Control Engineering TBKRT05E

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Lecture 2
ver. 1.5.2

Overview today

- Basics of modeling based on EL equations – Reader, Section 1.2
- Equations of motion for simple mechanical (and electrical) systems
- State space representations

The Euler-Lagrange equation 0

Preliminary facts

- A particle is a body whose dimensions may be neglected in describing its motion
- The position of a particle in the space is defined by a vector r in $\mathbb{R}^3$ given by its Cartesian coordinates

$$r = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

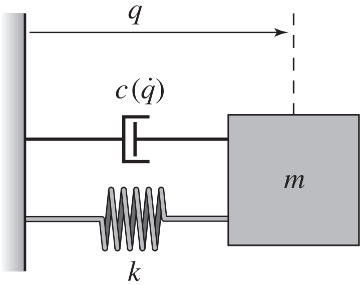
  Hence to define the position of a system of N particles, we need N vectors $r_1, r_2, \ldots, r_N$
- The number n of independent quantities $q_1, q_2, \ldots, q_n$ which must be specified in order to define uniquely the position of any particle is called the number of degrees of freedom and the quantities $q_1, q_2, \ldots, q_n$ generalized coordinates.
- The generalized coordinates need not be the Cartesian coordinates of the particles and n need not be equal to N because coordinates might be constrained by relations of the form

$$f_j(r_1, \ldots, r_N) = 0, \quad j = 1, 2, \ldots, m \qquad (n = 3N - m)$$

Landau, Lifshitz (1976). *Mechanics*. Butterworth-Heinemann.

Example mass-spring-damper system

Mass-spring-damper system (no gravity) (Lecture 1)



$m\ddot{q} + c(\dot{q}) + kq = F$
m mass
k spring coefficient
q cart displacement from rest pos.
$c(\dot{q}) = c\dot{q}$ linear damper
F external force or control acting on the mass along the direction of motion

Particle	=	the mass m
Space	=	$\mathbb{R}$
Cartesian reference	=	horizontal line pointing to the right with origin at the rest position of the spring
Position in space	=	$r = q \in \mathbb{R}$
Generalized coordinate	=	q (in this case, same as Cartesian coordinate) $n = 1$ no constraints

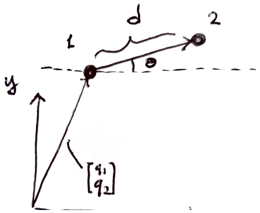
Example 2-particle system at fixed distance

- Two particles ($N = 2$) in $\mathbb{R}^2$ whose positions in the Cartesian coordinates are given by
$$r_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad r_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$
Hence a total of 4 variables define the positions of the two particles.
- The two particles are constrained to be at a constant distance $d > 0$
$$\|r_1 - r_2\| = d$$
In this case $m = 1$ (number of constraints) and
$$f_1(r_1, r_2) = \|r_1 - r_2\| - d$$
The two particles are constrained to evolve on the surface
$$M = \{(r_1, r_2) \in \mathbb{R}^4 : \|r_1 - r_2\| = d\}$$
- Each particle obeys Newton's equation of motion
$$m_i \ddot{r}_i = F_i + F_i^{(c)} \quad i = 1, 2$$
where $F_i^{(c)}$, $i = 1, 2$, are forces that constrain the two particles to be at the constant distance d: they are unknown but such that the vector $\begin{bmatrix} F_1^{(c)\top} & F_2^{(c)\top} \end{bmatrix}^\top$ is orthogonal to the tangent space to M at each point.

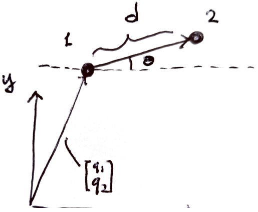
Example 2-particle system at fixed distance

- Because of the constraint, the positions of the two particles are determined by 3 independent variables: 2 are the position variables of particle 1, and 1 is the angle θ that the vector of length d pointing from particle 1 to particle 2 forms with the horizontal axis. Hence, $n = 3$ and

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} := \begin{bmatrix} x_1 \\ y_1 \\ \theta \end{bmatrix}$$



Example 2-particle system at fixed distance



- The position variables r_1, r_2 and the generalized coordinates q_1, q_2, q_3 are related by a so-called immersion formula:

$$r := \begin{bmatrix} r_1 \\ \hline r_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \hline \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + d \begin{bmatrix} \cos q_3 \\ \sin q_3 \end{bmatrix} \end{bmatrix} =: X(q)$$

