Functions of matrices

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We will study matrix functions that return matrix results

$$\left(f : (\mathbb{R}^{n\times n})^m \longrightarrow \mathbb{R}^{n\times n} \right)$$

Recall:

If A is an $p\times p$ matrix

- $A^2 = A\cdot A$
- $A^n = \underbrace{A\cdot A\cdot \ldots \cdot A}_{n \text{ times}}$
- $(A^n)^m = A^{n\cdot m}$
- $A^n A^m = A^{n+m}$
- $(A^{-1})^n = A^{-n}$ (when invertible)

- If $A, \hat{A}$ are similar $(A = M\hat{A}M^{-1})$

  Then $A^2 = M\hat{A}^2M^{-1}$

  $A^n = MA^nM^{-1}$

We can also define polynomials of matrices:

$$p(A) = A^n + \alpha_{n-1}A^{n-1} + \ldots + \alpha_1 A + \alpha_0 \mathbb{I}$$

which can be factorized (in complex matices as)

$$p(A) = (A - p_1\mathbb{I}) \ldots (A - p_n\mathbb{I})$$

where $p_1, \ldots, p_n$ are such that

$$\lambda^n + \ldots \lambda^{n-1} \ldots + \alpha_0 = (\lambda - p_1)\cdot\ldots(\lambda - p_n)$$

$$p(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} \cdots + \alpha_0 = (\lambda - p_1) \cdots (\lambda - p_n)$$

Cayley-Hamilton Thm

If the characteristic polynomial of an $n \times n$ matrix A is $\varphi(\lambda)$,

$\varphi(\lambda) = |A - \lambda \mathbb{I}|$, then A satisfies that

$$\varphi(A) = 0.$$

(A solves the char. equation)

Corolary

$$A^n = -\varphi_{n-1} A^{n-1} - \cdots - \varphi_0 \mathbb{I}, \quad \varphi_{n-1}, \ldots, \varphi_0 \text{ coef. of char. poly of } A.$$

Proof Clear since

$$\varphi(A) = A^n + \varphi_{n-1} A^{n-1} + \cdots + \varphi_0 \mathbb{I} = 0$$

Observation: No need to explicitely compute powers of A larger than n.

Coro.

Let A be a non-singular $n \times n$ matrix, then

$$A^{-1} = -\frac{1}{\varphi_0}\left(\varphi_1 \mathbb{I} + \varphi_2 A + \cdots + A^{n-1}\right)$$

Proof Consider the char. polynomial of A:

Proof Consider the char. polynomial of A:

$$\varphi(\lambda) = \lambda^n + \varphi_{n-1}\lambda^{n-1} + \dots + \varphi_1\lambda + \varphi_0 = 0$$

$$= \prod_{i=1}^{n}(\lambda - \lambda_i)$$

Since A is non-singular $\lambda = 0$ is not an eigenvalue of A.

Moreover $\varphi_0 = \det(A) = \prod_{i=1}^{n}\lambda_i \neq 0$.

Since $A^n + \varphi_{n-1}A^{n-1} + \dots + \varphi_0 \mathbb{I} = 0$

$$\Rightarrow \mathbb{I} = -\frac{1}{\varphi_0}\left(\varphi_1 A + \dots + A^n\right)$$

Hence $A^{-1} = \dfrac{-1}{\varphi_0}\left(\varphi_1 \mathbb{I} + \dots + A^{n-1}\right)$

Recall

1) Scalar functions are said to be analytic if they can be represented locally by a convergent power series

2) A Taylor series Maclaurin [centered at 0] is a power series of the form:

$$f(\lambda) = \sum_{i=0}^{\infty} \frac{1}{i!} \left.\frac{d^i f}{d\lambda^i}\right|_{\lambda=0} \lambda^i$$

We can define:

$$\Big(\sum^{\infty} \frac{1}{\quad} \left.\frac{d^i f}{\quad}\right| \quad \lambda^i \Big)$$

We can define:

$$f(A) = \left(\sum_{i=0}^{\infty} \frac{1}{i!} \left. \frac{d^i f}{d\lambda^i} \right|_{\lambda=0} \cdot A^i \right)$$

for an analytic function A.

Ex:

- $\sin(A) = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \dots$

- $\cos(A) = \mathbb{I} - \frac{A^2}{2!} + \frac{A^4}{4!} - \dots$

- $e^{At} = \mathbb{I} + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$

Obs

We could construct approx. of these functions by truncating the series, but we can also use C-H's Thm to create simplifications:

Recall that $\varphi(A) = 0$

$\hookrightarrow$ Every power of A above or equal to n can be calc. in terms of $A^0, \dots, A^{n-1}$

Since the Taylor series converges, for f analytic in A then

$$f(A) = \alpha_{n-1} A^n + \dots + \alpha_0 \mathbb{I}$$

for some appropriate $\alpha_{n-1}, \dots, \alpha_0$.

