Introduction to Logic (CS)

Answers Exam of 8 November 2019

1. (a) $D \wedge (\neg R \to \neg I)$
   D: It is dark outside.
   I: I will remain inside.
   R: It is raining.

   (b) $S \to (G \wedge \neg N)$
   S: We call it a success.
   G: There are many guests.
   N: The neighbours complain.

2. (a) $(a = b(c) \vee d = b(c)) \wedge c = b(a) \wedge c = b(d)$

   (b) $\forall x(b(x) = a \to (x = b(a) \vee x = b(d) \vee x = b(c)))$

   (c) $\exists x \forall y(x = y \leftrightarrow b(y) = c)$
   or $\exists x(b(x) = c \wedge \forall y(b(y) = c \to x = y))$
   or $\exists x\ b(x) = c \wedge \forall x \forall y((b(x) = c \wedge b(y) = c) \to x = y)$

3. (a)

P	Q	$(P$	$\wedge$	$\neg$	$Q)$	$\leftrightarrow$	$(P$	$\to$	$Q)$
T	T	T	F	F	T	F	T	T	T
T	F	T	T	T	F	F	T	F	F
F	T	F	F	F	T	F	F	T	T
F	F	F	F	T	F	F	F	T	F
1	2	3	8	7	4	10	5	9	6

The numbers in the last row indicate the order in which the columns are computed. The final column (numbered 10) contains only the value F, so the sentence is a *contradiction*.

   (b)

P	Q	R	$((P$	$\vee$	$Q)$	$\leftrightarrow$	$\neg$	$R)$	$\Leftrightarrow$	$\neg$	$(R$	$\wedge$	$(P$	$\to$	$Q))$
T	T	T	T	T	T	F	F	T	T	F	T	T	T	T	T
T	T	F	T	T	T	T	T	F	T	T	F	F	T	T	T
T	F	T	T	T	F	F	F	T	F	T	T	F	T	F	F
T	F	F	T	T	F	T	T	F	T	T	F	F	T	F	F
F	T	T	F	T	T	F	F	T	T	F	T	T	F	T	T
F	T	F	F	T	T	T	T	F	T	T	F	F	F	T	T
F	F	T	F	F	F	T	F	T	F	F	T	T	F	T	F
F	F	F	F	F	F	F	T	F	F	T	F	F	F	T	F
1	2	3	4	10	5	12	11	6	16	15	7	14	8	13	9

The numbers in the last row indicate the order in which the columns are computed. The final column (numbered 16) contains the value F in the 3rd, 7th and 8th row, so the sentences are *not tautologically equivalent*.

4. (a)

1.	$(P \to Q) \vee (R \to \neg P)$	
2.	$P \wedge R$	
3.	P	$\wedge$ Elim: 2
4.	$P \to Q$	
5.	Q	$\to$ Elim: 4, 3
6.	$R \to \neg P$	
7.	R	$\wedge$ Elim: 2
8.	$\neg P$	$\to$ Elim: 6, 7
9.	$\perp$	$\perp$ Intro: 3, 8
10.	Q	$\perp$ Elim: 10
11.	Q	$\vee$ Elim: 1, 4–5, 6–10
12.	$(P \wedge R) \to Q$	$\to$ Intro: 2–11

(b)

1.	$\forall x \forall y(\neg P(x) \to \neg Q(y))$	
2.	$\exists x Q(x)$	
3.	$\boxed{a}$	
4.	$\boxed{b}\ Q(b)$	
5.	$\forall y(\neg P(a) \to \neg Q(y))$	$\forall$ Elim: 1
6.	$\neg P(a) \to \neg Q(b)$	$\forall$ Elim: 5
7.	$\neg P(a)$	
8.	$\neg Q(b)$	$\to$ Elim: 6, 7
9.	$\perp$	$\perp$ Intro: 4, 8
10.	$\neg\neg P(a)$	$\neg$ Intro: 7–9
11.	$P(a)$	$\neg$ Elim: 10
12.	$P(a)$	$\exists$ Elim: 2, 4–11
13.	$\forall x P(x)$	$\forall$ Intro: 3–12

