TT : Sorting Partially Ordered Sets

Setup

Our goal will be to talk about high dimensional convex geometry through an application towards sorting partially ordered sets

First some definitions...

1. A partially ordered set is a pair $(X, \preceq)$ where X is a set and $\preceq$ is a binary relation such that
   - $\preceq$ is reflexive : $x \preceq x \quad \forall x \in X$
   - $\preceq$ is transitive : if $x \preceq y$ and $y \preceq z$ ~~$\forall x,z$~~ then $x \preceq z$
   - $\preceq$ possesses weak anti-symmetry : if $x \preceq y$ and $y \preceq x$ then $x = y$.

Now note that $\preceq$ may be a partial ordering meaning $x \preceq y$ may not be defined $\forall x, y \in X$. Hence...

2. An ordering is linear if $\forall x, y \in X$ we have

$$x \preceq y \quad \text{or} \quad y \preceq x.$$

3. From an ordering $\precsim$ we can extend it in a consistent way. Call ~~another 'orde~~ a linear ordering $\leq$ a <u>linear extension</u> of $\precsim$. if

$$\underset{\precsim}{\text{pairs identified under}} \subseteq \underset{\leq}{\text{pairs identified under}}$$

that is to say the ordering $\precsim \xrightarrow[\text{implies}]{} \leq$.

- There could be <u>many</u> linear extensions of a $\precsim$.
  i.e. suppose for $X =$ ~~$\{1, 2, 3\}$~~ $\{a, b, c\}$
  we only know $a \precsim b$ ~~and b~~
  - $\leq$ defined via $a \leq b$ and $b \leq c$ is consistent
  - $\leq$ defined via $c \leq a$ and $a \leq b$ is fine.
- Let $E(\precsim) := \{ \leq : \leq \text{ is a linear extension of } \precsim \}$
- Let $e(\precsim) := |E(\precsim)|$

Now the ~~problem~~: <u>model</u>:

We are given a set X w/ a linear ordering $\leq$ and our goal is to sort X according to $\leq$

- We can get information by making binary comparisons of the form

  1. Choose $a, b \in X$
  2. Ask oracle if $a \leq b$ or $b \leq a$.

Our algorithm can be adaptive meaning it can change its heuristic according to previously asked queries to the oracle.

- To be more precise, the algorithm can change the order of pairs $a, b \in X$ sent to the oracle based on previous responses.

- In comparison, a non-adaptive strategy must fix the order of items it provides to the oracle.

~~For regr~~

The problem:

In usual sorting we start w/ no information

A well known lower bound on the # of comparisons required is $\Omega(n \log n)$ where $n = |X|$.

~~In today's setting, we are given partial information~~

Claim: Any binary comparison sort requires $\Omega(n \log n)$ comparisons.

Proof: Consider ~~that $\forall$ query a,~~ all possible ways to sort n items. A query of the form

$$a \leq b \quad \text{or} \quad b \leq a$$

will allow you to throw away at most half of the possible permutations.

- if $a \leq b$, throw away all those where $b \leq a$

this means at least ~~at~~ log (# permutations) comparisons are required to recover the sorted list.

$$\#\text{ permutations} = n! \quad \longrightarrow \quad \log(n!) = \Omega(n \log n).$$

What if we were given partial information?

The problem is as follows...

<u>Input</u>: $(X, \leq, \lesssim)$:

- X set of elements
- $\leq$ linear order on X
- $\lesssim$ partial order such that $\leq$ is a <u>linear extension</u> of $\lesssim$.

<u>Output</u>: ~~How many~~ sorted X

- <u>Critically</u> we ask how many comparisons are required to sort X.

<u>Building Intuition</u>

Okay ~~how do we normally sort?~~ What does sorting mean here?

