university of groningen / faculty of science and engineering

Mechatronics

Week 4 Day 2

Previously

- We studied motivation and significance of sampling and discretization in Mechatronics systems
- We studied approximations from s-plane to z-plane
- We studied how to obtain transfer function of discrete-time system represented by a difference equation

Today's lecture:
Discrete-time systems and control

Learning objectives

After today's lecture, you will be able to

- Determine stability of discrete-time linear systems
- Design discrete PID controllers

Stability of discrete-time linear systems

Internal stability

The state-space representation of a discrete-time linear system (with no input) is given by

$$x(k+1) = Ax(k), \quad x(k) \in \mathbb{R}^n, \quad A \in \mathbb{R}^{nxn} \qquad (1)$$

which has general solution

$$x(k) = A^k x(0)$$

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Then the system (1) is…

- Stable if $\lim_{k\to\infty} \|x(k)\| < \infty$
- Asymptotically stable if $\lim_{k\to\infty} \|x(k)\| = 0$
- Unstable otherwise

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- Unstable otherwise

How can we verify these conditions?

Internal stability

$$x(k+1) = Ax(k), \quad x(k) \in \mathbb{R}^n, \quad A \in \mathbb{R}^{nxn} \qquad (1)$$

Theorem. *Consider system (1), and let us asume that A has distinct eigenvalues.*

Then, A is diagonalizable, i.e., there exists an invertible matrix $P \in \mathbb{R}^{nxn}$ *such that*

$$A = P\Lambda P^{-1},$$

where Λ *is a diagonal matrix and contains eigenvalues of* A *on its diagonal.*

Internal stability

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Theorem. *Consider system (1), and let us asume that A has distinct eigenvalues. Then, A is diagonalizable, i.e., there exists an invertible matrix* $P \in \mathbb{R}^{nxn}$ *such that* $A = P\Lambda P^{-1}$*, where* Λ *is a diagonal matrix; and contains eigenvalues of* A *on its diagonal.*

Therefore,

$$\begin{aligned} \|x(k)\| &= \left\|A^k x(0)\right\| \\ &\leq \left\|A^k\right\| \cdot \|x(0)\| \\ &= \left\|(P\Lambda P^{-1})^k\right\| \cdot \|x(0)\| \\ &= \|(P\Lambda P^{-1})(P\Lambda P^{-1}) \ldots (P\Lambda P^{-1})\| \cdot \|x(0)\| \\ &= \left\|P\Lambda^k P^{-1}\right\| \cdot \|x(0)\| \end{aligned}$$

Multipled K times

Internal stability

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Therefore,

$$\|x(k)\| = \left\|A^k x(0)\right\| \leq \left\|P\Lambda^k P^{-1}\right\| \cdot \|x(0)\|$$

Note that $P\Lambda^k P^{-1} = \sum_{k=1}^{n} p_i \widetilde{p_i} {\lambda_i}^k$ where

- p_i is the ith column of matrix P (ith eigenvector)
- $\widetilde{p_i}$ is the ith row of matrix P^{-1}
- λ_i are the eigenvaues of A on the diagonal

Internal stability

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Therefore,

$$\|x(k)\| = \|A^k x(0)\| \leq \|P\Lambda^k P^{-1}\| \cdot \|x(0)\|$$

with $P\Lambda^k P^{-1} = \sum_{k=1}^{n} p_i \widetilde{p}_i {\lambda_i}^k$

Then to obtain the eigenvalues:

$$\lim_{k\to\infty} \|x_k\| = \lim_{k\to\infty} \left\|\sum_{k=1}^{n} p_i \widetilde{p}_i {\lambda_i}^k\right\| \cdot \|x(0)\| = 0 \Leftrightarrow |\lambda_i| < 1 \forall i = 1,2,\dots n$$

where $(\lambda_1, \lambda_2, \dots, \lambda_n)$ are the eigenvalues of A.

Internal stability

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We have concluded that:

$$\lim_{k\to\infty} \|x_k\| = \lim_{k\to\infty} \left\|\sum_{k=1}^{n} p_i \tilde{p}_i {\lambda_i}^k\right\| \cdot \|x(0)\| = 0 \Leftrightarrow |\lambda_i| < 1 \forall i = 1,2,\ldots n$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are the eigenvalues of A.

What does this mean?

The stability of a discrete-time linear system is characterized by the eigenvalues of matrix A.
In particular, the system (1) is asymptotically stable if an only if the eigenvalues of A lie inside the unitary circle of the complex plane

Bounded-output bounded-input (BIBO) stability

Consider a discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ \mathrm{y}(k) &= Cx(k) + Du(k) \end{aligned} \qquad (2)$$

where $x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^p, \mathrm{y}(k) \in \mathbb{R}^m$

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As we learned in the previous lecture, the transfer function is:

$$G(z) = C(zI - A)^{-1}B + D = \frac{CAdj(zI - A)B + D|zI - A|}{|zI - A|}$$

Bounded-output bounded-input (BIBO) stability

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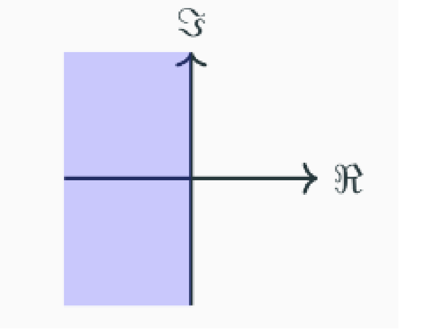
$$G(z) = C(zI - A)^{-1}B + D = \frac{CAdj(zI - A)B + D|zI - A|}{|zI - A|}$$

What does this mean?

The eigenvalues of A coincide with the poles of $G(z)$. Consequently, the system (2) is stable if an only if the poles of $G(z)$ lie inside the unitary circle in the complex plane

Mapping stable poles

Consider a stable continuous-time linear system. Its poles lie in the left-hand part of the complex plane.
Is the stability preserved after discretization?



