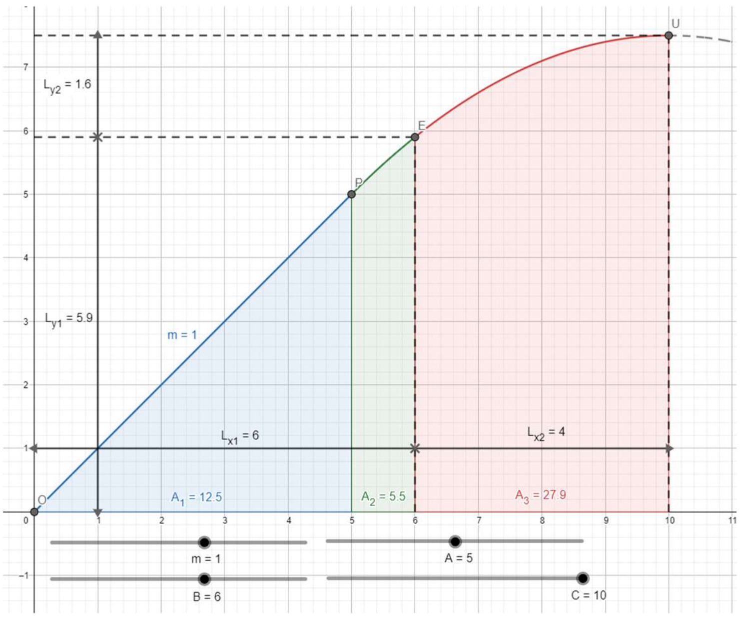
Young Modulus and Hooke’s Law

Each material possesses fixed properties under stress. These properties are dependent on the atoms of the material and their arrangement (the material itself), and the material’s dimensions and shape. One way to analyse the properties of a specific material is using a wire, modelled as a cylinder, a uniform shape with a cross-sectional area A and length l. There are three ways to apply a force to a material: Compression, the act of applying a vector force to an object (to the length of the cylinder) towards a fixed opposite edge such that the object has reduced length in that axis and usually higher density; tension, the act of applying a vector force to an object (to the length of the cylinder) away from a fixed opposite edge such that the object has increased length in that axis and usually lower density; and torsion, the act of applying a torque to an object (around the cross section of one circular edge of the cylinder) such that the object twists between two opposite ends. For this write-up, I will be using compression and tension on a wire of material to test its properties.

Plastic	Elasticity (Domain of strain in the elastic portion of σ-ε graph)	Elastic
Smooth	Roughness (Quantity of friction caused by material’s surface)	Rough
Weak	Strength (Range of stress in elastic and plastic portions of σ-ε graph)	Strong
Hard	Softness (Extent of local plastic deformation under compressive force)	Soft
Compliant/Flexible	Stiffness (Magnitude of stress to strain ratio in the elastic portion of σ-ε graph)	Stiff
Brittle	Malleability/Ductility (Domain of strain in the plastic portion of σ-ε graph [compressive/tensile])	Malleable /Ductile
Fragile/Brittle	Toughness (Area under σ-ε graph until fracture point)	Tough
Flimsy	Durability (Consistency within the σ-ε graph under repeated loads)	Durable

Of these properties, we can get the following from a graph of stress against strain: elasticity, the horizontal distance to the elastic limit; strength, the vertical distance to the breaking point; stiffness, the gradient of the line before the limit of proportionality; malleability/ductility, the horizontal distance between the elastic limit and breaking point; and toughness the area under the graph until the breaking point.



Here is a general graph of stress against strain of a material. When modelled as a cylinder, the stress, σ, on an object is equal to the force applied to one circular side divided by the cylinder’s cross-sectional area. This is measured in Pascals, and is a measure of the pressure on the surface of the object. The strain, ε, is equal to the extension, or change in length of an object divided by its original length. It has no units, as it is meters in one dimension divided by meters in the same dimension, and can be represented as a percentage change. The blue portion of the graph shows where the correlation between stress and strain is linear. In this portion, the gradient of the line is equal to the Young Modulus of the material, which represents its stiffness, or incapacity to extend under a given force. The green portion of the graph is where the extension is no longer acting proportionally to the applied force, as point P stands for the limit of proportionality. However, after unloading the material, it will return to its original size. Point E is the elastic limit, meaning any extension beyond that point correlates to additional length when the load is removed. The red portion of the graph, then, is the plastic region, where any load will result in extension after unloading, called a “permanent set”. Despite the green region not technically being linear, because points P and E are so close together, it can be treated as linear as part of a triangle containing the blue and green regions. This assumption, that proportionality implies elasticity, decreases the predicted elasticity and stiffness of the material, and increase its predicted malleability.

