Homework 0

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Kronecker Product

Problem 1 For a randomly generated $\mathbf{A} \in \mathbb{C}^{N\times N}$ and $\mathbf{B} \in \mathbb{C}^{N\times N}$, evaluate the computational performance (run time) of the following matrix inversion formulas:

(a) for $N \in \{2, 4, 8, 16, 32, 64\}$

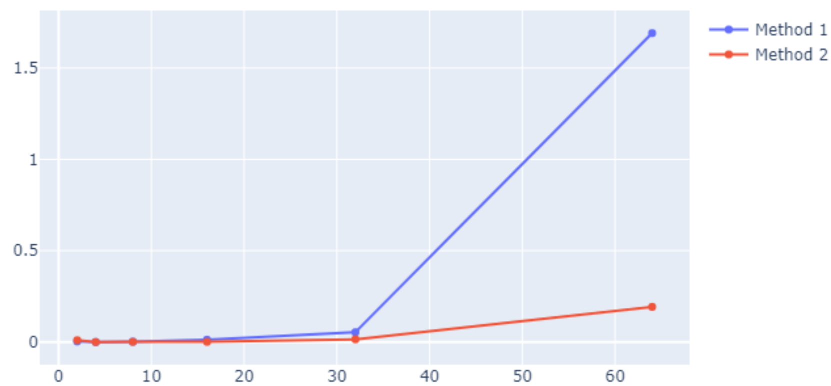
- Method 1: $(\mathbf{A}_{N\times N} \otimes \mathbf{B}_{N\times N})^{-1}$
  Solution:

```
N_list = [2, 4, 8, 16, 32, 64]
Method1_times = []

for n in N_list:
    A = randn(n, n)
    B = randn(n, n)

    start_time = time.time()

    A_x_B = kron(A, B)

    A_x_B_inv = inv(A_x_B)

    end_time = time.time()

    Method1_times.append(end_time-start_time)
```

- Method 2: $\mathbf{A}^{-1}_{N\times N} \otimes \mathbf{B}^{-1}_{N\times N}$
  Solution:

```
N_list = [2, 4, 8, 16, 32, 64]
Method2_times = []

for n in N_list:
    A = randn(n, n)
    B = randn(n, n)

    start_time = time.time()

    A_inv = inv(A)
    B_inv = inv(B)

    A_x_B = kron(A, B)

    end_time = time.time()

    Method2_times.append(end_time-start_time)
```

- Comparison of results
  The graph below shows the run time difference between the methods



(b) for $K \in \{2, 4, 6, 8, 10\}$

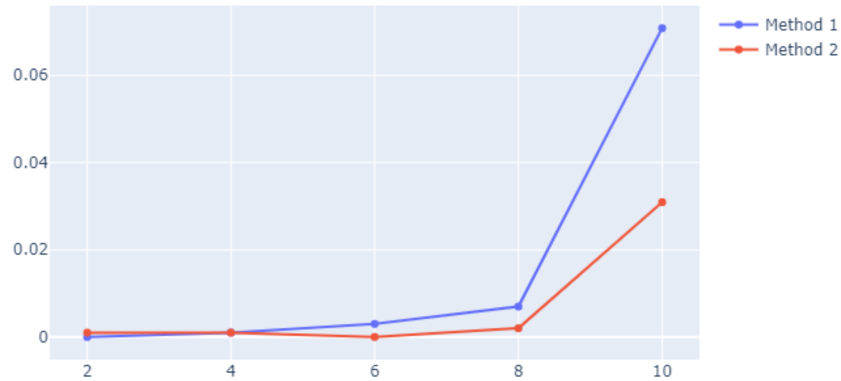
- Method 1: $\left(\mathbf{A}^{(1)}_{2\times 2} \otimes \mathbf{A}^{(2)}_{2\times 2} \otimes ... \otimes \mathbf{A}^{(K)}_{2\times 2}\right)^{-1} = \left(\otimes_{i=1}^{K} \mathbf{A}^{(i)}_{2\times 2}\right)^{-1}$
  Solution:

```
K_list = [2, 4, 6, 8, 10]
Method1_times = []

for k_value in K_list:

    A_start = randn(2, 2)

    start_time = time.time()
    for k in range(0, k_value):
        if k == 0:
            A = A_start
        else:
            A = kron(A, A_start)

    A = inv(A)
    end_time = time.time()

    Method1_times.append(end_time-start_time)
```

