrix} + \begin{pmatrix} 0 \\ C_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_1 + C_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \qquad \begin{matrix} C_1 = 3 \\ C_2 = 5 - C_1 = 2 \end{matrix}$

Así, se tiene $\vec{x} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2}$

3. Encuentre la solución general del sistema.

a) $\frac{dx}{dt} = -6x + 5y \qquad \frac{dy}{dt} = -5x + 4y$

$A = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$ → El autovalor es $\lambda = -1$ con multiplicidad 2

→ El autovector es $\vec{K}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

* $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$

$\rightarrow (A - \lambda I)\vec{P} = \vec{K}$

$$\begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -5 & 5 & 1 \\ -5 & 5 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} -5 & 5 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

V. libre $= P_2$

Si $P_2 = \frac{1}{5}$ $\quad -5P_1 + 5(1/5) = 1 \Rightarrow -5P_1 + 1 = 1$

$\Rightarrow P_1 = 0$

$$\Rightarrow \vec{P} = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$$

$$* \; X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} e^{-t}$$

$\rightarrow$ la solución general es $x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} e^{-t} \right)$

4. Hallar la solución general del sistema.

a) $\frac{dx}{dt} = x + y \qquad \frac{dy}{dt} = -2x - 2y$

$A = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$ $\rightarrow$ dos valores propios son $\lambda_1 = 0$, $\lambda_2 = -1$.

dos vectores propios son $K_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $K_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

da solución general es $\vec{X} = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t}$

6. Utilice el método de los coef. indeterminados

$$X' = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} X + \begin{pmatrix} -2t^2 \\ t+5 \end{pmatrix}$$

$$2\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \left[\begin{pmatrix} a \\ b \end{pmatrix} t^2 + \begin{pmatrix} c \\ d \end{pmatrix} t + \begin{pmatrix} k \\ f \end{pmatrix} \right] + \begin{pmatrix} -2t^2 \\ t+5 \end{pmatrix}$$

$$\begin{pmatrix} 2at + c \\ 2bt + d \end{pmatrix} = \begin{pmatrix} (a+3b)t^2 + (c+3d)t + (k+3F) - (2t^2) \\ (3a+b)t^2 + (3c+d)t + (3k+F) + (t+5) \end{pmatrix}$$

"t^2" $\rightarrow 0 = a + 3b - 2$ $\quad$ "t" $\rightarrow 2a = c + 3d$ $\quad$ "coef" $\rightarrow c = k + 3F$

"t^2" $\rightarrow 0 = 3a + b$ $\quad$ "t" $\rightarrow 2b = 3c + d + 1$ $\quad$ "coef" $\rightarrow d = 3k + f + 5$

