Introduction to Logic (AI)
Lecture 1

Gerard Renardel

13 November 2023

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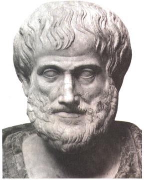

Introduction to Logic: Learning goals

Welcome!

Welcome!

What will we be learning about?

1. formal languages: first-order logic (FOL) and its sublanguage propositional logic (PL)
2. validity of arguments
3. how to build valid arguments: inferences and inference rules

Skills to acquire

1. *translate* English sentences to PL and FOL
2. check the *validity* of arguments
3. develop *formal proofs* for inferences
4. transform logical sentences to a *normal form*
5. apply the *Horn algorithm* to Horn sentences
6. apply the truth definition for first-order logic

Example exam questions

1. Check with a truth table whether the following sentence is a tautology.

$$((A \to B) \vee (A \leftrightarrow B)) \wedge (\neg A \vee B)$$

2. Give a formal proof of the following inference. Do not forget the justifications.

$$\begin{array}{|l} \forall x(\exists y \neg R(x,y) \to P(x)) \\ \forall x \forall y(x = y \to \neg R(x,y)) \\ \hline P(a) \end{array}$$

Who are we?

There is a team of 12 people taking care of you in this course!

Lectures: Davide Grossi and Gerard Renardel de Lavalette
Course coordinator: Davide Grossi
Tutorials:

- Jan van Houten (TAs coordinator)
- Diana Catana (TAs coordinator)
- Alexandra Stan
- Carolina Aranda Bassegoda
- Yoni Zuidinga
- Nora Meier
- Laura Quiros
- Thomas Zwartfeld
- Diana Todoran
- Aleksandar Todorov

On Brightspace you can register for your tutorial group.
E-mail your TA for practical questions, e.g., changing tutorial groups, software access

What are we going to do?

Introduction to Logic

- Lectures
- Tutorials/Practicals
- Homework assignments (formative)
- Midterm
- Final exam

See the **Weekly Schedule** on Brightspace for a detailed breakdown of activities (including deadlines for homeworks!).

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Rules and Regulations from the Board of Examiners

Fraud is an act or omission by the examinee designed to partly or wholly hinder the forming of a correct assessment of his or her knowledge, understanding and skills.

Lectures and tutorials/practicals

In the lectures, the contents of the book **Language, Proof and Logic** and the syllabus are presented

In the tutorials, you will make exercises from the book and the syllabus.

You will use both **pencil & paper**, and the computer tools **Tarski's World**, **Fitch** and **Boole**

You will get help from the teaching assistants.

Attendance is not enforced, but still *highly recommended*.

Weekly schedule & grading

Week 1 and 6: 2 hours lecture, 2 hours tutorials
Week 2: 2 hours lecture, 4 hours tutorials
Week 5: 4 hours lecture, 2 hours tutorials
Week 3, 4, 7: 4 hours lectures, 4 hours tutorials

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Items to be handed in for feedback (no grade)
Week 4: Homework assignment 1 to be submitted on paper (in pairs) by Monday, Dec 4, 13:00, in the lecture hall
Week 7: Homework assignment 2 to be submitted on paper (in pairs) by Monday, Jan 8, 13:00, in the lecture hall

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Week 5: midterm exam (25%), Tuesday, Dec 12, 18:30-20:30
Week 9: exam (75%), Wednesday, January 24th, 15:00-17:00

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Final grade: 25% midterm exam, 75% exam

Overview

Practical information

What is logic?

Valid and invalid arguments

Atomic sentences

Logic of atomic sentences

Overview

Logic

- Reasoning and arguing are typically human activities.
- "An argument is a connected series of statements intended to establish a definite proposition" (Monty Python)

- There are inference rules that determine whether an argument is correct or not.
- Logic is the study of these inference rules.

Language, Proof and Logic, p.1

... *all* rational inquiry depends on logic, on the ability of people to reason correctly most of the time, and, when they fail to reason correctly, on the ability of others to point out the gaps in their reasoning.