- w/ <u>no information</u>, it's like you're choosing <u>one</u> linear extension of the $\emptyset$ binary relation
  - the lower bound is log (# permutations) comparisons
  - Indeed $e(\emptyset) = \#$ permutations $= n!$
- w/ information such as $\lesssim$, ~~can we hope to sort w/~~

~~at most $\log(e(\preceq))$ comparisons?~~

~~Answer is yes...~~

we can run the same argument and derive a lower bound on # of comparisons to be..

$$\log(e(\preceq))$$

But is this lower-bound tight? Can we always sort in at most $\log(e(\preceq))$ comparisons?

Efficient Comparison Theorem

Yeah...

Theorem (Efficient Comparison): Let $(X, \preceq)$ be a poset and suppose $\preceq$ is NOT linear. $\exists\ a, b \in X$ s.t.

$$\delta \le \frac{e(\preceq + (a,b))}{e(\preceq)} \le 1 - \delta$$

where $\delta > 0$ is an absolute constant, $\preceq + (a,b)$ denotes the transitive closure of $\preceq \cup \{(a,b)\}$.

i.e. we know $a \preceq b$.

This means that we always make progress toward the right ordering if we choose a, b as in the theorem.

- Concretely what we can do is the following.

[1. At every step i choose (a_i, b_i) as in the algorithm.]

Let $\preceq_{i+1} = \preceq_i + (a_i, b_i)$. theorem then says...

$$\delta \leq \frac{e(\preceq_{i+1})}{e(\preceq_i)} \leq 1-\delta.$$

$$\longrightarrow e(\preceq_{i+1}) \leq (1-\delta)\, e(\preceq_i)$$

every iteration decreases # of linear completions by $(1-\delta)$ multiplicative factor...

- how large can i be until $e(\preceq_{i+1}) = 1$?

$$\begin{aligned} e(\preceq_{i+1}) &\leq (1-\delta)\, e(\preceq_i) \\ &\leq (1-\delta)^2 \cdot e(\preceq_{i-1}) \\ &\ \ \dots \\ &\leq (1-\delta)^i \cdot e(\preceq) \end{aligned}$$

For what value of i is $e(\preceq_{i+1}) \geq 1$?

$$1 \leq (1-\delta)^i \cdot e(\preceq)$$

$$\frac{1}{(1-\delta)^i} \leq e(\preceq)$$

$$\cancel{i \leq \log (\lceil \log}$$

$$i \leq \log_{1/(1-\delta)} (e(\preceq))$$

If $i = \lceil \log_{1/(1-\delta)} (e(\preceq)) \rceil$ then done.

- what is δ?
  - Some conjecture that $\delta = \frac{1}{3}$. Since it is tight w/ the poset

i.e. $\preceq$ only identifies a, b.

  - We'll use $\delta = \frac{1}{2e} \approx 0.184$

The Proof

~~1 Order Polytope.~~

Polytope: Convex hull of a set of points.

Convex hull: $\mathrm{Conv}(\{v_1, \dots, v_n\})$ take $v_i \in \mathbb{R}^n$

$$\sum_{i=1}^{n} \lambda_i v_i \quad \text{s.t.} \quad \sum_{i=1}^{n} \lambda_i = 1 \quad \lambda_i \geq 0 \ \forall i.$$

examples: polygons ... yeah cube.

Okay we said this had something to do w/ convex geometry ... umm? Because we'll model figuring out linear extensions as a polytope

1 The order Polytope

Let's assign a convex polytope to partial orderings...

(Order Polytope): ~~Let $(X, \preceq)$ be an n-element poset. Let coordinates in $\mathbb{R}^n$ be indexed by~~ element

1. Take a poset $(X, \preceq)$ w/ n-elements
2. Index ~~vector in~~ coordinates in $\mathbb{R}^n$ by elements of X.
3. The <u>order polytope</u> $P(\preceq)$ is the set of all $x \in [0,1]^n$ satisfying

$$x_a \leq x_b \quad \forall a, b \in X \text{ st. } a \preceq b.$$

Equivalently this can be defined as...

