Numerical Linear Algebra - Sheet 1

to be handed in until October 28, 2022, 2pm.

Problem 1. Review the following items and write down at least one of the definitions and one of the theorems with proof in detail.

- Definition of a projection
- Definition of an orthogonal projection
- Theorem: Orthogonal projection is uniquely determined by subspace

  Consider a finite-dimensional space V with inner product $\langle \cdot, \cdot \rangle$ and a subspace $W \subset V$. Then, there exists a unique orthogonal projection

  $$P_W : V \to W.$$

- Theorem: Best Approximation Theorem

  Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{x}$ be any vector in $\mathbb{R}^n$, and let $\tilde{\mathbf{x}}$ be the orthogonal projection of $\mathbf{x}$ onto W. Then $\tilde{\mathbf{x}}$ is the closest point in W to $\mathbf{x}$, in the sense that

  $$\|\mathbf{x} - \tilde{\mathbf{x}}\| < \|\mathbf{x} - \mathbf{w}\|$$

  for all $\mathbf{w}$ in W distinct from $\mathbf{x}$.

- Theorem: Orthogonal projection in orthonormal basis

  Let $B = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_p\}$ be an orthonormal basis of a subspace W of a finite-dimensional space V with inner product $\langle \cdot, \cdot \rangle$. Then, the orthogonal projection P_i of any vector $\mathbf{v} \in V$ onto $\mathbf{u}_i$, and the orthogonal projection P_W of any vector $\mathbf{v} \in V$ onto W have the following expressions, respectively:

  $$P_i(\mathbf{v}) = \langle \mathbf{v}, \mathbf{u}_i \rangle \mathbf{u}_i, \qquad i = 1, 2, ..., p,$$
  $$P_W(\mathbf{v}) = \sum_{i=1}^{p} \langle \mathbf{v}, \mathbf{u}_i \rangle \mathbf{u}_i,$$

  and

  $$\mathbf{v} = P_W(\mathbf{v}) + \mathbf{z}, \qquad \mathbf{z} \perp W.$$

- Theorem: Parseval identity

  Suppose that W is a finite-dimensional linear space with inner product $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{e}_i\}$, $i = 1, ..., n$ be an orthonormal basis of W. Then, for every $\mathbf{w} \in W$ it holds

  $$\sum_{i=1}^{n} |\langle \mathbf{w}, \mathbf{e}_i \rangle|^2 = \sum_{i=1}^{n} \|\mathbf{w}\|^2.$$

Problem 2. Proof that every finite-dimensional vector space with scalar product has an orthonormal basis.

Hint: Gram-Schmidt orthogonalization

Problem 3. Compute the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix},$$

for $\alpha \in \mathbb{R}$.

Problem 4 (Programming)**.** Write a program that:

- Generates an orthonormal basis from a given set of n n-dimensional complex vectors.
- Computes the Gram matrix of the obtained orthonormal basis

Test your program on the set of vectors: $\{\mathbf{v}_k\}$, $k = 1, ..., n$, $(\mathbf{v}_k)_j = e^{\frac{i\alpha_k j}{n}}$, $\alpha_k = 1 + \frac{1}{k}$. Investigate the obtained Gram matrix.

Numerical Linear Algebra - Sheet 1 Solutions

Solution 1.

- *Def projection:* Let V be a vectorspace (VS).
  A linear map $P : V \to V$ is called projection $\iff P^2 = P$

- *Def orthogonal projection:* Let V be a VS, $W \subseteq V$ a subspace and $\langle \cdot\ ,\ \cdot \rangle$ be an inner product on V.
  P_w is called orthogonal projection $\iff P_w$ is a projection and $P_w(v) - v \perp W \quad \forall v \in V$

- *Theorem:* Orthogonal projection is uniquely determined by the subspace
  Proof: Existance: Gram Schmidt
  Uniqueness: Let V finite dim VS with inner product $\langle \cdot\ ,\ \cdot \rangle$, $W \subseteq V$ subspace, P_w and P'_w two orthogonal projections on W.
  $\implies P_w(v) - v \perp W \forall v \in V$

$$\begin{aligned} \langle P_w(v) - P'_w(v), w\rangle &= \langle (P_w(v) - v) - (P_w(v) - v), w\rangle \\ &= \langle P_w(v) - v, w\rangle - \langle P'_w(v) - v, w\rangle \\ &= 0 - 0 \ = \ 0 \quad \forall v \in V,\ \forall w \in W \end{aligned}$$

