Difference Equations Pt1

jueves, 27 de enero de 2022 10:29 a. m.

Difference Eqs. 1

Difference and differential eqs are versatile tools for modelling dynamical systems

↳ They represent well some dynamic phenomena

↳ They have a rich mathematical framework (theory)

We will study initially the case where these eqs have a single state variable (one funct. of time)

Obs

- These models can be generalized to more variables
- Single variable models are very useful in many situations + Theory serves for a good background for multivariate generalizations

↳ Single variable models are a good 1st step.

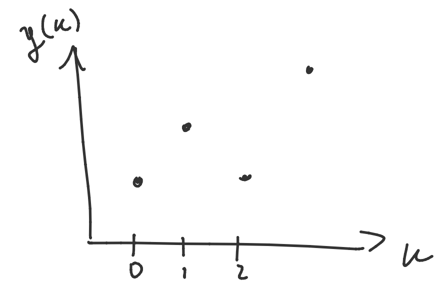
Difference Eqs.

sequence of points

Def Suppose there is a sequence of points - possibly representing equally-spaced time points - indexed by k, ($k \in \mathbb{N}^*$. or <u>any discrete set</u>)

Suppose there is a value $y(k) \in \mathbb{R}$ associated with each of these points

$$y : \mathbb{N}^* \longrightarrow \mathbb{R}$$
$$k \longmapsto y(k)$$



A difference equation is an equation relating $y(k)$ at point K to values at other (usually neighboring) points.

Def 2 Let $x : \mathbb{N} \longrightarrow \mathbb{R}$ ($\mathbb{N}^*$), a d.e is an eq. of the form

$$F(k, x(k), \dots, x(k+m)) = 0$$

Ex:

1) $y(k+1) = a\, y(k)$, $k = 0, 1, 2, \dots$

ctnt.

2) $k\, y(k+2)\, y(k+1) = \frac{1}{2}\sqrt{y(k)\, y(k-1)}$ $k = 1, 2, 3, \dots$

Obs A difference eq. can be seen as a "rule"

1)

Obs. A difference eq.

1) or as a set of eqs.

↳ one for each k.

2) Usually $k \in \mathbb{Z}$, $\underbrace{k \in \mathbb{N}^*}_{\text{most common.}}$, or $k \in \mathbb{N}$

3) Once the indexing set is define, one can substitute the k's to obtain explicit eqs. relating the y's.

Ex: Ex. 1 is equivalent to

$$y(1) = a\,y(0)$$
$$y(2) = a\,y(1)$$
$$\vdots$$

(How many initial cond. do we need to generate a solution? (A: 1))

Obs

- If the y(k) are known, the $\underbrace{\text{difference eq.}}_{\text{d.e}}$ is a relation among values (short description)

  y(0)

  Ex. 1, 3, 9, ...

  $y(k+1) = 3\,y(k)$

- If the y(k) are unknown the d.e can be used to solve for the unknown y(k) based be used to solve for the unknown y

on a set of I.C. ↳ More frequent.

(Relationship to be satisfied by solution ↳ function)

The name difference eq. comes from the possibility of expressing (explicitly) the differences:
↳ jumps or diff. of jumps

$$\Delta^0(k) = y(k)$$

$$\Delta^1(k) = \Delta^0(k+1) - \Delta^0(k) \qquad (y(k+1) - y(k))$$

$$\Delta^2(k) = \Delta^1(k+1) - \Delta^1(k)$$

$$((\Delta^0(k+2) - \Delta^0(k+1)) - (\Delta^0(k+1) - \Delta^0(k))$$

$$(y(k+2) - y(k+1)) - (y(k+1) - y(k))$$

$\vdots$

<u>Def</u> The order of a d.e is the difference between the highest and the lowest index that appears.

Ex. Ex 1 has order 1,

Ex 2 has order 3

Def A d.e is said to be linear if it has the form:

$$a_n(k)\, y(k+n) + a_{n-1}(k)\, y(k+n-1) + \dots + a_0(k)\, y(k) = g(k)$$

a_i's are coeff., can depend on k

no nonlinear oper. of $y(k)$'s

(unknown y appears linearly)

If the a_i's don't depend on k the d.e. the eq. is said to have constant coefficients (time-invariant)

$g(k)$ is known as the forcing term
if $g(k)=0$ the d.e is said to be homogeneous.

