The blow up section family

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Introduction

A *cluster of r sections* of a family $X \longrightarrow S$ is a sequence of sections $(\sigma_1, \dots, \sigma_r)$

$$\begin{tikzcd} S \arrow[r, bend left, "\sigma_1"] \arrow[rr, bend left, "\sigma_2"] & X \arrow[l] & \mathrm{bl}_{\sigma_1} X \arrow[l] & \cdots \arrow[l] \end{tikzcd} \quad \cdots$$

A *family of clusters of r sectoins* parametrised by a scheme T

$$\begin{tikzcd} T \times S \arrow[r, bend left, "t_1"] \arrow[rr, bend left, "t_2"] & T \times X \arrow[l] & \mathrm{bl}_{t_1}(T \times X) \arrow[l] & \cdots \arrow[l] \end{tikzcd} \quad \cdots$$

When $X \longrightarrow S$ is smooth, families of clusters of r sections form a functor

$$\mathcal{C}\ell_r \colon \mathsf{Sch}_{\mathbb{k}} \longrightarrow \mathsf{Set}.$$

Theorem 1 (Existence)

If X is quasiprojective and S is proper, then, for every $r > 0$, $\mathcal{C}\ell_r$ is representable by a scheme Cl_r.

Closed points $c \in \mathrm{Cl}_r$ correspond to clusters of r sections $(\sigma_1, \dots, \sigma_r)$. Observe that, for $r > 0$, there is a morphism,

$$F \colon \mathrm{Cl}_{r+1} \longrightarrow \mathrm{Cl}_r \times_{\mathrm{Cl}_{r-1}} \mathrm{Cl}_r$$

sending a closed point $c = (\sigma_1, \dots, \sigma_r, \sigma_{r+1}) \in \mathrm{Cl}_r$ to the couple

$$F(c) = \big((\sigma_1, \dots, \sigma_r),\ (\sigma_1, \dots, \sigma_{r-1}, b \circ \sigma_{r+1})\big),$$

where

$$b \colon \mathrm{bl}_{\sigma_r}(\dots \mathrm{bl}_{\sigma_1}(X) \dots) \longrightarrow \mathrm{bl}_{\sigma_{r-1}}(\dots \mathrm{bl}_{\sigma_1}(X) \dots)$$

is the blow up morphism.

Theorem 2 (Structure)

If X is quasiprojective and S is proper and smooth, then there is a stratification by locally closed subschemes

$$\mathrm{Cl}_r \times_{\mathrm{Cl}_{r-1}} \mathrm{Cl}_r = \sqcup_i D_i = \Delta_{\mathrm{Cl}_r} \sqcup (\sqcup_j D_j)$$

such that, for every irreducible component $C \subseteq \mathrm{Cl}_{r+1}$,

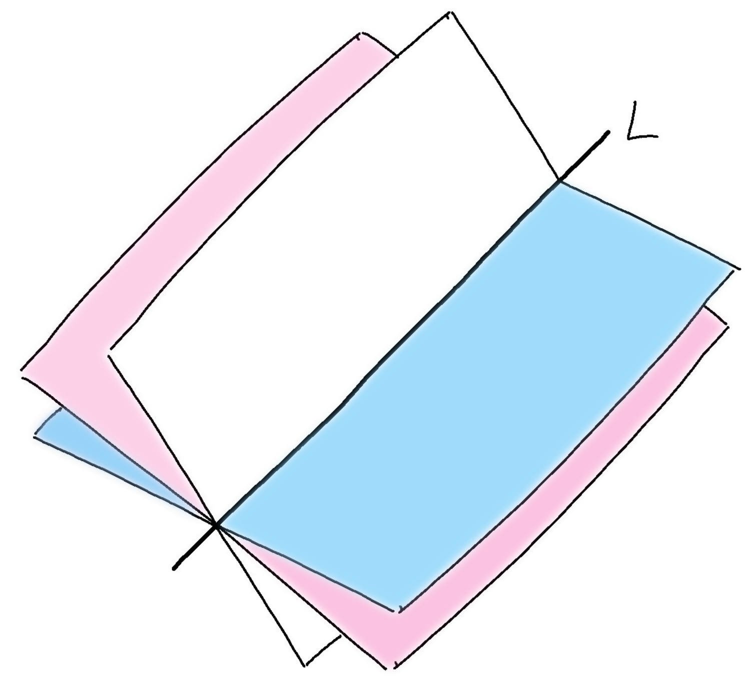
- $F(C) \subseteq \overline{D_i}$ for some (obviously unique) D_i,
- if $D_i \neq \Delta_{\mathrm{Cl}_r}$, then $F|_C \colon C \longrightarrow \overline{D_i}$ is a blow up along an ideal which fails to be Cartier only along the diagonal $\Delta_{\mathrm{Cl}_r} \cap \overline{D_i}$.

An example

The family (a pencil of planes):

$$\pi \ : \ \mathbb{P}^3_{\mathbb{k}} \setminus L \longrightarrow \mathbb{P}^1_{\mathbb{k}}$$
$$(x : y : z : w) \longmapsto (x : y)$$

where L is the line $V(x, y)$. Set $X = \mathbb{P}^3_{\mathbb{k}} \setminus L$.



