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SYLLABUS INTRODUCTION TO LOGIC

Rineke Verbrugge,
Joris IJsselmuiden &
Barteld Kooi

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Contents

Part A

1 Introduction

This syllabus complements the book “Language, Proof and Logic” by Barwise and Etchemendy. It provides some alternative notations, additional exercises and background information. These are mainly drawn from the following existing sources:

- Syllabus *Parvulae Logicales* by Phoebe Kradavis
- Syllabus *Elementaire Verzamelingenleer* by Riegholt Hilbrands

Special thanks goes out to Víctor Sánchez for providing some translation exercises, Erik Krabbe for his remarks about an earlier version of this syllabus, Jan Misker for providing solutions to some of the exercises in this syllabus, Ayla Kangur and Sybren Jansen for helping in translating the syllabus, and Jan van Houten and René Mellema for providing some new exercises about translations and Horn sentences.

LPL Introduction pp. 1 – 11

The introduction of “Language, Proof and Logic” describes what logic entails, why we study logic, and provides instructions on how to use the book properly.

2 Atomic sentences

Alternative definition of atomic sentences pp. 23 LPL

The book LPL (Language, Proof and Logic) is exceptional in the way that atomic sentences of propositional logic are defined. In most books the convention is to write atomic sentences of propositional logic with a single uppercase letter. Instead of writing `Taller(claire, max)` those books simply write it as H provided with the translation key:

H : Claire is taller than Max.

A similar notation with lowercase instead of uppercase letters often occurs as well. You will encounter several exercises in this syllabus in which you will learn to work with these more conventional atomic sentences.

Other books provide a clearer distinction between propositional and first-order logic than LPL does. In propositional logic, arguments are studied in which the validity depends on the meaning of connectives like *not*, *and*, *or*, *if ... then* and *if and only if*. Everything else is disregarded. Other books are therefore not interested in the construction of atomic sentences: abbreviating them with a single uppercase letter is sufficient.

In first-order logic, arguments of which the validity depends on the meaning of quantifiers, including *all* and *some*, is studied as well. In this case the structure of the atoms is of great importance. In LPL, logic is presented as a single system in which atomic propositions are decomposed using predicate symbols and constants. For the final exam it is important that you can distinguish between both logics. You should know that in propositional logic, atomic sentences are represented by a single uppercase letter and that in first-order logic, these sentences are further decomposed using predicate symbols and constants.

Formal proofs using Fitch

To create formal proofs you can make use of Fitch, software which is already installed on the university computers. There you will also find practice exercises and an extensive user manual.

EXERCISE 2.1

Given the premises SameColumn(a,b) and a = c, use Fitch to prove that SameColumn(c,b) is the case. Use the same premises to prove that c = c.

3 Propositional logic: conjunction, disjunction and negation

3.1 Logical possibility and logical truth

Spurious rows pp. 94 – 103 LPL

On page 94 of LPL, Barwise and Etchemendy make the distinction between a tautology and a logical truth. A row in a truth table is called a *spurious row* if that row cannot possibly be true because of the meaning of the atomic sentences in the sentence. Suppose that the atomic sentence A represents $d = d$. Assignments which make A false cannot really exist. All rows containing such assignments are spurious rows.

If you want to check whether a sentence is *logically true*, you first cross out all spurious rows and check whether the formula is true in all remaining rows. If you just want to know whether a formula is *logically possible*, you first cross out all the spurious rows and check whether the sentence is true in at least one remaining row.[1]

EXERCISE 3.1

Suppose A, B and C are atomic sentences.

a. Build a truth table for the sentence $\neg(A \wedge B) \leftrightarrow (\neg C \vee \neg B)$.

b. Is the sentence $\neg(A \wedge B) \leftrightarrow (\neg C \vee \neg B)$ a tautology? Explain your answer.

For c, d and e suppose that the atomic sentences A and C are logically equivalent, for example A stands for `Larger(e,d)` and C stands for `Smaller(d,e)`.

c. Specify which rows of the truth table constructed in part a are spurious rows.

d. Is the sentence $\neg(A \wedge B) \leftrightarrow (\neg C \vee \neg B)$ logically true? Explain your answer.

[1] In other logic books, the term 'satisfiability' is often used instead of 'possibility'.

e. Is the sentence $\neg(A \wedge B) \leftrightarrow (\neg C \vee \neg B)$ logically possible? If so, provide a truth assignment which makes the sentence true. If not, explain why.

3.2 Formal proofs

Previously proven theorems

During lectures and tutorials, we occasionally refer to previously proven theorems. It is important to understand these occurrences. During the final exam, however, it is not allowed to use them. With Fitch you can refer to these previously proven theorems by using the Taut Con rule. For propositional logic these are among others the following:

a. $P \vee \neg P$ without premises

b. $\neg(P \wedge Q)$ from premise $\neg P \vee \neg Q$

c. $\neg P \vee \neg Q$ from premise $\neg(P \wedge Q)$

d. $\neg(P \vee Q)$ from premise $\neg P \wedge \neg Q$

e. $\neg P \wedge \neg Q$ from premise $\neg(P \vee Q)$

f. Q from premises $P \vee Q$ and $\neg P$

g. $\neg\neg P$ from premise P

EXERCISE 3.2

Prove by natural deduction, if necessary use Fitch:

a. $\neg(A \wedge \neg A)$

b. B from premise $A \wedge \neg A$

EXERCISE 3.3

Prove by natural deduction, if necessary use Fitch:

a. $A \vee B \vee C$ from premise $A \wedge C$

b. B from premise $B \vee (A \wedge B)$

c. B from premise $B \wedge (A \vee B)$

d. $B \wedge (A \vee B)$ from premise B

e. C from premise $(A \wedge C) \vee (B \wedge C)$

3.3 Translations

EXERCISE 3.4

Translate the following sentences into propositional logic, using capital letters to represent atomic formulas. For each sentence, show as much structure as possible and provide a translation key.

a. You can't take the sky from me.

b. Mal is not interested in being liked by anyone, although he is very protective of his crew.

c. Inara did not say that barging into her shuttle was manly and impulsive, but Mal needs Kaylee in the engine room.

d. Even though Simon did a good job coming up with the plan, Jayne does not like him or want to keep him around.

e. Serenity is a deathtrap, or it will keep you out of reach of the alliance and living like real people.

EXERCISE 3.5

Translate the following sentences into first-order logic, providing one translation key giving the translations of constants and predicate symbols for the whole exercise. You do not have to use quantifiers in this exercise.

a. Bunny is Edmund.

b. Neither Richard nor Bunny is taller than Henry.

c. Unless Richard is mistaken, Julian is a wonderful person and Henry is a little strange.

d. Cloke is a friend of Bunny, who is a friend of Henry.

e. Henry may have pushed Bunny, but Richard did nothing to stop him.

f. Either Camilla and Charles really like each other, or they don't, and in that case they're very good actors.

