Introduction to Logic
Lecture 4

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29 November 2023

Overview

Tautologies and logical truths

Logical and tautological consequence

Logically equivalent sentences

Substitution

Negation Normal Form (NNF)

Other rules for simplification of sentences

Conjunctive normal form and disjunctive normal form

Truth-functional completeness

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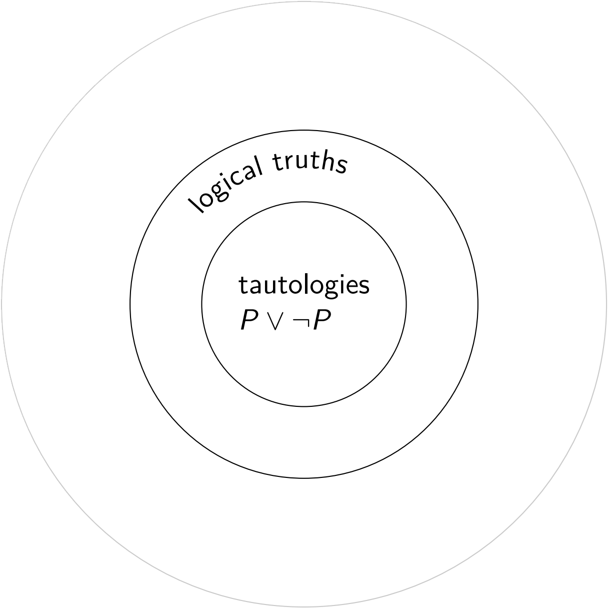
Recap Lecture 3: the big picture of necessities

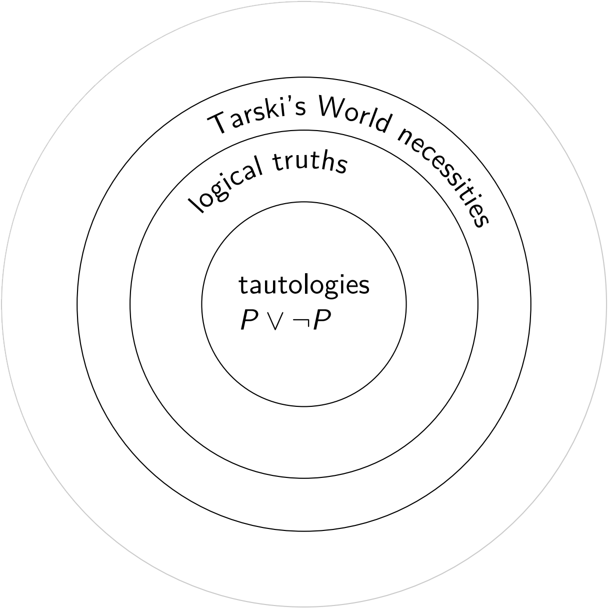
tautologies

$P \vee \neg P$

Recap Lecture 3: the big picture of necessities



Recap Lecture 3: the big picture of necessities



Recap Lecture 3: Tautologies

Tautology: Definition

A sentence S is a *tautology* if and only if (iff) in the truth table for S, there are only T's in the column under the *main connective* of S.

Example:

P	P	$\vee$	$\neg P$
T		T	F
F		T	T

Recap Lecture 3: Tautologies vs. logical truths

Logical truth: Definition

A sentence is a logical truth (or: logical necessity) iff it is true 'under all circumstances'.

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So every tautology is a logical truth.

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So every tautology is a logical truth. But not the other way. E.g.:

$a = a$	$a = b$	$a = a \vee a = b$
T	T	T
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$a = a$	$a = b$	$a = a \vee a = b$
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The last two rows are *spurious*!
It is *logically impossible* that $a = a$ is false.

Recap Lecture 3: Tarski's World necessities

Tarski's World necessary (TW-necessary)

A sentence is Tarski's World necessary if and only if it is true under all circumstances that we can construct using Tarski's World.

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Examples of Tarski's World necessities:

$\text{Tet(a)} \vee \text{Dodec(a)} \vee \text{Cube(a)}$

$(\neg\ \text{Small(a)} \wedge \neg\ \text{Large(a)}) \rightarrow \text{Medium(a)}.$

The above two sentences are not logical truths, and certainly not tautologies.

New material: Contradiction, TT-possibility

Contradiction: Definition

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$\boxed{\text{QUESTION}}$ Example?

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$\boxed{\text{QUESTION}}$ Example? $P \land \neg P$

TT-possible: Definition

A sentence S is *TT-possible* iff in the truth table (TT) for S, there is at least one T in the column under S's main connective.

