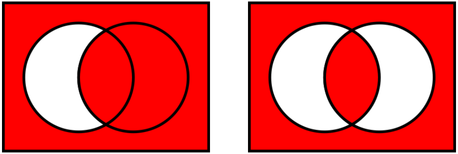
Introduction to Logic
Lecture 3: Conditionals and Truth Tables

Davide Grossi

27 November 2023



Review

You already know these symbols:

- P, Q, R, ...
- $P(a)$, $K(j, m)$, ...
- $\neg$
- $\land$
- $\lor$

You also know these formal proof rules:

$=$Intro, $=$Elim, $\land$Intro, $\land$Elim, $\lor$Intro, $\lor$Elim,

$\neg$Intro, $\neg$Elim, $\bot$Intro and $\bot$Elim.

Today we will add the symbols $\rightarrow$ and $\leftrightarrow$.

Overview

Material conditional: $\rightarrow$

Biconditional: $\leftrightarrow$

What a material conditional $\rightarrow$ does *not* capture

- Ordinary reading vs. logical reading
- Conversational implicature
- Causality and Counterfactuals

Hintikka game

Informal proofs

Formal proofs

Truth tables

Tautologies and logical truths

Overview

Material conditional: $\rightarrow$

Biconditional: $\leftrightarrow$

What a material conditional $\rightarrow$ does *not* capture

- Ordinary reading vs. logical reading
- Conversational implicature
- Causality and Counterfactuals

Hintikka game

Informal proofs

Formal proofs

Truth tables

Tautologies and logical truths

If . . . , then . . .

Example:

If you are an employee, then you get a discount.

This sentence is a **conditional**.

The sentence

you are an employee

is called the *antecedent*, and

you get a discount.

is called the *consequent*.

Notation and semantics for the material conditional

$$\rightarrow$$

Notation and semantics for the material conditional

$$\longrightarrow$$

A sentence $P \rightarrow Q$ is a *material conditional*, with this truth table:

Truth table for the material conditional

P	Q	$P \rightarrow Q$
T	T	
T	F	
F	T	
F	F	

Notation and semantics for the material conditional

$$\longrightarrow$$

A sentence $P \rightarrow Q$ is a *material conditional*, with this truth table:

Truth table for the material conditional

P	Q	$P \rightarrow Q$
T	T	T
T	F	
F	T	
F	F	

Notation and semantics for the material conditional

$$\rightarrow$$

A sentence $P \rightarrow Q$ is a *material conditional*, with this truth table:

Truth table for the material conditional

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	
F	F	

Notation and semantics for the material conditional

$$\rightarrow$$

A sentence $P \rightarrow Q$ is a *material conditional*, with this truth table:

Truth table for the material conditional

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	

Notation and semantics for the material conditional

$$\rightarrow$$

A sentence $P \rightarrow Q$ is a *material conditional*, with this truth table:

Truth table for the material conditional

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Notation and semantics for the material conditional

$$\rightarrow$$

A sentence $P \rightarrow Q$ is a *material conditional*, with this truth table:

Truth table for the material conditional

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Note: $P \rightarrow Q$ has the same truth conditions* as $\neg P \vee Q$.

*True in exactly the same circumstances

Notation and semantics for the material conditional

$$\rightarrow$$

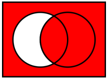
A sentence $P \rightarrow Q$ is a *material conditional*, with this truth table:

Truth table for the material conditional

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Note: $P \rightarrow Q$ has the same truth conditions* as $\neg P \vee Q$.

*True in exactly the same circumstances



Necessary and sufficient conditions in conditionals

- If $P \rightarrow Q$ is true, then
  - P is a sufficient condition for Q and
  - Q is a necessary condition for P.

Necessary and sufficient conditions in conditionals

- If $P \rightarrow Q$ is true, then
  - P is a sufficient condition for Q and
  - Q is a necessary condition for P.

- *If you are an employee, then you get a discount.*
  Being an employee is a *sufficient* condition to get a discount.

Necessary and sufficient conditions in conditionals

- If $P \rightarrow Q$ is true, then
  - P is a sufficient condition for Q and
  - Q is a necessary condition for P.

- *If you are an employee, then you get a discount.*
  Being an employee is a *sufficient* condition to get a discount.

- *If you have the right to vote, then you are an adult.*
  Being an adult is a *necessary* condition for the right to vote.

Variations of "If . . . , then . . . "

- If n is a natural number, n is an integer.
- You staying quiet implies that you are guilty.
- You can take the exam, provided that you are enrolled.

Variations of "If . . . , then . . . "

- If n is a natural number, n is an integer.
- You staying quiet implies that you are guilty.
- You can take the exam, provided that you are enrolled.

P implies Q	corresponds to	If P then Q
if P, Q	corresponds to	If P then Q
provided that P, Q	corresponds to	If P then Q

Variations of "If . . . , then . . ."

- If n is a natural number, n is an integer.
- You staying quiet implies that you are guilty.
- You can take the exam, provided that you are enrolled.

P implies Q	corresponds to	If P then Q
if P then Q	corresponds to	If P then Q
if P, Q	corresponds to	If P then Q
provided that P, Q	corresponds to	If P then Q

Variations of "If . . . , then . . . "

- If n is a natural number, n is an integer.
- You staying quiet implies that you are guilty.
- You can take the exam, provided that you are enrolled.

P implies Q	corresponds to	If P then Q
if P then Q	corresponds to	If P then Q
if P, Q	corresponds to	If P then Q
Q if P	corresponds to	If P then Q
provided that P, Q	corresponds to	If P then Q

Variations of “If . . . , then . . . ”

- If n is a natural number, n is an integer.
- You staying quiet implies that you are guilty.
- You can take the exam, provided that you are enrolled.