For each $q \in \mathbb{R}^3$ the resulting vector $r = X(q)$ belongs to the surface M

$$\|r_1 - r_2\| = \|d \begin{bmatrix} \cos q_3 \\ \sin q_3 \end{bmatrix} \| = d((\cos q_3)^2 + (\sin q_3)^2)^{1/2} = d$$

Example 2-particle system at fixed distance

- Given the point $r \in M$ with corresponding coordinates q related by $r = X(q)$, the columns of the matrix

$$\frac{\partial X}{\partial q} = \begin{bmatrix} \frac{\partial X}{\partial q_1} & \frac{\partial X}{\partial q_2} & \frac{\partial X}{\partial q_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -d \sin q_3 \\ 0 & 1 & d \cos q_3 \end{bmatrix}$$

  span the tangent space to the surface M at the point r.

- By the orthogonality of $\begin{bmatrix} F_1^{(c)\top} & F_2^{(c)\top} \end{bmatrix}^\top$ to the tangent space to M at each point, we have

$$\begin{bmatrix} F_1^{(c)\top} & F_2^{(c)\top} \end{bmatrix} \begin{bmatrix} \frac{\partial X}{\partial q_1} & \frac{\partial X}{\partial q_2} & \frac{\partial X}{\partial q_3} \end{bmatrix} = \begin{bmatrix} F_1^{(c)\top} & F_2^{(c)\top} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -d \sin q_3 \\ 0 & 1 & d \cos q_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

  which returns

$$F_1^{(c)} = -F_2^{(c)} \qquad F_2^{(c)} \perp \begin{bmatrix} -\sin q_3 \\ \cos q_3 \end{bmatrix}$$

The Euler-Lagrange equation I

- Consider Newton's equations of motion for the two particles

$$\begin{bmatrix} m_1 \ddot{r}_1 \\ m_2 \ddot{r}_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \begin{bmatrix} F_1^{(c)} \\ F_2^{(c)} \end{bmatrix}$$

and multiply both sides by $\frac{\partial X}{\partial q_i}^\top$, $i = 1, 2, 3$, to obtain

$$\frac{\partial X}{\partial q_i}^\top \begin{bmatrix} m_1 \ddot{r}_1 \\ m_2 \ddot{r}_2 \end{bmatrix} = \frac{\partial X}{\partial q_i}^\top \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \qquad \text{where} \qquad \frac{\partial X}{\partial q_i}^\top \begin{bmatrix} F_1^{(c)} \\ F_2^{(c)} \end{bmatrix} = 0$$

- It can be shown that

$$\frac{\partial X}{\partial q_i}^\top \begin{bmatrix} m_1 \ddot{r}_1 \\ m_2 \ddot{r}_2 \end{bmatrix} = \frac{d}{dt} \frac{\partial T^*}{\partial \dot{q}_i} - \frac{\partial T^*}{\partial q_i} \quad \text{where} \quad T^*(q, \dot{q}) = \frac{1}{2} (m_1 \dot{r}_1^\top \dot{r}_1 + m_2 \dot{r}_2^\top \dot{r}_2)_{\dot{r} = \frac{\partial X}{\partial q} \dot{q}}$$

- Assume that the forces $F_i(r)$ are given by the gradient of a potential function $\hat{V}(r)$

$$F_i(r) = -\frac{\partial \hat{V}(r)}{\partial r_i} \quad i = 1, 2$$

Then $\hat{V}(r)$ in the q-coordinates $V(q) = \hat{V}(r)|_{r=X(q)}$ satisfies (chain rule)

$$\frac{\partial V}{\partial q_i} = \left. \frac{\partial \hat{V}}{\partial r} \right|_{r=X(q)}^\top \frac{\partial X}{\partial q_i} = -\begin{bmatrix} F_1^\top & F_2^\top \end{bmatrix} \frac{\partial X}{\partial q_i}$$

