1a) $g'(x) = -\frac{\pi}{2}\cos(\pi x)$ $|g'(2)| = \frac{\pi}{2} > 1$ no convergence

$x_0 = 2,\ x_1 = \frac{3}{2},\ x_2 = 2,\ x_3 = \frac{3}{2}$, etc no divergence

b) $x_{n+1} = x_n - \dfrac{(\sin(\pi x_n) + 2x_n - 3)}{\pi\cos(\pi x_n) + 2}$ $x_0 = 1.8$

$x_1 = 1.797310\ldots$

c) $x_{n+1} = x_n + \alpha(\sin(\pi x_n) + 2x_n - 3)$

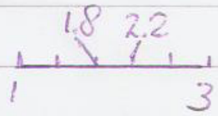
$g'(x) = 1 + \alpha(\pi\cos(\pi x) + 2)$

$g'(1.8) = 0 \rightarrow \alpha = \dfrac{-1}{\pi\cos(1.8\pi) + 2} = -0.220186\ldots$

d) $K = \dfrac{x_5 - x_4}{x_4 - x_3} = -0.132478\ldots$

(1) $\varepsilon_5 \leq \dfrac{K}{1-K}|x_5 - x_4| = 1.0879\text{ E-}7$

(2) $2x_5 - x_4 = 1.797304\ldots$

2a) 

$T = \frac{0.4}{2}\left(\frac{1}{1.8} + \frac{1}{2.2}\right) = \frac{20}{99} = 0.202020\ldots$

(1) $S = \frac{0.4}{6}\left(\frac{1}{1.8} + \frac{4}{2} + \frac{1}{2.2}\right) = 0.200673\ldots$

(2) $f''(x) = \frac{2}{x^3}$, max at $1.8 \rightarrow M = 2(1.8)^{-3}$

$\varepsilon \leq \dfrac{0.4}{12}(0.4)^2 M = 1.8289\text{ E-}3$

b) $q_{16} = \left|\dfrac{I_{16} - I_8}{I_{16} - I_{32}}\right| = 3.956126\ldots \approx 4$, according to theory

$T_2(16) = \frac{4}{3}T_{16} - \frac{1}{3}T_8 = 1.09860553$ $\}\ * -\frac{1}{15}$

$T_2(32) = \frac{4}{3}T_{32} - \frac{1}{3}T_{16} = 1.09861185$ $\}\ * \frac{16}{15}$ > 1.09861227

$\hookrightarrow$ error (exact) $1.34856\text{ E-}8$

c) $\varepsilon_{256} = \frac{1}{3}(T_{256} - T_{128}) = 2.2604\text{ E-}6$

$\left(\frac{1}{4}\right)^n * 2.2604\text{ E-}6 \rightarrow n \geq 4 \rightarrow 16 \times 256 = 4096$ segments for 1.0E-8

efficiency: 50 segments instead of 4096 segments for comparable accuracy

3a) $y(0.5) = 1 + 0.5(-0 * 1^2)$

(1) $y(1) = 1 + 0.5(-0.5 * 1^2) = 0.75$ Euler expl.

(2) $k_1 = 0.5(-0 * 1^2) = 0$
$k_2 = 0.5(-0.5 * (1+0)^2) = -0.25$ } $y(0.5) = 1 + \frac{1}{2}(0 - 0.25) = 0.875$ Heun

(3) $y(0.5) = 1 + 0.5(-0.5\, y^2(0.5)) = 1 - \frac{1}{4} y^2(0.5)$

$\rightarrow y(0.5) = \frac{-1 \pm \sqrt{1+1}}{2/4} = -2 + 2\sqrt{2}$ Euler impl OR $-2 - 2\sqrt{2}$ reject, see values Euler Heun

b) $q = \left| \frac{y^{1/16} - y^{1/8}}{y^{1/16} - y^{1/32}} \right|$ at 1.0 $\hat{=}$ 8.61 close enough to $2^3 = 8$ according to theory

(1)

(2) $\hat{y} = \frac{8}{7} y^{1/32} - \frac{1}{7} y^{1/16} = 0.6666\,6663$

(3) $\mathcal{E}(1.0) = \frac{1}{7}(y^{1/32} - y^{1/16})$ at 1.0 $\hat{=}$ 6.11 E-7

1.5 ... 1.5 9.08 E-7 $\rightarrow$ error accumulation when going from L to R

(4) solutions on all grids follow same pattern $\rightarrow$ no instability

4 a)