1. Call $U \subseteq X$ an <u>up set</u> if

$$a \in U \text{ and } a \preceq b \longrightarrow b \in U.$$

that is if $a \in U$ then U contains all b larger than a according to $\preceq$

2. The vertices of $P(\preceq)$ are ~~the~~ exactly the <u>characteristic vectors</u> of all upsets in $(X, \preceq)$

Example using defns...

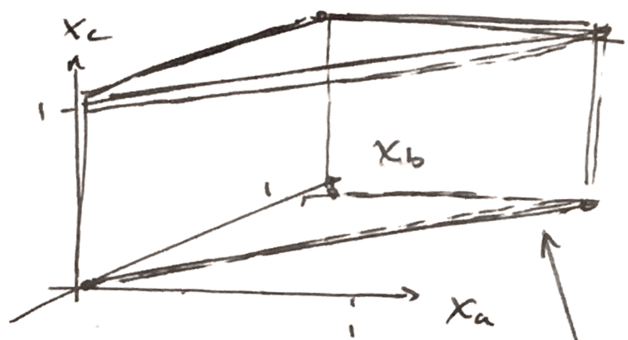
1. Take $\{a, b, c\}$ where $\preceq = \{(a,b)\}$. Then

$x \in [0,1]^n$ s.t.

$x_a \leq x_b$.

Suppose $x \in \mathbb{R}^3$ is indexed as...

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \quad \text{Then.}$$



this region is $P(\preceq)$.

$$P(\preceq) = \{ x \in [0,1]^3 : x_a \leq x_b \}$$

2. Using the vertex defn.

- Vertices of $P(\preceq)$ = characteristic vectors of all <u>upsets</u>

- Upset is $U \subseteq X$ s.t. if $a \in U$ and $b \succeq a$ $\rightarrow b \in U$.

- 3 cases to consider

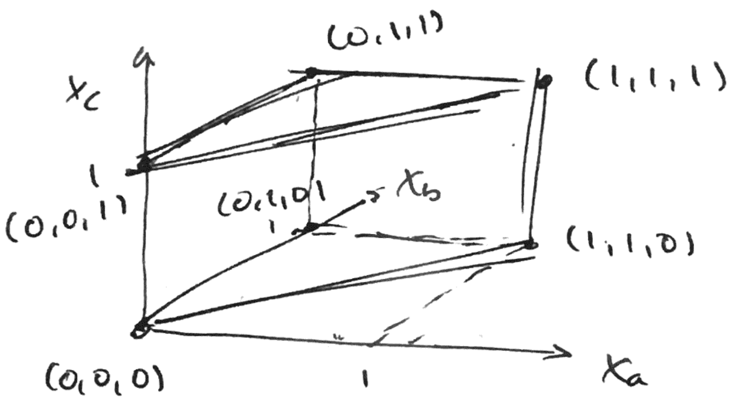
  - No ~~vertices~~ points in U
    → $\chi_U = \vec{0}$ → $\vec{0}$

  - $a \in U$. Since $a \preceq b$
    → $\{a,b\} \subseteq \chi_U$ → $\{a,b\} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $\{a,b,c\} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

  - $b \in U$. No requirements → $\{b\} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

    $\{b,c\} = \begin{pmatrix}0\\1\\1\end{pmatrix}$

  - $c \in U$. No requirements → $\{c\} = \begin{pmatrix}0\\0\\1\end{pmatrix}$



<u>Claim</u>: Def'n of $P(\preceq)$ as set of pts. and as polytope whose vertices are upset characteristic vectors are equiv.

Proof: Observe ...

1. Let U be an upset and χ_U its characteristic vector. Then claim $\chi_U \in P(\preceq)$.

   Consider any $(a,b) \in \preceq$. There are two cases...

1. $a \in U$: Then by defn $b \in U$ since $a \preceq b$.

this means

$$\chi_u(a) = \chi_u(b)$$

satisfying $\chi_u(a) \leq \chi_u(b) \rightarrow \chi \in P(\preceq)$

2. $a \notin U$. Then $\chi_u(a) = 0 \leq \chi_u(b)$ since ~~$b \in \{0,1\}$~~. $\chi_u(b) \in \{0,1\}$.

