Homogeneous continuous-time systems

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We will use a procedure analogue to discrete systems to generate solutions to continuous-time systems.

Def A linear cont. dynamical system is homogeneous or free if it has the form:

$$\dot{x} = A(t)\, x(t) \quad , t \in T \qquad (1)$$

(free of external forces)

Obs

1) If the elements of $A(t)$ are continuous functions of t, (1) will have a unique solution for every initial condition

initial state vector

2) An explicit expression of the solution (general form) in terms of the I.C is not possible

(unlike the discrete case)

3) The state-transition matrix & the fund. sets of solutions can also be defined and are useful

For relations (not existance) between solutions

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We assume that given $\tau \in ($, if $x(\tau)$ is specified there is a unique solution to (1). We define Φ as a matrix solution to (1) and use it for computing $x(t)$.

Def The s.t.m $\phi(t,x)$ corresponding to (1) is the $n \times n$ matrix satisfying:

i) $\frac{d}{dt}\phi(t,\tau) = \mathbb{A}(t)\,\phi(t,\tau)$

ii) $\phi(\tau,\tau) = \mathbb{I}$

$$\phi(t,\tau) = \begin{pmatrix} | & & | \\ x^1(t,\tau) & \cdots & x^n(t,\tau) \\ | & & | \end{pmatrix}$$

Interpretation

1) If τ is fixed, the n colums of ϕ satisfy (1)

(are vect. functions of t)

2) A $t=\tau$ the solutions must satisfy

$\phi(\tau,\tau) = \mathbb{I}$

(each solution vector must be one of standard unit basis vectors)

finding these n solutions yields $\phi(t,\tau)$

(We can do such a procedure $\forall \tau$)

Prop:

Prop:

Suppose $x(t)$ is a sol to (1), let $\tau \in T$ fixed then for any t

a known aux cond.

$$x(t) = \underbrace{\phi(t,\tau)\, x(\tau)}$$

Solution; linear comb. of cols of $\phi(t,\tau)$

Proof

$y(t) = \phi(t,\tau)x(\tau)$ is a solution to (1) (lin. comb. of cols of $\phi(t,x)$.

Also $y(\tau) = x(\tau)$.

Hence $y(t) = x(t)$.

Fundamental sets of solutions

We can express $\phi(t,\tau)$ in terms of an arbitrary fund. set of solutions.

Def Let $x^1(t), \ldots, x^m(t)$ be vector valued functions.

We say that they are lin. indep if

$$\alpha_1 x^1(t) + \ldots + \alpha^m x^m(t) = 0 \quad \forall t$$

$$\Rightarrow \alpha_1 = \cdots = \alpha_m = 0$$

Let $x^1(t), \ldots, x^n(t)$ be n lin. indep sols to (1). (there is no gen. method for obtaining these sols.)

Then these sols. form a fund. set of solutions

Let
$$\mathbb{X}(t) = \left(x^1(t) \,\Big|\, \ldots \,\Big|\, x^n(t) \right)$$

be the fund. matrix of solutions.

Then
$$\dot{\mathbb{X}}(t) = \mathbb{A}(t)\,\mathbb{X}(t).$$

Lemma

$\mathbb{X}(t)$ is non-singular for all t.

Proof Suppose that for a given τ, $\mathbb{X}(\tau)$ is singular

Then, there exists an $\alpha \neq 0$ such that

$$\mathbb{X}(\tau)\,\alpha = 0$$

Let
$$x(t) = \mathbb{X}(t)\,\alpha$$

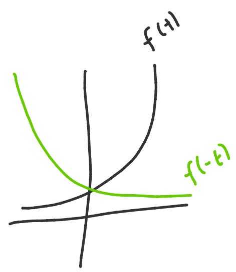
Then $x(t)$ is a sol. to (1) which is identically 0 for $t = \tau$.

Hence
$$x(t) = 0 \quad \text{for all } t \geq \tau.$$

Likewise, if $t' = -t$ (backwards time) then

$x(t')$ ← backwards solution is a solution of

$$\dot{x}(t') = -A\, x(t')$$



$$\left(\frac{dx}{dt'} \cdot \frac{dt'}{dt} = \frac{dx}{dt} \right)$$

and $x(t' = -\tau) = 0 \Rightarrow x(t') = 0 \quad t' > -\tau$

or $x(t) = 0 \quad t < \tau$.

Hence $\mathbb{X}(t)\alpha = 0 \;\; \forall t$ which is a contradiction since $x^1(t), \dots, x^n(t)$ are lin. indep

$\lightning$

Prop Let $\mathbb{X}(t)$ be a fund. matrix of sols. of (1).

Then
$$\phi(t, \tau) = \mathbb{X}(t)\, \mathbb{X}(\tau)^{-1}$$

Proof

Proof Let τ be fixed. Then

$\mathbb{X}(t)\mathbb{X}(\tau)^{-1}$ is a fund matrix of sols.

n lin. comb. of sols in $\mathbb{X}(t)$, with coef. $\mathbb{X}(\tau)^{-1}$.

