# Homogeneous discrete time systems & general solution

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**Def**

A linear dynamic system is homogeneous or free if there is no forcing term.

↳ Run by themselves w/o external control/inputs/forces

i.e if $\mathbb{x}(k+1) = \mathbb{A}(k)\, x(k)$ , $k \in I$ (1)

$\left( \dot{\mathbb{x}}(t) = \mathbb{A}(t)\, \mathbb{x}(t) \quad t \in T \text{ if system is continuous} \right)$

We will focus on the discrete case first.

**Solution to free system**

Consider the system

$$x(k+1) = \mathbb{A}(k)\, x(k), \quad k \in \mathbb{N}^*$$

We can find the solution recursively given the I.C (a vector);

$k=0$, $\mathbb{x}(1) = \mathbb{A}(0)\, \mathbb{x}(0)$

$k=1$, $\mathbb{x}(2) = \mathbb{A}(1)\, \mathbb{x}(1) = \mathbb{A}(1)\, \mathbb{A}(0)\, x(0)$

$\vdots$

In general:

$$x(k) = \mathbb{A}(k-1) \cdots \mathbb{A}(0)\, x(0) \quad , \quad k \geq 1.$$

We define

$$\phi(k,0) = \mathbb{A}(k-1) \cdots \mathbb{A}(0)$$

$$\phi(k,0) = A(k-1) \cdots A(0)$$

↳ state transition matrix

def - Multiplying an I.C by s.t. matrix $\phi(k,0)$ yields state at time k.

The general definition of $\phi$ is:

Def

transition between states

The state-transition matrix of a homogeneous system (1) is:

$$\phi(k,l) = A(k-1)A(k-2) \cdots A(l), \quad k > l$$

$$\phi(k,k) = \mathbb{I}$$

Alternatively (and equivalently)

$$\phi(k+1,l) = A(k)\phi(k,l), \quad k > l$$

$$\phi(l,l) = \mathbb{I}$$

Obs (state transition property)

The matrix can be used to compute $x(k)$ if $x(l)$ is known $(k \geq l)$;

$$x(k) = \phi(k,l)\, x(l)$$

Obs

The s.t.m has great notational advantages but can only be calculated explicitely and simply in few situations

(a shortcut)

## Fundamental sets of solutions

- The s.t.m can be used to represent solutions of systems
- Alternative interpretation based on fund. solutions.
  - Highlights linearity, is relevant later in cont. systems.

Suppose $A(k)$ nonsingular $\forall k$ (not essential but makes things simpler)

Consider a set of $n$ solutions of to eq. (1)

$\mathbf{x}^1(k), \ldots, \mathbf{x}^n(k)$

vectors of $n$ dimensions, function of $k$.

each sol. is a seq. of vectors satisfying (1)

We require lin. indep

Def A set of $m$ vector seq. $\mathbf{x}^1(k), \mathbf{x}^2(k), \ldots, \mathbf{x}^m(k)$ $k \in I$, is said to be lin. indep. if there is no nontrivial lin. comb. of them that is identically zero.

i.e.

no nontrivial lin. comb. of them.

i.e.

$$\alpha_1 \mathbb{x}^1(k) + \dots + \alpha_m \mathbb{x}^m(k) = 0 \quad \forall k$$

$$\Longrightarrow \quad \alpha_1 = \alpha_2 = \dots = \alpha_m = 0$$

Obs

- The definition must apply for all $k \in I$.
- for a fixed $k$ the vectors $\mathbb{x}^1(k), \dots, \mathbb{x}^m(k)$ could be linearly dependent
- It is sufficient that for at least one $k$ they are lin. indep

Def

Suppose $\mathbb{x}^1(k), \dots, \mathbb{x}^n(k)$ are $n$ lin. indep solutions to (1).

$$\mathbb{x}^i(k+1) = \mathbb{A}(k)\, \mathbb{x}^i(k) \qquad i = 1, \dots, n \qquad , k \in I.$$

The set $\mathbb{x}^1(k), \dots, \mathbb{x}^n(k)$ is known as a fundamental set of solutions

(We will show that every sol. can be expressed as a lin. comb. of the $n$ fund. sols.)

