Example exercises with solutions for semantics

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Dear students,

This is a semantics exercise, meant to help you apply the truth definition. For every exercise in this document, two forms of solution are given: a solution with extra explanation, useful when studying semantics, and a solution which contains just the necessary information for a correct exam answer.

Good luck!

1 The exercise

Let $\mathfrak{M}$ be a model such that $\mathfrak{M}(\forall) = \{1, 2, 3, 4\}$, $\mathfrak{M}(R) = \{\langle 1, 2\rangle, \langle 1, 4\rangle, \langle 3, 1\rangle, \langle 3, 4\rangle\}$ and $\mathfrak{M}(P) = \{2, 3\}$. Moreover, let h be an assignment such that $h(x) = 1$, $h(y) = 2$ and $h(z) = 3$.

Show whether the following three statements are true or not. In doing so, use the truth definition step by step.

a) $\mathfrak{M} \models R(x, y) \to \neg R(y, x)[h]$

b) $\mathfrak{M} \models \forall x(R(x, z) \vee x \neq z)[h]$

c) $\mathfrak{M} \models \exists x R(x, y) \leftrightarrow \forall y P(y)[h]$

2 Part a

2.1 Long solution

We are asked to show whether the statement

$$\mathfrak{M} \models R(x, y) \to \neg R(y, x)[h] \tag{1}$$

is true, using the truth definition.

Statement (1) is, according to the truth definition,[1] true if and only if at least one of the following two statements is true.

$$\mathfrak{M} \not\models R(x, y)[h] \text{ or} \tag{2}$$

$$\mathfrak{M} \models \neg R(y, x)[h] \tag{3}$$

Let's take a look at (3). According to the truth definition, this is true if and only if

$$\mathfrak{M} \not\models R(y, x)[h] \tag{4}$$

is true. This is the atomic case. According to the definition, $\mathfrak{M} \models R(y, x)[h]$ is true if and only if $\left\langle [\![y]\!]_h^{\mathfrak{M}}, [\![x]\!]_h^{\mathfrak{M}} \right\rangle \in \mathfrak{M}(R)$. However, $\left\langle [\![y]\!]_h^{\mathfrak{M}}, [\![x]\!]_h^{\mathfrak{M}} \right\rangle = \langle 2, 1\rangle \notin \mathfrak{M}(R)$ so $\mathfrak{M} \models R(y, x)[h]$ is false, which means that (4) is true. From this it follows that (3) is true, and so at least one of (2) and (3) is true, which means that (1) is true.

[1] The truth definition such as the one in Chapter 18 of the book says this in words. For any wff P, the notation $\mathfrak{M} \models P[h]$ signifies that "h satisfies P in $\mathfrak{M}$". The notation $\mathfrak{M} \not\models P[h]$ means that h does not satisfy P in $\mathfrak{M}$.

2.2 Short solution

$$\mathfrak{M} \models R(x,y) \to \neg R(y,x)[h] \Leftrightarrow$$
$$\mathfrak{M} \not\models R(x,y)[h] \text{ or } \mathfrak{M} \models \neg R(y,x)[h]$$

Apart from this,

$$\mathfrak{M} \models \neg R(y,x)[h] \Leftrightarrow \text{not } \mathfrak{M} \models R(y,x)[h] \Leftrightarrow \text{not } \left\langle [\![y]\!]_h^{\mathfrak{M}}, [\![x]\!]_h^{\mathfrak{M}} \right\rangle \in \mathfrak{M}(R) \Leftrightarrow$$
$$\langle 2,1\rangle \notin \{\langle 1,2\rangle, \langle 1,4\rangle, \langle 3,1\rangle, \langle 3,4\rangle\}.$$

The statement $\langle 2,1\rangle \notin \{\langle 1,2\rangle, \langle 1,4\rangle, \langle 3,1\rangle, \langle 3,4\rangle\}$ is true, so $\mathfrak{M} \models R(x,y) \to \neg R(y,x)[h]$.

3 Part b

3.1 Long solution

Here we'll use the fact that "=" is a special predicate: it holds that

$$\mathfrak{M}(=) = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle\}.$$

We are asked to show whether the statement

$$\mathfrak{M} \models \forall x (R(x,z) \lor x \neq z)[h] \tag{5}$$

is true, using the truth definition.