(c)

1.	$\forall x \exists y(\neg P(x) \leftrightarrow P(y))$	
2.	$\exists y(\neg P(a) \leftrightarrow P(y))$	$\forall$ Elim: 1
3.	$\boxed{b}$ $\neg P(a) \leftrightarrow P(b)$	
4.	$a = b$	
5.	$\neg P(b) \leftrightarrow P(b)$	$=$ Elim: 3, 4
6.	$P(b)$	
7.	$\neg P(b)$	$\leftrightarrow$ Elim: 5, 6
8.	$\bot$	$\bot$ Intro: 6, 7
9.	$\neg P(b)$	$\neg$ Intro: 6–8
10.	$P(b)$	$\leftrightarrow$ Elim: 5, 9
11.	$\bot$	$\bot$ Intro: 10, 9
12.	$\neg(a = b)$	$\neg$ Intro: 4–11
13.	$\exists y \neg(a = y)$	$\exists$ Intro: 12
14.	$\exists x \exists y \neg(x = y)$	$\exists$ Intro: 13
15.	$\exists x \exists y \neg(x = y)$	$\exists$ Elim: 2, 3–14

5. (a) $\mathfrak{M} \models (Q(a) \to \forall x R(b, x))[h]$

$\Leftrightarrow$ { definition of satisfaction for implication }

$\mathfrak{M} \not\models Q(a)[h]$ or $\mathfrak{M} \models \forall x R(b, x)[h]$

$\Leftrightarrow$ { definition of satisfaction for universal quantification }

$\mathfrak{M} \not\models Q(a)[h]$ or for all $d \in \mathfrak{M}(\forall)$ $\mathfrak{M} \models R(b, x)[h[x/d]]$

$\Leftrightarrow$ { definition of satisfaction for atomic formulae }

$[\![a]\!]_h^{\mathfrak{M}} \notin \mathfrak{M}(Q)$ or for all $d \in \mathfrak{M}(\forall)$ $\langle [\![b]\!]_{h[x/d]}^{\mathfrak{M}}, [\![x]\!]_{h[x/d]}^{\mathfrak{M}} \rangle \in \mathfrak{M}(R)$

$\Leftrightarrow$ { $[\![a]\!]_h^{\mathfrak{M}} = \mathfrak{M}(a) = 3$, $[\![b]\!]_{h[x/d]}^{\mathfrak{M}} = \mathfrak{M}(b) = 1$, $[\![x]\!]_{h[x/d]}^{\mathfrak{M}} = d$,
definition of $\mathfrak{M}(Q)$, $\mathfrak{M}(R)$ and $\mathfrak{M}(\forall)$ }

$3 \notin \{1, 3\}$ or for all $d \in \{1, 2, 3\}$ $\langle 1, d \rangle \in \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle\}$

$\Leftrightarrow$ { elementary set theory }

false or $\{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle\} \subseteq \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle\}$

$\Leftrightarrow$ { elementary set theory }

false or false $\Leftrightarrow$ false

(b) $\mathfrak{M} \models (R(b,a) \vee \exists x(P(x) \wedge Q(x)))[h]$

$\Leftrightarrow$ { definition of satisfaction for disjunction }

$\mathfrak{M} \models R(b,a)[h]$ or $\mathfrak{M} \models \exists x(P(x) \wedge Q(x))[h]$

$\Leftrightarrow$ { definition of satisfaction for existential quantification }

$\mathfrak{M} \models R(b,a)[h]$ or for some $d \in \mathfrak{M}(\forall)$ $\mathfrak{M} \models (P(x) \wedge Q(x))[h[x/d]]$

$\Leftrightarrow$ { definition of satisfaction for conjunction }

$\mathfrak{M} \models R(b,a)[h]$ or for some $d \in \mathfrak{M}(\forall)$ $(\mathfrak{M} \models P(x)[h[x/d]]$ and $\mathfrak{M} \models Q(x)[h[x/d]])$

