Measuring Gravity by Freefall

Deriving the 5 SUVAT equations of motion for uniform acceleration:

$$a = \frac{v - u}{t} \ (based\ on\ logic, change\ in\ velocity\ over\ time)$$

$$s = \frac{t(v + u)}{2} \ (based\ on\ logic, average\ velocity\ multiplied\ by\ time)$$

$$t = \frac{v - u}{a} = \frac{2s}{v + u} \ (rearranging\ each\ equation\ in\ terms\ of\ t\ then\ combining)$$

$$(v + u)(v - u) = 2as = v^2 - u^2 \ (multiplying\ out\ the\ fractions\ then\ expanding\ brackets)$$

$$v = \pm\sqrt{u^2 + 2as}\ or\ u = \pm\sqrt{v^2 - 2as} \ (rearranged\ for\ v\ and\ u\ respectively)$$

$$u = v - at = \frac{2s}{t} - v \ (rearranging\ each\ equation\ in\ terms\ of\ u\ then\ combining)$$

$$s = vt - 2^{-1}at^2 \ (combining\ like\ terms, then\ rearranging\ to\ find\ s)$$

$$v = u + at = \frac{2s}{t} - u \ (rearranging\ each\ equation\ in\ terms\ of\ v\ then\ combining)$$

$$s = ut + 2^{-1}at^2 \ (combining\ like\ terms, then\ rearranging\ to\ find\ s)$$

Method Comparison:

	Stopwatch	**Light Gate**	**Tickertape**	**Strobe Light**	**Camera**
Measurements	2 (initial & final)	2 ($final_1$ & $final_2$)	n (vibration frequency)	n (strobe frequency)	n (frame frequency)
Calculation	$\Delta v^2/2s$	$\Delta v^2/2s$	$\Delta f(t)/\Delta t$	$\Delta f(t)/\Delta t$	$\Delta f(t)/\Delta t$
Random Error	Start/Stop (t)	Vibration (t)	Vibration Lapse (t)	Strobe Lapse (t)	Frame Lapse (t)
	Height Measurement (l)	Length Measurement (l)	Marking Measurement (s)	Shutter Speed (s)	Shutter Speed (s)
Time	Time of Drop	Beam Break Time	Vibration Time	Strobe Time	Frame Time
Length	Distance Dropped	Length of Card	Marking Displacement	Mass Displacement	Calibration Length
Independent Var	Height	Height	Time	Time	Time
Dependent Var	Time	Velocity	Displacement	Displacement	Displacement
% Uncertainty	~101% (~1% trapdoor)	~3.0%	~4.0%	~2.3%	~0.8%

I listed the possible uncertainties for the 5 different methods I used in a table, making sure to consider as many factors as possible that could affect my results.

Stopwatch Random Errors:

Human error in the start and stop of the stopwatch (~±0.2s [average human reaction time] over ~0.8s)
Error [and zero error] in the measurement of the drop height (~±0.005m [cm scale division] over ~2m)

$$\frac{2 \cdot (\Delta s - u\Delta t)}{\Delta t^2} = \frac{2 \cdot \Delta s}{\Delta t^2} \; (t_{\%e} \cdot 4 + s_{\%e} \cdot 4 = \sim 100\% + \sim 1\% = \sim 101\%)$$

Light Gate Random Errors:

Possible vibration of environment affecting light gate position (negligible)
Error [and zero error] in the measurement of the card length (~±0.0005m [mm scale division] over 0.1m)
Error [and zero error] in the measurement of the drop height (~±0.005m [cm scale division] over ~2m)

$$\frac{\Delta v^2}{2 \cdot \Delta s} = \frac{\Delta \left(\frac{\Delta s}{\Delta t}\right)^2}{2 \cdot \Delta s} = \frac{\Delta s^2}{2 \cdot \Delta s \cdot \Delta t^2} \; (s_{1\%e} \cdot 4 + s_{2\%e} \cdot 4 = \sim 2.0\% + \sim 1.0\% = \sim 3.0\%)$$

Tickertape Random Errors:

Half of time when pointer is down (~±0.003s [frequency / 2] over ~0.6s)
Error [and zero error] in the length between marks (~±0.0005m [mm scale division] over 0.05m)

$$\frac{\Delta v}{\Delta t} = \frac{\Delta \left(\frac{\Delta s}{\Delta t}\right)}{\Delta t} = \frac{\Delta s}{\Delta t^2} \; (t_{\%e} \cdot 4 + s_{\%e} \cdot 2 = \sim 2.0\% + \sim 2.0\% = \sim 4.0\%$$