For some materials, the point U, or ultimate tensile stress, is not the same point as the ultimate tensile strain, and thus will keep extending past U without extra force applied. However, based on my results, there was not much additional strain between each added force, so this can likely be safely ignored for copper. With these assumptions in mind, and the variables on the example graph, I have created a list of formulae for the properties of a copper wire:

- $Elasticity = L_{x1}$
- $Strength = L_{y1} + L_{y2}$
- $Stiffness\ (Young\ Modulus, E) = m$
- $Malleability \backslash Ductility = L_{x2}$
- $Toughness = \int_0^{U_x} f(x)dx = A_1 + A_2 + A_3 \approx \frac{1}{2}L_{x1}L_{y1} + L_{x2}L_{y1} + \frac{1}{2}L_{x2}L_{y2}$

As for the other 3 properties, roughness does not have an effect on any materials' extension under a force; hardness is only a property under local deformation, though could have an effect on the unmeasured necking of the wire between the ultimate tensile stress and strain points; and durability is only distinguished under multiple loads and unloads, the change in a graph of stress against strain after multiple separate plastic deformations, whereas I was loading each wire to its breaking point.

For this investigation, I made two GeoGebra projects. One shows the general graph of a material with variable location for the proportionality and elastic limits, and fracture point. This model shows the values of the mathematically calculatable properties of a material, given the shape of its stress-strain graph. The other is a model of the correlation between force, stress, Young Modulus, strain, length and extension, and how this affects a diagram of a cylinder. Both of these models can be found published under https://www.geogebra.org/u/laurahannah44 as of 14/05/19.

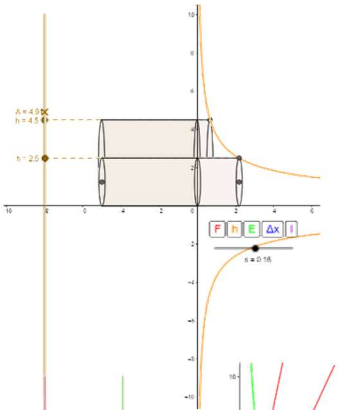
The Young Modulus, E of a material is defined as its incapacity to extend a certain percentage, ε, under some stress, σ in Pascals. Using the following equations, the correlation between each variable factor in the formula can be derived:

$$A = \frac{\pi h^2}{4} \qquad \sigma = \frac{4F}{\pi h^2} \qquad E = \frac{\sigma}{\varepsilon}$$

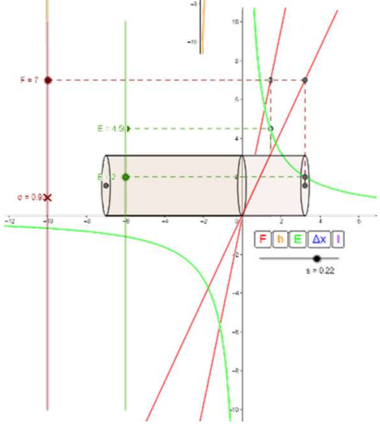
$$\sigma = \frac{F}{A} \qquad \varepsilon = \frac{\Delta x}{l} \qquad \therefore E = \frac{4Fl}{\pi h^2 \Delta x}$$

With this final equation in E, F, l, h and Δx, we can replace any two variables with x and y, and rearranging to make an equation y=f(x) for their correlation. For example, if we want to see how a change in the Young Modulus of a material used affects its extension:

$$x = \frac{4Fl}{\pi h^2 y} \qquad y = \frac{4Fl}{\pi h^2 x} \qquad y = \frac{4Fl}{\pi h^2} x^{-1}$$



This shows that the young modulus is inversely proportional to the change in length, under a constant force, original length, and height (diameter). For correlations with a visual input and output, it is helpful to model this on a graph of the function, like this example linking height to extension. For this, $y = \pm kx^{-0.5}$, the equation shown by the orange lines to the right of the cylinder. In this case, negative values for h just imply the same object upside down, hence the mirror on y=0. This method of correlation analysis can be very useful in planning systems with fixed parameters; how each variable can change to slightly adjust a systems state, position or rate of change.