The Euler-Lagrange equation I

- From Newton's equations of motion, we obtained

$$\frac{\partial X}{\partial q_i}^{\top} \begin{bmatrix} m_1 \ddot{r}_1 \\ m_2 \ddot{r}_2 \end{bmatrix} = \frac{\partial X}{\partial q_i}^{\top} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \qquad i = 1, 2, 3$$

  where

$$\frac{\partial X}{\partial q_i}^{\top} \begin{bmatrix} m_1 \ddot{r}_1 \\ m_2 \ddot{r}_2 \end{bmatrix} = \frac{d}{dt} \frac{\partial T^*}{\partial \dot{q}_i} - \frac{\partial T^*}{\partial q_i}$$

  and

$$\frac{\partial V}{\partial q_i} = \left. \frac{\partial \hat{V}}{\partial r} \right|_{r=X(q)}^{\top} \frac{\partial X}{\partial q_i} = - \begin{bmatrix} F_1{}^{\top} & F_2{}^{\top} \end{bmatrix} \frac{\partial X}{\partial q_i}$$

- Hence

$$\frac{d}{dt} \frac{\partial T^*}{\partial \dot{q}_i} - \frac{\partial T^*}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \qquad i = 1, 2, 3$$

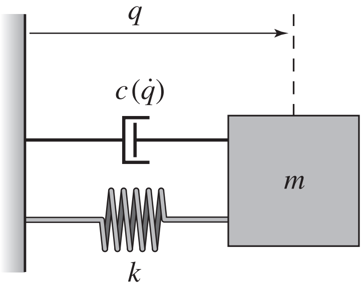
- Define the Lagrangian function $L(q, \dot{q}) = T^*(q, \dot{q}) - V(q)$. Then

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, 2, 3$$

  which are the so-called Euler-Lagrangian function.
- You can use these equations to derive the equations of motions of the 2 particles constrained to evolve at a constant distance and subject to conservative forces.

Example unforced mass-spring system

Mass-spring-damper system (no gravity)



$m\ddot{q} + c(\dot{q}) + kq = F$
m mass
k spring coefficient
q cart displacement from rest pos.
$c(\dot{q}) = 0$ (no damper)
$F = 0$ no external force or control acting on the mass along the direction of motion

Lagrangian function $\quad L(q, \dot{q}) = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} kq^2 \;\; (q \in \mathbb{R},\ n = 1)$

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} \;\Rightarrow\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = m\ddot{q}$$

$$\frac{\partial L}{\partial q} = -kq$$

The force exerted by the spring on the mass is $-kq$, which is a conservative force as it can be expressed as the derivative (gradient) of a potential function $V(q)$

$$-kq = -\frac{dV}{dq} \quad \Rightarrow \quad V(q) = k\frac{q^2}{2} \quad (V(0) = 0)$$

The EL equations in the presence of external forces

External forces

- Controls u and external disturbances d
- Dissipation

To take into account the action of these non-conservative forces, the Lagrangian is modified as

$$L_{NC}(q, \dot{q}) = L(q, \dot{q}) + (Bu + d)^\top q + \int_0^t \mathcal{D}(\dot{q}(s))ds$$

where

- $Bu + d$ is an n-dimensional vector (with $B \in \mathbb{R}^{n \times m}$ a full-column rank matrix) obtained by projecting the non-conservative forces $F_{NC} \in \mathbb{R}^N$ in the Cartesian coordinates onto the tangent space to M, that is

$$(Bu + d)_i := F_{NC}^\top \frac{\partial X}{\partial q_i} \quad i = 1, 2, \ldots, n$$

- $\mathcal{D}(\dot{q})$ is the so-called Rayleigh dissipation function, which by definition satisfies

$$\dot{q}^\top \frac{\partial \mathcal{D}}{\partial \dot{q}} = \sum_{i=1}^{n} \dot{q}_i \frac{\partial \mathcal{D}}{\partial \dot{q}_i} \geq 0, \quad \text{for all } \dot{q} \in \mathbb{R}^n$$

  In most of the practical cases (linear friction or constant resistances)

$$\mathcal{D}(\dot{q}) = \frac{1}{2} \dot{q}^\top R \dot{q} = \sum_{i=1}^{n} \frac{1}{2} R_i \dot{q}_i^2, \quad R \geq 0 \text{ and diagonal}$$

Ortega, R *et al* (1998). *Passivity-based control of Euler-Lagrange systems*, Springer-Verlag.

