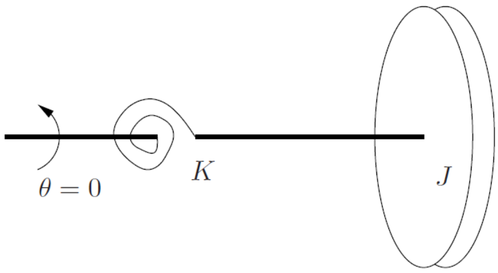
Control Engineering

Instruction Lecture 2

Basic concepts modeling | Chapter 2,3 + reader chapter 1

Exercise 1. Euler-Lagrange equations

Consider a rotating mass connected to a spring. The mass has an intertia J and the rotational spring has a spring constant K. The left part of the spring is fixed (angle $\theta = 0$). An external torque $\tau(t)$ acts on the rotating mass.



1. Give the Euler-Lagrange equation of the system.
2. Determine the state space model from the Euler-Lagrange equation with the angular velocity $\dot{\theta}$ as an output.

Exercise 2. State space models

Consider three masses evolving on a unitary circle (radius=1) which are connected by linear springs with coefficients a_{12}, a_{13}, a_{23} respectively, see Figure 1. A torque τ_1 is applied to mass 1, resulting in a circular motion of the mass. The variables $\theta_1, \theta_2, \theta_3$ are the angular displacements, $\omega_1, \omega_2, \omega_3$, the resulting angular velocities and J_1, J_2, J_3 the moment of inertia of the three rotating masses.

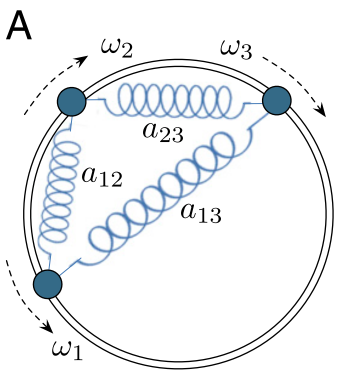


Figure 1: From Dörfler et al. Synchronization in complex oscillator networks and smart grids. PNAS 2013.

When neglecting the gravity and friction and assuming that the angle differences are small, the (linearized) equations of motion are given by

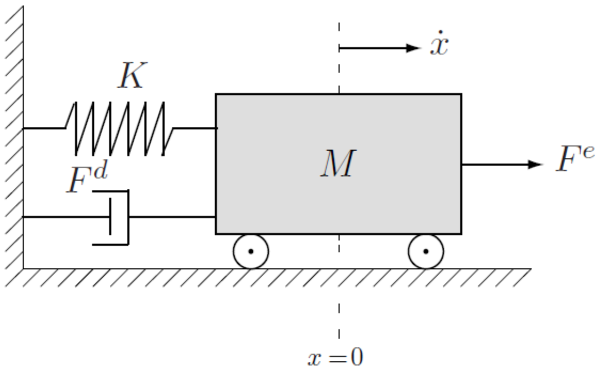
$$\begin{aligned} J_1\dot{\omega}_1 + a_{12}(\theta_1 - \theta_2) + a_{13}(\theta_1 - \theta_3) &= \tau_1 \\ J_2\dot{\omega}_2 - a_{12}(\theta_1 - \theta_2) + a_{23}(\theta_2 - \theta_3) &= 0 \\ J_3\dot{\omega}_3 - a_{13}(\theta_1 - \theta_3) - a_{23}(\theta_2 - \theta_3) &= 0. \end{aligned} \tag{1}$$

Derive a state space model for the system by introducing the state variables $x_1 = \theta_1, x_2 = \theta_2, x_3 = \theta_3$, $x_4 = \omega_1, x_5 = \omega_2$ and $x_6 = \omega_3$, with $u = \tau_1$ as an input, and the angular velocity $y = \omega_1$ as the output.

NOTE: $\dot{\theta}_i = \omega_i$, for $i = 1, 2, 3$.

Exercise 3. Modeling of a mass-spring damper system

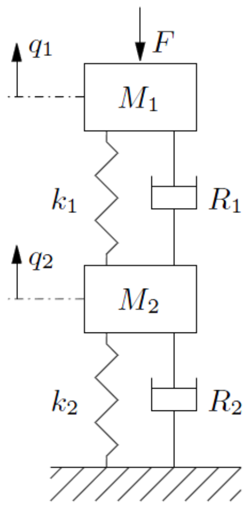
Consider a mechanical system with external force F^e. Mass $M > 0$ and spring constant $K > 0$ are constants. The damping force is given by F^d.



1. Give the Lagrangian for this system.
2. Assume $F^d = 0$ and give the Euler-Lagrange equations for this system.
3. Assume $F^d = -\dot{x} - 2\dot{x}^3$, and determine the Rayleigh dissipation function.
4. Derive the equations of motion from the Euler-Lagrage equations.
5. Derive a state space model from the equations of motion found above and take $\dot{x}$ as an output.

Exercise 4.

Consider the linear mass-spring-damper system with input force F depicted below. Throughout this exercise gravity may be neglected.



1. Determine the Lagrangian of the system.
2. Determine the Rayleigh dissipation function.
3. Determine the equations of motion, using the Lagragian.
4. Give a state space model with the positions of the two masses as the output.
5. Give the equivalent electrical circuit.

Exercise 5. (Book exercise 2.1)

Consider the linear ordinary differential equation[1]

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = u. \tag{2}$$

1. Show that by choosing a state space representation with $x_1 = y$, the dynamics can be written as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{3}$$

   with

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & \ldots & 0 & 0 \end{pmatrix}.$$

   This canonical form is called the *chain of integrators* form.

2. There exist other canonical forms as well. So see this, define $x_1 = \frac{dy^{n-1}}{dt}, \ldots, x_{n-1} = \frac{dy}{dt}, x_n = y$ and write the system (2) in the form (3) using this set of state variables.

[1] Equation (2.7) in the book.