Hence,

$$\frac{d}{dt}\mathbb{X}(t)\mathbb{X}(\tau)^{-1} = \mathbb{A}(t)\mathbb{X}(t)\mathbb{X}(\tau)$$

Also, for $t=\tau$

$$\mathbb{X}(t)\mathbb{X}(\tau)^{-1} = \mathbb{I}$$

Hence we must have by uniqueness thm.

$$\Phi(t,\tau) = \mathbb{X}(t)\mathbb{X}(\tau)^{-1}$$

Example 1

Consider the system

$$\dot{x}(t) = \begin{pmatrix} 0 & 0 \\ t & 1/t \end{pmatrix} x(t) \qquad t \geq 1.$$

$$\left(\begin{aligned} &\dot{x}_1(t) = 0 \\ &\dot{x}_2(t) = t\, x_1(t) + \frac{1}{t} x_2(t) \end{aligned} \right)$$

Then $x_1(t) = c$ (cnst)

substituting $\quad z(t) = \dfrac{x_2(t)}{t}$

$$t\dot{z}(t) + z(t) = ct + z(t)$$

$$\Rightarrow \quad \dot{z}(t) = c \quad \text{or} \quad x_2(t) = ct^2 + dt, \; c, d \in \mathbb{R}.$$

Hence we can costruct a fund. matrix of sols.

$$\mathbb{X}(t) = \begin{pmatrix} 1 & 0 \\ t^2 & t \end{pmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ c=1,\, d=0 & c=0,\, d=1 \end{array}$$

and

$$\phi(t,\tau) = \mathbb{X}(t)\mathbb{X}(\tau)^{-1} = \begin{pmatrix} 1 & 0 \\ t^2 & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\tau & 1/\tau \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ t(t-\tau) & t/\tau \end{pmatrix}$$

Ex. 2

Time invariant syst:

Consider a system:

$$\dot{x}(t) = \mathbb{A}\, x(t)$$

Then, the matrix satisfying

$$\dot{\mathbb{X}}(t) = \mathbb{A}\,\mathbb{X}(t)$$

$$\mathbb{X}(0) = \mathbb{I}$$

has to satisfy

converges $\forall \mathbb{A}, \forall t$
see ex. 20 ch. 4

has to satisfy

$$(*) \qquad \mathbb{X}(t) = \mathbb{I} + \mathbb{A}t + \overbrace{\frac{\mathbb{A}^2 t^2}{2!} + \frac{\mathbb{A}^3 t^3}{3!} + \dots}$$

matrix for each value of t

analog to Taylor series of exponential e^{at}

(element i,j of $\mathbb{X}(t)$ satisfies a series)

We define:

$$e^{\mathbb{A}t} = \mathbb{I} + \mathbb{A}t + \frac{\mathbb{A}^2 t^2}{2!} + \dots + \frac{\mathbb{A}^k t^k}{k!} + \dots$$

$\forall t, \forall \mathbb{A}$

To see $(*)$, let $t = 0$

then $\mathbb{X}(0) = \mathbb{I}$.

also

$$\frac{d\mathbb{X}}{dt} = \dot{\mathbb{X}}(t) = 0 + \mathbb{A} + \mathbb{A}^2 t + \frac{\mathbb{A}^3 t^2}{2!} + \dots$$

$$= \mathbb{A}\left(\mathbb{I} + \mathbb{A}t + \frac{\mathbb{A}^2 t^2}{2!} + \dots\right) = \mathbb{A}\,\mathbb{X}(t)$$

Hence if

$$\phi(t,\tau) = \mathbb{X}(t)\,\mathbb{X}(\tau)^{-1}$$

$$= e^{\mathbb{A}t}\, e^{-\mathbb{A}\tau} = e^{\mathbb{A}(t-\tau)}$$

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Since $\phi(t,\tau)$ depends on $t-\tau$ we normally write

$$\phi(t) = e^{\mathbb{A}t}$$

<u>Example 3</u>

Consider the eq:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \text{(Harmonic motion)}$$

In state-vector form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}_{\text{ctnt.}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Then

$$\phi(t) = e^{\mathbb{A}t}$$

now,

$$\mathbb{A}^2 = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} = -\omega^2 \mathbb{I}$$

hence,

$$\mathbb{A}^k = \begin{cases} (-1)^{k/2} \, \omega^k \, \mathbb{I}, & k \text{ even} \\ (-1)^{k-1/2} \, \omega^{k-1} \, \mathbb{A}, & k \text{ odd} \end{cases}$$

looking at each element of $e^{\mathbb{A}t}$ separately looking at each element of e^{At} separately

(remember $\cos(wt) = 1 - \frac{w^2t^2}{2!} + \frac{w^4t^4}{4!} \cdots$

$\sin(wt) = wt - \frac{w^3t^3}{3!} + \frac{w^5t^5}{5!} + \cdots$)

$$\Phi(t) = \begin{pmatrix} \cos wt & \frac{\sin wt}{w} \\ -w \sin wt & \cos wt \end{pmatrix}$$

E3 (Warfare, Lanchester 1916)

Each belligerent has a hitting power: average casualties per unit time of one soldier

$\hookrightarrow$ depends on military technology.

Suppose,

Bell. 1 : N_1 units of a force with hitting power α

Bell 2 : N_2 " " " " " β

$$\dot{N}_1(t) = -\beta N_2(t)$$

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$$A = \begin{pmatrix} 0 & -\beta \\ -\alpha & 0 \end{pmatrix}. \qquad A^2 = \begin{pmatrix} \alpha\beta & 0 \\ 0 & \alpha\beta \end{pmatrix} = \alpha\beta \, \mathbb{I}$$

$$A^k = \begin{cases} (\alpha\beta)^{k/2} \, \mathbb{I} & k \text{ even} \\ (\alpha\beta)^{k-1/2} \, A & k \text{ odd.} \end{cases}$$

$$e^{At} = \mathbb{I} + At + \frac{A^2 t}{2!} + \cdots$$

can be expressed in terms of $\sinh(\alpha\beta)$, $\cos(\alpha\beta)$