Statement (5) is, according to the definition, true if and only if

$$\text{for every } d \in \mathfrak{M}(\forall), \mathfrak{M} \models (R(x,z) \lor x \neq z)[h[x/d]], \tag{6}$$

which means iff the following four statements are all true:

$$\mathfrak{M} \models (R(x,z) \lor x \neq z)[h[x/1]] \tag{7}$$
$$\mathfrak{M} \models (R(x,z) \lor x \neq z)[h[x/2]] \tag{8}$$
$$\mathfrak{M} \models (R(x,z) \lor x \neq z)[h[x/3]] \tag{9}$$
$$\mathfrak{M} \models (R(x,z) \lor x \neq z)[h[x/4]] \tag{10}$$

Let's have a look at (9). According to the definition, that statement is true if and only if at least one of the following two statements is true.

$$\mathfrak{M} \models R(x,z)[h[x/3]] \tag{11}$$
$$\text{or } \mathfrak{M} \models (x \neq z)[h[x/3]] \tag{12}$$

The statement (11) is an atomic case and it is true if and only if $\left\langle [\![x]\!]_{h[x/3]}, [\![z]\!]_{h[x/3]} \right\rangle \in \mathfrak{M}(R)$. However, $\left\langle [\![x]\!]_{h[x/3]}, [\![z]\!]_{h[x/3]} \right\rangle = \langle 3,3\rangle \notin \mathfrak{M}(R)$ so (11) is false.

Because $x \neq z$ is short for $\neg(x = z)$, (12) is true if and only if the statement

$$\mathfrak{M} \not\models (x = z)[h[x/3]] \tag{13}$$

is true. This is the atomic case: $\mathfrak{M} \models (x = z)[h[x/3]]$ is true if and only if it holds that $\left\langle [\![x]\!]_{h[x/3]}, [\![z]\!]_{h[x/3]} \right\rangle \in \mathfrak{M}(=)$. We rewrite this as $\left\langle [\![x]\!]_{h[x/3]}, [\![z]\!]_{h[x/3]} \right\rangle = \langle 3,3\rangle \in \mathfrak{M}(=)$, which is indeed true; so $\mathfrak{M} \models (x = z)[h[x/3]]$ is true, which means that (13) is false. Therefore, the statement (12) is also false.

Neither of the statements (11) and (12) is true, which means that (9) is false. From this, it follows that the statements (7) to (10) are *not all* true. Therefore, the statement (5) is not true.

3.2 Short solution

$$\begin{aligned}&\mathfrak{M} \models \forall x(R(x,z) \vee x \neq z)[h] \Leftrightarrow\\&\mathfrak{M} \models (R(x,z) \vee x \neq z)[h[x/1]], \mathfrak{M} \models (R(x,z) \vee x \neq z)[h[x/2]],\\&\mathfrak{M} \models (R(x,z) \vee x \neq z)[h[x/3]] \text{ and } \mathfrak{M} \models (R(x,z) \vee x \neq z)[h[x/4]].\end{aligned}$$

Looking at the ingredients step by step, it holds that

$$\begin{aligned}&\mathfrak{M} \models (R(x,z) \vee x \neq z)[h[x/3]] \Leftrightarrow\\&\mathfrak{M} \models R(x,z)[h[x/3]] \text{ or } \mathfrak{M} \models (x \neq z)[h[x/3]],\end{aligned}$$

and that

$$\begin{aligned}&\mathfrak{M} \models R(x,z)[h[x/3]] \Leftrightarrow \left\langle [\![x]\!]_{h[x/3]}, [\![z]\!]_{h[x/3]} \right\rangle \in \mathfrak{M}(R) \Leftrightarrow\\&\langle 3,3 \rangle \in \{\langle 1,2 \rangle, \langle 1,4 \rangle, \langle 3,1 \rangle, \langle 3,4 \rangle\},\end{aligned}$$

and that

$$\begin{aligned}&\mathfrak{M} \models (x \neq z)[h[x/3]] \Leftrightarrow \left\langle [\![x]\!]_{h[x/3]}, [\![z]\!]_{h[x/3]} \right\rangle \notin \mathfrak{M}(=) \Leftrightarrow\\&\langle 3,3 \rangle \notin \{\langle 1,1 \rangle, \langle 2,2 \rangle, \langle 3,3 \rangle, \langle 4,4 \rangle\}.\end{aligned}$$

However, $\langle 3,3 \rangle \notin \{\langle 1,2 \rangle, \langle 1,4 \rangle, \langle 3,1 \rangle, \langle 3,4 \rangle\}$ and $\langle 3,3 \rangle \in \{\langle 1,1 \rangle, \langle 2,2 \rangle, \langle 3,3 \rangle, \langle 4,4 \rangle\}$ so $\mathfrak{M} \not\models \forall x(R(x,z) \vee x \neq z)[h]$.

4 Part c

4.1 Long solution

We are asked to show whether the statement

$$\mathfrak{M} \models \exists x R(x,y) \leftrightarrow \forall y P(y)[h] \tag{14}$$

is true, using the truth definition.

