$f(x) = e^{2x} + x^2 - 3$

1 a) $[0 \quad 1]$ $m_0 = 1/2$ $f(1/2) < 0$

(1) $[\frac{1}{2} \quad 1]$ $m_1 = 3/4$ $f(3/4) > 0$

$[\frac{1}{2} \quad \frac{3}{4}]$ $m_2 = 5/8$

(2) $x_{n+1} = x_n - \dfrac{e^{2x_n} + x_n^2 - 3}{2e^{2x_n} + 2x_n}$

$x_0 = 0.5$ $x_1 = 0.504927811$ $x_2 = 0.504903699$

b) $g(x) = \frac{1}{2} \ln(3 - x^2)$

(1) $g'(x) = \frac{1}{2} \dfrac{1}{3-x^2} \cdot (-2x)$ $|g'(\frac{1}{2})| = 0.18182 < 1$

Remark: $|g'(5)| = 0.22$
gives same conclusion

$\Rightarrow$ convergence
"relatively fast", 0.18 not
very close to zero

(2) $K = \dfrac{x_4 - x_3}{x_3 - x_2} = -0.183906$

$\varepsilon_4 \leq \dfrac{K}{1-K} |x_4 - x_3| = 5,5844207\mathcal{7}5 \; 10^{-6}$

(3) $K = -0.183906 \approx g'(1/2) = -0.18182$

linear convergence, with factor $\approx 0.18$
(1st order)

(4) $(0.18)^n \, 5.5844 \; 10^{-6} < 10^{-8}$ $n > 4 \rightarrow 4$ iterations more

$\rightarrow$ 8 iterations in total

2 a) (1) $f(x) = \frac{e^x}{x} = x^{-1}e^x \quad f'(x) = -\frac{1}{x^2}e^x + \frac{1}{x}e^x = \left(-\frac{1}{x^2}+\frac{1}{x}\right)e^x$

$f''(x) = \left(\frac{2}{x^3} - \frac{1}{x^2}\right)e^x + \left(-\frac{1}{x^2}+\frac{1}{x}\right)e^x = \left(\frac{2}{x^3} - \frac{2}{x^2} + \frac{1}{x}\right)e^x$

$f''(x)$ bounded between $[1,2] \rightarrow 2^{nd}$ order convergence

(2) 1 $\frac{3}{2}$ 2

$$\left.\begin{aligned} trap(1) &= \tfrac{1}{4}\left(f(1)+f(\tfrac{3}{2})\right) \\ trap(2) &= \tfrac{1}{4}\left(f(\tfrac{3}{2})+f(2)\right) \end{aligned}\right\} \tfrac{1}{4}f(1) + \tfrac{1}{2}f(\tfrac{3}{2}) + \tfrac{1}{4}f(2)$$

$$= \frac{1}{4}\frac{e}{1} + \frac{1}{2}\cdot\frac{2}{3}e^{3/2} + \frac{1}{4}\frac{e^2}{2} \approx 3{,}0971$$

$$\varepsilon < \frac{2-1}{12}\left(\frac{1}{2}\right)^2 M$$

$$\left.\begin{aligned} f''(1) &= e \\ f''(1.6) &\approx 1.64 \end{aligned}\right\} M = e$$

$$\varepsilon < \frac{1}{48}e \approx 0.056631$$

b) (1) $q_{128} = \left|\frac{I_{64} - I_{128}}{I_{128} - I_{256}}\right| = 3{,}99716 \approx 4$

$\rightarrow 2^{nd}$ order convergence as expected

(2) $\varepsilon_{256} = \left|\frac{1}{3}(I_{256} - I_{128})\right| \approx 2.35\,10^{-6}$

(3) $Extrap_{256} = \frac{4}{3}I_{256} - \frac{1}{3}I_{128} = 3.05911654$

$Extrap_{128} = \frac{4}{3}I_{128} - \frac{1}{3}I_{64} = 3.05911654666\ldots$

$I \approx \frac{16}{15}Extrap_{256} - \frac{1}{15}Extrap_{128} = 3.059116589555\ldots$