P implies Q	corresponds to	If P then Q
if P then Q	corresponds to	If P then Q
if P, Q	corresponds to	If P then Q
Q if P	corresponds to	If P then Q
provided that P, Q	corresponds to	If P then Q
Q provided that P	corresponds to	If P then Q

Variations of "If . . . , then . . . "

- If n is a natural number, n is an integer.
- You staying quiet implies that you are guilty.
- You can take the exam, provided that you are enrolled.

P implies Q	corresponds to	If P then Q
if P then Q	corresponds to	If P then Q
if P, Q	corresponds to	If P then Q
Q if P *	corresponds to	If P then Q
provided that P, Q	corresponds to	If P then Q
Q provided that P *	corresponds to	If P then Q

Note in which cases the order changes!

“Only If”

- You will get better only if you take this medicine.
-

“Only If”

- You will get better only if you take this medicine.
-

only if P, Q corresponds to If Q then P

Q only if P corresponds to If Q then P

"Only If"

- You will get better only if you take this medicine.
-

only if P, Q	corresponds to	If Q then P
Q only if P	corresponds to	If Q then P
if P, Q	corresponds to	If P then Q
Q if P	corresponds to	If P then Q

“Only If”

- You will get better only if you take this medicine.
- You will get better if you take this medicine.

only if P, Q	corresponds to	If Q then P
Q only if P	corresponds to	If Q then P
if P, Q	corresponds to	If P then Q
Q if P	corresponds to	If P then Q

"Only If"

- You will get better only if you take this medicine.
- You will get better if you take this medicine.

only if P, Q	corresponds to	If Q then P
Q only if P	corresponds to	If Q then P
if P, Q	corresponds to	If P then Q
Q if P	corresponds to	If P then Q

Recall:

- If $P \to Q$ is true, then
  - P is a sufficient condition for Q and
  - Q is a necessary condition for P.

"Only If"

- You will get better only if you take this medicine.
- You will get better if you take this medicine.

only if P, Q	corresponds to	If Q then P
Q only if P *	corresponds to	If Q then P
if P, Q	corresponds to	If P then Q
Q if P *	corresponds to	If P then Q

Recall:

- If $P \to Q$ is true, then
  - P is a sufficient condition for Q and
  - Q is a necessary condition for P.

* Note in which cases the order changes!

Logical translation of "unless"

- You may continue driving, unless the traffic light is red.
- Unless the traffic light is red, you may continue driving.

Logical translation of “unless”

- You may continue driving, unless the traffic light is red.
- Unless the traffic light is red, you may continue driving.

Possible logical translations of unless P, Q:

If not P, then Q $\quad \neg P \to Q$

Logical translation of "unless"

- You may continue driving, unless the traffic light is red.
- Unless the traffic light is red, you may continue driving.

Possible logical translations of unless P, Q:

If not P, then Q $\quad \neg P \to Q \equiv \neg\neg P \vee Q \equiv P \vee Q$

Logical translation of "unless"

- You may continue driving, unless the traffic light is red.
- Unless the traffic light is red, you may continue driving.

Possible logical translations of unless P, Q:

If not P, then Q	$\neg P \to Q \equiv \neg\neg P \vee Q \equiv P \vee Q$
P or Q	$P \vee Q$

Logical translation of "unless"

- You may continue driving, unless the traffic light is red.
- Unless the traffic light is red, you may continue driving.

Possible logical translations of unless P, Q:

If not P, then Q $\quad \neg P \to Q \equiv \neg\neg P \vee Q \equiv P \vee Q$

P or Q $\quad P \vee Q$

- You may continue driving, or the traffic light is red.

Overview

Material conditional: $\rightarrow$

Biconditional: $\leftrightarrow$

What a material conditional $\rightarrow$ does *not* capture

- Ordinary reading vs. logical reading
- Conversational implicature
- Causality and Counterfactuals

Hintikka game

Informal proofs

Formal proofs

Truth tables

Tautologies and logical truths

. . . if and only if . . .

*I will switch to Linux **if and only if** there are no costs.*

This sentence is a **biconditional** of two sentences:

I will switch to Linux.

There are no costs.

Biconditional: Notation and semantics

$$\leftrightarrow$$

truth table for the biconditional

P	Q	$P \leftrightarrow Q$

Biconditional: Notation and semantics

$$\leftrightarrow$$

truth table for the biconditional

P	Q	$P \leftrightarrow Q$
T	T	
T	F	
F	T	
F	F	

Biconditional: Notation and semantics

$$\leftrightarrow$$

truth table for the biconditional

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	
F	T	
F	F	

Biconditional: Notation and semantics

$$\leftrightarrow$$

truth table for the biconditional

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	
F	F	

Biconditional: Notation and semantics

$$\leftrightarrow$$

truth table for the biconditional

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	

Biconditional: Notation and semantics

$$\leftrightarrow$$

truth table for the biconditional

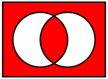
P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional: Notation and semantics

$$\leftrightarrow$$

truth table for the biconditional

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T



Exercise: check for yourself that $P \leftrightarrow Q$ has the same truth conditions as $(P \to Q) \wedge (Q \to P)$, and $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

Variations of "if and only if"

- n is even iff n^2 is even.
- A number is odd exactly if it is not even.
- In a triangle, all sides are equal
  just in case all angles are equal.
- Being at least 18 is necessary and sufficient for being an adult.

Overview

Material conditional: $\rightarrow$

Biconditional: $\leftrightarrow$

What a material conditional $\rightarrow$ does *not* capture

- Ordinary reading vs. logical reading
- Conversational implicature
- Causality and Counterfactuals

Hintikka game

Informal proofs

Formal proofs

Truth tables

Tautologies and logical truths

Ordinary reading vs. *logical* reading

Consider the following sentences:

Ordinary reading vs. *logical* reading

Consider the following sentences:

If J.R.R. Tolkien wrote *The Lord of the Rings*, then $2 + 2 = 4$.

