Introduction to Logic
Lecture 6

Gerard Renardel de Lavalette

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$\forall x(Need(you, x) \rightarrow Love(x))$

Overview

Quantifiers in English vs. FOL

History of quantifiers: from Aristotle to Frege

Translations of complex noun phrases

Semantics and the Hintikka game

Reminder: formal proofs, all rules so far

Universal quantifier rules
- Universal instantiation and $\forall$ Elim
- Universal generalization and $\forall$ Intro

Existential quantifier rules
- Existential generalization and $\exists$ Intro
- Existential instantiation and $\exists$ Elim

Proof strategies

Quantifiers in English

- Every
- Many
- Some
- No
- Few
- Most

Quantifiers in English

- Everyone is valuable.
- Every student is at least 18 years old.
- Many students work hard.
- Some exams are doable.
- No book is perfect.
- Few books are dull.
- Most students like logic.

Some complex quantified expressions in English

The following noun phrases in red are called *quantified expressions*, and the sentences containing them *quantified sentences*.

- At least two major national football teams have not participated in the last World Cup.
- Every Dutch person is deemed to know the law.
- Every natural number has a unique decomposition into prime factors.
- At least half of the times when Bob asks her a question, Alice knows the answer.

Extending the language of FOL

Ingredients of the language of FOL up to now:

- constants a, b, c, . . .

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- terms a, f(a), f(f(a)), g(b,c), g(g(f(c),c),f(g(a,b))), ...

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- predicates P, Q, A, B, ...
- connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- sentences P(a), Q(b)$\rightarrow$ R(a,b), ...

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To be added:

- variables x, y, z, ...
- quantifiers $\forall, \exists$
- well-formed formulas (wffs) P(x), A(y,a)$\vee$B(x), ...

Variables and atomic formulas

Variables: u, v, w, x, y, z (in Tarski's World)
Variables can also occur with subscripts: $y_1, y_2, y_3 \ldots, x_1, x_2, \ldots, y$
(not in Tarski's World; OK for Fitch)

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Atomic formulas (atomic well-formed formulas, wffs) in FOL

B(a), R(c,d), Q(b,f,e,d,a), a=b
No free variables. They are atomic *sentences*.

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P(x), a=y, Q(b,f,x,y,x), Cube(x)
There are free variables. They are well-formed formulas, but *not* sentences. They can be used together with quantifiers to build sentences.

Universal quantifier

$$\forall$$

Read $\forall x$ as: "For every object x, . . ."

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$\forall x$ Happy(x): Everyone is happy.
Universal quantification over Happy(x)

Existential quantifier

$$\exists$$

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Read $\exists x$ as: "There is at least one object x such that ..."

$\exists x$ Tall(x): Someone is tall.
Existential quantification over Tall(x)

(Well-formed) formulas vs. sentences

Well-formed formulas form a larger class containing the class of all sentences of FOL.

- A formula may contain variables;
- A sentence is a formula in which every variable is *bound* by a quantifier

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We are going to inductively define well-formed languages of FOL.

Remember that terms were inductively defined in Lecture 5.
For example, variables such as x, constants such as d, and more complex constructs involving function symbols such as *mother(deepika)* are terms of languages of FOL.

Inductive definition of well-formed formula of FOL

Well-formed formulas (wffs) can be defined *inductively* as follows:

1. If A is an n-place predicate symbol and each of $t_1, \ldots, t_n$ is a term, then $A(t_1, \ldots, t_n)$ is an *atomic wff*.

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2. If $P, Q, P_1, \ldots, P_n$ are well-formed formulas, then so are:
   - $\neg P$
   - $P_1 \wedge \ldots \wedge P_n$
   - $P_1 \vee \ldots \vee P_n$
   - $P \rightarrow Q$
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3. If P is a well-formed formula and ν is a variable (i.e. one of $u, v, w, x, y, z, x_1, x_2 \ldots$), then $\forall \nu P$ is a well-formed formula, and all occurrences of ν in $\forall \nu P$ are said to be *bound*.

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4. If P is a well-formed formula and ν is a variable (i.e. one of $u, v, w, x, y, z, x_1, x_2 \ldots$), then $\exists \nu P$ is a well-formed formula, and all occurrences of ν in $\exists \nu P$ are said to be *bound*.

Free and bound variables

Take a formula A. Then $\forall x$ *binds* all occurrences of variable x in $\forall x A$, that were free in A (if any).

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$\boxed{\text{NOTE}}$ The order of quantifiers matters! One single occurrence of a variable cannot be bound by two quantifiers. For example, $\forall x$ does *not* bind the last x in sentence

$$\forall x \exists x R(x)$$

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the x in $\exists x R(x)$ is not free, but has already been bound by $\exists x$. So, a sentence is a *formula without free variables*.

