# Geometry - Solutions to Exercises

As the course progresses, solutions to more exercises will be added to this document.

## Contents

# 1 Week 1

## Exercise VII.1

Let $\mathcal{C}$ be a conic in $\mathbb{R}^2$ and let $A$ be a point of $\mathcal{C}$. Let us perform a change of coordinates such that $A$ is the origin of our new coordinate system, that is $A = (0, 0)$. Let $\mathcal{D}_t$ be a line of slope $t$ passing through the point $A$. It will in general intersect $\mathcal{C}$ at two points (unless it is a tangent line). Let the other point of intersection between $\mathcal{D}_t$ and $\mathcal{C}$ be $M_t = (x(t), y(t))$.

A conic is given by the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{1}$$

where $a, b, c, d, e, f \in \mathbb{R}$, and at least one of $a, b, c$ are non-zero. Since $\mathcal{D}_t$ has slope $t$ and it runs through the origin, it can be written as $y = tx$. Then the intersection points between the conic and the line is given by (1) with $y = tx$:

$$(a + bt + ct^2)x^2 + (d + et)x + f = 0. \tag{2}$$

Note that one of the intersection points, $A$, is by assumption at the origin. This implies that $f = 0$. Let us now write (2) as

$$x(g(t)x + h(t)) = 0,$$

where $g(t) = a+bt+ct^2$ and $h(t) = d+et$. Again, the solution $x = 0$ corresponds to the point $A$. Thus, the other point must be

$$x = -\frac{h(t)}{g(t)}, \quad y = tx = -\frac{th(t)}{g(t)}.$$

Note that $g(t)$ is indeed non-zero by the definition of the conic.

Let us now consider the circle $x^2 + y^2 = 1$ minus a point, say $(-1, 0)$. The circle corresponds to the conic (1) with $a = c = 1$, $b = d = e = 0$ and $f = -1$. This time $A$ is not located at the origin. The equation of the line with slope $t$ running from $A = (-1, 0)$ to $M_t = (x(t), y(t))$ is given by $y = t(x + 1)$. The intersection points are given by the solutions to

$$x^2 + t^2(x + 1)^2 - 1 = 0.$$

This equation can be written

$$(1 + t^2)x^2 + 2t^2x + t^2 - 1 = 0.$$

Using the *abc*-formula gives

$$x = \frac{-2t^2 \pm \sqrt{4t^4 - 4(1 + t^2)(t^2 - 1)}}{2(1 + t^2)} = \frac{-2t^2 \pm \sqrt{4t^4 - 4(1 + t^2)}}{2(1 + t^2)} = \frac{-t^2 \pm 1}{1 + t^2}.$$

Note that the choice of minus-sign gives us the $x$-value for $A$, namely $x = -1$. Hence, we should use the plus-sign, which gives $x(t) = \frac{1-t^2}{1+t^2}$. Then $y(t) = t(x + 1) = \frac{2t}{1+t^2}$. This defines a parametrization of the circle.

## Exercise VII.3

Let $\mathcal{C}$ be the curve parametrized by

$$\begin{cases} x &= \frac{t}{1+t^4} \\ y &= \frac{t^3}{1+t^4}. \end{cases}$$

The curve is an inclined figure 8, as seen in Figure 1.

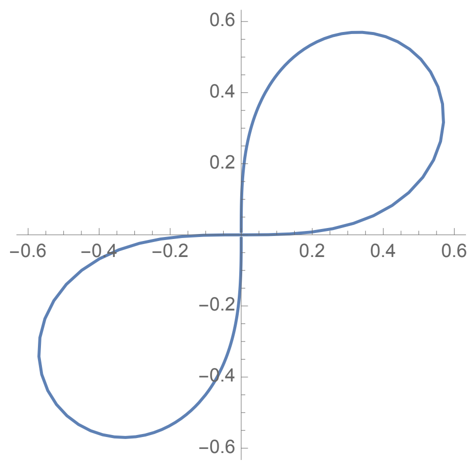



Figure 1: The curve from Exercise VII.3. The figure was created with Mathematica.

Note that $y = xt^2$, i.e. $t^2 = y/x$. Thus,

$$x = \frac{t}{1 + y^2/x^2} = \frac{x^2 t}{x^2 + y^2} \implies t = \frac{x^2 + y^2}{x}.$$

Plugging this back into $y = xt^2$ gives us the Cartesian equation:

$$xy = (x^2 + y^2)^2.$$

## Exercise VII.5

We consider the curve parametrized by

$$\begin{cases} x = t + \frac{1}{2t^2} \\ y = t^2 + \frac{2}{t}. \end{cases}$$

We find the singularities:

$$\frac{dx}{dt} = 1 - \frac{1}{t^3}, \quad \frac{dy}{dt} = 2t - \frac{2}{t^2}.$$

If we set them both to be equal to zero, we find that $t = 1$ is the only possibility. That is, $t = 1$ is the only singularity of the curve.

We see that when $|t|$ is big, then $x \approx t$ and $y \approx t^2$. In other words, $y = x^2$. This means that, when $|t|$ is large, then the curve approximates the parabola given by $y = x^2$. Likewise, when $|t|$ is small, then $x \approx \frac{1}{2t^2}$ and $y \approx \frac{2}{t}$. In other words, $y^2 = 8x$. This means that, when $|t|$ is small, then the curve approximates the parabola given by $y^2 = 8x$.

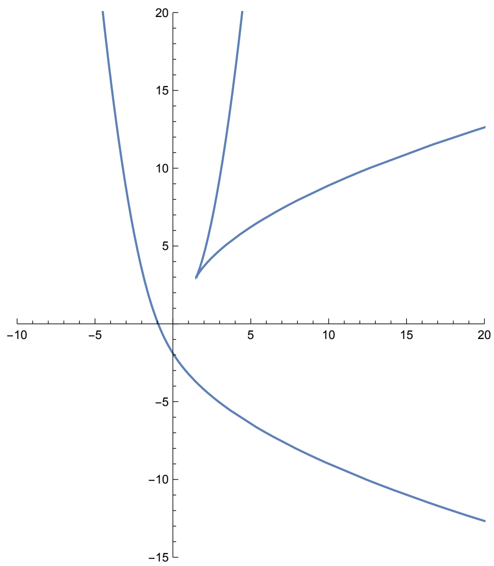



Figure 2: The curve from Exercise VII.5. The figure was created with Mathematica.

## Exercise VII.15

The exercise asks us to first find the equation of the line $D'$. In Figure 3, the path for the line $D$ as it is first reflected, and then refracted, and so becomes $D'$ is drawn in red.

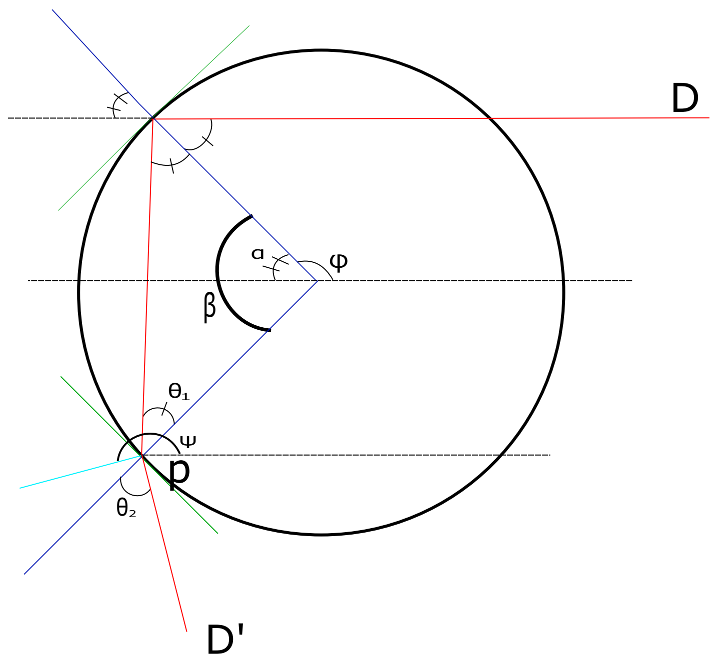



Figure 3: The figure is related to Exercise VII.15. Please see the problem solution for an explanation of the figure. The figure was created with Inkscape.

Let $D$ be a line parallel to the $x$-axis given by $y = \sin\varphi$. Thus, the line is reflected by the circle at the point $(\cos\varphi, \sin\varphi)$. Observe that the angle $\alpha$ in the figure is equal to $\pi - \varphi$. The angles identified by two short lines are indeed identical, as they are located between parallel lines, and a straight line through the parallel lines. Furthermore, these angles must also be identical to the angles identified by only one short line, as opposite angles are equal. Note also that the two angles at the top are equal since this is a reflection. Finally, the bottom angle, $\theta_1$, is equal to the top angles since we have an isosceles triangle. The blue lines extending to the border of the circle are both radii of the circle, i.e. of equal length. Thus, we have $\theta_1 = \alpha = \pi - \varphi$.

Snell tells us now how to find $\theta_2$:

$$\sin\theta_2 = n_{12}\sin\theta_1,$$

where $n_{12}$ is the refraction index. Note that, as $|\sin\theta| \leq 1$ for any $\theta$, we have, if $n_{12} > 1$ (between water and air $n_{12} = 4/3 > 1$), some restriction to what the angles $\varphi$, and hence$\theta_2$, may be. Precisely, we find

$$\theta_2 = \arcsin(n_{12}\sin\theta_1) = \arcsin(n_{12}\sin(\pi - \varphi)) = \arcsin(n_{12}\sin(\varphi)).$$

Thus, we must have $|n_{12}\sin(\varphi)| \leq 1$ (the domain of the function arcsin is $[-1, 1]$). In the case of refraction from water to air, where $n_{12} = 4/3$, we then have $\arcsin(-3/4) \leq \varphi \leq \arcsin(3/4)$.

Now, to find the equation of the line $D'$, we are going to use its normal vector, which has been coloured turquoise in Figure 3. Note that the $x$-value of the unit normal vector is given by $\cos\psi$ and the $y$-value is given by $\sin\psi$. Here $\psi$ is in fact a function of $\varphi$, $\psi = \psi(\varphi)$. So, what is then $\psi$? Note that the inner angle of the isosceles triangle mentioned above is equal to $\beta := \pi - 2\theta_1 = \pi - 2(\pi - \varphi) = 2\varphi - \pi$. Then, the angle between the $x$-axis and $D'$ is $\varphi + \beta + \theta_2$. To get to the outward normal vector (turquoise line) of $D'$ we need to subtract $\pi/2$ from this angle:

$$\psi = \varphi + \beta + \theta_2 - \pi/2 = \varphi + 2\varphi - \pi + \theta_2 - \pi/2 = 3\varphi - \pi + \theta_2 - \pi/2.$$

Thus, we have been able to find an expression for the unit normal vector to $D'$, given by $(\cos\psi, \sin\psi)$ (let us not fill in the expression we found for $\psi$, but only remember that it is indeed dependent on $\varphi$). The equation describing $D'$ is then given by

$$\cos(\psi)x + \sin(\psi)y + \omega = 0, \tag{3}$$

where $\omega$ is the point of intersection. Let us find an expression for $\omega$: Recall that the point $p$ (see Figure 3) where the line is refracted is given by $\beta + \varphi = 3\varphi - \pi$. Thus, the $x$-value is $\cos(3\varphi - \pi)$ and the $y$-value is $\sin(3\varphi - \pi)$. Plugging this into (3) gives us

$$\omega = -\cos(\psi)\cos(3\varphi - \pi) - \sin(\psi)\sin(3\varphi - \pi) = \cos(\psi)\cos(3\varphi) + \sin(\psi)\sin(3\varphi).$$