Ordinary reading vs. *logical* reading

Consider the following sentences:

If J.R.R. Tolkien wrote *The Lord of the Rings*, then $2 + 2 = 4$.

If J.R.R. Tolkien wrote *The Hobbit*, then $2 + 2 = 5$.

Ordinary reading vs. *logical* reading

Consider the following sentences:

If J.R.R. Tolkien wrote *The Lord of the Rings*, then $2 + 2 = 4$.

If J.R.R. Tolkien wrote *The Hobbit*, then $2 + 2 = 5$.

If Queen Elizabeth wrote *The Lord of the Rings*, then $2 + 2 = 4$.

Ordinary reading vs. *logical* reading

Consider the following sentences:

If J.R.R. Tolkien wrote *The Lord of the Rings*, then $2 + 2 = 4$.

If J.R.R. Tolkien wrote *The Hobbit*, then $2 + 2 = 5$.

If Queen Elizabeth wrote *The Lord of the Rings*, then $2 + 2 = 4$.

If Queen Elizabeth was lead singer for *The Beatles*, then $2 + 2 = 5$.

Ordinary reading vs. *logical* reading

Consider the following sentences:

If J.R.R. Tolkien wrote *The Lord of the Rings*, then $2+2=4$. $T \rightarrow T$

If J.R.R. Tolkien wrote *The Hobbit*, then $2+2=5$. $T \rightarrow F$

If Queen Elizabeth wrote *The Lord of the Rings*, then $2+2=4$. $F \rightarrow T$

If Queen Elizabeth was lead singer for *The Beatles*, then $2+2=5$. $F \rightarrow F$

Ordinary reading vs. *logical* reading

Consider the following sentences:

If J.R.R. Tolkien wrote *The Lord of the Rings*, then $2 + 2 = 4$.	$T \to T$: T
If J.R.R. Tolkien wrote *The Hobbit*, then $2 + 2 = 5$.	$T \to F$: F
If Queen Elizabeth wrote *The Lord of the Rings*, then $2 + 2 = 4$.	$F \to T$: T
If Queen Elizabeth was lead singer for *The Beatles*, then $2 + 2 = 5$.	$F \to F$: T

Logically OK, but rather unusual in English!

Logical symbols vs. English words

$\neg$ corresponds well with *not*

$\wedge$ corresponds well with *and*

$\vee$ corresponds rather well with inclusive *or*

Logical symbols vs. English words

$\neg$ corresponds well with *not*

$\wedge$ corresponds well with *and*

$\vee$ corresponds rather well with inclusive *or*

$\rightarrow$ corresponds only partially with if ... then ...
The material conditional does not cover any causal meaning!

Logical symbols vs. English words

$\neg$ corresponds well with *not*

$\wedge$ corresponds well with *and*

$\vee$ corresponds rather well with inclusive *or*

$\rightarrow$ corresponds only partially with if ... then ...
The material conditional does not cover any causal meaning!

Principle of truth-functionality

The truth value of a compound sentence is determined by the truth values of its parts.

In Logic, only truth values matter, but English cares about content!

Overview

Material conditional: $\rightarrow$

Biconditional: $\leftrightarrow$

What a material conditional $\rightarrow$ does *not* capture
- Ordinary reading vs. logical reading
- Conversational implicature
- Causality and Counterfactuals

Hintikka game

Informal proofs

Formal proofs

Truth tables

Tautologies and logical truths

Conversational implicatures (H.P. Grice (1913–1988))

Maybe you want to translate "Max is home unless Claire is at the library" as
$\neg Library(Claire) \leftrightarrow Home(max)$, instead of just
$\neg Library(Claire) \rightarrow Home(max)$?

Conversational implicatures (H.P. Grice (1913–1988))

Maybe you want to translate “Max is home unless Claire is at the library” as
$\neg \mathit{Library}(\mathit{Claire}) \leftrightarrow \mathit{Home}(\mathit{max})$, instead of just
$\neg \mathit{Library}(\mathit{Claire}) \rightarrow \mathit{Home}(\mathit{max})$?

It is useful to distinguish between the *truth conditions* of a sentence and *other things that follow from the assertion* of the sentence.

Conversational implicatures (H.P. Grice (1913–1988))

Maybe you want to translate "Max is home unless Claire is at the library" as
$\neg Library(Claire) \leftrightarrow Home(max)$, instead of just
$\neg Library(Claire) \rightarrow Home(max)$?

It is useful to distinguish between the *truth conditions* of a sentence and *other things that follow from the assertion* of the sentence.

For these "other things" Paul Grice (1913-1988) introduced the concept of conversational implicatures of a statement.

Conversational implicatures (H.P. Grice (1913–1988))

Maybe you want to translate “Max is home unless Claire is at the library” as
$\neg Library(Claire) \leftrightarrow Home(max)$, instead of just
$\neg Library(Claire) \rightarrow Home(max)$?

It is useful to distinguish between the *truth conditions* of a sentence and *other things that follow from the assertion* of the sentence.

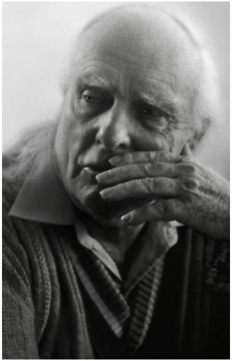
For these “other things” Paul Grice (1913-1988) introduced the concept of conversational implicatures of a statement.

Example (Conversational Implicature)

A doctor says “If you take the medicine, you will get better”.
This may have as conversational implicature the converse,
“If you do not take the medicine, you will not get better”.
But this *does not follow logically* from the doctor’s statement!