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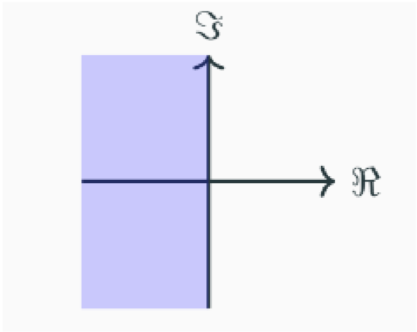
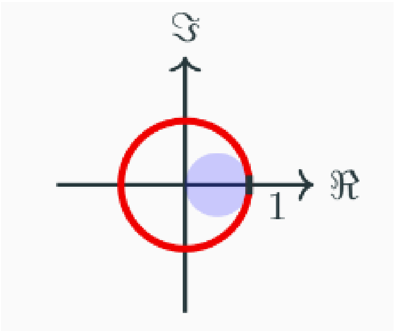


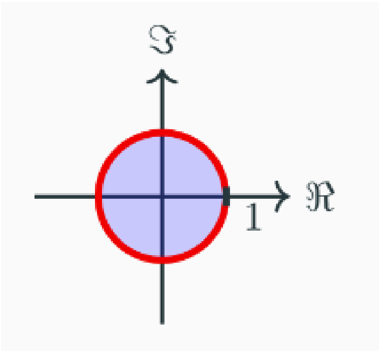
Euler backward

$$s \simeq \frac{z-1}{zT_s}$$



Bilinear

$$s \simeq \frac{2}{T_s}\frac{z-1}{z+1}$$



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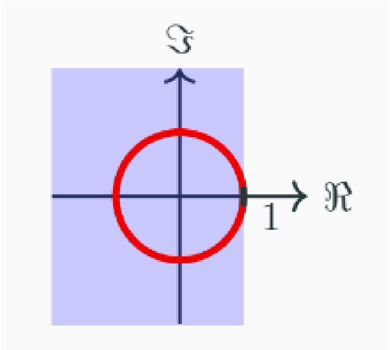

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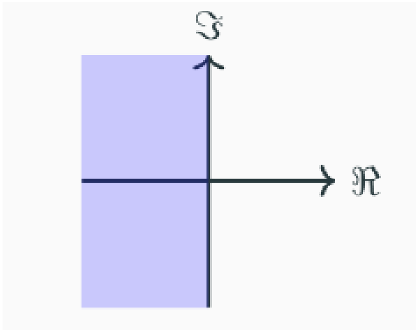

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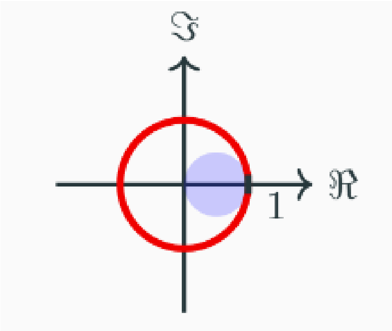

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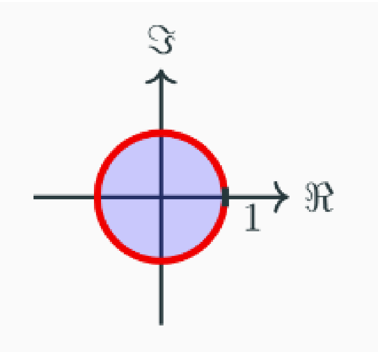
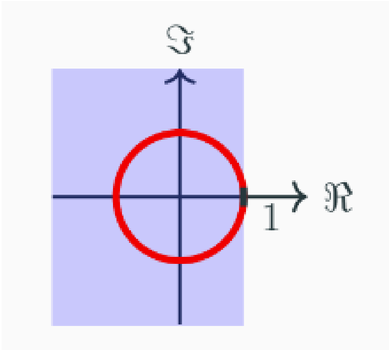


Always stable!!

Euler forward

$$s \simeq \frac{z-1}{T_s}$$



Mapping stable poles

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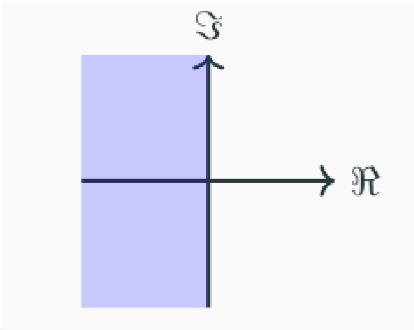

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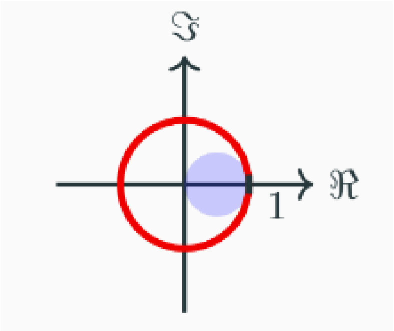

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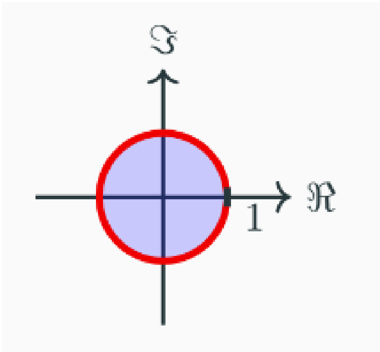

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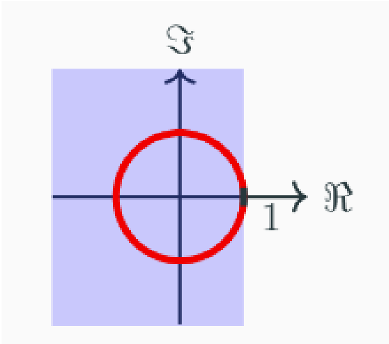

Large T_s might cause instability

Example

- Consider the system

$$\frac{dx(t)}{dt} = \underbrace{\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, D = 0$$

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- It follows that

$$C(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & -3 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+1 & 3 \\ 0 & s+2 \end{bmatrix} = \frac{3}{(s+1)(s+2)}$$

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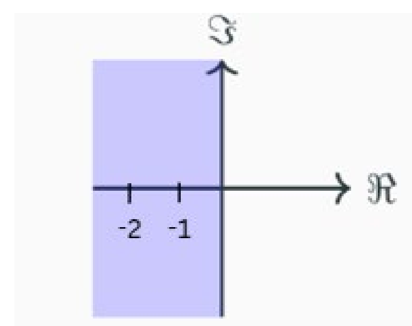
$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+1 & 3 \\ 0 & s+2 \end{bmatrix} = \frac{3}{(s+1)(s+2)}$$

which is stable since poles $s_1 = -1$ and $s_2 = -2$ are on the left hand plane



Example

- Using Euler forward approximation, s$\simeq \frac{z-1}{T_s}$, we get the TF in z:

$$\mathrm{G(z)} = \frac{3\mathrm{T_s}^2}{(\mathrm{z} + \mathrm{T_s} - 1)(\mathrm{z} + 2\mathrm{T_s} - 1)}$$

Example

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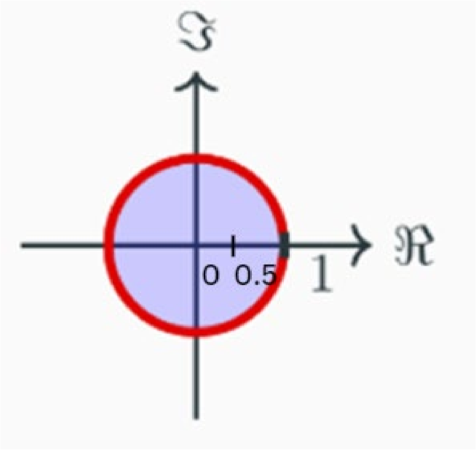
$$G(z) = \frac{3{T_s}^2}{(z + T_s - 1)(z + 2T_s - 1)}$$

- If we choose $\boldsymbol{T_s = 0.5\ s}$, the poles are $z_1 = 0.5$ and $z_2 = 0$.