To find the envelope, we will consider the line we just found, and the derivative of it,

$$\begin{cases} \cos(\psi)x + \sin(\psi)y + \omega = 0 \\ \frac{\mathrm{d}}{\mathrm{d}\varphi}(\cos(\psi))x + \frac{\mathrm{d}}{\mathrm{d}\varphi}(\sin(\psi))y + \frac{\mathrm{d}}{\mathrm{d}\varphi}(\omega) = 0. \end{cases}$$

Now, note that $\frac{\mathrm{d}}{\mathrm{d}\varphi}(\cos(\psi)) = -\sin(\psi)\psi'$ and $\frac{\mathrm{d}}{\mathrm{d}\varphi}(\sin(\psi)) = \cos(\psi)\psi'$, where $\psi' := \frac{\mathrm{d}\psi}{\mathrm{d}\varphi}$. We will also denote by $\omega' = \frac{\mathrm{d}\omega}{\mathrm{d}\varphi}$. Thus, the system of equations becomes

$$\begin{cases} \cos(\psi)x + \sin(\psi)y + \omega = 0 & \text{(I)} \\ -\sin(\psi)\psi' x + \cos(\psi)\psi' y + \omega' = 0 & \text{(II).} \end{cases}$$

Now we can find an expression for $x$ and $y$:

$$\begin{aligned} 0 &= \sin(\psi)\psi' \cdot \text{(I)} + \cos(\psi) \cdot \text{(II)} \\ &= (\sin^2(\psi) + \cos^2(\psi))\psi' y + \sin(\psi)\psi'\omega + \cos(\psi)\omega' \\ &= \psi' y + \sin(\psi)\psi'\omega + \cos(\psi)\omega'. \end{aligned}$$

Thus, after a similar computation for $x$,

$$\begin{aligned} x &= -\cos(\psi)\omega + \sin(\psi)\frac{\omega'}{\psi'} \\ y &= -\sin(\psi)\omega - \cos(\psi)\frac{\omega'}{\psi'}. \end{aligned}$$

This defines the envelope depicted in Figure 4, when the refraction index is 4/3 (between water and air).

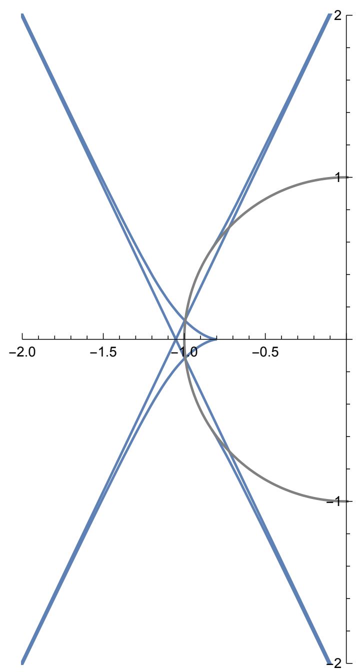



Figure 4: The envelope found in Exercise VII.15. The figure was created with Mathematica.

We could also find the equation for the line $D'$ in another way, which is useful for finding the asymptotes. That is, we could instead consider the angle between the $x - axis$ and the line $D'$; its largest value correspond to the asymptote. One can show that this angle is $\xi = \pi - \theta_2 - (\beta - \alpha) = \pi - 3\varphi - \theta_2$. Then the direction of $D'$ is given by $y = \tan \xi$.

One finds the maximum value for $\xi$ by computing the derivative with respect to $\varphi$, solving for $\varphi$, and plugging back into $\xi$. This yields the asymptotes to the envelope curve. The extrema of $\xi$ are at

$$\varphi_{\pm} = \pm \arccos\left(-\frac{3\sqrt{n^2 - 1}}{2\sqrt{2}n}\right), \tag{4}$$

and the asymptotes are parallel to $y = x \tan \varphi_{\pm}$.

For more on this, see http://people.uncw.edu/hermanr/Documents/Talks/Rainbow_History.pdf

## 2 Week 2

### Exercise VII.9

We are given a circle of radius $r$, which we assume is centred at the origin in the plane. Thus, it is given by $x^2 + y^2 = r^2$, and parametrised by $x = r\cos t$, $y = r\sin t$. Furthermore, we have a point $F = (x_F, y_F)$, we which assume lies outside the circle, which implies $x_F^2 + y_F^2 > r^2$. We are interested in finding the envelope of the perpendicular bisector $MF$. The perpendicular bisector is the line perpendicular to $MF$, which runs through the midpoint of $MF$, which we will denote by $A$, see Figure 5. Note that the point $M$ is given by $(r\cos t, r\sin t)$, and so we find $A$:

$$(r\cos t, y\sin t) + \frac{1}{2}(x_F - r\cos t, y_F - r\sin t) = \frac{1}{2}(x_F + r\cos t, y_F + r\sin t).$$

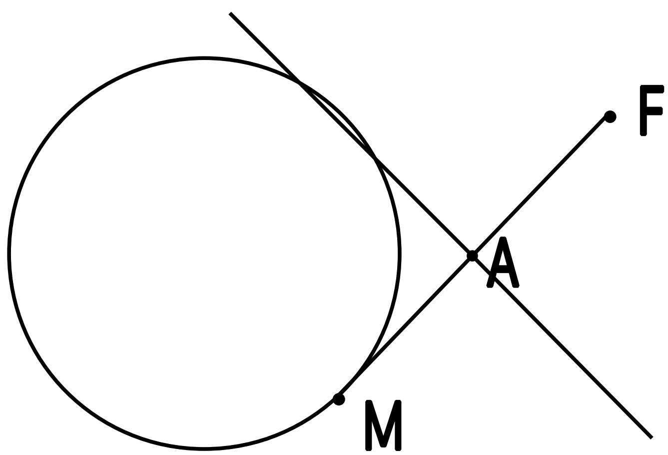



Figure 5: The perpendicular bisector is the line running through $A$, perpendicular to the line $MF$.

Let us now find the equation of the perpendicular bisectors. We first find its direction, using the fact that it is perpendicular to $MF$, and so their dot-product vanishes:

$$0 = (x_F - r\cos t, y_F - r\sin t)\cdot(x, y) = (x_F - r\cos t)x + (y_F - r\sin t)y.$$

The equation for the line should also include the intersection term $c$, i.e. the line is given by

$$(x_F - r\cos t)x + (y_F - r\sin t)y + c = 0.$$

We find $c$ by noting that the point $A$ lies on the line, and so

$$c = -\frac{1}{2}\left((x_F - r\cos t)^2 + (y_F - r\sin t)^2\right).$$

Now the envelope is given by the solution to the following system:

$$\begin{cases} (x_F - r\cos t)x + (y_F - r\sin t)y - \frac{1}{2}\left((x_F - r\cos t)^2 + (y_F - r\sin t)^2\right) = 0 \\ r\sin tx - r\cos ty - x_F r\sin t + y_F r\cos t = 0. \end{cases}$$

The solution is

$$\begin{cases} x = \frac{2y_F^2\cos t - r(1+x_F^2+y_F^2)\cos t + 2x_F(r - y_F\sin t)}{2r(r - x_F\cos t - y_F\sin t)} \\ y = \frac{2ty_F - 2x_F y_F\cos t + 2x_F^2\sin t - r(1+x_F^2+y_F^2)\sin t}{2r(r - x_F\cos t - y_F\sin t)}. \end{cases}$$

An example of the envelope is then given in Figure 6.

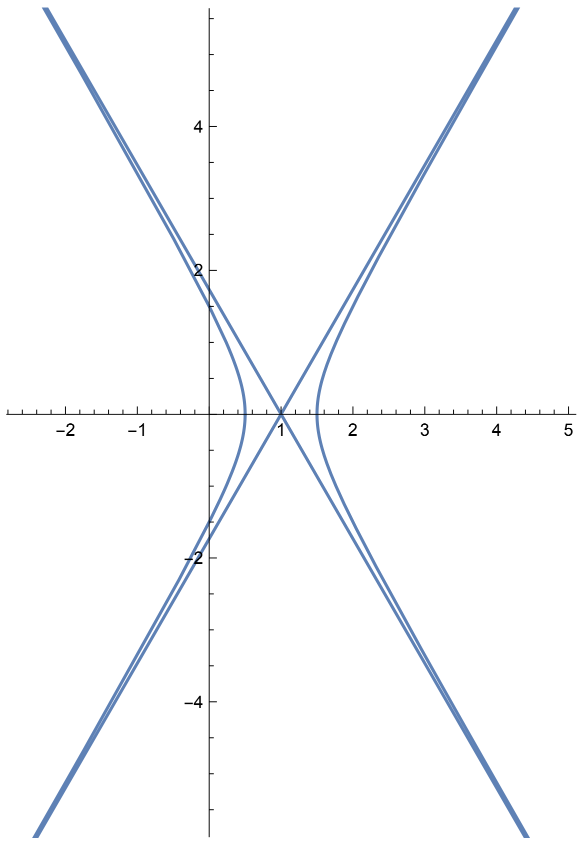



Figure 6: Here the circle is centred at $(0,0)$ and with radius 1. Furthermore, $(x_F, y_F) = (2,0)$.

## Exercise VII.23

Note that a singularity here means a point where the derivative vanishes. Sometimes this is also called a critical point.

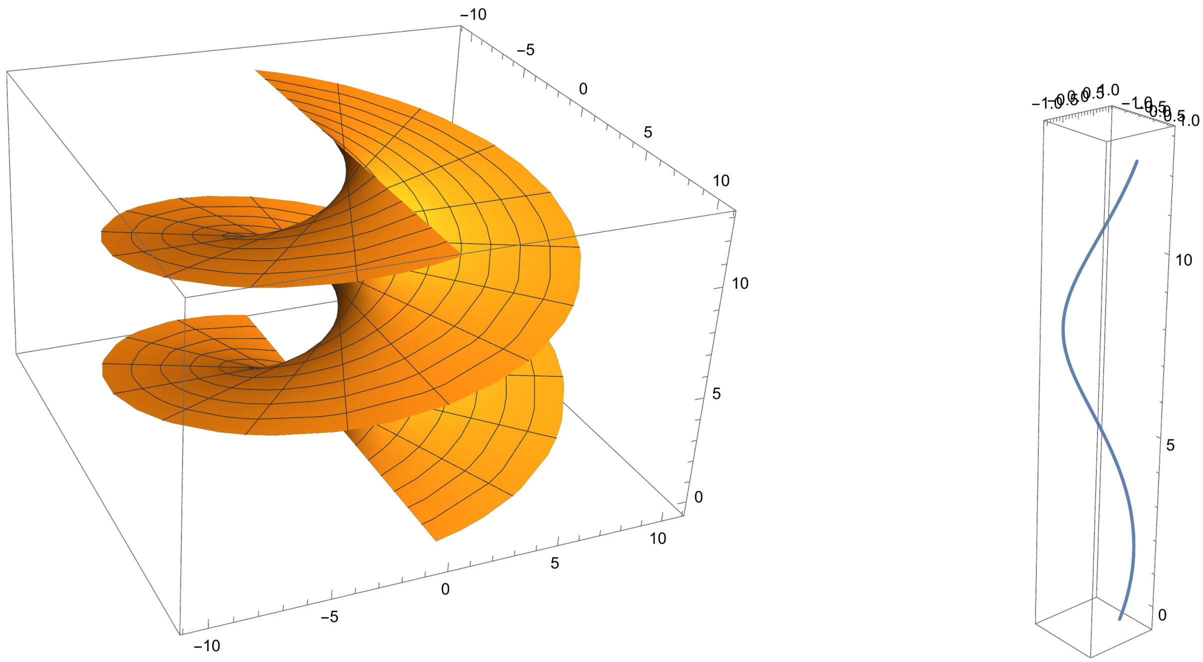
The evolute of a curve $\gamma(t)$ is given by $c(t) = \gamma(t) + n(t)/K(t)$, where $K(t)$ is the curvature, and $n(t)$ is the normal vector to $\gamma(t)$. Consider the differential,

$$\begin{aligned} c'(t) &= \gamma'(t) + \frac{n'(t)K(t) + n(t)K'(t)}{K(t)^2} \\ &= \tau(t) + \frac{-K(t)^2\tau(t) + n(t)K'(t)}{K(t)^2} = \frac{n(t)K'(t)}{K(t)^2} \end{aligned}$$

where $\tau(t) = \gamma'(t)$ is the tangent to $\gamma(t)$, and $n'(t) = -K(t)\tau(t)$. Thus, if $c'(t) = 0$, then so is $K'(t) = 0$.