4 Propositional logic: material conditional and biconditional

4.1 Formal proofs with conditionals

EXERCISE 4.1

Prove by natural deduction, if necessary use Fitch:

a. $A \to A$

b. $(A \wedge \neg A) \to B$

c. $(A \to \neg A) \to \neg A$

d. $((A \to B) \wedge (A \to \neg B)) \to \neg A$

EXERCISE 4.2

Prove by natural deduction, if necessary use Fitch:

a. $A \to (B \wedge C)$ from premises $A \to B$ and $A \to C$

b. $A \to \neg\neg A$, without premises

c. $A \to C$ from premises $(A \wedge B) \to C$ and B

EXERCISE 4.3

Prove by natural deduction, if necessary use Fitch:

a. $A \to (B \to C)$ from premise $B \to (A \to C)$

b. $A \to D$ from premises $A \to (B \vee C)$, $B \to D$ and $C \to D$

c. $\neg C$ from premises $\neg(A \wedge B)$, $C \to A$ and $C \to B$

d. $(A \vee B) \leftrightarrow A$ from premise $\neg B$

e. B from premises $A \to B$ and $\neg A \to B$

f. B from premises A, $(A \wedge C) \to B$ and $((A \to B) \to B) \to (A \to C)$

g. $(A \to B) \vee (A \wedge \neg B)$, without premises

4.2 Translations with conditionals

EXERCISE 4.4

Translate the following sentences into propositional logic, using capital letters to represent atomic formulas. For each sentence, show as much structure as possible and provide a translation key.

a. Donald wants to build a wall unless he has to pay for it.

b. If the wall is not built it's not Donald's fault.

c. Donald can only build a wall if Mexico and California don't try to stop him.

d. Donald doesn't know much about walls, but he does know a thing or two about business as well as about money.

e. The wall will be finished if and only if Donald starts in time, and if he starts in 2019, the wall will not be finished.

5 First-order logic: introduction to quantifiers

5.1 Translate

EXERCISE 5.1

Translate the following sentences into first-order logic. Provide as much structure as possible and give a translation key for each domain of discourse. First check which atomic sentences you will need per domain of discourse.

Domain of discourse 1: All animals.

a. Some primates are not old world monkeys.

b. No primate is an ape and not intelligent.

c. All flying mammals are bats.

d. If an animal is an ape, it is not an old world monkey.

e. Bats are flying mammals, but they are not intelligent.

Domain of discourse 2: All companies and licenses.

f. Not every company can finance themselves.

g. Companies that do not work with us are our competitors.

h. Companies that cooperate with us make profit. Some companies that do not cooperate with us also make profit.

i. There is a license that Microsoft has not granted to Sun.

j. Microsoft has granted no license to Sun.

Domain of discourse 3: All computer software.

k. Everything is open source or closed source.

l. Everything is open source or everything is closed source.

m. Some open source and closed source computer programs are affordable.

n. No open source computer program is unaffordable.

EXERCISE 5.2

Translate the following sentences into first-order logic. Provide as much structure as possible and give a translation key for each sentence. The domain of discourse is the set of integers: $\dots, -2, -1, 0, 1, 2, \dots$

a. 2 is an even number.

b. 2 is greater than 3.

c. If 5 is greater than 4, then 5 is also greater than 3.

d. Every number greater than 4 is also greater than 3.

e. There exists an even number greater than 3.

f. No number is greater than itself.

5.2 Formal proofs with quantifiers

EXERCISE 5.3

Prove by natural deduction, if necessary use Fitch:

a. $\forall x(P(x) \to R(x))$ from the premises $\forall x(P(x) \to Q(x))$ and $\forall x(Q(x) \to R(x))$

b. $\forall x P(x) \to \forall x Q(x)$ from the premise $\forall x(P(x) \to Q(x))$

c. $\forall x P(x) \land \forall x Q(x)$ from the premise $\forall x(P(x) \land Q(x))$

d. $\forall x(P(x) \land Q(x))$ from the premise $\forall x P(x) \land \forall x Q(x)$

e. $\exists x(P(x) \lor Q(x))$ from the premise $\exists x P(x) \lor \exists x Q(x)$

f. $\exists x P(x) \lor \exists x Q(x)$ from the premise $\exists x(P(x) \lor Q(x))$

g. $\exists x P(x) \land \exists x Q(x)$ from the premise $\exists x(P(x) \land Q(x))$
(Take care: the reverse is not valid.)

h. $\neg\exists x(P(x) \land R(x))$ from the premises $\forall x(P(x) \to Q(x))$ and $\neg\exists x(Q(x) \land R(x))$

EXERCISE 5.4

Prove by natural deduction, if necessary use Fitch:

a. $c = i$ from the premises $\forall x(O(b, x) \to x = i)$ and $O(b, c)$.

This is a possible formalization of the riddle “Brothers and sisters I have none, but that man’s father is my father’s son. Who am I?”, with the following translation key:
b : my father
c : that man’s father
i : me
$O(x, y)$: x is a parent of y

b. $\exists x H(a, x)$ from hypothesis $\forall x H(x, a)$

6 First-order logic: sentences with multiple quantifiers

EXERCISE 6.1

Translate the following sentences into first-order logic. Provide as much structure as possible and give a translation key for each domain of discourse. First check which atomic sentences you will need per domain of discourse.

Domain of discourse 1: All primates.

a. If a primate is not an ape, it is an old world monkey.

b. Kanzi acquires something from Washoe, but not from Matata.

c. Every ape loves another ape.

d. Kanzi does not steal food from another primate.

e. No ape is smarter than Washoe, except Kanzi.

Domain of discourse 2: All humans.

f. Every student knows a student that admires him/her.

g. Every student knows a student that admires himself/herself.

h. Some people are kind and some are unkind.

i. Nobody admires everyone.

j. No one admires someone who admires everyone who admires no one.

k. Someone whom no one sees is invisible.

l. Someone who sees no one is blind.

m. Every lecturer is satisfied if even one student is satisfied.

n. John loves nobody except himself.

Part B

7 Introduction

Part B of this syllabus explores some topics which are of special interest for students Artificial Intelligence, Computing Science and Mathematics. For example, normal forms are essential for logic programming, and the extra attention to models of first-order logic is useful for courses such as Advanced Logic, Artificial Intelligence, Logical Aspects of Multiagent Systems, Program Correctness and Discrete Structures.

8 Set theory

Informal introduction to sets pp. 37 – 38 LPL

You have encountered many different sets in mathematics classes in high school and university, for example $\mathbb{N}$, the set of natural numbers $0, 1, 2, \ldots$, and $\mathbb{Z}$, the set of integers $\ldots, -2, -1, 0, 1, 2, \ldots$. Strangely enough one cannot precisely define the concept of "set". However, since the end of the 19th century Cantor, Zermelo, Fraenkel and other mathematicians have made axiomatic descriptions of set theory. More on this topic can be found in Chapter 15.9 of LPL (not part of the course material). We provide a short but more intuitive introduction to the most common concepts encountered in set theory, including a number of examples and exercises.

An example of a set is $\{1, 2, 3\}$. This set contains the *members* (or *elements*) 1, 2 and 3. You write $1 \in \{1, 2, 3\}$ to denote that 1 is a member of the set $\{1, 2, 3\}$ (and similarly for the other members), but $4 \notin \{1, 2, 3\}$. Sets can have all sorts of members; for example $\{Aristoteles, Plato\}$ is the set that contains exactly two philosophers, namely Aristoteles and Plato, as its members. Sets can also contain other sets as their members. For example, the set $\{1, \{2, 3\}\}$ contains two members: $1 \in \{1, \{2, 3\}\}$ and $\{2, 3\} \in \{1, \{2, 3\}\}$.

8.1 Set operations and statements

In the descriptions below, A and B are both sets.

- A and B are *equal*, written as $A = B$, if A and B contain exactly the same members. This is formally written as: for all x it holds that $x \in A$ if and only if $x \in B$.

  Examples: $\{1,2\} = \{2,1\}$ and $\{\{1,2\},0,\{2,1\}\} = \{\{1,2\},0\}$, but it is not the case that $\{\{1,2\},1\} = \{1,2\}$.

- A is a *subset* of B, written as $A \subseteq B$, if every member of A is also a member of B. This is formally written as: for all x it holds that if $x \in A$ then also $x \in B$.

  Examples: $\{1\} \subseteq \{1,2\}$, and $\{1\} \subseteq \{1\}$, but not $1 \subseteq \{1,2\}$.