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Tautology: Definition (reminder)

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What do you think of the sentence $((P \to Q) \to P) \to P$?
Is it a tautology? A contradiction? A TT-possibility?

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What do you think of the sentence $((P \rightarrow Q) \rightarrow P) \rightarrow P$?
Is it a tautology? A contradiction? A TT-possibility?
Answers: Yes; No; Yes

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$\boxed{\text{NOTE}}$ For each sentence S, the following three are equivalent:

1) S is a tautology;

2) $\neg S$ is a contradiction;

3) $\neg S$ is not TT-possible.

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NOTE For each sentence S, the following three are equivalent:

1) S is a tautology;

2) $\neg S$ is a contradiction;

3) $\neg S$ is not TT-possible. QUESTION Why?

Logical possibility

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A sentence S is a logical possibility iff *there is a non-spurious row of the truth table* for S, with a T in the column under the main connective of S.

So every logical possibility is a TT-possibility.

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So every logical possibility is a TT-possibility. But not the other way. Consider $a = b \land \neg(b = a)$:

$a = b$	$b = a$	$a = b$	$\land$	$\neg(b = a)$
T	T		F	F
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TT-possibility (true in row 2) but not logically possible!

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TT-possibility (true in row 2) but not logically possible! Rows 2 and 3 are spurious.

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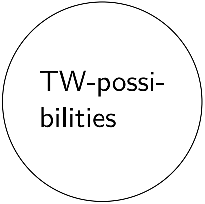
Example of a Tarski's World possibility (therefore also a logical possibility and a TT-possibility):

$\text{Tet(a)} \leftrightarrow \text{Dodec(a)}$

The following sentence is not Tarski's World possible (although it is logically possible and TT-possible):

$\neg\ (\text{Small(a)} \vee \text{Large(a)} \vee \text{Medium(a)})$.

The big picture of possibilities



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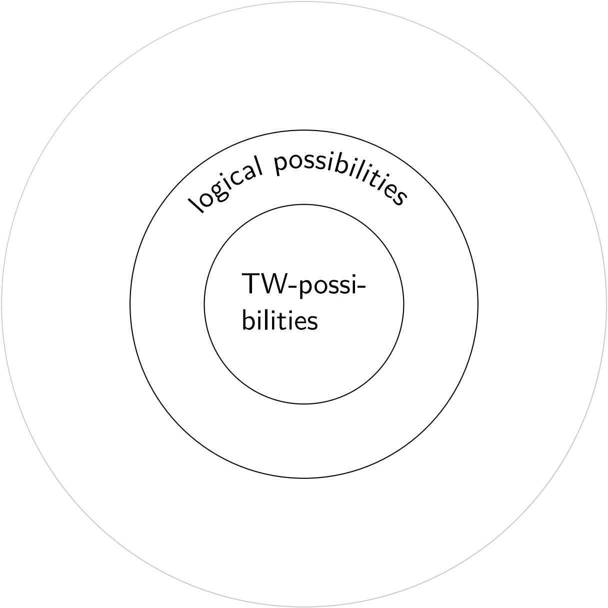

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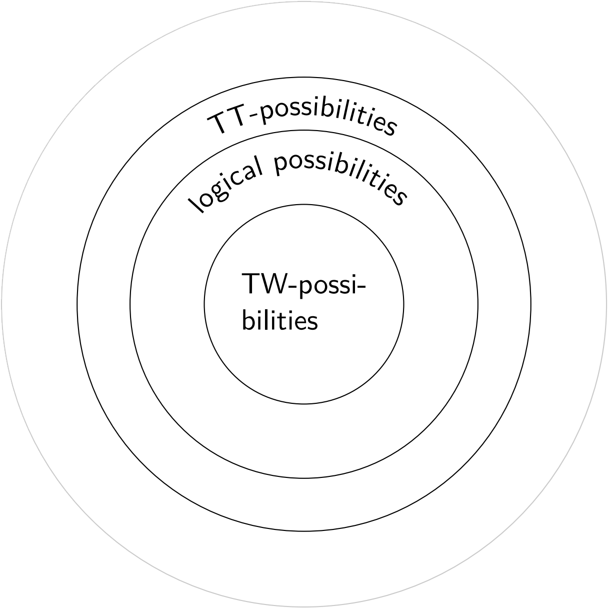

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Definitions validity and tautological consequence

Validity: reminder

An argument is *valid* if and only if it is impossible that the premises are true while the conclusion is false.
That is, an argument is *valid* if and only if in every situation in which all the premises are true, the conclusion is also true.

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For purely propositional sentences, the following are equivalent:

- Q is a tautological consequence of $P_1, \ldots, P_n$;
- $(P_1 \wedge \cdots \wedge P_n) \rightarrow Q$ is a tautology.