The Euler-Lagrange equations in the presence of external forces

Modified Lagrangian

$$L_{NC}(q,\dot q) = L(q,\dot q) + (Bu+d)^\top q + \int_0^t \mathcal{D}(\dot q(s))ds$$

As a result, the EL equations become

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L_{NC}}{\partial \dot q_i}\right) - \frac{\partial L_{NC}}{\partial q_i} \overset{(*)}{=} \quad & \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i} + \int_0^t \frac{\partial \mathcal{D}(\dot q)}{\partial \dot q_i} ds\right) - \left(\frac{\partial L}{\partial q_i} + (Bu+d)_i\right) \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) + \frac{\partial \mathcal{D}(\dot q)}{\partial \dot q_i} - \left(\frac{\partial L}{\partial q_i} + (Bu+d)_i\right) \\
&= 0, \quad i = 1,\ldots,n
\end{aligned}$$

that is

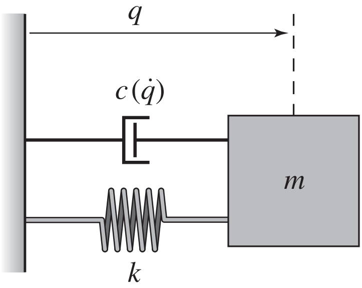
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial \mathcal{D}(\dot q)}{\partial \dot q_i} = (Bu)_i + d_i, \quad i = 1,\ldots,n$$

$(*)$ Let $c = Bu + d$; then $(Bu+d)^\top q = c^\top q = \sum_{j=1}^n c_j q_j$. Hence

$$\frac{\partial (Bu+d)^\top q}{\partial q_i} = \frac{\partial \sum_{j=1}^n c_j q_j}{\partial q_i} = c_i = (Bu+d)_i$$

Example mass-spring-damper system

Mass-spring-damper system (no gravity)



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F external force or control acting on the mass along the direction of motion

Lagrangian function $\quad L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \ (q \in \mathbb{R},\ n = 1)$

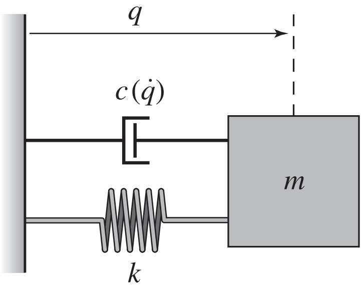
Modified Lagrangian $\quad L_{NC}(q, \dot{q}) = L(q, \dot{q}) + (Bu + d)^\top q + \int_0^t \mathcal{D}(\dot{q}(s))ds$

$(Bu + d)^\top q = (1 \cdot F + 0)q = Fq$
(work done by F to cause displ. q)

Rayleigh function $\quad \mathcal{D}(\dot{q}) = \frac{1}{2}c\dot{q}^2$

Example mass-spring-damper system

Mass-spring-damper system (no gravity)



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q cart displacement from rest pos.
$c(\dot{q}) = c\dot{q}$ linear damper
F external force or control acting on the mass along the direction of motion

$$L_{NC}(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 + Fq + \int_0^t \frac{1}{2}c\dot{q}^2(s)ds$$
$$\Downarrow$$
$$\frac{d}{dt}\left(\frac{\partial L_{NC}(q,\dot{q})}{\partial \dot{q}}\right) = \frac{d}{dt}\left(m\dot{q} + \int_0^t c\dot{q}(s)ds\right) = m\ddot{q} + c\dot{q}$$
$$\frac{\partial L_{NC}(q,\dot{q})}{\partial q} = -kq + F$$

Hence $0 = m\ddot{q} + c\dot{q} - (-kq + F)$, which is $m\ddot{q} + c\dot{q} + kq = F$

The Euler-Lagrange equation II

To recap, the approach to modeling adopted here rests on the use of the **Euler-Lagrange equations** of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = (Bu)_i + d_i, \qquad i = 1, \ldots, n.$$

where

- n is the number of degrees of freedom
- $q \in \mathbb{R}^n$, $q = (q_1 \ldots q_n)^T$ is the **generalized coordinate**
- L is the **Lagrangian** function

$$L(q, \dot{q}) = T^*(q, \dot{q}) - V(q)$$

- $T^*(q, \dot{q})$ is the total **kinetic (co-)energy**
- $V(q)$ is the total **potential energy**
- $(Bu)_i + d_i$ are the non-conservative forces (control + disturbances) acting on the system (the force F_i may also be 0).

