Introduction to Logic Summary

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Ingredients

First-Order Logic Ingredients

1. constants $a, b, c \ldots$
2. variables $\ldots x, y, z$
3. functions $f, g, h \ldots$
4. terms $a, f(a), f(g(a)) \ldots$
5. predicates $P, Q \ldots$
6. connectives $\neg, \wedge, \vee \ldots$
7. quantifiers $\forall, \exists$
8. sentences $P(a) \ldots$
9. well-formed formulas

Propositional Logic Ingredients

1. propositions $P, Q \ldots$
2. connectives $\neg, \wedge, \vee \ldots$
3. sentences $P \wedge Q$

Connectives

Negation $\neg$ — not

P	$\neg P$
T	F
F	T

Conjunction $\wedge$ — and, moreover, although, but, however

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction $\vee$ — or, unless

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional $\rightarrow$ — if

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional $\leftrightarrow$ — if and only if, precisely if

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Inference Schemes

∧ Intro

i.	P		P is true
j.	Q		Q is true
k.	$P \wedge Q$	$\wedge$Intro: i, j	Therefore, (P and Q) is true

∧ Elim

i.	$P \wedge Q$		(P and Q) is true
j.	P	$\wedge$Elim: i	Therefore, P is true

∨ Intro

i.	P		P is true
j.	$P \vee Q$	$\vee$Intro: i	Therefore, (P or Q) is true

∨ Elim

i.	$P \vee Q$		(P or Q) is true
j.	P		
	$\vdots$		P implies R
k.	R		
l.	Q		
	$\vdots$		Q implies R
m.	R		
n.	R	$\vee$Elim: i, j-k, l-m	Therefore, R is true

¬ Intro

i. P		
$\vdots$		P leads to a contradiction
j. $\bot$		
k. $\neg P$	$\neg$ Intro: i-j	Therefore, $\neg P$ is true

¬ Elim

i. $\neg\neg P$		(not not P) is true
j. P	$\neg$ Elim: i	Therefore, P is true

⊥ Intro

i. P		P is true
j. $\neg P$		(not P) is true
k. $\bot$	$\bot$ Intro: i,j	Therefore, there is a contradiction

⊥ Elim

i. $\bot$		1. contradiction
j. P	$\bot$ Elim: i	2. ???
		3. PROFIT

$\rightarrow$ Intro

i. P		
$\vdots$		Q follows from P
j. Q		
k. $P \rightarrow Q$	$\rightarrow$ Intro: i-j	Therefore, P implies Q

$\rightarrow$ Elim

i. $P \rightarrow Q$		P implies Q
j. P		P is true
k. Q	$\rightarrow$ Elim: i, j	Therefore, Q is true

$\leftrightarrow$ Intro

i. P		
$\vdots$		Q follows from P
j. Q		
k. Q		
$\vdots$		P follows from Q
l. P		
m. $P \leftrightarrow Q$	$\leftrightarrow$ Intro: i-j, k-l	Therefore, P if and only if Q

$\leftrightarrow$ Elim

i. $P \leftrightarrow Q$ (or $Q \leftrightarrow P$)		P is true if and only if Q is true
j. P		P is true
k. Q	$\leftrightarrow$ Elim: i, j	Therefore, Q is true

$\forall$ Elim

i. $\forall x P(x)$		For any x, $P(x)$ is true
j. $P(c)$	$\forall$ Elim: i	Therefore, $P(c)$ is also true

$\forall$ Intro

i. $\boxed{c}$		For an arbitrary c,
$\vdots$		
j. $P(c)$		$P(c)$ is true
k. $\forall x P(x)$	$\forall$ Intro: i–j	Therefore, $P(x)$ is true for every x