Some examples of formulas and sentences

- $(Cube(x) \lor Dodec(x) \lor Tet(x))$ is a formula with three free occurrences of x.
  $\boxed{\text{NOTE}}$ You may leave out the outer brackets:
  $Cube(x) \lor Dodec(x) \lor Tet(x)$

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- $\exists x Cube(x) \wedge \exists x Small(x)$ is a sentence.
  The first $\exists x$ has as scope $Cube(x)$ and binds the x in it;
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- $\forall x((Cube(x) \wedge Small(x)) \rightarrow \exists y LeftOf(x, y))$ is a sentence.
  The $\forall x$ binds all three occurrences of x in
  $((Cube(x) \wedge Small(x)) \rightarrow \exists y LeftOf(x, y))$.
  The $\exists y$ binds the y in $LeftOf(x, y)$.

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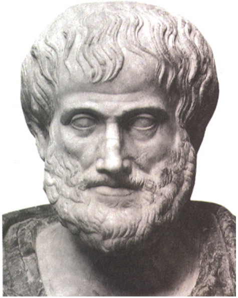
Aristotle of Stagira (384–322 v.C.): Syllogisms



From Aristotle (384-322 BC) until the 19th century, most works on logic were about *syllogisms*.

Example syllogism

Every human is mortal
Every Greek is human
Therefore, every Greek is mortal

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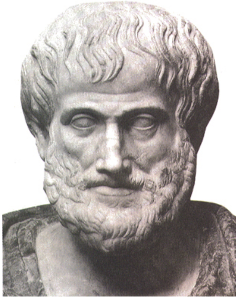

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Syllogisms are based on 4 types of premises:

- universal affirmative: Every S is a P
- particular affirmative: Some S is a P
- universal negative: No S is a P
- particular negative: Some S is not a P

These are all expressible with quantifiers.

First application of quantifiers: syllogisms

Every S is a P

Some S is a P

No S is a P

Some S is not a P

First application of quantifiers: syllogisms

Every S is a P	$\forall x(\mathrm{S}(x) \rightarrow \mathrm{P}(x))$
Some S is a P	
No S is a P	
Some S is not a P	

First application of quantifiers: syllogisms

Every S is a P	$\forall x(\mathsf{S}(x) \to \mathsf{P}(x))$
Some S is a P	$\exists x(\mathsf{S}(x) \wedge \mathsf{P}(x))$
No S is a P	
Some S is not a P	

First application of quantifiers: syllogisms

Every S is a P	$\forall x(\mathsf{S}(x) \rightarrow \mathsf{P}(x))$
Some S is a P	$\exists x(\mathsf{S}(x) \land \mathsf{P}(x))$
No S is a P	$\neg\exists x(\mathsf{S}(x) \land \mathsf{P}(x))$
	Also correct: $\forall x(\mathsf{S}(x) \rightarrow \neg\mathsf{P}(x))$
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These are our modern formulas for the “categorical sentences” used in syllogisms. At the time of Aristotle, there was no notation for quantifiers.

Gottlob Frege (1848–1925): inventor of first-order logic
Begriffschrift (1879)

Basic concept	Frege's notation	Modern notations
Judging	$\vdash A$, $\Vdash A$	$p(A) = 1$ $p(A) = i$
Negation	$\neg A$	$\neg A$; $\sim A$
Conditional (implication)	A B	$B \rightarrow A$ $B \supset A$
Universal quantification	$\overset{\mathfrak{y}}{\smile}\ \Phi(\mathfrak{y})$	$\forall y: \Phi(y)$
Existential quantification	$\neg\overset{\mathfrak{y}}{\smile}\neg\ \Phi(\mathfrak{y})$	$\exists y: \Phi(y)$
Content identity (equal sign)	$A \equiv B$	$A = B$

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Pay attention when translating “some P’s are Q’s”

Consider $\exists x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge Q(x))$.

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Consider $\exists x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge Q(x))$.
$\boxed{\text{QUESTION}}$ Is there any difference?

$\exists$ x (P(x) $\rightarrow$ Q(x)) is true if there is at least one object in the domain of discourse that does not have property P, and also if there is at least one object that has property Q.

E.g., $\exists$ x (Square(x) $\rightarrow$ Circle(x)) is true for a set of geometrical figures as soon as there is at least one circle among them.

The correct translation of “some Ps are Qs” is $\exists$ x (P(x) $\wedge$ Q(x))

Translating complex sentences with quantifiers

At least one cute small kitten was eating

$\exists$ x (C(x) $\wedge$ S(x) $\wedge$ K(x) $\wedge$ E(x))

Translating complex sentences with quantifiers

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All cute small kittens were eating
$\forall$ x ((C(x) $\wedge$ S(x) $\wedge$ K(x)) $\rightarrow$ E(x))

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In general, show as much logical structure as possible. First determine the domain of discourse, then determine the translation key, and finally translate the sentence.

$\boxed{\text{NOTE}}$ The order of the English sentence does not always correspond to the order of its FOL translation.

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Domain of discourse and objects satisfying a formula

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The *domain of discourse* is the (*nonempty*) collection of objects that the quantifiers quantify over.

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Definition of satisfaction

Given a formula $P(x)$, where x is the only unbound variable that occurs in $P(x)$. An object d *satisfies* $P(x)$ if d has the property expressed by $P(x)$.

