HOMEWORK 1.

1. Evaluate the following expressions:
    - $3t^4\delta(t-1)$

    Answer:

    The sampling property states that $f(t)\delta(t-a) = f(a)\delta(t)$. For this example, f(t) = $3t^4$ and a=1. Then,

    $$3t^4\delta(t-1) = 3 * 1^4\delta(t)$$

    - $\int_{-\infty}^{\infty} t\delta(t-2)dt$

    Answer:

    The shifting property states that $\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha)$. For this example, $f(t) = t$ and $\alpha = 2$. Then $\int_{-\infty}^{\infty} t\delta(t-2)dt = f(2) = 2$

2. Express the voltage waveform $v(t)$ shown in the figure 1. As a sum of unit steps functions for the time interval $-1 < t < 7s$.

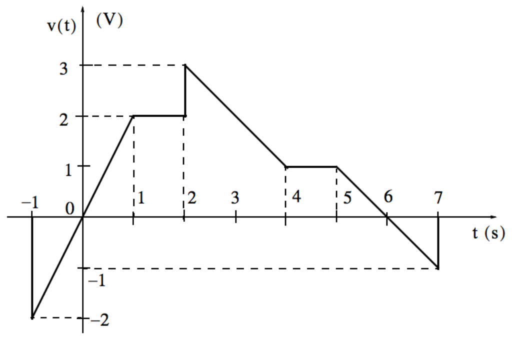
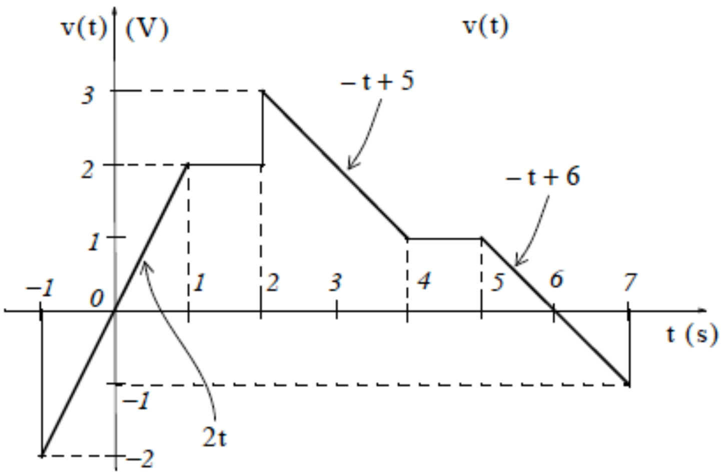


Figure 1. Waveform for problem 2.

Begin with the derivation of the equations for the linear segments of the given waveform as shown in the figure at next

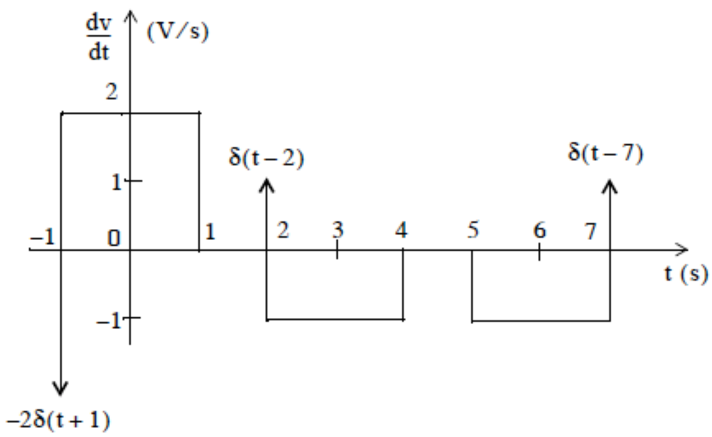


next express v(t) in terms of the unit step function $\hat{u}(t)$, and obtain

$$v(t) = 2t[\hat{u}(t+1) - \hat{u}(t-1)] + 2[\hat{u}(t-1) - \hat{u}(t-2)] + (-t+5)[\hat{u}(t-2) - \hat{u}(t-4)] + [\hat{u}(t-4) - \hat{u}(t-5)] + (-t+6)[\hat{u}(t-5) - \hat{u}(t-7)]$$

3. Using the results of problem 2., compute the derivative of v(t) and sketch its waveform.

$$\frac{dv}{dt} = 2\hat{u}(t+1) - 2\delta(t+1) - 2\hat{u}(t-1) - \hat{u}(t-2) + \delta(t-2) + \hat{u}(t-4) - \hat{u}(t-5) + \hat{u}(t-7) + \delta(t-7)$$