An example of this is the effect of Young Modulus on how force affects extension. Because force and extension are in different sections of the fraction while on the same side of the equation, they can be rearranged to create a linear equation:

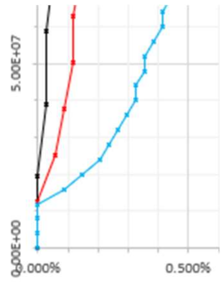
$$\Delta x = \frac{4l}{\pi h^2 E} F$$

Say we want to change the rate at which force affects extension. We know from this equation, that length affects the gradient linearly, height or diameter affects the gradient by inverse square, and young modulus affects the gradient inversely. We can then use this fact to tweak one of those three variables until we have our desired system for extension or compression.

I used copper wire for my experiment, specifically 3 different thicknesses, and thus SWG values. Because of the classroom environment I was in, I was able to allign 4 tables next to each other, and suspend the wire along its length, fixed at one end and the other on top of a pulley. I used masses tied to the pulley end of the wire, so that I could monitor the force applied to the wire, and a piece of blutack fashioned to a point close to the end of the pulley, so that I could see its extension from a base state. I wanted to maximise the extension of the wire, as I knew the extension would be very low, making my measurements of length quite inprecice. To do this, I tried to maximise the length, as it is the only non-variable parameter which is linearly proportional to extension (the other being force, a variable). The other options are minimising diameter and young modulus, however the material was already constant, and minimising the diameter brings separate problems that make it difficult to work with. I used three different diameters of wire as they were available to me.

Here's a list of the sources of inaccuracy and inprecision in my experiment, including how I was able to, or could have fixed or minimised them, and their effect on my results:

Inaccuracy:



- Zero-mass extension – Because I used copper wire, which was originally wound up in a coil, it could not be taut over the length of the ruler with no applied force, and thus I did not have a true value for the base extension. This issue also meant that my value for strain was offset for the smaller values of stress by some "slackness" function. These two issues resulted in the origin of the stress-strain graph looking like so, and was accounted for in my results by excluding the first few data points from the line-of-best-fit calculation.
- Wire mass per unit length – Along with this, the small amount of wire draped over the pulley also had a mass that wasn't accounted for, thus decreasing the predicted force, and decreasing the Young Modulus slightly. The amount of draped wire also increased over time, meaning there would be a concave effect on the graph of stress-strain produced.
- Pulley friction – The use of a pulley to convert masses under gravitational energy to tension force relies on the assumption that there is no friction in the pulley. Friction would cause our percieved force to be slightly higher than the wire was truly subject to, and thus increase our value for the Young Modulus. This isnt likely a significant factor, however, as the total extension was around 16cm maximum, and the pulley had a circumference of ~25cm, meaning there were only around 1.28π radians of rotation total. There wasn't a clear way I could account for this, though it is likely a small enough effect to be ignored.
- Ruler parallax – Because the wire was on a pulley system, it is elevated from the table. This elevation may be an issue for recording the change in length of the wire, as there is a parallax between the marker used on the wire, and the ruler on the table below. To measure the position of the marker, I tried to get vertically above the setup, to minimise the effect. Another way to fix parallax would be to suspend the rulers closer to the wire, however this would be difficult to do with my setup of tables, and needing to use 4 metre sticks. Along with this, if the wire was touching the ruler, there would be slight friction of the wire against the surface, slightly reducing the effective force.
- Ruler alignment – Over the 4m length of wire, I used 4 meter rules from one end of the table to find the position of the secured end of wire x_1. Despite the fact that these were rules rather than rulers, and thus ended exactly on the 0cm marking and 100cm marking, they likely were not held perfectly flat against each other, and were not perfectly parallel to the wire. This would have two opposing effects, and can likely be ignored: If the rules were angled slightly, more rule length would fit in a shorter space, leading to overassumption of length, and if the edges of the rules pressed against each other were slightly angled, there would be a slight gap between each measurement, meaning less rule length would fit along the table. I could have ensured this was accounted for by using a tape measure, although it likely
- Wire angle – I was very limited in my methods for securing the wire in place. In the end I used two blocks of wood secured by a G clamp, however I did not ensure the wire was secured at the same height as the pulley, and as such, it may have formed the hypotenuse of a triangle, with its actual length being $\sqrt{h^2 + l^2}$, where h is the difference between the heights of each end of the wire. However, as the wire was taut, it should have formed a straight line, and thus the extra ratio of length should be equal for the calculation of total length and extension, thus being cancelled out when calculating strain.