Principles of cooperative communication: Maxims of Grice

- Maxim of quality: Be honest.
- Maxim of quantity: Be informative.
- Maxim of relation: Be relevant.
- Maxim of manner: Be clear.



Conversational implicature

Grice distinguishes

- the literal meaning (truth conditions) and
- the implicit meaning (conversational implicature).

Example

Ada says "John or Bob is at home."

Conversational implicature

Grice distinguishes

- the literal meaning (truth conditions) and
- the implicit meaning (conversational implicature).

Example

Ada says "John or Bob is at home."

Literal meaning: $\mathit{IsHome}(\mathit{john}) \vee \mathit{IsHome}(\mathit{bob})$.

Conversational implicature: They are probably not both at home, as far as Ada knows, otherwise she would have said "John and Bob are at home" (by the maxim of quantity).

Cancelling conversational implicatures

How to test: logical meaning or conversational implicature?

If the assertion of a sentence carries with it a suggestion that the speaker could cancel again (without contradiction) by further elaboration, then the suggestion is a *conversational implicature*, and not part of the content of the original claim.

Example

'You may enter the plane only if you have been cleared by customs."

Logical Translation: $E \rightarrow C$

Cancelling conversational implicatures

How to test: logical meaning or conversational implicature?

If the assertion of a sentence carries with it a suggestion that the speaker could cancel again (without contradiction) by further elaboration, then the suggestion is a *conversational implicature*, and not part of the content of the original claim.

Example

'You may enter the plane only if you have been cleared by customs."

Logical Translation: $E \rightarrow C$

Conversational implicature: $\neg(\neg E \wedge C) \equiv C \rightarrow E$

Cancelling conversational implicatures

How to test: logical meaning or conversational implicature?

If the assertion of a sentence carries with it a suggestion that the speaker could cancel again (without contradiction) by further elaboration, then the suggestion is a *conversational implicature*, and not part of the content of the original claim.

Example

'You may enter the plane only if you have been cleared by customs."

Logical Translation: $E \to C$

Conversational implicature: $\neg(\neg E \wedge C) \equiv C \to E$
The speaker can cancel this implicature by elaborating:
"Even if you have been cleared by customs, you may be denied entry for other reasons.".

Overview

“Because” is not truth-functional

Principle of truth-functionality

The truth value of a compound sentence is determined by the truth values of its parts.

The earth is spinning because the laws of physics are working. OK, true

The earth is spinning because I am giving this lecture. ??, false?

“Because” is not truth-functional

Principle of truth-functionality

The truth value of a compound sentence is determined by the truth values of its parts.

The earth is spinning because the laws of physics are working. OK, true

The earth is spinning because I am giving this lecture. ??, false?

However, *the laws of physics are working* and *I am giving this lecture* have the same truth value!

“Because” is not truth-functional

Principle of truth-functionality

The truth value of a compound sentence is determined by the truth values of its parts.

The earth is spinning because the laws of physics are working. OK, true

The earth is spinning because I am giving this lecture. ??, false?

However, *the laws of physics are working* and *I am giving this lecture* have the same truth value!

“Because” is not truth-functional. Hence we should not even try to translate “because” into propositional logic or first-order logic.
In particular, the material conditional $\rightarrow$ cannot capture causality!

Counterfactuals cannot be translated using $\rightarrow$

If Deep Blue did not win their first chess match,
then Garry Kasparov did.

Counterfactuals cannot be translated using $\rightarrow$

If Deep Blue did not win their first chess match,
then Garry Kasparov did.
This conditional sentence is true, because the antecedent is false (Deep Blue did beat Garry Kasparov in their first match).

Counterfactuals cannot be translated using $\to$

If Deep Blue did not win their first chess match,
then Garry Kasparov did.
This conditional sentence is true, because the antecedent is false (Deep Blue did beat Garry Kasparov in their first match).

If Deep Blue *had not* won their first chess match,
then Garry Kasparov *would* have.

Counterfactuals cannot be translated using $\rightarrow$

If Deep Blue did not win their first chess match,
then Garry Kasparov did.
This conditional sentence is true, because the antecedent is false (Deep Blue did beat Garry Kasparov in their first match).

If Deep Blue *had not* won their first chess match,
then Garry Kasparov *would* have.
If Deep Blue *had not* won their first chess match,
then a magical unicorn *would* have.

Counterfactuals cannot be translated using $\rightarrow$

If Deep Blue did not win their first chess match,
then Garry Kasparov did.
This conditional sentence is true, because the antecedent is false (Deep Blue did beat Garry Kasparov in their first match).

If Deep Blue *had not* won their first chess match,
then Garry Kasparov *would* have.
If Deep Blue *had not* won their first chess match,
then a magical unicorn *would* have.

Such a counterfactual conditional

"If … *had/were* …, then … *would* …"

with a false antecedent is not truth-functional.
Therefore, we do not try to translate it into first-order logic or propositional logic.

Counterfactuals cannot be translated using $\rightarrow$

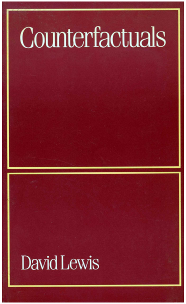
If Deep Blue did not win their first chess match,
then Garry Kasparov did.
This conditional sentence is true, because the antecedent is false (Deep Blue did beat Garry Kasparov in their first match).

If Deep Blue *had not* won their first chess match,
then Garry Kasparov *would* have.
If Deep Blue *had not* won their first chess match,
then a magical unicorn *would* have.

Such a counterfactual conditional

"If . . . *had/were* . . . , then . . . *would* . . . "

with a false antecedent is not truth-functional. Therefore, we do not try to translate it into first-order logic or propositional logic.



Overview

Hintikka game: Reminder of the rules

You can use the Hintikka game to find out the truth value of complex sentences in a given situation.