Strobe Light Random Errors:

Half of time when strobe is on (~±0.0006s [exposure time] over ~0.8s)
Error [and zero error] in the length between images (~±0.0005m [mm scale division] over 0.05m)

$$\frac{\Delta v}{\Delta t} = \frac{\Delta \left(\frac{\Delta s}{\Delta t}\right)}{\Delta t} = \frac{\Delta s}{\Delta t^2} \; (t_{\%e} \cdot 4 + s_{\%e} \cdot 2 = \sim 0.3\% + \sim 2.0\% = \sim 2.3\%)$$

Camera Random Errors:

Half of time when camera shutter is open (~±0.001s [average for camera] over ~0.8s)
Error [and zero error] in the calibration length (~±0.0005m [mm scale division] over 1m)

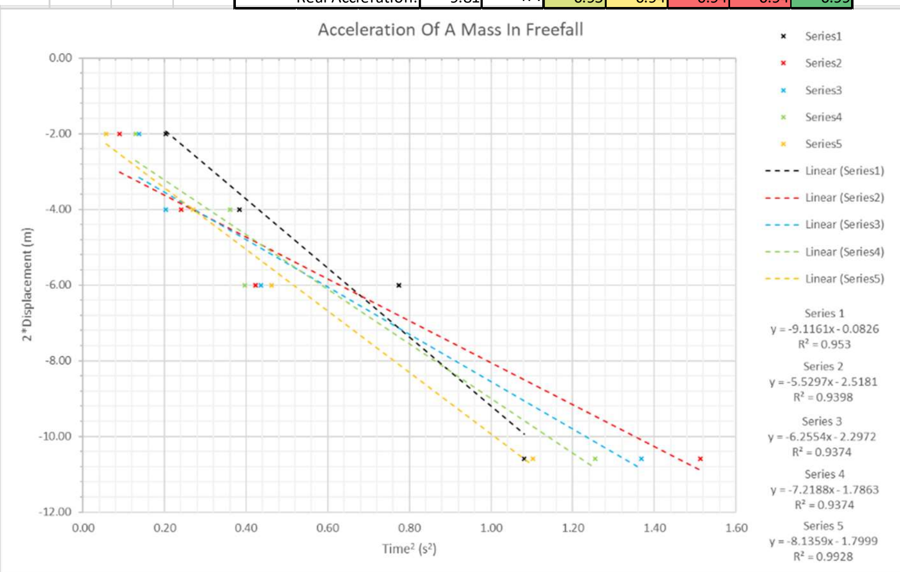
$$\frac{\Delta v}{t} = \frac{\Delta \left(\frac{\Delta s}{t}\right)}{t} = \frac{s}{t^2} \; (t_{\%e} \cdot 4 + s_{\%e} \cdot 2 = \sim 0.5\% + \sim 0.3\% = \sim 0.8\%)$$

Each method has its strengths and weakness, however, based on some arbitrary values, we can deduce a rough order of the precision of each one, provided the scientist can determine and negate or account for all types of systematic error. There are ways to reduce the random error in each investigation, and I will go over them in each method's section, but I tried to keep each to an average students' budget and time constraints for conducting each one.

Stopwatch:

This is the least precise of all 5 methods, almost solely because of human reaction time. I got my average value for human reaction time from http://censusatschool.ca/data-results/2007-08/average-reaction-time/ at 21/03/19 (Average reaction time, by age and dominant hand), which gave a large sample size of different ages, genders and dominant hands. Because for this method, a reaction must be made at the start and end of each reading, the human reaction time error had to be doubled immediately, dramatically decreasing precision. Along with this, the measurement for distance is often large, and cannot be calculated as precisely as the smaller measurements in other methods. Notably, in this and other methods with height as an independent variable, the formula used has 3 occurrences of displacement to calculate the acceleration, with the difference in displacement squared at the top of the fraction, and the doubled value at the bottom. This would also be a factor in the uncertainty of this method. There is an adaptation to this method, which uses a trapdoor and pressure pad to eliminate the human error. Interestingly, this timing method has one of the smallest random error out of any other method. My results showed two very different values for gravity. From my calculations, I found gravity to be -13.15ms^{-2}±6.64, showing an error percentage of 34.04%. From plotting a graph, I measured the value at -7.14ms^{-2}±3.61, with an error of 27.20%.

