

	(x_k)	(y_k)
	m_p	
1	2.5	30
2	3.0	25
3	3.5	22
4	4.0	18
5	4.5	15

$$\begin{aligned}
 l_1 &= y - \hat{y} = -(-5(2.5) + 40) - 27.5 = 2.5^2 = 6.25 \\
 l_2 &= y - \hat{y} = -(-5(3) + 40) - 25 = 0^2 = 0 \\
 l_3 &= y - \hat{y} = -(-5(3.5) + 40) - 22.5 = -0.5^2 = -0.25 \\
 l_4 &= y - \hat{y} = -(-5(4) + 40) - 20 = -2^2 = 4 \\
 l_5 &= y - \hat{y} = -(-5(4.5) + 40) - 17.5 = -2.5^2 = 6.25
 \end{aligned}$$

~~maximize $\sum l_i^2$~~ $\hat{y}_k = m x_k + b$

$$\text{Sum} = 16.75$$

$$\text{Loss} = 16.75 / 5 = 3.35$$

~~$$\begin{aligned}
 l &= \sum l_i^2 = 22.5 \\
 &\quad \text{Sum} = 16.75 \\
 &\quad \text{Loss} = 3.35
 \end{aligned}$$~~

1.2)

$$\begin{aligned}
 z &= y^2 + x^2 & \frac{\partial z}{\partial x} &= 2x & \frac{\partial z}{\partial t} &= 2t & \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \left(\frac{\partial x}{\partial t} \right) + \frac{\partial z}{\partial y} \left(\frac{\partial y}{\partial t} \right) \\
 y &= 2t & \frac{\partial z}{\partial y} &= 2y & \frac{\partial x}{\partial t} &= 2 & \Rightarrow \frac{\partial z}{\partial t} &= 2x(2t) + 2y(2) \\
 x &= t^2 & \frac{\partial z}{\partial y} &= 2y & & & &= 4xt + 4y
 \end{aligned}$$

$$L = L_1 + L_2 + L_3 + L_4 + L_S \Rightarrow L_K = \frac{1}{n} e_K^2$$

$$\Rightarrow \sum_k \frac{1}{n} e_K^2$$

$$1.3) \hat{y}_k = m x_k + b \quad e_k = (y_k - \hat{y}_k) \quad L_k = \left(\frac{1}{n} \right) e_k^2$$

A)

$$\frac{\partial L_k}{\partial e_k} = \frac{2}{n} e_k \left| \frac{\partial e_k}{\partial \hat{y}_k} \right| = -1 \quad \left| \frac{\partial \hat{y}_k}{\partial m} = x_k \right| \Rightarrow \frac{\partial L}{\partial m} = \left(\frac{2}{n} e_k \right) \cdot (-1) (x_k) = -\frac{2}{n} e_k x_k$$

$$k=1 \quad x_1 = 2.5 \quad y_1 = 30 \quad m = -5 \quad b = 40 \quad L_1 = 30 - (-5)(2.5) + 40 = 2.5$$

$$\frac{\partial L}{\partial m} = -\frac{2}{n} (2.5)(2.5) = -2.5$$

$$\begin{aligned}
 B) \frac{\partial L_k}{\partial b} &= \left(\frac{\partial L_k}{\partial e_k} \right) \left(\frac{\partial e_k}{\partial \hat{y}_k} \right) \left(\frac{\partial \hat{y}_k}{\partial b} \right) = \left(\frac{2}{n} e_k \right) (-1)(1) = -\frac{2}{n} (2.5) = -1 \\
 &\quad \downarrow \quad \downarrow \quad \downarrow \\
 &\quad \frac{2}{n} e_k \quad -1 \quad 1
 \end{aligned}$$

1.4)



$$\frac{\partial L_u}{\partial m} = -\frac{2}{n} (y_u - \hat{y}_u) x_u$$

$$\frac{\partial L_u}{\partial b} = -\frac{2}{n} (y_u - \hat{y}_u)$$

$$\frac{\partial L}{\partial m} = \frac{\partial L_1}{\partial m} + \frac{\partial L_2}{\partial m} + \frac{\partial L_3}{\partial m} + \frac{\partial L_u}{\partial m} + \frac{\partial L_s}{\partial m}$$

$$\frac{\partial L}{\partial b} = \frac{\partial L_1}{\partial b} + \frac{\partial L_2}{\partial b} + \frac{\partial L_3}{\partial b} + \frac{\partial L_u}{\partial b} + \frac{\partial L_s}{\partial b}$$

