Geometry 2023, mock exam

- Below you can find the mock exam questions. There are four questions summing up to 85 points. Note that you will be allowed to consult two A4 pages handwritten or typeset by you for formulas, results, etc. treated in this course.

QUESTIONS

1. $\boxed{10+15 = 25 \text{ pts}}$

   i) Consider the unit circle in $\mathbb{R}^2$. Let $D_t,\ t \in [0, 2\pi]$, be the family of lines in $\mathbb{R}^2$ defined as follows: D_t passes through the point $(\cos t, \sin t)$ and has direction given by the vector $(\cos t - \sin t, \cos t + \sin t)$. Compute the envelope of the family D_t.

   ii) Prove that there exists a regular curve $\gamma \colon \mathbb{R} \to \mathbb{R}^2$ whose evolute is a) everywhere regular and b) has exactly two singular points.

   Solution:

   i) The family of lines are given by the equation $x(\cos t + \sin t) + y(\sin t - \cos t) = 1$. Hence the envelope is obtained by solving the equation

$$\begin{pmatrix} \cos t + \sin t & \sin t - \cos t \\ \cos t - \sin t & \cos t + \sin t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

   from where

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \cos t + \sin t & -\sin t + \cos t \\ -\cos t + \sin t & \cos t + \sin t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \cos t + \sin t \\ -\cos t + \sin t \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin(t + \pi/4) \\ -\cos(t + \pi/4) \end{pmatrix}.$$

   ii) Recall that the singular points of the evolute correspond to singular points of the curvature. Hence, if the curvature $k(t)$ of a curve $\gamma \colon \mathbb{R} \to \mathbb{R}^2$ has everywhere non-vanishing derivative, the evolute of γ will be everywhere regular. An explicit example of this is given by Euler's or Cornu spiral, where the curvature is linear.

   To construct an example of a curve γ whose evolute has exactly two singular points, it suffices to let the curvature of γ admit exactly one minimum and one maximum. For example, we can let $k(s) = s^3 - s$, where s is the arc-length parameter. The corresponding curve γ which has curvature $k(s) = s^3 - s$ is obtained by solving the Frenet-Serret equations.

2. $\boxed{10+20 = 30\text{pts}}$ Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism.

i) Show that the image $f(S^2)$ of the unit sphere S^2 in $\mathbb{R}^3$ is a regular surface in $\mathbb{R}^3$.

ii) Prove that the integral

$$\frac{1}{2\pi}\int_{S^2} K \mathrm{d}S = 2,$$

where $K\colon S^2 \to \mathbb{R}$ is the Gaussian curvature of T^2 and $\mathrm{d}S$ is the area element.

Hint for part ii): subdivide S^2 into 4 geodesic triangles and use local Gauss-Bonnet theorem.

Solution: i) The image $f(S^2)$ is the level set of the equation

$$g \circ f^{-1}(\tilde{x}, \tilde{y}, \tilde{z}) = 0, \quad g(x, y, z) = x^2 + y^2 + z^2 - 1.$$

Since f is a diffeomorphism, $g \circ f^{-1}$ is smooth and has non-vanishing gradient along $f(S^2)$. Hence $f(S^2)$ is a regular surface by the IFT (as a solution to an implicit equation with non-vanishing gradient).

ii) Consider 4 points on S^2. By compactness, they can be connected by geodesics. These geodesics subdivide the sphere S^2 into 4 geodesic triangles. By the local Gauss-Bonnet theorem, for each of these triangles ABC, we have

$$\int_{ABC} K \mathrm{d}S = \alpha + \beta + \gamma - \pi.$$

But the total angle-sum of the 4 triangles is $4 \times 2\pi$. Hence

$$\int_{ABC} K \mathrm{d}S = 8\pi - 4\pi = 4\pi.$$

3. $\boxed{5+5+5=15 \text{ pts}}$ Consider the natural projection $\mathrm{Pr}\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}P^2$ and a regular curve γ in $\mathbb{R}P^2$. The preimage $M^2 = \mathrm{Pr}^{-1}(\gamma)$ determines a regular surface in $\mathbb{R}^3$.

i) Using homogeneous coordinates on $\mathbb{R}P^2$, write down a parametric equation for M^2;

ii) Compute the second fundamental form of M^2, as a function of γ;

iii) Compute the Gaussian curvature of M^2.