Thm Suppose $f(\lambda)$ is an arbitrary function, $g(\lambda)$ is $(n-1)$-order polynomial in λ.
If $f(\lambda_i) = g(\lambda_i)$ for every eigenvalue λ_i of an $n \times n$ matrix A, then $f(A) = g(A)$

Ex: Let $A = \begin{pmatrix} -3 & 1 \\ 0 & -2 \end{pmatrix}$.

Compute $\sin A$, A^5, e^{At}.

Sol

The eigenvalues of A are $\lambda_1 = -3$, $\lambda_2 = -2$.

The arbitrary $\overset{\lambda}{\widetilde{n-1}}$ polynomial is $g(\lambda) = \alpha_1 \lambda + \alpha_0$

and must satisfy

$$\sin(-3) = \alpha_1(-3) + \alpha_0$$

$$\sin(-2) = \alpha_1(-2) + \alpha_0$$

We can solve and $\alpha_1 = -0.768$, $\alpha_0 = -2.45$

Hence $g(\lambda) = -0.768\lambda - 2.45$.

Moreover,

$$f(A) = \sin(A) = g(A) = -0.768A - 2.45 \mathbb{I}$$

$f(A) = \sin(A) = g(A) = \ldots$

$$= \begin{pmatrix} -0.141 & -0.768 \\ 0 & -0.909 \end{pmatrix}$$

for A^5,

$$\begin{aligned} (-3)^5 &= \alpha_1(-3) + \alpha_0 \\ (-2)^5 &= \alpha_1(-2) + \alpha_0 \end{aligned} \Rightarrow \begin{aligned} \alpha_1 &= 211 \\ \alpha_0 &= 390 \end{aligned}$$

Then $f(A) = A^5 = (211)A + (390)\,\mathbb{I} = \begin{pmatrix} -243 & 211 \\ 0 & 32 \end{pmatrix}$

For e^{At},

$$e^{-3t} = \alpha_1(t)(-3) + \alpha_0(t)$$

$$e^{-2t} = \alpha_1(t)(-2) + \alpha_0(t)$$

(This is more difficult)

$$\begin{pmatrix} e^{-3t} \\ e^{-2t} \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} e^{-3t} \\ e^{-2t} \end{pmatrix} = \begin{pmatrix} -e^{-3t} + e^{-2t} \\ [illegible] \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} e^{-3t} \\ e^{-2t} \end{pmatrix} = \begin{pmatrix} -e^{-3t} + e^{-2t} \\ -2e^{-3t} + 3e^{-2t} \end{pmatrix}$$

Then
$$e^{At} = \left(-e^{-3t} + e^{-2t}\right) \begin{pmatrix} -3 & 1 \\ 0 & -2 \end{pmatrix} + \left(-2e^{-3t} + 3e^{-2t}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-3t} & -e^{-3t} + e^{-2t} \\ 0 & e^{-2t} \end{pmatrix}$$

Obs In the previous example A had real diff. eigenvalues

- then the above system was solvable

- If the eigenvalues are repeated (once) one must use

$$f(\lambda_i) = g(\lambda_i), \qquad \left.\frac{df}{d\lambda}\right|_{\lambda=\lambda_i} = \left.\frac{dg}{d\lambda}\right|_{\lambda=\lambda_i}$$

(More repetitions require higher order derivatives)

Ex Compute e^{At},

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Sol

The eig. values are $\lambda_1 = \lambda_2 = -1$

we get then

$$f(\lambda) = g(\lambda) = \alpha_1 \lambda + \alpha_0$$

$$\Rightarrow \quad e^{-t} = \alpha_1(t)\,\lambda + \alpha_0(t)$$

differentiating :

$$f'(\lambda) = g'(\lambda) = \alpha_1(t)$$

$$\Rightarrow te^{-t} = \alpha_1(t)$$

Hence, $\alpha_0 = te^{-t} + e^{-t}$, and

$$e^{At} = te^{-t} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} + \left(te^{-t} + e^{-t}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

Diagonal matrices and $\underbrace{\text{Jordan forms}}_{\text{later}}$

↳ later

Suppose A is a diagonalizable matrix, i.e

$$A = M\hat{A}M^{-1}, \quad \hat{A} \text{ diag.}$$

Then, using Taylor expansions (Maclaurin)

$$f(A) = \sum_{i=0}^{\infty} \frac{1}{i!} \left.\frac{d^i f}{d\lambda^i}\right|_{\lambda=0} A^i = \sum_{i=0}^{\infty} \frac{1}{i!} \left.\frac{d^i f}{d\lambda^i}\right|_{\lambda=0} (M\hat{A}M^{-1})^i$$

$$\downarrow$$

$$M\hat{A}^i M^{-1}$$

Obs powers of diagonal matrices are powers of entries

$$\Rightarrow f(A) = M\left(\sum_{i=0}^{\infty} \frac{1}{i!} \left.\frac{d^i f}{d\lambda^i}\right|_{\lambda=0} \hat{A}^i\right) M^{-1}$$

$$= M \underbrace{f(\hat{A})}_{\text{easy to compute}} M^{-1}$$

(This can be extended to non-diag. matrices as well using Jordan forms – as seen later)