Solutions of d.e's.

A solution is a function $y(k)$ that reduces the eq. to an identity (satisfies the d.e $\forall k$).

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<u>Ex 3</u> : $y(k+1) = a\,y(k)$

Then $y(k) = a^k$ is a sol. since

$$y(k+1) = a^{k+1} = a \cdot a^k = a\, y(k)$$

<u>Obs</u> A sequence can be seen as a sequence of numbers.

<u>Ex</u> For ex. 3 if $a = \frac{1}{2}$ a solution can be a seq.

$$1, \ 1/2, \ 1/4, \ 1/8, \ \ldots$$

which can be written as $\frac{1}{2}^k$, $k = 0, 1, \ldots$

$$\left\{\frac{1}{2}^k\right\}_{k \in \mathbb{N}^*}.$$

<u>Obs</u>

In general finding a simple representation for the sol. is unviable.

<u>Ex 1</u> Consider the linear diff. eq.

$$(k+1)\, y(k+1) - k\, y(k) = 1 \qquad k = 1, 2, \ldots$$

A solution is $y(k) = 1 - 1/k$

Verification: $y(k) = \frac{k-1}{k}$, $y(k+1) = \frac{k}{k+1}$

hence $(k+1)\,y(k+1) - k\,y(k) = k-(k-1) = 1$

There are other solutions, (only one?)

Indeed, $y(k) = 1 - A/k$, $A \neq 0$

is a solution

Ex 2: A nonlinear d.e (arising in Genetics) ? ch.10

$$y(k+1) = \frac{y(k)}{1+y(k)} \qquad k = 0, 1, 2, \ldots$$

A family of sols is.

$$y(k) = \frac{A}{1+Ak}, \quad A \in \mathbb{R}$$

Ex 3 Consider the nonlin. d.e

$$y(k+1)^2 + y(k)^2 = -1$$

The d.e has no real solutions

Existence and uniqueness of solutions

Like other eqs. there can be no sols. or multiple solutions.

We will examine these conceps more generally

Initial Conditions

Obs One essential characteristic of d.e's is that over a finite interval of time (indexed by k) there are more unknowns that eqs.

e.g. if we have

$$y(k+1) = 2y(k)$$

for $k = 0, 1,$

$$y(1) = 2\underline{y(0)}$$

$$\underline{y(2)} = 2\underline{y(1)}$$

3 unknowns
2 eqs.

$\Rightarrow$ To derive an explicit (unique) solution it is necessary to asign a value to one of the unknowns

If more eqs. are added 1 unknown added

If more eqs. are added 1 unknown added per equation.

$\Rightarrow$ No matter how long seq. is (# eqs)
unknowns is larger than # eqs.

In general,

if for a d.e, for each k the eq. has $n+1$ unknown variables (the y's) (successive points)
there are n more unknowns that eqs. in any finite set.

(Any new eq. adds $\underbrace{\text{one unknown}}_{\text{at least}}$ and one eq.
$\Rightarrow$ surplus is n)

The surplus of n allows the values of n var. to be specified arbitrarily (n degrees of freedom)

These values are set as arbitrary constants in the expression of the general solution.

<u>Obs</u> Any n arbitrary components of the sol. can

Any n arbitrary comp...

be specified

It is conventional to specify 1st n.

$$\big(\underbrace{y(0), \dots, y(n-1)}_{\text{Initial conditions}}\big)$$

Ex.

1) $y(k+1) = a\, y(k)$ (geom. growth.)

gen. sol: $y(k) = C \cdot a^k$

But $y(0) = C \cdot a^0 = C$, so gen. sol

is $y(k) = y(0)\, a^k$.

2) $y(k+2) = y(k)$ (order 2)

(can be written as $\underbrace{y(k+2) + 0 \cdot y(k+1)}_{\text{have successive values}} = y(k)$)

2 I.C are needed

(one for even # k's, one for odd k's)

Specifying $y(0), y(1)$

yields gen. sol:

$$y(k) = \left(\frac{y(0) + y(1)}{2}\right) + (-1)^k \left(\frac{y(0) - y(1)}{2}\right)$$

Existence and uniqueness Thm

- D.E. can have or not sols.
- One does not need to explicetly find sol. to show (infer) its existence
- Also, existence does not guarantee uniqueness. (part. sol. from I.C.)