- A is a *proper subset* of B, written as $A \subsetneq B$, if every member of A is also a member of B, but B contains at least one other member. This is formally written as: $A \subseteq B$ and there exists an x such that $x \in B$ and $x \notin A$. In LPL proper subsets are also written as $A \subset B$, but other articles or textbooks use '$\subset$' in place of $\subseteq$. To avoid confusion, only use $\subseteq$ and *subsetneq*.

- The *empty set* $\emptyset$ is the set containing no element. An odd but true property of the empty set is that it is a subset of all sets!

  Example: $\emptyset \subseteq \{1,2\}$. Don't you believe that for every x it holds that if $x \in \emptyset$ then also $x \in \{1,2\}$? Then name an object that is a member of $\emptyset$ but not a member of $\{1,2\}$!

- If $E(x)$ is a particular property of objects x, then the *abstraction* $\{x \mid E(x)\}$ is the set of all objects having the property E. Often you use an earlier given set A as a basis for an abstraction. Then $\{x \in A \mid E(x)\}$ is the set containing all members of A that share the property E.

  Example: $\{x \mid x \in \mathbb{N}$ and x is even$\}$ is the set containing the even natural numbers; this is equivalent to $\{x \in \mathbb{N} \mid x$ is even$\}$.

- The *intersection* $A \cap B$ is $\{x \mid x \in A$ and $x \in B\}$.

  Example: $\{1,2\} \cap \{1,3\} = \{1\}$.

- The *union* $A \cup B$ is $\{x \mid x \in A$ or $x \in B\}$.

  Example: $\{1,2\} \cup \{1,3\} = \{1,2,3\}$. The "or" in this definition is the inclusive or. If an object is a member of both A and B, then it is also definitely a member of $A \cup B$.

- The *difference* $A \setminus B$ is equal to $\{x \mid x \in A$ and $x \notin B\}$.

  Example: $\{1,2\} \setminus \{1,3\} = \{2\}$.

- The *complement* A^c is $\{x \mid x \notin A\}$. In general, you consider the complement not with respect to everything (numbers, sets, cookies, philosophers, etc.), but with respect to some given superset U, called the universe. In that case $A^c = U \setminus A$.

  Example: Let U be the set of natural numbers $\mathbb{N}$. Then $\{n \mid n \in \mathbb{N}$ and n is even $\}^c = \{n \mid n \in \mathbb{N}$ and n is odd$\}$.

EXERCISE 8.1

Which of the following statements are true, which are not? Explain your answers.

a. $1 \in \{1\}$

b. $1 \subseteq \{1\}$

c. $\{\{1\}\} \in \{1, \{1\}\}$

d. $\{\{1\}\} \subseteq \{1, \{1\}\}$

e. $\emptyset \in \{\emptyset\}$

f. $\emptyset \subseteq \{\emptyset\}$

For the next three exercises (8.2, 8.3 and 8.4), we make use of the following sets:

$A = \{a, b, c, 2, 3, 4\}$

$B = \{a, b\}$

$C = \{c, 2\}$

$D = \{b, c\}$

$E = \{a, b, \{c\}\}$

$F = \emptyset$

$G = \{\{a, b\}, \{c, 2\}\}$

We assume that a, b and c are *not* natural numbers and that they differ from one another (so $a \neq b$, $a \neq c$ and $b \neq c$).

EXERCISE 8.2

Which of the following statements are true, which are not? Explain your answers.

a. $c \in A$

b. $c \in F$

c. $c \in E$

d. $\{c\} \subseteq E$

e. $b \in G$

f. $B \subseteq G$

EXERCISE 8.3

Write down the following sets (listing all members):

a. $B \cup C$

b. $D \cup F$

c. $A \cap B$

d. $A \setminus B$

e. $B \setminus A$

f. $G \setminus B$

EXERCISE 8.4

As universe we take $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, a, b, c, d, e, f\}$. Write down the following sets (listing all members):

a. A^c

b. $(A \cap C)^c$

c. $((A \setminus D) \setminus C)$

d. $(A \cap B) \cup C$

e. $((C \cup D) \cap B)^c$

f. $((A \cup D) \cup B) \cap F$

g. $((A \setminus D) \setminus C)^c$

8.2 Ordered pairs, relations and functions

The members of a set have no order: $\{1, 2\} = \{2, 1\}$. Sometimes we do want to represent order in set theory. For this we use *ordered pairs* or more generally *ordered n-tuples*. We write them as $\langle 1, 2\rangle$, or $\langle 1, 1, 2, 3, 5, 8, 13\rangle$. In this situation, order does matter and repetitions as well: $\langle 1, 2\rangle \neq \langle 2, 1\rangle$ and $\langle 1, 1, 2, 3, 5, 8, 13\rangle \neq \langle 1, 2, 3, 5, 8, 13\rangle$.

We can now use sets of ordered pairs to model predicates of any arity. A (binary) *relation* is a set of ordered pairs. Consider the following set $R = \{\langle x, y\rangle \mid x \in \mathbb{N}, y \in \mathbb{N} \text{ and } x = y + 5\}$. A pair $\langle x, y\rangle$ is a member of this set if x is five greater than y. The set R is thus the relation "five greater than". If $\langle x, y\rangle \in R$ we write it as xRy.

Some relations have a special property: for every x, y, z: if xRy and xRz, then $y = z$. In other words, for every x there is a unique y such that xRy. Such a relation is called a *function*, which is often displayed in lowercase. Consider the set $f = \{\langle x, y\rangle \mid x \in \mathbb{N}, y \in \mathbb{N} \text{ and } x^2 = y\}$. This relation is a function, namely the square function. Instead of writing $\langle x, y\rangle \in f$, we write $f(x) = y$.

EXERCISE 8.5

Make a graph of the following relations; represent ordered pairs by an arrow between their members.

a. $R_1 = \{\langle 1, 1\rangle\}$

b. $R_2 = \{\langle 1, 2\rangle\}$

c. $R_3 = \{\langle 1, 1\rangle, \langle 2, 2\rangle\}$

d. $R_4 = \{\langle 1, 2\rangle, \langle 2, 2\rangle\}$

e. $R_5 = \{\langle 1, 2\rangle, \langle 2, 1\rangle\}$

f. $R_6 = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle\}$

g. $R_7 = \emptyset$

h. $R_8 = \{\langle 1, 1\rangle, \langle 2, 3\rangle\}$

EXERCISE 8.6

Which relations from Exercise 8.5 are functions?

9 Propositional logic: normal forms

LPL 4.6 pp. 122 – 125

EXERCISE 9.1

Put the following sentences in disjunctive normal form:

a. $\neg(P \leftrightarrow (Q \vee R))$

b. $P \leftrightarrow (Q \leftrightarrow R)$

c. $(\neg P \vee Q \vee \neg R) \wedge (P \vee \neg Q)$

EXERCISE 9.2

Put the following sentences in conjunctive normal form:

a. $\neg(P \vee Q) \vee \neg(\neg R \wedge S)$

b. $(P \wedge Q \wedge R) \vee \neg S$

c. $\neg((P \to Q) \wedge (R \to S))$

10 First-order logic: syntax and semantics

10.1 Syntax

LPL 1.5 pp. 31 – 34

LPL 9.1 pp. 230 – 231

LPL mentions many examples of terms, but does not give an inductive definition. The definition is provided below:

- constants are terms
- variables are terms
- If $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is also a term
- nothing else is a term

This shows that you can reapply function symbols over and over again (termed *"stacking"*). The term *father(mother(max))*, for example, refers to the maternal grandfather of Max. But also $(x + 5) * (y - 2)$ is a stacking of function symbols. This can be easily observed if you use the prefix notation instead of the infix notation: $*(+(x, 5), -(y, 2))$

Note: predicate symbols cannot be stacked. $Big(Dodec(a))$ is total nonsense. If you want to express in first-order logic that a is a big dodecahedron you will need a conjunction: $Big(a) \wedge Dodec(a)$.