Inprecision:

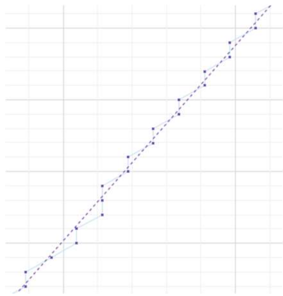
- Area – Given using SWG conversion to the nearest 0.0001mm^2 (A ± 0.00005mm^2/A ± 50µm^2). For the thickest wire, the most accurate result, the percentage uncertainty was ±0.2%.
- Length/Extension – For any of the measurements of location, I used a metre ruler with a precision of ±0.5mm, (x ± 0.0001m). This uncertainty will be added for every occurrence of a position in my calculations. There are a few different percentage uncertainties for position:

  $\frac{2*\pm 0.000}{3.369m} = \pm 0.01\%$ (total length of wire)

  $\frac{2*\pm 0.0001m}{0.023m} = \pm 0.87\%$ (extension to limit of proportionality)

  $\frac{2*\pm 0.000}{0.158m} = \pm 0.13\%$ (extension to fracture point)

  The values are multiplied by two, as there are two occurences of a position measurement for each length and extension.

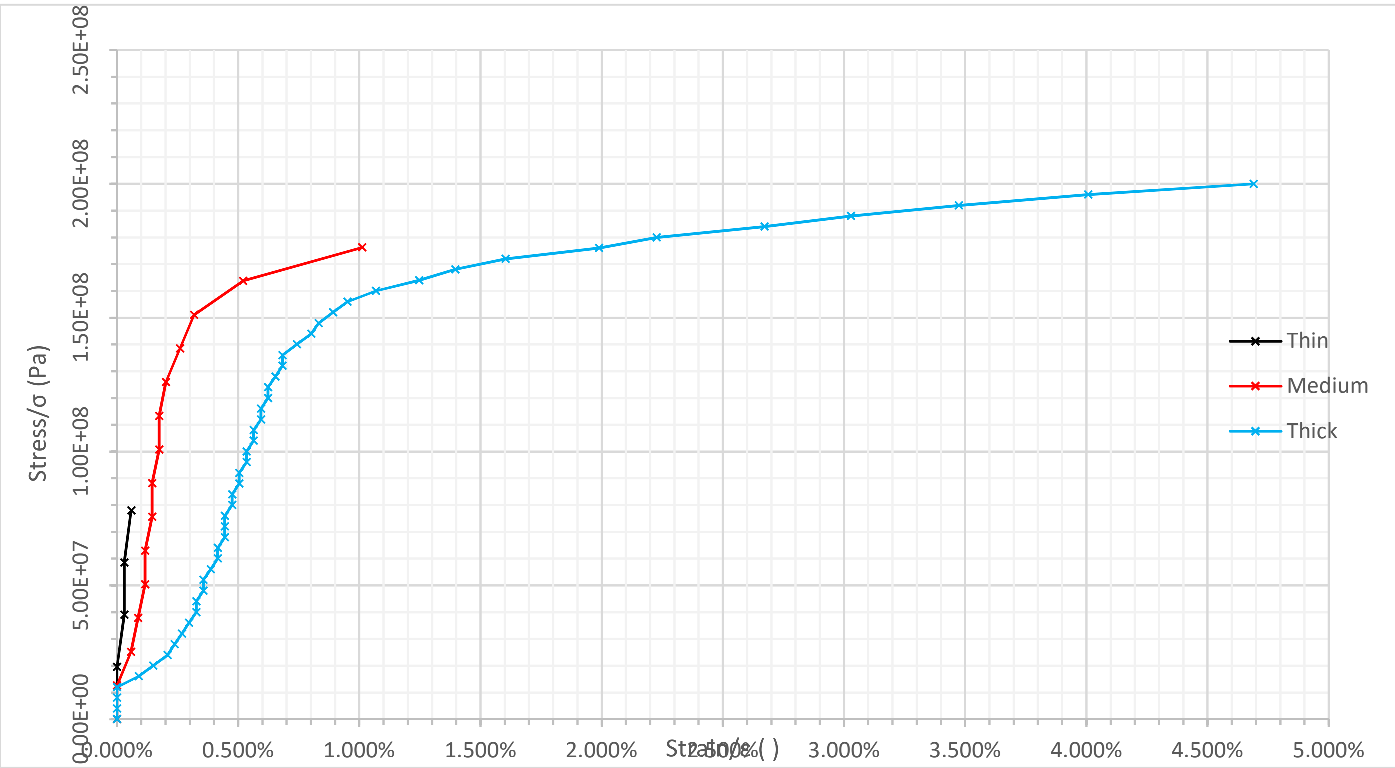


- Mass – I used classroom 0.1kg masses on the pulley end of the string. I did not measure the mass of one to calculate a percentage uncertainty, but as they are quoted as standard 100g masses, I'll use the assumption that they are precise to 1g (m ± 0.0005kg), which would have two different percentage uncertainties too.