Semantics of $\forall$, $\exists$

Now we are ready to define the semantics for quantified sentences:

Truth of universal sentence

A formula $\forall x \mathrm{P}(x)$ is true if and only if

every object in the domain of discourse satisfies P(x).

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$\boxed{\text{Note}}$: If every object in the domain of discourse has a name: a, b, c, ..., and the domain is finite, then

$\forall x\ P(x)$ *corresponds to* $P(a) \land P(b) \land P(c) \land \ldots$

$\exists x\ P(x)$ *corresponds to* $P(a) \lor P(b) \lor P(c) \lor \ldots$

Notation P(x) for complex formulas

We often refer to possibly complex formulas of FOL as Q(x) or P(x), e.g. P(x) may stand for:

$\exists$ y (LeftOf(x,y) $\wedge$ FrontOf(y,x))

Then P(b) would stand for the result of replacing all *free* occurrences of x by b:

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Note that such a variable displayed in Q(x) only stands for the free occurrences of x in Q. Example: suppose Q(x) is

$\exists$ y (LeftOf(x,y) $\wedge \exists$ x Cube(x))

Then Q(c) would stand for the result of replacing all *free* occurrences of x by c:

$\exists$ y (LeftOf(c,y) $\wedge \exists$ x Cube(x))

First-order logic and second-order logic

Where does the name first-order logic come from?

- First-order quantifiers quantify over *objects*.
- Second-order quantifiers quantify over *properties*, which can be seen as sets of objects: $\exists P P(a)$. Second-order logic is an advanced topic not treated in this introductory logic course

Hintikka game: Reminder of the rules

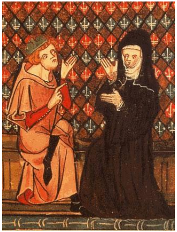
You can use the Hintikka game to find out the truth value of complex sentences in a given situation.

- There are two players: you and the opponent.
- There are two roles: Abelard (commit to false) and Eloise (commit to true).

Game rules for the connectives $\vee, \wedge, \neg$ and for atomic sentences:

$A \vee B$	Eloise (commit to true) chooses A or B and the game continues with that choice.
$A \wedge B$	Abelard (commit-to-false) chooses A or B and the game continues with that choice.
$\neg A$	The players swap roles; the game continues with A.
$A \rightarrow B$	is treated as abbreviation for $\neg A \vee B$
$A \leftrightarrow B$	is treated as abbreviation for $(\neg A \vee B) \wedge (\neg B \vee A)$
atomic sentence	For any atomic sentence such as Large(a), Eloise (commit-to-true) wins if the sentence is true; Abelard (commit-to-false) wins if the sentence is false.

Hintikka game: Reminder of winning strategies



It can be proven that:

Truth

A sentence is true if and only if Eloise can win the game, *no matter how Abelard plays*.

Falsehood

A sentence is false if and only if Abelard can win the game, *no matter how Eloise plays*.

Game rule for the universal quantifier $\forall$

$\forall x P(x)$ Abelard (commit-to-false) chooses an object (with the name c) and the game continues with $P(c)$

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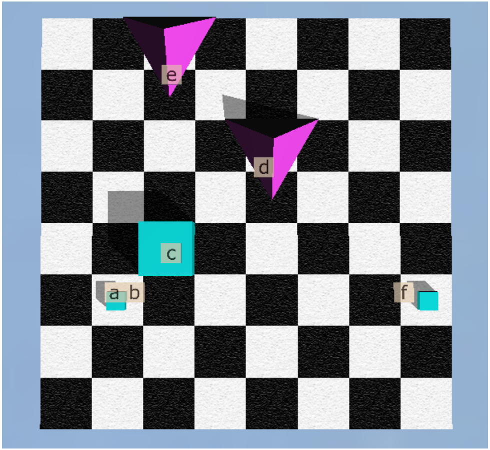
Mnemonics: $\forall$BELARD

Checking truth of a sentence with ∀ in a situation

Is the following sentence true or false in the world below?
Find out by playing the Hintikka game.

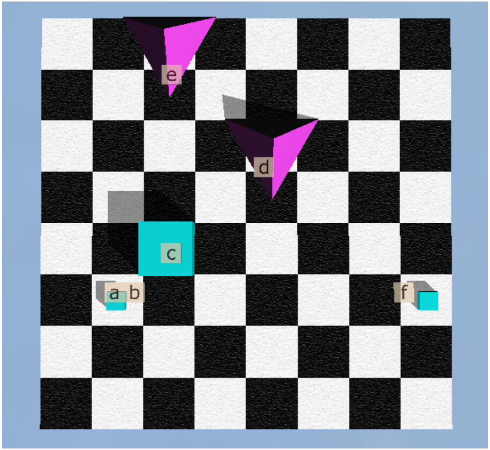
$$\forall x\, (\neg\, \mathrm{Cube}(x) \vee \mathrm{Between}(d,e,x) \vee \mathrm{Between}(c,d,x))$$