- Two players. In Tarski's World, the players are you and the computer.
- Two roles: Abelard (commit to false) and Eloise (commit to true).

Game rules for the connectives $\vee, \wedge, \neg$ and for atomic sentences:

$A \vee B$	Eloise (commit to true) chooses A or B and the game continues with that choice.
$A \wedge B$	Abelard (commit-to-false) chooses A or B and the game continues with that choice.
$\neg A$	The players swap roles, then continue with A.
atom	For any atomic sentence such as P or Large(a), Eloise (commit-to-true) wins if the sentence is true; Abelard (commit-to-false) wins if it is false.

Hintikka game: Reminder of winning strategies

Suppose the truth value of the atomic sentences is known.

Truth

A sentence is true iff Eloise has a winning strategy.

Falsehood

A sentence is false iff Abelard has a winning strategy.

Hintikka game rules for the conditional

$P \to Q$ The game continues with $\neg P \vee Q$.

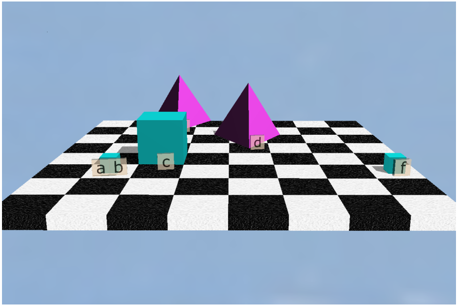
Recall: truth table for the conditional

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Checking truth of a conditional in a situation

Is the following sentence true or false in the world below?
Find out by playing the Hintikka game.

$$\text{Between}(c, a, d) \rightarrow \text{Between}(c, b, f)$$



We continue the game with $\neg\text{Between}(c, a, d) \vee \text{Between}(c, b, f)$.

Game play for the biconditional

$P \leftrightarrow Q$ The game continues with $(\neg P \vee Q) \wedge (\neg Q \vee P)$.

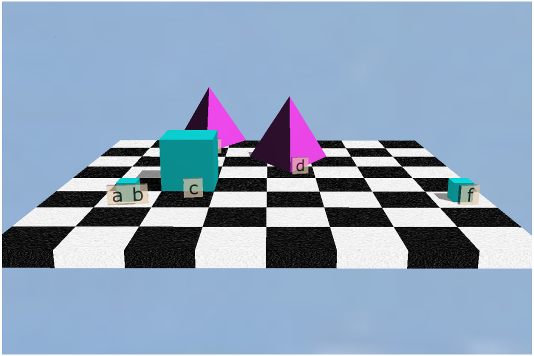
truth table for the biconditional

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Checking truth of a biconditional in a situation

Is the following sentence true or false in the world below?
Find out by playing the Hintikka game.

$$\text{Large}(d) \leftrightarrow a = b$$



Continue the game with
$(\neg\text{Large}(d) \lor a = b) \land (\neg(a = b) \lor \text{Large}(d))$.

Overview

Material conditional: $\rightarrow$

Biconditional: $\leftrightarrow$

What a material conditional $\rightarrow$ does *not* capture

- Ordinary reading vs. logical reading
- Conversational implicature
- Causality and Counterfactuals

Hintikka game

Informal proofs

Formal proofs

Truth tables

Tautologies and logical truths

Modus Ponens and biconditional elimination

Modus Ponens is a well-known valid reasoning rule:

If it rains, then the streets get wet.
It rains.

The streets get wet.

Modus Ponens and biconditional elimination

Modus Ponens is a well-known valid reasoning rule:

If it rains, then the streets get wet.
It rains.

The streets get wet.

Similarly to modus ponens, there is *biconditional elimination*:

Cube(a) if and only if Tet(b).
Tet(b).

Cube(a).

Contraposition

Contraposition is often used in informal proofs.
It occurs in several forms, among others:

$$\begin{array}{|l} \mathrm{P} \to \mathrm{Q} \\ \hline \neg Q \to \neg P \end{array}$$

$$\begin{array}{|l} \neg Q \to \neg P \\ \hline P \to Q \end{array}$$

Conditional proof

Claim:

If n is even, then $3n$ is even.

Proof.

Suppose that n is even.

So n is a multiple of 2, i.e. $n = 2m$ for some m.

Then $3n = 2 \cdot 3m$, so $3n$ is even, too.

So indeed: if n is even, then $3n$ is even.

Conditional proof

Claim:

If n is even, then $3n$ is even.

Proof.

Suppose that n is even.

So n is a multiple of 2, i.e. $n = 2m$ for some m.

Then $3n = 2 \cdot 3m$, so $3n$ is even, too.

So indeed: if n is even, then $3n$ is even.

General form of a conditional proof:

Suppose that P.

... [some reasoning] ...

Then Q.

Conclusion: if P then Q.

Hypothetical syllogism

$$\begin{array}{rcl} P & \rightarrow & Q \\ Q & \rightarrow & R \\ \hline P & \rightarrow & R \end{array}$$

Proving biconditionals by a circle of conditionals

If P_1 then P_2.
If P_2 then P_3.
If P_3 then P_4.
If P_4 then P_5.
If P_5 then P_1.

P_1, P_2, P_3, P_4, and P_5 are all equivalent.