  $\frac{2*\pm 0.0005kg}{3.4k} = \pm 0.03\%$ (mass to limit of proportionality)

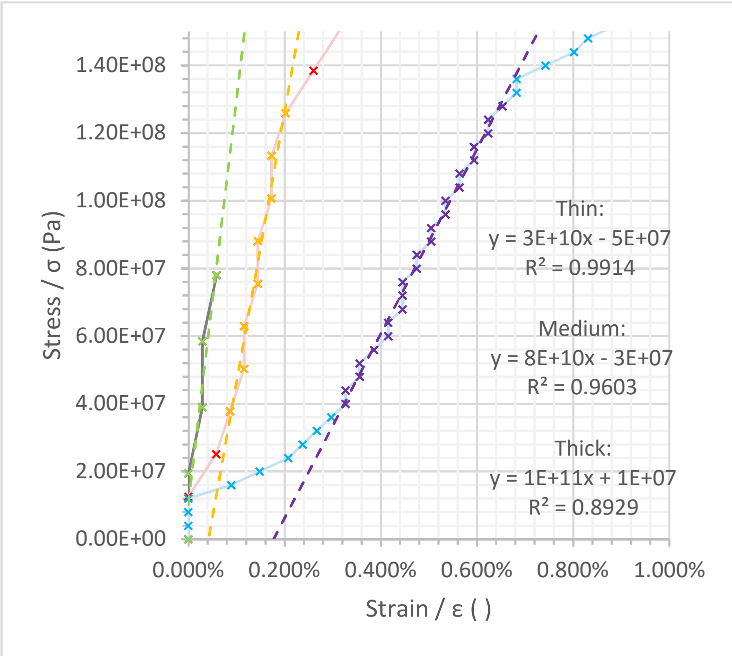
  $\frac{2*\pm 0.0005kg}{5.0kg} = \pm 0.02\%$ (mass to fracture point)

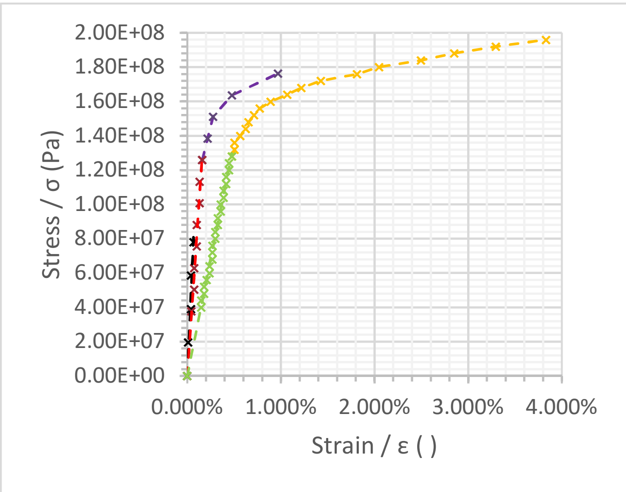
  For each of these values, the zero-error comes from whichever point I choose as the initial value, as the point where zero mass is known to be applied to the wire is skewed from the graph.
- Gravity – For my calculations of gravity, I used the value g = 9.80665ms^{-2}, so my uncertainty was a ± 0.00001%.
- Force – The force on the wire is mass * acc, so F_u = ±0.03% or ±0.02%. ($F_u = m_u + a_u$)
- Stress – The stress on a material is F/A, meaning the % uncertainties are added. Thus, σ_u = ±0.23% or ±0.22%.
- Strain – The strain on a material is Δx/l, so ε_u = ±0.88% or ±0.14%.
- Elasticity – L_{x1} = a ± 0.88%
- Strength – $L_{y1} + L_{y2}$ = b ± 0.45%
- Young Modulus – σ_1/ε_1 = E ± 1.11%
- Malleability – L_{x2} = c ± 0.14%
- Toughness – $\frac{1}{2}L_{x1}L_{y1} + L_{x2}L_{y1} + \frac{1}{2}L_{x2}L_{y2}$ = d ± 1.84%