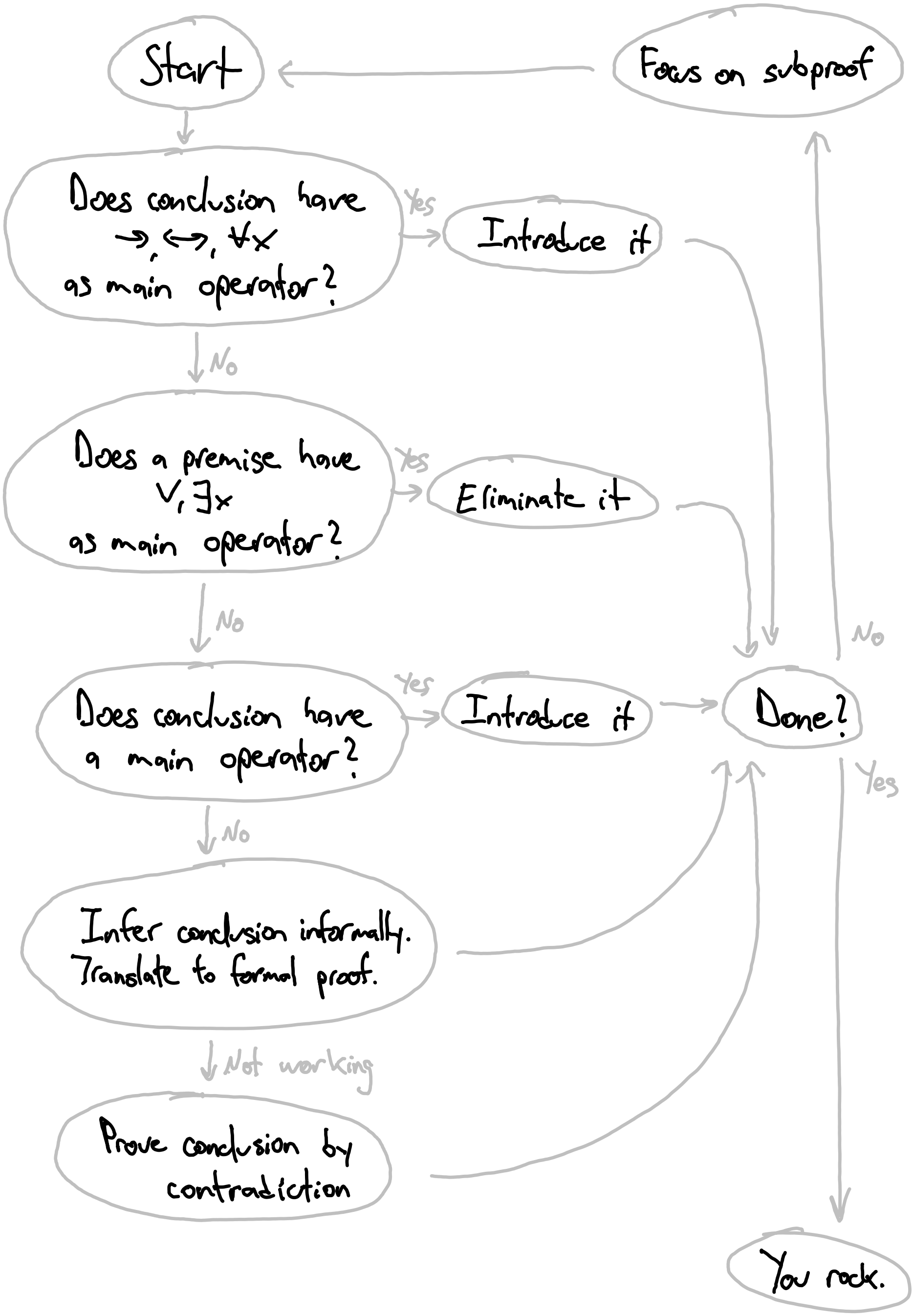
$\exists$ Intro

i. $P(c)$		$P(c)$ is true
j. $\exists x P(x)$	$\exists$ Intro: i	Therefore, there exists an x such that $P(x)$ is true

$\exists$ Elim

i. $\exists x P(x)$		There exists an x such that $P(x)$ is true
j. $\boxed{c}$ $P(c)$		For an arbitrary c,
$\vdots$		
k. Q		Q is true
l. Q	$\exists$ Elim: i, j–k	Therefore, Q is true

Proof Strategy



Truth Tables

Def. tautology → (sentence) always true

Def. contradiction → (sentence) always false

Def. logical truth/necessity → (sentence) always true on non-spurious rows

Def. truth-table possibility → (sentence) true on at least one row

Def. logical possibility → (sentence) true on at least one non-spurious row

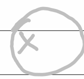
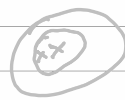
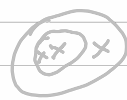
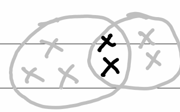
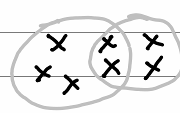
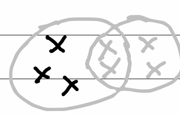
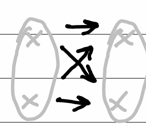
Def. tautological equivalence → (sentences) always have same truth values

Def. logical equivalence → (sentences) always have same truth values on non-spurious rows

Def. tautological consequence → (sentence) always true when all premises true

Def. logical consequence → (sentence) always true when all premises true on non-spurious rows