Overview

Material conditional: $\rightarrow$

Biconditional: $\leftrightarrow$

What a material conditional $\rightarrow$ does *not* capture
- Ordinary reading vs. logical reading
- Conversational implicature
- Causality and Counterfactuals

Hintikka game

Informal proofs

Formal proofs

Truth tables

Tautologies and logical truths

$\rightarrow$ Introduction

$\vdots$

$\vdots$

i. P

$\vdots$

j. Q (Justification)

$\vdots$

k. $P \rightarrow Q$ $\rightarrow$ Intro: i–j

$\vdots$

$\rightarrow$ Elimination

$\vdots$		
i.	$P \rightarrow Q$	Justification (or premise)
$\vdots$		
j.	P	Justification (or premise)
$\vdots$		
k.	Q	$\rightarrow$ Elim: i,j
$\vdots$		

$\leftrightarrow$ Introduction

$\vdots$

$\vdots$

i. P

$\vdots$

j. Q (Justification)

m. Q

$\vdots$

n. P (Justification)

k. $P \leftrightarrow Q$ $\leftrightarrow$ Intro: i–j, m–n

$\vdots$

$\leftrightarrow$ Elimination

$\vdots$		
i.	$P \leftrightarrow Q$ or: $Q \leftrightarrow P$	Justification (or premise)
$\vdots$		
j.	P	Justification (or premise)
$\vdots$		
k.	Q	$\leftrightarrow$ Elim: i,j
$\vdots$		

Proving $P \to R$ from $P \to Q$ and $Q \to R$

$$
\begin{array}{|l}
P \to Q \\
Q \to R \\
\hline
\vdots \\
\\
P \to R
\end{array}
$$

Proving $P \to R$ from $P \to Q$ and $Q \to R$

1.	$P \to Q$	
2.	$Q \to R$	
3.	P	
4.	Q	$\to$ Elim: 1, 3
5.	R	$\to$ Elim: 2, 4
6.	$P \to R$	$\to$ Intro: 3–5

Now we have proved the *hypothetical syllogism*.

Proving $P \to Q$ from $\neg P \lor Q$

$$
\begin{array}{|l}
\neg P \lor Q \\
\hline
\vdots \\
\\
P \to Q
\end{array}
$$

Proving $P \to Q$ from $\neg P \vee Q$

1. $\neg P \vee Q$	
2. $\neg P$	
3. P	
4. $\bot$	$\bot$ Intro: 3, 2
5. Q	$\bot$ Elim: 4
6. $P \to Q$	$\to$ Intro: 3–5
7. Q	
8. P	
9. Q	Reit: 7
10. $P \to Q$	$\to$ Intro 8–9
11. $P \to Q$	$\vee$ Elim: 1, 2–6, 7–10

A shorter proof for $P \to Q$ from $\neg P \vee Q$

1. $\neg P \vee Q$	
2. P	
3. $\neg P$	
4. $\bot$	$\bot$ Intro: 2,3
5. Q	$\bot$ Elim: 4
6. Q	
7. Q	Reit: 6
8. Q	$\vee$ Elim: 1, 3–5, 6–7
9. $P \to Q$	$\to$ Intro 2–8

Proving $\neg P \lor Q$ from $P \to Q$

$$
\begin{array}{|l}
P \to Q \\
\hline
\vdots \\
\\
\neg P \lor Q
\end{array}
$$

Proof of $\neg P \vee Q$ from $P \to Q$

1. $P \to Q$	
2. $\neg(\neg P \vee Q)$	
3. P	
4. Q	$\to$ Elim: 1, 3
5. $\neg P \vee Q$	$\vee$ Intro: 4
6. $\perp$	$\perp$ Intro: 5, 2
7. $\neg P$	$\neg$ Intro: 3–6
8. $\neg P \vee Q$	$\vee$ Intro: 7
9. $\perp$	$\perp$ Intro: 8, 2
10. $\neg\neg(\neg P \vee Q)$	$\neg$ Intro: 2–9
11. $\neg P \vee Q$	$\neg$ Elim: 10

Overview

Material conditional: $\rightarrow$

Biconditional: $\leftrightarrow$

What a material conditional $\rightarrow$ does *not* capture
- Ordinary reading vs. logical reading
- Conversational implicature
- Causality and Counterfactuals

Hintikka game

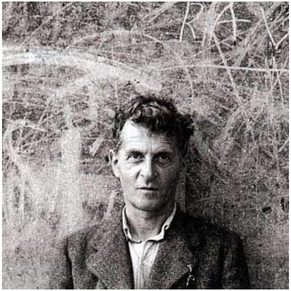
Informal proofs

Formal proofs

Truth tables

Tautologies and logical truths

Ludwig Wittgenstein (1899-1951)

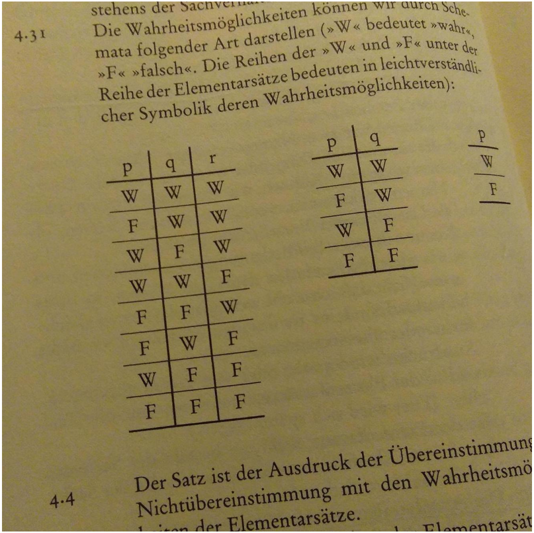


Credited with the invention of truth tables in his

Tractatus Logico-Philosophicus.

How to make a nice truth table — don't be Wittgenstein!