$\Leftrightarrow$ { definition of satisfaction for atomic formulae }

$\langle [\![b]\!]^{\mathfrak{M}}_h, [\![a]\!]^{\mathfrak{M}}_h \rangle \in \mathfrak{M}(R)$ or for some $d \in \mathfrak{M}(\forall)$ $([\![x]\!]^{\mathfrak{M}}_{h[x/d]} \in \mathfrak{M}(P)$ and $[\![x]\!]^{\mathfrak{M}}_{h[x/d]} \in \mathfrak{M}(Q)$)

$\Leftrightarrow$ { $[\![b]\!]^{\mathfrak{M}}_h = \mathfrak{M}(b) = 1$, $[\![a]\!]^{\mathfrak{M}}_h = \mathfrak{M}(a) = 3$, $[\![x]\!]^{\mathfrak{M}}_{h[x/d]} = d$,
definition of $\mathfrak{M}(P)$, $\mathfrak{M}(Q)$ and $\mathfrak{M}(R)$ }

$\langle 1,3 \rangle \in \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle\}$ or for some $d \in \mathfrak{M}(\forall)$ $(d \in \{2\}$ and $d \in \{1,3\})$

$\Leftrightarrow$ { elementary set theory, $\{2\} \cap \{1,3\} = \emptyset$ }

false or for some $d \in \mathfrak{M}(\forall)$ $d \in \emptyset$

$\Leftrightarrow$ { elementary set theory}

false or false $\Leftrightarrow$ false

(c) $\mathfrak{M} \models \forall x \exists y(R(x,y) \wedge \neg(x = y))[h]$

$\Leftrightarrow$ { definition of satisfaction for universal quantification }

for all $d \in \mathfrak{M}(\forall)$: $\mathfrak{M} \models \exists y(R(x,y) \wedge \neg(x = y))[h[x/d]]$

$\Leftrightarrow$ { definition of satisfaction for existential quantification }

for all $d \in \mathfrak{M}(\forall)$ there is an $e \in \mathfrak{M}(\forall)$ with $\mathfrak{M} \models (R(x,y) \wedge \neg(x = y))[h[x/d, y/e]]$

$\Leftrightarrow$ { definition of satisfaction for conjunction and for negation }

for all $d \in \mathfrak{M}(\forall)$ there is an $e \in \mathfrak{M}(\forall)$ with
$(\mathfrak{M} \models R(x,y)[h[x/d, y/e]]$ and $\mathfrak{M} \not\models x = y[h[x/d, y/e]])$

$\Leftrightarrow$ { definition of satisfaction for atomic formulae }

for all $d \in \mathfrak{M}(\forall)$ there is an $e \in \mathfrak{M}(\forall)$ with
$(\langle [\![x]\!]^{\mathfrak{M}}_{h[x/d,y/e]]}, [\![y]\!]^{\mathfrak{M}}_{h[x/d,y/e]]} \rangle \in \mathfrak{M}(R)$ and $[\![x]\!]^{\mathfrak{M}}_{h[x/d,y/e]} \neq [\![y]\!]^{\mathfrak{M}}_{h[x/d,y/e]}$

$\Leftrightarrow$ { $[\![x]\!]^{\mathfrak{M}}_{h[x/d,y/e]} = h[x/d, y/e](x) = d$, $[\![y]\!]^{\mathfrak{M}}_{h[x/d,y/e]} = h[x/d, y/e](y) = e$,
definition of $\mathfrak{M}(R)$ and $\mathfrak{M}(\forall)$ }

for all $d \in \{1,2,3\}$ there is an $e \in \{1,2,3\}$ with
$(\langle d,e \rangle \in \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle\}$ and $d \neq e$

