Geometry 2024, homework set 2

- Below you can find your second homework assignment. Please upload it to BrightSpace by **Tuesday March 19**. The deadline is strict, so late homework will not be graded.
- The number of points per question is given in a box. 15 extra points are given for a clear writing of solutions. Note that high marks for the homeworks contribute to the final grade.

Solutions

1. $\boxed{7+15+8 = 30 \text{ pts}}$ (**Riemannian metric on models of the hyperbolic plane**) Let

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\} \quad \text{and} \quad \mathbb{H} = \{z = (x + iy) \in \mathbb{C} \mid y > 0\}$$

be the open unit disk and the upper-half plane in the complex plane $\mathbb{C}$, respectively. Consider the so-called Cayley transform $f = \frac{z-i}{z+i} \colon \mathbb{C} \to \mathbb{C}$.

a) Show that f gives a complex-analytic diffeomorphism between $\mathbb{H}$ and $\mathbb{D}$, i.e. the restriction of f to $\mathbb{H}$ is a complex-analytic bijection between $\mathbb{H}$ and $\mathbb{D}$ with a complex analytic inverse.

b) Prove that the pull-back of the Riemannian metric

$$G(u, v) = \frac{4(\mathrm{d}u^2 + \mathrm{d}v^2)}{(1 - (u^2 + v^2))^2}, \quad z = u + iv,$$

on $\mathbb{D}$ under f has the form

$$(f^*G)(x, y) = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$

c) Conclude that $\mathbb{H}$ and $\mathbb{D}$ with these metrics are isometric.
(These are two famous models of planar hyperbolic geometry, called the *upper-half plane model* and the *Poincaré disk model.*)

Solutions:

a) First observe that f is invertible outside the point $z = -i$ (or as a map from $\mathbb{C} \cup \{\infty\} = P_1(\mathbb{C})$ to $\mathbb{C} \cup \{\infty\} = P_1(\mathbb{C})$): the inverse is given by

$$z = f^{-1}(w) = i\frac{1 + w}{1 - w}.$$

Since both maps are complex-analytic, f is a complex-analytic diffeomorphism between $\mathbb{C} \setminus \{-i\}$ and $\mathbb{C} \setminus \{1\}$ (in fact, f is a complex-analytic diffeomorphism of $P_1(\mathbb{C})$ to itself). It is left to observe that since

$$|f(z)| = 1 \quad \text{when} \quad \mathrm{Im}(z) = 0,$$

f maps $\mathbb{H}$ onto $\mathbb{D}$; hence it is a complex-analytic diffeomorphism from $\mathbb{H}$ onto $\mathbb{D}$.

b) Next we would like to show that $(f^*G)(x,y) = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$. Observing that

$$G(z) = \frac{2(\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z)}{(1-|z|^2)^2}$$

and

$$\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} = \frac{-2(\mathrm{d}w \otimes \mathrm{d}\bar{w} + \mathrm{d}\bar{w} \otimes \mathrm{d}w)}{(w-\bar{w})^2},$$

it is left to show that

$$f^*G(w) = \frac{2(\mathrm{d}f(w) \otimes \mathrm{d}\overline{f(w)} + \mathrm{d}\overline{f(w)} \otimes \mathrm{d}f(w))}{(1-|f(w)|^2)^2} = \frac{-2(\mathrm{d}w \otimes \mathrm{d}\bar{w} + \mathrm{d}\bar{w} \otimes \mathrm{d}w)}{(w-\bar{w})^2}.$$

The latter follows from a direct computation using the chain rule:

$$\mathrm{d}f(w) = \partial_w f \mathrm{d}w + \partial_{\bar{w}} f \mathrm{d}\bar{w} = \partial_w f \mathrm{d}w \quad \text{and} \quad \mathrm{d}\overline{f(w)} = 0 + \partial_{\bar{w}} \bar{f} \mathrm{d}\bar{w}.$$

c) Since f is a (complex-analytic) bijection between $\mathbb{H}$ and $\mathbb{D}$, the property

$$f^*G = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} = \frac{-2(\mathrm{d}w \otimes \mathrm{d}\bar{w} + \mathrm{d}\bar{w} \otimes \mathrm{d}w)}{(w-\bar{w})^2}, \quad w = x + iy,$$

means that $\mathbb{H}$ and $\mathbb{D}$ are isometric by definition.

2. $\boxed{\text{10+10 =20pts}}$ (**Gaussian curvature of Plücker's conoid**) Consider the following surface in affine Euclidean 3-space

$$M^2 = \{(x,y,z) \in \mathbb{R}^3 \mid z(x^2+y^2) = xy\}.$$

(It is called Plücker's conoid; cf. Exercise VIII.3 of M. Audin).

a) Prove that M^2 is a ruled surface. Determine at which points is this surface regular. Is M^2 orientable?

b) Compute the Gaussian curvature of Plücker's conoid at its regular points.

