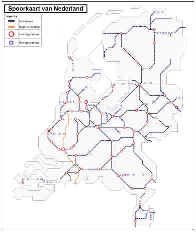
Introduction to Logic
Lecture 5: Sets, Relations, Functions

Davide Grossi

4 December 2023

Overview

Sets

Relations

Functions

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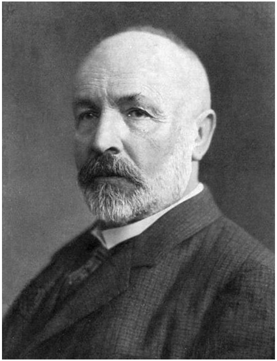
The beginning of Set Theory



Georg Cantor

Beiträge zur Begründung der transfiniten Mengenlehre (1869)

Contributions to the Founding of the Theory of Transfinite Numbers (1915)

https://archive.org/details/contributionstof00cant

Set (classical definition)

BY an "aggregate" (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects *m* of our intuition or our thought. These objects are called the "elements" of M.

Set theory (LPL Section 1.6 and Syllabus, Section 8.1)

Examples of finite sets:

- $\{1, 2, 3\}$
- $\{aristoteles, plato, socrates\}$
- $\{mars, merkel, mounteverest, 42\}$
- $\{1, \{2, 3\}, \{4, \{5, 6\}\}\}$

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Examples of infinite sets:

- $\mathbb{N}$, the set of natural numbers $0, 1, 2, \ldots$
- $\mathbb{Z}$, the set of integers $\ldots, -2, -1, 0, 1, 2, \ldots$

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Examples which are not sets:

- $\langle 1, 2, 3 \rangle$
- $\langle 2, 1, 3 \rangle$
- $P \land Q$
- Large( )

Membership in a set

$$\in$$

- Notation: $a \in b$.
- "a is in b" or "a is an element of b" or "a is a member of b"
- For "a is not in b" we write $a \notin b$ (or $\neg(a \in b)$).

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Giuseppe Peano

Arithmetices Prinicipia Nova Methodo Exposita (1889)

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Relations between sets: Equality

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- if $A \subseteq B$ and $B \subseteq A$ then $A = B$
- if $A \subseteq \emptyset$ then $A = \emptyset$
- $A \subsetneq A$ never holds
- $A \subsetneq \emptyset$ never holds

Operations on sets

Intersection $A \cap B$ is $\{x \mid x \in A \textit{ and } x \in B\}$

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We often work in a large set U called *universe* of discourse.
If U is given, then we can define:

Complement $A^c := U \setminus A = \{x \in U \mid x \notin A\}$

Overview of Symbols

You should know by now:

- $\in$
- $\notin$
- $\subseteq$
- $\subsetneq$
- $\emptyset$
- $\cap$
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You should know by now:

- $\in$ `\in`
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- $\cap$ `\cap`
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Next to the symbols you can see their LaTeX commands.

A very useful tool/website if you are looking for symbol commands: `https://detexify.kirelabs.org/`.

Building sets: Abstraction

Let E be any property, for example "being an even number", "being a mathematical entity", "being nice", ...

General abstraction says:

> *There exists a set A such that*
> for any x, $x \in A$ *if and only if* $E(x)$

We write this new set as:

$$\{x \mid E(x)\}$$

That is, the set of all objects with property E.

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Abstraction can also be relative to a set, for example:

$$\{x \in \mathbb{N} \mid \mathit{Even}(x)\}$$

Building sets: Abstraction examples

Examples of general abstraction:

- $\{x \mid \mathit{Loud}(x) \wedge \mathit{InGroningen}(x)\} = \{\mathit{martiniToren}, \mathit{forum}, \ldots\}$
- $\{x \mid x = \mathit{bob} \vee x = \mathit{alice}\} = \{\mathit{alice}, \mathit{bob}\}$

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Examples of relative abstraction:

- $\{x \in \mathbb{N} \mid \mathit{Odd}(x)\} = \{1, 3, 5, 7, \ldots\}$
- $\{x \in \{1, 23, 42\} \mid \mathit{Even}(x)\} = \{42\}$

Russell's paradox

Using general abstraction, let R be the set defined by the property "not being a member of itself"

$$R := \{x \mid x \notin x\}$$

Then we have for any x: $x \in R$ iff $x \notin x$

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Moral: General abstraction $\{x \mid \dots\}$ is "dangerous"!
Only *relative* abstraction is safe: $\{x \in A \mid \dots\}$.

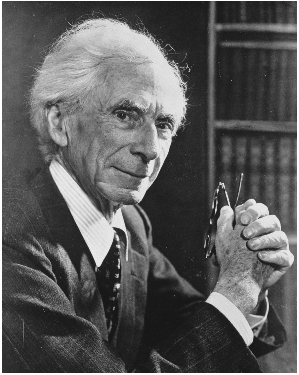
Bertrand Russell (1872–1970)

Either a thing is true or it isn't.