Set Theory

Def.	set $\{1\}$		collection of elements
Def.	empty set $\emptyset$		set with no elements
Def.	tuple $\langle 1,2 \rangle$		ordered pair
Def.	relation $\{\langle 1,2 \rangle\}$		set of tuples
Def.	is element of $\in$		is the element part of the set?
Def.	is subset of $\subseteq$		is every element in the left set an element of the right set?
Def.	is proper subset of $\subsetneq$		is every element in the left set an element of the right set and does the right set have elements which are not in the left set?
Def.	intersection $\cap$		set of all commonly shared elements of 2 sets
Def.	union $\cup$		set of all elements of 2 sets put together
Def.	difference $\setminus$		set of all elements which are part of the left set but are not part of the right set
Def.	cartesion product $\times$		set of all possible pairs composed of an element from the left set and an element of the right set

Relation Properties

① Reflexivity $x \in A \Rightarrow \langle x,x \rangle \in R$ 

② Symmetry $x,y \in A,\ \langle x,y \rangle \in R \Rightarrow \langle y,x \rangle \in R$ 

③ Transitivity $x,y,z \in A,\ \langle x,y \rangle, \langle y,z \rangle \in R \Rightarrow \langle x,z \rangle \in R$ 

④ Density $x,y,z \in A,\ \langle x,z \rangle \in R \Rightarrow \langle x,y \rangle, \langle y,z \rangle \in R$ 

⑤ Functionality $x,y,z \in A,\ \langle x,y \rangle, \langle x,z \rangle \in R \Rightarrow y=z$ 

Normal Forms

Def. negation normal form
- only $\wedge, \vee, \neg$ connectives
- negations directly in front of atoms

Def. conjunction normal form
- negation normal form conditions
- conjunction of disjunctions of literals

Def. disjunction normal form
- negation normal form conditions
- disjunction of conjunctions of literals

Equivalences

1. Double negation elimination $\neg\neg P \Leftrightarrow P$
2. Idempotency of $\wedge$ and $\vee$ $P \wedge P \Leftrightarrow P$, $P \vee P \Leftrightarrow P$
3. Distributivity of $\wedge$ and $\vee$ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$, $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$
4. Commutativity of $\wedge$ and $\vee$ $P \wedge R \Leftrightarrow R \wedge P$, $P \vee R \Leftrightarrow R \vee P$
5. Associativity of $\wedge$ and $\vee$ $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$, $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$
6. DeMorgan Laws $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$, $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
7. Definition of $\rightarrow$ and $\leftrightarrow$ $P \rightarrow Q \Leftrightarrow \neg P \vee Q$, $P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

Semantics of FOL

Def. model → like a dictionary for FOL "words"
→ a mapping of symbols to the objects they signify
→ a mapping of a language to a world

	language		world		
	c	→	M(c)	a constant	objects
letters / Symbols / signs	f	→	M(f)	a function	objects
	P	→	M(P)	a predicate	objects

Def. M(c) → "interpretation" of c in M

Def. variable assignment → like a dictionary for FOL "names"
→ a mapping of variable symbols to their values
→ designed to hold temporary interpretations

Def. h(x) → "interpretation" of x under h

Def. h[x/d] → new variable assignment in which variable x is assigned value d

Def. $h_\emptyset$ → empty variable assignment

Def. $[\![t]\!]^M_h$ → interpretation of term t in model M under variable assignment h

Def. $M \models A[h]$ → formula A is satisfied in model M by variable assignment h

Def. $M \not\models A[h]$ formula A is not satisfied in model M by variable assignment h

Equivalences

1. $M \models P(t_1, \ldots t_n)[h] \iff \langle [\![t_1]\!]^M_h, \ldots [\![t_n]\!]^M_h \rangle \in M(P)$

2. $M \models \neg A[h] \iff M \not\models A[h]$

3. $M \models (A \wedge B)[h] \iff M \models A[h]$ and $M \models B[h]$

4. $M \models (A \vee B)[h] \iff M \models A[h]$ or $M \models B[h]$

5. $M \models (A \rightarrow B)[h] \iff M \not\models A[h]$ or $M \models B[h]$

6. $M \models (A \leftrightarrow B)[h] \iff M \models A[h]$ and $M \models B[h]$ or $M \not\models A[h]$ and $M \not\models B[h]$

7. $M \models \forall x A[h] \iff$ for all $d \in M(\forall)$, $M \models A[h[x/d]]$

8. $M \models \exists x A[h] \iff$ there is a $d \in M(\forall)$ such that $M \models A[h[x/d]]$

Quantifier Normal Forms

Def. Prenex Normal Form → $\underbrace{Q_1}_{\text{quantifiers}} \overbrace{v_1}^{\text{variables}} \dots \underbrace{Q_n}_{\text{quantifiers}} \overbrace{v_n}^{\text{variables}} \underbrace{P}_{\text{quantifier-free formula}}$

Def. Skolem Normal Form
- → prenex normal form conditions
- → only universal quantifiers