  $\implies P_w(v) - P'_w(v) = 0 \quad \forall v \in V$
  $\implies P_w = P'_w$

- *Theorem (Best Approximation):* Let $W \subseteq \mathbb{R}^n$ subspace, $\widetilde{x}$ the orthogonal projection of $x \in \mathbb{R}^n$ in W
  $\implies \|x - \widetilde{x}\| \leqslant \|x - w\| \forall w \in W$
  Proof:

$$\begin{aligned} \|x - w\|^2 =& \|(x - \widetilde{x}) + (\widetilde{x} - w)\|^2 \\ \overset{\text{Pythagoras}}{=} & \|(x - \widetilde{x})\| + \|(\widetilde{x} - w)\|^2 \leqslant \|(\widetilde{x} - w)\|^2 \end{aligned}$$

- *Theorem:* Orthogonal projection in orthonormal basis
  Proof: Just compute and see P_w is an orthogonal projection

- *Theorem (Parseval identity):* Let V VS, $\{e_1, ..., e_n\}$ an orthogonal basis. Then $\forall x \in V$:

$$x = \sum \langle x, e_i \rangle e_i \tag{1}$$

  Proof: $x = \sum a_i e_i$
  $\implies \langle x, e_j\rangle = \sum a_i \langle e_i, e_j\rangle = a_j$
  $\implies x = \sum \langle x, e_i\rangle e_i$
  $\implies \|x\|^2 = \langle x, x\rangle = \langle \sum \langle a, e_i\rangle e_i, \sum \langle a, e_j\rangle e_j\rangle$
  $= \sum \sum \langle x, e_i\rangle \langle x, e_j\rangle \langle e_i, e_j\rangle$
  $= \sum |\langle x, e_i\rangle|^2$

Solution 2. *Claim:* Every finite-dim VS with a scalar product has an orthonormal basis.
Proof: Use Gram-Schmidt on any basis of the VS.

Solution 3. Let $M = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \in \mathbb{R}^{2\times 2}$ with $c = cos(\alpha),\ s = sin(\alpha),\ \alpha \in \mathbb{R}$

- *Eigenvalues:* $0 = det(M - \lambda I) = \lambda^2 - 2c\lambda + 1$
  $\implies \lambda_{1,2} \ = \ c \pm is \ = \ e^{\pm i\alpha}$

- *Eigenvectors:* $(M - \lambda I)x = 0$
  $\lambda_1 = e^{i\alpha}: \quad x_1 = \alpha\begin{pmatrix} 1 \\ i \end{pmatrix}$
  $Eig(\lambda_1) = span\{x_1\}\lambda_2 = e^{-i\alpha}: \quad x_2 = \alpha\begin{pmatrix} 1 \\ -i \end{pmatrix}$
  $Eig(\lambda_2) = span\{x_2\}$

- $\alpha \in \pi\mathbb{Z} \implies M = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$
  $\implies \lambda_{1,2} = \pm 1,\ Eig(\lambda_{1,2}) = \mathbb{C}^2$

Solution 4 (Programming)**.**

Numerical Linear Algebra - Sheet 2

to be handed in until November 04, 2022, 2pm.

Problem 1. Show that a matrix is normal, if and only if it is unitarily similar to a normal matrix. Namely, matrix $\mathbf{A}$ is normal, if and only if there exists a matrix $\mathbf{Q}$ and a normal matrix $\mathbf{B}$ such that

$$\mathbf{A} = \mathbf{Q}^{-1}\mathbf{B}\mathbf{Q}.$$

Problem 2. Show that for normal matrices the left and right eigenvectors for a given eigenvalue coincide.

Problem 3. Construct a counterexample that the problem of finding eigenvectors is *not* well-posed, if the eigenspaces are almost parallel.

Problem 4. Consider the following matrix

$$\mathbf{A} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}^T \begin{pmatrix} 1 & \\ & c \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

with parameters $\varphi \in [0, 2\pi]$ and $c \in (0, 1)$.

1. Compute the eigenvalues and eigenvectors of $\mathbf{A}$.
2. (Programming) Write a program which computes the sequence $\mathbf{x}^{(n)} \in \mathbb{R}^2$ defined as

$$\begin{aligned} \mathbf{x}^{(n)} &= \mathbf{A}\mathbf{x}^{(n-1)}, \\ \mathbf{x}^{(0)} &= \mathbf{x}^*, \end{aligned}$$

   for $\mathbf{x}^* = (1,\ 0)^T$, $c = 0.1$, and $\varphi = \frac{\pi}{4}$. Try playing with different values of those parameters.
3. Is there a limit of $\mathbf{x}^{(n)}$? What is about the case $c = 1$?
4. Compute the limit: $\lim_{n\to\infty} \mathbf{A}^n$.