# 3 Week 3

### Exercise VIII.2




(a) The helicoid with $a = 2$, $t \in \{-10, 10\}$ and $\theta \in \{0, 2\pi\}$.

(b) The helicoid with $a = 2$, $t = 1$ and $\theta \in \{0, 2\pi\}$.

Figure 7: The figures related to Exercise VIII.2. Both figures were created with Mathematica.

The surface (curve) parametrized by

$$(\theta, t) \mapsto (t\cos\theta, t\sin\theta, a\theta)$$

for $t \in (-10, 10)$ ($t = 1$) is drawn in Figure 7a (Figure 7b). We notice that the curves $t = \textit{constant}$ are circular helices.

Let us consider the curve parametrized by $\theta \mapsto (0, 0, a\theta)$, i.e. the $z$-axis. At any point $p = (0, 0, a\theta_0)$ on this curve, we associate a vector $w(p) = (\cos\theta_0, \sin\theta_0, a\theta_0)$. Then $w(p) - p$ defines a straight line given by

$$y\sin\theta_0 + x\cos\theta_0 = 0.$$

We can extend this line in the $x$- and $y$-directions by considering $v(p) = (t\cos\theta_0, t\sin\theta_0, a\theta_0)$ for $t \in \mathbb{R}$. Thus, the lines extending from $p$ draws the surface, it is a ruled surface.

**Exercise VIII.10**

The Whitney umbrella parametrized by $(u, v) \overset{f}{\mapsto} (uv, v, u^2)$ is drawn in Figure 8.

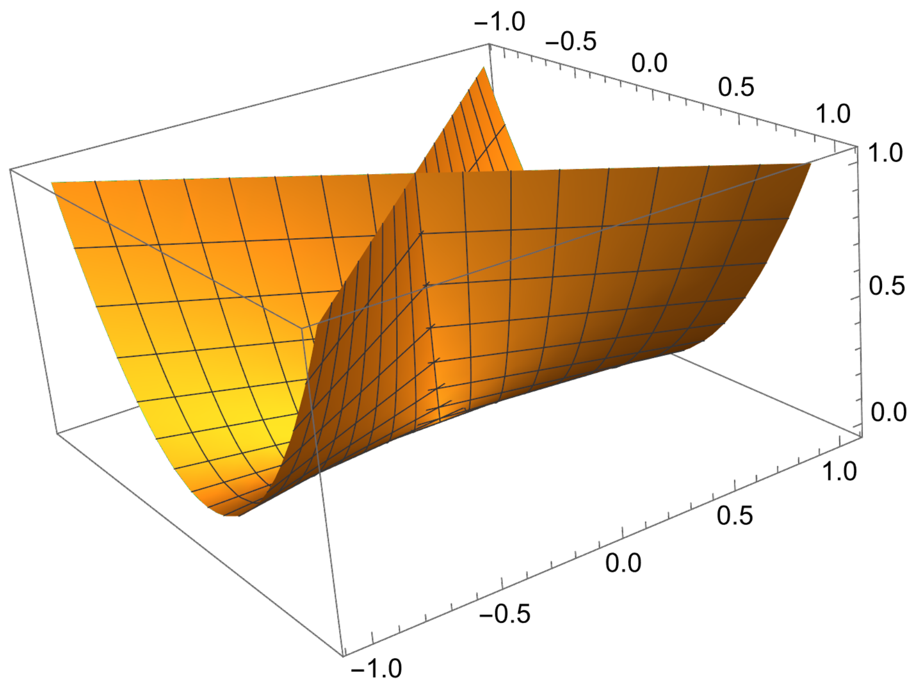



Figure 8: The Whitney umbrella. Figure created with Mathematica.

Note that

$$df|_{u,v} = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix} = \begin{pmatrix} v & u \\ 0 & 1 \\ 2u & 0 \end{pmatrix}.$$

This matrix has rank less than 2 only when $u = v = 0$, the origin in $\mathbb{R}^3$. Thus, the origin is a singular point (of rank 1), and it is the only one.

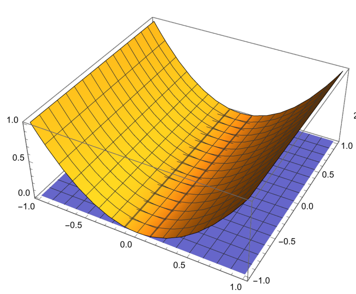
Furthermore, we see that the half-line $(x = y = 0, z > 0)$ is given by $u \neq 0$ and $v = 0$. Thus, there is always two choices giving the same point $(0, 0, u^2)$ on the half-line, namely $\pm u$. Outside the origin, everything is smooth, and so the two values $\pm u$ has to be approached by two different "branches" of the surface. Thus, the half-line consists of only self-intersecting points, in other words, of only double-points. Furthermore, Note that $N(u,v)$ is normal to each direction of $df|_{u,v}$:

$$(v,0,2u)\left(-2\frac{u}{v}, 2\frac{u^2}{v}, 1\right)^T = 0,$$
$$(u,1,0)\left(-2\frac{u}{v}, 2\frac{u^2}{v}, 1\right)^T = 0.$$

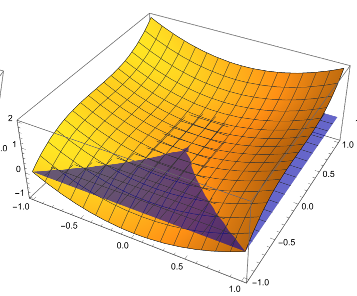
(We could of course also have computed the vector product $(v,0,2u)\times(u,1,0)$.) Thus, $N(u,v)$ is normal to the surface for all $v \neq 0$. As $(u,v)$ tends to 0, then the first column of $df|_{0,0}$ vanishes so all lines are normal to that direction, whilst all lines of the form $(x,0,z)$ are normal to the direction given by the second column. But the normal vector cannot go in multiple directions, only two (also the "negative" direction).

**Exercise VIII.11**

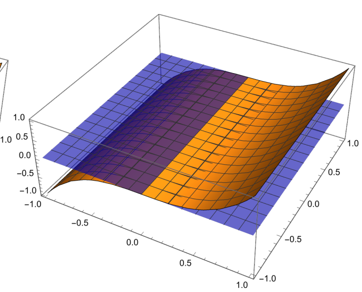
Here, we can find the tangent planes by parametrizing the surfaces by $s$ and $t$. Set $x=s$, $y=t$, and $z=z(s,t)$, i.e. the surface is parametrized by $f(s,t) = (s,t,z(s,t))$. In **(a)**, we then have $z=s^2$. Computing the derivatives with respect to $s$ and $t$, we find that the tangent plane is spanned by $\frac{\partial f}{\partial s} = (1,0,2s)$ and $\frac{\partial f}{\partial t} = (0,1,0)$. We are interested in the origin, i.e. $(s,t)=(0,0)$, and so the tangent plane is the $xy$-plane. The calculation is the same in all cases.



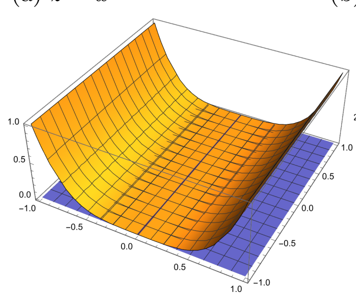
(a) $z = x^2$



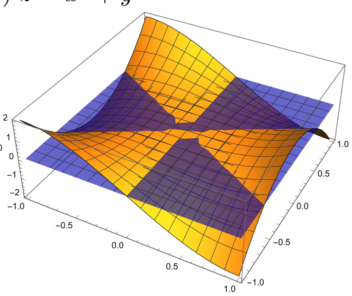
(b) $z = x^2 + y^3$



(c) $z = x^3$



(d) $z = x^4$



(e) $z = x^3 - 3xy^2$

**Exercise VIII.15**

Consider the cone in Figure 10.

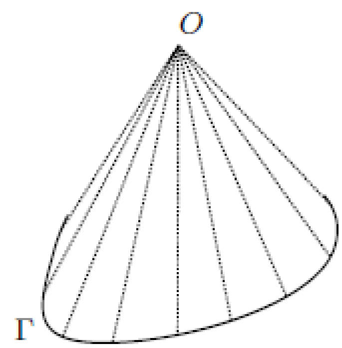



Figure 10: The cone from Audin, VIII Figure 3

This is a ruled surface. Clearly, the normal vector along the lines between $\Gamma$ and $O$ does not change, so the curvature along that line does not change either. More precisely, let us parametrize the curve $\Gamma$ by $t \mapsto \gamma(t)$, where $\gamma : \mathbb{R} \to \mathbb{R}^3$. If we assume $O$ is at the origin, then $\gamma(t)$ also gives us the direction from the curve to $O$, which we denote by $w(t) = \gamma(t)$. If $O$ is not at the origin, then one simply needs to translate $w(t)$, so that $w(t) = O - \gamma(t)$. This will make no difference in the following, so we assume $O = (0, 0, 0)$. Then the cone is parameterized by $(t, u) \mapsto \gamma(t) + uw(t) = (1 + u)\gamma(t) =: f(t, u)$, where $u$ is positive (since $\Gamma$ is the end of the cone). Then

$$\frac{\partial f}{\partial t} = (1+u)\gamma'(t), \quad \frac{\partial f}{\partial u} = \gamma(t).$$

The normal vector is normal to both of these, so we may consider the cross/vector-product:

$$n(t,u) := \frac{\frac{\partial f}{\partial t} \times \frac{\partial f}{\partial u}}{\left|\left|\frac{\partial f}{\partial t} \times \frac{\partial f}{\partial u}\right|\right|} = \frac{(1+u)\gamma'(t) \times \gamma(t)}{||(1+u)\gamma'(t) \times \gamma(t)||} = \frac{\gamma'(t) \times \gamma(t)}{||\gamma'(t) \times \gamma(t)||},$$

where the $1 + u$ part cancels since it is just a positive constant. Thus, $n(t, u)$ does not depend on $u$, and so $\frac{\partial n}{\partial u} = 0$. Now, the Gaussian curvature is defined as $K(p) = \det d_p n(t, u)$. Since the column given by $\frac{\partial n}{\partial u}$ of $d_p n(t, u)$ is zero, the determinant is zero, so the curvature is zero. In fact, this is true for all ruled surfaces.

# 4 Week 4

## Exercise VIII.16

We will find the Gaussian curvature as the determinant of the second fundamental form divided by the determinant of the first funamental form.

