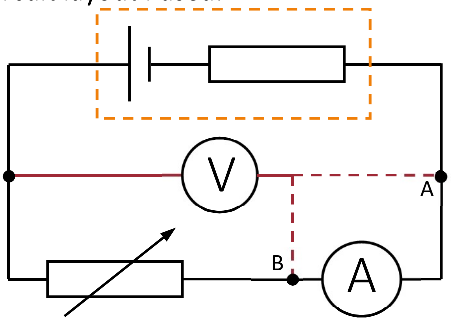
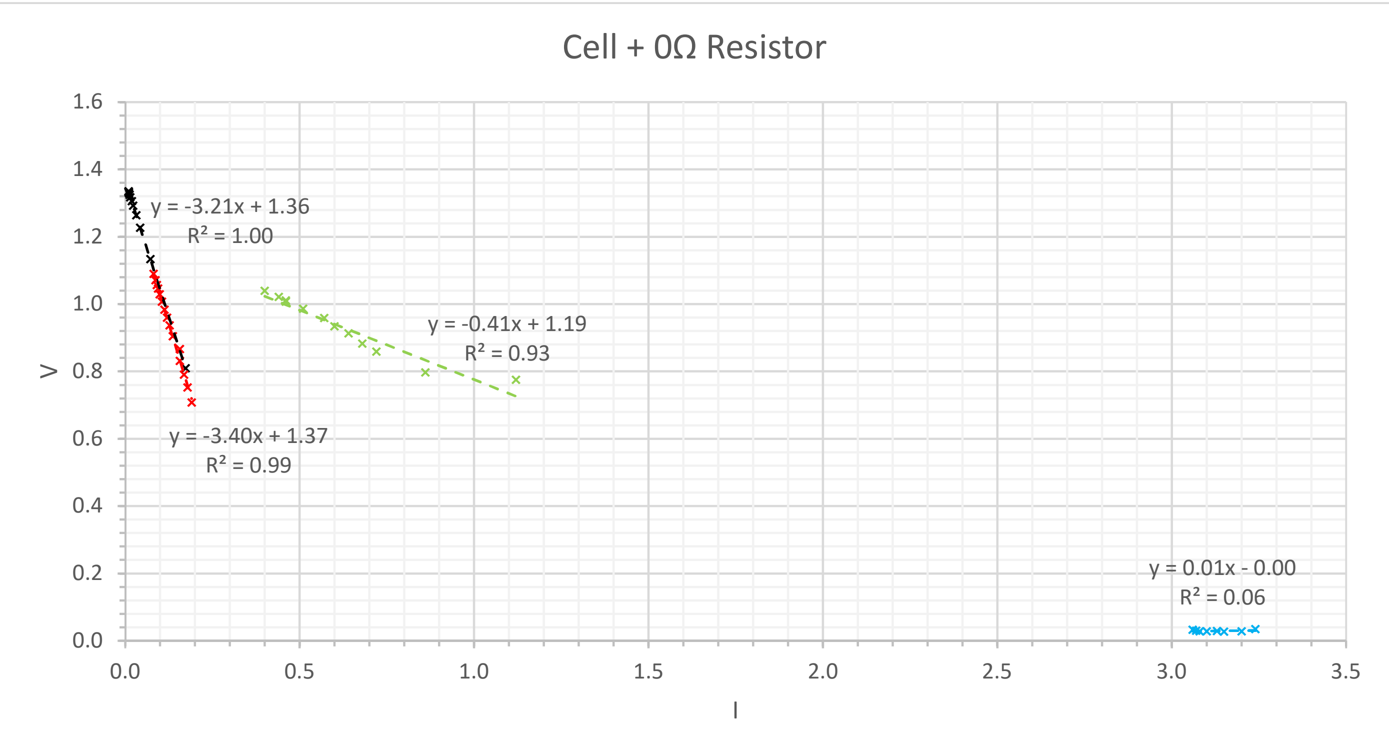
Emf and Internal Resistance

The aim for this investigation was to be able to take accurate measurements from an electrical circuit, and use these to calculate the internal resistance of different cells and the power and EMF of the circuit, while varying its load resistance. I set up a simple circuit with a cell, two multimeters, a rheostat and a resistor to act as additional internal resistance to the cell. This was the circuit layout I used:

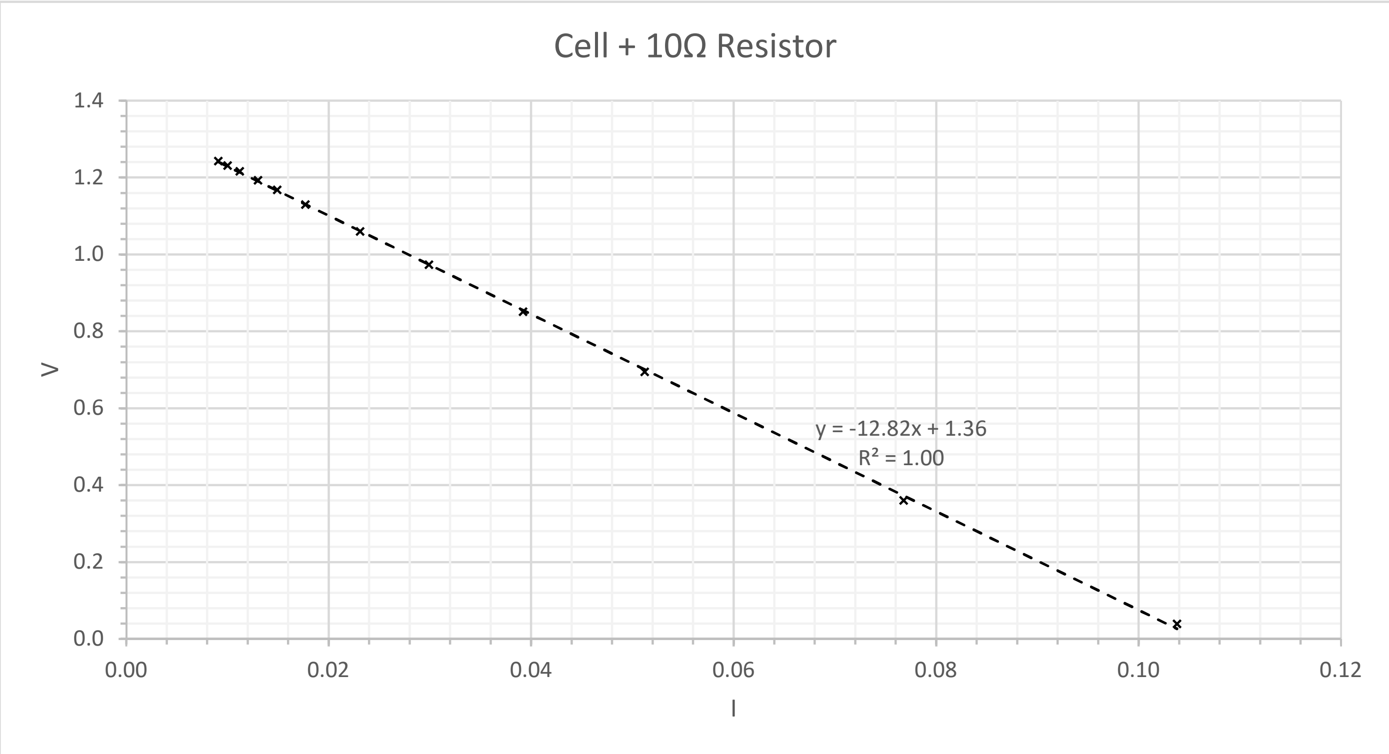


I used multimeters for the voltmeter and ammeter, as they provided more consistent precision than the regular devices. The voltmeter measures the voltage across the entire load of the circuit (preferably including the wires, as they do produce resistance), and the ammeter measures the current in the main branch, including the rheostat. Interestingly, for the voltmeter to get the most accurate value, it should include the ammeter in its partition, as the ammeter has a small amount of resistance. However, for the ammeter reading to be most accurate, it should be on the main undivided branch, as the current is shared via a low ratio between the (red) branch with the voltmeter and the branch with the rheostat. This means for the most accurate voltmeter reading, the (dotted) wire should connect at point A, and for the most accurate ammeter reading, the wire should connect at point B. As the readings were only to 3 significant figures, I just kept the wire connected at point B to save time and get more readings. I chose to vary the resistor that was acting as part of the internal resistance at 0Ω, 10Ω, and 100Ω, to see how that would affect the precision of available measurements. I also varied the rheostat over its full range so I could plot a graph of the results to calculate a value for both the internal resistance and the EMF of the cell and circuit respectively.