"Calculemus!" G.W. Leibniz (1646–1716)

"The only way to rectify our reasonings is to make them as tangible as those of the Mathematicians, so that we can find our error at a glance, and when there are disputes among persons, we can simply say: *Let us calculate*, without further ado, to see who is right"

Example argument in physics

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

Example argument in politics

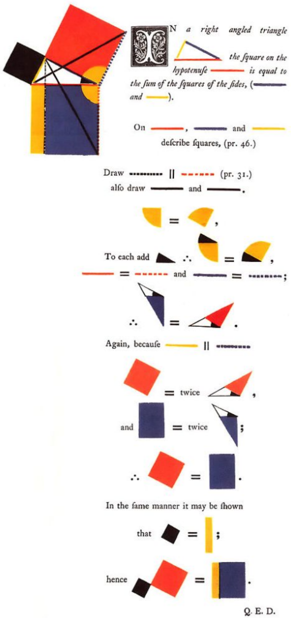
Donald J. Trump
@realDonaldTrump

Some or all of the content shared in this Tweet is disputed and might be misleading about an election or other civic process. Learn more

I easily WIN the Presidency of the United States with LEGAL VOTES CAST. The OBSERVERS were not allowed, in any way, shape, or form, to do their job and therefore, votes accepted during this period must be determined to be ILLEGAL VOTES. U.S. Supreme Court should decide!

Overview

Pythagoras' theorem: $a^2 + b^2 = c^2$



$\sqrt{2}$ is irrational (Hippasus of Metapontum, plm 500 BC)

Suppose $\sqrt{2}$ is rational. Then there must be a fraction that cannot be further simplified, let's say $\frac{a}{b}$, such that $\sqrt{2} = \frac{a}{b}$. From this it follows that $2 = (\frac{a}{b})^2 = \frac{a^2}{b^2}$ and thus $2b^2 = a^2$.

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Therefore a^2 is even, but then a is also even, because the square of an odd number is odd. So there is a number c such that $2c = a$. If we now substitute $2c$ for a, then we obtain $2b^2 = (2c)^2 = 4c^2$.

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Therefore a^2 is even, but then a is also even, because the square of an odd number is odd. So there is a number c such that $2c = a$. If we now substitute $2c$ for a, then we obtain $2b^2 = (2c)^2 = 4c^2$. Thus $b^2 = 2c^2$. But then also b is even. But if a and b are both even, then $\frac{a}{b}$ can be simplified. This is in contradiction with our previous assumption. Thus $\sqrt{2}$ is irrational.

The birth of logic: Aristotle of Stagira (384–322 BC)

Prior Analytics
First then take a universal negative with the terms A and B.
If no B is A, neither can any A be B.
For if some A (say c) were B, it would not be true that no B is A;
for c is a B which is A.

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Four examples of arguments: argument 1

If you are a member, then you get a discount.
You are a member.

You get a discount.

Four examples of arguments: argument 1

If you are a member, then you get a discount. **Premise**

You are a member.

You get a discount.

Four examples of arguments: argument 1

If you are a member, then you get a discount.	**Premise**
You are a member.	**Premise**
You get a discount.	

Four examples of arguments: argument 1

If you are a member, then you get a discount.	**Premise**
You are a member.	**Premise**
You get a discount.	**Conclusion**

Another argument: argument 2

If I miss the train, then I will be late.
I miss the train.

I will be late.

And another argument: argument 3

If there is asbestos in the building, then you may not enter.
There is asbestos in the building.

You may not enter.

We abstract from the contents of premises and conclusion, and only look at the form of the argument: *the inference scheme*.
Arguments 1, 2, 3 all follow the same inference scheme

Inference schemes

If P, then Q.
P

Q

Inference schemes

If P, then Q.
P

Q

In symbolic notation:

$P \to Q$
P

Q

Inference schemes

If P, then Q.
P

Q

In symbolic notation:

$P \rightarrow Q$
P

Q

This scheme is valid: “Modus Ponens”

A different argument

If you are a member, then you get a discount.
You are not a member.

You do not get a discount.

A different inference scheme

If P, then Q.
It is not the case that P.

It is not the case that Q.

A different inference scheme

If P, then Q.
It is not the case that P.

It is not the case that Q.