We construct the stacked matrix:

$$\underset{n \times n}{\mathbb{X}(k)} = \Big( \underset{n\times 1}{\mathbb{x}^1(k)} \,\Big|\, \mathbb{x}^2(k) \,\Big|\, \dots \,\Big|\, \underset{n \times 1}{\mathbb{x}^n(k)} \Big)$$

$$\underbrace{\underset{n\times n}{\mathbb{X}(k)}}_{\text{Fundamental matrix of solutions.}} = \Big( \underset{n\times 1}{x^1(k)} \,\Big|\, x^2(k) \,\Big|\, \cdots \,\Big|\, \underset{n\times 1}{x^n(k)} \Big)$$

**Lemma** The fundamental matrix of sols. satisfies

$$\mathbb{X}(k+1) = \mathbb{A}(k)\,\mathbb{X}(k)$$

**Proof**

Clear since each column of $\mathbb{X}(k)$ is a sol. of (1)

**Lemma** A fundamental matrix of solutions $\mathbb{X}(k)$ is nonsingular for every value of $k$.

**Proof.**

Suppose that for $k_0$, $\mathbb{X}(k_0)$ is singular.

Then, $\exists\, \alpha \neq 0 : \mathbb{X}(k_0)\alpha = 0$.

$$\Rightarrow \quad \mathbb{X}(k_0+1)\alpha = \mathbb{A}(k_0)\,\mathbb{X}(k_0)\alpha = 0$$

hence $\mathbb{X}(k_0+1)$ is singular

Also, when multiplying by $\mathbb{A}(k_0-1)^{-1}$

$$\mathbb{X}(k_0-1)\alpha = \mathbb{A}(k_0-1)^{-1}\,\mathbb{X}(k_0)\alpha = 0$$

$$\mathbb{X}(k_0-1)\,\alpha = \mathbb{A}(k_0-1)^{-1}\,\mathbb{X}(k_0)\,\alpha = 0$$

$\Rightarrow$ $\mathbb{X}(k_0-1)$ singular.

Continuing forward and backwards

$$\mathbb{X}(k)\,\alpha = 0 \qquad \forall\, k.$$

Then

$$\alpha_1 x^1(k) + \dots + \alpha_n x^n(k) = 0 \qquad \forall k$$

But $x^1(k), \dots, x^n(k)$ are lin indep

then $\alpha = 0$ $\lightning$.

Now, let $x(k)$ be a solution to (1) with an initial condition $x(0)$.

Consider the sequence $\bar{x}(k)$ where

$$\bar{x}(k) = \mathbb{X}(k)\,\mathbb{X}(0)^{-1}\,x(0)$$

Then, $\bar{x}(0) = x(0)$

Also, if $\alpha = \mathbb{X}(0)^{-1}\,x(0)$

$$\Rightarrow \quad \bar{x}(k) = \mathbb{X}(k)\,\alpha.$$

($\bar{x}(k)$ is a lin. comb. of the sols.)

Moreover,
$$\bar{x}(k+1) = \mathbb{X}(k+1)\,\alpha = \mathbb{A}(k)\,\mathbb{X}(k)\,\alpha$$
$$= \mathbb{A}(k)\,\bar{x}(k)$$

(1).

$= A(k)\, x(k)$

So $\bar{x}(k)$ is a sol. to (1).