Checking truth of a sentence with ∀ in a situation

Answer: Abelard can win the game for the sentence below by choosing object *c*, so the sentence is false.

$$\forall x\ (\neg\ \text{Cube}(x) \lor \text{Between}(d,e,x) \lor \text{Between}(c,d,x))$$



Game rule for the existential quantifier *exists*

$\exists x P(x)$ Eloise (commit-to-true) chooses an object (with the name c) and the game continues with $P(c)$

Game rule for the existential quantifier *exists*

$\exists x P(x)$ Eloise (commit-to-true) chooses an object (with the name c) and the game continues with $P(c)$

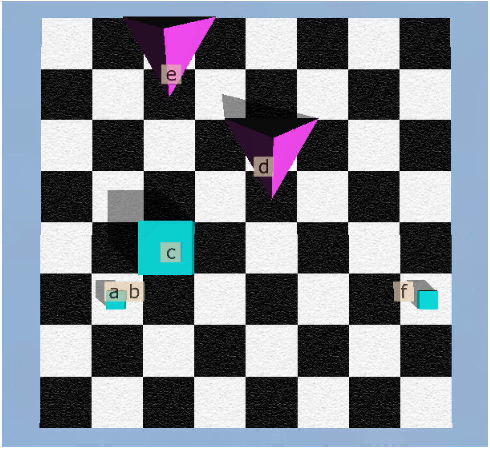
Mnemonics: $\exists$LOISE

Checking truth of a sentence with ∃ in a situation

Are the following sentences true or false in the world below?
Find out by playing the Hintikka game

$$\exists x\ ((\mathrm{Large}(x) \vee \mathrm{Medium}(x)) \wedge \mathrm{Tet}(x) \wedge \neg(a{=}b))$$

$$\exists x\ ((\mathrm{Large}(x) \vee \mathrm{Medium}(x)) \wedge \mathrm{Tet}(x) \wedge \neg(x{=}e))$$



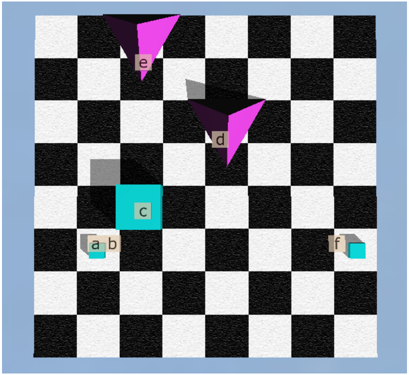
Checking truth of a sentence with ∃ in a situation

Abelard has a winning strategy because $\neg$ (a=b) is false:

$$\exists x\ ((\text{Large}(x) \vee \text{Medium}(x)) \wedge \text{Tet}(x) \wedge \neg(a=b))$$

Eloise wins the game for the following sentence by choosing *d*:

$$\exists x\ ((\text{Large}(x) \vee \text{Medium}(x)) \wedge \text{Tet}(x) \wedge \neg(x=e))$$



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Proof strategies

Summary of proof rules: Reiteration

For a summary of all proof rules, see also the textbook pp. 573–577.

⋮		
j.	P	Justification (or premise)
⋮		
k.	P	Reit: j
⋮		

$=$ Introduction

$$\begin{array}{l|lll} & \vdots & & \\ & \text{k.} & a = a & = \text{Intro} \\ & \vdots & & \end{array}$$

$=$ Elimination

$\vdots$

i. $P(a)$ Justification (or premise)

$\vdots$

j. $a = b$ Justification (or premise)

$\vdots$

k. $P(b)$ $=$Elim: i,j

$\vdots$

$\boxed{\text{Note}}$ You may replace a by b in one place in $P(a)$, or in more places if it occurs more times.

$\boxed{\text{Note}}$ The order of justifications matters.

$\wedge$ Introduction (general)

$\vdots$		
i.	P_1	Justification (or premise)
$\Downarrow$		
j.	P_n	Justification (or premise)
$\vdots$		
k.	$P_1 \wedge \ldots \wedge P_n$	$\wedge$ Intro: i,. . . , j
$\vdots$		

$\wedge$ Elimination (general)

$\vdots$		
i.	$P_1 \wedge \ldots \wedge P_n$	Justification (or premise)
$\vdots$		
k.	P_j	$\wedge$ Elim: i
$\vdots$		

¬ Introduction

⋮

⋮

i. P

⋮

j. $\bot$ (Justification)

k. $\neg P$ $\neg$ Intro: i–j

⋮

$\neg$ Elimination

$\vdots$

i. $\neg\neg P$ Justification (or premise)

$\vdots$

k. P $\neg$ Elim: i

$\vdots$

$\perp$ Introduction

$\vdots$		
i.	P	Justification (or premise)
$\vdots$		
j.	$\neg P$	Justification (or premise)
$\vdots$		
k.	$\perp$	$\perp$ Intro: i, j
$\vdots$		

$\perp$ Elimination

$\vdots$		
i.	$\perp$	Justification (or premise)
$\vdots$		
k.	P	$\perp$ Elim: i
$\vdots$		