10.2 Semantics

LPL 18.1, 18.2 pp. 511 – 522

EXERCISE 10.1

This exercise is about the three musketeers. Suppose that we have a structure $\mathfrak{M}$ with the domain of discourse $D = \{Aramis, Athos, Porthos\}$.

We know the following facts about the predicates and constants:

- $\mathfrak{M}(Tall) = \{Porthos\}$,
- $\mathfrak{M}(Rich) = \{Aramis\}$,
- $\mathfrak{M}(Fast) = \{Aramis, Porthos\}$,
- $\mathfrak{M}(Old) = \{Athos\}$,
- $(a)^{\mathfrak{M}} = Aramis$

Let the variable assignment h be defined as

- $h(x_1) = Aramis$
- $h(x_2) = Athos$
- $h(x_3) = Porthos$

Which of the following statements are true, which are not? Explain your answers for c, f, j and o.

a. $\mathfrak{M} \models Fast(a)\ [h]$

b. $\mathfrak{M} \models Fast(x_2)\ [h]$

c. $\mathfrak{M} \models Fast(x_3)\ [h[x_3/Aramis]]$
Explain your answer

d. $\mathfrak{M} \models Fast(x_3)\ [h[x_3/Porthos]]$

e. $\mathfrak{M} \models Rich(x_2)\ [h]$

f. $\mathfrak{M} \models Fast(x_1) \leftrightarrow \neg Old(x_1)\ [h[x_1/Aramis]]$
Explain your answer

g. $\mathfrak{M} \models Fast(x_1) \leftrightarrow \neg Old(x_1)\ [h[x_1/Athos]]$

h. $\mathfrak{M} \models Fast(x_1) \leftrightarrow \neg Old(x_1)\ [h[x_1/Porthos]]$

i. $\mathfrak{M} \models \forall x_1(Fast(x_1) \leftrightarrow \neg Old(x_1))\ [h]$

j. $\mathfrak{M} \models \forall x_1(Fast(x_1) \rightarrow Rich(x_1))\ [h]$
Explain your answer

k. $\mathfrak{M} \models \forall x_1(Rich(x_1) \rightarrow Fast(x_1))\ [h]$

l. $\mathfrak{M} \models \exists x_1 \forall x_2(\neg Tall(x_2) \leftrightarrow x_1 = x_2)\ [h]$

m. $\mathfrak{M} \models \forall x_1(Fast(x_1) \rightarrow (Rich(x_1) \leftrightarrow Tall(x_1)))\ [h]$

n. $\mathfrak{M} \models \forall x_1(Tall(x_1) \rightarrow (Fast(x_1) \leftrightarrow Rich(x_1)))\ [h]$

o. $\mathfrak{M} \models \exists x_2 \forall x_1(x_1 \neq x_2 \rightarrow Fast(x_1))\ [h]$
Explain your answer

EXERCISE 10.2

Which of the statements of Exercise 10.1 change truth value if we add D'Artagnan to the domain of discourse D? It is known that D'Artagnan is not tall. He is rich and old, but not fast.

11 First-order logic: normal forms

11.1 Prenex normal form

LPL 11.7 pp. 320 – 323

A formula is in prenex normal form if either of the following holds:

Definition prenex normal form LPL p. 320

- it contains no quantifiers at all, or
- it is of the form $Q_1 v_1 \ldots Q_n v_n P$, where Q_i is either $\forall$ or $\exists$, each v_i is a variable, and P is a quantifier-free formula.

Formulas that are in prenex normal form are, for example:

- $P(c)$, and
- $\forall x \exists y(P(x,y) \rightarrow R(x,y))$.

Not in prenex normal form are:

- $\neg\exists x P(x)$ (due to the $\neg$ at the beginning of the formula), and
- $\exists x P(x) \rightarrow P(c)$ (this formula does not conform to the form quantifier + quantifier-free formula, because $\exists x$ only applies to $P(x)$).

Equivalences for prenex normal form

LPL p. 285

In addition to the equivalences provided on page 285, there are some other equivalences that you can use to convert formulas to prenex normal form. They are all listed below. See also LPL 10.3 and 10.4 (pp. 277–285).

To create an alphabetical variant.
Note: y is free in $P(y)$ exactly where x is free in $P(x)$.

a. $\forall x P(x) \Leftrightarrow \forall y P(y)$

b. $\exists x P(x) \Leftrightarrow \exists y P(y)$

To push negation signs past a quantifier (LPL p. 281):

c. $\neg\forall x P(x) \Leftrightarrow \exists x \neg P(x)$

d. $\neg\exists x P(x) \Leftrightarrow \forall x \neg P(x)$

To push quantifiers to the front in formulas containing $\wedge, \vee$:

e. $\forall x P \wedge \forall x Q \Leftrightarrow \forall x(P \wedge Q)$

f. $\exists x P \vee \exists x Q \Leftrightarrow \exists x(P \vee Q)$

g. $\forall x P \vee Q \Leftrightarrow \forall x(P \vee Q)$ if x is not free in Q.

h. $\exists x P \wedge Q \Leftrightarrow \exists x(P \wedge Q)$ if x is not free in Q.

i. $\forall x P \wedge Q \Leftrightarrow \forall x(P \wedge Q)$ if x is not free in Q.

j. $\exists x P \vee Q \Leftrightarrow \exists x(P \vee Q)$ if x is not free in Q.

k. $Q \Leftrightarrow \forall x Q$ if x is not free in Q.

l. $Q \Leftrightarrow \exists x Q$ if x is not free in Q.

The equivalences i and j are convenient because they provide more efficiency. Besides, i follows from e and k, and j follows from f and l.

To push quantifiers to the front in formulas containing $\rightarrow$:

m. $\forall x P \rightarrow Q \Leftrightarrow \exists x(P \rightarrow Q)$ if x is not free in Q.

n. $\exists x P \rightarrow Q \Leftrightarrow \forall x(P \rightarrow Q)$ if x is not free in Q.

o. $P \rightarrow \forall x Q \Leftrightarrow \forall x(P \rightarrow Q)$ if x is not free in P.

p. $P \rightarrow \exists x Q \Leftrightarrow \exists x(P \rightarrow Q)$ if x is not free in P.

Here follows an example how to put a formula in prenex normal form using the above-mentioned equivalences. Suppose we start with the following formula:

$$\neg\exists x P(x) \vee \forall x Q(x).$$

We first push the negation sign past the quantifier, by rule *d*. This leads to

$$\forall x \neg P(x) \vee \forall x Q(x).$$

We canno push all the quantifiers to the front by using rules *e*–*l* at this moment: we first have to make sure that both disjuncts have different quantifiers as main connective. Using rule *a* we can create an alphabetical variant

$$\forall x \neg P(x) \vee \forall y Q(y).$$

By rule *g* we get

$$\forall x(\neg P(x) \vee \forall y Q(y)),$$

and applying rule *g* again gives us

$$\forall x \forall y(\neg P(x) \vee Q(y)).$$

This formula is in prenex normal form and is equivalent to the formula we started with.