I plotted three graphs of V against I with varying load resistances, for the 3 different internal resistances:



This first graph was of the circuit with no extra internal resistance added. The black set was the original set of data measured, and interestingly had the highest precision of any other set. This means it would also likely be the closest to the actual value of the internal resistance, which was measured to be ~3.21Ω, and EMF, which was ~1.36V. The actual value on the cell was stated to be 2.9Ω, meaning I had a percentage error of 10.7%The second recorded set was red, which was recorded using the same rheostat, with a range of 0-135Ω, in which I tried to record more values towards the lower resistances, to get a larger range of data, and more points in the area which originally contained less, due to the reciprocal relationship between resistance, voltage and current. The red data set was slightly less precise than the black set, likely due to spending less time getting the results as correct as possible. The blue and green sets were very likely all anomalous, as I used a very different rheostat, of 0-14Ω resistance, and focussed my results especially on the lower area of this rheostat, to try to find the point of maximum power in the circuit. Such a low resistance in a circuit very likely caused heating due to the current being so high, invalidating my readings. The calculated internal resistance decreased the lower the circuit's resistance was, either being due to the perceived current being higher than reality, or the perceived voltage being lower. As the value for EMF also decreased at lower resistances, it's likely that the internal resistance change was due to the voltage as EMF is the potential maximum voltage over the entire circuit.



The second dataset was the most consistent and produced the neatest graphs, likely due to its mid-range resistance, meaning there was not too much of either voltage or current, causing anomalies. This graph shows the trend produced the best out of all of them. Essentially, as the load resistance increased, the measured value for load voltage increased, and the measured value for circuit current decreased. The EMF for the circuit is the forecast voltage if the circuit had 0 amps of current going through it, meaning it is given as the intercept of the line of best fit through the results. Along with this, the gradient of the line is the internal resistance of the circuit multiplied by -1, due to the equation for EMF:

$$\mathcal{E} = IR_{total}, \qquad R + r = R_{total}, \qquad V = IR$$

$$\mathcal{E} = I(R + r)$$

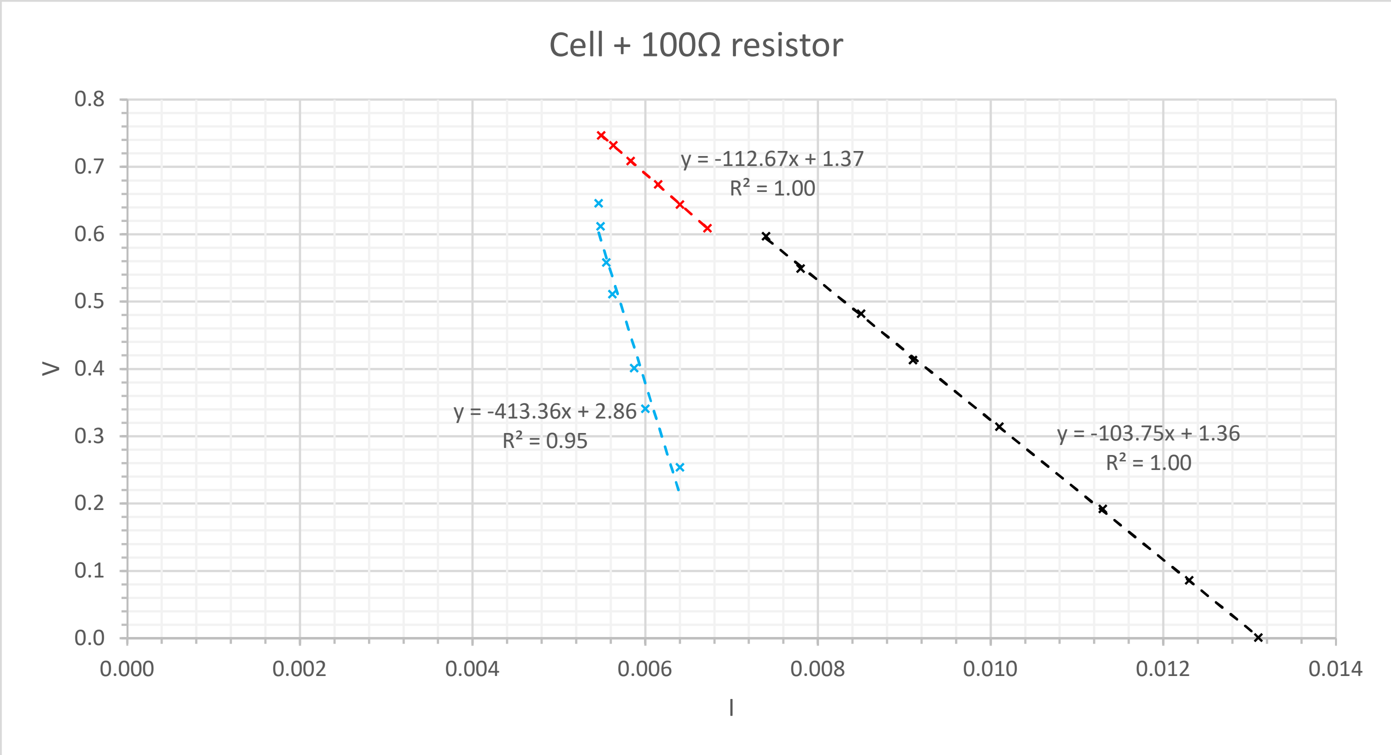
$$= IR + Ir$$

$$= V + Ir$$

$$V = -rI + \mathcal{E}$$

$$y = mx + c, \qquad \text{where } x = I, \qquad y = V, \qquad m = -r, \qquad c = \mathcal{E}$$

The value for internal resistance was 12.82Ω, at 2.8% compared to the real stated value of 2.9Ω, if I assume the 10Ω resistor to be exactly 10Ω. For calculations, it is safe to assume this, as it shouldn't affect the resistance of the actual cell, however, from an uncertainty perspective, subtracting the unknown value like this would count as double a "zero error" of uncertainty.



With the 100Ω resistor, I had the opposite problem to the 0Ω one. Over time during the experiment, the values I read for current were decreasing, resulting in a higher gradient (and higher calculated internal resistance) for each discrete set of data I measured. The black data set was the first I measured, also being the set with the gradient closest to what I predicted, having a percentage error of 2.9%. However, as the load resistance increased, the current decreased to the point where I had to change the multimeter range from ±2A (black set) to ±200mA (red set). From this point onwards, the current being displayed on the ammeter was gradually decreasing, as if the internal resistance was increasing over time. The blue set was recording the same points on the rheostat as the red set, just further on in time. Time being a factor implies heating has occurred, which is backed up by the fact that the perceived resistance increases over time. This is likely just because there's such a high resistance in the circuit that ion collisions are very likely. The red set has a percentage error of 336.9%, and the blue set has a percentage error of 10,705.5%. This effect only began to happen at higher set resistances on the rheostat (0-135Ω), as the circuit resistance was close to 135Ω + 100Ω + 2.9Ω + n, where n is the resistance increase due to heating.

Taking an average of the original results of each set, and ignoring factors such as heating, gives these values for internal resistance of the cell, and EMF of the circuit:

$$\frac{(3.21-0)+(12.82-10)+(103.75-100)}{3}=3.26\Omega$$

$$\frac{|3.26-2.9|}{2.9}=12.4\%$$

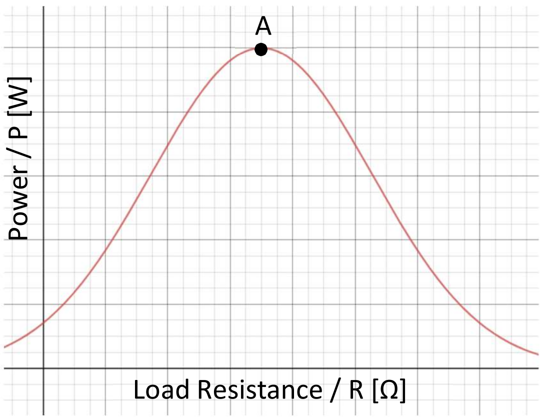
$$\frac{1.36+1.36+1.36}{3}=1.36\text{V}$$

The values for the EMF of the circuit was 1.36 to 3 significant figures, and the correlation coefficient for each line of best fit was 1.00 to 3 significant figures on every graph, showing my experiment was very precise. However, I had a percentage error of 12.4% for the average of my results, due to the actual value being so small. This is likely due to heating in general, as I had the circuit running at quite high resistances for long periods of time, causing heating in the wires and components. It would be beneficial to have used the circuit in short sessions, quickly recording a range of load resistances before leaving the components to cool, and doing this in staggered measurements, i.e. 100, 80, 60, 40, 20Ω, then 90, 70, 50, 30, 10Ω, etc.