The ellipsoid is parametrized by $\gamma(u,v) = (\sqrt{\alpha_1}\cos u \sin v, \sqrt{\alpha_2}\sin u \sin v, \sqrt{\alpha_3}\cos v)$. We need the following derivatives:

$$\begin{aligned}
\frac{\partial \gamma}{\partial u} &= (-\sqrt{\alpha_1}\sin u \sin v, \sqrt{\alpha_2}\cos u \sin v, 0),\\
\frac{\partial \gamma}{\partial v} &= (\sqrt{\alpha_1}\cos u \cos v, \sqrt{\alpha_2}\sin u \cos v, -\sqrt{\alpha_3}\sin v),\\
\frac{\partial^2 \gamma}{\partial u^2} &= -(\sqrt{\alpha_1}\cos u \sin v, \sqrt{\alpha_2}\sin u \sin v, 0),\\
\frac{\partial^2 \gamma}{\partial u \partial v} &= (-\sqrt{\alpha_1}\cos v \sin u, \sqrt{\alpha_2}\cos u \cos v, 0),\\
\frac{\partial^2 \gamma}{\partial v^2} &= -(\sqrt{\alpha_1}\cos u \sin v, \sqrt{\alpha_2}\sin u \sin v, \sqrt{\alpha_3}\cos v).
\end{aligned}$$

Furthermore, the unit normal is given by

$$\begin{aligned}
n(u,v) &= \frac{\frac{\partial \gamma}{\partial u} \times \frac{\partial \gamma}{\partial v}}{\left\|\frac{\partial \gamma}{\partial u} \times \frac{\partial \gamma}{\partial v}\right\|}\\
&= -\frac{(\sqrt{\alpha_2\alpha_3}\cos u \sin^2 v, \sqrt{\alpha_1\alpha_3}\sin u \sin^2 v, \sqrt{\alpha_1\alpha_2}\cos v \sin v)}{\sqrt{\alpha_1\alpha_2\cos^2 v \sin^2 v + \alpha_3(\alpha_2\cos^2 u + \alpha_1 \sin^2 u)\sin^4 v}}.
\end{aligned}$$

Now, let us find the first fundamental form, with the following matrix elements:

$$\begin{aligned}
E &= \frac{\partial \gamma}{\partial u}\cdot\frac{\partial \gamma}{\partial u} = (\alpha_1 \sin^2 u + \alpha_2\cos^2 u)\sin^2 v,\\
F &= \frac{\partial \gamma}{\partial u}\cdot\frac{\partial \gamma}{\partial v} = (\alpha_2 - \alpha_1)\cos u \sin u \cos v \sin v,\\
G &= \frac{\partial \gamma}{\partial v}\cdot\frac{\partial \gamma}{\partial v} = (\alpha_1 \cos^2 u + \alpha_2\sin^2 u)\cos^2 v + \alpha_3 \sin^2 v.
\end{aligned}$$

The elements of the second fundamental form is found next:

$$\begin{aligned}
L &= n \cdot \frac{\partial^2 \gamma}{\partial u^2} = \frac{\sqrt{\alpha_1\alpha_2\alpha_3}\sin^2 v}{\sqrt{\alpha_1\alpha_2\cos^2 v \sin^2 v + \alpha_3(\alpha_2\cos^2 u + \alpha_1 \sin^2 u)\sin^2 v}}.\\
M &= n \cdot \frac{\partial^2 \gamma}{\partial u \partial v} = 0,\\
N &= n \cdot \frac{\partial^2 \gamma}{\partial v^2} = \frac{\sqrt{\alpha_1\alpha_2\alpha_3}}{\sqrt{\alpha_1\alpha_2\cos^2 v \sin^2 v + \alpha_3(\alpha_2\cos^2 u + \alpha_1 \sin^2 u)\sin^2 v}}.
\end{aligned}$$

Then, the Gaussian curvature of the ellipsoid is given by

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\alpha_1\alpha_2\alpha_3}{(\alpha_1\alpha_2 \cos^2 v \sin^2 v + \alpha_3(\alpha_2 \cos^2 u + \alpha_1 \sin^2 u) \sin^2 v)^2}.$$

## Exercise VIII.17

We consider a surface of revolution parametrized by

$$\gamma(s, \theta) := (g(s) \cos\theta, g(s) \sin\theta, h(s)).$$

Let us find the partial derivatives:

$$\frac{\partial\gamma}{\partial s} = (g'(s)\cos\theta, g'(s)\sin\theta, h'(s)), \quad \frac{\partial\gamma}{\partial\theta} = (-g(s)\sin\theta, g(s)\cos\theta, 0).$$

Let us find the normal vector by computing the cross-product:

$$\begin{aligned}\tilde{n}(s,\theta) &= \begin{vmatrix} i & j & k \\ g'(s)\cos\theta & g'(s)\sin\theta & h'(s) \\ -g(s)\sin\theta & g(s)\cos\theta & 0 \end{vmatrix} \\ &= (-h'(s)g(s)\cos\theta, -h'(s)g(s)\sin\theta, g'(s)g(s)).\end{aligned}$$

The length is given by

$$||\tilde{n}(s,\theta)|| = \sqrt{h'^2 g^2 \cos^2\theta + h'^2 g^2 \sin^2\theta + g'^2 g^2} = g(s),$$

and hence the unit normal vector is given by

$$n(s,\theta) = \frac{\tilde{n}(s,\theta)}{||\tilde{n}(s,\theta||} = (-h'(s)\cos\theta, -h'(s)\sin\theta, g'(s)).$$

Let us now compute its partial derivatives (we could also have proceeded as in the previous exercise):

$$\begin{aligned}\frac{\partial n}{\partial s} &= (-h''(s)\cos\theta, -h''(s)\sin\theta, g''(s)) = -\frac{h''(s)}{g'(s)}\frac{\partial\gamma}{\partial s}, \\ \frac{\partial n}{\partial\theta} &= (h'(s)\sin\theta, -h'(s)\cos\theta, 0) = -\frac{h'(s)}{g(s)}\frac{\partial\gamma}{\partial\theta}.\end{aligned}$$

Recall that the tangent vector of the normal vector lies in the tangent plane to the surface, and so the coefficients of the matrix $d|_p n$, where $p$ is some point, are the coefficients of the derivatives of the parametrization $\gamma$ (where I is the first fundamental form, which in this case is 1):

$$d|_p n = \frac{1}{\mathrm{I}} \begin{pmatrix} \frac{\partial n}{\partial s} \cdot \frac{\partial\gamma}{\partial s} & \frac{\partial n}{\partial\theta} \cdot \frac{\partial\gamma}{\partial\theta} \\ \frac{\partial n}{\partial s} \cdot \frac{\partial\gamma}{\partial s} & \frac{\partial n}{\partial\theta} \cdot \frac{\partial\gamma}{\partial\theta} \end{pmatrix}.$$

Then, the Gaussian curvature is given by

$$K = \det \begin{pmatrix} -\frac{h''(s)}{g'(s)} & 0 \\ 0 & -\frac{h'(s)}{g(s)} \end{pmatrix} = \frac{h''(s)h'(s)}{g'(s)g(s)}.$$

Let us now differentiate ${g'}^2 + {h'}^2 = 1$ to obtain $2g'g'' + 2h'h'' = 0$, or in other words $h''(s)h'(s) = -g''(s)g'(s)$, so that

$$K = -\frac{g''(s)}{g(s)}.$$

Now, assume $K$ is constant.

1. $K = 0$: This means $g''(s) = 0$, so that we may write $g(s) = as+b$ for some coefficients $a, b$, i.e., $g(s)$ is a straight line. Furthermore, ${g'}^2 + {h'}^2 = 1$ tells us that $a^2 + {h'}^2 = 1$, so that $h(s) = \pm \int_0^s \sqrt{1-a^2}du = \pm\sqrt{1-a^2}s$. If $a = 0$, then $\gamma(s, \theta)$ draws a cylinder, if $0 < a < 1$, then $\gamma(s, \theta)$ draws a cone, and if $a = 1$, then $\gamma(s, \theta)$ draws a plane.

2. $K > 0$: We solve the following ODE: $g''(s) + Kg(s) = 0$. This has as solutions $g(s) = a\cos(\sqrt{K}s) + b\sin(\sqrt{K}s)$. It follows that $h(s) = \pm \int_0^s \sqrt{1 + a^2K\sin^2(\sqrt{K}u) - b^2K\cos^2(\sqrt{K}u)}du$. If $a = 0$, and $b^2K = 1$, then $h(s) = \pm \int_0^s \sin(\sqrt{K}u)du = \frac{1}{\sqrt{K}}(1-\cos(\sqrt{K}s)$, and $g(s) = \frac{1}{\sqrt{K}}\sin(\sqrt{K}s)$. Plugging this into the equation for $\gamma(s, \theta)$ gives us the standard parameterization for a sphere of radius $\frac{1}{\sqrt{K}}$ and center $(0, 0, \frac{1}{\sqrt{K}})$.

3. $K < 0$: We solve the same ODE as above, but since $K$ is negative, we have as solutions $g(s) = ae^{\sqrt{-K}s} + be^{-\sqrt{-K}s}$. Figure 11 shows the surface as $a = 1$, $b = 0$, $K = -1$, i.e. $g(s) = e^s$ and $h(s) = \pm \int_0^s \sqrt{1-e^{2t}}dt$.

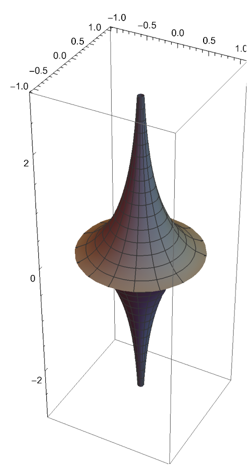



Figure 11: A pseudosphere

## Exercise VIII.19

Recall that the Gaussian curvature is given as the fraction of the determinant of the second fundamental form divided by the determinant of the first fundamental form. We also now that the first fundamental form is positive definite, and so the sign of the Gaussian curvature is decided by the sign of $\det Q_{ij}$, $Q_{ij}$ being the matrix of the second fundamental form. Furthermore, by Corollary 8.2 in chapter 6 of Audin, we may always choose an orthonormal basis, $\{X_i\}$, such that $Q_{ij}$ is a diagonal matrix. Thus, only the diagonal entries, $\mathrm{II}(X_i, X_i)$, of $Q_{ij}$ is necessary to find find, as the determinant is then $\prod_i \mathrm{II}(X_i, X_i)$. Furthermore, $\mathrm{II}(X_i, X_i)$ is just the normal curvature in the direction of $X_i$, $K_{X_i}$. Thus, if we can show that all $K_{X_i}$ are positive, then we are done.

Let $O$ be some point outside of $\Sigma$, and let $x$ be some point on $\Sigma$. Let us define the function $f(x) = ||x||^2$, which is the Euclidean distance from $O$ to $x$ squared. Since $\Sigma$ is compact, $f(x)$ will attain a local maximum on $\Sigma$. Let $p$ be a point for which $f(p)$ is a local maximum.

Note that $df_x = 2x$ (for example, if $\Sigma$ is 3-dimensional and $O = (0, 0, 0)$, then $x = (x_1, x_2, x_3)$, $f(x) = x_1^2 + x_2^2 + x_3^2$, and $df_x = \nabla_x f = 2(x_1, x_2, x_3)$). Let $X$ be a unit vector in $T_p\Sigma$. By the virtue of $p$ being a local maximum, we have that $df_p(X) = 2p \cdot X = 0$, and so $T_p\Sigma = p^\perp$, i.e. the tangent space consists of vectors perpendicular to $p$. Hence, a normal vector is parallel to $p$. We will choose the normal vector $n(p)$ such that it has the opposite direction to $p$, i.e. we may write $p = -\,||p||\, n(p)$.

Let $\gamma(s)$ be a curve on $\Sigma$ generated by intersecting the plane spanned by $X$ and $n(p)$ with $\Sigma$. We take $X$ to be the tangent to $\gamma(0) = p$, i.e. $\gamma'(0) = X$. As we assumed $X$ is a unit vector, this means that $\gamma(s)$ is a parameterisation by arc length. Taylor expansion up to order 2 in $s$ of $\gamma(s)$, and using that $\gamma''(s) = \tau'(s) = K_X(s)n(s)$, yields

$$\begin{aligned}
\gamma(s) &= \gamma(0) + s\gamma'(0) + \frac{s^2}{2}\gamma''(0) + o(s^2) \\
&= p + sX + \frac{s^2}{2}K_X(0)n(0) + o(s^2) \\
&= sX + \left(\frac{s^2}{2}K_X(0) - ||p||\right) n(0) + o(s^2).
\end{aligned}$$

Now, recalling that $X$ and $n(0)$ satisfies $\langle X, X\rangle = 1 = \langle n(0), n(0)\rangle$ and $\langle X, n(0)\rangle = 0$, we find

$$\begin{aligned}
||\gamma(s)||^2 = \langle \gamma(s), \gamma(s)\rangle &= s^2 + \frac{s^4}{4}K_X(0)^2 - s^2\,||p||\,K_X(0) + ||p||^2 \\
&= ||p||^2 + s^2(1 - ||p||\,K_X(0)) + o(s^2)
\end{aligned}$$

At $s = 0$ this is just $f(0)$, which we said is a local maximum. Then, as $s$ increases or decreases, $||\gamma(s)||^2$ must decrease. Thus $(1 - ||p||\,K_X(0)) < 0$, or in other words, $K_X(0) > \frac{1}{||p||} > 0$.