u	x_u	y_u	\hat{y}_u	$y_u - \hat{y}_u$	$\frac{\partial L_u}{\partial m}$	$\frac{\partial L_u}{\partial b}$
1	2.5	30	27.5	2.5	$-\frac{2}{5}(2.5)(2.5) = -2.8$	$-\frac{2}{5}(2.5) = -1$
2	30	25	25	0	0	0
3	3.5	22	22.5	-0.5	$-\frac{2}{5}(-.5)(3.5) = .7$	$-\frac{2}{5}(-.5) = 2$
4	4.0	18	20	-2	$-\frac{2}{5}(-2)(4) = 3.2$	$-\frac{2}{5}(-2) = .8$
5	4.5	15	17.5	-2.5	$-\frac{2}{5}(-2.5)(4.5) = 4.5$	$-\frac{2}{5}(-2.5) = 1$
				Sum = 5.9		Sum = 1

$$\frac{\partial L}{\partial m} = 5.9 \quad \frac{\partial L}{\partial b} = 1.0$$

to produce less we shrink both m & b

$$m_{\text{new}} = m - \alpha \frac{\partial L}{\partial m} = -5 - .1(5.9) = \boxed{-5.59}$$

$$b_{\text{new}} = b - \alpha \frac{\partial L}{\partial b} = \boxed{39.9}$$

Homework 3

Linear Regression with Gradient Descent (from scratch)

```
In [1]: import numpy as np # Only for data loading and plotting
import matplotlib.pyplot as plt

class LinearRegressionGD:
    """
    Linear Regression using Gradient Descent
    Model:  $y_{\text{hat}} = m * x + b$ 

    This implementation follows the mathematical framework from Part 1:
    - Compute gradient contribution from each data point
    - Sum all contributions to get total gradient
    - Update parameters
    """

    def __init__(self, learning_rate=0.01, num_iterations=1000):
        """
        learning_rate: step size for gradient descent ( $\alpha$ )
        num_iterations: number of iterations to run
        """

        self.learning_rate = learning_rate
        self.num_iterations = num_iterations
        self.m = 0 # slope
        self.b = 0 # intercept
        self.loss_history = [] # track loss over iterations

    def predict(self, x):
        """
        Make a prediction for a single input value

        Input:
            x: a single input value (scalar)
        Returns:
            y_hat: predicted value (scalar)
        """

        return self.m * x + self.b

    def optimize(self, X, y):
        """
        Train the model using gradient descent

        Inputs:
            X: list or array of input values
            y: list or array of true output values
        Returns:
            self (for method chaining)
        """

        n = len(X)
        self.loss_history = []

        for iteration in range(self.num_iterations):
            grad_m = 0.0
            grad_b = 0.0
            total_loss = 0.0

            # Loop over each data point individually (no vectorized gradients)
            for x_i, y_i in zip(X, y):
                y_hat = self.predict(x_i)
```

```

        error = y_hat - y_i

        grad_m += (2.0 / n) * error * x_i
        grad_b += (2.0 / n) * error
        total_loss += error ** 2

    self.m -= self.learning_rate * grad_m
    self.b -= self.learning_rate * grad_b

    self.loss_history.append(total_loss / n)

    return self

```

In [2]:

```

# Data from Problem 1.1
X_small = [2.5, 3.0, 3.5, 4.0, 4.5]
y_small = [30, 25, 22, 18, 15]

# Train model
model = LinearRegressionGD(learning_rate=0.01, num_iterations=100)
model.optimize(X_small, y_small)

print(f"\nFinal parameters:")
print(f"m (slope): {model.m:.4f}")
print(f"b (intercept): {model.b:.4f}")
print(f"Final Loss: {model.loss_history[-1]:.4f}")

```

Final parameters:
m (slope): 4.4800
b (intercept): 4.7464
Final Loss: 73.3700

In [3]:

```

# Verification: 1 iteration from m=-5, b=40
verify_model = LinearRegressionGD(learning_rate=0.01, num_iterations=1)
verify_model.m = -5
verify_model.b = 40
verify_model.optimize(X_small, y_small)

print("After 1 iteration with lr=0.01 from m=-5, b=40:")
print(f"m: {verify_model.m:.4f}")
print(f"b: {verify_model.b:.4f}")
print("Compare these values to your hand calculation from Problem 1.4 Part B Q3.")