I then set out to prove the max power theorem using both algebra, and my sets of data.

The Max Power Theorem

Let $R = load\ resistance$, $\quad r = internal\ resistance$, $\quad R + r = R_{total}$
Let $V = voltage\ over\ load$, $\quad I = current\ in\ circuit$, $\quad P = power\ of\ circuit$
Let $V = IR$, $\quad P = IV$, $\quad \mathcal{E} = IR_{total}$
Aim: Find the load resistance that maximises the power of the circuit.



At point A of maximum power, $\frac{dP}{dR} = 0$

$P(R) = IV = IIR = I^2R$, $\quad$ (substituting $V = IR$ into $P = IV$)

$\mathcal{E} = I(R + r)$, $\quad$ (substituting $R + r = R_{total}$ into $\mathcal{E} = IR_{total}$)

$I = \frac{\mathcal{E}}{R + r}$

$P(R) = \frac{\mathcal{E}^2 R}{(R + r)^2}$, $\quad$ (final equation for graph)

$\frac{dP}{dR} = P'(R) = 0$ at maximum power

when $f(x) = \frac{g(x)}{h(x)}$, $\quad f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h^2(x)}$

$f(x) = P(R)$, $\quad \therefore g(R) = \mathcal{E}^2 R$, $\quad g'(R) = \mathcal{E}^2$, $\quad h(R) = (R + r)^2 = R^2 + 2Rr + r^2$, $\quad h'(R) = 2R + 2r$

$\mathrm{P}'(\mathrm{R}) = \frac{\mathcal{E}^2(R + r)^2 - \mathcal{E}^2 R(2R + 2r)}{(R + r)^4} = 0$, $\quad$ (substituting into quotient rule)

either $\cancel{(R + r)^4 = \infty}$, $\quad$ (for this to be the case, either the internal or load resistance must be $\pm\infty$)

or $\mathcal{E}^2(R + r)^2 - \mathcal{E}^2 R(2R + 2r) = 0$

$\therefore \cancel{\mathcal{E}^2}(R + r)^2 = \cancel{\mathcal{E}^2}R(2R + 2r)$, $\quad$ (cancel out $\mathcal{E}^2$ values multiplicatively)

$(\cancel{R^2} + \cancel{2Rr} + r^2) = (\cancel{2}R^2 + \cancel{2Rr})$, $\quad$ (cancel out R^2 and $2Rr$ values additively)