Here is a more difficult example with less explanation (the rules that we applied can be found directly after the $\Leftrightarrow$ sign).

$$\forall x A(x) \rightarrow [\exists y B(y) \rightarrow \forall x B(x)] \Leftrightarrow (a)$$

$$\forall x A(x) \rightarrow [\exists y B(y) \rightarrow \forall z B(z)] \Leftrightarrow (n)$$

$$\forall x A(x) \rightarrow \forall y[B(y) \rightarrow \forall z B(z)] \Leftrightarrow (o)$$

$$\forall x A(x) \rightarrow \forall y \forall z[B(y) \rightarrow B(z)] \Leftrightarrow (o)$$

$$\forall y(\forall x A(x) \rightarrow \forall z[B(y) \rightarrow B(z))] \Leftrightarrow (o)$$

$$\forall y \forall z(\forall x A(x) \rightarrow [B(y) \rightarrow B(z)]) \Leftrightarrow (m)$$

$$\forall y \forall z \exists x(A(x) \rightarrow [B(y) \rightarrow B(z)])$$

Note that we are using square brackets here. Square brackets are interpreted in the same way as normal parentheses, but in some situations provide more clarity.

EXERCISE 11.1

Put the following formulas into prenex normal form:

a. $\forall x A(x) \wedge B(c)$

b. $B \vee \neg\exists x A(x)$

c. $\forall x A(x) \rightarrow B(x)$

d. $\forall x A(x, y) \rightarrow \exists x A(x, y)$

e. $B \rightarrow \forall x(\exists y(A(y) \rightarrow C(x, y)) \rightarrow D(x))$

f. $\forall x(\forall y(R(x, y) \rightarrow A(y)) \rightarrow A(x)) \rightarrow \forall x A(x)$

11.2 Skolem normal form

LPL 18.5 pp. 530 – 532

Definition Skolem normal form LPL p. 530

A formula is in Skolem normal form if it is in prenex normal form and only contains universal quantifiers.

LPL provides only the simplest case of the Skolem normal form: A Skolem normal form of the formula $\forall x \exists y P(x, y)$ is given by the (non-equivalent) formula $\forall x P(x, f(x))$.

Here are two other cases:

- The formula $\forall x \forall y \exists z P(x, y, z)$ has as its Skolem normal form the formula
  $\forall x \forall y P(x, y, g(x, y))$; note that g is a binary function! With more universal quantifiers you will need corresponding n-ary Skolem functions.

- The formula $\exists z P(z)$ has as its Skolem normal form the formula $P(d)$; in this case the constant d can be viewed as a nullary Skolem function.

EXERCISE 11.2

Skolemize the following formulas. If necessary, put them in prenex normal form first.

a. $\forall x \exists y \forall z P(x, y, z)$

b. $\forall x \exists y \forall z \exists u Q(x, y, z, u)$

c. $\forall x \forall y (A(x, y) \rightarrow \exists z (B(z) \wedge \exists u C(z, y, u)))$

11.3 Horn sentences

LPL 17.3 (partly) pp. 495 - 500

Section 17.3 of LPL up to the point where "correctness of algorithm" is shown in the margin, is part of the course material.

Now that you know prenex and Skolem normal forms, we can extend the definition of a Horn sentence. A Horn sentence:

- is in prenex normal form,

- is in Skolem normal form (so only with universal quantifiers), and

- its quantifier-free part is in Horn form, as defined in LPL: it is in conjunctive normal form with the additional property: every disjunction of literals in the sentence contains *at most one* positive literal.

Examples of Horn sentences are:

- $\forall x (\neg R(x) \vee T(x))$

- $\forall x R(x)$

- $\forall x ((\neg A(x) \vee B(f(x))) \wedge (\neg B(f(x)) \vee \neg C(x) \vee A(x)))$.

The following are not Horn sentences:

- $\forall x (A(x) \vee B(x) \vee C(x))$ (this formula has too many positive disjuncts.)

- $\forall x \exists y R(x, y)$ (this formula contains an existential quantifier.)

Universal quantifiers are often left out of Horn sentences, like in the formula $(\neg A(x) \vee B(f(x))) \wedge (\neg B(f(x)) \vee \neg C(x) \vee A(x)))$; in these cases you must think of all the free variables as being universally quantified.

Here follows an example to put a formula in Horn form:

$$\neg\forall x \forall y R(x,y) \vee \forall z A(z) \Leftrightarrow (c)$$

$$\exists x \neg\forall y R(x,y) \vee \forall z A(z) \Leftrightarrow (c)$$

$$\exists x \exists y \neg R(x,y) \vee \forall z A(z) \Leftrightarrow (o)$$

$$\exists x \exists y \forall z(\neg R(x,y) \vee A(z)).$$

This formula is now in prenex normal form. The next step is to Skolemize this formula to create a (non-equivalent) Skolem normal form:

$$\forall z(\neg R(c,d) \vee A(z)).$$

This sentence is already in Horn form, because the quantifier-free part is a conjunction of a single disjunction, which indeed contains at most one positive disjunct, namely $A(z)$.

EXERCISE 11.3

Put the following sentences in Horn form.

a. $(A \to B) \to \neg(C \wedge \neg A)$

b. $\neg(A \to ((A \wedge B \wedge \neg C) \vee (C \wedge B)))$

c. $\neg((\neg((\neg A \wedge B) \vee (A \wedge \neg C)) \to (B \wedge \neg D)) \vee C)$

EXERCISE 11.4

Put the following formulas in Horn form; if necessary use Skolem normal forms

a. $\forall x \forall y(R(x,y) \to \forall z(R(y,z) \to R(x,z)))$

b. $\forall x \forall y(R(x,y) \to \neg R(y,x))$

c. $\forall x(A(x) \to B(x)) \to (\forall y A(y) \to \forall z B(z))$ (this last one is a trick question)

Selected exercises: Part A

LPL ex. 1.9 p. 30

1. $Owned(claire, folly, \text{2:00})$

2. $Gave(claire, pris, max, \text{2:05})$

3. $Student(max)$

4. $Fed(claire, carl, \text{2:00})$

5. $Owned(max, folly, \text{3:05})$

6. $\text{2:00} < \text{2:05}$

LPL ex. 3.23 p. 88

1. $Student(max) \land \neg Pet(max)$

2. $Fed(claire, folly, \text{2:00}) \land Gave(claire, folly, max, \text{2:00})$
The second 2:00 is open to discussion.

3. $Owned(max, folly, \text{2:05}) \lor Owned(claire, folly, \text{2:05})$

4. $\neg Fed(max, folly, \text{2:00}) \land \neg Fed(max, folly, \text{2:05}) \land$
$\neg Fed(claire, folly, \text{2:00}) \land \neg Fed(claire, folly, \text{2:05})$.
Alternatively:
$\neg(Fed(max, folly, \text{2:00}) \lor Fed(max, folly, \text{2:05})) \land$
$\neg(Fed(claire, folly, \text{2:00}) \lor Fed(claire, folly, \text{2:05}))$

5. $\text{1:55} < \text{2:00} \land \text{2:00} < \text{2:05}$
Warning: it is not possible to translate this sentence as
$\text{1:55} < \text{2:00} < \text{2:05}$.