	Calculate											Graph								
s (m)	t (1)	t (2)	t (3)	t (4)	t (5)	t (6)	t (7)	t (8)	t (av)	a (ms^{-2})	%Diff	t^2 (1)	t^2 (2)	t^2 (3)	t^2 (4)	t^2 (5)	t^2 (6)	t^2 (7)	t^2 (8)	2s (m)
-1.00	0.45	0.3	0.37	0.36	0.24	0.31	0.38	0.35	0.35	-16.80	27.83%	0.20	0.09	0.14	0.13	0.06	0.10	0.14	0.12	-2.00
-2.00	0.62	0.49	0.45	0.6	0.52	-	-	-	0.54	-13.92	5.92%	0.38	0.24	0.20	0.36	0.27	-	-	-	-4.00
-3.00	0.88	0.65	0.66	0.63	0.68	0.68	0.67	0.57	0.68	-13.07	0.56%	0.77	0.42	0.44	0.40	0.46	0.46	0.45	0.32	-6.00
-5.30	1.04	1.23	1.17	1.12	1.05	1.03	1.05	-	1.10	-8.78	33.18%	1.08	1.51	1.37	1.25	1.10	1.06	1.10	-	-10.60
									Calculated Accleration:	-13.15	34.04%	Acceleration From Graph								
									Graph Acceleration:	-7.14	27.20%	-9.12	-5.53	-6.26	-7.22	-8.14				
									Real Accleration:	-9.81	R^2:	0.95	0.94	0.94	0.94	0.99				

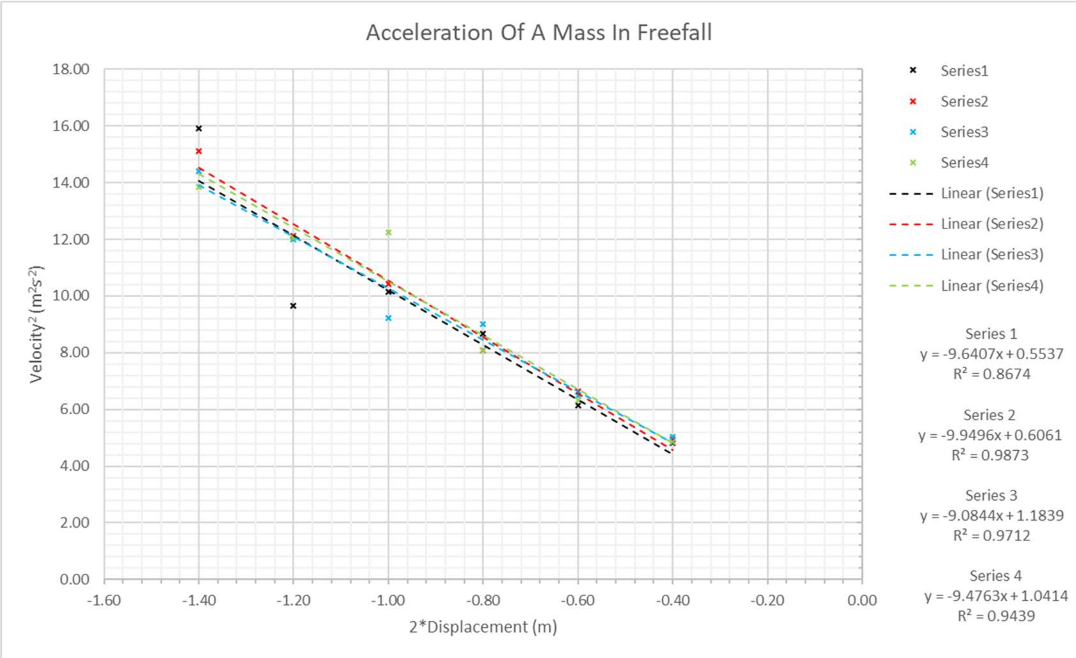


There are many things wrong with how I conducted this investigation. To improve my results, I should have taken readings from a much larger range of heights and mark the height against a plain surface to ensure I always drop from the exact height measured. I could also have used the trapdoor pressure pad method given more time and budget or taken more time to ensure the timing with the stopwatch is as accurate as humanly possible. Despite these improvements, this method is one of the least precise or accurate, and it would not be as worth making improvements to this method as the others. One interesting point is that my graph calculation and table calculations fell either side of the real value for gravity, showing a lack of precision, but a higher accuracy.