Solutions:

a) First observe that for every value of z, the equation $z(x^2+y^2) = xy$ represents a degenerate conic, which is a pair of lines for $|z| < \frac{1}{2}$, a single line for $z = \pm\frac{1}{2}$ and a point for $|z| > \frac{1}{2}$. The single point has necessarily the x, y coordinates equal to zero, thus the whole z-axis belongs to the surface. This proves Plücker's conoid is a ruled surface: obtained by the union of the z-axis with lines $z(x^2 + y^2) = xy$ in the affine planes of height z with $|z| \leq \frac{1}{2}$.

The singular points of Plücker's conoid are obtained by equating the gradient of the function $f(x, y, z) = z(x^2 + y^2) - xy$ to zero:

$$\mathrm{grad} f(x, y, z) = (2xz - y, 2yz - x, x^2 + y^2) = 0$$

implies $x = y = 0$. Thus the whole z-axis consists of singular points of the surface.

The surface is not orientable as it contains a Möbius band; see Figure **??** below. (4)

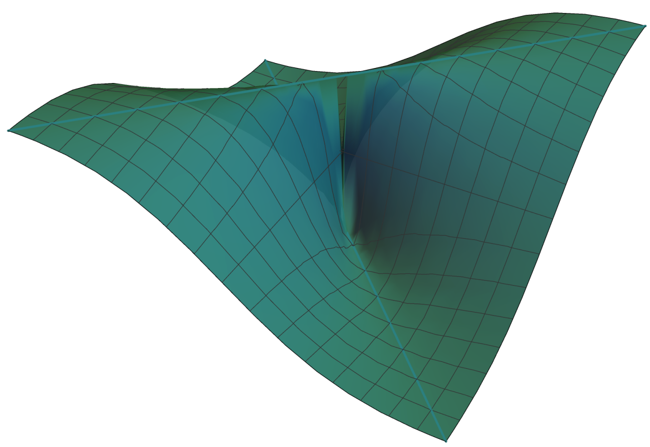

Figure 1: Plücker's conoid.

b) To compute the Gaussian curvature, first observe that at its regular points, Plücker's conoid looks like the graph of the function $z = \frac{xy}{x^2+y^2)}$ (1). One can now compute the Gaussian curvature as the quotient of the determinants of the second and first fundamental forms (3). Specifically, the normal vector to the surface has the expression

$$\vec{n}(x, y) = \frac{(-z_x, -z_y, 1)}{\sqrt{1 + z_x^2 + z_y^2}}, \quad (1)$$

hence the second fundamental form is given by

$$\begin{pmatrix} \frac{z_{xx}}{\sqrt{1+z_x^2+z_y^2}} & \frac{z_{xy}}{\sqrt{1+z_x^2+z_y^2}} \\ \frac{z_{xx}}{\sqrt{1+z_x^2+z_y^2}} & \frac{z_{yy}}{\sqrt{1+z_x^2+z_y^2}} \end{pmatrix}. \quad (1)$$

The matrix of the Riemannian metric is computed as

$$\begin{pmatrix} 1+z_x^2 & z_x z_y \\ z_x z_y & 1+z_y^2 \end{pmatrix}. \quad (1)$$

It follows that

$$K = \frac{z_{xx}z_{yy} - z_{xy}^2}{(1+z_x^2+z_y^2)^2}. \quad (2)$$

Substituting $z = \frac{xy}{x^2+y^2}$ into this formula gives

$$K = -\frac{(x^4-y^4)^2}{(x^6+y^4+y^6+x^2y^2(-2+3y^2)+x^4(1+3y^2))^2} \quad (1).$$

3. $\boxed{10+10+8+7 = 35\text{pts}}$ (**Geodesics**)

   Consider the two models of hyperbolic geometry given in exercise 1 (the upper-half plane $\mathbb{H}$ and the Poincaré disk $\mathbb{D}$ models).

   a) Write down the geodesic equations for these models;

   b) Verify that circular arcs in $\mathbb{H}$, respectively, $\mathbb{D}$, meeting the boundary of $\mathbb{H}$, resp., $\mathbb{D}$, orthogonally are geodesics (when parametrised by constant speed).

   Consider now an ellipsoid in $\mathbb{R}^3$:

$$E^2 = \{(x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$$

   c) Prove that the intersections of E^2 with coordinate xy, xz, yz 2-planes are geodesics on E^2, that is, $\gamma_x = E^2 \cap \{x=0\}$, $\gamma_y = E^2 \cap \{y=0\}$, and $\gamma_z = E^2 \cap \{z=0\}$, are geodesics (when parametrised by constant speed).

   d) Does it follow[1] from part c) that γ_x, γ_y, and γ_z are the only *closed* geodesics of E^2?