Example

Question: Is $P \to R$ a tautological consequence of premises $P \to Q$ and $Q \to R$?

P	Q	R	$P \to Q$	$Q \to R$	$P \to R$

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Thus the argumentation scheme is *valid*. It is a *tautological consequence*, because in all 4 rows in the truth table in which the premises $P \to Q$ and $Q \to R$ are both true, the conclusion $P \to R$ is true as well.

Definition

Logical consequence

Conclusion Q is a *logical consequence* of premises $P_1, \ldots, P_n$ if and only if for *every non-spurious row* in the truth table in which $P_1, \ldots, P_n$ are all true, Q is also true.

Example: $a = a \wedge a = b$ is a logical consequence of $a = b$.

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Tautological consequence: reminder

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So, if Q is a tautological consequence of premises $P_1, \ldots, P_n$, then Q is also a logical consequence of $P_1, \ldots, P_n$.
The reverse does not hold: $a = a \wedge a = b$ is a logical consequence of $a = b$, but not its tautological consequence.

Some notation for logical consequence

Reminder: ‘iff’ is an abbreviation for ‘if and only if’.

Logical consequence

Q is a *logical consequence* of P iff the argument

$$\begin{array}{|l} P \\ \hline Q \end{array}$$

is valid.

Notation: $P \Rightarrow Q$

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The following are equivalent (for purely propositional sentences):

- $P_1, \ldots, P_n \Rightarrow Q$
  (i.e. Q is a logical consequence of $P_1, \ldots, P_n$);
- the sentence $(P_1 \wedge \cdots \wedge P_n) \rightarrow Q$ is a logical truth;
- there is a proof in Fitch of Q from the premises $P_1, \ldots, P_n$.

Validity and proof: soundness and completeness

Let $\mathcal{F}$ be the *natural deduction* system introduced in lectures 1-3:

There exists a proof of Q in $\mathcal{F}$ from premises $P_1, \ldots, P_n$ $\overbrace{\Longrightarrow}^{\text{soundness}}$

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Overview

Logical equivalence (LPL Section 4.5)

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Q is *logically equivalent* with P iff both $P \Rightarrow Q$ and $Q \Rightarrow P$.
In words: Q is a logical consequence of P and *vice versa*.
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Two sentences S and T are *tautologically equivalent* iff in every row of the truth table the formulas receive the same truth value.

For purely propositional sentences, the following are equivalent:

- $S \Leftrightarrow T$;
- S and T are tautologically equivalent;
- $S \leftrightarrow T$ is a tautology;
- there is a proof in Fitch of $S \leftrightarrow T$.

Examples of logical equivalences

In the lectures 1–3, we have already seen the following logical equivalences:

law of double negation	P	$\Leftrightarrow$	$\neg\neg P$
definability of $\rightarrow$	$P \rightarrow Q$	$\Leftrightarrow$	$\neg P \vee Q$
definability of $\leftrightarrow$	$P \leftrightarrow Q$	$\Leftrightarrow$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$
	$P \leftrightarrow Q$	$\Leftrightarrow$	$(\neg P \vee Q) \wedge (\neg Q \vee P)$
	$P \leftrightarrow Q$	$\Leftrightarrow$	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$

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Substitution of sentences

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Example: $S(P) = \neg(P \vee (A \to (B \leftrightarrow P)))$.

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(Note: "substitute Q for P" means the same as "replace P by Q".)
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In the example: $S(Q) = \neg(Q \vee (A \rightarrow (B \leftrightarrow Q)))$.

You can also substitute more complex sentences for P; and also P itself may be a complex sentence.
In the example:
$S(P \rightarrow Q) = \neg((P \rightarrow Q) \vee (A \rightarrow (B \leftrightarrow (P \rightarrow Q))))$.

Principle of substitution of logical equivalents

For logical equivalence, we have the following property:

Principle of substitution of logical equivalents

If $P \Leftrightarrow Q$, then $S(P) \Leftrightarrow S(Q)$

In words:
If P and Q are logically equivalent, then so are $S(P)$ and $S(Q)$.

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For logical equivalence, we have the following property:

Principle of substitution of logical equivalents

If $P \Leftrightarrow Q$, then $S(P) \Leftrightarrow S(Q)$

In words:
If P and Q are logically equivalent, then so are $S(P)$ and $S(Q)$.