In symbolic notation:

$$\begin{array}{|l} P \to Q \\ \neg P \\ \hline \neg Q \end{array}$$

A different inference scheme

If P, then Q.
It is not the case that P.

It is not the case that Q.

In symbolic notation:

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QUESTION: Is this scheme valid?

A different inference scheme

If P, then Q.
It is not the case that P.

It is not the case that Q.

In symbolic notation:

$$\begin{array}{|l} P \to Q \\ \neg P \\ \hline \neg Q \end{array}$$

QUESTION: Is this scheme valid?

This scheme is *not* valid: "Denying the antecedent". This invalid scheme is a famous fallacy.

The definition of validity

Validity: from Language, Proof and Logic, p. 44

An argument is *valid* if and only if the conclusion must be true in any circumstance in which the premises are true.
We say that the conclusion of a logically valid argument is a *logical consequence* of its premises.

An argument is *sound* if it is valid and the premises are all true.

Logic focuses on **validity** rather than soundness.

Note that an argument is *valid* if and only if it is impossible that the premises are true while the conclusion is false.

Question: How to show that an argument is valid?

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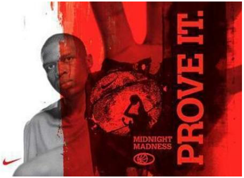

Language, Proof and Logic p. 46-47

A proof is a step-by-step demonstration that a conclusion (say S) follows from some premises (say P, Q, R). The way a proof works is by establishing a series of intermediate conclusions, each of which is an obvious consequence of the original premises and the intermediate conclusions previously established. The proof ends when we finally establish S as an obvious consequence of the original premises and the intermediate conclusions.

Question: How to show that an argument is *not* valid?

Question: How to show that an argument is *not* valid?

Answer: Give a counterexample!

Language, Proof and Logic, p. 63

To show that a sentence Q is **not** a consequence of premises P_1,...,P_n, we must show that the argument with premises P_1 , . . . , P_n and conclusion Q is invalid. This requires us to demonstrate that it is possible for P_1 , . . . , P_n to be true while Q is simultaneously false. That is, we must show that there is a possible situation or circumstance in which the premises are all true while the conclusion is false. Such a circumstance is said to be a *counterexample* to the argument.

One counterexample suffices

All swans are white

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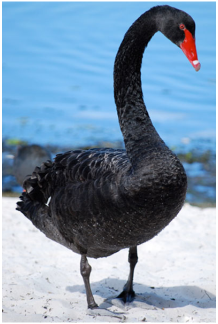
All swans are white



Affirming the consequent

$$\begin{array}{l|l} & P \rightarrow Q \\ & Q \\ \hline & P \end{array}$$

Affirming the consequent

$$\begin{array}{|l} P \to Q \\ Q \\ \hline P \end{array}$$

This scheme "Affirming the consequent" is *not* valid. It is a famous fallacy.

Affirming the consequent

$$\begin{array}{|l}
P \to Q \\
Q \\
\hline
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QUESTION: Can you give me a counterexample?

Affirming the consequent

$$\begin{array}{|l} P \to Q \\ Q \\ \hline P \end{array}$$

This scheme "Affirming the consequent" is *not* valid. It is a famous fallacy.
QUESTION: Can you give me a counterexample?

P: You are in Groningen.
Q: You are in the Netherlands.

Overview

Practical information

What is logic?

Valid and invalid arguments

Atomic sentences
- Names
- Predicates
- Identity

Logic of atomic sentences

The meaning of names in first-order logic: Two names for one object is OK, as well as objects without name

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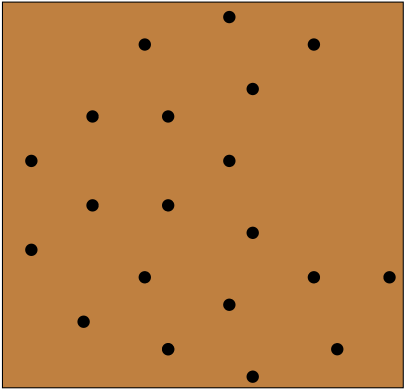