Then the system is stable since the poles are both inside the unitary circle



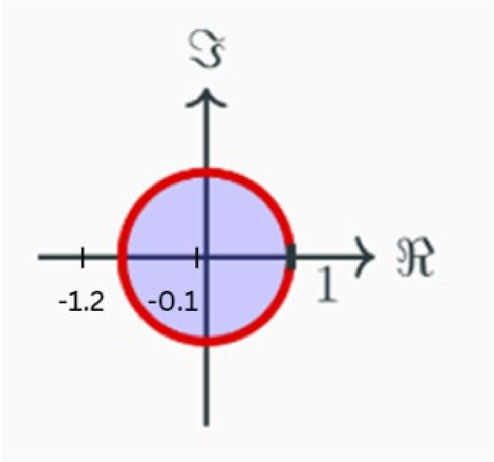
Example

- Using Euler forward approximation, s$\simeq \frac{z-1}{T_s}$, we get the TF in z:

$$G(z) = \frac{3{T_s}^2}{(z + T_s - 1)(z + 2T_s - 1)}$$

- If instead we choose $\boldsymbol{T_s = 1.1\ s}$, the poles are $z_1 = -0.1$ and $z_2 = -1.2$.

Then the system becomes unstable since the $z_2 = -1.2$ is outside the unitary circle



Discrete PID control

Discretization of PID controller

In continuous time, a PID controller has form

$$u(t) = K_p e(t) + K_d \frac{d}{dt} e(t) + K_i \int_0^t e(\tau) d\tau$$

with corresponding transfer function

$$C(s) = \frac{U(s)}{E(s)} = K_p + K_d s + K_i \frac{1}{s}$$

It is possible to obtain a discrete-time transfer function of the PID controller by using one of the approximation relations from s to z

$$s \simeq \frac{z-1}{T_s} \qquad s \simeq \frac{z-1}{zT_s} \qquad s \simeq \frac{2}{T_s}\frac{z-1}{z+1}$$

Discretization of PID controller

Euler Forward: $s \simeq \frac{z-1}{T_s}$

$$C(z) = \frac{K_d z^2 + \left(K_p T_s - 2K_d\right)z + K_i {T_s}^2 - K_p T_s + K_d}{T_s(z-1)}$$

Euler Backward: $s \simeq \frac{z-1}{T_s z}$

$$C(z) = \frac{\left(K_i {T_s}^2 + K_p T_s + K_d\right)z^2 - \left(2K_d + K_p T_s\right)z + K_d}{T_s z(z-1)}$$

Bilinear: $s \simeq \frac{2}{T_s}\frac{z-1}{z+1}$

$$C(z) = \frac{\left(K_i {T_s}^2 + 2K_p T_s + 4K_d\right)z^2 + \left(2K_i {T_s}^2 - 8K_d\right)z + K_i {T_s}^2 - 2K_p T_s + 4K_d}{2T_s(z^2-1)}$$

Plant discretization using Zero Order Hold

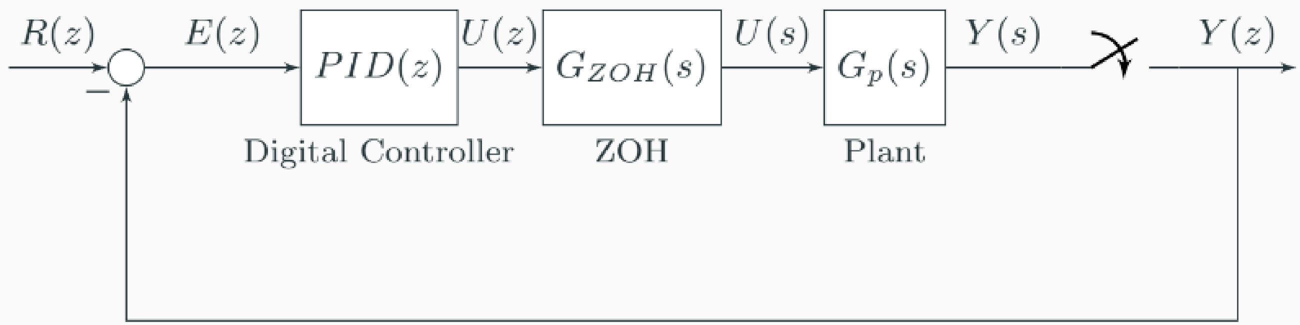
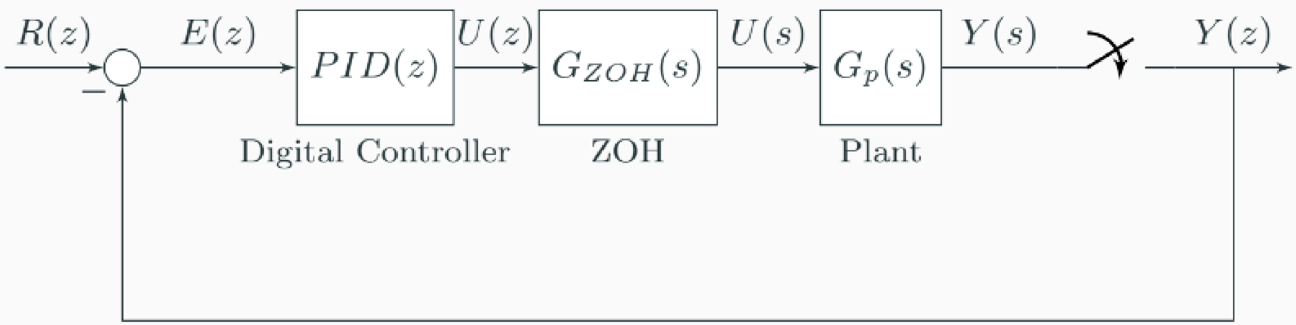


Figure 1: Diagram of a closed-loop digital controller. Here, the switch represents a digital encoder.