Numerical Linear Algebra - Sheet 2 Solutions

Solution 1. *Claim:* A normal $\iff A = Q^{-1}BQ$ with Q unitary, B normal
Proof: " $\implies$ " : A is normal, $A \sim A$
" $\Leftarrow$ " : Let A be a unitarily similar to B, B normal
$\implies BB^* = B^*B, \quad \exists Q \in U(n) : A = Q^{-1}BQ$
$\implies$

$$\begin{aligned} AA^* &= Q^{-1}BQ(Q^{-1}BQ)^* \\ &= Q^{-1}BQQ^*B^*(Q^{-1})^* \\ &= Q^{-1}BB^*Q \\ &= Q^{-1}B^*BQ \\ &= Q^{-1}B^*QQ^{-1}BQ \\ &= (Q^*B^*Q)(Q^{-1}BQ) \\ &= (Q^*BQ)^*A \\ &= A^*A \end{aligned}$$

$\implies A$ normal

Solution 2. *Claim:* A normal, $Av = \lambda v \implies v^*A = \lambda v^*$
Proof: Let A be a normal matrix.
note:

$$\begin{aligned} \|Ax\| = 0 &\iff \|Ax\|^2 = 0 \quad \iff \langle Ax, Ax\rangle = 0 \\ &\iff \langle x, A^*Ax\rangle = 0 \iff \langle x, AA^*x\rangle = 0 \\ &\iff \langle A^*x, A^*x\rangle = 0 \iff \|A^*x\|^2 = 0 \\ &\iff \|A^*x\| = 0 \end{aligned}$$

$\implies$

$$(A - \lambda\mathbb{I})^* = (\overline{\mathrm{A}} - \overline{\lambda\mathbb{I}})^T = (A^* - \overline{\lambda}\mathbb{I})$$

$\implies$

$$\begin{aligned} (A - \lambda\mathbb{I})(A - \lambda\mathbb{I})^* &= (A - \lambda\mathbb{I})(A^* - \overline{\lambda}\mathbb{I}) \\ &= AA^* - \lambda A^*\mathbb{I} - \overline{\lambda}A\mathbb{I} - \lambda\overline{\lambda}\mathbb{I} \\ &= A^*A - \overline{\lambda}A\mathbb{I} - \lambda A^*\mathbb{I} - \lambda\overline{\lambda}\mathbb{I} \\ &= (A^* - \overline{\lambda}\mathbb{I})(A - \lambda\mathbb{I}) \\ &= (A - \lambda\mathbb{I})^*(A - \lambda\mathbb{I}) \end{aligned}$$

$\implies A - \lambda\mathbb{I}$ is normal

Let v be a (right) eigenvector to λ

$$
\begin{aligned}
\implies\quad & (A-\lambda\mathbb{I})v = 0 \\
\implies\quad & (A-\lambda\mathbb{I})^* v = 0 \\
\implies\quad & A^* v = \overline{\lambda}\mathbb{I} v \\
\implies\quad & v^T (A^*)^T = \overline{\lambda} v^T \\
\implies\quad & v^T \overline{A} = \overline{\lambda} v^T \\
\implies\quad & v * A = \lambda v^*
\end{aligned}
$$

Solution 3. $M = \left(\begin{smallmatrix} \eta & 1 \\ \eta & \eta \end{smallmatrix}\right)$ with $|\eta| << 1$

$$
\begin{aligned}
\implies\quad & \lambda_{1,2}(M) = \eta \pm \sqrt{\eta}, \quad v_1 = \left(\begin{smallmatrix} \sqrt{\pm\eta} \\ 1 \end{smallmatrix}\right) \\
\implies\quad & v_1 \text{almost parralel to} v_2
\end{aligned}
$$

Take

$$\widetilde{M} := M + \Delta M$$

so $\widetilde{M}$ is a small change to M.

$$
\begin{aligned}
e.g.\quad & \Delta M = -(\eta)\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \\
& \implies \widetilde{M} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \\
& \implies \lambda_{1,2}(\widetilde{M}) = 0, \quad v_{1,2}(\widetilde{M}) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \implies \text{not conituous}
\end{aligned}
$$

Solution 4.