- Method 2: $\left(\mathbf{A}^{(1)}\right)^{-1}_{2\times 2} \otimes \left(\mathbf{A}^{(2)}\right)^{-1}_{2\times 2} \otimes ... \otimes \left(\mathbf{A}^{(K)}\right)^{-1}_{2\times 2} = \otimes_{i=1}^{K} \left(\mathbf{A}^{(i)}\right)^{-1}_{2\times 2}$
  Solution:

```
K_list = [2, 4, 6, 8, 10]
Method2_times = []

for k_value in K_list:

    A_start = randn(2, 2)

    start_time = time.time()
    for k in range(0, k_value):
        if k == 0:
            A = inv(A_start)
        else:
            A = kron(A, inv(A_start))

    end_time = time.time()

    Method2_times.append(end_time-start_time)
```

- Comparison of results
  The graph below shows the run time difference between the methods



Problem 2 Let $eig(\mathbf{X})$ be the function that returns the matrix $\mathbf{\Sigma}_{K\times K}$ of eigenvalues of $\mathbf{X}$. Show algebraically that $eig(\mathbf{A}\otimes\mathbf{B}) = eig(\mathbf{A})\otimes eig(\mathbf{B})$.

<u>Hint</u>: Use the property

$$(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = \mathbf{AC}\otimes\mathbf{BD} \tag{1}$$

Solution: Let $\mathbf{A}\in\mathbb{C}^{M\times M}$ with eigenvector matrix $\mathbf{V}$ and eigenvalue matrix $\mathbf{\Sigma}_A$, and $\mathbf{B}\in\mathbb{C}^{N\times N}$ with eigenvector matrix $\mathbf{U}$ and eigenvalue matrix $\mathbf{\Sigma}_B$. Their respective eigendecompositions are:

$$\mathbf{AV} = \mathbf{V\Sigma}_A \rightarrow \mathbf{A} = \mathbf{V\Sigma}_A\mathbf{V}^{-1} \tag{2}$$

$$\mathbf{BU} = \mathbf{U\Sigma}_B \rightarrow \mathbf{B} = \mathbf{U\Sigma}_B\mathbf{U}^{-1} \tag{3}$$

Taking the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ and applying property (1) and substitutions (2) and (3)

$$\begin{aligned}\mathbf{A}\otimes\mathbf{B} &= (\mathbf{V\Sigma}_A\mathbf{V}^{-1})\otimes(\mathbf{U\Sigma}_B\mathbf{U}^{-1})\\ &= (\mathbf{V\Sigma}_A\otimes\mathbf{U\Sigma}_B)(\mathbf{V}^{-1}\otimes\mathbf{U}^{-1})\\ &= (\mathbf{V}\otimes\mathbf{U})(\mathbf{\Sigma}_A\otimes\mathbf{\Sigma}_B)(\mathbf{V}^{-1}\otimes\mathbf{U}^{-1})\end{aligned}$$

We can make use of the property $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$ to have

$$\mathbf{A} \otimes \mathbf{B} = (\mathbf{V} \otimes \mathbf{U})(\mathbf{\Sigma}_A \otimes \mathbf{\Sigma}_B)(\mathbf{V} \otimes \mathbf{U})^{-1}$$

This is the eigendecomposition of $\mathbf{A} \otimes \mathbf{B}$, where $(\mathbf{\Sigma}_A \otimes \mathbf{\Sigma}_B)$ is its eigenvalue matrix $\mathbf{\Sigma}_{\mathbf{A} \otimes \mathbf{B}}$. Thus, $eig(\mathbf{A} \otimes \mathbf{B}) = eig(\mathbf{A}) \otimes eig(\mathbf{B})$.