MAT102: The knight & the king chess problem May 2020

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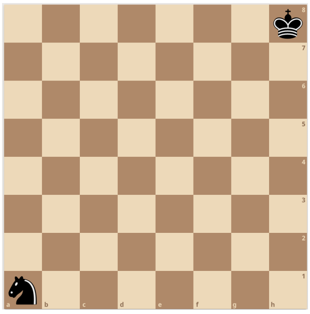
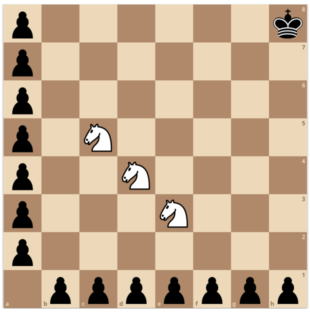
Problem . This is a chess board, with a knight in the bottom left, and a king in the top right. The king does not move. The knight can make any number of "L moves" (two-right/one-up, or two-up/one-right) but it must always move up and to the right at least one unit. What are the possible squares the Knight can reach? In particular, can the Knight reach the King's square?



Proposition 1. Assume that the bottom left square is (1,1). The Knight cannot reach positions $(1, n)$ or $(n, 1)$ for any natural number $n > 1$.

Proof. Since the only way the Knight can move is by simultaneously increasing both its x-coordinate and its y-coordinate, all positions $(1, n)$ or $(n, 1)$ other than (1,1) are unreachable. *(Demonstrated by the black pawns)* □



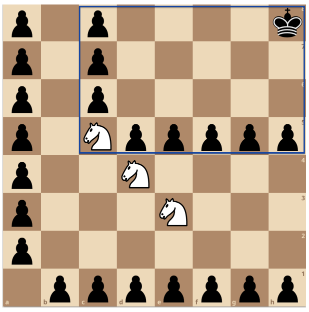
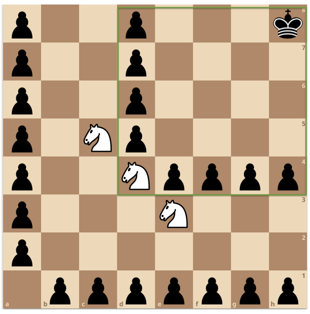
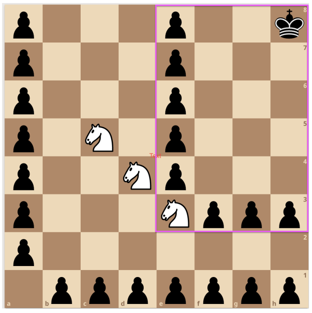
Proposition 2. If the Knight can reach position (a, b) then the Knight can reach position $(a + 3, b + 3)$

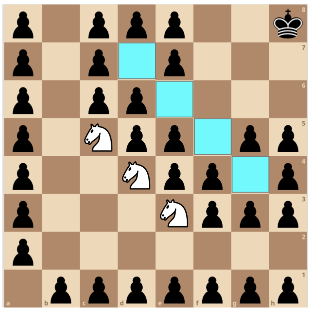
Proof. From position (a, b) the Knight can use move 2 to go to position $(a + 2, b + 1)$. Then it can use move 3 to go to position (a+2+1, b+1+2) = (a+3,b+3). *(Demonstrated by middle white Knight)*

The side Knights represent an edge case where move 3 repeats move 2. As such we get $(a + 4, b + 2)$ or $(a + 2, b + 4)$ □

Proposition 3. The knight's possible position can be represented as $(3n - 2, 3n - 2)$ if $n \in \mathbb{N}$.

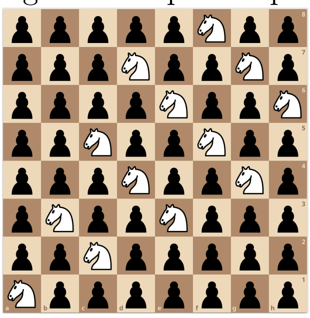
Proof. Let the 3 Knights represent the new position (a, b). Starting with the top knight on the blue box and using proposition 1, positions $(1, n)$ or $(n, 1)$ are unreachable as demonstrated by the black pawns. Proposition 1 is also applied to the middle Knight of



the green box and the bottom Knight of the pink box. This concludes that the Knight is unable to reach a position to either of its sides (Vertical or Horizontal)

Collectively, the board results in the image below, where the blue diagonal represents all possible first moves by applying the first part of proposition 2 to each Knight.



For the second part of proposition 2, we get ALL possible positions for the Knight. This concludes that diagonally there is always a difference of 3 from the previous Knight to the next.

This pattern is perfectly illustrated in the diagonal of the chess board and this relationship is illustrated through the arithmetic sequence: $X_n = a + d(n-1)$ where a represents the first term, d represents the difference and n represents the number of moves.

$$\begin{aligned} X_n = a + d(n-1) &= 1 + 3(n-1) \\ \implies X_n &= 1 + 3n - 3 \\ \implies &\boxed{X_n = 3n - 2} \end{aligned}$$

By then applying proposition 3 into the problem and assuming that the King's position (8,8) is possible for the Knight, $8 = 3n - 2$

Finally, $n = \frac{10}{3}$ and since proposition 3 is broken, we can conclude that it is impossible for the Knight to reach the King. □