1	3	5
1	5	25

$\rightarrow \hat{x} = \frac{x-3}{2} \rightarrow$

-1	0	1
0	ln(5)	ln(25)

$M_0 = 3,\ M_1 = 0,\ M_2 = 2$

$F_0 = \ln(5) + \ln(25) = 3\ln(5),\quad F_1 = 0 + 0 + \ln(25) = 2\ln(5)$

$\begin{array}{cc|c} 3 & 0 & 3\ln(5) \\ 0 & 2 & 2\ln(5) \end{array}$ $\hat{a} = \ln(5)$ $a = e^{\ln(5)} = 5$; $\hat{b} = \ln(5)$

$\hat{y} = 5e^{\ln(5)\hat{x}} \quad \left(= 5 * 5^{\hat{x}}\right)$

$y = 5e^{\ln(5)\frac{x-3}{2}} = 5e^{-\frac{3}{2}\ln(5)} e^{\frac{1}{2}\ln(5)x}$

$= 5 * 5^{-3/2} e^{\frac{1}{2}\ln(5)x} = 5^{-1/2} e^{\frac{1}{2}\ln(5)x}$

b) $f(0) = 5^{-1/2} \cdot e^0 = \frac{1}{\sqrt{5}} \rightarrow$ this is the exact value the curve goes exactly through the data

5 a) $x_1 = \frac{1}{1}(1-0) = 1$

$x_2 = \frac{1}{-5}(2-(-1)(1)-(2)(1)) = \frac{-1}{5}$

(1) $x_3 = \frac{1}{-5}(3-(1)(1)-(1)(1)-(-2)(1)) = \frac{-3}{5}$

$x_4 = \frac{1}{5}(4-(-1)(1)) = 1$

$\hat{x}_1 = \frac{1}{1}(1-0) = 1$ | $x_1 = 5(1) - 4(1) = 1$

$\hat{x}_2 = \frac{1}{-5}(2-(-1)(1)-(2)(1)) = \frac{-1}{5}$ | $x_2 = 5(-\frac{1}{5}) - 4(1) = -5$

(2) $\hat{x}_3 = \frac{1}{-5}(3-(1)(1)-(1)(-5)-(-2)(1)) = \frac{-9}{5}$ | $x_3 = 5(-\frac{9}{5}) - 4(1) = -13$

$\hat{x}_4 = \frac{1}{5}(4-(-1)(1)) = 1$ | $x_4 = 5(1) - 4(1) = 1$

b) (1) $A\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-4\\-5\\4\end{pmatrix} \rightarrow A\begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \begin{pmatrix}0\\-6\\-8\\0\end{pmatrix} \rightarrow \|r^{(0)}\|_\infty = 8$

(2) row1: 0, row2: $\frac{3}{5}$, row3: $\frac{4}{5}$, row4: $\frac{1}{5}$

$(0.8)^n < 0.1 \rightarrow n \geqq 11$ iterations

c) use $A(1,1)$ to eliminate $A(3,1)$ and $A(4,1)$

6 a) $\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + \alpha \phi_i^n = \Gamma \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{(\Delta x)^2}$ $\quad R = \frac{\Gamma \Delta t}{(\Delta x)^2}$

(1) $\phi_i^{n+1} = R\phi_{i+1}^n + \underbrace{(1-2R-\alpha\Delta t)}_{>0}\phi_i^n + R\phi_{i-1}^n$

$R = \frac{10^{-4}}{10^{-4}} \cdot \Delta t \quad \rightarrow 1 - 2\Delta t - 0.5\Delta t > 0 \Rightarrow \Delta t < 0.4$

(2) $A = 0 \rightarrow \phi(x,0) = 20$ not sol of pde $\rightarrow$ errors will build up, no effect on stab., still $\Delta t < 0.4$

b) $\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + \alpha \phi_i^{n+1} = \Gamma \frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{(\Delta x)^2}$

(1) $\phi_i^{n+1} = \phi_i^n + R\phi_{i+1}^{n+1} + (-2R-\alpha\Delta t)\phi_i^{n+1} + R\phi_{i-1}^{n+1}$

$\Longrightarrow R\phi_{i-1}^{n+1} + \underbrace{(-1-2R-\alpha\Delta t)}_{=D}\phi_i^{n+1} + R\phi_{i+1}^{n+1} = -\phi_i^n$

$A(i,:) = (0 \cdots 0 R D R 0 \cdots 0)$; $b(i) = -\phi_i^n$

(2) α term enhances sign $\rightarrow$ always stable ($\alpha > 0$)