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}^T \begin{pmatrix} 1 & \\ & c \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} =: R^T A R$$

1. R orthogonal
   $\overset{\text{Lemma 1.1.13}}{\implies} A$ diagonalizable, R orthogonal basis of eigenvectors
   $\implies \lambda_1 = 1,\ \lambda_2 = c, \quad v_1 = \left(\begin{smallmatrix} \cos\varphi \\ \sin\varphi \end{smallmatrix}\right) v_2 = \left(\begin{smallmatrix} -\sin\varphi \\ \cos\varphi \end{smallmatrix}\right)$
2. (Programming)
3. $c = 1 \implies A = \mathbb{I} \implies x^{(i)} = x^{(0)}\ \forall i.$
   Since A^n converges (see (4)), $A^n x$ converges, too.
4.

$$
\begin{aligned}
lim_{n\to\infty} A^n &\overset{\text{R orthogonal}}{=} R^T lim_{n\to\infty} \left(\begin{smallmatrix} 1 & \\ & c \end{smallmatrix}\right) R \\
&= R^T \left(\begin{smallmatrix} 1 & \\ & 0 \end{smallmatrix}\right) R \\
&= \left(\begin{smallmatrix} (\cos\varphi)^2 & -\sin\varphi\cos\varphi \\ -\sin\varphi\cos\varphi & (\sin\varphi)^2 \end{smallmatrix}\right)
\end{aligned}
$$

Numerical Linear Algebra - Sheet 3

to be handed in until November 11, 2022, 2pm.

Problem 1. Consider a Hermitian matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ and a unitary linear operator $\mathbf{Q} \in \mathbb{C}^m \to \mathbb{C}^n$, $m < n$. Prove, that an eigenvalue $\lambda_k(\mathbf{B})$ of the matrix $\mathbf{B} = \mathbf{Q}^*\mathbf{A}\mathbf{Q} \in \mathbb{C}^{m\times m}$ is either equal to 0, or the following estimate holds

$$|\lambda_{\min}(\mathbf{A})| \leq |\lambda_k(\mathbf{B})| \leq |\lambda_{\max}(\mathbf{A})|,$$

where $\lambda_{\min}(\mathbf{A})$ and $\lambda_{\max}(\mathbf{A})$ denote the smallest and largest eigenvalues of $\mathbf{A}$ measured by their magnitude.

Problem 2. A diagonalizable real matrix $\mathbf{A}$ has the following spectrum:

$$\sigma(\mathbf{A}) = \{-2,\ 1-2i,\ 1+2i,\ 1,\ -i,\ i,\ 2\}\,.$$

Consider using the inverse power method (*vector iteration*) to compute its eigenvalues.

(a) Find a set of all shift parameters for which the inverse power method may *not* converge. Draw a sketch.

(b) For every *real* eigenvalue find a *real* range of shifts that, if used in the inverse power method, will reduce the error of approximation of the eigenvalue by a factor of 10 in each iteration.

Problem 3. Propose shift parameters that will allow you to compute *all* eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 100 & 15 & 3 \\ 15 & 20 & 5 \\ 3 & 5 & 65 \end{pmatrix}.$$

Prove that your choice is correct. *Hint:* Gershgorin circle theorem

Problem 4 (Programming)**.** Write a program that computes all eigenvectors and eigenvalues of the matrix

$$\mathbf{A}_\varepsilon = \begin{pmatrix} 100 & 15 & 3 & 0 & 0 & \varepsilon \\ 15 & 20 & 5 & 0 & \varepsilon & 0 \\ 3 & 5 & 65 & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & 110 & 20 & 5 \\ 0 & \varepsilon & 0 & 20 & 80 & 4 \\ \varepsilon & 0 & 0 & 5 & 4 & 30 \end{pmatrix}$$

using the shifted (inverse) power method by observing the following steps:

(a) Consider the matrix $\mathbf{A}_0$ with $\varepsilon = 0$ and examine the structure of the eigenvalue problem.

(b) Compute all eigenvalues and eigenvectors of $\mathbf{A}_0$.

(c) Would the strategy you proposed in Problem 3 allow you to compute all 6 eigenvalues?

Rewrite: Does the algorithm work, if you use Gerschgorin for ...

(d) Make an *educated* guess of the eigenvalues of $\mathbf{A}_1$ and try computing those.

Can you use part (a) to get something better

Numerical Linear Algebra - Sheet 3 Solutions

Solution 1.