Cl_1 parametrises rational curves of degree one, that is lines $R \subseteq X$.

$$\mathrm{Cl}_1 = \mathbb{G}(1, 3) \setminus \{\text{lines intersecting } L\} \cong \mathbb{A}^4$$

Cl_2 parametrises couples of sections (σ, τ) where

- $\sigma \colon \mathbb{P}^1_{\mathbb{k}} \longrightarrow X$ parametrises a line $R \subseteq X$.
- $\tau \colon \mathbb{P}^1_{\mathbb{k}} \longrightarrow \mathrm{bl}_\sigma X$ is a section of $\mathrm{bl}_\sigma X \longrightarrow X \longrightarrow \mathbb{P}^1_{\mathbb{k}}$

Notation:

$$Z \subseteq \mathrm{Cl}_1 \times \mathrm{Cl}_1 \ \text{ cl. subscheme parametrising lines that intersect.}$$
$$E_\sigma \subseteq \mathrm{bl}_\sigma X \text{ the exceptional divisor.}$$

- Observation: when τ *does* factorises through E_σ, it is a section of
$$\mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}} \cong E_\sigma \longrightarrow \mathrm{bl}_\sigma X \longrightarrow X \longrightarrow \mathbb{P}^1_{\mathbb{k}}.$$
It corresponds to a curve
$$C_n \subseteq \mathbb{P}^1_{\mathbb{k}} \times \mathbb{P}^1_{\mathbb{k}} \text{ of bidegree } (1, n).$$
Such curves are parametrised by open subschemes
$$V_n \longrightarrow \mathbb{P}^1 \times \mathbb{P}^n$$
- Observation: for $n = 0, 1$, the curves C_n are respectively flat limit of two intersecting lines or two non-intersecting lines.
$$\mathrm{Cl}_2 = W \sqcup U \sqcup (\sqcup_{n \geq 2} V_n)$$
with
$$W = \big(Z \setminus \Delta_{\mathrm{Cl}_1}\big) \cup V_0 \longrightarrow Z \qquad \text{and}$$
$$U = \big((\mathrm{Cl}_1 \times \mathrm{Cl}_1) \setminus Z\big) \cup V_1 \longrightarrow \mathrm{Cl}_1 \times \mathrm{Cl}_1 .$$

The blow up section family

Consider schemes X, S and a closed subscheme Z of $X \times S$. The *blow up section family of the projection $X \times S \longrightarrow S$ along Z* is a morphism $b\colon \mathfrak{X} \longrightarrow X$ such that $(b \times \mathrm{Id}_S)^{-1}(Z) \subseteq \mathfrak{X} \times S$ is an effective Cartier divisor

$$\begin{tikzcd} (b \times \mathrm{Id}_S)^{-1}(Z) \arrow[r, "\text{Cartier}"] \arrow[d] & \mathfrak{X} \times S \arrow[r] \arrow[d, "b \times \mathrm{Id}_S"] & \mathfrak{X} \arrow[d, "b"] \\ Z \arrow[r, "\text{cl.emb.}"] & X \times S \arrow[r] & X \end{tikzcd}$$

and b satisfies the following universal property: For every morphism $f\colon X' \longrightarrow X$ such that $(f \times \mathrm{Id}_S)^{-1}(Z) \subseteq X' \times S$ is an effective Cartier divisor, there is a unique morphism $g \colon X' \longrightarrow \mathfrak{X}$ such that $f = g \circ b$.

Theorem 3 (Existence)

If X is quasiprojective and S is projective and integral, then the blow up section family of $X \times S \longrightarrow S$ along Z exists.

Theorem 4 (Structure)

If X is quasiprojective and S is projective, integral and smooth, then there is a stratification by locally closed subschemes with a distinguished stratum X' (which is a closed subscheme)

$$X = \sqcup_i X_i = X' \sqcup (\sqcup_j X_j)$$

such that the morphism

$$b|_{\mathfrak{X} \setminus b^{-1}(X')} \colon \mathfrak{X} \setminus b^{-1}(X') \longrightarrow X \setminus X'$$

factorises through $\sqcup_j X_j$ and such a corestriction is an isomorphism with its image.

Examples

- When S is the base field, we retrieve the classic blow up and Theorem 4 (Structure) just says that a blow up is an isomorphism away of its centre.
- From the example, the morphism
$$b \colon W \sqcup U \longrightarrow \mathrm{Cl}_1 \times \mathrm{Cl}_1$$
is the blow up sections family of $\mathbb{P}^1_{\mathbb{k}} \times (\mathrm{Cl}_1 \times \mathrm{Cl}_1) \longrightarrow \mathbb{P}^1_{\mathbb{k}}$ along a suitable closed subscheme. The distinguished stratum is Δ_{Cl_1} and
$$(W \sqcup U) \setminus b^{-1}(\Delta_{\mathrm{Cl}_1}) = \big(Z \setminus \Delta_{\mathrm{Cl}_1}\big) \sqcup \big((\mathrm{Cl}_1 \times \mathrm{Cl}_1) \setminus Z\big)$$