Given $G_p(s)$, the combined discrete transfer function of the plant and ZOH can be written as

$$H(z) = Z\{G_{ZOH}(s)G_p(s)\} = Z\left\{\frac{1 - e^{-sT_s}}{s} G_p(s)\right\}$$

Plant discretization using Zero Order Hold



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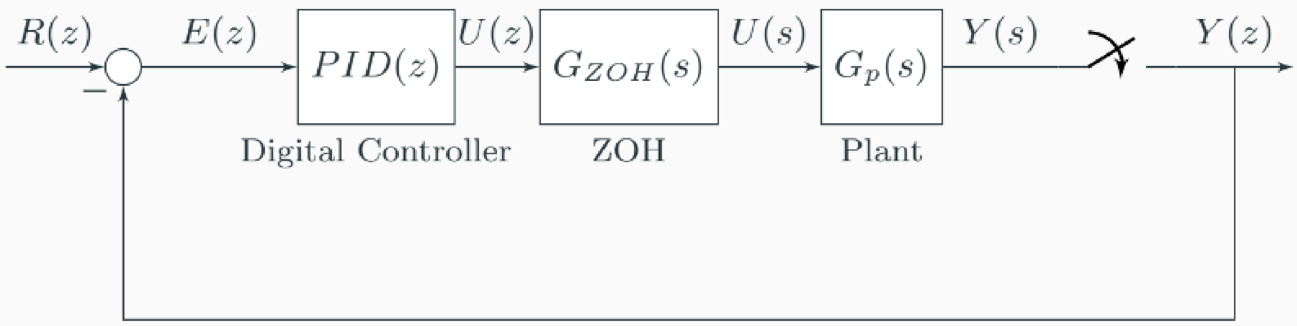
Recalling $z = e^{sT_s}$, we have

$$\boxed{H(z) = (1-z^{-1})Z\left\{\frac{G_p(s)}{s}\right\}}$$

Example



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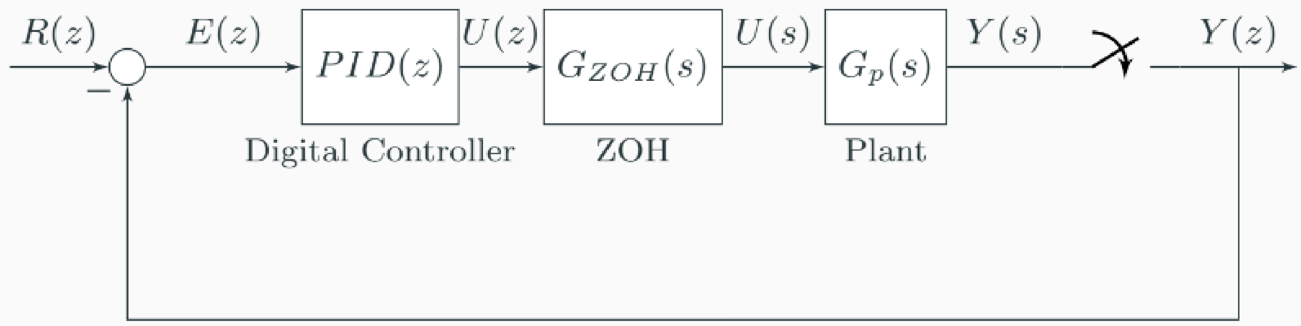
- Take a plant with transfer function $G_p(s) = \frac{1}{2s+1}$

Example



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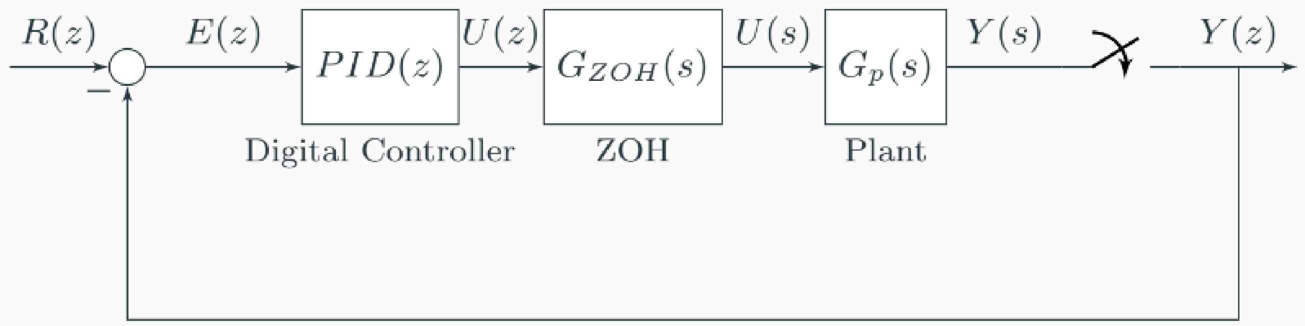
- Take a plant with transfer function $\boldsymbol{G_p(s) = \frac{1}{2s+1}}$
- The digital to analog (DAC) conversion is achieved through a zero order hold $G_{ZOH}(s) = \frac{1-e^{-sT_s}}{s}$ with sampling time $\boldsymbol{T_s = 1\ sec}$.

Example



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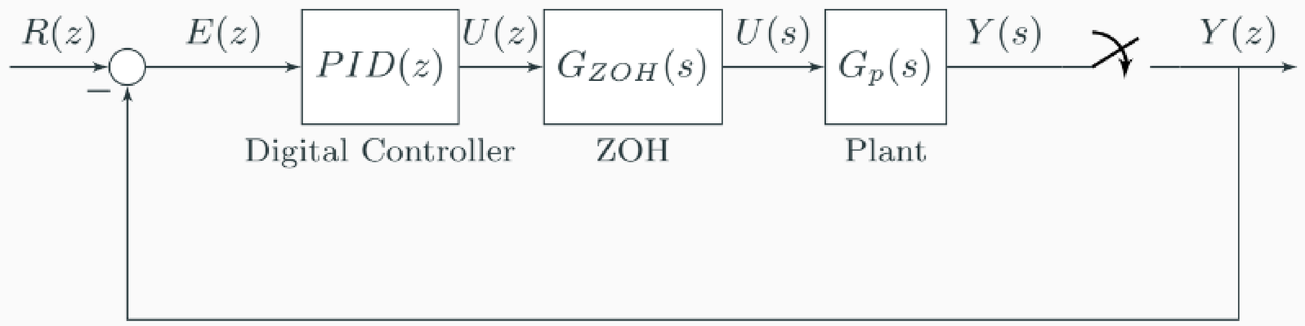
- You want the poles of the closed loop system to be located at

$$z_{1,2} = 1 \pm 2i$$

Example



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- The digital to analog (DAC) conversion is achieved through a zero order hold $G_{ZOH}(s) = \frac{1-e^{-sT_s}}{s}$ with sampling time $\boldsymbol{T_s = 1}$ **sec.**
- You want the poles of the closed loop system to be located at
$$\boldsymbol{z_{1,2} = 1 \pm 2i}$$
- Design a PI controller (compute values of K_p and K_i) that achieves this.