$\Leftrightarrow$ { for $d = 1$ take $e = 2$; for $d = 2$, take $e = 3$; for $d = 3$, take $e = 1$ }

true

6. (a) $\neg((P \wedge \neg Q) \to (R \wedge S))$

$\Leftrightarrow$ { eliminate $\to$ }

$\neg(\neg(P \wedge \neg Q) \vee (R \wedge S))$

$\Leftrightarrow$ { apply De Morgan }

$\neg\neg(P \wedge \neg Q) \wedge \neg(R \wedge S)$

$\Leftrightarrow$ { eliminate $\neg\neg$, apply De Morgan }

$P \wedge \neg Q \wedge (\neg R \vee \neg S)$

$\Leftrightarrow$ { apply distribution }

$(P \wedge \neg Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg S)$

(b) $\forall x \exists y R(x, y) \to \forall z \neg \forall x Q(x, z)$

$\Leftrightarrow$ { rename x in the right hand part }

$\forall x \exists y R(x, y) \to \forall z \neg \forall w Q(w, z)$

$\Leftrightarrow$ { eliminate $\to$ }

$\neg \forall x \exists y R(x, y) \vee \forall z \neg \forall w Q(w, z)$

$\Leftrightarrow$ { move $\neg$ inside, using De Morgan }

$\exists x \forall y \neg R(x, y) \vee \forall z \exists w \neg Q(w, z)$

$\Leftrightarrow$ { move $\exists x \forall y$ outside }

$\exists x \forall y(\neg R(x, y) \vee \forall z \exists w \neg Q(w, z))$

$\Leftrightarrow$ { move $\forall z \exists w$ outside }

$\exists x \forall y \forall z \exists w(\neg R(x, y) \vee \neg Q(w, z))$

The Skolem normal form for this prenex formula is

$$\forall y \forall z(\neg R(c, y) \vee \neg Q(f(y, z), z))$$

(c) Horn algorithm for the Horn sentence

$((A \wedge D) \to H) \wedge (G \to A) \wedge (C \to D) \wedge ((A \wedge B \wedge H) \to \bot) \wedge C \wedge ((C \wedge D) \to G) \wedge ((A \wedge G) \to B)$

A	B	C	D	G	H	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
4	5	1	2	3	6	7

In step 1, we assign $\top$ to C, because C is a conjunct of the Horn sentence.
In step 2, we observe that the premiss of conjunct $C \to D$ is true, so we assign $\top$ to D.
In step 3, we observe that the premiss of conjunct $(C \wedge D) \to G$ is true, so we assign $\top$ to G.
In step 4, we observe that the premiss of conjunct $G \to A$ is true, so we assign $\top$ to A.
In step 5, we observe that the premiss of conjunct $(A \wedge G) \to B$ is true, so we assign $\top$ to B.
In step 6, we observe that the premiss of conjunct $(A \wedge D) \to H$ is true, so we assign $\top$ to H.
In step 7, we observe that the premiss of conjunct $(A \wedge B \wedge H) \to \bot$ is true, so we must assign $\top$ to $\bot$. We conclude that the Horn formula is *not satisfiable*.

7.

1. $\forall x P(x) \to \exists x Q(x)$	
2. $\neg \exists x (P(x) \to Q(x))$	
3. $\boxed{a}$	
4. $\neg P(a)$	
5. $P(a)$	
6. $\bot$	$\bot$ Intro: 5, 4
7. $Q(a)$	$\bot$ Elim: 6
8. $P(a) \to Q(a)$	$\to$ Intro: 5–7
9. $\exists x (P(x) \to Q(x))$	$\exists$ Intro: 8
10. $\bot$	$\bot$ Intro: 9, 2
11. $\neg\neg P(a)$	$\neg$ Intro: 4–10
12. $P(a)$	$\neg$ Elim: 11
13. $\forall x P(x)$	$\forall$ Intro: 3–12
14. $\exists x Q(x)$	$\to$ Elim: 1, 13
15. $Q(b)$	
16. $P(b)$	
17. $Q(b)$	Reit: 15
18. $P(b) \to Q(b)$	$\to$ Intro: 16–17
19. $\exists x (P(x) \to Q(x))$	$\exists$ Intro: 18
20. $\exists x (P(x) \to Q(x))$	$\exists$ Elim: 14, 15–19
21. $\bot$	$\bot$ Intro: 20, 2
22. $\neg\neg \exists x (P(x) \to Q(x))$	$\neg$ Intro: 2–21
23. $\exists x (P(x) \to Q(x))$	$\neg$ Elim: 22