```

After 1 iteration with lr=0.01 from m=-5, b=40:
m: -5.0590
b: 39.9900
Compare these values to your hand calculation from Problem 1.4 Part B Q3.

In [4]:

```

from urllib.request import urlretrieve
import pandas as pd

url = "https://archive.ics.uci.edu/ml/machine-learning-databases/auto-mpg/auto-mpg.data"
urlretrieve(url, "auto-mpg.data")

# Only the numeric columns (ignore car_name because it has spaces)
column_names_num = ['mpg', 'cylinders', 'displacement', 'horsepower', 'weight',
                     'acceleration', 'model_year', 'origin']

data = pd.read_csv(
    "auto-mpg.data",
    sep=r"\s+",
    header=None,
    names=column_names_num,
    usecols=list(range(8)),  # read only the first 8 columns
    na_values="?",
    engine="python",
)

```

```

data = data.dropna()

X_full = (data["weight"].astype(float).to_numpy() / 1000)
y_full = data["mpg"].astype(float).to_numpy()

print(f"Dataset size: {len(X_full)} vehicles")

```

Dataset size: 392 vehicles

```

In [5]: # Train model on full dataset
# Start with the same learning rate as before and adjust if needed
model_full = LinearRegressionGD(learning_rate=0.01, num_iterations=1000)
model_full.optimize(X_full, y_full)

print(f"\nFull dataset - Final parameters:")
print(f"m (slope): {model_full.m:.4f}")
print(f"b (intercept): {model_full.b:.4f}")
print(f"Final Loss: {model_full.loss_history[-1]:.4f}")

x_test = 3.0
print(f"Predicted MPG for 3000 lbs (x=3.0): {model_full.predict(x_test):.4f}")

```

Full dataset - Final parameters:
 m (slope): -4.1255
 b (intercept): 34.9596
Final Loss: 28.2222
Predicted MPG for 3000 lbs ($x=3.0$): 22.5832

Problem 2.2 Questions (fill in after running)

- Slope interpretation: For every 1000 lb increase in vehicle weight, MPG changes by approximately m MPG.
- Relationship sense-check: Heavier cars usually get lower MPG, so a negative slope is expected.
- Learning rate note: Keep 0.01 if loss decreases smoothly; reduce it if the loss oscillates/diverges.

```

In [6]: class LinearRegressionGDMomentum:
    """
    Linear Regression using Gradient Descent with Momentum

    Momentum update rule (from Going Beyond 1.4):
        v_m = β * v_m + gradient_m
        m = m - α * v_m
    """

    def __init__(self, learning_rate=0.01, num_iterations=1000, momentum=0.9):
        """
        learning_rate: step size α
        num_iterations: number of iterations
        momentum: momentum coefficient β (typically 0.9)
        """

        self.learning_rate = learning_rate
        self.num_iterations = num_iterations
        self.momentum = momentum
        self.m = 0
        self.b = 0
        self.v_m = 0 # velocity for m
        self.v_b = 0 # velocity for b
        self.loss_history = []

    def predict(self, x):
        """
        Same as before
        """
        return self.m * x + self.b

```

```

def optimize(self, X, y):
    """
    Train using gradient descent with momentum
    """

    Inputs:
        X: list or array of input values
        y: list or array of true output values
    Returns:
        self
    """

    n = len(X)
    self.loss_history = []

    for _ in range(self.num_iterations):
        grad_m = 0.0
        grad_b = 0.0
        total_loss = 0.0

        for x_i, y_i in zip(X, y):
            y_hat = self.predict(x_i)
            error = y_hat - y_i

            grad_m += (2.0 / n) * error * x_i
            grad_b += (2.0 / n) * error
            total_loss += error ** 2

        self.v_m = self.momentum * self.v_m + grad_m
        self.v_b = self.momentum * self.v_b + grad_b

        self.m -= self.learning_rate * self.v_m
        self.b -= self.learning_rate * self.v_b

        self.loss_history.append(total_loss / n)

    return self