Thm: Let a difference equation of the form:

$$y(k+n) + f\left(y(k+n-1), \ldots, y(k), k\right) = 0 \qquad (1)$$

where f is an arbitrary function $(\mathrm{Im}(f) \subseteq \mathbb{R})$.

Suppose the d.e is defined over a finite or infinite sequence of integer values
$(k = k_0, k_0+1, k_0+2, \ldots)$

The eq. has one and only one solution corresponding to each arbitrary specification of n initial values $y(k_0), \ldots, y(k_0+n-1)$.

Proof (uses the axiom of choice)

Suppose the values $y(k_0), \dots, y(k_0+n-1)$ are specified.

Then the d.e with $k = k_0$ can be solved uniquely for $y(k_0+n)$ $\left(y(k_0+n) = -f(\dots\ \dots) \right)$

Then the d.e can be used to solve for $y(k_0+n+1)$, with $k = k_0+1$.

Continuing inductively one can generate the successive values of the seq. $\square$

Obs

* No restrictions are required on f.
* Linear eqs. have form of (1)

  (f is a lin. comb.)

  $\Rightarrow$ Lin. d.e have a unique sol. for every set of I.C.

—/——

Generating sols. of eq. of order 1.

Consider the eq.

$y(k+1) = a\,y(k) + b$. (linear with constant coeff.)

To find general sol. suppose $k_0 = 0$

$y(0) = y_0$ (order 1 $\rightarrow$ 1 I.C)

$y(1) = a\,y(0) + b = a y_0 + b$

$y(2) = a\,y(1) + b = a(a y_0 + b) + b = a^2 y_0 + ab + b$

$y(3) = a\,y(2) + b = a(a^2 y_0 + ab + b) + b = a^3 y_0 + a^2 b + ab + b$

$\vdots$

$y(k) = a^k y_0 + b(a^{k-1} \cdots + a + 1)$

$$= \begin{cases} y_0 + b \cdot K, & a = 1 \\ a^k y_0 + \dfrac{1-a^k}{1-a} b, & a \neq 1. \end{cases}$$

↳ $D a^k + \dfrac{b}{1-a}$, $D = y_0 - \left(\dfrac{b}{1-a}\right)$

Chain letters and amortization (Examples)

<u>Ex1</u>.

Suppose you receive a chain letter with 6 names and addresses

- You have to send 10 ¢ to 1st person
- You can then create a new letter with the first name deleted and your name

the first name deleted and you ...
added at the bottom
↳ Send letter to 5 friends

You are promised \$1562.50.

Sol.

Let $y(k)$: # letters in kth generation

(letter you receive is $y(0) = 1$)

Each letter introduces 5 more in next gen.

$$y(k+1) = 5y(k).$$

Solution is $y(k) = 5^k$

Since all recipients of 6th gen. send you 10¢, you get $10 \cdot y(6) = 10 \cdot \underbrace{15,625}_{5^6} = 1562.5$ \$.

Ex 2 Accummulation of bank deposits with simple interest. (Deposits made each year at the beg

$y(k)$: Amount at beginning of year k.

If no interest:

$$y(k+1) = y(k) + \underbrace{b(k)}_{\text{deposit}}$$ made at start of year.

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if $b(k) = b$, growth is linear.

If interest i compounded annually:

$$y(k+1) = y(k)(1+i) + \underbrace{b}_{\text{constant deposits}}$$

Ex 3 Amortization

If payment B is made at the end of each year

$$d(k+1) = (1+i)\, d(k) - B$$

where $d(0) = D$; initial debt

If we want to pay in n years we need B such that $d(n) = 0$

Using gen. sol.

$$d(n) = D(1+i)^n - \frac{1-(1+i)^n}{1-(1+i)} B$$

$$d(n) = 0 \Rightarrow B = \frac{iD}{1-(1+i)^{-n}}$$