Example: P and $\neg\neg P$ are logically equivalent, so we have

$$\neg(P \vee (A \rightarrow (B \leftrightarrow P)) \Leftrightarrow \neg(\neg\neg P \vee (A \rightarrow (B \leftrightarrow \neg\neg P))$$

Elimination of $\rightarrow$ and $\leftrightarrow$

Claim: Every sentence is logically equivalent to a sentence that contains no $\rightarrow$ and no $\leftrightarrow$.

Elimination of $\to$ and $\leftrightarrow$

Claim: Every sentence is logically equivalent to a sentence that contains no $\to$ and no $\leftrightarrow$. $\boxed{\text{QUESTION}}$ True or false?

Elimination of $\rightarrow$ and $\leftrightarrow$

Claim: Every sentence is logically equivalent to a sentence that contains no $\rightarrow$ and no $\leftrightarrow$. $\boxed{\text{QUESTION}}$ True or false?

True! To see this, use the *substitution property* and the *logical laws* about the definability of $\rightarrow$ and $\leftrightarrow$:

$$P \rightarrow Q \quad \Leftrightarrow \quad \neg P \vee Q$$

$$P \leftrightarrow Q \quad \Leftrightarrow \quad (\neg P \vee Q) \wedge (\neg Q \vee P)$$

Elimination of $\rightarrow$ and $\leftrightarrow$

Claim: Every sentence is logically equivalent to a sentence that contains no $\rightarrow$ and no $\leftrightarrow$. $\boxed{\text{QUESTION}}$ True or false?

True! To see this, use the *substitution property* and the *logical laws* about the definability of $\rightarrow$ and $\leftrightarrow$:

$$P \rightarrow Q \quad \Leftrightarrow \quad \neg P \vee Q$$

$$P \leftrightarrow Q \quad \Leftrightarrow \quad (\neg P \vee Q) \wedge (\neg Q \vee P)$$

Example:

$A \rightarrow (B \leftrightarrow C)$

$\Leftrightarrow \qquad \{ \text{ definability of } \rightarrow \}$

$\neg A \vee (B \leftrightarrow C)$

$\Leftrightarrow \qquad \{ \text{ definability of } \leftrightarrow \}$

$\neg A \vee ((\neg B \vee C) \wedge (\neg C \vee B))$

Riddle: The portrait and the three boxes

There is a portrait hidden in one of three boxes: gold, silver, lead.

Each box contains two statements.

No box contains more than one false statement.

Gold box

1. The portrait is not in here. 2. The artist is from Venice.

Silver box

1. The portrait is not in the gold box.
2. The artist is from Florence.

Lead box

1. The portrait is not in here. 2. The portrait is in the silver box.

In which box is the portrait to be found?

Overview

Tautologies and logical truths

Logical and tautological consequence

Logically equivalent sentences

Substitution

Negation Normal Form (NNF)

Other rules for simplification of sentences

Conjunctive normal form and disjunctive normal form

Truth-functional completeness

Augustus De Morgan (1806–1871)

De Morgan was the first to prove . . .

. . . the duality of $\wedge$ and $\vee$

For every sentence P and Q:

$$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$$

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$

These *De Morgan laws* are useful for "pushing negations inwards" to apply only to atomic sentences.

Negation Normal Form (NNF) (LPL Section 4.5)

Negation Normal Form (NNF)

A sentence is in Negation Normal Form (NNF) iff it is built out of atomic sentences using the three connectives $\wedge, \vee, \neg$, and all negations in it occur directly in front of atomic sentences.

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Examples. Let A, B, C be atomic sentences.

- $(\neg A \land \neg B) \lor C$

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- $(\neg A \wedge \neg B) \vee C$ is in negation normal form
- $\neg A \vee \neg B \vee \neg C$

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- $(\neg A \wedge \neg B) \vee C$ is in negation normal form
- $\neg A \vee \neg B \vee \neg C$ is in negation normal form
- $\neg(A \vee B)$

Negation Normal Form (NNF) (LPL Section 4.5)

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A sentence is in Negation Normal Form (NNF) iff it is built out of atomic sentences using the three connectives $\wedge, \vee, \neg$, and all negations in it occur directly in front of atomic sentences.

Examples. Let A, B, C be atomic sentences.

- $(\neg A \wedge \neg B) \vee C$ is in negation normal form
- $\neg A \vee \neg B \vee \neg C$ is in negation normal form
- $\neg(A \vee B)$ is *not* in negation normal form: the negation does not occur directly in front of the atomic sentence A.

Rewriting sentences to Negation Normal Form (NNF)

$\boxed{\text{QUESTION}}$ Do you remember what is a literal?

Rewriting sentences to Negation Normal Form (NNF)

$\boxed{\text{QUESTION}}$ Do you remember what is a literal? A *literal* is an atomic formula (A) or a negation of an atomic formula ($\neg A$).