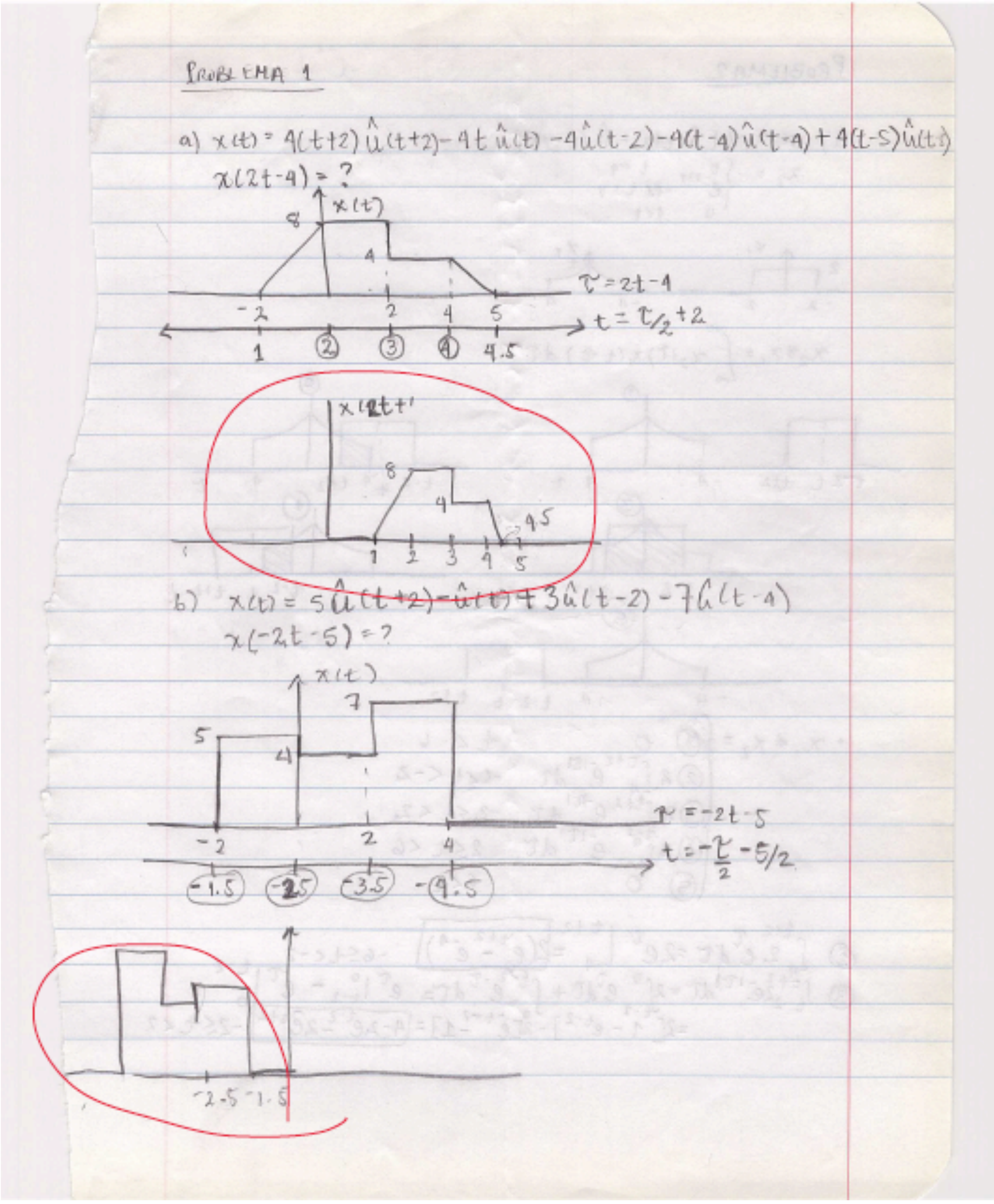
4. Given the signal

$$x(t) = 4(t+2)\hat{u}(t+2) - 4t\hat{u}(t) - 4\hat{u}(t-2) - 4(t-4)\hat{u}(t-4)$$
$$+ 4(t-5)\hat{u}(t-5)$$

Plot the figure for $x(2t-4)$

5. Given the signal $x(t) = 5\hat{u}(t+2) - \hat{u}(t) + 3\hat{u}(t-2) - 7\hat{u}(t-4)$

   Plot the figure for $x(-2t-5)$



6. Find the fundamental periods (T for continuous-time signals, N for discrete – time signals of the following periodic signals