Example

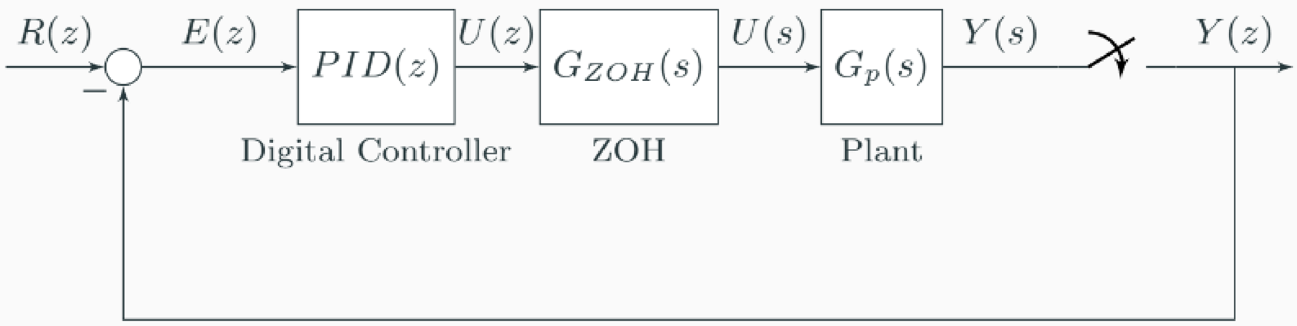
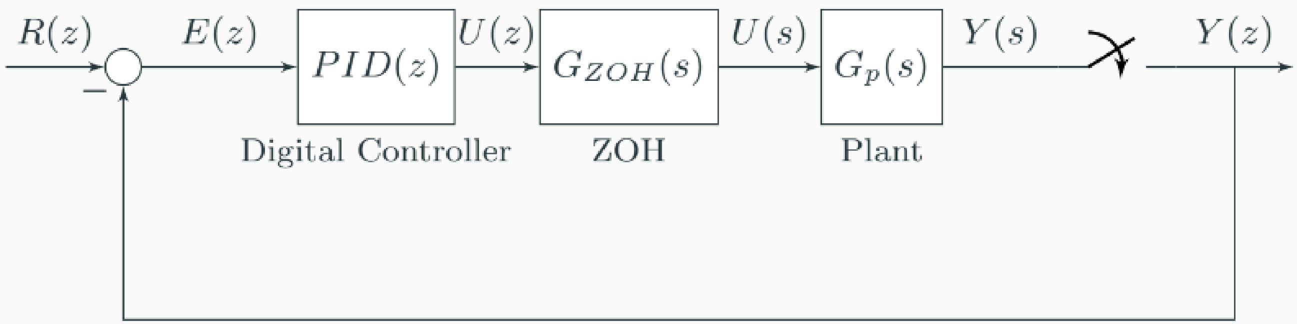


Figure 1: Diagram of a closed-loop digital controller. Here, the switch represents a digital encoder.

The steps to follow to achieve this are

1. Combine the plant and ZOH to get the discretized plant

Example



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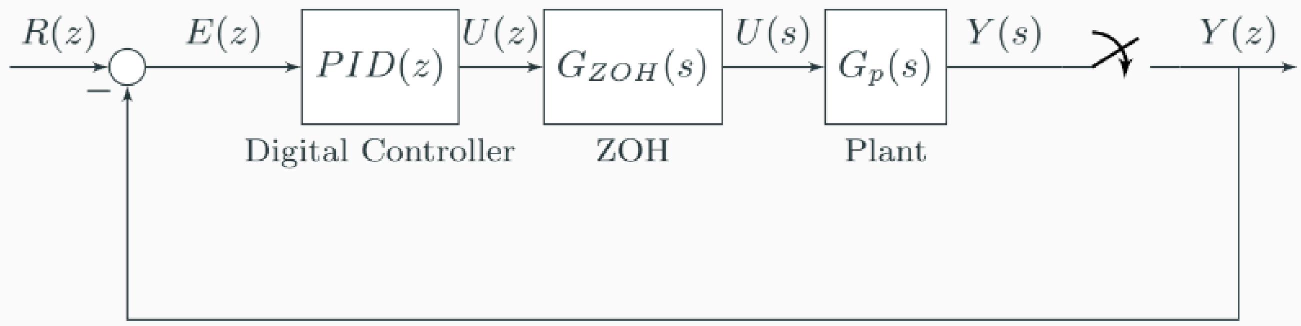
The steps to follow to achieve this are

1. Combine the plant and ZOH to get the discretized plant
2. Discretize the PI controller

Example



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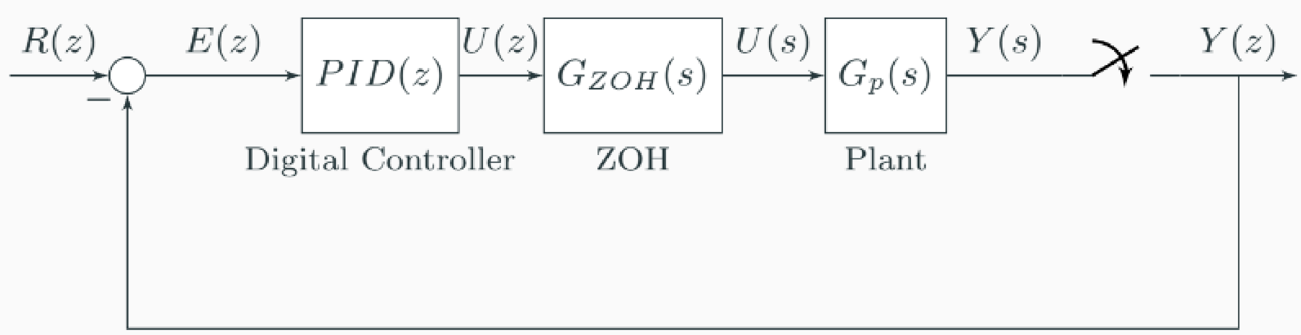
The steps to follow to achieve this are

1. Combine the plant and ZOH to get the discretized plant
2. Discretize the PI controller
3. Obtain closed-loop polynomial of the discretized system