$\vee$ Introduction (general)

$\vdots$		
i.	P_j	Justification (or premise)
$\vdots$		
k.	$P_1 \vee \ldots \vee P_n$	$\vee$ Intro: i
$\vdots$		

$\vee$ Elimination (general)

$\vdots$

i. $P_1 \vee \ldots \vee P_n$ Justification (or premise)

j. P_1

$\vdots$

l. R (Justification)

$\Downarrow$

m. P_n

$\vdots$

n. R (Justification)

k. R $\vee$ Elim: i, j–l,. . . ,m–n

$\vdots$

$\to$ Introduction

$\vdots$

$\vdots$

i. P

$\vdots$

j. Q (Justification)

$\vdots$

k. $P \to Q$ $\to$ Intro: i–j

$\vdots$

$\rightarrow$ Elimination

$\vdots$		
i.	$P \rightarrow Q$	Justification (or premise)
$\vdots$		
j.	P	Justification (or premise)
$\vdots$		
k.	Q	$\rightarrow$ Elim: i, j
$\vdots$		

$\leftrightarrow$ Introduction

$\vdots$

$\vdots$

i. P

$\vdots$

j. Q (Justification)

m. Q

$\vdots$

n. P (Justification)

k. $P \leftrightarrow Q$ $\leftrightarrow$ Intro: i–j,m–n

$\vdots$

$\leftrightarrow$ Elimination

$\vdots$		
i.	$P \leftrightarrow Q$ or: $Q \leftrightarrow P$	Justification (or premise)
$\vdots$		
j.	P	Justification (or premise)
$\vdots$		
k.	Q	$\leftrightarrow$ Elim: i,j
$\vdots$		

Riddle: Chess and strategy

Claim:
In the game of chess, one of the players can avoid losing no matter how the other player plays.

True? false? It depends?

About subproofs

When a subproof has ended:

- you may cite the subproof *as a whole* in a $\neg$ Introduction or in an $\vee$ elimination or in a $\rightarrow$ Intro or in a $\leftrightarrow$ Intro ...

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- but you may **not** use individual steps from it!

About subproofs

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- you may cite the subproof *as a whole* in a $\neg$ Introduction or in an $\vee$ elimination or in a $\rightarrow$ Intro or in a $\leftrightarrow$ Intro ...
- but you may **not** use individual steps from it!

So you may use only steps from (sub)proofs that are still open (i.e. not ended), and if you still are "in" that subproof.

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Existential quantifier rules

Informal proof idea: Universal instantiation

Everyone has overslept at least once.

Carlo has overslept at least once.

Translation key (we abstract away from the temporal details):

Domain: all people

$S(x)$: x has overslept at least once.
c: Carlo

Informal proof idea: Universal instantiation

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Informal rule of universal instantiation

Suppose we have $\forall x S(x)$.
Let c be an individual constant, that is, a name of an object in the domain of discourse.

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Suppose we have $\forall x S(x)$.
Let c be an individual constant, that is, a name of an object in the domain of discourse.
Then we may conclude $S(c)$.

Formal proof rule: Universal quantifier elimination / ∀ Elim

$\vdots$

i. $\forall x P(x)$ Justification (or premise)

$\vdots$

k. $P(c)$ $\forall$ Elim: i

$\vdots$

Compare this with general $\wedge$-Elimination:

$P(a) \wedge P(b) \wedge P(c) \wedge P(d) \wedge \ldots$

$P(c)$ $\wedge$ Elim

Example proof using $\forall$ Elim: formalizing an informal argument

We want to prove:

Alma loves everyone who loves her
Alma does not love Mahler

Mahler does not love Alma

Let's first translate this argument into FOL:

$\forall x(L(x,a) \rightarrow L(a,x))$
$\neg L(a,b)$

$\neg L(b,a)$

Translation key:
L(x,y): x loves y
a: Alma
b: Mahler

Example using $\forall$ Elim, continued

To prove:

$$\begin{array}{|l} \forall x(L(x,a) \rightarrow L(a,x)) \\ \neg L(a,b) \\ \hline \neg L(b,a) \end{array}$$

Let us make a proof using the rules that we know so far:

$$\begin{array}{|l} 1.\ \forall x(L(x,a) \rightarrow L(a,x)) \\ 2.\ \neg L(a,b) \\ \hline \\ \\ \\ \\ \\ \\ \\ \text{j. } \neg L(b,a) \end{array}$$

Example using $\forall$ Elim, continued

To prove:

$$\forall x(L(x,a) \to L(a,x))$$
$$\neg L(a,b)$$
$$\neg L(b,a)$$

Let us make a proof using the rules that we know so far:

1. $\forall x(L(x,a) \to L(a,x))$
2. $\neg L(a,b)$

   3. $L(b,a)$

   j-1. $\bot$

j. $\neg L(b,a)$ — $\neg$ Intro: 3–(j-1)

Example using $\forall$ Elim, continued

To prove:

$$
\begin{array}{|l}
\forall x(L(x,a) \to L(a,x)) \\
\neg L(a,b) \\
\hline
\neg L(b,a)
\end{array}
$$

Let us make a proof using the rules that we know so far:

1.	$\forall x(L(x,a) \to L(a,x))$	
2.	$\neg L(a,b)$	
3.	$\quad L(b,a)$	
4.	$\quad L(b,a) \to L(a,b)$	$\forall$ Elim: 1
5.	$\quad L(a,b)$	$\to$ Elim: 4,3
6.	$\quad \bot$	$\bot$ Intro: 5,2
7.	$\neg L(b,a)$	$\neg$ Intro: 3–6

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Informal proof idea: Universal generalization

Informal rule of universal generalization

How to prove $\forall x S(x)$?

