Control Engineering TBKRT05E

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Lecture 3
ver. 1.5.1.2

Last lectures

- Euler-Lagrange equations of motion
- Motion equations for simple mechanical systems (assigned reading of Chapter 2 of the reader for more examples, e.g. the 2 DOF manipulator)

Objectives last lecture

- To be able to model simple physical systems via classical methods.
- To work out simple yet meaningful examples of feedback control systems
- For more examples of feedback control systems, please refer to Chapter 2 and 3 of the textbook

Today

- **Linear versus nonlinear systems**
- Linearization

Linear versus nonlinear

Nonlinear (time-invariant) system

$$
\begin{aligned}
\dot{x} &= f(x,u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \\
y &= h(x,u) \quad y \in \mathbb{R}^p
\end{aligned}
$$

Example (1 DOF frictionless robot manipulator) $M\ddot{q} + mgl\sin q = u$, $M = ml^2$. Choosing the state variables $x_1 = q$, $x_2 = \dot{q}$, one arrives at

$$
\begin{aligned}
\frac{d}{dt}\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} &= \underbrace{\begin{pmatrix} x_2 \\ -\frac{mg\,l}{ml^2}\sin x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix} u}_{f(x,u)} = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin x_1 + \frac{1}{ml^2}u \end{pmatrix} =: \begin{pmatrix} f_1(x_1,x_2,u) \\ f_2(x_1,x_2,u) \end{pmatrix} \\
y &= \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{h(x)}
\end{aligned}
$$

Why nonlinear systems are important?

- Many physical systems exhibit nonlinear behaviour
- Linear models and their analysis is often only valid in neighborhood of the operating point, based on linearization
- For behaviour and/or excitation of system outside that neighborhood, more advanced tools are necessary

Linear versus nonlinear

Phenomena that only occur in the presence of nonlinearities

- Finite escape time
- Nonunique solutions
- Multiple isolated equilibria (slide 16)
- Limit cycles
- Bifurcations
- Chaos

Nonlinear phenomena

Finite escape time

Consider the system (with no input, $u = 0$)

$$\dot{x} = f(x) = x^2, \quad x(0) = 1$$

Its solution* (check by differentiation) is

$$x(t) = \frac{1}{1 - t}.$$

The system has a finite escape time, i.e. $x(t) \to +\infty$ as $t \to 1^-$.

*(integration by separation of variables)
$x^{-2}dx = dt \Leftrightarrow \int x^{-2}dx = \int dt \Leftrightarrow \frac{x^{-1}}{-1} = t + c \Leftrightarrow x(t) = \frac{-1}{t+c} \Leftrightarrow x(t) = \frac{-1}{t - \frac{1}{x(0)}}$

Nonlinear phenomena

Nonunique solution

Consider the system (with no input, $u = 0$)

$$\dot{x} = 2\sqrt{x}, \quad x(0) = 0.$$

Then both

$$x(t) = t^2 \quad \text{and} \quad x(t) = 0 \quad t \geq 0$$

are solutions to the Cauchy problem above.

$$x(0) = 0^2, \quad \dot{x}(t) = 2t, \quad 2\sqrt{x(t)} = 2\sqrt{t^2} = 2t \quad \Rightarrow \quad \dot{x}(t) = 2\sqrt{x(t)}, \quad t \geq 0$$

$$x(0) = 0, \quad \dot{x}(t) = 0, \quad 2\sqrt{x(t)} = 2\sqrt{0} = 0 \quad \Rightarrow \quad \dot{x}(t) = 2\sqrt{x(t)}, \quad t \geq 0$$

To prevent this to happen, one should ask the vector field $f(x)$ to be **locally** Lipschitz: for each $\bar{x} \in \mathbb{R}^n$ there exists a neighborhood $I_{\bar{x}}$ of x and a constant $L > 0$ such that for all $x, y \in I_{\bar{x}}$

$$||f(x) - f(y)|| < L||x - y||$$

where $|| \cdot ||$ denotes the Euclidean norm of a vector.

Qualitative analysis of nonlinear systems - Phase portrait

Vector field

For the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n,$$

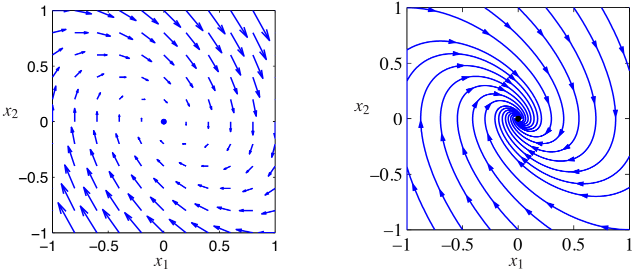
for each $x \in \mathbb{R}^n$, $f(x)$ is a vector representing the velocity of the system at that point.