$a = -1/4 \quad c = 1/4 \quad k = -2$

$b = 3/4 \quad d = -1/4 \quad f = -3/4$

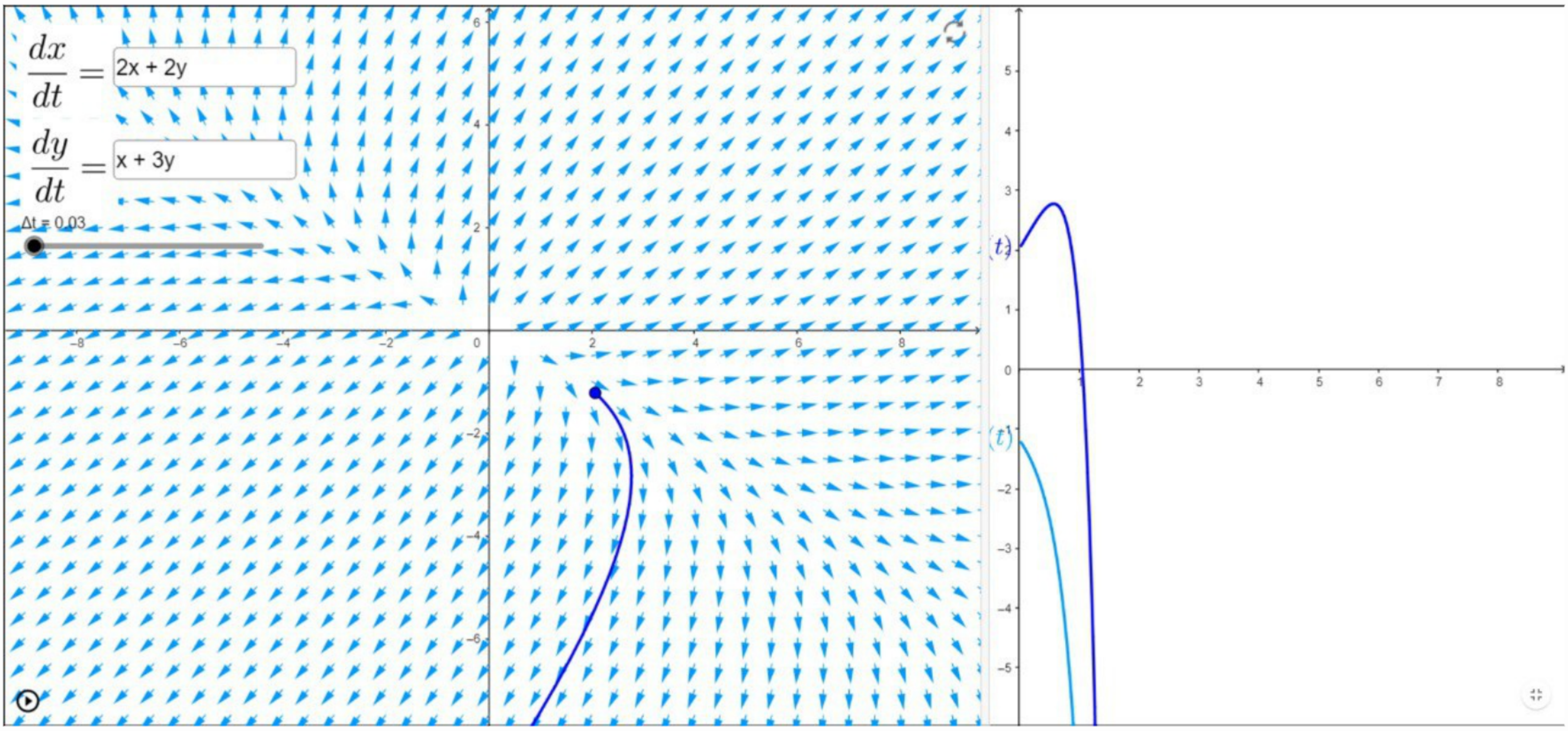
$\rightarrow X_p = \begin{pmatrix} -1/4 \\ 3/4 \end{pmatrix} t^2 + \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix} t + \begin{pmatrix} -2 \\ 3/4 \end{pmatrix}$

$\rightarrow$ Solución general:

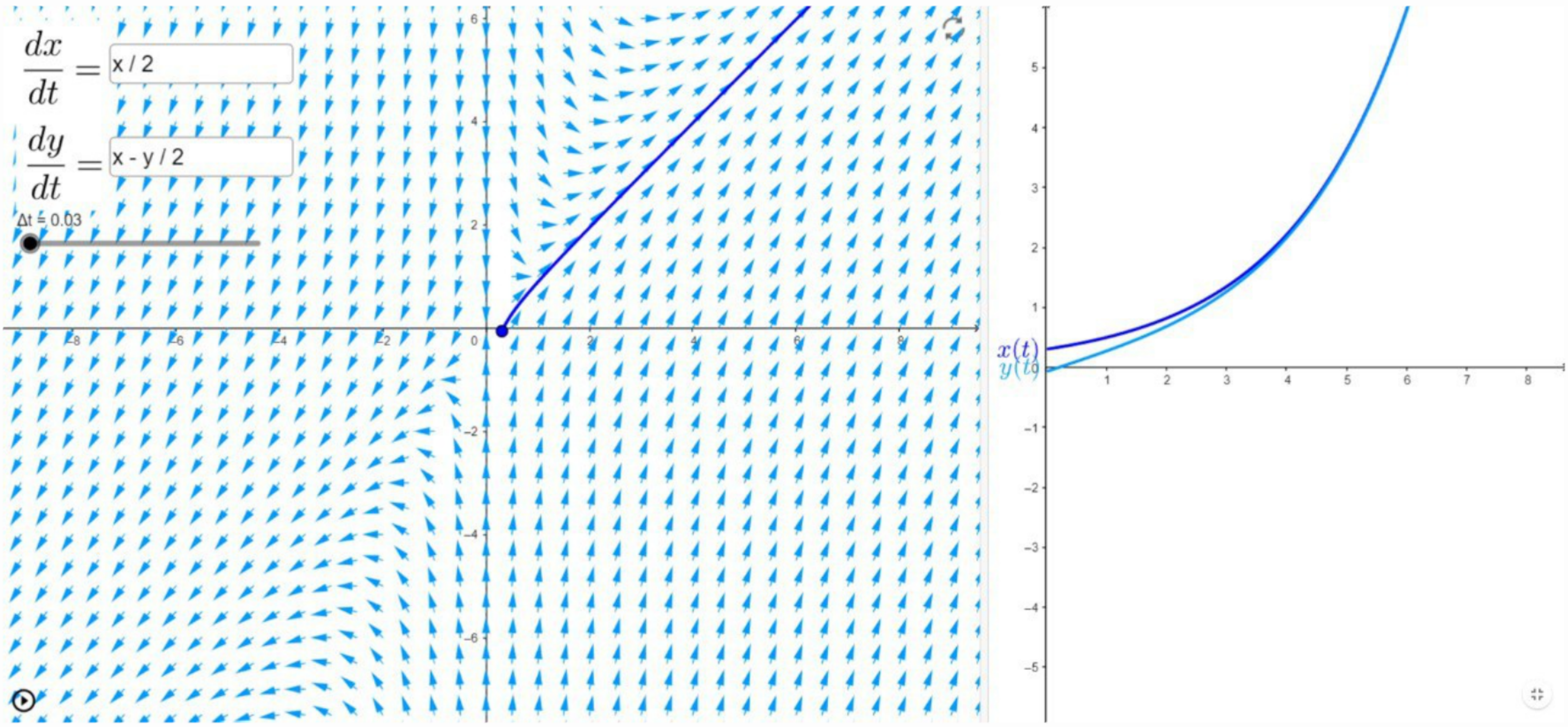
$$X = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} -1/4 \\ 3/4 \end{bmatrix} t^2 + \begin{bmatrix} 1/4 \\ -1/4 \end{bmatrix} t + \begin{bmatrix} -2 \\ 3/4 \end{bmatrix}$$