Next observe that these are the only integral vectors in $P(\preceq)$.

- Any other integral vector must have

$$\chi_a(a) = 1 \geq \chi_u(b) = 0$$

but then since $a \preceq b$, ~~$\chi_u(a)$~~ $\chi \notin P(\preceq)$

Finally observe that all vertices of $P(\preceq)$ are integral

- Any vertex is the intersection of n, $(n-1)$-dimensional hyperplanes.
- These hyperplanes must satisfy constraints like

$$x_a = 1 \quad \text{or} \quad x_a = 0 \quad \text{or} \quad x_a = x_b.$$

- Thus they can only intersect at integral pts. $\square$

Using this object we can say the following about partial and linear orders... (Before defn simplex: Conv hull of a set of affine ind pts... $\{v_1 \ldots v_n\}$ s.t. $\{v_1 - v_n \ldots v_{n-1} - v_n\}$ lin ind.)

Claim: Let X be an n-element set...

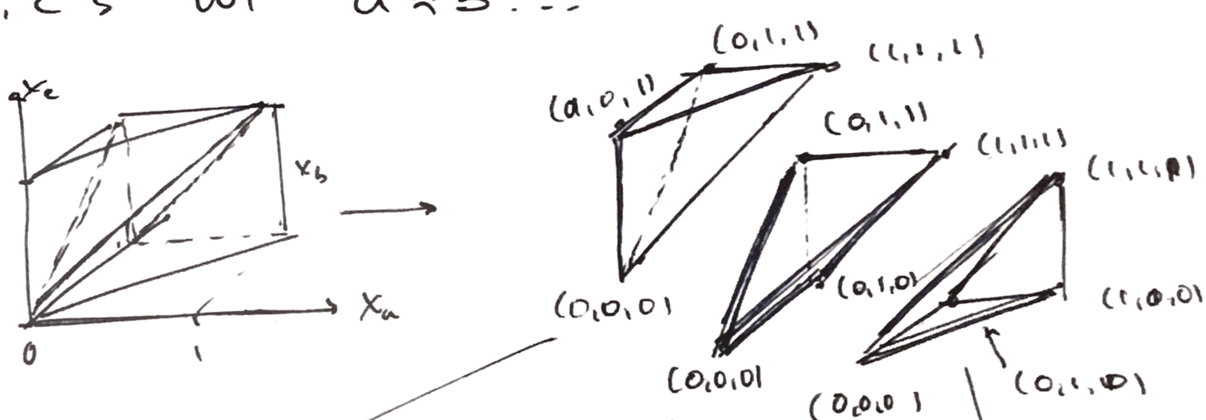
1. If $\leq$ is a linear ordering on X then $P(\leq)$ has a volume of $\frac{1}{n!}$.

2. For any partial ordering $\precsim$ on X, the simplices of the form $P(\leq)$ where $\leq \in E(\precsim)$. cover $P(\precsim)$ and have disjoint interiors. Thus...

$$\mathrm{Vol}(P(\precsim)) = \frac{1}{n!} \cdot e(\precsim).$$

To get a feel for ① observe...

1. $\{a, b, c\}$ w/ $a \precsim b$...



$b \precsim c$

$a \precsim b \precsim c$

$b \precsim c$

$a \precsim b \precsim c$.

$\boxed{a \precsim c \text{ and } c \precsim b.}$

$c \precsim a$

$c \precsim a \precsim b$

Proof: We prove each claim... but a claim first.

~~(i): WLOG assume $1 \leq \dots \leq n$. Then upset characteristic vectors have the form~~

~~$(0, \dots, 0, 1, \dots, 1)$~~

First: Observe $P(\leq)$ is a simplex... WLOG assume

$$\leq \; := \; 1 \leq 2 \leq \dots \leq n.$$

then upset charackersistic vectors take the form...

$$\left.\mathcal{X}_u = (0 \dots 0, \underset{i}{1}, \dots 1)\ \right] \rightarrow \text{there are } n \text{ of these.}$$

provided $1 \in U$ is the smallest elem. Furthermore, each $\mathcal{X}_u$ is affinely independent of one another

i.e. no way to write

$\Leftrightarrow$ $$\sum \alpha_i \mathcal{X}_u(i) = 0 \quad w/ \quad \sum \alpha_i = 0$$

and $\alpha_i > 0$ for some i.

no need if already defined.