Example



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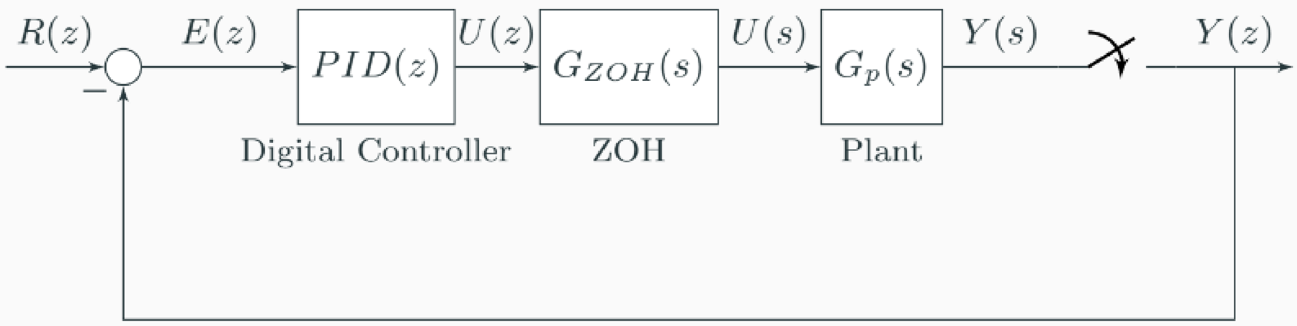
1. Combine the plant and ZOH to get the discretized plant
2. Discretize the PI controller
3. Obtain closed-loop polynomial of the discretized system
4. Obtain target polynomial from desired poles in the z-plane

Example



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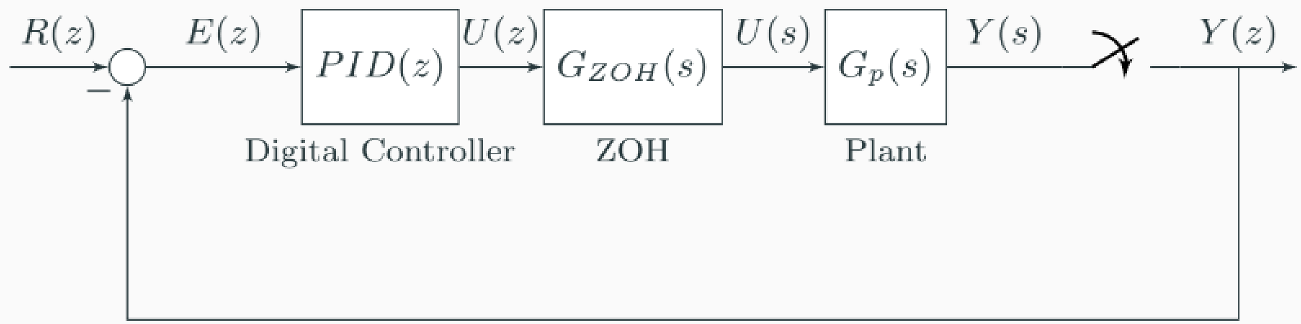

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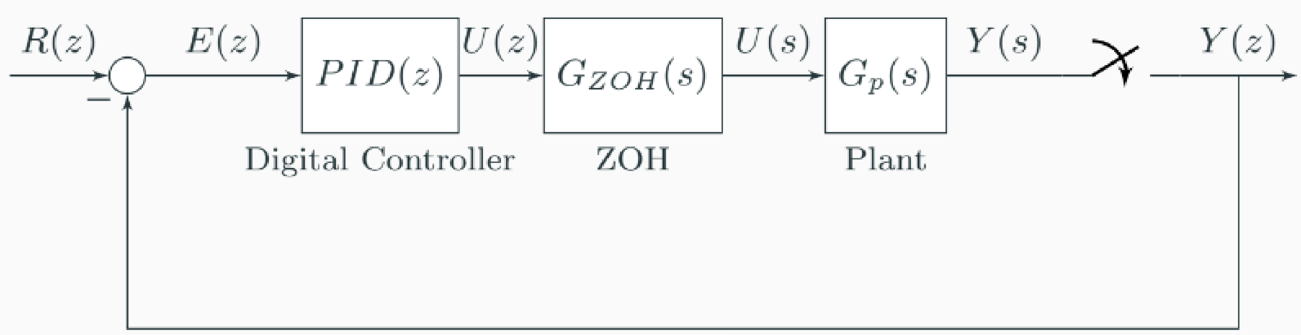
Step 1. Combine the plant and ZOH to get the discretized plant

- The combined discretized transfer function $H(z) = \frac{Y(z)}{U(z)}$ of the plant and zero-order hold is defined to be $H(z) = (1 - z^{-1})Z\left\{\frac{G_p(s)}{s}\right\}$

Example



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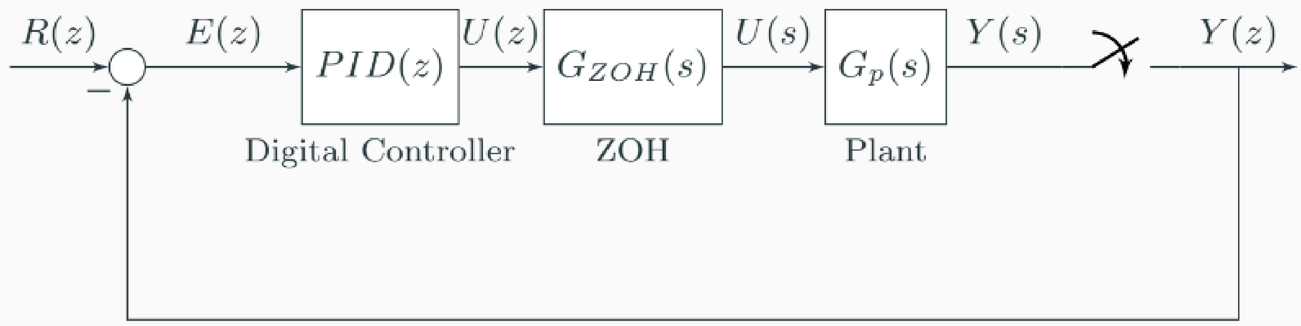
- To compute $Z\left\{\frac{G_p(s)}{s}\right\}$, we use backward Euler's approximation ($s \simeq \frac{z-1}{T_s z}$) with $T_s = 1\ sec$ on the continuous time plant $G_p(s) = \frac{1}{2s+1}$
- Then $\boxed{H(z) = \frac{z{T_s}^2}{z(2+T_s)-2}}$ with $T_s = 1\ sec$ is the discretized plant

Example



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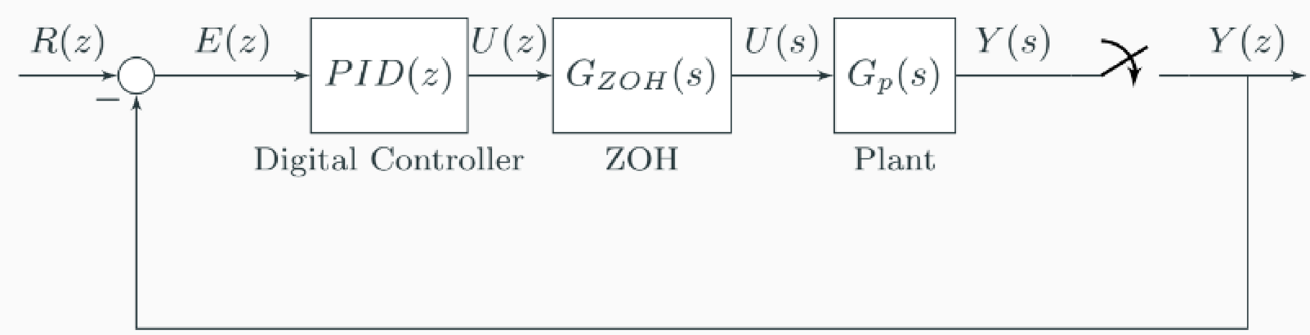
Step 2. Discretize the PI controller