```

```

In [7]: # Standard gradient descent
model_standard = LinearRegressionGD(learning_rate=0.01, num_iterations=1000)
model_standard.optimize(X_full, y_full)

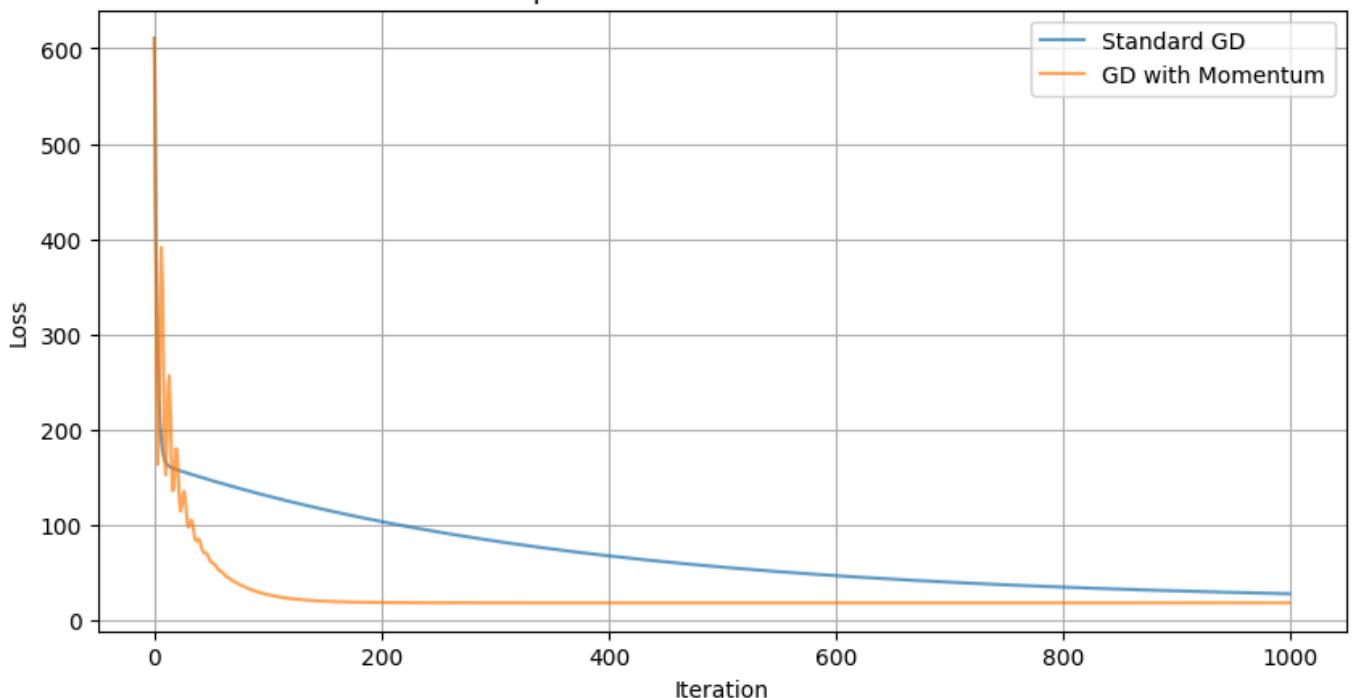
# Gradient descent with momentum
model_momentum = LinearRegressionGDMomentum(learning_rate=0.01, num_iterations=1000,
model_momentum.optimize(X_full, y_full)

# Plot comparison
plt.figure(figsize=(10, 5))
plt.plot(model_standard.loss_history, label='Standard GD', alpha=0.7)
plt.plot(model_momentum.loss_history, label='GD with Momentum', alpha=0.7)
plt.xlabel('Iteration')
plt.ylabel('Loss')
plt.title('Comparison: Standard vs Momentum')
plt.legend()
plt.grid(True)
plt.show()

print(f"Standard GD final loss: {model_standard.loss_history[-1]:.4f}")
print(f"Momentum GD final loss: {model_momentum.loss_history[-1]:.4f}")