Rewriting to Negation Normal Form (NNF)

Any sentence built out of atomic sentences using the three connectives $\wedge, \vee, \neg$ is logically equivalent to one built from *literals* using just $\vee$ and $\wedge$.

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Any sentence built out of atomic sentences using the three connectives $\wedge, \vee, \neg$ is logically equivalent to one built from *literals* using just $\vee$ and $\wedge$.

How to make an NNF from a sentence? First eliminate $\rightarrow$ and $\leftrightarrow$. Then you are left with a sentence built up of $\wedge, \vee$ and $\neg$. Rewrite it as a sentence in which $\neg$ only applies to atomic sentences.
Use the *principle of substitution with logical equivalents* to generate the NNF, by the laws of De Morgan:

$$\neg(P \wedge Q) \quad \Leftrightarrow \quad \neg P \vee \neg Q$$

$$\neg(P \vee Q) \quad \Leftrightarrow \quad \neg P \wedge \neg Q$$

and the law of double negation: $\neg\neg P \Leftrightarrow P$.

Example of rewriting a sentence to NNF

$$
\begin{array}{ll}
 & \neg((A \vee B) \wedge \neg C) \\
\Leftrightarrow & \quad \{ \text{ De Morgan: } \neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q \ \} \\
 & \neg(A \vee B) \vee \neg\neg C \\
\Leftrightarrow & \quad \{ \text{ Law of double negation: } \neg\neg P \Leftrightarrow P \ \} \\
 & \neg(A \vee B) \vee C \\
\Leftrightarrow & \quad \{ \text{ De Morgan: } \neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q \ \} \\
 & (\neg A \wedge \neg B) \vee C
\end{array}
$$

The result is a sentence in **negation normal form**

Overview

Simplification of complex sentences: rules

A number of further logical equivalences are useful when we wish to simplify complex sentences:

idempotency of $\wedge$	$P \wedge P$	$\Leftrightarrow$	P
idempotency of $\vee$	$P \vee P$	$\Leftrightarrow$	P
commutativity of $\wedge$	$P \wedge Q$	$\Leftrightarrow$	$Q \wedge P$
commutativity of $\vee$	$P \vee Q$	$\Leftrightarrow$	$Q \vee P$
associativity of $\wedge$	$(P \wedge Q) \wedge R$	$\Leftrightarrow$	$P \wedge (Q \wedge R)$
associativity of $\vee$	$(P \vee Q) \vee R$	$\Leftrightarrow$	$P \vee (Q \vee R)$

Applying the new rules in an example

$$((C \wedge A) \wedge (B \wedge (C \wedge B)) \wedge C)$$

$$\Leftrightarrow \quad \{ \text{ associativity: } (P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R) \ \}$$

$$C \wedge A \wedge B \wedge C \wedge B \wedge C$$

$$\Leftrightarrow \quad \{ \text{ commutativity: } P \wedge Q \Leftrightarrow Q \wedge P \ \}$$

$$A \wedge B \wedge B \wedge C \wedge C \wedge C$$

$$\Leftrightarrow \quad \{ \text{ idempotency: } P \wedge P \Leftrightarrow P \ \}$$

$$A \wedge B \wedge C$$

Annotated linear chains of equivalences

General form:

$$
\begin{array}{ll}
 & A \\
\Leftrightarrow & \quad \{ \text{ reason why } A \Leftrightarrow B_1 \ \} \\
 & B_1 \\
\Leftrightarrow & \quad \{ \text{ reason why } B_1 \Leftrightarrow B_2 \ \} \\
 & B_2 \\
 & \vdots \\
 & B_n \\
\Leftrightarrow & \quad \{ \text{ reason why } B_n \Leftrightarrow C \ \} \\
 & C
\end{array}
$$

Example of using the logical rules

$\boxed{\text{QUESTION}}$: which rule is used in each step? And from which step onwards is the formula in NNF?

Chain of equivalences

Suppose A, B, C are arbitrary atomic sentences

$$(A \vee B) \wedge C \wedge (\neg(\neg B \wedge \neg A) \vee B) \Leftrightarrow$$

$$(A \vee B) \wedge C \wedge ((\neg\neg B \vee \neg\neg A) \vee B) \Leftrightarrow$$

$$(A \vee B) \wedge C \wedge ((B \vee A) \vee B) \Leftrightarrow$$

$$(A \vee B) \wedge C \wedge (B \vee A \vee B) \Leftrightarrow$$

$$(A \vee B) \wedge C \wedge (A \vee B \vee B) \Leftrightarrow$$

$$(A \vee B) \wedge C \wedge (A \vee B) \Leftrightarrow$$

$$(A \vee B) \wedge (A \vee B) \wedge C \Leftrightarrow$$

$$(A \vee B) \wedge C$$

Overview

- Tautologies and logical truths
- Logical and tautological consequence
- Logically equivalent sentences
- Substitution
- Negation Normal Form (NNF)
- Other rules for simplification of sentences
- **Conjunctive normal form and disjunctive normal form**
- Truth-functional completeness

Normal forms and use of principles from propositional logic

Normal forms are restrictions to the form of a sentence S.