Phase portrait

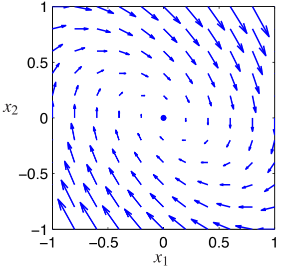
Graphical representation of the qualitative behavior of a *planar* dynamical system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^2$$

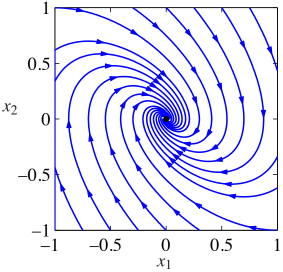
For each $x \in \mathbb{R}^2$, draw an arrow with tail in the point x, whose length is given by $||f(x)||$ and whose direction is given by $f(x)$ itself



Qualitative analysis of nonlinear systems - Phase portrait



(a) Vector field



(b) Phase portrait

You can draw phase portraits like the one above on-line

`https://www.wolframalpha.com`

For instance, to draw the phase portrait of

$$\dot{x} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix}$$

on the interval $[-2\pi, 2\pi] \times [-2\pi, 2\pi] \subset \mathbb{R}^2$, where $g = 9.8$ and $l = 1$, type in

```
StreamPlot[{y,-9.8sin(x)},{x,-6.28,6.28},{y,-6.28,6.28}]
```

And now?

Nonlinear vs. linear models

Nonlinear (control) systems are more difficult to analyze and deal with.
In this course, we focus on linear control systems

First: How to “reduce” systems that are nonlinear to linear ones?

Via **linearization**

Linearization I

Procedure:

- **Step 1:** Determine a solution (for instance, a point)
- **Step 2:** Approximate functions around this solution via truncated Taylor's expansion
- **Step 3:** Convert the system to local coordinates around this solution
- **Step 4:** Obtain an approximate linear model

Linearization I - Solutions of a system

Solution

Consider the system

$$\dot{x} = f(x, u)$$

Given a function of time $\overline{u}(t) : \mathbb{R} \to \mathbb{R}^m$ and the initial condition $x_0 \in \mathbb{R}^n$, a **solution** to the system from the initial condition x_0 is any function of time $\overline{x}(t)$ that solves the Cauchy problem, that is

$$\dot{\overline{x}}(t) = f(\overline{x}(t), \overline{u}(t)), \quad \overline{x}(0) = x_0$$

for all t where the solution $\overline{x}(t)$ is defined.

Example $\dot{x} = x^2 - 1 + u$, with $x_0 = 1$ and $\overline{u}(t) = 1$. Then a solution is given by

$$\overline{x}(t) = \frac{1}{1 - t}$$

which is defined on the interval $[0, 1)$.

Linearization I - Solutions of a system

Operating solution

In the context of linearization, we consider the state-input pair $(\bar{x}(t), \bar{u}(t))$, for all $t \geq 0$ where the pair is defined, such that $\bar{x}(t)$ is a **solution** of the system, i.e.

$$\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t))$$

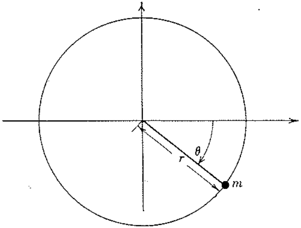
Interpretation

If the system $\dot{x} = f(x, u)$ is initially at the point $\bar{x}(0)$ ($x(0) = \bar{x}(0)$) and an input $\bar{u}(t)$ is applied, then the state of the system $x(t)$ will evolve as $\bar{x}(t)$ for all time t ($x(t) = \bar{x}(t)$ for all $t \geq 0$ where $\bar{x}(t)$ is defined)

From now on, when it is clear that the solution $(\bar{x}(t), \bar{u}(t))$ depends on time t, we might simply drop the explicit dependence on t, i.e. we simply write $(\bar{x}, \bar{u})$.

Linearization I - Solutions of a system

Example (Satellite problem – Brockett 1970) Point mass in an inverse square law force field



$$
\begin{aligned}
\ddot{r} = & \quad r\dot{\theta}^2 - \frac{k}{r^2} + u_1 \\
\ddot{\theta} = & \quad -\frac{2\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2
\end{aligned}
$$

u_1 radial thrust, u_2 tangential thrust

Solution $\overline{u}_1(t) = 0$, $\overline{u}_2(t) = 0$, $\overline{r}(t) = \overline{r} = const$, $\overline{\theta}(t) = \overline{\theta}t$

In fact: $\dot{\overline{r}}(t) = 0$, $\ddot{\overline{r}}(t) = 0$, $\dot{\overline{\theta}}(t) = \overline{\theta}$, $\ddot{\overline{\theta}}(t) = 0$. Hence

$$
\begin{aligned}
0 = & \quad \overline{r}\overline{\theta}^2 - \frac{k}{\overline{r}^2} \\
0 = & \quad -\frac{2\overline{\theta}\,0}{\overline{r}} + \frac{1}{\overline{r}}0
\end{aligned}
$$

The solution is an equilibrium solution if $\overline{r}^3\overline{\theta}^2 = k$.