Light Gate:

The light gate is another way of almost negating the time errors, by recording a short portion of time electronically to calculate final velocity. In this case, the only random thing that could affect the perceived time would be minute vibrations in the environment affecting the position of the gate as the object goes through it. I deemed that this change would be so small that it wouldn't be a factor in the percentage error of the investigation. One downside of the light gate method is that two measurements must be made, one for the drop height and one for the length of the object dropped. Despite these factors, this is one of the most precise methods I tried and is the best that doesn't require post analysis of the data collected. The values of my results were much closer to the real value of gravity this time, falling at $-10.69ms^{-2}\pm0.32$ off the table, with an error of 8.99%, and $-9.31ms^{-2}\pm0.28$ from the graph, having an error of 5.08%.

	Calculation							Graph				
s (m)	v (1)	v (2)	v (3)	v (4)	v (av)	a (ms^{-2})	%Diff	v^2 (1)	v^2 (2)	v^2 (3)	v^2 (4)	2s
-0.20	-2.20	-2.22	-2.24	-2.20	-2.22	-12.28	14.86%	4.83	4.95	5.02	4.84	-0.40
-0.30	-2.48	-2.58	-2.55	-2.52	-2.53	-10.68	0.10%	6.15	6.64	6.50	6.34	-0.60
-0.40	-2.94	-2.85	-3.00	-2.84	-2.91	-10.58	1.04%	8.66	8.10	9.02	8.09	-0.80
-0.50	-3.19	-3.23	-3.04	-3.50	-3.24	-10.49	1.86%	10.15	10.44	9.23	12.24	-1.00
-0.60	-3.11	-3.48	-3.46	-3.47	-3.38	-9.53	10.86%	9.66	12.12	11.99	12.06	-1.20
-0.70	-3.99	-3.89	-3.80	-3.72	-3.85	-10.58	1.00%	15.93	15.12	14.40	13.85	-1.40
			Calculated Accleration:			-10.69	8.99%	Acceleration From Graph				
			Graph Acceleration:			-9.31	5.08%	-9.64	-9.95	-9.08	-9.48	
			Real Accleration:			-9.81	R^2:	0.8674	0.9873	0.9712	0.9439	

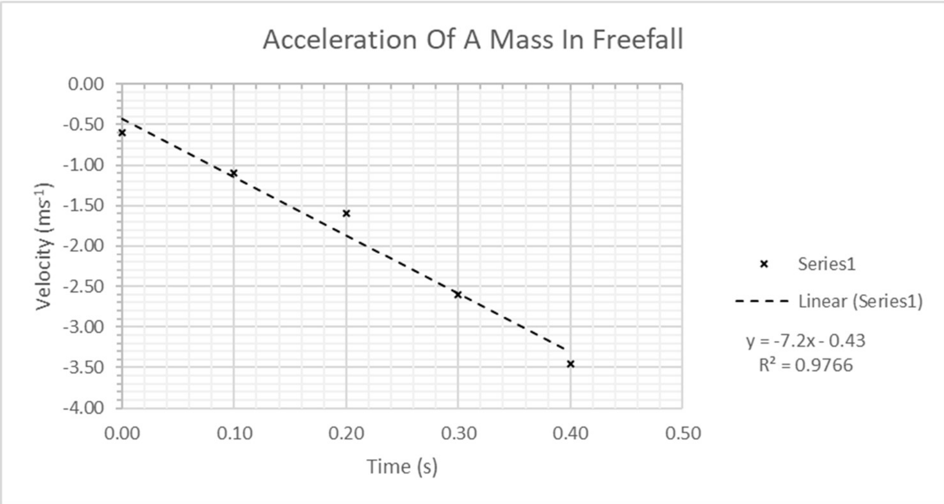


The values for gravity also fell either side of the true -9.8, suggesting that the precision is still low, and the accuracy is quite high. This can also be seen in the graph, where some points lie far away from the line, yet each line has a very similar gradient. The real value of g was outside both bounds calculated from this experiment, which greatly suggests that one or multiple systematic errors were made in getting the results. The main issue with this method was getting the falling object to move straight through the light gate without tilting. In the end I settled on using a curtain rail to guide the mass down through the beam, however this likely reduced the measured acceleration because of drag from friction on the sides of the rail.