Euler-Lagrange equations III

The Euler-Lagrange equations can be derived following these steps:

1. Identify a generalized displacement vector $q \in \mathbb{R}^n$ (basic independent variables of the physical components)
2. Determine the kinetic co-energy $T^*(q, \dot{q})$ and the potential energy $V(q)$ associated with elements respectively
3. Determine the Lagrangian function $L(q, \dot{q}) = T^*(q, \dot{q}) - V(q)$
4. Differentiate $L(q, \dot{q})$ with respect to q_i and $\dot{q}_i$
5. Write the EL equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i, \qquad i = 1, \ldots, n.$$

Euler-Lagrange equations IV

Euler-Lagrange equations can be written in compact (matrix) form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial \mathcal{D}}{\partial \dot{q}} = F,$$

where

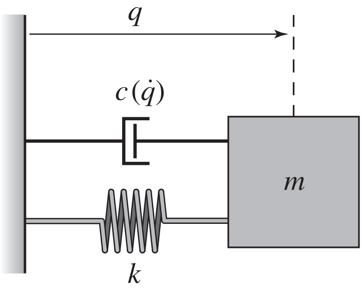
$$\frac{\partial L}{\partial \dot{q}} = \begin{pmatrix} \frac{\partial L}{\partial \dot{q}_1} \\ \vdots \\ \frac{\partial L}{\partial \dot{q}_n} \end{pmatrix}, \quad \frac{\partial L}{\partial q} = \begin{pmatrix} \frac{\partial L}{\partial q_1} \\ \vdots \\ \frac{\partial L}{\partial q_n} \end{pmatrix}, \quad \frac{\partial \mathcal{D}}{\partial \dot{q}} = \begin{pmatrix} \frac{\partial \mathcal{D}}{\partial \dot{q}_1} \\ \vdots \\ \frac{\partial \mathcal{D}}{\partial \dot{q}_n} \end{pmatrix} \quad F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$$

and the derivation operator $\frac{d}{dt}$ is applied component-wise

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \begin{pmatrix} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) \\ \vdots \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) \end{pmatrix}$$

Example mass-spring-damper system

Autonomous or unforced system (no external force on the system) (Lecture 1)



$m\ddot{q} + c(\dot{q}) + kq = F$
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F external force or control acting on the mass along the direction of motion

Equations obtainable via

$$\text{EL equations} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial \mathcal{D}}{\partial \dot{q}} = F$$

where

Kinetic (co-)energy (mass)	$T^*(\dot{q}) = \frac{1}{2} m\dot{q}^2$
Potential energy (spring)	$E(q) = \frac{1}{2} kq^2$
Rayleigh dissipation function	$\mathcal{D}(\dot{q}) = \frac{1}{2} c\dot{q}^2$
Lagrangian function	$L(q, \dot{q}) = T^*(\dot{q}) - E(q)$

Linear Euler-Lagrange systems

The mass-spring system is a special case of an **ideal linear system** (Reader, Section 1.2.2) for which the kinetic (co-)energy is

$$T^*(\dot{q}) = \frac{1}{2}\,\dot{q}^T M \dot{q}$$

with $M \geq 0$ (M $n \times n$ **mass inertia** matrix). Often $M > 0$.
Also, for the potential energy, we often have

$$V(q) = \frac{1}{2}\,q^T Q q,$$

with $Q \geq 0$.

- Definition An $n \times n$ symmetric matrix M is positive semidefinite, denoted as $M \geq 0$, if

$$x^T M x \geq 0, \text{for all } x \in \mathbb{R}^n$$

- Definition An $n \times n$ symmetric matrix M is positive definite, denoted as $M > 0$, if

$$x^T M x > 0, \text{ for all } x \in \mathbb{R}^n \text{ such that } x \neq 0$$

Linear Euler-Lagrange systems

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Also, for the potential energy, we often have

$$V(q) = \frac{1}{2}\,q^T Q q,$$

with $Q \geq 0$.

- An $n \times n$ symmetric matrix M is positive semidefinite if and only if all its eigenvalues are non-negative and at least one is equal to zero.
- An $n \times n$ symmetric matrix M is positive definite if and only if all its eigenvalues are positive

Linear Euler-Lagrange systems

The Lagrangian function is

$$L_{NC}(q,\dot q) = \frac{1}{2}\,\dot q^T M\dot q - \frac{1}{2}\,q^T Qq + (Bu+d)^\top q + \int_0^t \frac{1}{2}\dot q(s)^\top C\dot q(s)ds$$

with $M, Q, C \geq 0$.