3 a) $y' = f(x,y) = y^2 - \frac{1}{2}x$

(1) Euler $y_{\frac{1}{2}} = 1 + \frac{1}{2}\left(1^2 - \frac{1}{2} \cdot 0\right) = \frac{3}{2}$

$y_1 = \frac{3}{2} + \frac{1}{2}\left(\left(\frac{3}{2}\right)^2 - \frac{1}{2} \cdot \frac{1}{2}\right) = \frac{5}{2}$

(2) Heun $k_1 = \frac{1}{2}\left(1^2 - \frac{1}{2} \cdot 0\right) = \frac{1}{2}$

$k_2 = \frac{1}{2}\left(\left(1 + \frac{1}{2}\right)^2 - \frac{1}{2} \cdot \frac{1}{2}\right) = 1$ $\quad y_{\frac{1}{2}} = 1 + \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{7}{4}$

(3) Implicit Euler $y_{\frac{1}{2}} = 1 + \frac{1}{2}\left(y_{\frac{1}{2}}^2 - \frac{1}{2} \cdot \frac{1}{2}\right)$

$\Rightarrow \frac{1}{2} y_{\frac{1}{2}}^2 - y_{\frac{1}{2}} + \frac{7}{8} \Rightarrow y_{\frac{1}{2}} = \frac{1 \pm \sqrt{1 - \frac{14}{8}}}{1}$ problem: complex numbers

b) (1) $q = \left|\frac{3.39\ldots - 3.40\ldots}{3.40\ldots - 3.41\ldots}\right| \approx 3{,}7974 \approx 4$
$2^{nd}$ order as expected

(2) $\varepsilon_{128} = \frac{1}{3}(3.41\ldots - 3.40\ldots) \approx 8.1997\ 10^{-4}$

(3) $\left(\frac{1}{4}\right)^n 8.1997\ 10^{-4} < 10^{-8} \rightarrow n > 9 \rightarrow 128 * 2^9 = 65536$ segments

(4) $\frac{4}{3}(3.41\ldots - 3.40\ldots) = 3.41098971$

4 a) $\hat{x} = 2(x - \frac{1}{2})$

| -1 | 0 | 1 |
|---|---|---|
| 0.74194 | 1.1314 | 1.7405 |

b) $M_0 = 3 \quad M_1 = 0 \quad M_2 = 2$

$F_0 = 3.6138 \quad F_1 = 0.99856$

$3\hat{a} + 0\hat{b} = 3.6130 \qquad \hat{a} = 1.2046$

$0\hat{a} + 2\hat{b} = 0.99856 \qquad \hat{b} = 0.49928$

$a = e^{\hat{a}} = 3.3354$

$b = \hat{b} = 0.49928$

$y(x) = 3.3354\, e^{0.49928 * 2 * (x - \frac{1}{2})}$

$= 3.3354\, e^{0.99856(x - 1/2)}$

5 a) $\rho$ factor $= \frac{2}{2.5} = 0.8 \qquad 0.8^n < 0.01 \Rightarrow \underline{n \geq 21}$ iterations

b) Jacobi $\quad x_1 = \frac{1}{1}(1-0) = 1$

(1) $\quad x_2 = \frac{1}{2.5}(4-0) = 1.6$

$x_3 = \frac{1}{2.5}(4-0) = 1.6$

$x_4 = \frac{1}{1}(1-0) = 1$

(2) SOR ($\omega = 1.5$) $\quad x_1 = \frac{1}{1}(1-0) = 1 \rightarrow x_1 = 3/2 * 1 - 1/2 * 0 = 3/2$

$x_2 = \frac{1}{2.5}(4 - (-1 * \frac{3}{2} - 0)) = 2.2 \rightarrow x_2 = \frac{3}{2} * 2.2 - \frac{1}{2} * 0 = 3.3$