Tickertape:

This is the first method of the three that use vibrations to mark as many points to measure from as possible. In this one, the vibrations are spaced evenly, which is problematic because random time errors are defined by the exposure, or "on-time" of the reading. This means that the possible time each mark could be placed is larger. Other than this, the measurement of the final marks is also the cause of human error, and because it's over a much smaller range, the error is higher than others would be. My results for this investigation weren't as accurate as others and had a very low precision compared to most of the methods. I calculated -7.25ms^{-2}±0.29, with quite a high error percent at 26.07%, and I measured -7.20ms^{-2}±0.29 from the graph, 26.58% off the true value.

t (s)	s (m)	v (ms^{-1})	a (ms^{-2})	%Diff	%Error
0.00	-0.01	-0.60	-		
0.10	-0.02	-1.10	-5.00	31.03%	49.01%
0.20	-0.03	-1.60	-7.50	3.45%	23.52%
0.30	-0.05	-2.60	-9.25	27.59%	5.68%
0.40	-0.07	-3.45	-		
Calculated acceleration:			-7.25		26.07%
Graph Acceleration:			-7.20		26.58%
Real Acceleration:			-9.81		

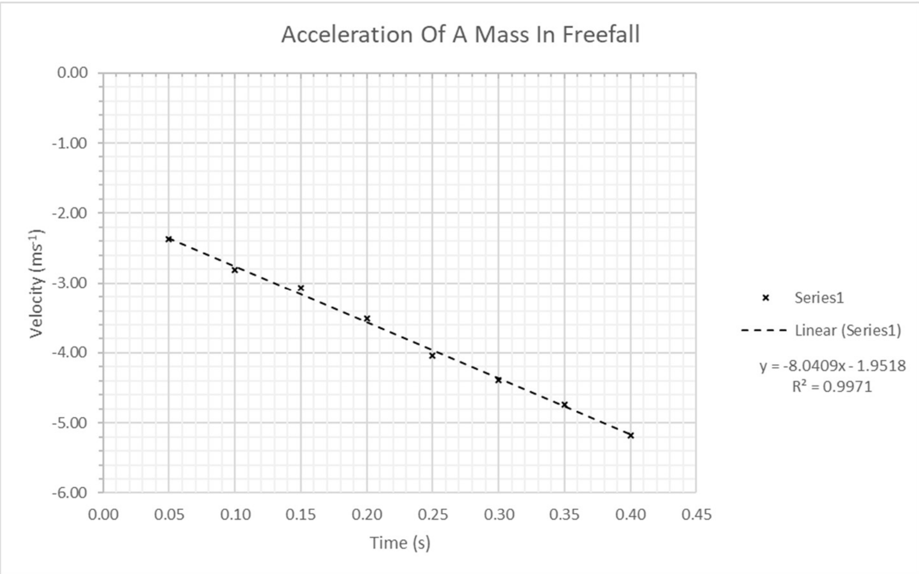


My results weren't precise or accurate, likely due to a lack of repeat trials for the former, and the needle and feedthrough contraption producing a lot of friction for the latter. This could be reduced by lubricating the surface the tape slides over and holding the tape vertically over the slot to allow it to drop without gravity counteracting the weight on the masses of the other end. This method is known to have high accuracy but low precision, which is likely because the needle that draws over the tape as it falls would limit the velocity of the masses, and the contraption that holds the tape in place would also produce a very large amount of friction. With this considered, this method would not be worth using to accurately calculate gravity and is the method with the fewest adaptations and potential for calculating gravity.

Stroboscope:

This method has many benefits that others do not. It can produce a high quality, low size image which shows all necessary data with a high contrast between the points of interest and the background. It is done by setting a camera to a very low shutter speed in a dark room and point a strobe light at a reflective mass as you drop it. Strobe lights are generally much more effective then video recording, for a multitude of reasons: For one, the light can be directed at the mass as it falls, to make it stand out much more against the environment. The flashes of light are also able to be much more precise and frequent, as with a camera, each frame must be processed after being received, and a camera shutter cannot be as precise as a light starting and stopping, meaning there will be much less blur in the position of the object. Each image of the object is also compounded onto a single picture, making it easier to measure the change in displacement without the use of outside programs. There is a downside to this, however, because when the object is travelling at a lower speed, the images of it can overlap, losing some initial results. Despite this, the initial velocity is not needed for this method, so measurements can be started and stopped at any point. My results for this method weren't good at all, however. I calculated -7.89ms^{-2}±0.18 (19.50% error) and measured -8.04ms^{-2}±0.18 from the graph (18.01% error). This is almost certainly because of systematic error, as the results on a graph followed a logarithmic curve when I plotted $2s/t^2$, and the error bounds of the measured and calculated values don't encompass the real value.