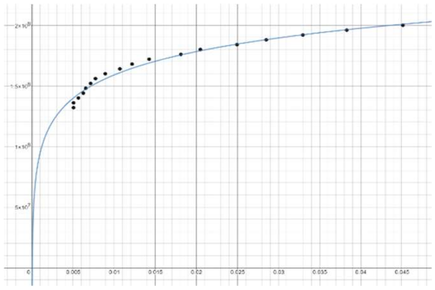
My results formed the graph on the previous page. Without any systematic error, each set of results should completely overlap. However, due to the zero-error mentioned before, the thicker wires are offset where they take some force to become fully taut. To account for this issue, I can choose a portion of the data that is proportional, plot a line of best fit, and subtract its x-intercept from every data point, discarding the points from before this trend is formed.

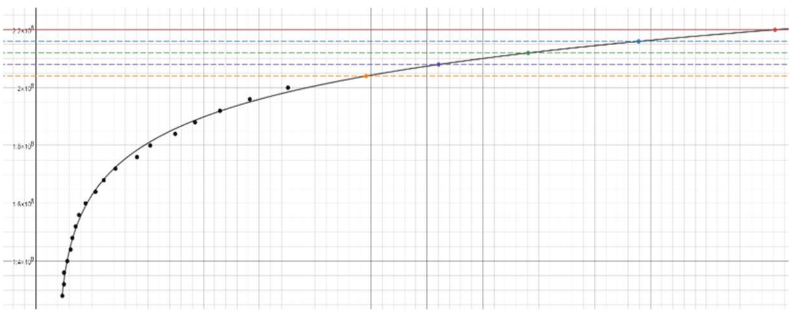
For the thinnest wire, there is a negative x-intercept (-0.009%). I will still subtract this from the results, as it just shows that the initial mass had more of an effect on the wire than the slackness. The medium wire had an intercept of 0.042% "false extension", and the thickest had 0.177%. With these corrections in place, the graph now looks like so:





This graph makes it quite clear that at least toughness (vertical range) and strength (integral), are linked to cross-sectional area. Along with this, malleability and elasticity changed significantly with cross-sectional area, although this may be due to other factors mentioned. The necking in the wire likely caused around 2mm for the thickest wire, so around 0.05%. There are still two more things I need to account for before I can get values for the properties: The final point in each of these data sets is the point before the wire broke, as I was unable to record the extension for the force that caused the wire to snap. To account for this slight extra force, I need to create an approximation of the equation for the plastic region of the graph, then project it out until it is equal to the highest force I applied, and find the x-coordinate, or extension, that corresponds to that fracture point on the graph. For the thinnest wire, I will make the assumption that the entire unbroken region of the thin wire is elastic, as both the medium and thick wire had identical elastic ranges (higher than the fracture strain of the thin wire). As excel's trend approximation is very limited, I decided to do this in Desmos, using a logarithmic curve. With a natural logarithmic curve, I cannot match both the final and initial gradients. To account for this, I added another two variables to the approximation, to turn it from $\mathrm{y} = \mathrm{a} * \ln(\mathrm{x}) + \mathrm{b}$ to $\mathrm{y} = \mathrm{a} * \log_{\mathrm{b}}(x - c) + d$. This allowed me to tweak the parameters to get the following trend:





$\mathrm{y} = 1.02\mathrm{E}^{7} \log_{1.71}(x - 3.7E^{-3}) + 2.59E^{8}$

so therefore:

$\sigma = 1.02\mathrm{E}^{7} \log_{1.71}(\varepsilon - 3.7E^{-3}) + 2.59E^{8}$ and

for the final stress σ = 2.20E^{8}Pa:

$2.20\mathrm{E}^{8} = 1.02\mathrm{E}^{7} \log_{1.71}(\varepsilon - 3.7E^{-3}) + 2.59E^{8}$

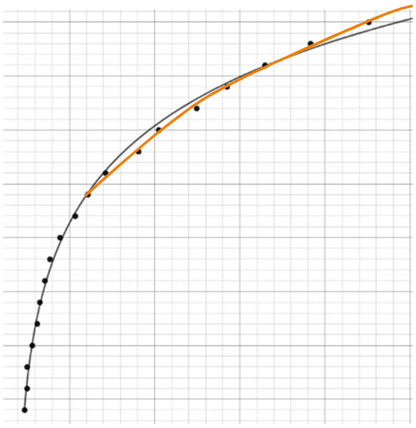
and gave a value of ε as 0.1323 or 13.23%.