6. $Gave(max, folly, claire, \text{2:00}) \land \neg Hungry(folly, \text{2:00}) \land$
$Hungry(folly, \text{3:00})$

LPL ex. 7.18 p. 189

1. $Gave(claire, folly, max, \text{2:03}) \rightarrow$
$(Owned(claire, folly, \text{2:00}) \land Owned(max, folly, \text{2:05}))$

2. $Fed(max, folly, \text{2:00}) \land$
$(Gave(max, folly, claire, \text{2:00}) \rightarrow \neg Hungry(folly, \text{2:05}))$

3. $(\neg Fed(max, folly, \text{2:00}) \land \neg Fed(claire, folly, \text{2:00})) \rightarrow$
$Hungry(folly, \text{2:00})$

4. $Angry(max, claire, \text{2:05}) \rightarrow$
$(Fed(claire, folly, \text{2:00}) \lor Fed(claire, silly, \text{2:00}))$

5. $Student(max) \leftrightarrow \neg Student(claire)$

Syllabus ex. 5.2 p. 7

a. Translation key:
Even(x) : x is an even number
two : the number 2

Translation: Even(two)

b. Translation key:
Greater(x, y) : x is greater than y
two : the number 2
three : the number 3

Translation: Greater(two, three)

c. Translation key:
four : the number 4
three : the number 3
five : the number 5
Greater(x, y) : x is greater than y

Translation: Greater(five, four) $\rightarrow$ Greater(five, three)

d. Translation key:
four : the number 4
three : the number 3
Greater(x, y) : x is greater than y

Translation: $\forall x$(Greater(x, four) $\rightarrow$ Greater(x, three))

e. Translation key:
three : the number 3
Even(x) : x is an even number
Greater(x, y) : x is greater than y

Translation: $\exists x$(Even(x) $\wedge$ Greater(x, three))

f. Translation key:
Greater(x, y) : x is greater than y

Translation: $\forall x \neg$Greater(x, x), or: $\neg \exists x$ Greater(x, x)

Solutions to selected exercises: Part B

Syllabus ex. 8.1 p. 11

1. True: 1 is a member of the set $\{1\}$.

2. False: 1 is not a set, so definitely not a subset of $\{1\}$.

3. False: the set $\{1, \{1\}\}$ contains only the members 1 and $\{1\}$, it does not contain the member $\{\{1\}\}$.

4. True: all members of the set $\{\{1\}\}$ (which is only one, namely $\{1\}$) occur in the set $\{1, \{1\}\}$. The set $\{\{1\}\}$ is thus a subset of the set $\{1, \{1\}\}$.

5. True: the empty set is a member of the set $\{\emptyset\}$.

6. True: the empty set is a subset of all sets, because all members of the empty set (none) occur in all sets.

Syllabus ex. 8.3 p. 11

a. $\{a, b, c, 2\}$

b. $\{b, c\}$

c. $\{a, b\}$

d. $\{c, 2, 3, 4\}$

e. $\emptyset$

f. $\{\{a, b\}, \{c, 2\}\}$

Additional proof exercises with solutions

ADD. EXERCISE 1

Prove by natural deduction:

a. $P \vee Q$ from premise $P \wedge Q$

b. $a = c$ from premise $(a = b) \wedge (b = c)$

c. $C \vee B$ from premise $(A \wedge B) \vee C$

d. $(A \wedge B) \vee (A \wedge C)$ from premises A and $B \vee C$

ADD. EXERCISE 2

Prove by natural deduction:

a. $A \to (B \to A)$ without premises

b. $(A \to (B \to C)) \leftrightarrow ((A \wedge B) \to C)$ without premises

c. $C \wedge D$ from premises $A \vee (B \wedge C)$, $\neg E$, $(A \vee B) \to (D \vee E)$ and $\neg A$

ADD. EXERCISE 3

Prove by natural deduction:

$$(P \to Q) \leftrightarrow (\neg P \vee Q)$$

ADD. EXERCISE 4

Prove by natural deduction:

$$\exists x Dodec(x)$$

from premises:

$$\forall y(Cube(y) \vee Dodec(y)),$$

$$\forall x(Cube(x) \to Large(x)),$$

$$\exists x \neg Large(x)$$

ADD. EXERCISE 5

Prove by natural deduction:

a. $\forall x P(x)$ from premise $\forall y P(y)$

b. $\exists x P(x)$ from premise $\exists y P(y)$

ADD. EXERCISE 6

Prove by natural deduction:

a. $\neg\forall x P(x)$ from premise $\exists x \neg P(x)$

b. $\neg\exists x P(x)$ from premise $\forall x \neg P(x)$

c. $\forall x \neg P(x)$ from premise $\neg\exists P(x)$

ADD. EXERCISE 7

Prove by natural deduction:

a. $\forall x \exists y Likes(x,y)$ from premise $\forall x \forall y Likes(x,y)$

b. $\exists x Cube(x)$ from premises $\forall x(Small(x) \rightarrow Cube(x))$ and $\exists x \neg Cube(x) \rightarrow \exists x Small(x)$

c. $\exists x Likes(x, carl)$ from premises $Likes(carl, max)$ and $\forall x(\exists y(Likes(y,x) \vee Likes(x,y)) \rightarrow Likes(x,x))$

d. $\forall x \exists y Likes(x,y)$ from premises $\forall x \forall y(Likes(x,y) \rightarrow Likes(y,x))$ and $\exists x \forall y Likes(x,y)$

ADD. EXERCISE 8

Prove by natural deduction:

a. $P \wedge (Q \vee R)$ from premise $(P \wedge Q) \vee (P \wedge R)$

b. $(P \wedge Q) \vee (P \wedge R)$ from premise $P \wedge (Q \vee R)$

c. $P \vee (Q \wedge R)$ from premise $(P \vee Q) \wedge (P \vee R)$

d. $(P \vee Q) \wedge (P \vee R)$ from premise $P \vee (Q \wedge R)$

Solutions to the additional proof exercises

Add. exercise 1

a.

1. P ∧ Q	
2. P	∧ Elim: 1
3. P ∨ Q	∨ Intro: 2

b.

1. (a = b) ∧ (b = c)	
2. a = b	∧ Elim: 1
3. b = c	∧ Elim: 1
4. a = c	= Elim: 2,3

c.

1. (A ∧ B) ∨ C	
2. A ∧ B	
3. B	∧ Elim: 2
4. C ∨ B	∨ Intro: 3
5. C	
6. C ∨ B	∨ Intro: 5
7. C ∨ B	∨ Elim: 1, 2–4, 5–6

d.

1. A	
2. B ∨ C	
3. B	
4. A ∧ B	∧ Intro: 1, 3
5. (A ∧ B) ∨ (A ∧ C)	∨ Intro: 4
6. C	
7. A ∧ C	∧ Intro: 1, 6
8. (A ∧ B) ∨ (A ∧ C)	∨ Intro: 7
9. (A ∧ B) ∨ (A ∧ C)	∨ Elim: 2, 3–5, 6–8

Add. exercise 2

a.

$\vert$ 1. A	
$\vert\ \vert$ 2. B	
$\vert\ \vert$ 3. A	Reit: 1
$\vert$ 4. $\mathsf{B} \to \mathsf{A}$	$\to$ Intro: 2–3
5. $\mathsf{A} \to (\mathsf{B} \to \mathsf{A})$	$\to$ Intro: 1–4

b.

$\vert$ 1. $\mathsf{A} \to (\mathsf{B} \to \mathsf{C})$	
$\vert\ \vert$ 2. $\mathsf{A} \wedge \mathsf{B}$	
$\vert\ \vert$ 3. A	$\wedge$ Elim: 2
$\vert\ \vert$ 4. $\mathsf{B} \to \mathsf{C}$	$\to$ Elim: 1, 3
$\vert\ \vert$ 5. B	$\wedge$ Elim: 2
$\vert\ \vert$ 6. C	$\to$ Elim: 4, 5
$\vert$ 7. $(\mathsf{A} \wedge \mathsf{B}) \to \mathsf{C}$	$\to$ Intro: 2–6
$\vert$ 8. $(\mathsf{A} \wedge \mathsf{B}) \to \mathsf{C}$	
$\vert\ \vert$ 9. A	
$\vert\ \vert\ \vert$ 10. B	
$\vert\ \vert\ \vert$ 11. $\mathsf{A} \wedge \mathsf{B}$	$\wedge$ Intro: 9, 10
$\vert\ \vert\ \vert$ 12. C	$\to$ Elim: 8, 11
$\vert\ \vert$ 13. $\mathsf{B} \to \mathsf{C}$	$\to$ Intro: 10–12
$\vert$ 14. $\mathsf{A} \to (\mathsf{B} \to \mathsf{C})$	$\to$ Intro: 9–13
15. $(\mathsf{A} \to (\mathsf{B} \to \mathsf{C})) \leftrightarrow ((\mathsf{A} \wedge \mathsf{B}) \to \mathsf{C})$	$\leftrightarrow$ Intro: 1–7, 8–14

c.