Solutions:

i) Let $(x : y : z)$ be homogeneous coordinates for $\mathbb{R}P^2$ and let γ be given by $(x(t) : y(t) : z(t))$. Then, assuming that one of the variables, say $z = z(t)$, is non-zero, we can write the same curve as

$$\gamma = (u(t), v(t), 1),$$

where $u = \frac{x(t)}{z(t)}$ and $v = \frac{x(t)}{z(t)}$. Then M^2 is parametrised by

$$r(t,s) = (su(t), sv(t), s), \quad s \neq 0$$

(on the domain, where $z(t) \neq 0$).

ii) First observe that the direction normal vector to the surface is given by

$$\begin{aligned} n(t,s) &= [(u(t), v(t), 1), (su'(t), sv'(t), 0)]/\|[(u(t), v(t), 1), (su'(t), sv'(t), 0)]\| \\ &= \mathrm{sing}(s)(-v'(t), u'(t), u(t)v'(t) - v(t)u'(t))/\sqrt{u'^2(1+v^2) + v'^2(1+u^2) - 2uvu'v'}. \end{aligned}$$

Denoting the square root by $c(t)$, the second fundamental form becomes

$$Q = \begin{pmatrix} \langle r_{tt}, n\rangle & \langle r_{ts}, n\rangle \\ \langle r_{st}, n\rangle & \langle r_{ss}, n\rangle \end{pmatrix} = \frac{|s|}{c(t)} \begin{pmatrix} -u''v' + v''u' & 0 \\ 0 & 0 \end{pmatrix}.$$

iii) Recall that the Gaussian curvature is the determinant of the operator $G^{-1}Q$, where G is the first fundamental form and Q is the second. From the above, $\det Q = 0$, hence the Gaussian curvature of M^2 is identically zero.

4. $\boxed{\text{5+10=15pts}}$

1. Let $\alpha\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a non-zero linear function. Prove that the kernel Ker α gives a compact subset of $\mathbb{R}P^n$, which can be identified with $\mathbb{R}P^{n-1}$.
2. Let $f \in \mathrm{PSL}(2,\mathbb{C})$. Show that f can have either 1, 2 or infinitely many fixed points in $\mathbb{C}P^1$. Give explicit examples of f with theses numbers of fixed points.

Solutions: i) This basically follows from the definition and the fact that an $(n-1)$-sphere S^{n-1} is compact. Indeed, since α is a non-zero linear function, its kernel is an n-dimensional linear subspace of $\mathbb{R}^{n+1}$ and is thus isomorphic to $\mathbb{R}^n$. Hence the projective space $P(\text{Ker } \alpha)$ can be identified with $\mathbb{R}P^{n-1}$. Since it is the quotient of S^{n-1} by the equivalence relation $x \sim -x$, which is compact, $\mathbb{R}P^{n-1}$ is itself compact (as the image of compact space S^{n-1} under the projection map $\mathrm{Pr}\colon S^{n-1} \to \mathbb{R}P^{n-1},\ x \mapsto \pm x$).

ii) Recall that every element $f \in \mathrm{PSL}(2,\mathbb{C})$ can be represented as

$$f = \frac{az+b}{cz+d}, \qquad \text{where } a,b,c,d \in \mathbb{C} \quad \text{and} \quad ad - bc = 1.$$

Looking for fixed points, $f(z_0) = z_0$ implies $az_0 + b = cz_0^2 + dz_0$, which is a quadratic equation. First let $c \neq 0$. Then $f(\infty) = \frac{a}{c} \neq \infty$, that is, ∞ is not a fixed point of f. Over $\mathbb{C}$ we then have at most two solutions.

If $c = 0$, then $f = \frac{a}{d}z + \frac{b}{d}$ is linear and infinity is a fixed point. Over $\mathbb{C}$, we have either one solution or no solutions (when $a = d$ and $b \neq 0$) or infinitely many (when $a = d$ and $b = 0$).

A concrete examples of f realising theses numbers of fixed points is for example the following

- $f = z - 1$ – one fixed point at infinity;
- $f = 1/z$ – two fixed points $z_0 = \pm 1$;
- $f = z$ – infinitely many fixed points.

End of exam