Examples we have seen:

- S contains only $\neg$, $\wedge$, $\vee$ and not $\rightarrow$, $\leftrightarrow$
- $\neg$ occurs only in front of atomic sentences (NNF)

Normal forms and use of principles from propositional logic

Normal forms are restrictions to the form of a sentence S.

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Logical equivalences are what allow us to simplify complex sentences.

We have already seen at work: the law of double negation, the definability of $\rightarrow$ and $\leftrightarrow$, the De Morgan laws, and the laws of associativity, commutativity and idempotence.

We will now additionally introduce the **laws of distributivity**.

Distributivity of $\wedge$ over $\vee$ and vice versa

$\boxed{\text{IDEA}}$ Like distributivity of multiplication over addition:
$a \times (b + c) = (a \times b) + (a \times c)$

In PL and FOL, $\wedge$ is distributive over $\vee$ and vice versa.

1. Distribution of $\wedge$ over $\vee$:

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$$

2. Distribution of $\vee$ over $\wedge$:

$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

Disjunctive Normal Form: DNF (LPL Section 4.6)

Reminder: A *literal* is an atomic formula (A) or a negation of an atomic formula ($\neg A$).

Disjunctive Normal Form

A sentence is in *disjunctive normal form* (DNF) iff it is a disjunction of (one or more) conjunctions of (one or more) literals.

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Using the first distributive law (the distribution of $\wedge$ over $\vee$) and commutativity, we can **turn any sentence that is already in negation normal form into a sentence that is in DNF**.

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Example of DNF

$(P \wedge \neg Q \wedge R) \vee (\neg P \wedge S)$ is in DNF.

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Example of DNF

$(P \wedge \neg Q \wedge R) \vee (\neg P \wedge S)$ is in DNF.

Sentences *not* in DNF:

$$P \wedge (Q \vee \neg R \vee \neg P)$$
$$\neg\neg A$$
$$\neg(A \wedge B)$$

Example: Rewriting a NNF into DNF

- Procedure: start from an NNF and use the first distributive law and commutativity to generate this DNF.

$$(A \vee B) \wedge (C \vee D)$$

$$\Leftrightarrow [(A \vee B) \wedge C] \vee [(A \vee B) \wedge D]$$

$$\Leftrightarrow (A \wedge C) \vee (B \wedge C) \vee [(A \vee B) \wedge D]$$

$$\Leftrightarrow (A \wedge C) \vee (B \wedge C) \vee (A \wedge D) \vee (B \wedge D)$$

Conjunction normal form: CNF (LPL Section 4.6)

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A sentence is in *conjunctive normal form* (CNF) iff it is a conjunction of (one or more) disjunctions of (one or more) literals.

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A sentence is in *conjunctive normal form* (CNF) iff it is a conjunction of (one or more) disjunctions of (one or more) literals.

Using the second distributive law (the distribution of $\vee$ over $\wedge$) and commutativity, we can **turn any sentence that is already in negation normal form into a sentence that is in CNF**.

Conjunction normal form: CNF (LPL Section 4.6)

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A sentence is in *conjunctive normal form* (CNF) iff it is a conjunction of (one or more) disjunctions of (one or more) literals.

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Example and non-example of CNF

$(P \vee \neg Q \vee R) \wedge (\neg P \vee S)$ is in CNF.
$P \vee (Q \wedge \neg R \wedge \neg P)$ is not in CNF. (But it is in DNF)

Example: Rewriting a NNF into CNF

- Procedure: start from an NNF and use the second distributive law to generate this CNF.

$$(A \wedge B) \vee (C \wedge D)$$

$$\Leftrightarrow [(A \wedge B) \vee C] \wedge [(A \wedge B) \vee D]$$

$$\Leftrightarrow (A \vee C) \wedge (B \vee C) \wedge [(A \wedge B) \vee D]$$

$$\Leftrightarrow (A \vee C) \wedge (B \vee C) \wedge (A \vee D) \wedge (B \vee D)$$

Another example

First generate the NNF and then the CNF

$\neg((A \vee B) \wedge \neg C)$

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$\Leftrightarrow (\neg A \wedge \neg B) \vee C$

Another example

First generate the NNF and then the CNF

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$\Leftrightarrow \neg(A \vee B) \vee \neg\neg C$

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Some sentences are CNF and DNF simultaneously

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$\boxed{\text{QUESTION}}$ How is that possible?