Def. Horn Form
- → Skolem normal form conditions
- → quantifier-free formula
  - → in CNF
  - → max. 1 positive literal per conjunct

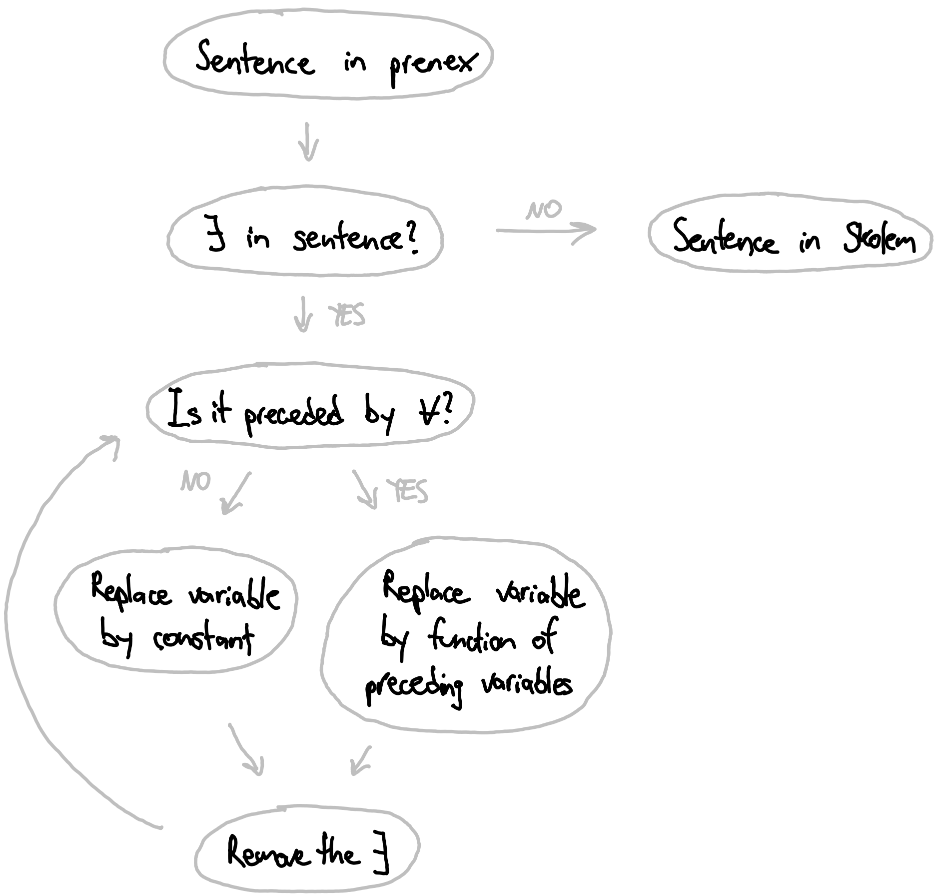
Equivalences

1. Vacuous quantification: $QxP \Leftrightarrow P$ if no free x in P
2. Renaming: $QxP(x) \Leftrightarrow QyP(y)$
3. De Morgan:
   $\neg\forall x P(x) \Leftrightarrow \exists x \neg P(x)$
   $\neg\exists x P(x) \Leftrightarrow \forall x \neg P(x)$
4. $\forall/\exists$ and $\wedge/\vee$:
   $QxP \wedge QxR \Leftrightarrow Qx(P \wedge R)$
   $QxP \vee QxR \Rightarrow Qx(P \vee R)$
   $QxP \wedge R \Leftrightarrow Qx(P \wedge R)$, no free x in R
   $QxP \vee R \Leftrightarrow Qx(P \vee R)$, no free x in R
5. $\forall/\exists$ and $\rightarrow$:
   $\forall x P \rightarrow R \Leftrightarrow \exists x(P \rightarrow R)$, no free x in R
   $\exists x P \rightarrow R \Leftrightarrow \forall x(P \rightarrow R)$, no free x in R
   $P \rightarrow QxR \Leftrightarrow Qx(P \rightarrow R)$, no free x in P

Normal Forms Feature Comparison

	NNF	CNF	DNF	Prenex	Skolem	Horn
only uses $\wedge, \vee, \neg$	✓	✓	✓	✗	✗	✓
$\neg$ only occurs in front of atomic sentences	✓	✓	✓	✗	✗	✓
conjunction of disjunctions	✗	✓	✗	✗	✗	✓
disjunction of conjunctions	✗	✗	✓	✗	✗	✗
quantifiers in front	✗	✗	✗	✓	✓	✓
only universal quantifiers	✗	✗	✗	✗	✓	✓
max. 1 positive literal per disjunction	✗	✗	✗	✗	✗	✓

Skolemization



Horn's algorithm