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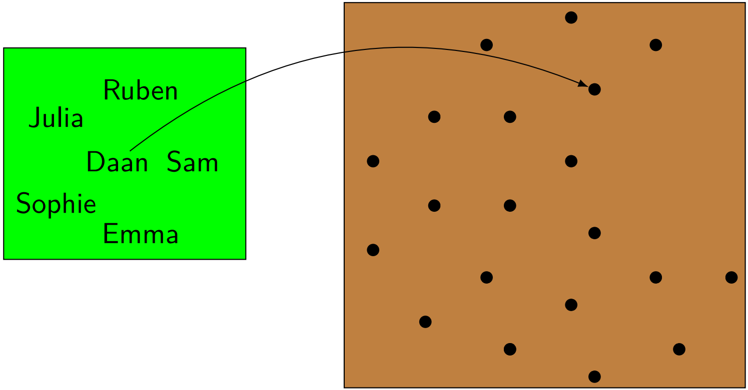

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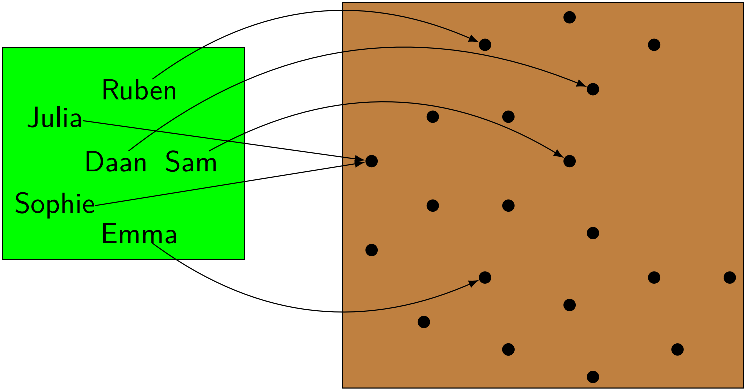
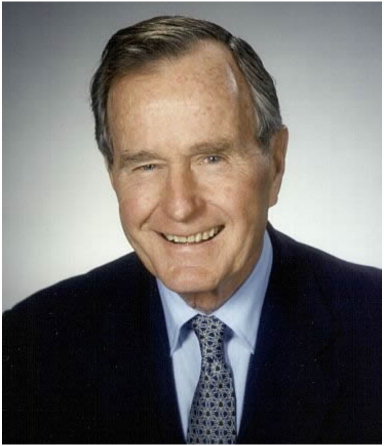


Not one name for two objects: Who is President Bush?



No name without an object: Who is Sinterklaas ($\approx$ St. Nicolas)?

Do's and don'ts for names in first-order logic

In summary:

Ok	✔	Something with one name.
	✔	Something with two or more names.
	✔	Something without a name.
Wrong	✖	One name for two things.
	✖	A name without a thing.

Riddle

A man is looking at a picture and says:
"Brothers and sons I have none,
but this person's father is my father's son."

Question: Who could be in the picture?

Names in first-order logic

Individual constants

In first-order logic, names are represented as starting with small letters. Examples: a, b, c, max, claire

Overview

Predicate symbols (also known as relation symbols)

Predicate symbols, from Language, Proof and Logic, p. 20

Predicate symbols are symbols used to express some property of objects or some relation between objects.

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These names (of objects) are the *arguments* of the predicate.

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These names (of objects) are the *arguments* of the predicate.

$\boxed{\text{WARNING}}$: 'argument' has two different meanings in logic:

- series of statements in which the conclusion is supposed to follow from the others;
- a name (of an object) to which a predicate symbol is applied.

Unary predicates: properties

Unary predicate symbols, having one argument, refer to properties.

Examples of properties:

- blue (x is blue)
- honest (x is honest)
- human (x is human)
- tree (x is a tree)

Binary predicates: relations

Binary predicate symbols have two arguments and refer to binary relations.

Examples of binary relations:

- to love (x loves y)
- being larger than (x is larger than y)
- to know (x knows y)
- being divisible by (x is divisble by y)
- to hit (x hits y)

n-ary predicates: n-ary relations ($n > 2$)

n-ary predicate symbols refer to n-ary relations ($n > 2$).