We have now found the normal curvature. In particular, let $X = e_k$, where $e_k$'s span an orthonormal basis for $T_p\Sigma$. Then each $K_{e_k}(0)$ is positive, and the product of all of them, which is equal to the Gaussian curvature $K(0)$, is also positive.

## Exercise VIII.23

The geodesics has parametrizations $f(s)$ such that $f''(s)$ is proportional to the normal vector at $f(s)$.

Let us first consider the cylinder. The cylinder is parameterised by $\gamma(u,v) = (\cos u, \sin u, v)$, so a parameterisation of a curve on the cylinder can be given by $f(s) = (\cos u(s), \sin u(s), v(s))$. The geodesics are given by solutions to $f(s)$ satisfying $f''(s)$ being parallel to the normal vector. Let us then find these elements. The second derivative is found

$$\begin{aligned}
&f'(s) = (-u'(s)\sin u(s), u'(s)\cos u(s), v'(s)) \\
&f''(s) = (-u''(s)\sin u(s) - (u'(s))^2\cos u(s), u''(s)\cos u(s) - (u'(s))^2\sin u(s), v''(s)),
\end{aligned}$$

and then the normal vector

$$\begin{aligned}
\frac{\partial f}{\partial u} &= (-\sin u(s), \cos u(s), 0), \quad \frac{\partial f}{\partial v} = (0,0,1) \\
n(s) &= \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} = (\cos u(s), \sin u(s), 0).
\end{aligned}$$

Note that $|f'(s)|^2 = u'(s)^2 + v'(s)^2$, and so we need to require $u'(s)^2 + v'(s)^2 = 1$ for our parameterisation to be of arc length. Comparing $f''(s)$ with $n(s)$, we see that $u''(s) = 0 = v''(s)$. Thus, $u(s) = a_1 s + b_1$ and $v(s) = a_2 s + b_2$. Furthemore, $u'(s)^2 + v'(s)^2 = a_1^2 + a_2^2$, and so we must have $a_1 = \pm\sqrt{1 - a_2^2}$. The geodesics are given by

$$f(s) = (\cos(a_1 s + b_1), \sin(a_1 s + b_1), (a_2 s + b_2)).$$

These are in general helices. If $a_2 = 0$, i.e. $a_1 = \pm 1$, they are circles, and if $a_1 = 0$, i.e. $a_2 = \pm 1$, they are straight lines.

We can do similarly for the sphere and the plane. The geodesics for the sphere are (parts of) great circles (the sphere intersected with a plane), and for the plane are straight lines.

Let us consider the sphere again. Instead of using the condition that $f''(s)$ is parallel to the normal vector, we can also use the equivalent condition of the geodesic equations, making use of the Christoffel symbols (of the second kind)

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0,$$

where $x^i$ is the $i$-parameter in the parameterisation, for example $x^i = u(s)$. In this notation, each possible value of $i$ will give an equation, and implicitely a

summation over the possible values of $j$ and $k$ is assumed.
(As soon as an index appears twice, once as an upper index and once as a lower one, a summation over the possible values of this index is implied; when an index appears in every term in the same position, we just have multiple equations: one for each value of the index. This way of writing is known as the *Einstein summation convention.*)
In this case, all vectors have two components, so we would get

$$\begin{cases} \ddot{x}^1 + \Gamma^1_{11}(\dot{x}^1)^2 + (\Gamma^1_{12} + \Gamma^1_{21})\dot{x}^1\dot{x}^2 + \Gamma^1_{22}(\dot{x}^2)^2 = 0, \\ \ddot{f}^2 + \Gamma^2_{11}(\dot{x}^1)^2 + (\Gamma^2_{12} + \Gamma^2_{21})\dot{x}^1\dot{x}^2 + \Gamma^2_{22}(\dot{x}^2)^2 = 0. \end{cases} \tag{5}$$

Now, recall that

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left( \frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right),$$

where we are again using the Einstein summation convention. Note that the index $m$ is repeated, we are summing over $m$. Furthermore, $(g_{jk})_{j,k}$ is the matrix of the first fundamental form, with $g_{jk}$ the element in row $j$ and column $k$. If the indices instead are in the upper position, we consider the inverse of the matrix, i.e. $(g^{im})_{i,m} = ((g_{im})_{i,m})^{-1}$. Thus, we need to find the first fundamental form for the sphere, parameterised by

$$\gamma(u, v) = (\cos u \sin v, \cos u \cos v, \sin u).$$

Recall that the first fundamental form is given by

$$(g_{jk})_{j,k} = \begin{pmatrix} \frac{\partial \gamma}{\partial u} \cdot \frac{\partial \gamma}{\partial u} & \frac{\partial \gamma}{\partial u} \cdot \frac{\partial \gamma}{\partial v} \\ \frac{\partial \gamma}{\partial v} \cdot \frac{\partial \gamma}{\partial u} & \frac{\partial \gamma}{\partial v} \cdot \frac{\partial \gamma}{\partial v} \end{pmatrix}.$$

We find

$$\frac{\partial \gamma}{\partial u} = (-\sin u \sin v, -\sin u \cos v, \cos u), \quad \frac{\partial \gamma}{\partial v} = (\cos u \cos v, -\cos u \sin v, 0),$$

and so

$$(g_{jk})_{j,k} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}, \quad ((g_{jk})_{j,k})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^{-2} u \end{pmatrix}.$$

Now we may find the Christoffel symbols. Let $x^1 = u$ and $x^2 = v$. Then, as the matrix of the first fundamental form and its inverse are diagonal, we find

$$\Gamma^1_{11} = \frac{1}{2}\sum_{m=1}^{2} g^{1m}\left(\frac{\partial g_{1m}}{\partial x^1} + \frac{\partial g_{1m}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^m}\right) = \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial u} + \frac{\partial g_{11}}{\partial u} - \frac{\partial g_{11}}{\partial u}\right) = 0,$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}\sum_{m=1}^{2} g^{1m}\left(\frac{\partial g_{1m}}{\partial x^2} + \frac{\partial g_{2m}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^m}\right) = \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial v} + \frac{\partial g_{12}}{\partial u} - \frac{\partial g_{12}}{\partial u}\right) = 0,$$

$$\Gamma^1_{22} = \frac{1}{2}\sum_{m=1}^{2} g^{1m}\left(\frac{\partial g_{2m}}{\partial x^2} + \frac{\partial g_{2m}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^m}\right) = \frac{1}{2}g^{11}\left(\frac{\partial g_{21}}{\partial v} + \frac{\partial g_{21}}{\partial v} - \frac{\partial g_{22}}{\partial u}\right) = -\frac{1}{2}\frac{\partial g_{22}}{\partial u} = \sin u \cos u,$$

$$\Gamma^2_{11} = \frac{1}{2}\sum_{m=1}^{2} g^{2m}\left(\frac{\partial g_{1m}}{\partial x^1} + \frac{\partial g_{1m}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^m}\right) = \frac{1}{2}g^{22}\left(\frac{\partial g_{12}}{\partial u} + \frac{\partial g_{12}}{\partial u} - \frac{\partial g_{11}}{\partial v}\right) = 0,$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}\sum_{m=1}^{2} g^{2m}\left(\frac{\partial g_{1m}}{\partial x^2} + \frac{\partial g_{2m}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^m}\right) = \frac{1}{2}g^{22}\left(\frac{\partial g_{12}}{\partial v} + \frac{\partial g_{22}}{\partial u} - \frac{\partial g_{12}}{\partial v}\right) = \frac{1}{2}g^{22}\frac{\partial g_{22}}{\partial u} = -\tan u,$$

$$\Gamma^2_{22} = \frac{1}{2}\sum_{m=1}^{2} g^{2m}\left(\frac{\partial g_{2m}}{\partial x^2} + \frac{\partial g_{2m}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^m}\right) = \frac{1}{2}g^{22}\left(\frac{\partial g_{22}}{\partial v} + \frac{\partial g_{22}}{\partial v} - \frac{\partial g_{22}}{\partial v}\right) = 0.$$

Now, we may plug this into the equations (5):

$$\begin{cases} \ddot{u} + \Gamma^1_{22}\dot{v}^2 = \ddot{u} + \dot{v}^2 \sin u \cos u = 0 \\ \ddot{v} + (\Gamma^2_{12} + \Gamma^2_{21})\dot{u}\dot{v} = \ddot{v} - 2\dot{u}\dot{v}\tan u = 0. \end{cases}$$

Note that, for example, the choices $u(s) = s$ and $v(s) = 0$ indeed satisfy the geodesic equations. Furthermore, they satisfy

$$f(s) = (0, \cos s, \sin s),$$

i.e. a parameterisation of a circle. One can show that in fact, circles are the only options (*expand on this*).

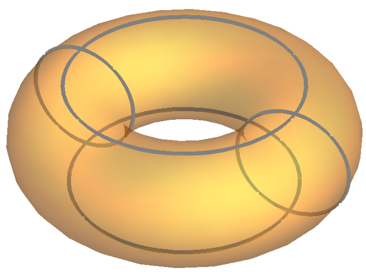
# 5 Week 5

## Coxeter Exercises 20.4.1-2

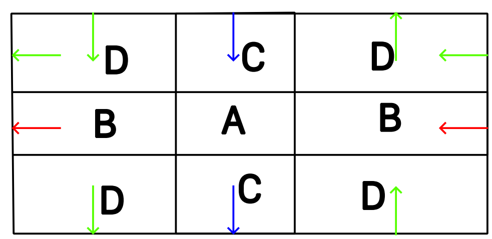
The integral curvature of the torus can be found by considering the geodesics given by the two circles generating the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Then the number of geodesic edges is $E = 2$, vertices $V = 1$, and faces $F = 1$. Thus,

$$\int_{\mathbb{T}^2} K\,dS = 2\pi(V + F - E) = 0.$$

This is independent of the choice of $V, F, E$, as long as each face is simply connected, i.e. each face can be contracted to a point. We can choose four geodesic arcs such that $V = 4$, $F = 4$, and $E = 8$ as in Figure 12.



(a) The torus as $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.




(b) The torus as a rectangle, where the letters show which faces are identified, and the arrows how each face is connected.

Figure 12: Four geodesics on a torus.

## Coxeter Exercise 20.7.1

We will find the geodesics of the pseudosphere by finding the the curves satisfying $\gamma''(s) \parallel n(s)$.

