university of groningen

faculty of science and engineering

Mechatronics

Week 2 Day 2

Previous lecture

- You were introduced to modeling of dynamical systems
- We discussed
  - Differential equations
  - Transfer functions
  - State-space equations

… to represent a dynamical system's behaviour

Today's lecture:

State-space representation and A, T, and D-type variables

Learning objectives

After today's lecture, you will be able to

- Establish analogies in the modeling of different physical domains

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- Describe the main differences and properties of A, T, and D-type elements in physical systems

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After today's lecture, you will be able to

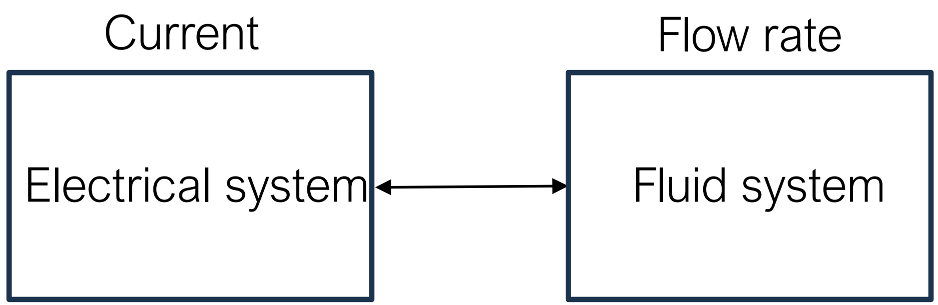
- Establish analogies in the modeling of different physical domains
- Describe the main differences and properties of A, T, and D-type elements in physical systems
- Identify the A,T, and D-type elements, and the A,T-type variables in different physical domains

Analogies among different physical domains

Analogy between Electrical and Fluid Systems

It is possible to establish analogies among different physical domains

- The current (defined as rate of flow of electric charge) is analogous to the flow rate of fluid in fluid systems



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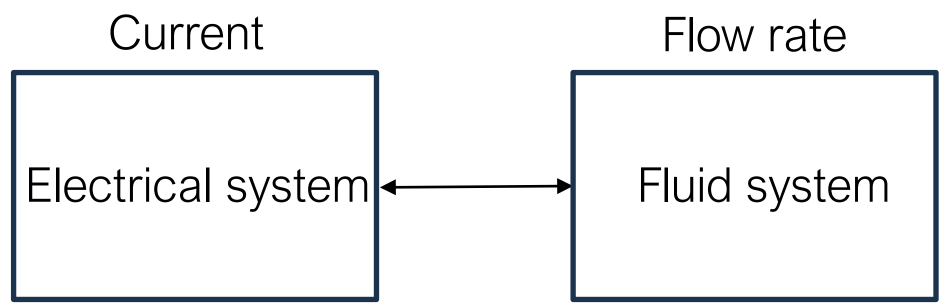
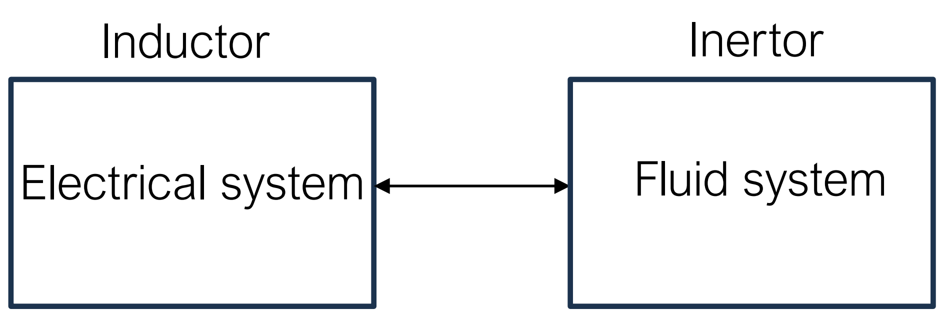


- An inertor in a fluid system behaves as an inductor in an electrical system. So we say that the inertor is the analogue of the inductor



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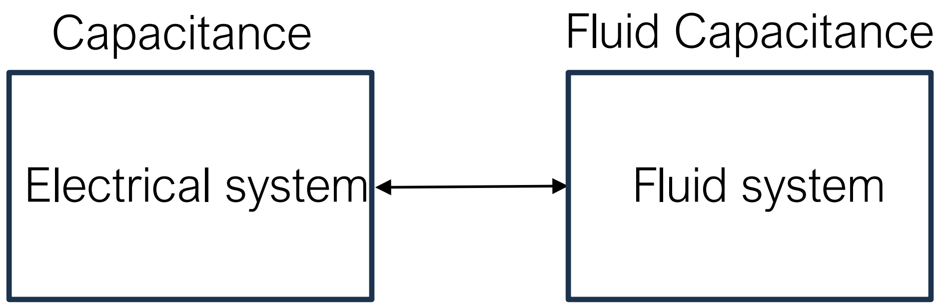
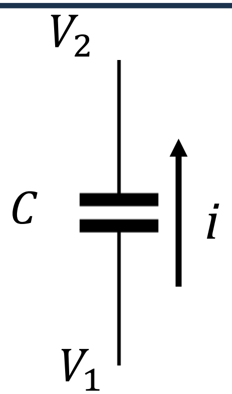