Informal proof idea: Universal generalization

Informal rule of universal generalization

How to prove $\forall x S(x)$?

Let c be the name of an *abitrarily chosen* object of the domain of discourse.

Then $\ldots$ [some reasoning] $\ldots S(c)$.

Since c was chosen arbitrarily, we conclude $\forall x S(x)$.

Informal proof idea: Universal generalization

Informal rule of universal generalization

How to prove $\forall x S(x)$?

Let c be the name of an *abitrarily chosen* object of the domain of discourse.

Then ... [some reasoning] ... $S(c)$.

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'Object c is arbitrarily chosen' means:
We do not make any assumptions about the properties of c
(only that it belongs to the domain of discourse).

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Then ... [some reasoning] ... $S(c)$.

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'Object c is arbitrarily chosen' means:
We do not make any assumptions about the properties of c
(only that it belongs to the domain of discourse).

Example:

- Take an arbitrary object. Let's call it c.
- c is identical to itself.
- So every object is identical to itself.

Formal proof rule:
Universal quantifier introduction / ∀ Introduction

$$
\begin{array}{l|l}
 & \\
\hline
 & \vdots \\
 & \quad \begin{array}{l|l} & \text{i. } \boxed{c} \\ \hline & \vdots \\ & \text{j. } P(c) \end{array} \\
 & \forall x P(x) \qquad\qquad\qquad\qquad\qquad\qquad \forall \text{ Intro: i–j}
\end{array}
$$

c does not occur outside the subproof where it is introduced.
This warrants that c is indeed chosen arbitrarily.

Example: The informal argumentation formalized

- Take an arbitrary object. Let's call it c.
- c is identical to itself.
- So every object is identical to itself.

In a proof an object is said to be arbitrary if we do not make any assumptions about the properties of the object.

Example: The informal argumentation formalized

- Take an arbitrary object. Let's call it c.
- c is identical to itself.
- So every object is identical to itself.

In a proof an object is said to be arbitrary if we do not make any assumptions about the properties of the object.

Formalizing the above argument:

$$
\begin{array}{|l}
\hline
\begin{array}{|l}
\boxed{\mathrm{c}} \\ \hline
c = c
\end{array} \\
\forall x (x = x)
\end{array}
$$

Example using $\forall$ Elim and $\forall$ Intro

To prove:

$$
\begin{array}{|l}
\forall x(P(d) \to Q(x)) \\
\underline{P(d)} \\
\forall y Q(y)
\end{array}
$$

Let us make a proof using the rules that we know so far:

Example using $\forall$ Elim and $\forall$ Intro

To prove:

$$\begin{array}{|l} \forall x(P(d) \to Q(x)) \\ P(d) \\ \hline \forall y Q(y) \end{array}$$

Let us make a proof using the rules that we know so far:

$$\begin{array}{|l} 1.\ \forall x(P(d) \to Q(x)) \\ 2.\ P(d) \\ \hline \\ \\ \\ \\ \\ \\ \\ j.\ \forall y Q(y) \end{array}$$

Example using ∀ Elim and ∀ Intro

To prove:

$$\begin{array}{|l} \forall x(P(d) \to Q(x)) \\ P(d) \\ \hline \forall y Q(y) \end{array}$$

Let us make a proof using the rules that we know so far:

1. $\forall x(P(d) \to Q(x))$	
2. $P(d)$	
3. $\boxed{c}$	
4. $P(d) \to Q(c)$	$\forall$ Elim: 1
5. $Q(c)$	$\to$ Elim: 4, 2
6. $\forall y Q(y)$	$\forall$ Intro: 3–5

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Informal proof idea: Existential generalization

Informal rule of existential generalization

Suppose we have $S(c)$, where c is an individual constant, i.e. a name of an object in the universe of discourse.

Informal proof idea: Existential generalization

Informal rule of existential generalization

Suppose we have $S(c)$, where c is an individual constant, i.e. a name of an object in the universe of discourse.

Then we may conclude $\exists x S(x)$.

Informal proof idea: Existential generalization

Informal rule of existential generalization

Suppose we have $S(c)$, where c is an individual constant, i.e. a name of an object in the universe of discourse.

Then we may conclude $\exists x S(x)$.

Example

a is a small cube

There exists a small cube.