For the thickest wire, it actually extended past my ruler measurement for the final five readings, which are the horizontal lines on the graph. Using this, I know the wire's fracture point was between the top two lines, or between 10.79% and 13.23%. I can also add ±1.22 uncertainty to my value, using the midpoint as the fracture point's strain: 12.01% ± 10.16% (without the uncertainty I calculated earlier for each individual point. This uncertainty will be much lower for the thinner wires, as they did not extend nearly as much at higher stresses.

I followed the same process for the other two thicknesses, and got results for all of them. For all points of interest, I found the two points that are certainly either side of it, and found their midpoint and half range as the point's value and uncertainty respectively. I also used the uncertainty of the function approximation for the logarithmic curves to find the uncertainty for the extension retrieved using it. My function was in the form $\mathrm{y} = \mathrm{alogb}(\mathrm{x} - \mathrm{c}) + \mathrm{d}$, which, in inverse, is x = b^(y-d)/a + c. I know the uncertainty for each of y, a, b, c and d, and thus I can calculate the uncertainties of the base and exponent, as (b) and (y+d+a) given that they are all in percentage form. With exponential uncertainty, it is helpful to list all possible combinations of bounds, and take the maximum and minimum as the bounds of the output.

There are a few assumptions I have had to make to get these measurements from my data. In reality, a graph of the plastic region of a stress-strain curve does not act logarithmically. For one, it will tend towards a zero-gradient "yield point", where any additional stress would cause fracture, and continuous stress results in the material being drawn into wire. This yield point is where necking occurs, meaning extra force would break the atoms apart, but no additional force would cause the neck to spread over the whole wire, resulting in a decreased uniform surface area. At this point on the graph, a distinction must be made between the "engineering stress" and true stress. As necking causes a reduction in the average surface area of the wire, the true stress increases as the wire is drawn, despite no increase in force. The engineering stress, then, is the stress on a material given that it is constantly modelled as the original wire. This is the value I had for stress, as I could not measure the cross-sectional area of the wire at or past the yield point as that would require a dynamic micrometer. Necking also does not occur instantly at the yield point, but instead gradually occurs in the upper area of the plastic region. This means the true ultimate tensile stress is higher than that calculated from my results, and does not necessarily occur once the plastic region reaches a gradient of zero. The necking of the wire also effects the strain, as an engineering strain only accounts for the length of wire given its shape as a uniform cylinder (thus with a uniform distribution of atoms), when necking causes this to not be the case. I measured the actual length of the wire, meaning my graphs were of engineering stress against true strain.

These differences are very important to account for, as they affect the model that I should use to approximate the trend. As I only have access to engineering stress, and real strain, I can only model the y-axis of the graph up until the maximum force I applied, and not the change in cross sectional area causing this to increase further. This means this point must be a maximum, on my graph, and thus have zero gradient. As a logarithm only tends towards zero, and only on the y-axis, a different model is needed, which can have a local gradient of zero. Along with this, based on my results, it appears that necking doesn't affect true strain evenly up until the yield point, as my graph seems to decrease then increase in slope as the force increases. This may have been the product of irregular mass placement, however, as the extension was occurring moderately slowly at that point, and thus may have been slightly increasing for some of my readings. This factor may have also caused an increase in the perceived ultimate tensile engineering stress, as the wire may have been able to fracture under less mass, but just taking time to do so. Under the time constraints of a school period, this was unavoidable, however I would make sure to account for it in a future experiment. If the slight s-curve in my results was due to the use of true strain, I could account for it using a polynomial approximation, which would also deal with the zero-gradient at the yield point. However, this would double the number of parameters required to approximate the trend, complicating the calculation significantly. Based on the irregular shape of the curve, it is likely I will not be able to confirm any approximation to be more accurate than another, rendering this method useless under my results.



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