Then, since $\bar{x}, x$ have the same I.Cs.

$$\bar{x}(k) = x(k)$$

Hence

$$x(k) = \mathbb{X}(k)\, \mathbb{X}(0)^{-1}\, x(0)$$

In general,

$$x(k) = \mathbb{X}(k)\, \mathbb{X}(l)^{-1} x(l) \quad \text{for any } l.$$

Proposition

Let $\mathbb{X}(k)$ be a fund. matrix of sols. for

$$x(k+1) = A(k)\, x(k)$$

The state transition matrix is given by

$$\Phi(k, l) = \mathbb{X}(k)\, \mathbb{X}(l)^{-1}, \quad k \geq l.$$

Obs: Consider the relation:

$$\Phi(k, 0) = \mathbb{X}(k)\, \mathbb{X}(0)^{-1}$$

Since $\mathbb{X}(k)$ is a fund. matrix of sols.

$\underbrace{\mathbb{X}(k)\, \mathbb{X}(0)^{-1}}_{Y(k)}$ is also a fund. matrix of sols.

$$\mathbb{X}(k+1)\, \mathbb{X}(0)^{-1} = A(k)\, \mathbb{X}(k)\, \mathbb{X}(0)^{-1}$$

$\mathbb{Y}(k)$

$$\underbrace{\mathbb{X}(k+1)\,\mathbb{X}(0)^{-1}}_{\mathbb{Y}(k+1)} = \mathbb{A}(k)\,\underbrace{\mathbb{X}(k)\,\mathbb{X}(0)^{-1}}_{\mathbb{Y}(k)}$$

columns of $\mathbb{Y}(k)$ are lin. comb. of columns of $\mathbb{X}(k)$

Therefore

$\phi(k,0)$ is a fund. matrix of sols. which satisfies $\phi(0,0) = \mathbb{I}$.

In other words:

$\phi(0,0)$ has cols $\quad \mathbb{x}^i(0) = \begin{array}{c} \left(\begin{array}{c} 0 \\ | \\ 1 \\ | \\ 0 \end{array}\right) \end{array} \begin{array}{l} \\ \\ \leftarrow i\text{th} \\ \\ \\ \end{array} \qquad i = 1, \dots, n.$

This set of sols can be used to construct a sol. corresponding to an I.C. $\mathbb{x}(0)$.

This, since

$$\mathbb{x}(k) = \overbrace{\phi(k,0)}^{\text{columns are } \mathbb{x}^i(k)}\,\mathbb{x}(0)$$

<u>Example</u> (time-invariant system)

Consider the system

$$\mathbb{x}(k+1) = \mathbb{A}\,\mathbb{x}(k), \qquad k \in I.$$

$\uparrow$ indep. of $k$.

$\llcorner$ indep. of k.

It can be seen that:

$$\phi(k,0) = \mathbb{A}^k \quad , \quad k \geq 0.$$

and,
$$\phi(k,\ell) = A^{k-\ell} \quad k \geq \ell.$$

(If $\mathbb{A}$ is invertible the expression is also valid for $k < \ell$)

In this case one usually denotes $\phi(k,0)$ by $\phi(k)$

since $\phi(k,\ell) = \phi(k-\ell)$.

Example 2 (non-cnst system)

Consider the system (lin. hom.)

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \quad , \quad k \geq 0.$$

A fund. set of sol. can be found with 2 lin. indep I.Cs.

Let $\mathbb{X}^1(k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , $k \geq 0$ be our 1st sol.

The second sol. can be constructed from:

$$\mathbb{X}^2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence $\mathbb{X}^2(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbb{X}^2(2) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbb{X}^2(3) = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$, $\mathbb{X}^2(4) = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$, ...

$$x^2(k) = \begin{pmatrix} k(k+1)/2 \\ 1 \end{pmatrix}$$

From the two sols the fund. matrix

$$\mathbb{X}(k) = \begin{pmatrix} 1 & k(k+1)/2 \\ 0 & 1 \end{pmatrix}$$

Given an I.C

$$\mathbb{x}(0) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

the solution will be

$$\mathbb{x}(k) = \mathbb{X}(k) \overbrace{\mathbb{X}(0)^{-1}}^{\mathbb{I}} \mathbb{x}(0)$$

$$= \begin{pmatrix} \alpha_1 + \alpha_2 (k(k+1))/2 \\ \alpha_2 \end{pmatrix}.$$