The pseudosphere is parameterised by

$$(u, v) \mapsto \Gamma(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, u - \tanh u),$$

and any curve on the pseudosphere is parameterised by $s \mapsto \gamma(s) = \Gamma(u(s), v(s))$. We find that

$$\begin{aligned}
\frac{\partial \Gamma}{\partial u} &= (-\operatorname{sech} u \tanh u \cos v, -\operatorname{sech} u \tanh u \sin v, 1 - \operatorname{sech}^2 u), \\
\frac{\partial \Gamma}{\partial v} &= (-\operatorname{sech} u \sin v, \operatorname{sech} u \cos v, 0).
\end{aligned}$$

Then we find a normal vector

$$\tilde{n} = \frac{\partial \Gamma}{\partial u} \times \frac{\partial \Gamma}{\partial v} = \left(-\operatorname{sech} u \tanh^2 u \cos v, -\operatorname{sech} u \tanh^2 u \sin v, -\operatorname{sech}^2 u \tanh u\right).$$

Notice that all terms multiply $-\operatorname{sech} u \tanh u$, and so we may simply scale by this, as we are only interested in its direction:

$$n = (\tanh u \cos v, \tanh u \sin v, \operatorname{sech} u).$$

Furthermore, we find (where the dependence on $s$ is assumed implicit)

$$\begin{aligned}\gamma'' = \Big( &\operatorname{sech} u \big(2u'^2 \tanh^2 u - u'^2 - v'^2 - u'' \tanh u\big) \cos v + \big(2u'v' \tanh u - v''\big) \sin v, \\ &\operatorname{sech} u \big(2u'^2 \tanh^2 u - u'^2 - v'^2 - u'' \tanh u\big) \sin v - \big(2u'v' \tanh u - v''\big) \cos v, \\ &\big(2u'^2 \operatorname{sech}^2 u + u'' \tanh u\big) \tanh u \Big).\end{aligned}$$

Comparing $\gamma''$ with $n$, we see that in the first component the term multiplying $\sin v$ must vanish. Likewise, the term multiplying $\cos v$ in the second component must vanish. This yields, in both cases,

$$v'' - 2u'v' \tanh u = 0.$$

Furthermore, if we assume $\gamma''$ has been scaled so that it is equal to $n$, then we must have that the term multiplying $\cos v$ in the first component, and the term multiplying $\sin v$ in the second component satisfy

$$\operatorname{sech} u \big(2u'^2 \tanh^2 u - u'^2 - v'^2 - u'' \tanh u\big) = \tanh u.$$

The third component must be

$$\big(2u'^2 \operatorname{sech}^2 u + u'' \tanh u\big) \tanh u = \operatorname{sech} u.$$

Multiplying both sides in the last equality by $\sinh u$ makes the right hand side into $\tanh u$, the same as the right hand side in the previous equation, and we may compare them:

$$\operatorname{sech} u \big(2u'^2 \tanh^2 u - u'^2 - v'^2 - u'' \tanh u\big) = \big(2u'^2 \operatorname{sech}^2 u + u'' \tanh u\big) \tanh u \sinh u.$$

Since $\operatorname{sech} u \sinh u = \tanh u$, the first term on both sides cancel, and we are left with, after dividing by $\operatorname{sech} u$ on both sides,

$$-u'^2 - v'^2 - u'' \tanh u = u'' \tanh u \sinh^2 u.$$

This can be written

$$u'' \tanh u (1 + \sinh^2 u) = -u'^2 - v'^2,$$

or, since $1 + \sinh^2 u = \cosh^2 u$,

$$u'' \tanh u = -\operatorname{sech}^2 u(u'^2 + v'^2).$$

Thus, the geodesics of the pseudosphere are given by solutions to

$$\begin{cases} u'' \tanh u + \operatorname{sech}^2 u(u'^2 + v'^2) = 0, \\ v'' - 2u'v' \tanh u = 0. \end{cases}$$

This system of differential equations is not easy to solve. We can instead try to show that the pseudosphere is isometric to another model, for which we can solve the geodesic equations. We find the metric for the pseudosphere:

$$\frac{\partial \Gamma}{\partial u} \cdot \frac{\partial \Gamma}{\partial u} = \tanh^2 u, \quad \frac{\partial \Gamma}{\partial u} \cdot \frac{\partial \Gamma}{\partial v} = 0 = \frac{\partial \Gamma}{\partial v} \cdot \frac{\partial \Gamma}{\partial u}, \quad \frac{\partial \Gamma}{\partial v} \cdot \frac{\partial \Gamma}{\partial v} = \operatorname{sech}^2 u.$$

Thus, the metric is

$$g_{\mathrm{ps}}(u, v) = \tanh^2 u \, du^2 + \operatorname{sech}^2 u \, dv^2 = \frac{\sinh^2 u \, du^2 + dv^2}{\cosh^2 u}.$$

We consider the right-half plane model in hyperbolic geometery, $\mathbb{H} = \{x + iy \in \mathbb{C} : x > 0\}$, for which the metric is given by $g_{\mathbb{H}}(x, y) = \frac{1}{x^2}(dx^2 + dy^2)$. Consider the function $f : \mathbb{R}^2 \to \mathbb{H}$ defined by $f(u, v) = (\cosh u, v)$. Then

$$f^* g_{\mathbb{H}}(u, v) = \frac{1}{\cosh^2 u}(d(\cosh u)^2 + dv^2) = \frac{1}{\cosh^2 u}(\sinh^2 u \, du^2 + dv^2) = g_{\mathrm{ps}}(u, v).$$

Thus, $f$ defines an isometry between the pseudosphere and the right-half plane hyperbolic model. Now, one can show (homework) that the geodesics in the right-half plane model are either

1. lines parameterised by $(s, c)$, where $c$ is a constant and $s \in (0, \infty)$, or
2. half-circles parameterised by $(\cos s, \sin s)$ where $s \in (-\pi/2, \pi/2)$.

Thus, the same geodesics on $\mathbb{R}^2$ are given by

1. $f^{-1}(s, c) = (\operatorname{arccosh} s, c)$, or
2. $f^{-1}(\cos s, \sin s) = (\operatorname{arccosh}(\cos s), \sin s)$.

Finally, on the pseudosphere, the geodesics are given by

1. $\gamma(s) = (\operatorname{sech}(\operatorname{arccosh} s) \cos c, \operatorname{sech}(\operatorname{arccosh} s) \sin(c), \operatorname{arccosh} s - \tanh(\operatorname{arccosh} s))$, or
2. $\gamma(s) = (\operatorname{sech}(\operatorname{arccosh}(\cos s)) \cos(\sin s), \operatorname{sech}(\operatorname{arccosh}(\cos s)) \sin(\sin s), \operatorname{arccosh}(\cos s) - \tanh(\operatorname{arccosh}(\cos s)))$.

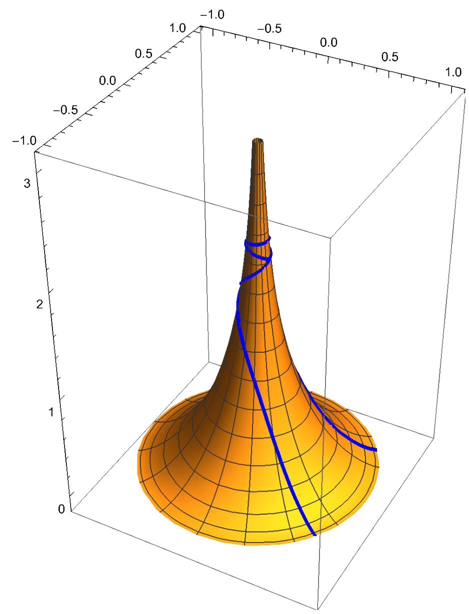


Figure 13: A geodesic on the pseudosphere.

# 6 Week 6

## Exercise I.9

Let $A$ be an $(m \times n)$-matrix with entries in a field $\mathbb{K}$ and $B$ a $(m \times 1)$-matrix (a column vector) in $\mathbb{K}^m$. Define the set $\mathcal{F} = \{X \in \mathbb{K}^n : AX = B\}$, and a function $f : \mathbb{K}^n \to \mathbb{K}^m$ by $f(X) = AX$. If $f(X) = AX = B$, then $X \in \mathcal{F}$. In particular, $\mathcal{F} = f^{-1}(B)$, and as is shown in Example 1.8.2 in chapter 1 of Audin, the level sets, $f^{-1}(B)$, are affine subspaces (of $\mathbb{K}^n$) directed by $\ker f$. Note that, for any $B \in \mathbb{K}^m$ $f^{-1}(B) = A^{-1}B$. However, $A^{-1}$ does not always exist. A necessary condition (but not sufficient) for the matrix $A$ to have a left inverse is that $n \leq m$. So, if $n > m$, then $\mathcal{F}$ is empty. Finally, we find the dimension of $\mathcal{F}$, which is equal to the dimension of its direction, using the rank-nullity theorem, $\dim \mathcal{F} = \dim \ker f = m - \operatorname{rank} A$.

## Exercise I.11

We consider the following systems of equations:

$$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \end{cases} \quad \text{and} \quad \begin{cases} a_1'x + b_1'y + c_1'z = 0 \\ a_2'x + b_2'y + c_2'z = 0. \end{cases}$$

Each equation defines a plane through the origin, and their intersection, also through the origin, is a line in $\mathbb{K}^3$ (as long as the planes are not the same). The planes are different only when their normal vectors are not parallel. Furthermore, the normal vectors are given by the coefficients of $x$, $y$ and $z$ (note that a curve on the plane is given by $ax(t) + by(t) + cz(t) = 0$, and so the derivative satisfies $ax'(t) + by'(t) + cz'(t) = (a, b, c) \cdot (x'(t), y'(t), z'(t)) = 0$, where $(x'(t), y'(t), z'(t))$ is a tangent vector). Thus, we need that $(a_1^{(\prime)}, b_1^{(\prime)}, c_1^{(\prime)}) \neq \lambda(a_2^{(\prime)}, b_2^{(\prime)}, c_2^{(\prime)})$ for some $\lambda \in \mathbb{K}$.

Let us now find an equation for the lines, i.e. the intersection of the planes. We find an expression for, say, the $z$-coordinate of the second equation for each system, and insert it into the first equation in the same system:

$$\left( a_1^{(\prime)} - a_2^{(\prime)} \frac{c_1^{(\prime)}}{c_2^{(\prime)}} \right) x + \left( b_1^{(\prime)} - b_2^{(\prime)} \frac{c_1^{(\prime)}}{c_2^{(\prime)}} \right) y = 0.$$

Let $A^{(\prime)}$ denote the coefficient in front of $x$ and $B^{(\prime)}$ denote the coefficient in front of $y$. If $\frac{A}{B} = \frac{A'}{B'}$, then the two lines will have the same slope, and so the systems define the same lines.

Now consider the following systems of equations:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases} \quad \text{and} \quad \begin{cases} a_1'x + b_1'y + c_1'z = d_1' \\ a_2'x + b_2'y + c_2'z = d_2'. \end{cases}$$

If the underlying vector space defines a line, then it can also be viewed as an affine line. We follow the same reasoning as above, and arrive at the same conclusion, i.e. we need that $(a_1^{(\prime)}, b_1^{(\prime)}, c_1^{(\prime)}) \neq \lambda(a_2^{(\prime)}, b_2^{(\prime)}, c_2^{(\prime)})$ for some $\lambda \in \mathbb{K}$.

Two affine lines are parallel if they have the same direction. This happens when the planes defines the same lines. Thus, using the same method as above, now with constant terms $d_i^{(\prime)}$, $i = 1, 2$, we get

$$\left(a_1^{(\prime)} - a_2^{(\prime)} \frac{c_1^{(\prime)}}{c_2^{(\prime)}}\right) x + \left(b_1^{(\prime)} - b_2^{(\prime)} \frac{c_1^{(\prime)}}{c_2^{(\prime)}}\right) y = d_1^{(\prime)} - d_2^{(\prime)} \frac{c_1^{(\prime)}}{c_2^{(\prime)}}.$$

However, the slope determines the line, so we still only need $\frac{A}{B} = \frac{A'}{B'}$.

## Exercise I.12

We are given an affine line $\mathcal{D} \subset \mathcal{P}$ where $\mathcal{P}$ is an affine plane. Without loss of generality, we can choose an origin $O$ for $\mathcal{P}$ in $\mathcal{D}$, and take $\mathcal{D}$ to be the $x$-axis of the resulting plane. Then we want to find a linear map $f : P \to D$ such that $f(\overrightarrow{OM}) = \overrightarrow{\pi(O)\pi(M)}$. By construction, $O \in \mathcal{D}$, so that $\pi(O) = O$. It follows that we can take $f : P \to D \subset P$ to be the linear map of vector spaces which projects a vector $v \in P$ to $D$ in the direction of $d'$. In particular, if the slope of $d' = a$, then, for some point $(x, y) \in P$, we take $f(x, y) = (x - ay, 0)$.