   - $x(t) = \cos(13\pi t) + 2\sin(4\pi t)$

   Answer:

$$x(t+T) = \cos(13\pi t + 13\pi T) + 2\sin\,(4\pi t + 4\pi T)$$

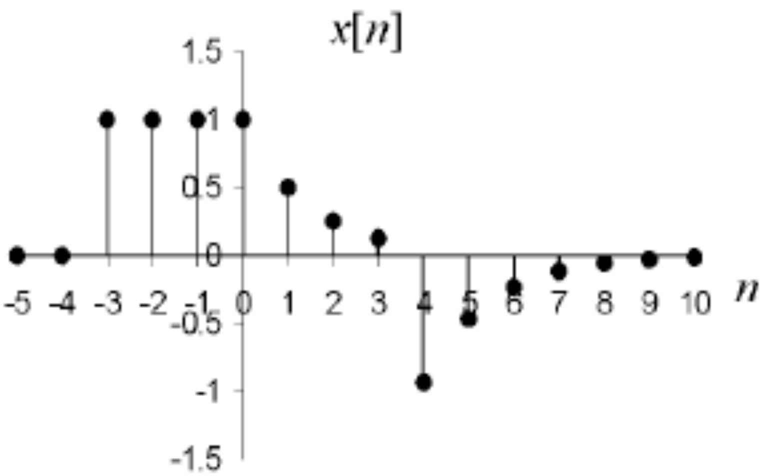
   will equal $x(t)$ if $\exists k, p \in \mathbb{Z}$ such that $13\pi t = 2\pi k, 4\pi T = 2\pi p$, which yields $T = \frac{2k}{13} = \frac{p}{2} \Rightarrow \frac{p}{k} = \frac{4}{13}$. The numerator and the denominator are coprime (no common divisor except 1); thus we take $p = 4, k = 13$, and the fundamental period is $T = \frac{p}{2} = 2$

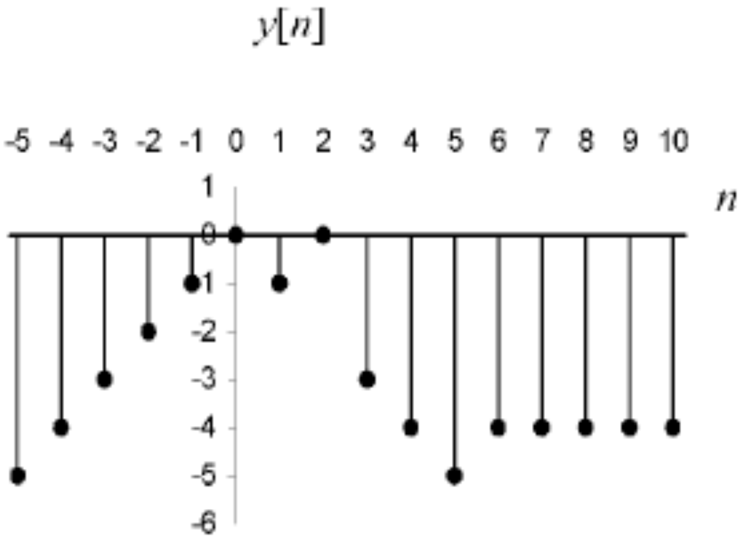
   - $x[n] = e^{j7.351\pi n}$

   Answer:

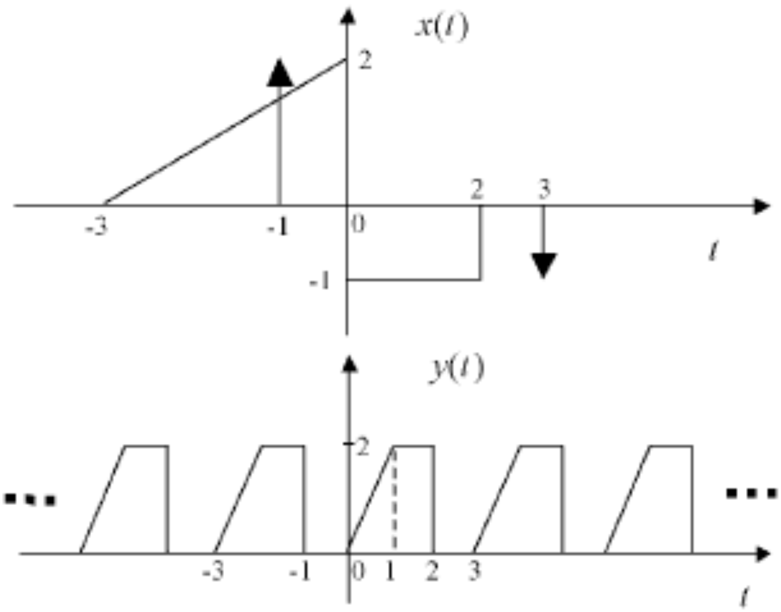
   $x[n] = e^{j7.351\pi n} = e^{j\frac{7351}{1000}\pi n}$; thus the frequency is $\omega_0 = \frac{7351}{1000}\pi = \frac{7351}{2000}2\pi$ and the number 7351 is prime, so the fundamental period is N=20001