Examples of n-ary relations:

- x is between y and z (ternary)
- x likes y more than z (ternary)
- x gives y to z for w (4-ary)
- x hears from y that z loves w more than v (5-ary)

Predicate symbols in the language of first-order logic (FOL)

- In first-order logic, properties and *n*-ary relations are all represented with capitalised words.
- Examples: Blue, Larger, S

Exception: some well-known binary predicates are represented by common symbols:
$=, <, >, \leq, \geq$

Order of arguments in an atomic sentence

The following sentences mean something completely different:

- Loves(romeo,juliet)
- Loves(juliet,romeo)

Similarly for the following two sentences:

- $2 < 4$
- $4 < 2$

Atomic sentences in propositional logic (PL) and first-order logic (FOL)

The languages of first-order logic (FOL) and propositional logic (PL) are different.
In PL, atomic sentences have no inner structure. They are represented by capital letters:

Atomic formulas in propositional logic

P, Q, R, S

Atomic sentences in propositional logic (PL) and first-order logic (FOL)

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In FOL, atomic sentences are constructed from predicate symbols and individual constants:

Atomic formulas in first-order logic

B(a), R(c,d), Q(b,f,e,d,a), Loves(romeo,julia)
or in infix notation: $a = b$, $c < d$, $e \geq f$

Analysis in propositional logic and first-order logic

John knows Mary.

Analysis in propositional logic and first-order logic

Propositional logic

This sentence expresses a single thought. It is an atomic sentence. Let's not analyze it any further.

John knows Mary.

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Propositional logic

This sentence expresses a single thought. It is an atomic sentence. Let's not analyze it any further.

First-order logic

This sentence expresses a single thought. It is an atomic sentence. Let's analyze what exactly is being said about the objects that are named.

John knows Mary.

Analysis in propositional logic and first-order logic

Translation key:
P: John knows Mary.

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Analysis in propositional logic and first-order logic

$\boxed{P}$

Translation key:
P: John knows Mary.

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Translation key:
j: John
m: Mary
K(x,y): x knows y.

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Analysis in propositional logic and first-order logic

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K(j,m)

Translation key:
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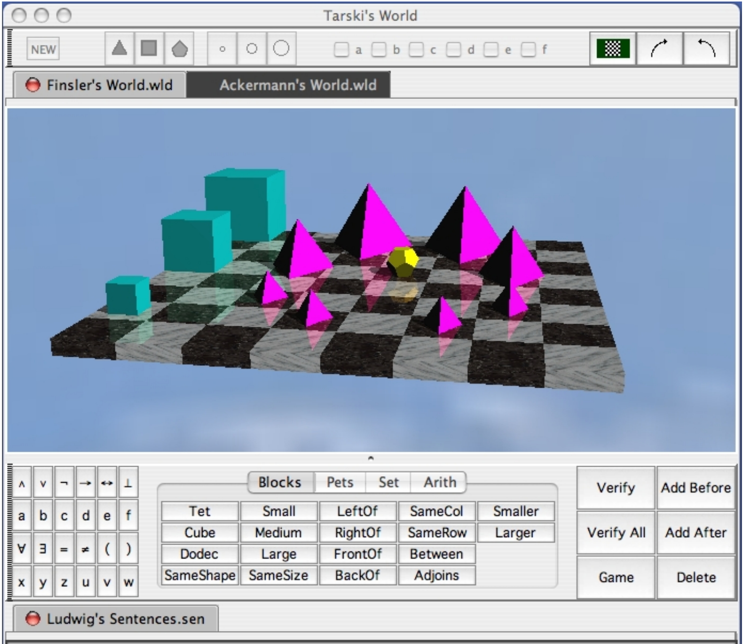
The language of Tarski's World



Unary predicates in Tarski's World

Tet(a)	a is a tetrahedron.
Cube(a)	a is a cube.
Dodec(a)	a is a dodecahedron.
Small(a)	a is small.
Medium(a)	a is medium.
Large(a)	a is large.