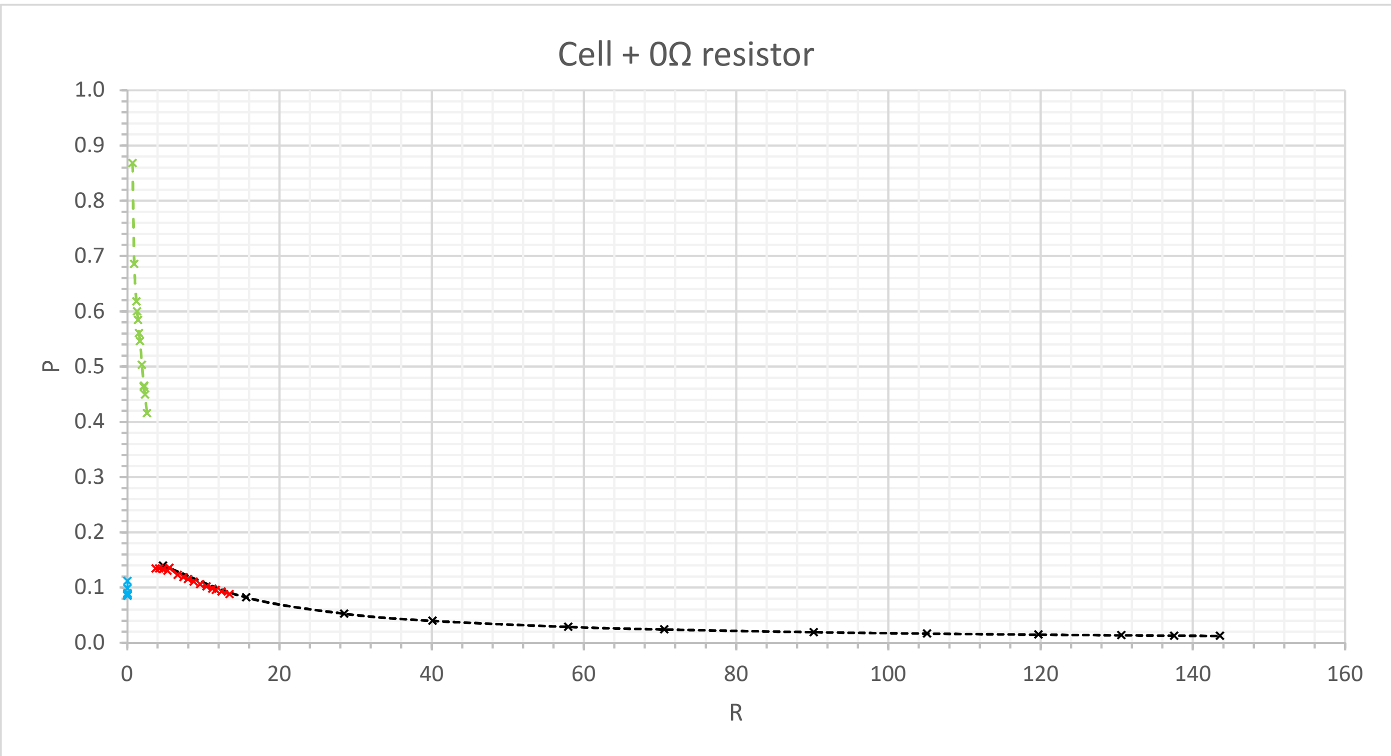
$R^2 - r^2 = 0$

$(R + r)(R - r) = 0$, $\quad$ (difference of two squares)

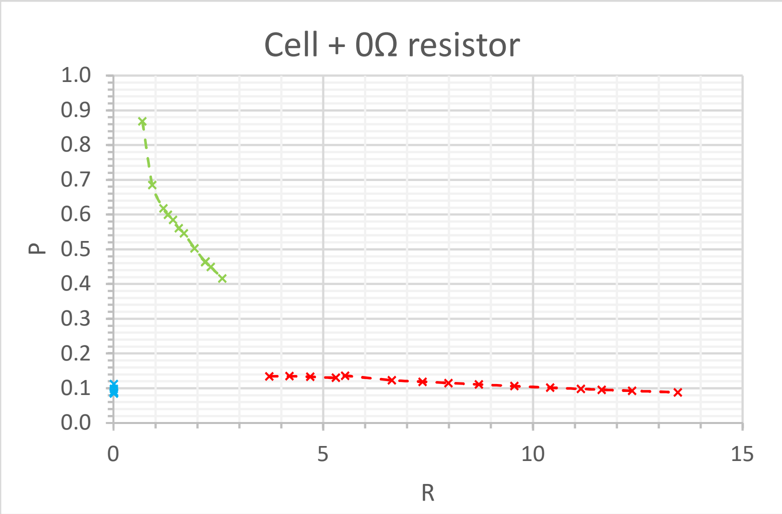
$\therefore$ either $\cancel{R = -r}$, $\quad$ (resistances must be positive to be valid in a circuit)