5. Diagrama de fase

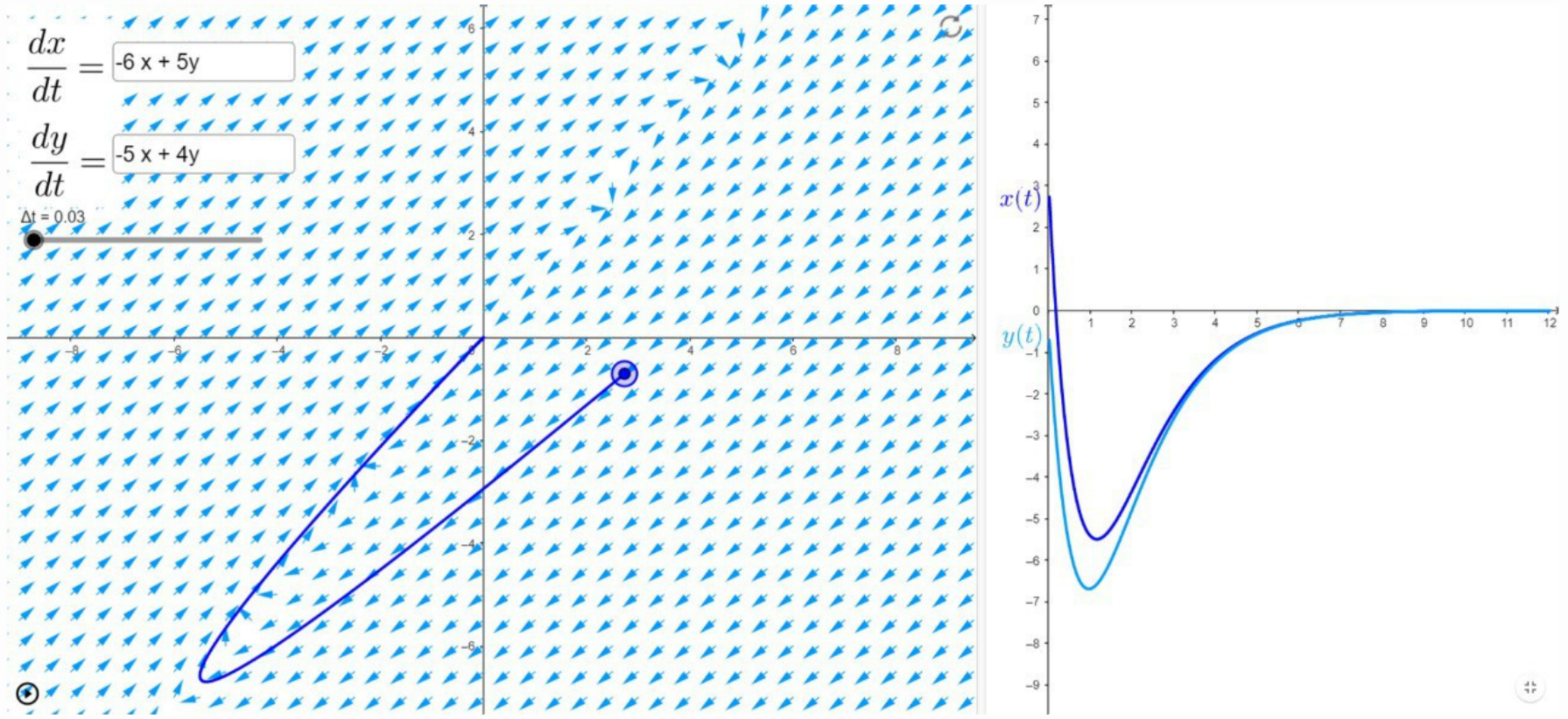
a) $\dfrac{dx}{dt} = 2x + 2y \qquad \dfrac{dy}{dt} = x + 3y$