If it is true, you should believe it.

And if it isn't you shouldn't.

— Bertrand Russell



Logicomix: An Epic Search for Truth by Papadimitriou et al.
`https://www.logicomix.com/`

Overview

Ordered pairs

Ordered pair of a and b is written as $\langle a, b \rangle$.

Main property: $\langle a, b \rangle = \langle c, d \rangle$ if and only if $a = c$ and $b = d$

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Order matters:

$$\langle 1, 2\rangle \neq \langle 2, 1\rangle$$

Recall that order does not matter for sets:

$$\{1, 2\} = \{2, 1\}$$

Relations

Definition

A (binary) relation is a set of ordered pairs.

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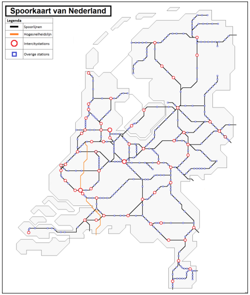
Example:
$\{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$



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A (binary) relation is a set of ordered pairs.

The domain of R is the set
$\{x \mid \langle x, y\rangle \in R,\ \text{for some } y\}$.
The codomain (range) of R is the set
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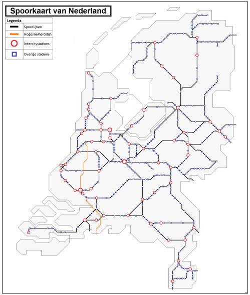

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The Cartesian product of A and B is
$A \times B := \{\langle x, y\rangle \mid x \in A \text{ and } y \in B\}$.
This is the largest relation with domain A and codomain B.

A relation on A is a subset of $A \times A$.
Notation: $R \subseteq A \times A$.

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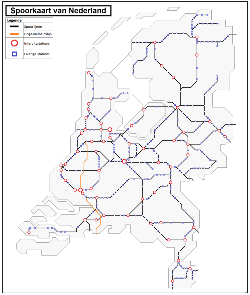

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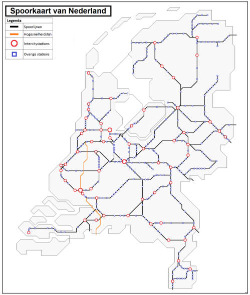

Some relations on $\mathbb{N}$ are $=$, $\leq$, $\geq$, $<$ and $>$.

Properties of relations

Let R be a relation on some set A, i.e. $R \subseteq A \times A$.

We define the following properties:

reflexivity for all $x \in A$: $\langle x, x \rangle \in R$

symmetry for all $x, y \in A$: if $\langle x, y \rangle \in R$, then $\langle y, x \rangle \in R$.

transitivity for all $x, y, z \in A$:
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density for all $x, y \in A$: if $\langle x, y \rangle \in R$,
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- $f : A \to B$ means: f is a (total) function with domain A and range $\subseteq B$.

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Let $f : A \to B$ be a (total) function with domain A and range $\subseteq B$. Such a function f my have some additional properties:

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Note: This was not in past exams, but may be used now!

Functions in First-order logic

We can add function symbols to the language of first-order logic.

Examples:

- father(x): the father of x
- mother(x): the mother of x
- $x + y$: the sum of x and y

Note: function symbols, just like constants, start with a lower-case letter, except special functions in infix notation such as $+$.

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(Constants, for example 5 or mary, are 0-place function symbols.)

Inductive definition of FOL terms (Syllabus Section 10.1)

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Complex terms are used just like names/constants,
to refer to objects in the domain of discourse.

Function symbols give more expressivity to FOL

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$\text{father}(\text{mother}(\text{max}))$ is a term.
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- You can express more properties using function symbols.
  Example: "Everyone's paternal grandmother is older and nicer than one's father"
  $\forall \text{x}(\text{OlderThan}(\text{mother}(\text{father}(\text{x})), \text{father}(\text{x})) \land$
  $\text{NicerThan}(\text{mother}(\text{father}(\text{x})), \text{father}(\text{x})))$

Example exercise from the 2019 exam

Given are the following five sets:

$$A = \{1, 2\},\ \ B = \{\{1\}, \{2\}\},\ \ C = \{1, \{1\}\},\ \ R = \{\langle 1, 2\rangle, \langle 2, 1\rangle\}$$
$$\text{and } S = \{1, \langle 2, 2\rangle\}.$$

For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

1. $C \subseteq A$
2. $\emptyset \in B$
3. $R \subseteq A$
4. $A \cap C \subseteq S$
5. $(A \cap C) \in (B \cap C)$
6. $C \subseteq A \cup B$
7. $A \cap S = \emptyset$
8. R is transitive

Next lecture

Wednesday 6 December 15:00 to 17:00

Coming soon

Quantification: meaning, translation, proof rules