1. $\mathsf{A} \vee (\mathsf{B} \wedge \mathsf{C})$	
2. $\neg \mathsf{E}$	
3. $(\mathsf{A} \vee \mathsf{B}) \to (\mathsf{D} \vee \mathsf{E})$	
4. $\neg \mathsf{A}$	
$\vert$ 5. A	
$\vert$ 6. $\perp$	$\perp$ Intro: 5, 4
$\vert$ 7. $\mathsf{C} \wedge \mathsf{D}$	$\perp$ Elim: 6
$\vert$ 8. $\mathsf{B} \wedge \mathsf{C}$	
$\vert$ 9. B	$\wedge$ Elim: 8
$\vert$ 10. C	$\wedge$ Elim: 8
$\vert$ 11. $\mathsf{A} \vee \mathsf{B}$	$\vee$ Intro: 9
$\vert$ 12. $\mathsf{D} \vee \mathsf{E}$	$\to$ Elim: 3, 11
$\vert\ \vert$ 13. D	
$\vert\ \vert$ 14. D	Reit: 13
$\vert\ \vert$ 15. E	
$\vert\ \vert$ 16. $\perp$	$\perp$ Intro: 15, 2
$\vert\ \vert$ 17. D	$\perp$ Elim: 16
$\vert$ 18. D	$\vee$ Elim: 12, 13–14, 15–17
$\vert$ 19. $\mathsf{C} \wedge \mathsf{D}$	$\wedge$ Intro: 10, 18
20. $\mathsf{C} \wedge \mathsf{D}$	$\vee$ Elim: 1, 5–7, 8–19

Add. exercise 3

1. $\mathsf{P} \to \mathsf{Q}$	
$\quad$ 2. $\neg(\neg\mathsf{P} \lor \mathsf{Q})$	
$\quad\quad$ 3. P	
$\quad\quad$ 4. Q	$\to$ Elim: 1, 3
$\quad\quad$ 5. $\neg\mathsf{P} \lor \mathsf{Q}$	$\lor$ Intro: 4
$\quad\quad$ 6. $\perp$	$\perp$ Intro: 5, 2
$\quad$ 7. $\neg\mathsf{P}$	$\neg$ Intro: 3–6
$\quad$ 8. $\neg\mathsf{P} \lor \mathsf{Q}$	$\lor$ Intro: 7
$\quad$ 9. $\perp$	$\perp$ Intro: 8, 2
10. $\neg\neg(\neg\mathsf{P} \lor \mathsf{Q})$	$\neg$ Intro: 2–9
11. $\neg\mathsf{P} \lor \mathsf{Q}$	$\neg$ Elim: 10
12. $\neg\mathsf{P} \lor \mathsf{Q}$	
$\quad$ 13. P	
$\quad\quad$ 14. $\neg\mathsf{P}$	
$\quad\quad$ 15. $\perp$	$\perp$ Intro: 13,14
$\quad\quad$ 16. Q	$\perp$ Elim: 15
$\quad\quad$ 17. Q	
$\quad\quad$ 18. Q	Reit: 17
$\quad$ 19. Q	$\lor$ Elim: 12, 14–16, 17–18
20. $\mathsf{P} \to \mathsf{Q}$	$\to$ Intro: 13–19
21. $(\mathsf{P} \to \mathsf{Q}) \leftrightarrow (\neg\mathsf{P} \lor \mathsf{Q})$	$\leftrightarrow$ Intro: 1–11, 12–20

Add. exercise 4

1. $\forall y[\mathsf{Cube}(y) \lor \mathsf{Dodec}(y)]$	
2. $\forall x[\mathsf{Cube}(x) \to \mathsf{Large}(x)]$	
3. $\exists x \neg\mathsf{Large}(x)$	
$\quad$ 4. $\boxed{a}$ $\neg\mathsf{Large}(a)$	
$\quad$ 5. $\mathsf{Cube}(a) \lor \mathsf{Dodec}(a)$	$\forall$ Elim: 1
$\quad$ 6. $\mathsf{Cube}(a) \to \mathsf{Large}(a)$	$\forall$ Elim: 2
$\quad\quad$ 7. $\mathsf{Cube}(a)$	
$\quad\quad$ 8. $\mathsf{Large}(a)$	$\to$ Elim: 6, 7
$\quad\quad$ 9. $\perp$	$\perp$ Intro: 8, 4
$\quad\quad$ 10. $\mathsf{Dodec}(a)$	$\perp$ Elim: 9
$\quad\quad$ 11. $\mathsf{Dodec}(a)$	
$\quad\quad$ 12. $\mathsf{Dodec}(a)$	Reit: 11
$\quad$ 13. $\mathsf{Dodec}(a)$	$\lor$ Elim: 5, 7–10, 11–12
$\quad$ 14. $\exists x\mathsf{Dodec}(x)$	$\exists$ Intro: 13
15. $\exists x\mathsf{Dodec}(x)$	$\exists$ Elim: 3, 4–14

Add. exercise. 5

a.

1. $\forall y\mathsf{P}(y)$	
$\quad$ 2. $\boxed{a}$	
$\quad$ 3. $\mathsf{P}(a)$	$\forall$ Elim: 1
4. $\forall x\mathsf{P}(x)$	$\forall$ Intro: 2–3

b.

1. $\exists y P(y)$	
2. $\boxed{a}$ $P(a)$	
3. $\exists x P(x)$	$\exists$ Intro: 2
4. $\exists x P(x)$	$\exists$ Elim: 1, 2–3

Add. exercise 6

a.

1. $\exists x \neg P(x)$	
2. $\boxed{a}$ $\neg P(a)$	
3. $\forall x P(x)$	
4. $P(a)$	$\forall$ Elim: 3
5. $\perp$	$\perp$ Intro: 4, 2
6. $\neg\forall x P(x)$	$\neg$ Intro: 3–5
7. $\neg\forall x P(x)$	$\exists$ Elim: 1, 2–6

b.

1. $\forall x \neg P(x)$	
2. $\exists x P(x)$	
3. $\boxed{a}$ $P(a)$	
4. $\neg P(a)$	$\forall$ Elim: 1
5. $\perp$	$\perp$ Intro: 3, 4
6. $\perp$	$\exists$ Elim: 2, 3–5
7. $\neg\exists x P(x)$	$\neg$ Intro: 2–6

c.

1. $\neg\exists x P(x)$	
2. $\boxed{a}$	
3. $P(a)$	
4. $\exists x P(x)$	$\exists$ Intro: 3
5. $\perp$	$\perp$ Intro: 4, 1
6. $\neg P(a)$	$\neg$ Intro: 3–5
7. $\forall x \neg P(x)$	$\forall$ Intro: 2–6

Add. exercise 7

a.

1. $\forall x \forall y \mathsf{Likes}(x, y)$	
2. $\boxed{a}$	
3. $\forall y \mathsf{Likes}(a, y)$	$\forall$ Elim: 1
4. $\mathsf{Likes}(a, b)$	$\forall$ Elim: 3
5. $\exists y \mathsf{Likes}(a, y)$	$\exists$ Intro: 4
6. $\forall x \exists y \mathsf{Likes}(x, y)$	$\forall$ Intro: 2–5

b.