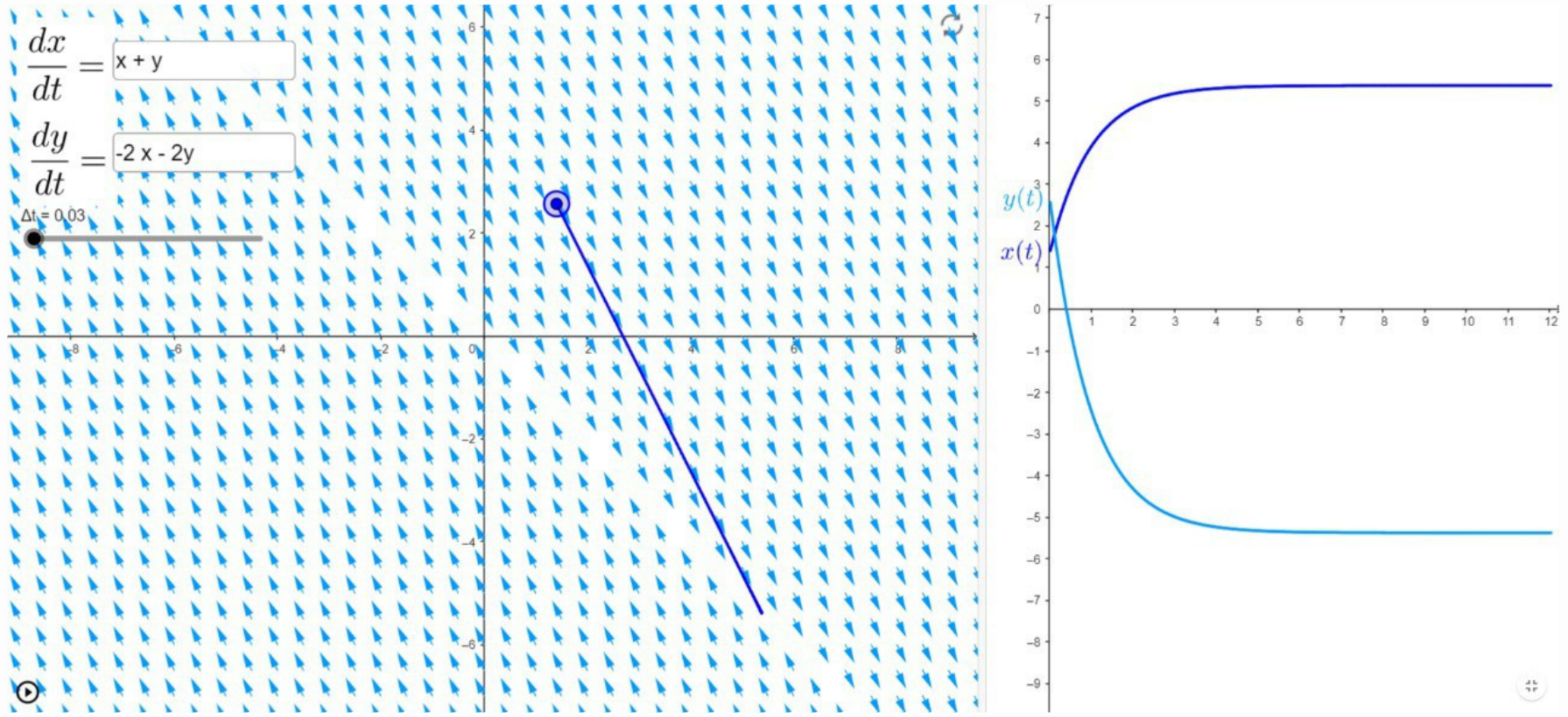
b) $X' = \begin{pmatrix} 1/2 & 0 \\ 1 & -1/2 \end{pmatrix} X \quad , \quad X(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$



c) $\dfrac{dx}{dt} = -6x + 5y \qquad \dfrac{dy}{dt} = -5x + 4y$



d) $\frac{dx}{dt} = x + y \qquad \frac{dy}{dt} = -2x - 2y$



7. Utilice variación de parámetros para resolver el sistema dado.

a) $X' = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} X + \begin{pmatrix} \operatorname{sen} 2t \\ 2\cos 2t \end{pmatrix} e^{2t}$

Si $A = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}$ → dos valores propios son $\lambda = 2 \pm 2i$

→ dos vectores propios son $K = \begin{pmatrix} i \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} i$

Con lo cual $\vec{X_c} = C_1 e^{2t}\left[\begin{pmatrix} 0 \\ 2 \end{pmatrix}\cos(2t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\operatorname{sen}(2t)\right] + C_2 e^{2t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(2t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\operatorname{sen}(2t)\right]$

$$\vec{X_c} = C_1 e^{2t}\begin{pmatrix} -\sin(2t) \\ 2\cos(2t)\end{pmatrix} + C_2 e^{2t}\begin{pmatrix} \cos(2t) \\ 2\sin(2t)\end{pmatrix}$$

Así $\Phi(t) = \begin{pmatrix} -e^{2t}\sin(2t) & e^{2t}\cos(2t) \\ 2e^{2t}\cos(2t) & 2e^{2t}\sin(2t)\end{pmatrix}$

Para invertir $|\Phi(t)| = \begin{vmatrix} -e^{2t}\sin(2t) & e^{2t}\cos(2t) \\ 2e^{2t}\cos(2t) & 2e^{2t}\sin(2t)\end{vmatrix} = -2e^{4t} \neq 0$