Binary predicates in Tarski's World

SameSize(a,b)	a is the same size as b.
SameShape(a,b)	a has the same shape as b.
Larger(a,b)	a is larger than b.
Smaller(a,b)	a is smaller than b.
SameCol(a,b)	a is in the same column as b.
SameRow(a,b)	a is in the same row as b.

More binary predicates in Tarski's World

Adjoins(a,b)	a and b are in adjacent squares (*not* diagonally).
LeftOf(a,b)	a is located nearer to the left edge of the grid than b.
RightOf(a,b)	a is located nearer to the right edge of the grid than b.
FrontOf(a,b)	a is located nearer to the front of the grid than b.
BackOf(a,b)	a is located nearer to the back of the grid than b.

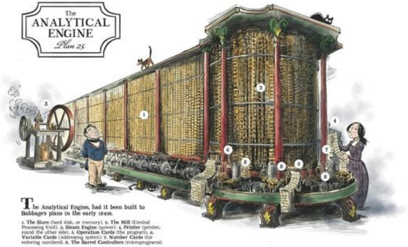
Ternary predicates in Tarski's World

Between(a,b,c)	a, b, and c are in the same row, column, or diagonal and a is between b and c.

Overview

How to translate ‘is’ to FOL?

1. Augusta Ada Byron is Ada Lovelace
2. Augusta Ada Byron is a computer scientist



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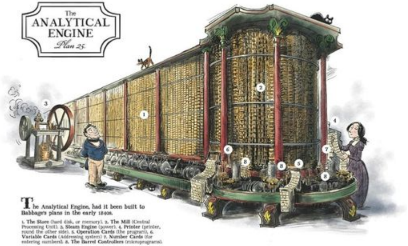

1. a=l, with the translation key

   a: Augusta Ada Byron;
   l: Ada Lovelace
2. C(a), with the translation key

   C(x): x is a computer scientist;
   a: Augusta Ada Byron

Identity a=b: when two names refer to the same object
G.W. Leibniz (1646–1716)

"Eadem sunt quorum unum potest substitui alteri salva veritate"

"Those things are identical of which one can be substituted for the other without loss of truth."

Example

A. A. Byron and Ada Lovelace are the same person
A. A. Byron wrote the first algorithm

Ada Lovelace wrote the first algorithm

Example

A. A. Byron and Ada Lovelace are the same person
A. A. Byron wrote the first algorithm

$\vdash$ Ada Lovelace wrote the first algorithm

An important general property:

The indiscernibility of identicals

If a is identical to b, then a and b have all the same properties

First-order logic (FOL) does not have a single language

In general, we design a specific language of FOL for each problem context:

Guideline from Language, Proof and Logic, p. 29

Usually, the overall goal is to come up with a language that can say everything you want, but that uses the smallest "vocabulary" possible. Picking the right names and predicates is the key to doing this.

Overview

Overview

Reminder: the definition of validity of an argument

Validity

An argument is *valid* if and only if it is impossible that the premises are true while the conclusion is false.

Examples of valid arguments

If that is a real Rolex, then the moon is made of blue cheese.
That is a real Rolex.

The moon is made of blue cheese.

No human is a daffodil.

No human is a daffodil.

A. Lovelace is the same person as A. Lovelace.

Definition formal proofs

Formal proof

A formal proof is a proof in a formal language (such as first-order logic), in which it is predetermined which argumentation steps are allowed.

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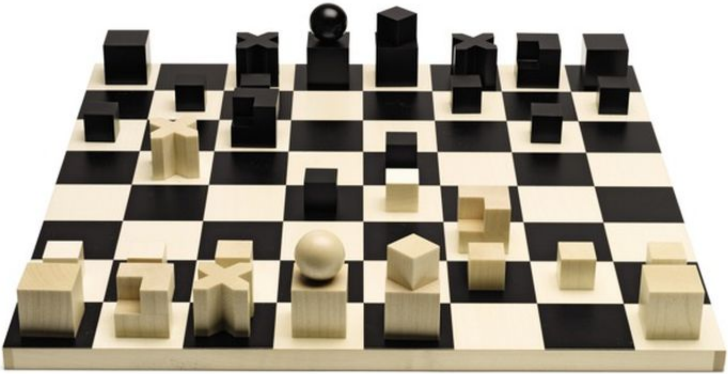
A system called $\mathcal{F}$ for *natural deduction* is developed in the book.