According to the truth definition, (14) is true if and only if the following two statements are either both true or both false.

$$\mathfrak{M} \models \exists x R(x,y)[h] \tag{15}$$

$$\mathfrak{M} \models \forall y P(y)[h] \tag{16}$$

Statement (15) is true if and only if for some $d \in \mathfrak{M}(\forall)$ it holds that

$$\mathfrak{M} \models R(x,y)[h[x/d]], \tag{17}$$

so if and only if at least one of the following four statements is true:

$$\mathfrak{M} \models R(x,y)[h[x/1]] \tag{18}$$

$$\mathfrak{M} \models R(x,y)[h[x/2]] \tag{19}$$

$$\mathfrak{M} \models R(x,y)[h[x/3]] \tag{20}$$

$$\mathfrak{M} \models R(x,y)[h[x/4]] \tag{21}$$

Let's have a look at (18). That statement is true if and only if $\left\langle [\![x]\!]^{\mathfrak{M}}_{h[x/1]}, [\![y]\!]^{\mathfrak{M}}_{h[x/1]} \right\rangle \in \mathfrak{M}(R)$. This is the case, because indeed $\left\langle [\![x]\!]^{\mathfrak{M}}_{h[x/1]}, [\![y]\!]^{\mathfrak{M}}_{h[x/1]} \right\rangle = \langle 1,2 \rangle \in \mathfrak{M}(R)$. Therefore, statement (18) is true, so at least one of the statements (18) to (21) is true. This means that the statement (15) is true.

Statement (16) is true if and only if for every $d \in \mathfrak{M}(\forall)$ it holds that

$$\mathfrak{M} \models P(y)[h[y/d]], \tag{22}$$

so iff the following four statements are all true:

$$\mathfrak{M} \models P(y)[h[y/1]] \tag{23}$$

$$\mathfrak{M} \models P(y)[h[y/2]] \tag{24}$$

$$\mathfrak{M} \models P(y)[h[y/3]] \tag{25}$$

$$\mathfrak{M} \models P(y)[h[y/4]] \tag{26}$$

Let's have a look at (23). That statement is true if and only if $[\![y]\!]^{\mathfrak{M}}_{h[y/1]} \in \mathfrak{M}(P)$. This is not the case, since $[\![y]\!]^{\mathfrak{M}}_{h[y/1]} = 1 \notin \mathfrak{M}(P)$. This means that statement (23) is not true, so that the statements (23) to (26) are *not all* true. Therefore, the statement (16) is false.

This means that one of the two statements (15) and (16) is true while the other is false. Since the statements are not both true or both false, statement (14) is not true.

4.2 Short solution

$$\begin{aligned}&\mathfrak{M} \models \exists x R(x,y) \leftrightarrow \forall y P(y)[h] \Leftrightarrow\\ &(\mathfrak{M} \models \exists x R(x,y)[h] \text{ and } \mathfrak{M} \models \forall y P(y)[h]) \text{ or } (\mathfrak{M} \not\models \exists x R(x,y)[h] \text{ en } \mathfrak{M} \not\models \forall y P(y)[h]).\end{aligned}$$

Looking at the ingredients step by step, it holds that

$$\begin{aligned}&\mathfrak{M} \models \exists x R(x,y)[h] \Leftrightarrow\\ &\mathfrak{M} \models R(x,y)[h[x/1]], \mathfrak{M} \models R(x,y)[h[x/2]], \mathfrak{M} \models R(x,y)[h[x/3]] \text{ or } \mathfrak{M} \models R(x,y)[h[x/4]]\end{aligned}$$

and that

$$\begin{aligned}&\mathfrak{M} \models \forall y P(y)[h] \Leftrightarrow\\ &\mathfrak{M} \models P(y)[h[y/1]], \mathfrak{M} \models P(y)[h[y/2]], \mathfrak{M} \models P(y)[h[y/3]] \text{ en } \mathfrak{M} \models P(y)[h[y/4]].\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\mathfrak{M} \models R(x,y)[h[x/1]] \Leftrightarrow \left\langle [\![x]\!]^{\mathfrak{M}}_{h[x/1]}, [\![y]\!]^{\mathfrak{M}}_{h[x/1]} \right\rangle \in \mathfrak{M}(R) \Leftrightarrow\\ &\langle 1,2 \rangle \in \{\langle 1,2 \rangle, \langle 1,4 \rangle, \langle 3,1 \rangle, \langle 3,4 \rangle\},\end{aligned}$$

from which it follows that $\mathfrak{M} \models \exists x R(x,y)[h]$ is true and

$$\mathfrak{M} \models P(y)[h[y/1]] \Leftrightarrow [\![y]\!]^{\mathfrak{M}}_{h[y/1]} \in \mathfrak{M}(P) \Leftrightarrow 1 \in \{2,3\},$$

from which we conclude that $\mathfrak{M} \not\models \forall y P(y)[h]$. Therefore, it holds that
$\mathfrak{M} \not\models \exists x R(x,y) \leftrightarrow \forall y P(y)[h]$