The Euler-Lagrange equation becomes (Tutorial 1, Exercise 3, for the calculations when $n=2$) :

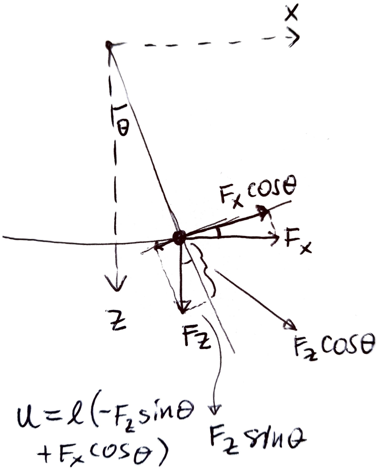
$$\begin{aligned} 0 = \quad & \frac{d}{dt}\left(M\dot q + \int_0^t C\dot q(s)ds\right) - (-Qq + Bu + d) \\ = \quad & M\ddot q + C\dot q + Qq - (Bu+d) \end{aligned}$$

that is

$$M\,\ddot q + C\dot q + Q\,q = B\,u + d$$

Example: 1 DOF robot manipulator I

Frictionless robot manipulator schematically:



Example: 1 DOF robot manipulator II

Cartesian coordinates $r = \begin{bmatrix} z \\ x \end{bmatrix}$; generalized coordinate $q = \theta$

Immersion function $\begin{bmatrix} z \\ x \end{bmatrix} = \ell \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} =: X(q)$

Kinetic energy $T^*(\dot{r}) = \frac{1}{2} m(\dot{z}^2 + \dot{x}^2)$. In the generalized coordinates it becomes

$$T^*(\dot{q}) = T^*(\dot{r})|_{\dot{r} = \frac{\partial X}{\partial q}\dot{q}} = \frac{1}{2} M\dot{q}^2, \quad \text{with } M = m\ell^2$$

The gravity force $F(r) = \begin{bmatrix} mg \\ 0 \end{bmatrix}$ is a conservative force because

$$F(r) = -\frac{\partial \hat{V}}{\partial r}, \quad \text{with } \hat{V}(r) = -mgz$$

In the generalized coordinates it becomes $V(q) = \hat{V}(r)|_{r=X(q)} = -mg\ell\cos\theta$

Hence, $L(q, \dot{q}) = T^*(\dot{q}) - V(q) = \frac{1}{2} M\dot{q}^2 + mg\ell\cos\theta$ and

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = M\ddot{q} - (-mg\ell\sin\theta) = M\ddot{q} + mg\ell\sin\theta$$

Example: 1 DOF robot manipulator II

Euler-Lagrange equations in the presence of non-conservative forces F_{NC}

- Denote by u the projection of the non-conservative force $F_{NC} \in \mathbb{R}^2$ in the Cartesian coordinates onto the tangent space to M, that is

  $$u := F_{NC}^\top \frac{\partial X}{\partial q}$$

  See next slide for an explicit expression of u and its geometric interpretation.

  The modified Lagrangian function L_{NC} can then be written as:

  $$L_{NC}(q, \dot{q}) = L(q, \dot{q}) + uq = \frac{1}{2} M \dot{q}^2 + mg\,\ell \cos(q) + uq$$

  and the Euler-Lagrange equations become

  $$0 = \frac{d}{dt} M\dot{q} - (-mg\ell \sin(q) + u)$$

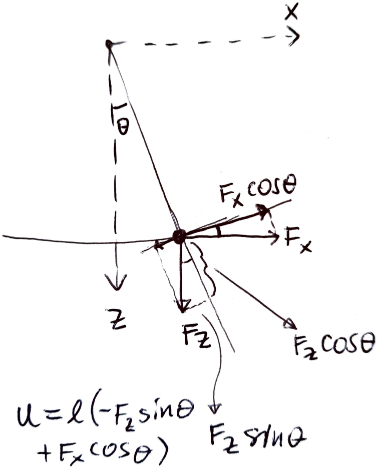
  that is,

  $$M\ddot{q} + mg\ell \sin(q) = u$$

- 2 DOF (degree of freedom) robot manipulator in reader. Calculate T^* expressing kinetic (co-)energy $T^*(v) = \frac{1}{2} \sum_{i=1,2} m_i v_i^2$ in terms of the generalized displacement vector $q = (\theta_1, \theta_2)$ and its derivative $\dot{q}$. It is a useful exercise.