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Sentences that are both CNF and DNF

A sentence can be simultaneously in CNF and DNF.

$$Home(claire) \wedge \neg Home(max)$$

This sentence is a disjunction of 1 conjunction of literals (namely the sentence itself), so it is in DNF.

This sentence is also in CNF because it is a conjunction of two disjunctions, each consisting of 1 literal.

Some sentences that are in CNF and DNF:

$$A \vee \neg B \vee C$$

$$\neg A \wedge B$$

$$A$$

Tests for DNF and CNF

Reminder: CNF and DNF are named for the main operator:

Conjunctive Normal Form

A sentence is in Conjunctive Normal Form (CNF) if it is a *conjunction* of *disjunctions* of literals.

Disjunctive Normal Form

A sentence is in Disjunctive Normal Form (DNF) if it is a *disjunction* of *conjunctions* of literals.

DNF test for sentences solely built up from $\wedge$,$\neg$,$\vee$

- Are all the negations directly in front of atomic sentences?
- Are all conjunction signs in between literals?

CNF test for sentences solely built up from $\wedge$,$\neg$,$\vee$

- Are all the negations directly in front of atomic sentences?
- Are all disjunction signs directly in between literals?

Overview

Truth-functional completeness (LPL Section 7.4)

The connectives $\neg, \vee, \wedge \rightarrow$ and $\leftrightarrow$ can be seen as *truth value* functions: you put in one or two truth values, and the result is a truth value.

The truth table of the connective specifies the truth value function.

Truth-functional completeness (LPL Section 7.4)

The connectives $\neg, \vee, \wedge \rightarrow$ and $\leftrightarrow$ can be seen as *truth value* functions: you put in one or two truth values, and the result is a truth value.

The truth table of the connective specifies the truth value function.

More generally, any sentence with atoms $P_1, \ldots, P_n$ can be seen as an n-ary truth function.

Truth-functional completeness: binary connectives

$\boxed{\text{QUESTION}}$ How many different binary connectives are there?

In other words: how many binary truth value functions are there?

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truth table

P	Q	$P * Q$
T	T	
T	F	
F	T	
F	F	

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T	F	T/F
F	T	T/F
F	F	T/F

In every row there are 2 choices: T or F, so you end up with $2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$ different binary connectives (truth functions).

16 possible binary connectives: examples

We already know four of the possible binary connectives: $\vee, \wedge, \rightarrow, \leftrightarrow$. Thus there are 12 other possible binary connectives left.

Let us consider some others.
Example: xor , the exclusive disjunction.

16 possible binary connectives: examples

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T	F	T
F	T	T
F	F	F

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F	T	T
F	F	F

xor is definable: we have P xor $Q \Leftrightarrow (P \vee Q) \wedge \neg(P \wedge Q)$

Also, P xor $Q \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Two interesting binary connectives

Two other binary connectives: nor and nand .

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Two other binary connectives: nor and nand .

P	Q	P nor Q	P nand Q
T	T	F	F
T	F	F	T
F	T	F	T
F	F	T	T

Two interesting binary connectives

Two other binary connectives: nor and nand .

P	Q	P nor Q	P nand Q
T	T	F	F
T	F	F	T
F	T	F	T
F	F	T	T

They are definable, too: we have

$$P \text{ nor } Q \Leftrightarrow \neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$$

$$P \text{ nand } Q \Leftrightarrow \neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$$

Also,

$$P \text{ nand } Q \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$$

Definability of connectives

We see a pattern:

- We come up with a new binary connective, defined by a truth table;
- it appears that this connective is definable using only $\neg$, $\wedge$ and $\vee$.

We shall show that this holds for *all* new binary connectives.

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We will show even more:
Every n-ary connective defined by a truth table can be defined by a sentence $S(P_1, \ldots, P_n)$ that uses only $\neg$, $\wedge$ and $\vee$.
This may be surprising, for there are *many* n-ary connectives: 2^{2^n}.

$$\begin{array}{rlcl} \textit{unary} & 2^{2^1} & = & 4 \\ \textit{binary} & 2^{2^2} & = & 16 \\ \textit{ternary} & 2^{2^3} & = & 256 \\ \textit{quaternary} & 2^{2^4} & = & 65536 \\ \textit{quintary} & 2^{2^5} & = & 4294967296 \end{array}$$

Defining ternary connectives

We consider an 'arbitrary' ternary connective $\clubsuit$:

P	Q	R	$\clubsuit(P, Q, R)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

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F	T	F	F
F	F	T	T
F	F	F	F

We want to find a sentence $S(P, Q, R)$ with as connectives only $\neg, \wedge, \vee$, such that $\clubsuit(P, Q, R) \Leftrightarrow S(P, Q, R)$.
How?