```

Comparison: Standard vs Momentum



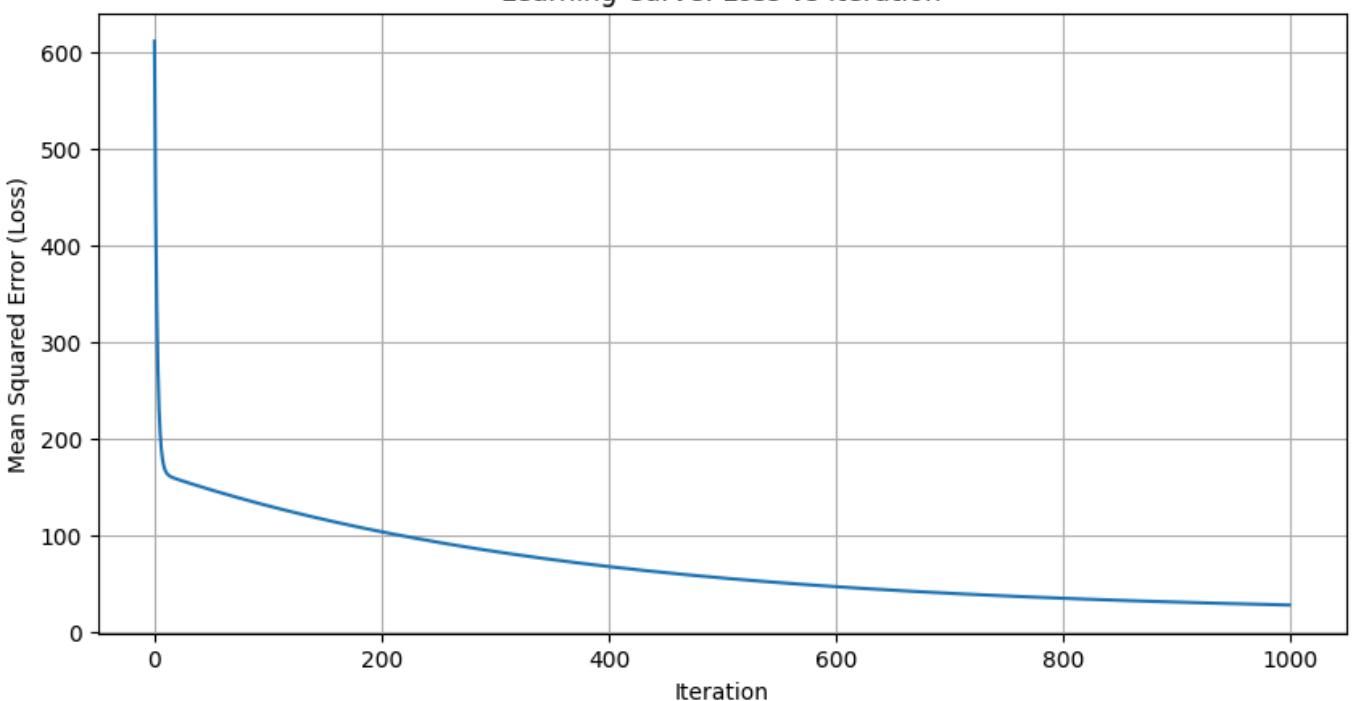
Standard GD final loss: 28.2222

Momentum GD final loss: 18.6766

In [8]: # Part 3.1: Learning curve

```
plt.figure(figsize=(10, 5))
plt.plot(model_full.loss_history)
plt.xlabel('Iteration')
plt.ylabel('Mean Squared Error (Loss)')
plt.title('Learning Curve: Loss vs Iteration')
plt.grid(True)
plt.show()
```

Learning Curve: Loss vs Iteration



In [9]: # Part 3.2: Model fit visualization

```
plt.figure(figsize=(12, 6))

# Plot 1: Small dataset
plt.subplot(1, 2, 1)
plt.scatter(X_small, y_small, color='blue', s=100, alpha=0.6, label='Data')
# Plot the fitted line
x_line = np.linspace(min(X_small), max(X_small), 100)
```

```

y_line = [model.predict(x) for x in x_line]
plt.plot(x_line, y_line, 'r-', linewidth=2, label=f'Fit: y = {model.m:.2f}x + {model.b:.2f}')
plt.xlabel('Weight (1000s lbs)')
plt.ylabel('MPG')
plt.title('Small Dataset (5 points)')
plt.legend()
plt.grid(True, alpha=0.3)

# Plot 2: Full dataset
plt.subplot(1, 2, 2)
plt.scatter(X_full, y_full, color='blue', s=20, alpha=0.4, label='Data')
x_line = np.linspace(min(X_full), max(X_full), 100)
y_line = [model_full.predict(x) for x in x_line]
plt.plot(x_line, y_line, 'r-', linewidth=2,
          label=f'Fit: y = {model_full.m:.2f}x + {model_full.b:.2f}')
plt.xlabel('Weight (1000s lbs)')
plt.ylabel('MPG')
plt.title(f'Full Dataset ({len(X_full)} vehicles)')
plt.legend()
plt.grid(True, alpha=0.3)

plt.tight_layout()
plt.show()

```