1.	$\forall x(\mathsf{Small}(x) \to \mathsf{Cube}(x))$	
2.	$\exists x \neg \mathsf{Cube}(x) \to \exists x \mathsf{Small}(x)$	
3.	$\quad \neg \exists x \mathsf{Cube}(x)$	
4.	$\quad\quad \neg \mathsf{Cube}(a)$	
5.	$\quad\quad \exists x \neg \mathsf{Cube}(x)$	$\exists$ Intro: 4
6.	$\quad\quad \exists x \mathsf{Small}(x)$	$\to$ Elim: 2, 5
7.	$\quad\quad\quad \boxed{b}\ \mathsf{Small}(b)$	
8.	$\quad\quad\quad \mathsf{Small}(b) \to \mathsf{Cube}(b)$	$\forall$ Elim: 1
9.	$\quad\quad\quad \mathsf{Cube}(b)$	$\to$ Elim: 8, 7
10.	$\quad\quad\quad \exists x \mathsf{Cube}(x)$	$\exists$ Intro: 9
11.	$\quad\quad\quad \bot$	$\bot$ Intro: 10, 3
12.	$\quad\quad \bot$	$\exists$ Elim: 6, 7–11
13.	$\quad \neg\neg \mathsf{Cube}(a)$	$\neg$ Intro: 4–12
14.	$\quad \mathsf{Cube}(a)$	$\neg$ Elim: 13
15.	$\quad \exists x \mathsf{Cube}(x)$	$\exists$ Intro: 14
16.	$\quad \bot$	$\bot$ Intro: 15, 3
17.	$\neg\neg \exists x \mathsf{Cube}(x)$	$\neg$ Intro: 3–16
18.	$\exists x \mathsf{Cube}(x)$	$\neg$ Elim: 17

c.

1.	$\mathsf{Likes}(carl, max)$	
2.	$\forall x[\exists y(\mathsf{Likes}(y,x) \lor \mathsf{Likes}(x,y)) \to \mathsf{Likes}(x,x)]$	
3.	$\mathsf{Likes}(max, carl) \lor \mathsf{Likes}(carl, max)$	$\lor$ Intro: 1
4.	$\exists y(\mathsf{Likes}(y, carl) \lor \mathsf{Likes}(carl, y))$	$\exists$ Intro: 3
5.	$\exists y(\mathsf{Likes}(y, carl) \lor \mathsf{Likes}(carl, y)) \to \mathsf{Likes}(carl, carl)$	$\forall$ Elim: 2
6.	$\mathsf{Likes}(carl, carl)$	$\to$ Elim: 5, 4
7.	$\exists x \mathsf{Likes}(x, carl)$	$\exists$ Intro: 6

d.

1.	$\forall x \forall y[\mathsf{Likes}(x,y) \to \mathsf{Likes}(y,x)]$	
2.	$\exists x \forall y \mathsf{Likes}(x,y)$	
3.	$\quad \boxed{a}$	
4.	$\quad\quad \boxed{b}\ \forall y \mathsf{Likes}(b,y)$	
5.	$\quad\quad \mathsf{Likes}(b, a)$	$\forall$ Elim: 4
6.	$\quad\quad \forall y \mathsf{Likes}(b,y) \to \mathsf{Likes}(y,b)$	$\forall$ Elim: 1
7.	$\quad\quad \mathsf{Likes}(b,a) \to \mathsf{Likes}(a,b)$	$\forall$ Elim: 6
8.	$\quad\quad \mathsf{Likes}(a, b)$	$\to$ Elim: 7, 5
9.	$\quad\quad \exists y \mathsf{Likes}(a,y)$	$\exists$ Intro: 8
10.	$\quad \exists y \mathsf{Likes}(a,y)$	$\exists$ Elim: 2, 4–9
11.	$\forall x \exists y \mathsf{Likes}(x,y)$	$\forall$ Intro: 3–10

Add. exercise 8

a.

1.	$(P \wedge Q) \vee (P \wedge R)$	
2.	$\quad \mid P \wedge Q$	
3.	$\quad \mid P$	$\wedge$ Elim: 2
4.	$\quad \mid Q$	$\wedge$ Elim: 2
5.	$\quad \mid Q \vee R$	$\vee$ Intro: 4
6.	$\quad \mid P \wedge (Q \vee R)$	$\wedge$ Intro: 3, 5
7.	$\quad \mid P \wedge R$	
8.	$\quad \mid P$	$\wedge$ Elim: 7
9.	$\quad \mid R$	$\wedge$ Elim: 7
10.	$\quad \mid Q \vee R$	$\vee$ Intro: 9
11.	$\quad \mid P \wedge (Q \vee R)$	$\wedge$ Intro: 8, 10
12.	$P \wedge (Q \vee R)$	$\vee$ Elim: 1, 2–6, 7–11

b.

1.	$P \wedge (Q \vee R)$	
2.	P	$\wedge$ Elim: 1
3.	$Q \vee R$	$\wedge$ Elim: 1
4.	$\quad \mid Q$	
5.	$\quad \mid P \wedge Q$	$\wedge$ Intro: 2, 4
6.	$\quad \mid (P \wedge Q) \vee (P \wedge R)$	$\vee$ Intro: 5
7.	$\quad \mid R$	
8.	$\quad \mid P \wedge R$	$\wedge$ Intro: 2, 7
9.	$\quad \mid (P \wedge Q) \vee (P \wedge R)$	$\vee$ Intro: 8
10.	$(P \wedge Q) \vee (P \wedge R)$	$\vee$ Elim: 3, 4–6, 7–9

c.

1.	$(P \vee Q) \wedge (P \vee R)$	
2.	$P \vee Q$	$\wedge$ Elim: 1
3.	$P \vee R$	$\wedge$ Elim: 1
4.	$\quad \mid P$	
5.	$\quad \mid P \vee (Q \wedge R)$	$\vee$ Intro: 4
6.	$\quad \mid Q$	
7.	$\quad \mid \quad \mid P$	
8.	$\quad \mid \quad \mid P \vee (Q \wedge R)$	$\vee$ Intro: 7
9.	$\quad \mid \quad \mid R$	
10.	$\quad \mid \quad \mid Q \wedge R$	$\wedge$ Intro: 6, 9
11.	$\quad \mid \quad \mid P \vee (Q \wedge R)$	$\vee$ Intro: 10
12.	$\quad \mid P \vee (Q \wedge R)$	$\vee$ Elim: 3, 7–8, 9–11
13.	$P \vee (Q \wedge R)$	$\vee$ Elim: 2, 4–5, 6–12

d.

1.	$\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R})$	
2.	P	
3.	$\mathsf{P} \lor \mathsf{Q}$	$\lor$ Intro: 2
4.	$\mathsf{P} \lor \mathsf{R}$	$\lor$ Intro: 2
5.	$(\mathsf{P} \lor \mathsf{Q}) \land (\mathsf{P} \lor \mathsf{R})$	$\land$ Intro: 3, 4
6.	$\mathsf{Q} \land \mathsf{R}$	
7.	Q	$\land$ Elim: 6
8.	R	$\land$ Elim: 6
9.	$\mathsf{P} \lor \mathsf{Q}$	$\lor$ Intro: 7
10.	$\mathsf{P} \lor \mathsf{R}$	$\lor$ Intro: 8
11.	$(\mathsf{P} \lor \mathsf{Q}) \land (\mathsf{P} \lor \mathsf{R})$	$\land$ Intro: 9, 10
12.	$(\mathsf{P} \lor \mathsf{Q}) \land (\mathsf{P} \lor \mathsf{R})$	$\lor$ Elim: 1, 2–5, 6–11