Please do not confuse your reader / teaching assistant with your own creative order of "T"s and "F"s in your truth tables.


stehens der Sachver

4.31 Die Wahrheitsmöglichkeiten können wir durch Schemata folgender Art darstellen (»W« bedeutet »wahr«, »F« »falsch«. Die Reihen der »W« und »F« unter der Reihe der Elementarsätze bedeuten in leichtverständlicher Symbolik deren Wahrheitsmöglichkeiten):

p	q	r
W	W	W
F	W	W
W	F	W
W	W	F
F	F	W
F	W	F
W	F	F
F	F	F

p	q
W	W
F	W
W	F
F	F

p
W
F

4.4 Der Satz ist der Ausdruck der Übereinstimmung
Nichtübereinstimmung mit den Wahrheitsmö
der Elementarsätze.
Elementarsät

Conventions for truth tables with two or three atoms

Truth table for two atoms

P	Q

Conventions for truth tables with two or three atoms

Truth table for two atoms

P	Q
T	T
T	F
F	T
F	F

Conventions for truth tables with two or three atoms

Truth table for two atoms

P	Q
T	T
T	F
F	T
F	F

Truth table for three atoms

P	Q	R

Conventions for truth tables with two or three atoms

Truth table for two atoms

P	Q
T	T
T	F
F	T
F	F

Truth table for three atoms

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Size of truth table in relation to number of atoms

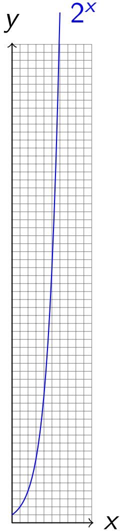
Size of truth tables

Number of atoms	Number of rows
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
n	2^n

Size of truth table in relation to number of atoms

Size of truth tables

Number of atoms	Number of rows
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
n	2^n



Making a truth table for a sentence with two atoms

So far we used truth tables to *define* connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, ...

We can also *compute* truth tables for more complex sentences.
Example:

P	Q	$(\neg(P{\rightarrow}Q)$	$\vee$	$\neg P)$	$\wedge$	Q

Making a truth table for a sentence with two atoms

So far we used truth tables to *define* connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, ...

We can also *compute* truth tables for more complex sentences.
Example:

P	Q	$(\neg$	$(P \rightarrow Q)$	$\vee$	$\neg P)$	$\wedge$	Q
T	T	F	T	F	F	F	
T	F	T	F	T	F	F	
F	T	F	T	T	T	T	
F	F	F	T	T	T	F	

Result: $(\neg(P \rightarrow Q) \vee \neg P) \wedge Q$ is true precisely in the third row, where P is false and Q is true.

Making a truth table for a sentence with three atoms

A	B	C	$\neg(A$	$\wedge$	$(\neg A$	$\vee$	$(B \wedge C)$	$))$	$\vee$	B

Making a truth table for a sentence with three atoms

A	B	C	$\neg(A$	$\wedge$	$(\neg A$	$\vee$	$(B \wedge C)$	$))$	$\vee$	B
T										
T										
T										
T										
F										
F										
F										
F										
(1)										

Making a truth table for a sentence with three atoms

A	B	C	$\neg(A$	$\wedge$	$(\neg A$	$\vee$	$(B \wedge C)$	$))$	$\vee$	B
T	T									
T	T									
T	F									
T	F									
F	T									
F	T									
F	F									
F	F									
(1)	(2)									

Making a truth table for a sentence with three atoms

A	B	C	$\neg(A$	$\wedge$	$(\neg A$	$\vee$	$(B \wedge C)$	$))$	$\vee$	B
T	T	T								
T	T	F								
T	F	T								
T	F	F								
F	T	T								
F	T	F								
F	F	T								
F	F	F								
(1)	(2)	(3)								

Making a truth table for a sentence with three atoms

A	B	C	$\neg(A$	$\wedge$	$(\neg A$	$\vee$	$(B \wedge C)$	$))$	$\vee$	B
T	T	T			F					
T	T	F			F					
T	F	T			F					
T	F	F			F					
F	T	T			T					
F	T	F			T					
F	F	T			T					
F	F	F			T					
(1)	(2)	(3)			(4)					

Making a truth table for a sentence with three atoms

A	B	C	$\neg(A$	$\wedge$	$(\neg A$	$\vee$	$(B \wedge C)$	$))$	$\vee$	B
T	T	T			F		T			
T	T	F			F		F			
T	F	T			F		F			
T	F	F			F		F			
F	T	T			T		T			
F	T	F			T		F			
F	F	T			T		F			
F	F	F			T		F			
(1)	(2)	(3)			(4)		(5)			

Making a truth table for a sentence with three atoms

A	B	C	$\neg(A$	$\wedge$	$(\neg A$	$\vee$	$(B \wedge C)$	$))$	$\vee$	B
T	T	T			F	T	T			
T	T	F			F	F	F			
T	F	T			F	F	F			
T	F	F			F	F	F			
F	T	T			T	T	T			
F	T	F			T	T	F			
F	F	T			T	T	F			
F	F	F			T	T	F			
(1)	(2)	(3)			(4)	(6)	(5)			

Making a truth table for a sentence with three atoms

A	B	C	$\neg(A$	$\wedge$	$(\neg A$	$\vee$	$(B \wedge C)$	$))$	$\vee$	B
T	T	T		T	F	T	T			
T	T	F		F	F	F	F			
T	F	T		F	F	F	F			
T	F	F		F	F	F	F			
F	T	T		F	T	T	T			
F	T	F		F	T	T	F			
F	F	T		F	T	T	F			
F	F	F		F	T	T	F			
(1)	(2)	(3)		(7)	(4)	(6)	(5)			

Making a truth table for a sentence with three atoms

A	B	C	$\neg(A$	$\wedge$	$(\neg A$	$\vee$	$(B \wedge C)$	$))$	$\vee$	B
T	T	T	F	T	F	T	T			
T	T	F	T	F	F	F	F			
T	F	T	T	F	F	F	F			
T	F	F	T	F	F	F	F			
F	T	T	T	F	T	T	T			
F	T	F	T	F	T	T	F			
F	F	T	T	F	T	T	F			
F	F	F	T	F	T	T	F			
(1)	(2)	(3)	(8)	(7)	(4)	(6)	(5)			