Example: 1 DOF robot manipulator II

Physical interpretation of u. Recall that $(Bu + d)_i := F_{NC}^\top \frac{\partial X}{\partial q_i}$, $i = 1, 2, \ldots, n$ (for this example, $n = 1$ and $d = 0$)



$r = \begin{bmatrix} z & x \end{bmatrix}^\top \in \mathbb{R}^2$, $q = \theta \in \mathbb{R}$

$F_{NC} = \begin{bmatrix} F_z & F_x \end{bmatrix}^\top \in \mathbb{R}^2$

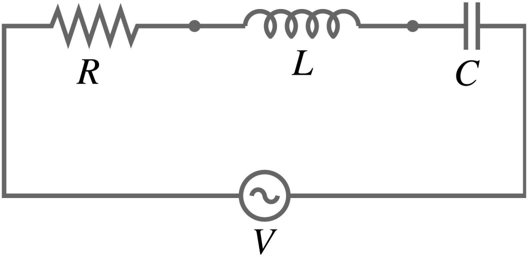
Immersion function $\begin{bmatrix} z \\ x \end{bmatrix} = \ell \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} =: X(q)$

In this case $n = 1$, $B = 1$, $d = 0$ and u is

$$\begin{aligned} u = &\ \begin{bmatrix} F_z & F_x \end{bmatrix} \ell \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \\ = &\ \ell(-F_z \sin\theta + F_x \cos\theta) \end{aligned}$$

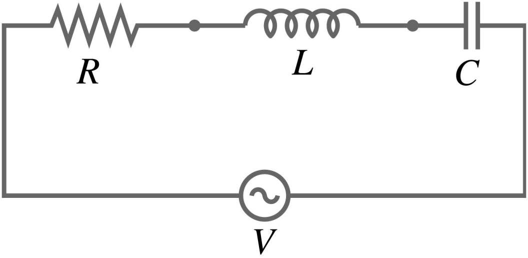
which is the non-conservative force (the input) acting on the pendulum along the direction of the tangent space (which in this case is a line) at each point of the surface (the circle or radius ℓ) where the motion is evolving – see figure on the left

Example: RLC circuit



"Particle"	$=$	the electron
"Position in space"	$=$	$q \in \mathbb{R}$ is the electric charge (transported by the current $i = \dot{q}$ in the circuit in a period of time)
Magnetic energy of the inductor	$=$	$T^*(\dot{q}) = \frac{1}{2} L \dot{q}^2$
Electrical energy of the capacitor		$E(q) = \frac{1}{2} \frac{1}{C} q^2$
Rayleigh function associated to the resistor	$=$	$\mathcal{D}(\dot{q}) = \frac{1}{2} R \dot{q}^2$
Energy required to move charge q through potential V	$=$	qV

Example: RLC circuit



$$L_{NC}(q, \dot{q}) = \frac{1}{2}L\dot{q}^2 - \frac{1}{2}\frac{1}{C}q^2 + qV + \int_0^t \frac{1}{2}R\dot{q}(s)^2 ds$$

RLC circuit dynamic equations - EL systems with dissipation

$$0 = \frac{d}{dt}\left(L\dot{q} + \int_0^t R\dot{q}(s)ds\right) - \left(-\frac{1}{C}q + V\right) \Rightarrow L\ddot{q} + R\dot{q} + \frac{1}{C}q = V$$

Remarks

- How EL modelling works for general electrical circuits is explained in the reader. Reading assignment: Chapter 1. Section 1.2 is the most important.

State space systems

There is a systematic way to convert an nth order linear differential equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = u$$

where

- t is the **time** variable
- y is the **output** variable
- u is the is the **input** variable
- $a_1, \ldots, a_n \in \mathbb{R}$ are constant parameters
- $\frac{d^i y}{dt^i}$ denotes the ith derivative wrt time

into a system of first order linear differential equations.