$x_3 = \frac{1}{2.5}(4 - (-1 * 3.3 - 0)) = 2.92 \rightarrow x_3 = \frac{3}{2} * 2.92 - \frac{1}{2} * 0 = 4.38$

$x_4 = \frac{1}{1}(1-0) \rightarrow x_4 = \frac{3}{2} * 1 - \frac{1}{2} * 0 = 3/2$

c) $y'' + by = f(x) \qquad \Delta x = 1/2$

$\frac{y_{i+1} - 2y_i + y_{i-1}}{(1/2)^2} + by_i = f_i \Rightarrow -y_{i+1} + 2y_i - y_{i-1} - \frac{b}{4} y_i = -\frac{1}{4} f_i$

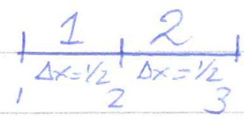
$\Rightarrow -1 \quad 2 - \frac{b}{4} \quad -1 \quad \Big| \quad -\frac{1}{4} f_i \qquad \Rightarrow -\frac{b}{4} = \frac{1}{2}, \quad -\frac{1}{4} f_i = 4$

$\Rightarrow b = -2, \quad f_i = -16$

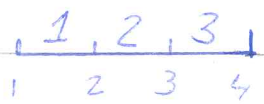
$y''(x) - 2y(x) = -16$ diff eqn

$y(0) = 1, \quad y(1) = 1$ bound. cond.

Remark: two segments (mentioned in questions) actually gives 3*3 system




three segments gives 4*4 system with $\Delta x = 1/3 \Rightarrow$




$\frac{y_{i+1} - 2y_i + y_{i-1}}{(1/3)^2} + by_i = f_i \Rightarrow y''(x) - \frac{9}{2} y(x) = -36$

6 a) 501 points, 500 segments for $[0,5] \rightarrow \Delta x = 0.01$

$D = 10^{-4}$

$$\phi_i^{n+1} = \phi_i^n + R(\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n) - \underbrace{\left(\frac{\Delta t\, u}{\Delta x}\right)}_{=\eta}(\phi_i^n - \phi_{i-1}^n)$$

coeff $\phi_{i+1}^n : R$

$\phi_{i-1}^n : R + \eta$

$\phi_i^n : 1 - 2R - \eta$

$$R = \frac{\Delta t D}{\Delta x^2} = \Delta t \frac{10^{-4}}{(10^{-2})^2} = \Delta t$$

$u = 0 \rightarrow \eta = 0 \quad R < \frac{1}{2} \quad \frac{\Delta t\, 10^{-4}}{(10^{-2})^2} < \frac{1}{2} \quad \Delta t < 1/2$

$u = 0.01 \rightarrow \eta = \Delta t \frac{10^{-2}}{10^{-2}} = \Delta t \qquad 1 - 2R - \eta \geq 0 \quad 2R + \eta < 1$

$3\Delta t < 1 \quad \Delta t < 1/3$

b) $$\phi_i^{n+1} = \phi_i^n + R(\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n) - \frac{\eta}{2}(\phi_{i+1}^n - \phi_{i-1}^n)$$

coeff $\phi_{i+1}^n : R - \frac{\eta}{2}$

$\phi_{i-1}^n : R + \frac{\eta}{2}$

$\phi_i^n : 1 - 2R$

$$R = \frac{\Delta t\, 10^{-4}}{10^{-2}} = 0.01\,\Delta t$$

$$\eta = \Delta t \frac{0.01}{0.1} = 0.1\,\Delta t$$

$R < 1/2 \Rightarrow \Delta t < 50$

$R - \frac{\eta}{2} > 0 \Rightarrow 0.01\,\Delta t - 0.05\,\Delta t > 0$ not possible

always unstable (for every $\Delta t$)

not faster