Así $\Phi^{-1}(t) = -\dfrac{1}{2e^{4t}}\begin{pmatrix} 2e^{2t}\sin(2t) & -e^{2t}\cos(2t) \\ -2e^{2t}\cos(2t) & -e^{2t}\sin(2t)\end{pmatrix} = \begin{pmatrix} -e^{-2t}\sin(2t) & \frac{1}{2}e^{-2t}\cos(2t) \\ e^{-2t}\cos(2t) & \frac{1}{2}e^{-2t}\sin(2t)\end{pmatrix}$

$$X_p = \begin{pmatrix} -e^{2t}\sin(2t) & e^{2t}\cos(2t) \\ 2e^{2t}\cos(2t) & 2e^{2t}\sin(2t)\end{pmatrix}\int \begin{pmatrix} -e^{-2t}\sin(2t) & \frac{1}{2}e^{-2t}\cos(2t) \\ e^{-2t}\cos(2t) & \frac{1}{2}e^{-2t}\sin(2t)\end{pmatrix}\begin{pmatrix} \sin 2t \\ 2\cos 2t\end{pmatrix} e^{2t}\,dt$$

$$= \begin{pmatrix} -e^{2t}\sin(2t) & e^{2t}\cos(2t) \\ 2e^{2t}\cos(2t) & 2e^{2t}\sin(2t)\end{pmatrix}\int \begin{pmatrix} -\sin^2(2t) + \cos^2(2t) \\ \sin(2t)\cos(2t) + \sin(2t)\cos(2t)\end{pmatrix} dt$$

$$\int \cos 4t\,dt = \frac{1}{4}\int \cos u\,du = \frac{\sin u}{4} = \frac{\sin(4t)}{4}$$

$\int 2\,\text{sen}\,2t\cos 2t\,dt$ $\quad u = \text{sen}\,2t \rightarrow dt = \dfrac{1}{2\cos 2t}du$

$$\int 2\,\text{sen}\,2t\cos 2t\,dt = \int 2u\cos 2t\,\frac{1}{2\cos 2t}\,du = \int u\,du = \frac{u^2}{2} = \frac{\text{sen}^2\,2t}{2} = -\frac{\cos 4t}{4}$$

Ahora la sol. particular

$$X_p = \begin{pmatrix} -e^{2t}\,\text{sen}\,2t & e^{2t}\cos 2t \\ 2e^{2t}\cos 2t & 2e^{2t}\,\text{sen}\,2t\end{pmatrix}\int \begin{pmatrix} \cos 4t \\ \hline 2\,\text{sen}\,2t\cos 2t\end{pmatrix} dt$$

$$= \begin{pmatrix} -e^{2t}\,\text{sen}\,2t & e^{2t}\cos 2t \\ 2e^{2t}\cos 2t & 2e^{2t}\,\text{sen}\,2t\end{pmatrix}\begin{pmatrix} \frac{\text{sen}\,4t}{4} \\ \frac{-\cos 4t}{4}\end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\text{sen}\,2t\,\text{sen}\,4t}{4} - \frac{\cos 2t\cos 4t}{4} \\ \frac{\cos 2t\,\text{sen}\,4t}{2} - \frac{\text{sen}\,2t\cos 4t}{2}\end{pmatrix} e^{2t}$$

Finalmente, la sol. General es

$$x(t) = X_c + X_p$$

$$= C_1\begin{pmatrix} -\text{sen}\,2t \\ 2\cos 2t\end{pmatrix}e^{2t} + C_2\begin{pmatrix} \cos 2t \\ 2\,\text{sen}\,2t\end{pmatrix}e^{2t} + \begin{pmatrix} -\frac{\text{sen}\,2t\,\text{sen}\,4t}{4} - \frac{\cos 2t\cos 4t}{4} \\ \frac{\cos 2t\,\text{sen}\,4t}{2} - \frac{\text{sen}\,2t\cos 4t}{2}\end{pmatrix}e^{2t}$$