Example Mass-spring-damper system $m\ddot{q} + c\dot{q} + kq = F$ is a special case of the system above, with $y = q$, $n = 2$, $a_1 = \frac{c}{m}$, $a_2 = \frac{k}{m}$, $u = \frac{F}{m}$ (recall the notation $\dfrac{dq}{dt} = \dot{q}$, $\dfrac{d^2 q}{dt^2} = \ddot{q}$).

State space systems

The nth order differential equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = u$$

can be written as a Linear Time Invariant (LTI) system, that is, a system of n linear differential equations of the first order (derivative of order 1).
Set

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{d^{n-1}y}{dt^{n-1}} \\ \frac{d^{n-2}y}{dt^{n-2}} \\ \vdots \\ \frac{dy}{dt} \\ y \end{pmatrix}$$

State space systems

The nth order differential equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_n y = u \quad (*)$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{d^{n-1}y}{dt^{n-1}} \\ \frac{d^{n-2}y}{dt^{n-2}} \\ \vdots \\ \frac{dy}{dt} \\ y \end{pmatrix} \Rightarrow \begin{aligned} \dot{x}_1 &= \frac{d}{dt}\frac{d^{n-1}y}{dt^{n-1}} = \frac{d^n y}{dt^n} \\ &\overset{(*)}{=} -a_1 \frac{d^{n-1}y}{dt^{n-1}} - \ldots - a_n y + u \\ &= -a_1 x_1 - \ldots - a_n x_n + u \\ \\ \dot{x}_2 &= \frac{d}{dt}\frac{d^{n-2}y}{dt^{n-2}} = \frac{d^{n-1}y}{dt^{n-1}} = x_1 \\ &\vdots \\ \dot{x}_n &= \dot{y} = \frac{dy}{dt} = x_{n-1} \\ y &= x_n \end{aligned}$$

State space systems

Then

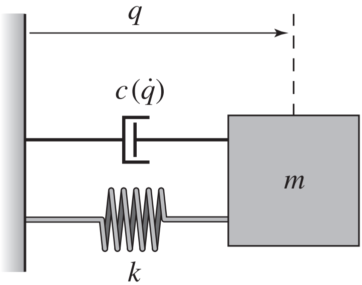
$$\dot{x} = \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -a_1 x_1 - \ldots - a_n x_n \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} + \begin{pmatrix} u \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad y = x_n$$

or,

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -a_1 & -a_2 & \ldots & -a_{n-1} & -a_n \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \end{pmatrix} x \end{aligned}$$

which is the familiar system $\dot{x} = Ax + Bu$, $y = Cx + Du$ $(D = 0)$

Example: mass-spring-damper system



$m\ddot{q} + c\dot{q} + kq = F$
m mass
k spring coefficient
q cart displacement from rest pos.
F external force

Define **output** $y(t) = q(t)$, **input** $u(t) = F(t)$, **states** $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}$. Then

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x(t)$$

State space systems

The state space representation is not unique and different choices of coordinates lead to a different state space representation.
For instance what state representation is obtained if one sets

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} y \\ \frac{dy}{dt} \\ \vdots \\ \frac{d^{n-2}y}{dt^{n-2}} \\ \frac{d^{n-1}y}{dt^{n-1}} \end{pmatrix}?$$

Other examples in Tutorial 2.
We will see in one of the next lectures that these state space models are equivalent.

For a more detailed discussion on the canonical forms of linear systems, read

A.D. Lewis, "A Mathematical Approach to Classical Control", Section 2.5

State space systems – Examples

Analogous state space representations can be found also for some nonlinear systems that we considered.

1 DOF robot manipulator $M\ddot{q} + mgl\sin(q) = F$

This is a second-order non-linear o.d.e. that can be transformed in the state space form. Define the state, input and output variables

$$x_1 = q, \quad x_2 = \dot{q}, \quad u = F, \quad y = q.$$

Then the state space equations are (recall that $M = ml^2$)

$$\begin{aligned}
\frac{d}{dt}\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} &= \underbrace{\begin{pmatrix} x_2 \\ -\frac{mg\,l}{ml^2}\sin x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix} u}_{f(x,u)} \\
y &= \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{h(x)}
\end{aligned}$$

Today

- Basic facts about Euler-Lagrange modeling
- Examples of mechanical, electrical systems
- Linear and nonlinear systems in state space

Next

Reading assignment
Chapter 2 of the textbook for additional information on modelling
Chapter 1 of the reader for the Euler-Lagrange equations