Linearization I - Equilibrium of a system

Operating point

A special case is the **constant** state-input pair $(\bar{x}, \bar{u})$ which is a **solution** of the system (algebraic equations)

$$0 = f(\bar{x}, \bar{u})$$

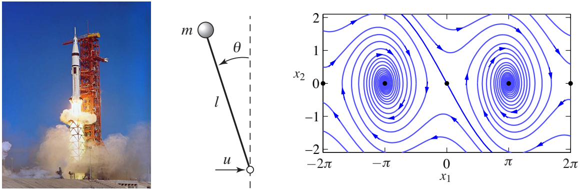
Interpretation

if the system $\dot{x} = f(x, u)$ is initially at the point $\bar{x}$ ($x(0) = \bar{x}$) and an input $\bar{u}$ is applied, then the state of the system will remain in $\bar{x}$ for all time t ($x(t) = \bar{x}$ for all $t \geq 0$, where $x(t)$ is a solution to the system)

Linearization I - Equilibrium of a system

Example Consider the inverted pendulum

$$\dot{x} = f(x,u) = \begin{pmatrix} x_2 \\ \sin x_1 - c x_2 + u \cos x_1 \end{pmatrix}$$



(a) (b) (c)

Equilibrium points for $\bar{u} = 0$ are

$$\bar{x} = \begin{pmatrix} \pm k\pi \\ 0 \end{pmatrix}, \quad k \in \mathbb{N}$$

given by the solution of the system of nonlinear algebraic equations

$$\begin{cases} 0 = & \overline{x}_2 \\ 0 = & \sin \overline{x}_1 - c\overline{x}_2 + \overline{u} \cos \overline{x}_1 \end{cases}$$

Linearization II

Determine an **Operating point** $\overline{x}, \overline{u}$ for the nonlinear system

$$\dot{x} = f(x,u), \quad y = h(x,u), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m\ y \in \mathbb{R}^p$$

and consider the Taylor series expansion around $\overline{x}, \overline{u}$:

$$f(x,u) = f(\overline{x}, \overline{u}) + \left.\frac{\partial f}{\partial x}\right|_{\overline{x},\overline{u}} (x - \bar{x}) + \left.\frac{\partial f}{\partial u}\right|_{\overline{x},\overline{u}} (u - \bar{u}) + \text{h.o.t.}(x - \bar{x}, u - \bar{u})$$

where

$$f(x,u) = \begin{bmatrix} f_1(x,u) \\ \vdots \\ f_n(x,u) \end{bmatrix} \quad \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x} \\ \vdots \\ \frac{\partial f_n}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad \frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}$$

and h.o.t. denotes higher order terms, e.g., for $n = m = 1$ (Lagrange remainder)

$$\begin{aligned} \text{h.o.t.}(x - \bar{x}, u - \bar{u}) = \quad & \left.\tfrac{\partial f^2}{\partial x^2}\right|_{\hat{x},\hat{u}} (x - \bar{x})^2 + \left.\tfrac{\partial f^2}{\partial u \partial x}\right|_{\hat{x},\hat{u}} (x - \bar{x})(u - \bar{u}) \\ & + \left.\tfrac{\partial f^2}{\partial x \partial u}\right|_{\hat{x},\hat{u}} (u - \bar{u})(x - \bar{x}) + \left.\tfrac{\partial f^2}{\partial u^2}\right|_{\hat{x},\hat{u}} (u - \bar{u})^2 \end{aligned}$$

with $\hat{x}, \hat{u}$ a point on the segment connecting (x,u) and $(\overline{x}, \overline{u})$.

Linearization III

Define the new variable (small perturbation around the equilibrium)

$$\Delta x = x - \overline{x}, \quad \Delta u = u - \overline{u}.$$

Since $\dot{x} = \dot{\overline{x}} + \Delta\dot{x}$, we have that

$$\dot{\overline{x}} + \Delta\dot{x} = f(\overline{x}, \overline{u}) + \frac{\partial f}{\partial x}\Big|_{\overline{x},\overline{u}} \Delta x + \frac{\partial f}{\partial u}\Big|_{\overline{x},\overline{u}} \Delta u + \text{h.o.t.}(\Delta x, \Delta u)$$

Same procedure for $h(x, u)$.