Solution 2.

Solution 3.

Solution 4 (Programming)**.**

Numerical Linear Algebra - Sheet 4

to be handed in until November 18, 2022, 2pm.

Problem 1. Let $\mathbf{A}$ be a symmetric tridiagonal matrix. Show that the QR-iteration (see Algorithm 1.4.3 in the lecture notes) preserves the tridiagonal structure of the matrix, i.e., all iterates $\mathbf{A}^{(n)}$ generated by the QR-iteration are tridiagonal.

Problem 2. Show that a (complex) symmetric matrix can be transformed to a tridiagonal matrix by using similarity transformations (this proves Theorem 1.4.14 in the lecture notes).

Problem 3. Rewrite the QR factorization of a tridiagonal (complex) symmetric matrix such that its complexity is of order $O(n)$ (this proves the second part of Corollary 1.4.13 in the lecture notes).

Problem 4. Find an example of a matrix with a real spectrum for which the QR method will *not* converge to an upper triangular matrix.

Problem 5 (Programming)**.**

(a) Implement the Hessenberg QR step (Algorithm 1.4.12 in the lecture notes) in real arithmetic.

(b) Test your code with the tridiagonal matrix $\mathbf{A}_n = \mathrm{tridiag}(-1., 2., -1.)$ in dimension $n = 4$ and check, if your results are correct.

(c) Use your implementation to run several steps of the QR iteration (Algorithm 1.4.3 in the lecture notes) for the matrix $\mathbf{A}_{10}$.

(d) Discuss the observed convergence of the off-diagonal and diagonal entries, respectively.

Numerical Linear Algebra - Sheet 4 Solutions

Solution 1.

Solution 2.

Solution 3.

Solution 4.

Solution 5 (Programming)**.** The eigenvalues of $\mathbf{A}_n$ are

$$\lambda_{n,j} = 4\sin^2\left(\frac{j\pi}{2n+2}\right)$$

https://math.stackexchange.com/questions/3875168/eigenvalues-of-a-tridiagonal-matrix-with-1-2-1-as-entries

https://math.stackexchange.com/questions/177957/eigenvalues-of-tridiagonal-symmetric-matrix-with-diagonal-entries-2-and-subdiago?rq=1

Numerical Linear Algebra - Sheet 5

to be handed in until November 25, 2022, 2pm.

Problem 1. Show that a normal triangular matrix is diagonal. *Hint:* look at the norms of $\mathbf{A}\mathbf{e}_i$ and $\mathbf{A}^*\mathbf{e}_i$.

Problem 2. Prove that in case of a normal real matrix, for each complex eigenvalue pair there is a 2×2 matrix with according invariant subspace.

(a) Show that complex eigenvalues of a real matrix come in complex conjugate pairs.

(b) Show that the eigenvectors are of the form $\mathbf{u} \pm i\mathbf{v}$.

(c) Choose real linear combinations of these vectors to obtain the 2×2 block.

Problem 3. Provide the following steps of Lemma 1.5.11 in the lecture notes (explicit double shift).

(a) Show that $\mathbf{Q}_1\mathbf{Q}_2\mathbf{R}_2\mathbf{R}_1$ represents the QR factorization of a real matrix $\mathbf{M} = (\mathbf{H} - \sigma_1\mathbb{I})(\mathbf{H} - \sigma_2\mathbb{I})$.

(b) Show that $\mathbf{Q}_1\mathbf{Q}_2$ is the orthogonal matrix that implements the similarity transformation of $\mathbf{H}$ to obtain $\mathbf{H}_2$.

Problem 4 (Programming)**.** Implement the symmetric QR step with implicit shift (Algorithm 1.5.6 in lecture notes) for a symmetric, unreduced, tridiagonal matrix $\mathbf{T}$ by observing the following steps:

(a) Store $\mathbf{T}$ in two vectors, one for the diagonal, and one for the subdiagonal entries.

(b) Implement the Givens rotation $\mathbf{G}_{12}$ and think about where to store the additional non-zero entry t_{31}.

(c) Implement the additional Givens rotations for this data structure.

(d) Use this to compute the eigenvalues of the matrix $\mathbf{A}_n = \mathrm{tridiag}(-1., 2., -1.)$ in dimension $n = 10$.

Numerical Linear Algebra - Sheet 5
Solutions

Solution 1.

Solution 2.

Solution 3.

Solution 4 (Programming)**.**