MAT102: Proof of the Quadratic Formula May 2020

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Proof. Let us begin with a quadratic equation in the form: $\mathbf{ax^2 + bx + c = 0}$, where $a, b, c \in N$ and $a \neq 0$. To find the solutions of the equation we must complete the square. Since the first coefficient must be 1 and a is a variable, divide both sides by a.

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = 0$$

To complete the square, a new term where the second coefficient is halved and squared,is added and subtracted.

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

$$x^2 + \frac{bx}{a} + \left(\frac{\mathbf{b}}{\mathbf{a}} \cdot \frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{2}} - \left(\frac{\mathbf{b}}{\mathbf{a}} \cdot \frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{2}} + \frac{c}{a} = 0$$

$$\mathbf{x^2} + \frac{\mathbf{b}}{\mathbf{a}}\mathbf{x} + \left(\frac{\mathbf{b}}{\mathbf{2a}}\right)^{\mathbf{2}} = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

Use the identity: $\mathbf{(x + y)^2 = x^2 + 2xy + y^2}$ where y $= \frac{b}{2a}$ to simplify the left side.

$$\left(\mathbf{x} + \frac{\mathbf{b}}{\mathbf{2a}}\right)^{\mathbf{2}} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(\mathbf{x} + \frac{\mathbf{b}}{\mathbf{2a}}\right)^{\mathbf{2}} = \frac{b^2 - 4ac}{4a^2}$$

Since the square roots cannot be applied to negative numbers, 3 cases are considered:

1. If $\mathbf{b^2 - 4ac < 0}$, then $\frac{b^2-4ac}{4a^2} < 0$ and considering that $\left(x + \frac{b}{2a}\right)^2 \geq 0$ for any real number x, there are **no real solutions** in this case.
2. If $\mathbf{b^2 - 4ac = 0}$, then $\frac{b^2-4ac}{4a^2} = 0$ so this produces **one unique solution**: $\mathbf{x = \frac{-b}{2a}}$
3. If $\mathbf{b^2 - 4ac > 0}$, use square roots and get **two distinct solutions:**

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

□