t (s)	s (m)	v (ms^{-1})	a (ms^{-2})	%Diff	%Error
0.00	-1.75	-	-		
0.05	-1.65	-2.37	-		
0.10	-1.52	-2.81	-7.02	11.11%	28.44%
0.15	-1.37	-3.07	-7.02	11.11%	28.44%
0.20	-1.21	-3.51	-9.65	22.22%	1.61%
0.25	-1.02	-4.04	-8.77	11.11%	10.55%
0.30	-0.81	-4.39	-7.02	11.11%	28.44%
0.35	-0.58	-4.74	-7.89	0.00%	19.50%
0.40	-0.33	-5.18	-		
0.45	-0.06	-	-		
Calculated Acceleration:			-7.89		19.50%
Graph Acceleration:			-8.04		18.01%
Real Acceleration:			-9.81		

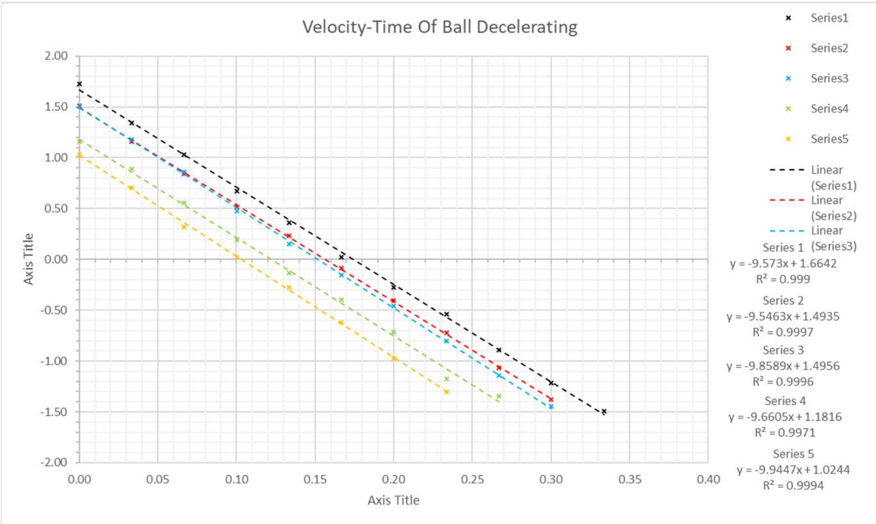


Camera:

This was my most accurate method, likely because I spent much more time perfecting it. My results were:

Measured Gravitiational Acceleration:	-9.72
Without Anomaly:	-9.74
Real Gravitational Acceleration:	-9.81
Percentage Error:	0.726%

	Period 1	Period 2	Period 3	Period 4	Period 5	Gravity	%Error
m:	-9.57	-9.55	-9.86	-9.66	-9.94	-9.72	0.917%
R^2:	0.9990	0.9997	0.9996	0.9971	0.9994		



Each period was a section of the velocity time graph, where m was the gradient of the line, and R^2 was given by excel as based on the average square area of space between each point and the line of best fit. I used the R^2 value in the average to calculate gravity based on this formula:

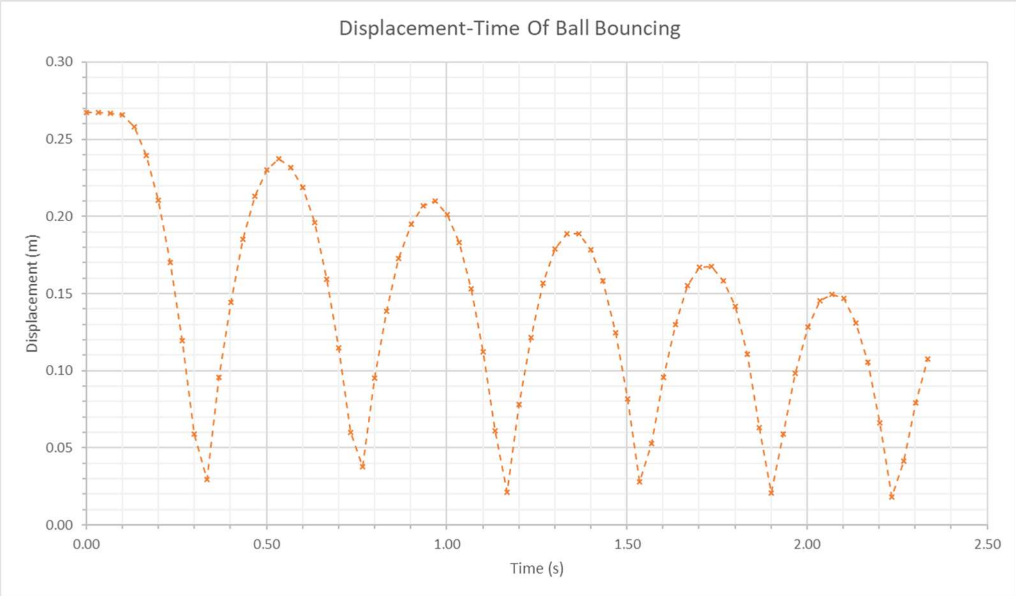
$$a = \frac{\sum_{a=1}^{5}(m_a \cdot R_a^2)}{\sum_{b=1}^{5}(R_b^2)}$$