$\Leftrightarrow$ Fix ~~on~~ $\mathcal{X}_u(n)$ then $\forall i \neq n$.

$$\{\mathcal{X}_u(i) - \mathcal{X}_u(n) : i \neq n\}$$

is linearly ind.

this means all simplices are congruent!

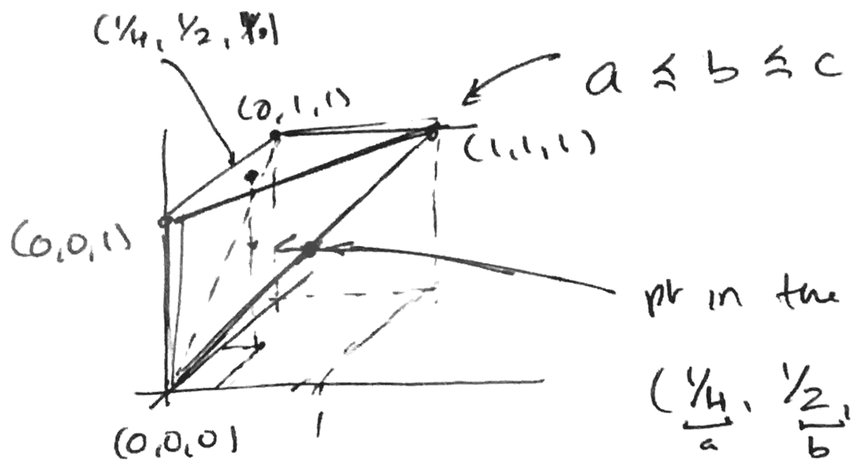
By defn. that's a simplex $\longrightarrow$ you can always rearrange coordinates to form this linear ordering

Now to show our claims.

<u>(2)</u> : Observe that any point $(x_1 \dots x_n) \in P(\precsim)$

~~with~~ with <u>distinct</u> coordinates determines a <u>unique</u> linear order...

- the one determined by the natural ordering of its coordinates as real #s.



$(\underbrace{1/4}_{a}, \underbrace{1/2}_{b}, \underbrace{1}_{c})$

Ordering is

$a \precsim b \precsim c$.

Furthermore $P(\leq) \subseteq P(\precsim)$ by defn $\forall \leq \in E(\precsim)$

- Each point belongs to exactly one linear ordering simplex. unless on boundary where a coord is =.
- $P(\leq) \subseteq P(\precsim) \ \forall \leq \in E(\precsim)$

$\Rightarrow$ the simplices of $E(\precsim)$ subdivide $P(\precsim)$.

(1): To see all ~~simplices~~ linear order simplices have $\text{Vol} = \frac{1}{n!}$ ~~taken~~

we want to utilize congruence and construct the shape that has all total order simplices as subparts (via part 1 of (2)) then divide by # of possible ~~total~~ linear orders $(n!)$

- Take $\preceq$ to be the discrete ~~linear~~ partial order

$$\preceq = \{(1,1) \ldots (n,n)\}$$

- Its polytope is bounded by literally all $x \in \{0,1\}^n$

$\Longrightarrow$ unit hypercube $\Rightarrow \text{Vol} = 1$

- $e(\preceq) = \{\text{all possible orders}\} = n!$

- Volume of linear order simplex $= \frac{1}{n!}$

(2 pt 2): To see $\text{Vol}(P(\preceq))$ observe that each

$$P(\leq) \text{ for } \leq \in E(\preceq)$$

has disjoint interiors by 2 pt 1. each w/ $\text{vol} = \frac{1}{n!}$ thus

$$\text{Vol}(P(\preceq)) = \frac{1}{n!} \cdot e(\preceq). \quad \square$$

Height + Centre of Gravity

Take X to be a finite set and $\leq$ a <u>linear</u> order

we define the height of a in $\leq$.