In the above construction, we did not make use of the dimensionality of $\mathcal{D}$ or $\mathcal{P}$. The same argument works for projection to $\mathcal{F} \subset \mathcal{E}$ in the direction of $G$. One simply takes $d' = G$, $\mathcal{D} = \mathcal{F}$ and $\mathcal{P} = \mathcal{E}$.

If $\pi^2 = \pi$, then the corresponding linear mapping $f$ satisfies $f^2 = f$. From linear algebra, we know that this means $f$ is a projection. It follows that $\pi$ is an affine projection in the above sense.

## Exercise I.22

Suppose we are given an affine map $f : \mathcal{E} \to \mathcal{E}'$. We change our affine frame in $\mathcal{E}$ by an affine transformation $\phi : \mathcal{E} \to \mathcal{E}$, and in $\mathcal{E}'$ by $\psi : \mathcal{E}' \to \mathcal{E}'$. Then we want to find an expression for $g$, in the diagram below:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "f"] \arrow[d, "\phi"'] & \mathcal{E}' \arrow[d, "\psi"] \\ \mathcal{E} \arrow[r, "g"'] & \mathcal{E}' \end{tikzcd}$$

Evidently, $g = \psi \circ f \circ \phi^{-1}$. Now we need to translate this into a coordinate expression. We take

$$\begin{aligned} f(X) &= A_f X + B_f, \\ \phi(X) &= A_\phi X + B_\phi, \\ \psi(X') &= A_\psi X' + B_\psi. \end{aligned}$$

We need to find an inverse to $\phi$. Suppose $\phi^{-1}(Y) = A_{\phi^{-1}}Y + B_{\phi^{-1}}$. Then $\phi^{-1} \circ \phi(X) = A_{\phi^{-1}}(A_\phi X + B_\phi) + B_{\phi^{-1}}$. We deduce that $A_{\phi^{-1}} = A_\phi^{-1}$, and $B_{\phi^{-1}} = -A_{\phi^{-1}}B_\phi = -A_\phi^{-1}B_\phi$. In other words, $\phi^{-1}(Y) = A_\phi^{-1}(Y) - A_\phi^{-1}B_\phi$. We conclude that

$$g(X) = A_\psi(A_f(A_\phi^{-1}(X) - A_\phi^{-1}B_\phi) + B_f) + B_\psi.$$

## Exercise I.23

If a linear function $f : E \to E$ is collinear, then it means that, for any $x \in E$, then $f(x)$ lies on the same line as $x$. Written in mathematical symbols, $\forall x \in E$, $\exists \lambda \in \mathbb{K}$ such that $f(x) = \lambda x$. Here $\mathbb{K}$ is the field on which $E$ is defined.

A linear dilation is simply "expanding" the vector. In mathematical symbols: $\exists \lambda \in \mathbb{K}$ such that $\forall x \in E$ we have $f(x) = \lambda x$.

The difference is that for a dilation there exists $\lambda$ which is the same for all $x \in E$. We will now show that collinear functions are in fact dilations. Let us first consider the case when $\dim E = 1$. Then, if $x, y \in E$ we can write $y = \alpha x$ for some $\alpha \in \mathbb{K}$. Then we get the following:

$$\begin{aligned} f(x+y) &= \lambda_{x+y}(x+y) = (\alpha+1)\lambda_{x+y}x \\ f(x) + f(y) &= f(x) + \alpha f(x) = (\alpha+1)\lambda_x x. \end{aligned}$$

Since $f(x+y) = f(x)+f(y)$ we must have $\lambda_{x+y} = \lambda_x$, and so $\lambda_x$ cannot depend on $x$. Now, consider the case when $\dim E \geq 2$, so that there exists vectors $x, y \in E$ which are linearly independent. By a similar calculation as above, we get $(\lambda_{x+y} - \lambda_x)x + (\lambda_{x+y} - \lambda_y)y = 0$. Since $x$ and $y$ are linearly independent we must have $\lambda_{x+y} = \lambda_x$ and $\lambda_{x+y} = \lambda_y$. In particular, we have $\lambda_x = \lambda_y$, so, again, $\lambda_x$ does not depend on $x$. Thus, $f$ is a dilation.

# 7 Week 7

## Exercise III.7

One identifies $\mathbb{R}^2 \cong \mathbb{C}$. Take some $\mathbb{C} \ni z = x + iy$.

Clearly, if one translates some point in $(x, y) \in \mathbb{R}^2$ by $(b_1, b_2)$, then the translation of $z$ is simply $z \mapsto z + b$, where $b = b_1 + ib_2$. This is indeed an isometry, as $||(z_1 + b) - (z_2 + b)|| = ||z_1 - z_2||$.

A rotation about the origin in $\mathbb{R}^2$ is given by the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

which has determinant 1. The rotation of $(x, y)$ about the origin is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}.$$

In complex numbers this can be written $z \mapsto e^{i\theta} z$, and $\left|e^{i\theta}\right| = 1$. If we want to rotate about any point, we may first translate to the origin, rotate about the origin, and then translate back, i.e. $z \mapsto z - b' \mapsto a(z - b') \mapsto a(z - b') + b' = az + b$, where $a$ is the rotation with $|a| = 1$, and $b = b' - ab'$. This is indeed an isometry:

$$||az_1 + b - (az_2 + b)|| = ||az_1 - az_2|| = |a|\,||z_1 - z_2|| = ||z_1 - z_2||\,.$$

A reflection about a general line can be done by following these steps:

1. Translate the reflection line so that it passes through the origin.
2. Rotate the reflection line so that it coincide with the $x$-axis.
3. Conjugate the point you want to reflect.
4. Apply the inverse rotation from point 2.
5. Translate back to the start.

This gives us, using the notation from above,

$$z \mapsto z - b' \mapsto a'(z - b') \mapsto \overline{a'(z - b')} \mapsto a'^{-1}\overline{a'(z - b')} \mapsto a'^{-1}\overline{a'(z - b')} + b' = a\overline{z} + b.$$

where $a = a'^{-1}\overline{a'}$ with $|a| = 1$, and $b = b' - a\overline{b'}$. Note that $a'^{-1} = \overline{a'}$, as $(e^{i\theta})^{-1} = e^{-i\theta} = \overline{e^{i\theta}}$. Note also that doing the same reflection twice is just the identity. Thus,

$$\begin{aligned} a\overline{(a\overline{z} + b)} + b &= z \\ a(\overline{a}z + \overline{b}) + b &= z \\ |a|^2 z + a\overline{b} + b &= z \\ a\overline{b} + b &= 0. \end{aligned}$$

This is also an isometry:

$$||a\overline{z}_1 + b - (a\overline{z}_2 + b)|| = ||\overline{z}_1 - \overline{z}_2|| = ||z_1 - z_2|| \, ,$$

as $||\overline{z}|| = \sqrt{\overline{z}z} = \sqrt{z\overline{z}} = ||z||$.

Glide reflections are reflections composed with translations in the direction of the axis of reflection, and so we get $z \mapsto a\overline{z} + b$. By the discussion about reflections, and the fact that applying this twice is not the identity, we must have $a\overline{b} + b \neq 0$. Clearly this is also an isometry, by the same computation as above.

Every isometry can be written as a composition of an orthogonal matrix and a translation. The orthogonal matrices are of either of the following shapes:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Note that these corresponds to the following equations in complex numbers: $e^{i\theta}z$ and $e^{i\theta}\overline{z}$. Together with the translations, we find exactly the isometries discussed above.

## Exercise III.8

The affine transformations are given by $f(X) = AX + B$, where $A \in \mathrm{GL}(2, \mathbb{R})$, and $B \in \mathbb{R}^2$. In matrix form it is

$$f(X) = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1x + a_3y + b_1 \\ a_2x + a_4y + b_2 \end{pmatrix}.$$

Note that we make the following identification when going to complex numbers:

$$z = x + iy \cong \begin{pmatrix} x \\ y \end{pmatrix},$$

and so we want

$$f(z) = a_1x + a_3y + b_1 + i(a_2x + a_4y + b_2). \tag{6}$$

Furthermore, note that $2x = z + \bar{z}$ and $2y = -i(z - \bar{z})$. Hence,

$$a_1x = \frac{a_1}{2}(z + \bar{z}), \quad a_3y = -i\frac{a_3}{2}(z - \bar{z}),$$
$$ia_2x = i\frac{a_2}{2}(z + \bar{z}), \quad ia_4y = \frac{a_4}{2}(z - \bar{z}).$$

Thus, (6) can be written

$$\begin{aligned} f(z) &= \frac{1}{2}\left((a_1 + a_4) + i(a_2 - a_3)\right)z + \frac{1}{2}\left((a_1 - a_4) + i(a_2 + a_3)\right)\bar{z} + b_1 + ib_2 \\ &= c_1z + c_2\bar{z} + d. \end{aligned}$$

Note that $|c_1|^2 = (a_1 + a_4)^2 + (a_2 - a_3)^2$ and $|c_2|^2 = (a_1 - a_4)^2 + (a_2 + a_3)^2$, and so

$$|c_1|^2 - |c_2|^2 = 4a_1a_4 - 4a_2a_3 = 4\det A \neq 0,$$

since $A$ is invertible.

## Exercise III.61

We will show that, if we fix two circles $C$ and $C'$, with radius $r$ and $r'$ respectively, then we may always find an inversion that maps either the two circles to two lines, or they are mapped to two concentric circles.

We recall the definition of an inversion. Let $O$ be the centre of the circle of inversion, which has radius $R$. Then the inversion maps some point $P$ to

$$I(O,R)(P) = O + \frac{R^2}{\left|\left|\overrightarrow{OP}\right|\right|^2}\overrightarrow{OP}.$$

Let us first show that this maps a circle to either another circle or to a line.

First, we assume that the centre of inversion $O$ is not located on $C$. In this case, we will show that the inversion is a circle. It is convenient to use complex coordinates $z \in \mathbb{C}$, in which the equation of a circle with centre $a$ and radius $r$ becomes $||z-a|| = r$. This can be expanded as follows:

$$||z-a||^2 = r^2 \Leftrightarrow (z-a)\overline{(z-a)} = r^2 \Leftrightarrow ||z||^2 - z\bar{a} - \bar{z}a + ||a||^2 - r^2 = 0. \quad (7)$$

Note also that the inversion of a complex number, with the centre of inversion $z_0 \in \mathbb{C}$, is

$$I(z_0,R)(z) = z_0 + \frac{R^2}{||z-z_0||^2}(z-z_0) = z_0 + \frac{R^2}{\overline{z-z_0}}.$$

In fact, we will simply assume that $z_0 = 0$, and translate $C$ if necessary. Thus, $I(0,R)(z) = \frac{R^2}{\bar{z}} =: w$, and $\bar{w} = \frac{R^2}{z}$. Now we multiply the equation of the circle (7) by $\frac{R^2}{||z||^2}$, which yields

$$\begin{aligned} R^2\left(1 - \frac{\bar{a}}{\bar{z}} - \frac{a}{z} + \frac{||a||^2 - r^2}{||z||^2}\right) &= 0 \\ \Rightarrow R^2 - \bar{a}w - a\bar{w} + \frac{(||a||^2 - r^2)}{R^2}\,||w||^2 &= 0. \end{aligned}$$

This is again of the same form as the equation for a circle (7), and so the inverted points $w$ define a circle.