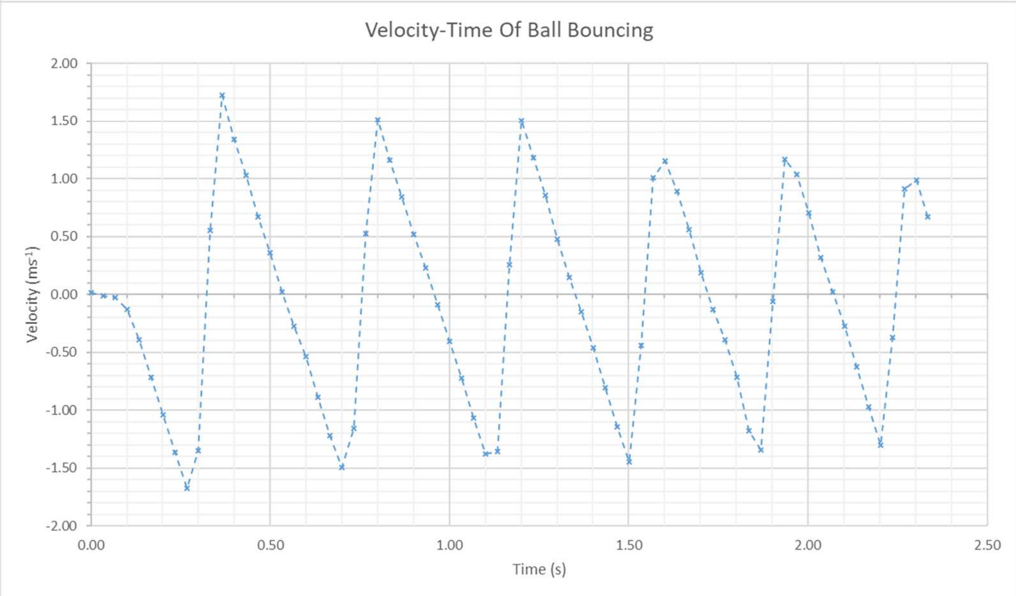
The percentage error was much lower on this experiment than any other, because I planned out my method and ensured several things:

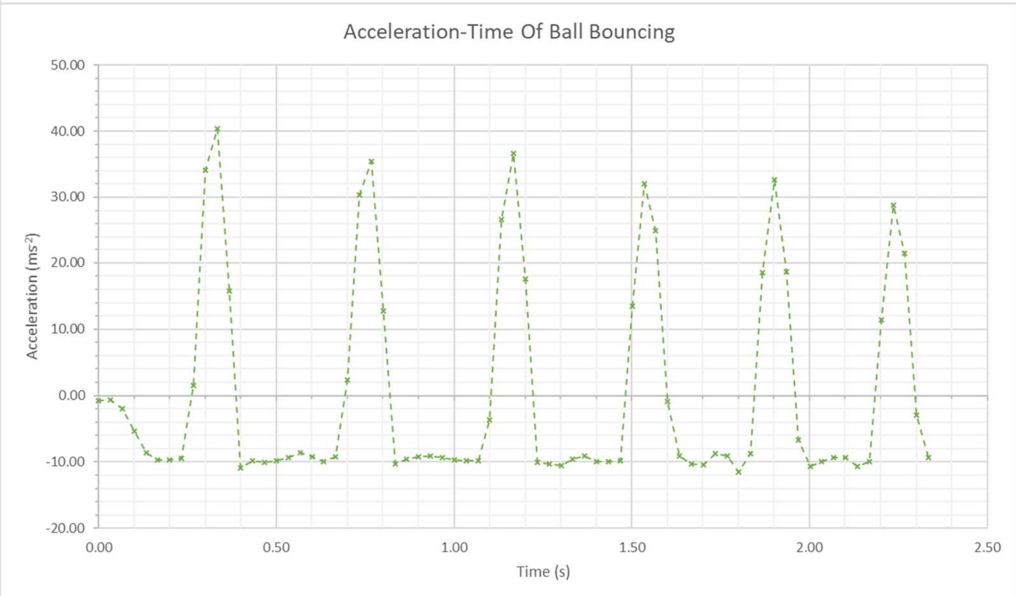
- The bounce surface was as flat as possible (spirit level, solid smooth material)
- The tracked plane was as close to the plane of motion of the ball as possible (two camera perspectives, marked horizontal and vertical planes, average horizontal position of ball)
- The calibration stick was as close to the real length as possible (two 10cm pieces of cotton parallel 10cm away from each other on both walls, measured based on plane of motion)
- Equal frame rates and positions on both cameras (same camera model, started at the same time)
- As smooth a ball and centre of mass as possible (spherical, equal density throughout)
- Highest visibility of mass for tracking (direct light on ball from both angles, high shutter-speed)