[1] Though not hard, this question is much related to so-called integrable dynamical systems and also to topology (Lusternik-Schnirelmann category).

Solution:

a) Recall that the geodesic equations have the form

$$\ddot{u}^i + \Gamma^i_{jk}(u^1, \ldots, u^{n-1})\dot{u}^j \dot{u}^k = 0,$$

where $\Gamma^i_{jk}(u^1, \ldots, u^{n-1})$ are Christoffel symbols of the second kind, given in terms of the first fundamental form g_{ij} as follows

$$\Gamma^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\partial g_{js}}{\partial u^k} + \frac{\partial g_{ks}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^s}\right).$$

For the upper-half plane, the matrix $G = (g_{ij})$ has the form

$$G = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}.$$

Here $u^1 = x, u^2 = y$. It follows that

$$\Gamma^1_{11} = 0,\ \Gamma^1_{22} = 0,\ \Gamma^2_{12} = \Gamma^2_{21} = 0; \quad \Gamma^1_{12} = \frac{-1}{y},\ \Gamma^2_{11} = \frac{-1}{y}, \ \text{ and } \Gamma^2_{22} = \frac{-1}{y}.$$

Hence the geodesic equations have the form

$$\ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0, \tag{1}$$

$$\ddot{y} + \frac{1}{y}(\dot{x}^2 - \dot{y}^2) = 0. \tag{2}$$

A similar computation for the Poincaré disk model gives

$$\ddot{u} + \frac{2}{1-u^2-v^2}\left(u\dot{u}^2 + 2v\dot{u}\dot{v} - u\dot{v}^2\right) = 0,$$
$$\ddot{v} + \frac{2}{1-u^2-v^2}\left(v\dot{v}^2 + 2u\dot{u}\dot{v} - v\dot{u}^2\right) = 0.$$

b) That circular arcs in $\mathbb{H}$ meeting the boundary of $\mathbb{H}$ orthogonally follows from the form of the geodesic equation (1-2). One can see this by eliminating t from equations (1-2), getting

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{2}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \frac{1}{y}\left(1 - (\frac{\mathrm{d}y}{\mathrm{d}x})^2\right) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \frac{1}{y} = 0.$$

But circles $y^2 + (x-c)^2 = r^2$ and lines $x = c,\ c \in \mathbb{R}$, are solutions of $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \frac{1}{y} = 0$ since $y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 1 = \frac{\mathrm{d}}{\mathrm{d}x}\left(y\frac{\mathrm{d}y}{\mathrm{d}x} + x\right)$, from where the result follows.

An alternative solution is based on the following geometric argument. First observe that lines $x = c$ are geodesics of $\mathbb{H}$; this follows readily from (1-2). Then use the (orientation-preserving) isometry group $\mathrm{PSL}(2, \mathbb{R})$ of $\mathbb{H}$ to map such lines to circles meeting $\partial\mathbb{H}$ orthogonally.

Yet another (and perhaps the quickest!) solution relies on the observation that inversions in circles $y^2 + (x - c)^2 = r^2$ and lines $x = c,\ c \in \mathbb{R}$, are isometries of $\mathbb{H}$. It is then left to apply the existence and uniqueness theorem for ODEs: Geodesics are specified by an initial point and a velocity vector. Therefore, if a circle $y^2 + (x-c)^2 = r^2$ or line $x = c$ was not a geodesic, applying the reflection in this circle or line, we would get two geodesics contradicting the existence and uniqueness theorem.

The corresponding property for $\mathbb{D}$ is deduced from that of $\mathbb{H}$ using the map $f = \frac{z-i}{z+i}$ from Exercise 1, which

i) maps circles or lines to circles or lines and

ii) preserves orthogonality being an isometry.

c) Consider the curves $\gamma_x = E^2 \cap \{x = 0\}$, $\gamma_y = E^2 \cap \{y = 0\}$, and $\gamma_z = E^2 \cap \{z = 0\}$. Observe that these are planar curves such that their normal vector is also normal to the ellipsoid. Thus, letting $\vec{n}(q)$ be a (unit) normal to E^2 at a point $q \in E^2$, we get

$$\vec{n}(\gamma_d(s)) \,||\, \gamma_d''(s)$$

are parallel (here $d = x, y$, or z and s is a constant multiple of the arc-length). This means that the curves $\gamma_x = E^2 \cap \{x = 0\}$, $\gamma_y = E^2 \cap \{y = 0\}$, and $\gamma_z = E^2 \cap \{z = 0\}$.

are geodesics of E^2.

d) No, as these are very special curves on the ellipsoid. For example, one gets infinitely many closed geodesics when two of the parameters a, b and c are equal by rotational symmetry[2].

End of homework

[2] In fact, one can show that for any choice of parameters a, b and c there are always infinitely many closed geodesics; this is linked to the so-called *integrability* of the problem.