Geometry 2024, homework set 2

- Below you can find your second homework assignment. Please upload it to BrightSpace by **Tuesday March 19**. The deadline is strict, so late homework will not be graded.
- The number of points per question is given in a box. 15 extra points are given for a clear writing of solutions. Note that high marks for the homeworks contribute to the final grade.

QUESTIONS

1. $\boxed{7+15+8 = 30 \text{ pts}}$ (**Riemannian metric on models of the hyperbolic plane**) Let

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\} \quad \text{and} \quad \mathbb{H} = \{z = (x + iy) \in \mathbb{C} \mid y > 0\}$$

be the open unit disk and the upper-half plane in the complex plane $\mathbb{C}$, respectively. Consider the so-called Cayley transform $f = \frac{z-i}{z+i} \colon \mathbb{C} \to \mathbb{C}$.

a) Show that f gives a complex-analytic diffeomorphism between $\mathbb{H}$ and $\mathbb{D}$, i.e. the restriction of f to $\mathbb{H}$ is a complex-analytic bijection between $\mathbb{H}$ and $\mathbb{D}$ with a complex analytic inverse.

b) Prove that the pull-back of the Riemannian metric

$$G(u, v) = \frac{4(\mathrm{d}u^2 + \mathrm{d}v^2)}{(1 - (u^2 + v^2))^2}, \quad z = u + iv,$$

on $\mathbb{D}$ under f has the form

$$(f^*G)(x, y) = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$

c) Conclude that $\mathbb{H}$ and $\mathbb{D}$ with these metrics are isometric.
(These are two famous models of planar hyperbolic geometry, called the *upper-half plane model* and the *Poincaré disk model.*)

2. $\boxed{10+10 = 20\text{pts}}$ (**Gaussian curvature of Plücker's conoid**) Consider the following surface in affine Euclidean 3-space

$$M^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z(x^2 + y^2) = xy\}.$$

(It is called Plücker's conoid; cf. Exercise VIII.3 of M. Audin).

a) Prove that M^2 is a ruled surface. Determine at which points is this surface regular. Is M^2 orientable?

b) Compute the Gaussian curvature of Plücker's conoid at its regular points.

3. $\boxed{10+10+8+7 = 35\text{pts}}$ (**Geodesics**)

Consider the two models of hyperbolic geometry given in exercise 1 (the upper-half plane $\mathbb{H}$ and the Poincaré disk $\mathbb{D}$ models).

a) Write down the geodesic equations for these models;

b) Verify that circular arcs in $\mathbb{H}$, respectively, $\mathbb{D}$, meeting the boundary of $\mathbb{H}$, resp., $\mathbb{D}$, orthogonally are geodesics (when parametrised by constant speed).

Consider now an ellipsoid in $\mathbb{R}^3$:

$$E^2 = \{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$$

c) Prove that the intersections of E^2 with coordinate xy, xz, yz 2-planes are geodesics on E^2, that is, $\gamma_x = E^2 \cap \{x = 0\}$, $\gamma_y = E^2 \cap \{y = 0\}$, and $\gamma_z = E^2 \cap \{z = 0\}$, are geodesics (when parametrised by constant speed).

d) Does it follow[1] from part c) that γ_x, γ_y, and γ_z are the only *closed* geodesics of E^2?

End of homework

[1] Though not hard, this question is much related to so-called integrable dynamical systems and also to topology (Lusternik-Schnirelmann category).