7. Sketch the signals $x[n] = \hat{u}[n+3] - \hat{u}[n] + 0.5^n\hat{u}[n] - 0.5^{n-4}\hat{u}[n-4]$ and $y[n] = n\hat{u}[-n] - \delta[n-1] - n\hat{u}[n-3] + (n-4)\hat{u}[n-6]$.





8. Find the expressions for the signals shown in Figure 2.



Answer:

$$x(t) = 2/3\,(t+3)\hat{u}(t+3) - 2/3\,(t+3)\hat{u}(t) - \hat{u}(t) + \hat{u}(t-2) + 2\delta(t+1) - \delta(t-3)$$

$$y(t) = \sum_{k=-\infty}^{\infty} 2(t-3k)\hat{u}(t-3k) - 2(t-3k-1)\hat{u}(t-3k-1)$$
$$- 2\hat{u}(t-2-3k)$$

9. Determine whether the following systems are: (1) memoryless, (2) time-invariant, (3) linear, (4) causal, or (5) BIBO stable. Justify your answers.

- $y[n] = x[1-n]$

  Answer:

  1. Memoryless? No. For example, the output y[0]=x[1] depends on a future value of the input.
  2. Time-invariant? No.
  $$y_1[n] = Sx[n-N] = x[1-n-N] \neq x[1-(n-N)]$$
  $$= x[1-n+N] = y[n-N]$$
  3. Linear? Yes. Let $y_1[n] \coloneqq Sx_1[n] = x_1[1-n],\ y_2[n] \coloneqq Sx_2[n] = x_2[1-n]$. Then, the output of the system with $x[n] \coloneqq \alpha x_1[n] + \beta x_2[n]$ is given by:
  $$y = Sx:$$
  $$y[n] = x[1-n] = \alpha x_1[1-n] + \beta x_2[1-n]$$
  $$= \alpha y_1[n] + \beta y_2[n]$$
  4. Causal? No. For example, the output $y[0] = x[1]$ depends on a future value of the input.
  5. Stable? Yes. $|x[n]| < B, for\ all\ n\ \ then\ |y[n]| = |x[1-n]| < B, for\ all\ n$.

- $y(t) = \frac{x(t)}{1+x(t-1)}$

Answer:

1. memoryless? No. The system has memory since at time t, it uses the value of the input x(t-1).
2. Time-invariant? Yes. $y_1(t) = Sx(t-T) = \frac{x(t-T)}{1+x(t-1-T)} = y(t-T)$.
3. Linear? No. The system *S* is nonlinear since it does not have the superposition property:

   For $x_1(t), x_2(t)$, let $y_1(t) = \frac{x_1(t)}{1+x_1(t-1)}, y_2(t) = \frac{x_2(t)}{1+x_2(t-1)}$

   Define $x(t) = ax_1(t) + bx_2(t)$.

   Then $y(t) = \frac{ax_1(t)+bx_2(t)}{1+ax_1(t-1)+bx_2(t-1)} \neq \frac{ax_1(t)}{1+x_1(t-1)} + \frac{bx_2(t)}{1+x_2(t-1)} = ay_1(t) + by_2(t)$
4. Causal? Yes. The system is causal, as the output is a function of the past and current values of the input $x(t-1)$ and input $x(t)$ only.
5. Stable? No. Fort he bounded input $x(t) = -1, for\ all\ t\ \Rightarrow |y(t)| = \infty$, that is, the output is unbounded.

- $y(t) = tx(t)$

1. Memoryless? Yes. The output at time *t* depends only on the current value of the input $x(t)$.
2. Time-invariant? No. $y_1(t) = Sx(t-T) = tx(t-T) \neq (t-T)x(t-T) = y(t-T)$.
3. Linear? Yes. Let $y_1(t) \coloneqq Sx_1(t) = tx_1(t),\quad y_2(t) \coloneqq Sx_2(t) = tx_2(t)$. Then, $y(t) = S[ax_1(t) + bx_2(t)] = atx_1(t) + btx_2(t) = ay_1(t) + by_2(t)$
4. Causal? Yes. The output at time *t* depends on the present value of the input only.