The tool Fitch is based on $\mathcal{F}$; Fitch is more flexible than $\mathcal{F}$.

Take care: when you are asked to give a formal proof, you have to do it in $\mathcal{F}$, and not in Fitch. (Therefore, avoid using **Ana Con** and **Taut Con** from Fitch.)

Making formal proofs can be seen as a game like chess



What does a formal proof look like?

Let P_2, P_2, P_3 be the premises of an argument, and let C be the conclusion. Then a formal proof of C from P_1, P_2, P_3 looks like this:

1.	P_1	
2.	P_2	
3.	P_3	
4.	T_1	Justification
5.	T_2	Justification
$\vdots$		$\vdots$
k.	C	Justification

Reiteration

$\vdots$

j. P Justification (or premise)

$\vdots$

k. P Reit: j

$\vdots$

Reiteration

$\vdots$

j. P Justification (or premise)

$\vdots$

k. P Reit: j

$\vdots$

The Reiteration proof rule is not technically necessary, but it may help make proofs look more natural.

$=$ Introduction

$$\begin{array}{|lll}
\vdots & & \\
\text{k.} & a = a & = \text{Intro} \\
\vdots & &
\end{array}$$

This proof rule $=$ Introduction is based on the fact that identity is *reflexive*:
every object is identical to itself,
i.e. for every object a we have that $a = a$.

$=$ Elimination

$$\begin{array}{lll} \vdots & & \\ \text{i.} & P(a) & \text{Justification (or premise)} \\ \vdots & & \\ \text{j.} & a = b & \text{Justification (or premise)} \\ \vdots & & \\ \text{k.} & P(b) & {=}\text{Elim : i, j} \\ \vdots & & \end{array}$$

Note that you may replace a by b in one place in $P(a)$, or in more places if it occurs more times. The choice is yours.
Note that in the justification, lines i, j appear in that order (order matters in justifications).

Examples of $=$ Elimination

1. $R(a, a)$	
2. $a = b$	
3. $R(a, b)$	$=$ Elim: 1, 2

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$$

In the above proofs, you view the sentence $R(a, a)$ as the formula $P(a)$ in order to apply $=$Elim, each time replacing a different choice of one or more occurrences of a by b.

Example: symmetry of identity

1.	$a = b$	
2.	$a = a$	$=$ Intro
3.	$b = a$	$=$ Elim: 2, 1

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QUESTION: What is playing the role of $P(a)$ for $=$Elim here?

Example: symmetry of identity

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2. $a = a$	$=$ Intro
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QUESTION: What is playing the role of $P(a)$ for $=$Elim here?

In the above proof, you view the sentence $a = a$ as the formula $P(a)$ in order to apply $=$Elim, replacing only the first a of $a = a$ by b.

Example: transitivity of identity

1. $a = b$	
2. $b = c$	
3. $a = c$	$=$ Elim: 1, 2

Example: transitivity of identity

1. $a = b$
2. $b = c$

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3. $a = c$ $\qquad$ $=$ Elim: 1, 2

In the above proof, you view the sentence $a = b$ as the formula $P(b)$ in order to apply $=$Elim, replacing the b of $a = b$ by c; the conclusion can then be seen as $P(c)$.

Properties of identity

Identity satisfies three properties:

reflexivity: $a = a$ for all a

symmetry: if $a = b$ then $b = a$

transitivity: if $a = b$ and $b = c$ then $a = c$

A problem

How to formally prove that the following argument is valid?

$$
\begin{array}{|l}
1.\ R(a) \\
2.\ b = a \\
\hline
3.\ R(b)
\end{array}
$$

Attempted solution?

1. $R(a)$
2. $b = a$

3. $R(b)$ $\quad = $ Elim: 1, 2 **??**

Attempted solution?