$\mathrm{Cube}(a) \wedge \mathrm{Small}(a)$

$\exists x(\mathrm{Cube}(x) \wedge \mathrm{Small}(x))$

Formal proof rule: Existential quantifier introduction / $\exists$ Intro

$$
\begin{array}{|lll}
\vdots & & \\
\text{i.} & P(c) & \text{Justification (or premise)} \\
\vdots & & \\
\text{k.} & \exists x P(x) & \exists \text{ Intro: i} \\
\vdots & &
\end{array}
$$

Formal proof rule:
Existential quantifier introduction / $\exists$ Intro

$\vdots$

i.	$P(c)$	Justification (or premise)
$\vdots$		
k.	$\exists x P(x)$	$\exists$ Intro: i
$\vdots$		

Compare with

$$\begin{array}{l|l} & P(c) \\ \hline & P(a) \vee P(b) \vee P(c) \vee P(d) \vee \dots \qquad \vee\ \text{Intro} \end{array}$$

Example using $\exists$ Intro: formalizing an informal argument

We want to prove

| If anyone can catch the murderer, Holmes can
| Holmes cannot
|—
| Watson cannot

Let's first translate this argument into FOL:

$$\begin{array}{|l}
\exists x C(x, b) \to C(h, b) \\
\neg C(h, b) \\
\hline
\neg C(a, b)
\end{array}$$

Translation key:
C(x,y): x can catch y
a: Watson
b: the murderer
h: Holmes

Example using $\exists$ Intro, continued

To prove:

$$\begin{array}{|l} \exists x C(x,b) \rightarrow C(h,b) \\ \neg C(h,b) \\ \hline \neg C(a,b) \end{array}$$

Let us make a proof using the rules that we know so far:

$$\begin{array}{|l} 1.\ \exists x C(x,b) \rightarrow C(h,b) \\ 2.\ \neg C(h,b) \\ \hline \\ \\ \\ \\ \\ \\ \\ j.\ \neg C(a,b) \end{array}$$

Example using $\exists$ Intro, continued

To prove:

$$\begin{array}{|l} \exists x C(x,b) \to C(h,b) \\ \neg C(h,b) \\ \hline \neg C(a,b) \end{array}$$

Let us make a proof using the rules that we know so far:

1.	$\exists x C(x,b) \to C(h,b)$	
2.	$\neg C(h,b)$	
3.	$C(a,b)$	
j-1.	$\bot$	
j.	$\neg C(a,b)$	$\neg$ Intro: 3–(j-1)

Example using ∃ Intro, continued

To prove:

$$\begin{array}{|l}
\exists x C(x,b) \to C(h,b) \\
\neg C(h,b) \\
\hline
\neg C(a,b)
\end{array}$$

Let us make a proof using the rules that we know so far:

1. $\exists x C(x,b) \to C(h,b)$	
2. $\neg C(h,b)$	
3. $C(a,b)$	
4. $\exists x C(x,b)$	$\exists$ Intro: 3
5. $C(h,b)$	$\to$ Elim: 1,4
6. $\bot$	$\bot$ Intro: 5, 2
7. $\neg C(a,b)$	$\neg$ Intro: 3–6

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Informal proof idea: Existential instantiation

Informal rule of existential instantiation

Suppose we have $\exists x S(x)$. How can we use this in a proof?

Informal proof idea: Existential instantiation

Informal rule of existential instantiation

Suppose we have $\exists x S(x)$. How can we use this in a proof?

Let c be the name of an *arbitrarily chosen* object of the universe of discourse. Assume that $S(c)$ holds.

Then ... [some reasoning] ... P.

So we have $S(c) \to P$ for an arbitrarily chosen object c.

But we know that $\exists x S(x)$, and we conclude: P.

Example argumentation based on existential instantiation

- Someone put the empty milk carton back in the fridge, finished the cheese and didn't put the lid back on the butter.

Example argumentation based on existential instantiation

- Someone put the empty milk carton back in the fridge, finished the cheese and didn't put the lid back on the butter.
- Whoever did that has to go to the store to buy new milk, cheese and butter.

Example argumentation based on existential instantiation

- Someone put the empty milk carton back in the fridge, finished the cheese and didn't put the lid back on the butter.
- Whoever did that has to go to the store to buy new milk, cheese and butter.
- For simplicity, let's call this person "the slob".

Example argumentation based on existential instantiation

- Someone put the empty milk carton back in the fridge, finished the cheese and didn't put the lid back on the butter.
- Whoever did that has to go to the store to buy new milk, cheese and butter.
- For simplicity, let's call this person "the slob".
- So the slob has to go to the store to buy new milk, cheese and butter.

Example argumentation based on existential instantiation

- Someone put the empty milk carton back in the fridge, finished the cheese and didn't put the lid back on the butter.
- Whoever did that has to go to the store to buy new milk, cheese and butter.
- For simplicity, let's call this person "the slob".
- So the slob has to go to the store to buy new milk, cheese and butter.
- Therefore, someone has to go to the store to buy new milk, cheese and butter.

Here, "the slob" is only used for simplicity, as a temporary marker (just like "Jack the Ripper").