- The transfer function in continuous time of a PI controller is $C(s) = \frac{K_p s + K_i}{s}$

Example



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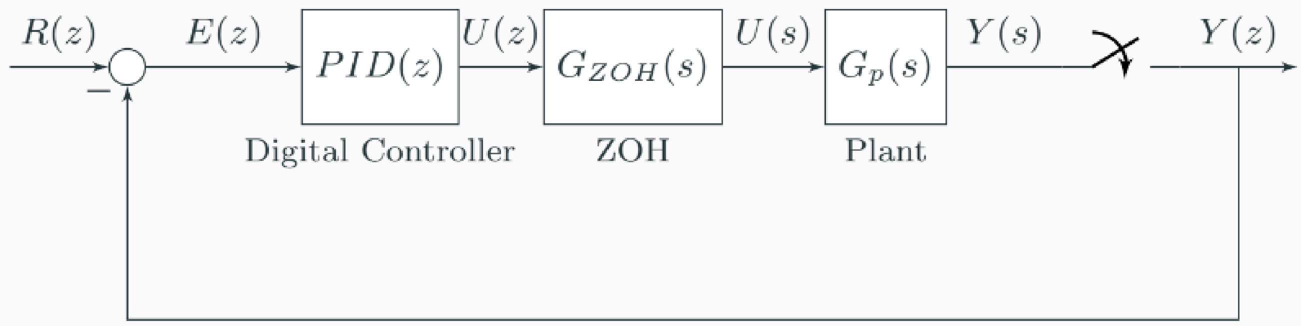
Step 2. Discretise the PI controller

- The transfer function in continuous time of a PI controller is $C(s) = \frac{K_p s + K_i}{s}$
- We compute $Z\left\{\frac{C(s)}{s}\right\}$ by using backward Euler's approximation ($s \simeq \frac{z-1}{T_s z}$) with $T_s = 1\ sec$ and obtain the discretized PI controller

$$\boxed{C(z) = \frac{z(K_p + K_i T_s) - K_p}{z-1}} \text{ with } T_s = 1\ sec$$

Example



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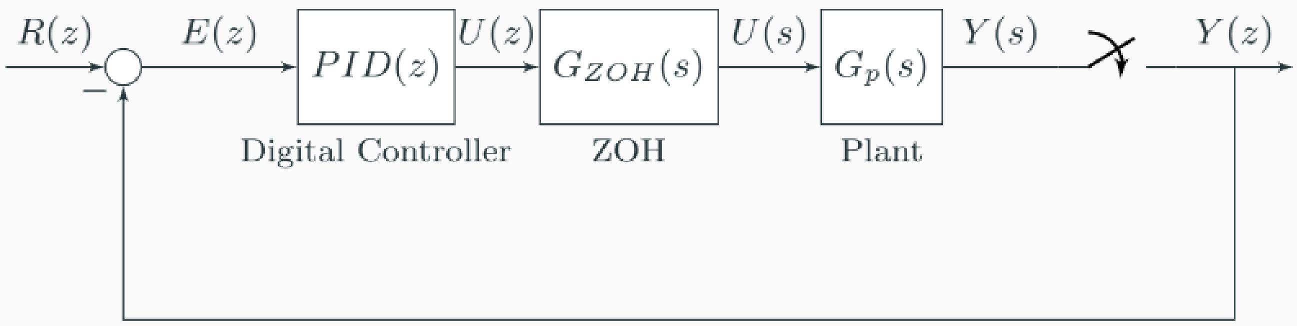
Step 3. Obtain closed-loop polynomial of the discretised system

- To obtain the closed loop polynomial, we can use the expression $1 + C(z)H(z) = 0$ with the discretized plant $C(z)$ and discrete PID controller $H(z)$:

Example



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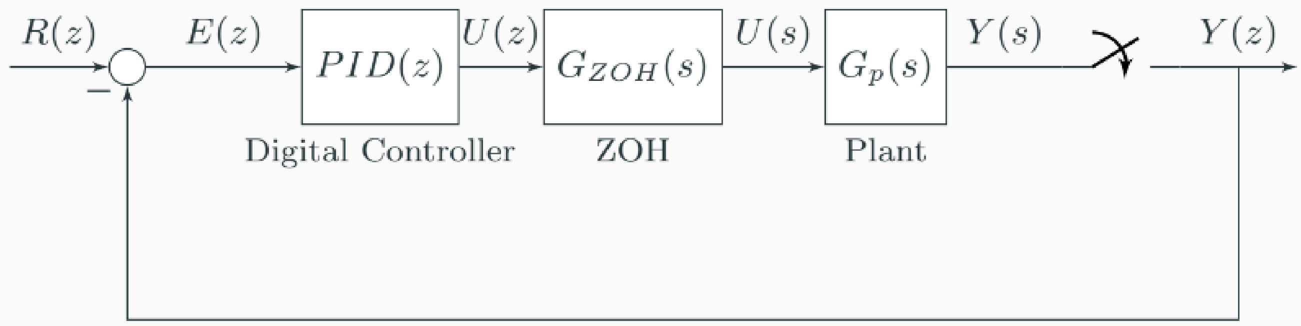
- After performing the calculations the closed-loop polynomial is obtained

$$\boxed{z^2 - \frac{z\left(4 + T_s + K_p T_s{}^2\right)}{2 + T_s + K_p T_s{}^2 + K_i T_s{}^3} + \frac{2}{2 + T_s + K_p T_s{}^2 + K_i T_s{}^3} = 0}$$

with $T_s = 1\ sec$

Example



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Step 4. Obtain target polynomial from desired poles in the z-plane

- From $z_{1,2} = 1 \pm 2i$, we can get the target polynomial computing $(z - 1 - 2i)(z - 1 + 2i)$