1. $R(a)$
2. $b = a$

---

3. $R(b)$ $\qquad = \text{Elim: } 1, 2$ **??**

Not allowed: $b = a$ only allows us to replace b by a, not the other way round

Solution

1. $R(a)$
2. $b = a$

---

3. $b = b$ $\quad\quad\quad\quad$ $=$ Intro

Solution

1.	$R(a)$	
2.	$b = a$	
3.	$b = b$	$=$ Intro
4.	$a = b$	$=$ Elim: 3, 2

Note that to derive line 4, we view $b = b$ as $P(b)$ and replace the first occurrence of b in it by a. The conclusion of that step, $a = b$, can now be seen as $P(a)$.

Solution

$$
\begin{array}{|ll}
1. & R(a) \\
2. & b = a \\
\hline
3. & b = b \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \text{Intro} \\
4. & a = b \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \text{Elim: } 3, 2 \\
5. & R(b) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \text{Elim: } 1, 4
\end{array}
$$

Note that to derive line 4, we view $b = b$ as $P(b)$ and replace the first occurrence of b in it by a. The conclusion of that step, $a = b$, can now be seen as $P(a)$.

Example: replacing one or more occurrences

1. $Q(a, b, c, a, d)$	
2. $a = d$	
3. $c = b$	
4. $Q(d, b, c, a, d)$	$=$Elim: 1, 2
5. $Q(d, b, c, d, d)$	$=$Elim: 4, 2
6. $Q(d, b, b, d, d)$	$=$Elim: 5, 3

QUESTION: Can it be proved faster, in less lines?

Example: replacing one or more occurrences

$$
\begin{array}{l|ll}
 & 1.\ Q(a, b, c, a, d) & \\
 & 2.\ a = d & \\
 & 3.\ c = b & \\
\hline
 & 4.\ Q(d, b, c, d, d) & =\text{Elim: } 1, 2 \\
 & 5.\ Q(d, b, b, d, d) & =\text{Elim: } 4, 3
\end{array}
$$

In step 4, two occurrences of a are replaced by d

In step 5, one occurrence of c is replaced by b

Reminder: The definition of validity

Validity

An argument is *valid* if and only if it is impossible that the premises are true while the conclusion is false.

In the above definition, “impossible” stands for “logically impossible”.

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If an argument is *not* valid, you can show this by providing a counterexample.

Counterexample

Is the following argument valid?

$$\begin{array}{|l} \text{Cube(a)} \\ \text{Cube(b)} \\ \hline \text{a=b} \end{array}$$

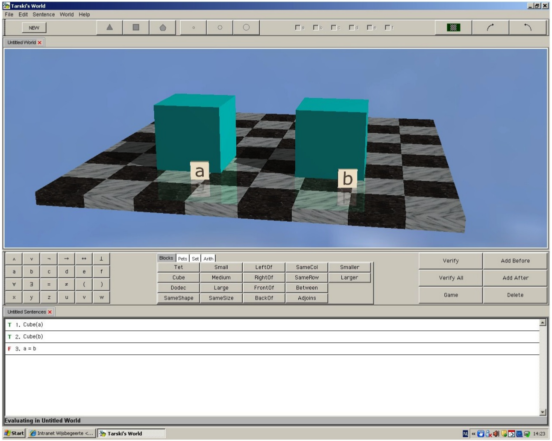
i.e., is $a = b$ a logical consequence from Cube(a), Cube(b)?

Counterexample

Is the following argument valid?

$$\begin{array}{|l} \text{Cube(a)} \\ \text{Cube(b)} \\ \hline \text{a=b} \end{array}$$

i.e., is $a = b$ a logical consequence from Cube(a), Cube(b)?



Completeness and Soundness

Soundness of the rules for identity

Every argument with identity that you can prove using only the rules

> $=$ *Introduction*
> $=$ *Elimination*

is valid.

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Completeness of the rules for identity

Using only the rules

$=$ *Introduction*

$=$ *Elimination*

you can provide a proof for *every* valid argument with identity.

Next time

Sentences with “and”, “or” and “not” and their proofs

read	LPL: 3.1-3.5,3.7, 5.1-5.4, 6.1-6.6 Syllabus 3.2
do	LPL: 1.8, 1.9, 1.10 YTI pp. 69, 72, 76, 144