or $R = r$ at maximum power

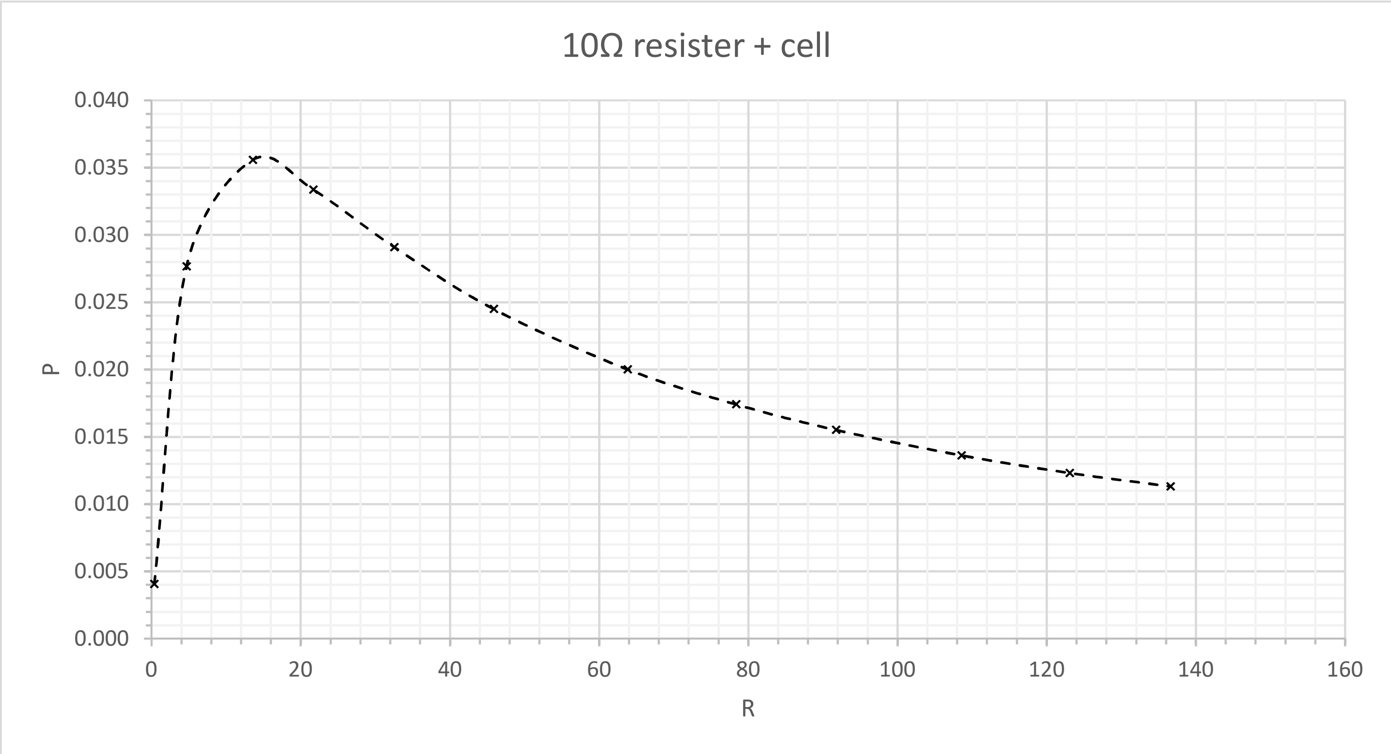
Based on this, the peak of the graph of power against load resistance should fall where the load resistance is exactly equal to the internal resistance of the cell. To get power and load resistance easily from my data, I used the two equations $P = IV$ and $V = IR$ as I had used them in the above derivation.



The first graph was hindered by the internal resistance being so low. Because of this, the peak of the graph happened very close to 0. I tried using a lower-range rheostat to get the blue and green datasets, with resistances as close to 0 as possible, however the current was so high in that range that it invalidated those two sets of data. Zooming in on the left portion of the graph gives this peak:



The green set of data on this graph should hold the peak, at roughly 2.9Ω, however the calculated power just keeps rising at lower resistances, stongly hinting that the current was too high to be accurately measured by the multimeter.

