university of groningen

Exam Numerical Methods

November 10th 2016   18.30-21.30

It is allowed to use a book (paper version only) and lecture notes, as well as a (graphical) pocket calculator. The use of electronic devices (tablet, laptop, mobile phone, etc.) is not allowed.

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: $\boxed{10}$

Practica: $\boxed{18}$ For the 6 computer practica a maximum of 6*3=18 points can be earned.

1. Consider the equation $e^{2x} + x^2 = 3$, with solution $x \approx 0.5$.

   (a) $\boxed{6}$ (1) Perform two iterations with the Bisection method (i.e. compute m_1 and m_2), with $I_0 = [0, 1]$ (and hence $m_0 = 0.5$) as initial search interval.
   (2) Compute two iterations with Newton's method, starting with $x_0 = 0.5$.

   (b) $\boxed{8}$ Someone uses the iterative method $x_{n+1} = \frac{1}{2}\ln\{3 - x_n^2\}$, with $x_0 = 5$.
   (1) Will this method give fast convergence? Explain why.
   (2) The first 4 iterations are given by

n	x_n
0	0.50000000
1	0.50580046
2	0.50473858
3	0.50493406
4	0.50489811

   Determine an error estimate for x_4
   (3) Describe how the errors decrease (which order, which factor, ...).
   (4) How many iteration are needed (approximately) for an accuracy of 1.0E-8?

2. Consider the integral

$$I = \int_1^2 \frac{e^x}{x}\,dx$$

   (a) $\boxed{8}$ (1) Explain that the Trapezoidal method will give optimal 2nd order convergence.
   (2) Use the Trapezoidal method on a grid with two segments to approximate I and give an error estimate using the global error theorem.
   Hint: you may use that $f''(x)$ has extreme values at $x = 1$ and $x \approx 1.6$.

   (b) $\boxed{6}$ With the Trapezoidal method on finer grids the following results are obtained

n	$I(n)$
16	3.05971773
32	3.05926686
64	3.05915412
128	3.05912594
256	3.05911889

   $I(n)$ is the approximation of the integral on a grid with n sub-intervals.

   (1) Compute the q-factor. What can you conclude?
   (2) Give an error estimate for $I(256)$ based on $I(n)$ values.
   (3) Compute improved solutions for $I(256)$ and $I(128)$ by means of extrapolation. Combine these extrapolations into a highly accurate approximation of I.

P.T.O.

3. Consider the differential equation $y'(x) = y^2(x) - \frac{1}{2}x$, with boundary condition $y(0) = 1$.

(a) $\boxed{7}$ (1) Use the Euler method to compute $y(x)$ at $x=1$ on a grid with $\Delta x=0.5$ (2 steps).
(2) Use Heun's method (RK2) to compute $y(x)$ at $x=0.5$ on a grid with $\Delta x=0.5$.
(3) Why does the Implicit(!) Euler method on a grid with $\Delta x=0.5$ lead to problems?

(b) $\boxed{7}$ With an explicit 2nd order method the solution is determined on 3 grids with $N = 32, 64, 128$ segments. The result at a selection of x locations is as follows

x_n	$N = 32$	$N = 64$	$N = 128$
0.0	1.00000000	1.00000000	1.00000000
0.25	1.31376417	1.31391441	1.31395291
0.5	1.88536590	1.88636234	1.88661946
0.75	3.39836847	3.40770982	3.41016974
1.0	18.80427323	20.52616796	21.16133630

(1) Compute the q-factor for $x = 0.75$. What can you conclude?
(2) Give an error estimate for the solution at $x = 0.75$ on the finest grid.
(3) How many segments (use powers of 2) are required for an accuracy of 1.0E-8?
(4) Compute an improvement for the solution at $x = 0.75$ by means of extrapolation.

4. We will fit an exponential curve through the data

x_i	0	0.5	1.0
f_i	2.1	3.1	5.7

(a) $\boxed{2}$ Apply a coordinate transformation, such that the $\hat{x}_i$ points are centered around $x=0$ as follows: $\hat{x}_1=-1$, $\hat{x}_2=0$, $\hat{x}_3=1$. Then apply the ln-function to the measured data.

(b) $\boxed{7}$ Determine the straight-line through the transformed data and finally the exponential curve through the original data.

5. Consider $A\vec{x} = \vec{b}$, with

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2.5 & -1 & 0 \\ 0 & -1 & 2.5 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ 4 \\ 4 \\ 1 \end{pmatrix}$$

(a) $\boxed{3}$ When the Jacobi method is used, how many iterations are required to reduce the initial error in $\vec{x}_0 = (0\ 0\ 0\ 0)^T$ with a factor 100?

(b) $\boxed{5}$ Compute $\vec{x}_1$, the result after 1 iteration with the initial vector $\vec{x}_0 = (0\ 0\ 0\ 0)^T$
(1) when the Jacobi method is used.
(2) when the SOR method is used with $\omega = 1.5$.

(c) $\boxed{4}$ The system $A\vec{x} = \vec{b}$ resembles a 2nd order linear differential eqn. $y''(x) + b\,y(x) = f$ for x in [0,1] (with b and f constant), on a grid with two segments.
Specify the differential equation with corresponding boundary conditions.

6. Consider for x in $[0, 5]$ the partial differential equation $\dfrac{\partial \phi}{\partial t} + U\dfrac{\partial \phi}{\partial x} = 10^{-4}\dfrac{\partial^2 \phi}{\partial x^2}$,
with initial condition $\phi(x, 0) = 100\sin(\pi x)$ and boundary conditions $\phi(0, t) = \phi(5, t) = 0$.
For $\partial^2/\partial x^2$ the standard [1 -2 1]-formula is applied. The gradient term at the lhs can be discretised as

$$\frac{\partial \phi}{\partial x} \text{ at } x_i \approx \frac{\phi_i - \phi_{i-1}}{\Delta x} \text{ (Method1) or } \frac{\partial \phi}{\partial \mathrm{x}} \text{ at } \mathrm{x_i} \approx \frac{\phi_{\mathrm{i+1}} - \phi_{\mathrm{i-1}}}{2\Delta \mathrm{x}} \text{ (Method2)}$$

(a) $\boxed{5}$ Determine the time step limit for the Explicit Euler method if Method 1 is used on an equidistant grid containing 501 grid points, for $U=0$ (diffusion only) and $U=0.01$.

(b) $\boxed{4}$ Method 2 has 2nd order accuracy in Δx, method 1 only 1st order. Suppose that a grid with 51 points yields the same spatial accuracy as method 1 with 501 grid points. Does this lead to faster computing times? Explain.

Total: $\boxed{100}$