Making a truth table for a sentence with three atoms

A	B	C	$\neg(A$	$\wedge$	$(\neg A$	$\vee$	$(B \wedge C)$	$))$	$\vee$	B
T	T	T	F	T	F	T	T		T	
T	T	F	T	F	F	F	F		T	
T	F	T	T	F	F	F	F		T	
T	F	F	T	F	F	F	F		T	
F	T	T	T	F	T	T	T		T	
F	T	F	T	F	T	T	F		T	
F	F	T	T	F	T	T	F		T	
F	F	F	T	F	T	T	F		T	
(1)	(2)	(3)	(8)	(7)	(4)	(6)	(5)		(9)	

Making a truth table for a sentence with three atoms

A	B	C	$\neg(A$	$\wedge$	$(\neg A$	$\vee$	$(B \wedge C)$	$))$	$\vee$	B
T	T	T	F	T	F	T	T		T	
T	T	F	T	F	F	F	F		T	
T	F	T	T	F	F	F	F		T	
T	F	F	T	F	F	F	F		T	
F	T	T	T	F	T	T	T		T	
F	T	F	T	F	T	T	F		T	
F	F	T	T	F	T	T	F		T	
F	F	F	T	F	T	T	F		T	
(1)	(2)	(3)	(8)	(7)	(4)	(6)	(5)		(9)	

Observe that $\neg(A \wedge (\neg A \vee (B \wedge C))) \vee B$ always has truth value T.
We call such a sentence a **tautology**.

Overview

Material conditional: $\rightarrow$

Biconditional: $\leftrightarrow$

What a material conditional $\rightarrow$ does *not* capture
- Ordinary reading vs. logical reading
- Conversational implicature
- Causality and Counterfactuals

Hintikka game

Informal proofs

Formal proofs

Truth tables

Tautologies and logical truths

Tautologies

Definition (Tautology)

A sentence S is a *tautology* if and only if in the truth table for S, there are only T's in the column under the main connective of S.

Example of another tautology:

P	P	$\vee$	$\neg P$
T		T	F
F		T	T

Tautologies

Definition (Tautology)

A sentence S is a *tautology* if and only if in the truth table for S, there are only T's in the column under the main connective of S.

Example of another tautology:

P	P	$\vee$	$\neg P$
T		T	F
F		T	T

Exercise for you: Is $((P \to Q) \to P) \to P$ a tautology?

Tautologies and logical truths

Definition: Tautology

A sentence S is a tautology iff in the truth table for S, there are only T's in the column under the main connective of S.

Definition: Logical truth (also called: logical necessity)

A sentence is a logical truth iff it is true under all circumstances.

Tautologies and logical truths

Definition: Tautology

A sentence S is a tautology iff in the truth table for S, there are only T's in the column under the main connective of S.

Definition: Logical truth (also called: logical necessity)

A sentence is a logical truth iff it is true under all circumstances.

Every tautology is a logical truth, but not the other way! Example:

$a = a$	$a = b$	$(a = a) \vee (a = b)$
T	T	T
T	F	T
F	T	T
F	F	F

This is a logical truth but not a tautology. The last two rows are spurious, because it is *logically impossible that $a = a$ is false*.

Spurious rows

Spurious rows: Definition (Syllabus, 3.1)

A row in a truth table of a sentence is called a *spurious row* if that row cannot possibly be true because of the meaning of the atomic sentences in the sentence.

We have:

Checking for logical truths (logical necessity)

A sentence S is a logical truth iff *in all non-spurious rows of the truth table* for S, there are only T's in the column under the main connective of S.

So indeed $a = a$ is a logical truth, but not a tautology.
Also $a = a \ \vee \ a = b$ is a logical truth, but not a tautology.

Tarski's World necessities

Definition (Tarski's World necessary (TW-necessary))

A sentence is Tarski's World necessary iff it is true under all circumstances that we can construct in Tarski's World.

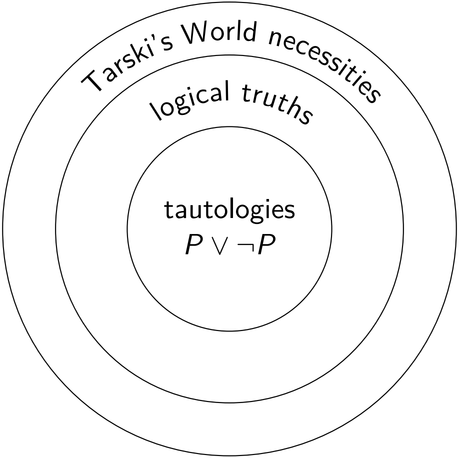
Examples of Tarski's World necessities:

$$\mathsf{Tet}(a) \lor \mathsf{Dodec}(a) \lor \mathsf{Cube}(a)$$

$$(\neg\mathsf{Small}(a) \land \neg\mathsf{Large}(a)) \to \mathsf{Medium}(a)$$

These sentences are not logical truths, and certainly not tautologies.

From tautologies to logical truths to Tarski's World necessities



Summary

Material conditional: $\rightarrow$

Biconditional: $\leftrightarrow$

What a material conditional $\rightarrow$ does *not* capture
- Ordinary reading vs. logical reading
- Conversational implicature
- Causality and Counterfactuals

Hintikka game

Informal proofs

Formal proofs

Truth tables

Tautologies and logical truths

Homework assignment 1 and next lecture

Formative Homework assignment 1

Available on Nestor, under *Course material/Homeworks*.

Hand in a physical copy, at the start of lecture 5 on **Monday 4 December 13:00**, work in pairs.

Next lecture

Propositional logic: logical equivalence and normal forms.

- Logical equivalence
- De Morgan's laws
- Negation normal form
- Distributive laws
- Conjunctive normal form
- Disjunctive normal form