If I were to repeat the investigation with a higher budget, I would ensure a few things: I would ensure the lenses of the cameras are as close to the ground as possible. This would decrease the effects of the ball changing plane as it bounces and may help to determine the exact axis that the camera is recording. I would also make sure I record at as high and consistent a frame rate as possible. This would increase the precision of readings and make averages for calculations. It would also help detect anomalies, such as the small error at $t=3.87$ that is almost undetectable on the displacement-time graph but shows up at least -1ms^{-2} off the predicted value for the acceleration-time graph. I would also get a rubber ball custom-made to be seamless and have a consistent density, so the centre of mass is the exact centre of the ball.

I made 3 other graphs to accurately analyse my data, each one the derivative of the next:







The displacement-time graph is the derivative of the velocity-time graph, and the velocity-time graph is the derivative of the acceleration-time graph. This means the gradient of each graph is determined by the height above the x axis of the next graph along, and the area under the graph between two points is determined by the change in height of those two points on the graph prior, in the chain:

$$displacement_{(ms^0)} - velocity_{(ms^{-1})} - acceleration_{(ms^{-2})} \left(-jerk_{(ms^{-3})} - jounce_{(ms^{-4})}\right)$$

This sequence shows the range of n from 0 to 4, resulting in units of ms^{-n}. This is because each next unit is the previous one divided by s, allowing each one to form the differential equation with the previous in the sequence:

$$f_{[n+1]}(t) = \frac{f_{[n]}(t + dt) - f_{[n]}(t)}{dt}$$

Each spike in the acceleration-time graph represents the moment the ball hits the ground, as it is almost instantly accelerated to travelling with a high velocity in the opposite direction. Outside of the spikes, the acceleration is shown as just under -$10ms^{-2}$, gravitational acceleration on earth. The velocity-time graph has a gradient of just under -10, as the ball's velocity decreases due to gravity alone. At the moments the ball hits the ground, the velocity-time graph spikes back up to just below the absolute value it was at before hitting the ground, showing that some energy has been lost. When the velocity crosses 0, the ball is at its stationary point in the air. This is roughly half way between the highest velocity after the previous bounce, and the highest absolute velocity before the next bounce. The displacement-time graph shows curves of decreasing height for each bounce. The height of each curve decreases due to the loss of energy the moment the ball bounces. The point where the ball bounces is synchronised with the velocity-time graph rapidly changing from very low to very high, causing the gradient of the displacement-time graph to change. Because of the changing velocity of the ball, the gradient of the displacement is always changing, forming an arc. The area underneath the velocity time graph between two points is the displacement of the ball between those two points, based on the rule of integration. Also, the area under the acceleration time graph between two points is equal to the change in velocity between those two points.

Each time the ball bounces, the measured displacement is always just above 0. This is for 3 reasons: First, the ball is a full 3D object with a diameter. It is treated as a centre of mass, so it isn't considered to be touching the x-axis at the lowest point. Second, the ball does not always bounce on the same plane, and because the camera isn't on the ground, the ball being closer or further on hitting the ground is interpreted as being lower down or higher up. Lastly, the camera could only record at 30 frames per second, so in most cases, the lowest frame the ball was captured on was not exactly on the ground.

Because I used the tracker program, the only significant uncertainties were of displacement, not time, due to the regular interval of frames taken by the 2 cameras. There could be an issue of lag of the cameras causing a slight delay to a few frames, however this is almost undetectable at 30 frames per second, so can be ignored when the scale of measurements provides enough uncertainty to make the small frame lags redundant.

Experimental write-up by Laura Hannah – Prince William School