Formal proof rule: Existential quantifier elimination / $\exists$ Elim

$\vdots$

i. $\exists x P(x)$

  j. $\boxed{c}$ $P(c)$

  $\vdots$

  k. Q

Q $\qquad\qquad$ $\exists$ Elim: i, j–k

c does not occur outside the subproof where it is introduced.
(So in particular, c does not occur in Q.)

Formal proof rule:
Existential quantifier elimination / ∃ Elim

$\vdots$

i. $\exists x P(x)$

j. $\boxed{c}$ $P(c)$

$\vdots$

k. Q

Q ∃ Elim: i, j–k

c does not occur outside the subproof where it is introduced.
(So in particular, c does not occur in Q.)

This warrants that c is indeed chosen arbitrarily.

$\exists$ Elimination reminds of $\vee$ Elimination

⋮

i. $\exists x P(x)$

  j. $\boxed{c}\ P(c)$

  ⋮

  k. Q

Q

⋮

i. $P(a) \vee P(b)$

  j. $P(a)$

  ⋮

  k. Q

  m. $P(b)$

  ⋮

  n. Q

Q

Example using $\exists$ Elim and $\forall$ Elim

To prove:

$$\begin{array}{|l} \exists x \neg A(x) \\ \hline \neg \forall y A(y) \end{array}$$

Let us make a proof using the rules that we know so far:

Example using $\exists$ Elim and $\forall$ Elim

To prove:

$$\begin{array}{|l} \exists x \neg A(x) \\ \hline \neg \forall y A(y) \end{array}$$

Let us make a proof using the rules that we know so far:

$$\begin{array}{|l} 1.\ \exists x \neg A(x) \\ \hline \\ \\ \\ \\ \\ \\ j.\ \neg \forall y A(y) \end{array}$$

Example using ∃ Elim and ∀ Elim, continued

To prove:

$$\begin{array}{|l} \exists x \neg A(x) \\ \hline \neg \forall y A(y) \end{array}$$

Let us make a proof using the rules that we know so far:

1. $\exists x \neg A(x)$	
2. $\boxed{c}, \neg A(c)$	
3. $\forall y A(y)$	
4. $A(c)$	$\forall$ Elim: 3
5. $\bot$	$\bot$ Intro: 4, 2
6. $\neg \forall y A(y)$	$\neg$ Intro: 3–5
7. $\neg \forall y A(y)$	$\exists$ Elim: 1, 2–6

Example using $\exists$ Elim and $\forall$ Elim, continued

To prove:

$$\begin{array}{|l} \exists x \neg A(x) \\ \hline \neg \forall y A(y) \end{array}$$

Another correct proof of the same, as constructed by the participants in the lecture:

1. $\exists x \neg A(x)$	
2. $\forall y A(y)$	
3. $\boxed{\text{c}}\ \neg A(c)$	
4. $A(c)$	$\forall$ Elim: 2
5. $\bot$	$\bot$ Intro: 4, 3
6. $\bot$	$\exists$ Elim: 1, 3–5
7. $\neg \forall y A(y)$	$\neg$ Intro: 2–6

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  - Existential generalization and $\exists$ Intro
  - Existential instantiation and $\exists$ Elim
- Proof strategies

General tips for making formal proofs

- Keep the goal in sight;
- Determine which rules you can apply to the premises and assumptions you have made; some "easy" rules are $\wedge$ elimination, $\perp$ introduction, $\forall$ elimination, $\exists$ introduction;
- Develop your intuition by thinking of informal proofs.

A step-by-step guide to planning your proofs

1. Is $\rightarrow$, $\leftrightarrow$ or $\forall x$ the main operator of the conclusion? Use the rule that *introduces* this operator. If not, continue to step 2.

A step-by-step guide to planning your proofs

1. Is $\rightarrow$, $\leftrightarrow$ or $\forall x$ the main operator of the conclusion? Use the rule that *introduces* this operator. If not, continue to step 2.
2. Is $\vee$ or $\exists x$ the main operator of one of the premises? Use the rule that *Eliminates* this operator. If not, continue to step 3.

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3. Does the conclusion have a main operator? If so, try to introduce it. Continue to step 4 otherwise.

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4. Try to infer the conclusion from the premises *in*formally. Then try to translate your informal proof into a formal proof.

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3. Does the conclusion have a main operator? If so, try to introduce it. Continue to step 4 otherwise.
4. Try to infer the conclusion from the premises *in*formally. Then try to translate your informal proof into a formal proof.
5. If everything else fails: Prove the conclusion (B) by contradiction.
   Start a new subproof and suppose that $\neg B$ is true. Try to infer $\bot$. You can now end the subproof and introduce a $\neg$. You now have $\neg\neg B$. Eliminate these two negations, and voilà.

Next time

Monday, December 11th (13:00-15:00)

Questions and Answers session for the midterm exam on all the material of lectures 1-5

Wednesday, December 13th, 15:00-17:00

Formal semantics (Lecture 7): First-order structures and the truth definition.