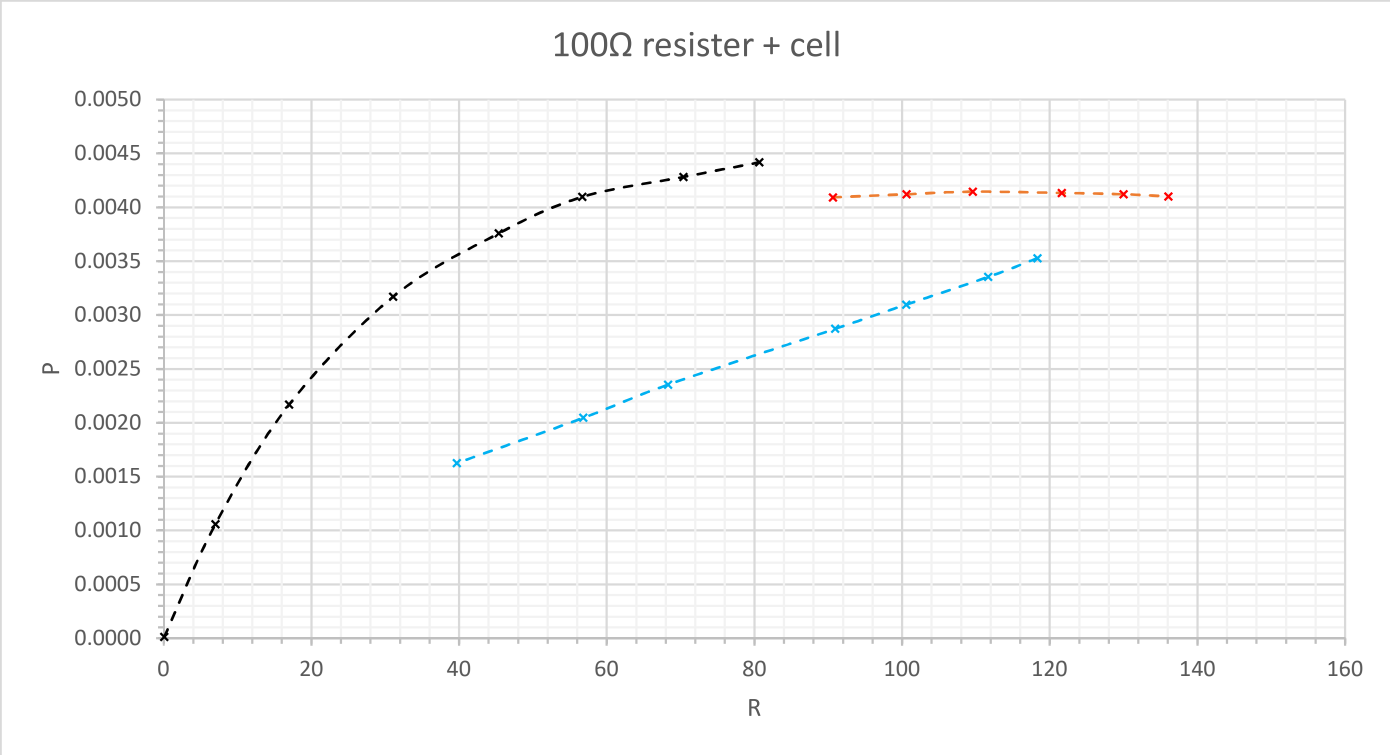


Like before, the second set of data provided the clearest trend, showing a peak around the point at 13.6Ω, which is close to the value of the internal resistance, at 12.9Ω. This graph demonstrates the max power theorem well, showing the power of the circuit increase at lower resistances, until the point where the load resistance is equal to the internal resistance, before gradually decreasing at higher and higher resistances after that point.

It is important to think about the range of data when plotting a graph like this. If you have points over too large a distance, the point of importance is small and doesn't cover much of the graph, however, with too few points surrounding the focus, or points around the focus spread out too much, the trend that defines the point is not well defined, and will be hard to pinpoint exactly. Both cases lead to a high uncertainty on the actual point, and detriment an experiment such as this. Another thing I could do would be to take the derivative of the equation that links power to resistance, and find where that graph crosses the x axis, at that would be more easily and well determined than a maximum point. The problem with finding an equation to take the derivative of, is that all other terms in the equation must be constants that do not rely on the load resistance to change. The closest equation that fits this is:

$$P(R) = \frac{\mathcal{E}^2 R}{(R+r)^2}$$

However, this equation is hard to differentiate, and my values for $\mathcal{E}$ and r are not well defined by my data set, meaning there would be a lot of uncertainty in this method.



The third power graph shows clearly how the heating effect reduces power in the graph, as the black set produces a clear curve that would reach a maximum point, but at the higher resistances, the power gradually reduces, as the actual circuit resistance is higher than is being calculated. There is a maximum point in the red dataset, around the point at 110Ω, which is around where the internal resistance was calculated to be based on the red set.

There is some use in making the internal resistance high in comparison to the load resistance, as seen towards the left edge of the final graph. It can be helpful to do this to decrease the allowed power in a circuit for a laboratory, or school, as it limits the power, and thus terminal potential difference in cells, reducing the chance of dangerous shocks.

My value for the internal resistance was higher than the value given by the manufacturer in most cases, likely due to the conditions under which I conducted the experiment. The voltmeter was not connected directly over the cell, meaning the resistance of the wires and the ammeter were not accounted for. Despite being small, at high currents they can affect a circuit's total resistance due to heating.

I was unable to calculate the percentage error of the EMF of the circuit, as I did not record the cell's actual value.

Experimental write-up by Laura Hannah – Prince William School