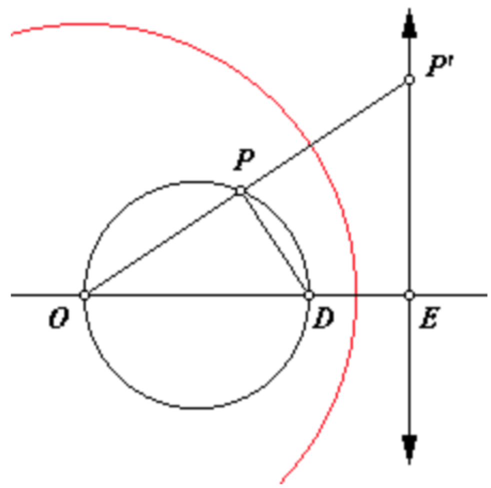



Figure 14

Next, assume that the centre of inversion is a point on the circle that is being inverted, cf. 14. We show that the inversion of the circle is a line. Indeed, by definition of the inversion, $\overrightarrow{OD} \cdot \overrightarrow{OE} = R^2 = \overrightarrow{OP} \cdot \overrightarrow{OP'}$, and since $O, D, E$ and $O, P, P'$ are collinear, we have $\left\|\overrightarrow{OD}\right\| \left\|\overrightarrow{OE}\right\| = \left\|\overrightarrow{OP}\right\| \left\|\overrightarrow{OP'}\right\|$ by the scalar product formula. This yields

$$\frac{\left\|\overrightarrow{OD}\right\|}{\left\|\overrightarrow{OP}\right\|} = \frac{\left\|\overrightarrow{OP'}\right\|}{\left\|\overrightarrow{OE}\right\|},$$

i.e. the length of two sides in the two triangles $\Delta POD$ and $\Delta P'OE$ are proportional to each other. Furthermore, the angle $POD$ is shared between the two triangles. Thus, the two triangles are similar, and one can show that the angle $OPD$ has to be the same as $OEP'$. Furthermore, as the angle $OPD$ is inscribed in a semi-circle, it is a right angle, and so the angle $OEP'$ is a right angle. But this means that the curve given by all $P'$ must be a straight line, as we wanted.

Finally, we need to consider the case of two circles. For the inversion to yield two lines, we must have that the two circle $C$ and $C'$ intersect, and the centre of inversion be one of the intersection points. All that remains is to show that we may choose the centre of inversion such that the inversion yields two concentric circles. Let us choose a coordinate system which puts the centre of $C$ at $(0, 0)$, and the centre of $C'$ at $(d, 0)$. It is clear that also the centre of inversion must lie on the same line as the centre of $C$ and $C'$, i.e. on the $x$-axis. We denote the centre of inversion by $(x_0, 0)$. Then the inversion of some point $(x, y)$ is given by

$$I(x_0, R)(x, y) = x_0 + \frac{R^2(x - x_0)}{(x - x_0)^2 + y^2}.$$

Let $x_C^-$ be the point on $C$ furthest away from $x_0$, and $x_C^+$ the point on $C$ closest to $x_0$. Note that $x_C^+$ will be mapped to the point on the inverted circle closest to $x_0$, and $x_C^-$ to the point on the invreted circle furthest away from $x_0$. Furthermore, $x_C^+ = x_C^- + 2r$, $r$ being the radius of $C$. The inversion, where we omit writing the $y$-component, as everything takes place on the $x$-axis, is

$$\begin{aligned} I(x_0, R)(x_C^-) &= x_0 + \frac{R^2}{x_C^- - x_0}, \\ I(x_0, R)(x_C^+) &= I(x_0, R)(x_C^- + 2r) = x_0 + \frac{R^2}{x_C^- + 2r - x_0}. \end{aligned}$$

The centre of the inverted circle is then the midpoint of those two points:

$$\begin{aligned} &\frac{1}{2}\left(I(x_0, R)(x_C^-) + I(x_0, R)(x_C^+)\right) \\ &= x_0 + \frac{1}{2}\frac{R^2(x_C^- + 2r - x_0)}{(x_C^- - x_0)(x_C^- + 2r - x_0)} + \frac{1}{2}\frac{R^2(x_C^- - x_0)}{(x_C^- + 2r - x_0)(x_C^- - x_0)} \\ &= x_0 + \frac{R^2(x_C^- + r - x_0)}{(x_C^- + r - x_0) - r^2}. \end{aligned}$$

Now, we may simply translate the setup such that we may always choose $x_C^- = 0$. We do similarly for $C'$, and choose $x_{C'}^- = d$. Then we have

$$\begin{aligned} I(x_0, R)(C)_{\text{centre}} &= x_0 - \frac{R^2 x_0}{x_0^2 - r^2}, \\ I(x_0, R)(C')_{\text{centre}} &= x_0 + \frac{R^2(d - x_0)}{(d - x_0)^2 - r'^2}. \end{aligned}$$

The centres should be the same, so we solve the following equation $I(x_0, R)(C)_{\text{centre}} = I(x_0, R)(C')_{\text{centre}}$ for $x_0$, and find

$$x_0 = \frac{d^2 + r^2 - r'^2 \pm \sqrt{(d^2 + r^2 - r'^2)^2 - 4d^2r^2}}{2d}.$$

Note that this is a real number as long as what is under the square root is non-negative:

$$\begin{aligned} d^2 + r^2 - r'^2 &\geq 2dr \\ d^2 - 2dr + r^2 &\geq r'^2 \\ (d - r)^2 &\geq r'^2. \end{aligned}$$

This simply means that we must have that $C$ and $C'$ do not intersect, in which case the inversion is two lines.

## Exercise III.24

Suppose we have found a solution $PQR$ to the problem. Then we can deduce what this solution should look like.
Define (like the suggestion in the book) the reflections $Q_1$ and $R_1$ of $P$ in the lines $AC$ and $AB$. Because we assume the perimeter of $PQR$ to be minimal, the sum of the distances $|Q_1R| + |RQ| + |QR_1| = |PR| + |RQ| + |QP|$ should be minimal, which implies that $Q$ and $R$ are on a straight line between $Q_1$ and $R_1$. (Otherwise, another choice of $Q$ and $R$ with the same $P$ would give smaller perimeter.)
This, together with $R_1$ being the reflection of $P$, implies that $\angle RQA = \angle R_1QB = \angle PQB$. In the same way, we find $\angle QRA = \angle Q_1RC = \angle PRC$, but also similarly $\angle QPB = \angle RPC$.
Furthermore, we have that $|AR_1| = |AP| = |AQ_1|$, and therefore $\angle AR_1Q_1 = \angle AQ_1R_1$.
Now we can write

$$\begin{aligned}
\angle APC &= \angle AQ_1C \quad \text{(by reflection)} \\
&= \angle AQ_1R_1 + \angle RQ_1C \quad (Q_1,\ R,\text{ and } R_1 \text{ are on a straight line}) \\
&= \angle AQ_1R_1 + \angle RPC \quad \text{(by reflection)} \\
&= \angle AR_1Q_1 + \angle QPB \\
&= \angle AR_1Q_1 + \angle QR_1B \quad \text{(by reflection)} \\
&= \angle AR_1B \quad (R_1,\ Q,\text{ and } Q_1 \text{ are on a straight line}) \\
&= \angle APB.
\end{aligned}$$

At the same time, $\angle APC + \angle APB = \pi$, therefore $\angle APC = \angle APB = \frac{\pi}{2}$.
(And in the same way $\angle BRA = \angle BRC = \frac{\pi}{2}$, and $\angle CQB = \angle CQA = \frac{\pi}{2}$.)

# 8 Week 8

## Exercise V.4

The vector space underlying a projective line is 2-dimensional, it is a plane. We will now consider the projective plane without one projective line, and we will take this line to be the one at infinity. We may always choose which line (or in general, hyperplane) to be the line at infinity. Now, the projective plane is just an affine plane union the line at infinity, so the complement we are intersted in is just an affine plane. Fix some origin in the affine plane, $A$. Then, for any other $B$ in the affine plane, $\overrightarrow{AB}$ is a vector in the underlying vector space. Of course, the vector $\overrightarrow{AB}$ can be connected to the origin in the vector space, and so it is path-connected. This also implies that the affine plane, and so the complement of a projective line in a projective plane, is path-connected.

Note that $P_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ is a circle, and that $P_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ is a 2-sphere. Hence, the complement $P_1(\mathbb{C}) \setminus P_1(\mathbb{R})$ is two semi-spheres, which are two connected components.

## Exercise V.10

Take a line, $l$, in $P^*$. This is mapped to a point, $m$, in $P$. Conversely, a point in $P^*$ is mapped to a line in $P$. Hence, any point on the same line $l$, must be mapped to a line running through the same point $m$.

## Exercise V.11

Let $P = P(E)$ be a projective plane defined over a 3-dimensional vector space $E$, $m \in P$ a point, $D \subset P$ a line not through $m$, and $p : E \to P$ a projection. Let $\{e_1, e_2, e_3\}$ be a basis of $E$, and $\{e_1^*, e_2^*, e_3^*\}$ a basis of $E^*$ such that $e_i^*(e_j) = \delta_{ij}$. Let the bases be such that $m = p(e_1)$ and $D = P(\text{span}(e_2, e_3))$. By projective dualtiy, $m^*$ is a line in $P^*$, which is the projective image of $\text{span}(e_2^*, e_3^*)$.

Let $ae_2^* + be_3^* = 0$ be a line. This intersects the plane $\text{span}(e_2, e_3)$ (whose "projectivization" is the line $D$) along the line $be_2 - ae_3 = 0$. We define the map

$$\begin{aligned} f : \text{span}(e_2^*, e_3^*) &\to \text{span}(e_2, e_3) \\ (a, b) &\mapsto (b, -a), \end{aligned}$$

which is clearly an isomorphism. It also induces a map

$$i : m^* \cong \text{span}(e_2^*, e_3^*) \to \text{span}(e_2, e_3) \cong D,$$

which it follows is a projective transformation.

## Exercise V.12

All lines in projective space intersect, either before or at infinity.

We want to prove that $g : H \to H'$, which maps $x \in H$ to the point of $\overrightarrow{mx}$ that intersects $H'$, is a projective transformation. We need to show that $g$ is a bijection induced by an isomorphism $f : F \to F'$, where $F, F' \subset E$ are subspaces such that $P(F) = H$ and $P(F') = H'$. Let $f$ take a point $x \in F$ and project it onto $F'$ in the direction $\overrightarrow{mx}$. This is an isomorphism, as $f(a+b) = \overrightarrow{m(x+y)} \cap H' = (\overrightarrow{mx} + \overrightarrow{my}) \cap H' = f(a) + f(b)$. It is clearly bijective, and it extends to the projective space $P$, defining the projective transformation $g$.

We have a line $D$ in $P$, which corresponds to pencil of lines in $P^*$, $n^*$. By the previous exercise, there exists a line $d$, such that $i_1 : n^* \to d$ is an incidence. Let us choose the line $d$, in $P^*$, which corresponds to the pencil of lines $m^*$ in $P$, which is given by the incidence $i_1 : D \cong n^* \to d$. Now we just compose with the incidence $i_2 : m^* \to D'$, to get the perspectivity of centre $m$ from $D$ to $D'$, namely $i_2 \circ i_1 : D \to D'$.

## Exercise V.25

Do something like this, for example.

## Exercise V.31

In a projective space $P(E)$, we consider three hyperplanes $H, H_1, H_2$, and two points $m_1, m_2$, such that $m_i \notin H \cup H_i$, for $i \in \{1, 2\}$. We also consider the mappings $g_i : H \to H_i$, $i \in \{1, 2\}$, defined such that they are the perspectives with centre $m_i$. In Exercise V.12 we showed that such mappings are projective transformations. The compostition $g_2 \circ g_1^{-1} : H_1 \to H_2$ is another projective transformation.

When $E$ is a three-dimensional space, then a projective frame of $P(E)$ can be given by taking 4 points in $P(E)$, where three of the points are projections of linearly independent (basis) vectors in $E$. By Proposition 5.6 in Audin, chapter V, any projective mapping from $H_1$ to $H_2$ maps a projective frame of $H_1$ onto a projective frame of $H_2$, e.g. $g_2 \circ g_1^{-1}$, providing the sought after gluing.