5. Stable? No. Consider the constant input $x(t) = B \Longrightarrow$ for any $K, \exists T$ such that $|y(T)| = |TB| > K$, namely, $T > \frac{K}{|B|}$; that is, the output is unbounded.

- $y[n] = \sum_{k=-\infty}^{0} x[n-k]$

Answer:

1. Memoryless?. No. The output y[n] is computed using all future values of the input.
2. Time-invariant? Yes. $y_1[n] = Sx[n-N] = \sum_{k=-\infty}^{0} x[n-N-k] = y[n-N]$.
3. Linear? Yes. Let $y_1[n] \coloneqq Sx_1[n] = \sum_{k=-\infty}^{0} x_1[n-k]$, $y_2[n] \coloneqq Sx_2[n] = \sum_{k=-\infty}^{0} x_2[n-k]$. Then, the outputof the system with $x[n] \coloneqq \alpha x_1[n] + \beta x_2[n]$ is given by

$y[n] = \sum_{k=-\infty}^{0} \alpha x_1[n-k] + \beta x_2[n-k]$=$\alpha \sum_{k=-\infty}^{0} x_1[n-k] + \beta \sum_{k=-\infty}^{0} x_2[n-k] = \alpha y_1[n] + \beta y_2[n]$

4. Causal? No. The output y[n] depends on future values of the input $x[n+|k|]$.
5. Stable? No. For he input signal $x[n] = B$, for all n, then $|y[n]| = \left|\sum_{k=-\infty}^{0} x[n-k]\right| = \left|\sum_{k=-\infty}^{0} B\right| = +\infty$; that is, the output is unbounded.

10. Properties of even and odd signals

- Show that if $x[n]$ is an odd signal, then $\sum_{-\infty}^{+\infty} x[n] = 0$.

Answer:

For an odd signal ,

$$x[n] = -x[-n] \Longrightarrow x[0]$$
$$= 0 \; and \; \sum_{-\infty}^{\infty} x[n]$$
$$= x[0]$$
$$+ \sum_{n=1}^{\infty} (x[n] + x[-n]) = \sum_{n=1}^{\infty} (x[n] - x[n]) = 0$$

- Show that if x_1 is odd and $x_2[n]$ is even, then their product is odd.

  Answer:

  $$x_1[n] = -x_1[-n], \quad x_2[n] = x_2[-n] \Longrightarrow x_1[-n]x_2[-n]$$
  $$= -x_1[n]x_2[n]$$

- Let $x[n]$ be an arbitrary signal with even and odd parts $x_e[n], x_o[n]$. Show that $\sum_{-\infty}^{+\infty} x^2[n] = \sum_{-\infty}^{+\infty} x_e^2[n] + \sum_{-\infty}^{+\infty} x_o^2[n]$.

  Answer:

  $$\sum_{n=-\infty}^{+\infty} x^2[n] = \sum_{n=-\infty}^{\infty} (x_e[n] + x_o[n])^2$$
  $$= \sum_{n=-\infty}^{\infty} x_e^2[n] + 2 \sum_{n=-\infty}^{\infty} x_e[n]x_o[n] + \sum_{n=-\infty}^{\infty} x_o^2[n]$$
  $$= \sum_{n=-\infty}^{+\infty} x_e^2[n] + \sum_{n=-\infty}^{+\infty} x_o^2[n]$$

11. Evaluate the following functions:

    Use sampling property and shifting property of the $\delta(t)$ function

    - $\sin(t)\,\delta(t - \frac{\pi}{6})$

Answer

$$\sin(t)\,\delta\left(t - \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right)\delta(t) = 0.5\delta(t)$$

- $\cos(2t)\,\delta(t-\frac{\pi}{4})$

Answer

$$\cos(2t)\,\delta\left(t-\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{2}\right)\delta(t)=0$$

- $cos^2(t)\,\delta(t-\frac{\pi}{2})$

Answer

$$cos^2(t)\,\delta\left(t-\frac{\pi}{2}\right)=(\frac{1}{2}+\frac{1}{2}cos(\pi))\delta(t)=0$$

- $\tan(2t)\,\delta(t-\frac{\pi}{8})$

Answer

$$\tan(2t)\,\delta\left(t-\frac{\pi}{8}\right)=\tan\left(\frac{\pi}{4}\right)\delta(t)=\delta(t)$$

- $\int_{-\infty}^{\infty} t^2e^{-t}\,\delta(t-2)dt$

Answer

$$\int_{-\infty}^{\infty} t^2e^{-t}\,\delta(t-2)dt=2^2e^{-2}=0.54$$