Definability of ternary connectives

P	Q	R	$\clubsuit(P, Q, R)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
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F	F	F	F

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T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

The truth table says: $\clubsuit(P, Q, R)$ is true iff

- P and Q and R are all true, or
- P and Q are both true and R is false, or
- P is false and Q and R are both true, or
- P and Q are both false and R is true.

Definability of ternary connectives

P	Q	R	$\clubsuit(P,Q,R)$	
T	T	T	T	$(P \wedge Q \wedge R) \vee$
T	T	F	T	$(P \wedge Q \wedge \neg R) \vee$
T	F	T	F	
T	F	F	F	
F	T	T	T	$(\neg P \wedge Q \wedge R) \vee$
F	T	F	F	
F	F	T	T	$(\neg P \wedge \neg Q \wedge R)$
F	F	F	F	

Definability of ternary connectives

P	Q	R	$\clubsuit(P, Q, R)$	
T	T	T	T	$(P \wedge Q \wedge R) \vee$
T	T	F	T	$(P \wedge Q \wedge \neg R) \vee$
T	F	T	F	
T	F	F	F	
F	T	T	T	$(\neg P \wedge Q \wedge R) \vee$
F	T	F	F	
F	F	T	T	$(\neg P \wedge \neg Q \wedge R)$
F	F	F	F	

and we get

$$\clubsuit(P, Q, R) \Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$$

Definability of the n-ary connective $\spadesuit$

You can use the following procedure to construct a sentence $S(P_1, \ldots, P_n)$ with as connectives only $\neg, \wedge, \vee$,
such that $S(P_1, \ldots, P_n)$ is logically equivalent to $\spadesuit(P_1, \ldots, P_n)$.

- First define the conjunctions $C_1, \ldots.. C_m$ where $m = 2^n$, corresponding to the situations described by the 2^n rows of the truth table.
- Construct a disjunction $S(P_1, \ldots, P_n)$ in which you include C_k as one of the disjuncts, if and only if row k in the truth table has value T.
- But if all rows get value F, then we equate $S(P_1, \ldots, P_n)$ to $P_1 \wedge \neg P_1$, or simply the logically equivalent $\bot$.

We have just sketched a proof that any truth function, of any arity whatsoever, can be expressed using just the Boolean connectives $\neg, \wedge$ and $\vee$, even in DNF.

Truth-functional completeness

Definition of "truth-functionally complete"

We say that a set of connectives is **truth-functionally complete** if the connectives in the set allow us to express any truth function.

So the set $\{\neg, \wedge$ and $\vee\}$ is truth-functionally complete.
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Economizing on connectives

We can use the De Morgan's Laws to obtain the following:

The set of operators $\{\neg, \vee\}$ is truth-functionally complete.

The set of operators $\{\neg, \wedge\}$ is also truth-functionally complete.

Is it possible to find one operator $\downarrow$ that is in itself truth-functionally complete?

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The same for $\rightarrow$, $\leftrightarrow$ and xor .

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But look at nor . We have

$$P \text{ nor } P \Leftrightarrow \neg(P \vee P) \Leftrightarrow \neg P$$

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And now we can define $\vee$, too, for

$$P \vee Q \Leftrightarrow \neg(P \text{ nor } Q) \Leftrightarrow (P \text{ nor } Q) \text{ nor } (P \text{ nor } Q)$$

So nor does it all: it is truth-functionally complete.

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The same holds for nand .

Disadvantages of economizing on connectives

As we saw, nor is truth-functionally complete, and so is nand. So we can write every sentence using only the connective nor (or nand).

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- it becomes very hard to *understand* a sentence written with only nor .

The second reason also applies to some extent to $\{\neg, \wedge\}$ and to $\{\neg, \vee\}$.

For good understanding of a sentence, the collection of connectives $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ is most often a good basis.

Therefore, in the rest of LPL the authors do not skimp on connectives.

Next time: Monday, December 4

Formative Homework assignment 1

Available on Nestor, under *Tests/Homeworks*.

Hand in a physical copy, at the start of lecture 5 on **Monday 4 December 13:00**, work in pairs.

Topics of the lecture of Monday December 5

Set theory and functions.