Since $\dot{\overline{x}} = f(\overline{x}, \overline{u})$, then

$$\Delta\dot{x} = \frac{\partial f}{\partial x}\Big|_{\overline{x},\overline{u}} \Delta x + \frac{\partial f}{\partial u}\Big|_{\overline{x},\overline{u}} \Delta u + \text{h.o.t.}(\Delta x, \Delta u)$$

If $\Delta x, \Delta u \approx 0$ (very small), then

$$\Delta\dot{x} \approx \frac{\partial f}{\partial x}\Big|_{\overline{x},\overline{u}} \Delta x + \frac{\partial f}{\partial u}\Big|_{\overline{x},\overline{u}} \Delta u$$

Linearization IV

We can consider the approximation

$$\begin{aligned}\Delta\dot{x} &= A\Delta x + B\Delta u, \quad \Delta x(t_0) = \Delta x_0 \\ \Delta y &= C\Delta x + D\Delta u,\end{aligned}$$

where

$$A = \frac{\partial f}{\partial x}\Big|_{\overline{x},\overline{u}},\ B = \frac{\partial f}{\partial u}\Big|_{\overline{x},\overline{u}},\ C = \frac{\partial h}{\partial x}\Big|_{\overline{x},\overline{u}},\ D = \frac{\partial h}{\partial u}\Big|_{\overline{x},\overline{u}}.$$

The system is the linearized system associated with the original system $\dot{x} = f(x,u)$, $y = h(x,u)$.

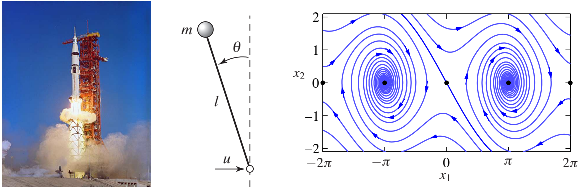
Interpretation

As far as Δx, Δu are small, the function $\bar{x} + \Delta x$ approximates the solution x to the original nonlinear system when the input $u = \bar{u} + \Delta u$ is applied.

Linearization – Example

Consider the inverted pendulum

$$\dot{x} = f(x,u) = \begin{pmatrix} x_2 \\ \sin x_1 - cx_2 + u\cos x_1 \end{pmatrix}$$



(a) (b) (c)

An equilibrium point for $\bar{u} = 0$ is $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Then

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}_{(\bar{x},\bar{u})} = \begin{pmatrix} 0 & 1 \\ \cos x_1 - u\sin x_1 & -c \end{pmatrix}_{(\bar{x},\bar{u})} = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix}_{(\bar{x},\bar{u})} = \begin{pmatrix} 0 \\ \cos x_1 \end{pmatrix}_{(\bar{x},\bar{u})} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Linearization – Example

For this example, $\frac{\partial^2 f_1}{\partial(x,u)^2} = 0$ and

$$\frac{\partial^2 f_2}{\partial(x,u)^2} = \begin{bmatrix} -\sin x_1 - u\cos x_1 & 0 & -\sin x_1 \\ 0 & 0 & 0 \\ -\sin x_1 & 0 & 0 \end{bmatrix}$$

Hence, the higher order term (the remainder) is given by

$$h.o.t.(x,u) = \frac{1}{2}\begin{bmatrix} 0 \\ -x_1^2(\sin\hat{x}_1 + \hat{u}\cos\hat{x}_1) - 2x_1 u \sin\hat{x}_1 \end{bmatrix}$$

where $(\hat{x}_1, \hat{u})$ is any point on the segment connecting the origin with the point (x_1, u).

Exercise Compute the linearization of the point mass satellite around the solution $\overline{u}_1(t) = 0$, $\overline{u}_2(t) = 0$, $\overline{r}(t) = \overline{r} = const$, $\overline{\theta}(t) = \overline{\theta}t$.

Linearization – Example

Consider the nonlinear system

$$\begin{array}{rcll} \dot{x}_1 & = & x_2 & =: f_1(x,u) \\ \dot{x}_2 & = & -x_1x_2 - x_2 + u & =: f_2(x,u) \\ y & = & x_1 & =: h(x,u) \end{array}$$

An equilibrium point is (check this!)

$$\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \bar{u} = 0.$$

Then

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}_{(\bar{x},\bar{u})} = \begin{pmatrix} 0 & 1 \\ -x_2 & -x_1 - 1 \end{pmatrix}_{(\bar{x},\bar{u})} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix}_{(\bar{x},\bar{u})} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{(\bar{x},\bar{u})} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \end{pmatrix}_{(\bar{x},\bar{u})} = \begin{pmatrix} 1 & 0 \end{pmatrix}_{(\bar{x},\bar{u})} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = \frac{\partial h}{\partial u}\Big|_{(\bar{x},\bar{u})} = 0$$

Today

- Linear versus nonlinear systems
- Linearization
- Reading assignment: Textbook Sections 4.1-4.2 and pp. 107-109, Reader Section 2.1

Next lecture

- Stability
- LTI system block diagram
- System response