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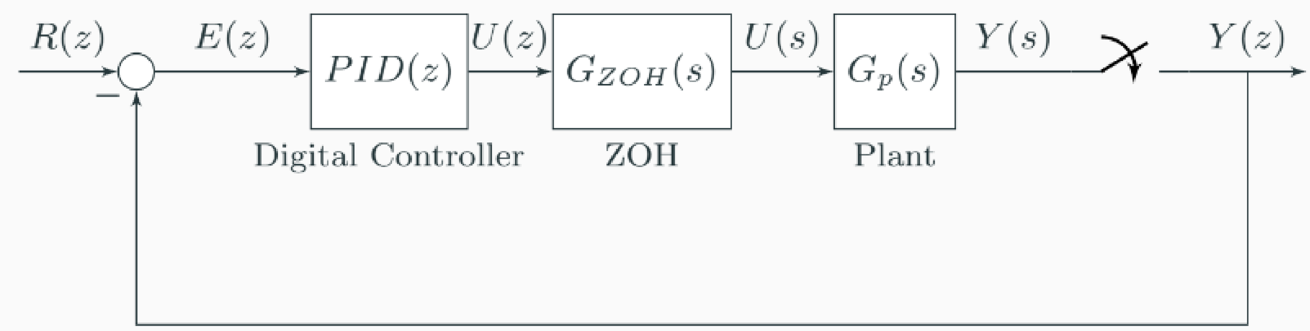

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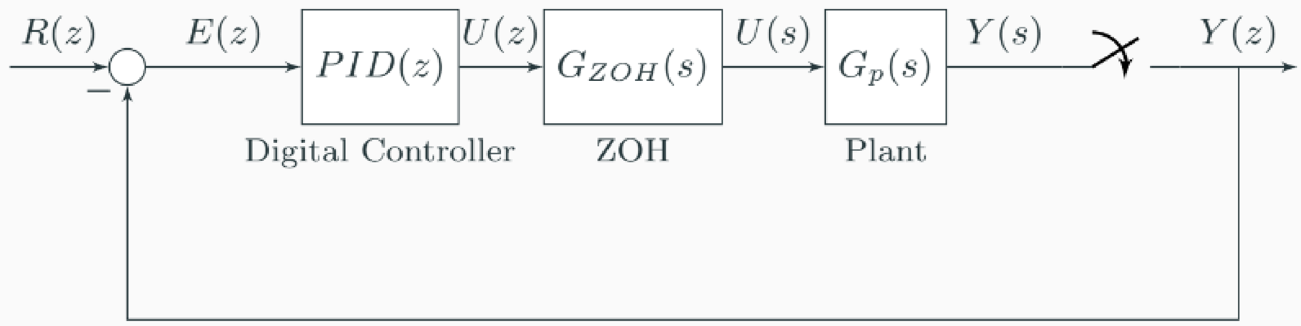
$$\boxed{z^2 - 2z + 5 = 0}$$

Example



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Step 5. Compute $\boldsymbol{K_p}$ and $\boldsymbol{K_i}$ to get to the target polynomial

- The closed-loop polynomial is $z^2 - \frac{z(4+T_s+K_pT_s{}^2)}{2+T_s+K_pT_s{}^2+K_iT_s{}^3} + \frac{2}{2+T_s+K_pT_s{}^2+K_iT_s{}^3} = 0$ with $T_s = 1$
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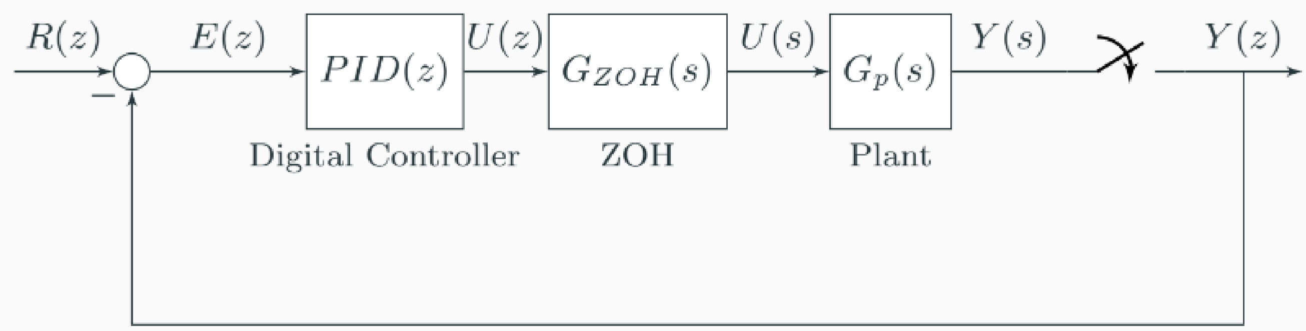

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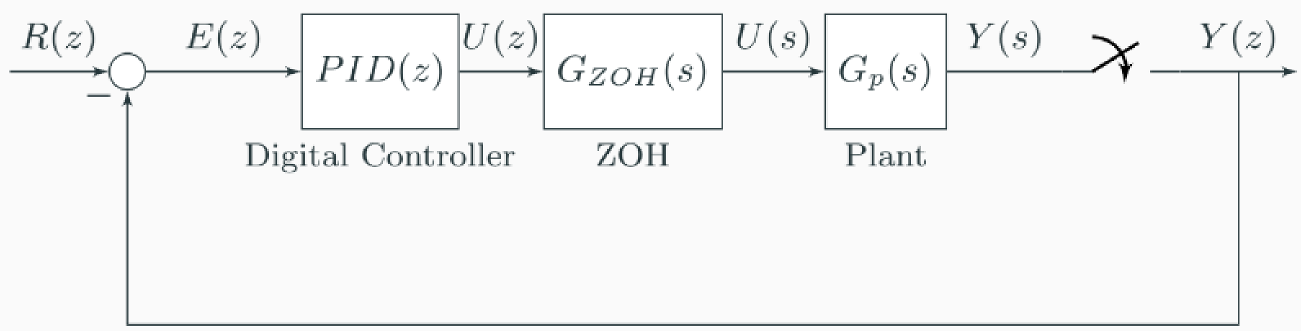
- The target polynomial from the desired poles is: $z^2 - 2z + 5 = 0$
- They need to be equal so that the poles of the system lie in the desired place in the z-plane, so we get the system of equations

$$\begin{cases} \dfrac{(4 + T_s + K_pT_s{}^2)}{2 + T_s + K_pT_s{}^2 + K_iT_s{}^3} = 2 \\ \dfrac{2}{2 + T_s + K_pT_s{}^2 + K_iT_s{}^3} = 5 \end{cases}$$

Example



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- Solving we get $\boxed{K_p \simeq -4.2 \text{ and } K_i \simeq 1.6}$

Summary

- A discrete-time system in state-space is stable if and only if the eigenvalues of A lie inside the unitary circle
- The transfer function of a discrete-time system is stable if and only if its poles lie inside the unitary circle
- Discretization can affect stability properties of a system
- We studied PID controllers in discrete-time

Next lecture:

Optimal control as information processing
in mechatronics Systems