$$h_{\leq}(a) := \underbrace{|\{x \in X : x \leq a\}|}_{\text{\# of elems smaller than } a.}$$

On a $\precsim$ partial order...

$$h_{\precsim}(a) := \operatorname*{Avg}_{\leq \in E(\precsim)} \{h_{\leq}(a)\}$$

$$= \frac{1}{e(\precsim)} \cdot \sum_{\leq \in E(\precsim)} h_{\leq}(a)$$

<u>Lemma 1:</u> ~~For any distinct a,b~~ Given $\precsim$, there exists distinct $a, b \in X$ s.t.

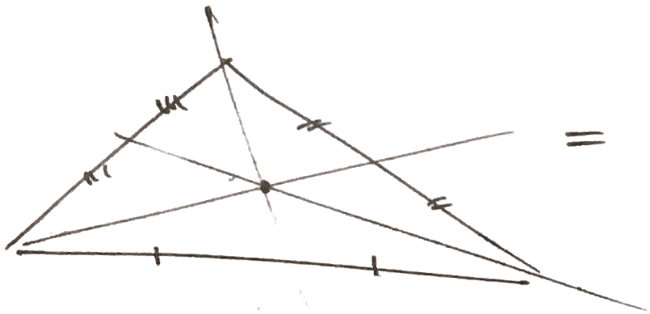
$$|h_{\precsim}(a) - h_{\precsim}(b)| < 1.$$

<u>Lemma 2</u>: For any n-element poset $(X, \precsim)$, the centre of gravity of the order polytope $P(\precsim)$ is.

$$c = (c_a : a \in X)$$

$$c_a = \frac{1}{n+1} h_{\precsim}(a).$$

What's the center of gravity? Probably saw it in Math 53.



= avg point of all points in a region

__Simplex__: $\frac{1}{n+1}\sum_{i=1}^{n+1} x_i$ $\quad x_i$ vertex of simplex.

~~In general~~ :

__Proof of lemma__: Since $P(\leq)$ for $\leq \in E(\preceq)$ cover $P(\preceq)$, the center of gravity for $P(\preceq)$ is the avg of centers of gravity for $P(\leq)$.

$$\text{Centroid}(P(\preceq)) = \frac{1}{e(\preceq)} \sum_{\leq \in E(\preceq)} \text{Centroid}(P(\leq)).$$

To compute centroid of $P(\leq)$ remember, we can always permute coordinates to get the total order $1 \leq 2 \leq \dots \leq n$.

the polytope for this has vertex coordinate

$(0,\dots,0)$ $\quad \left.\begin{array}{l}(0,\dots,1) \\ (0,\dots,1,1) \\ \dots\end{array}\right\}$ n times

Observe ...

for a^{th} coordinate

$$\begin{array}{c}
(0 \dots 0, 0, 0) \\
(0, \dots 0, 0, 1) \\
(0 \dots 0, 1, 1, 1) \\
(0 \dots 1, 1, 1, 1).
\end{array}$$

$n-a$ pts w/ a^{th} coord $=1$.

$$\left[\text{Centroid}\left(P(\leq)\right)\right]_a = \underbrace{\frac{1}{n+1}}_{\#\text{ pts}} \cdot \underbrace{\sum_{i=a}^{n} 1}$$

of $b \in X$ s.t. $b \geq a$.

$=$ height of a.

$$= \frac{1}{n+1} h_{\geq}(a)$$

thus ... $$\left[\text{Centroid}\left(P(\preceq)\right)\right]_a = \frac{1}{e(\preceq)} \sum_{\preceq \in E(\preceq)} \frac{1}{n+1} \cdot h_{\geq} a$$

$$= \frac{1}{n+1} h_{\preceq}(a). \ \square$$