Example 1

Electrical system		Fluid system
Current	⟷	Flow rate



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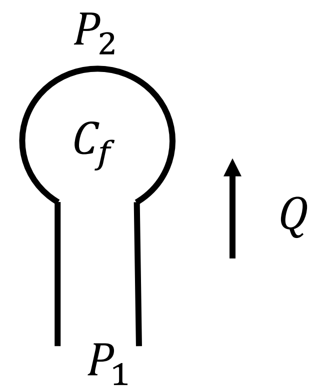


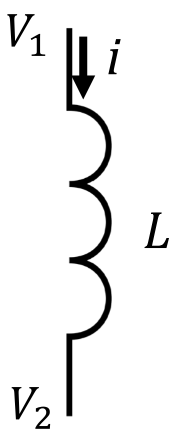
$$i = C\frac{dV_{12}}{dt}$$

$$Q = C_f\frac{dP_{12}}{dt}$$

Voltage (V_{12}) ≈ Pressure (P_{12})
Capacitance (C) ≈ Fluid capacitance (C_f)
Current (i) ≈ Flow rate(Q)

Example 2

Electrical system		Fluid system
Inductor	⟷	Inertor



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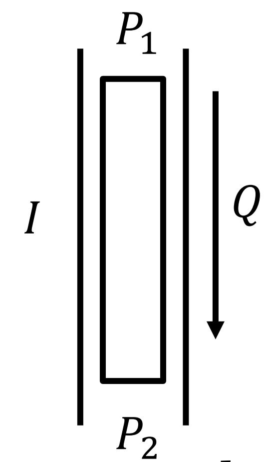


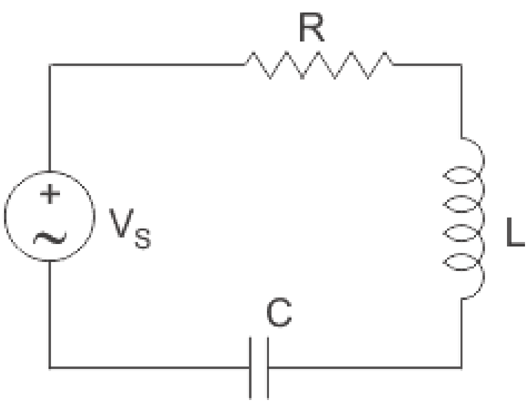
$$V_{12} = L\frac{di}{dt}$$

$$P_{12} = I\frac{dQ}{dt}$$

Voltage (V_{12}) ≈ Pressure (P_{12})
Inductance (L) ≈ Inertance (I)
Current (i) ≈ Flow rate(Q)

Force-Voltage Analogy

Electrical system



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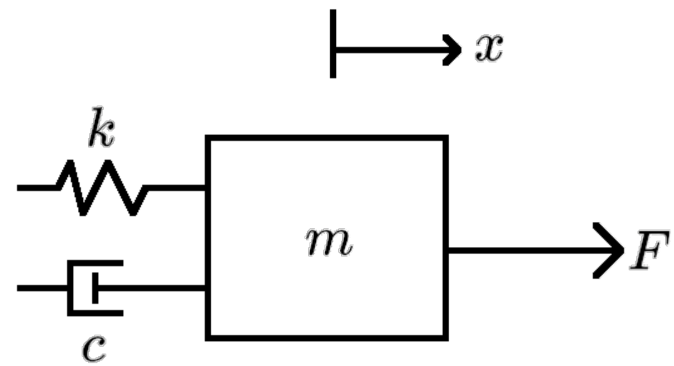
Translational Mechanical System



$$V_s = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt$$

Now, $i(t) = \frac{dq}{dt}$ $\rightarrow$

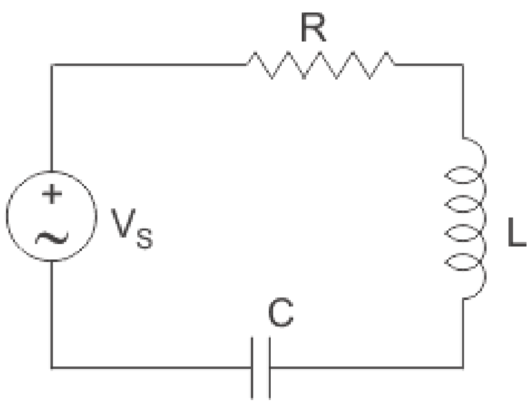
$$\rightarrow \quad V_s = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{1}{C}q \rightarrow$$

$$\rightarrow \quad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_s \quad (1)$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F \qquad (1')$$

Force-Voltage Analogy

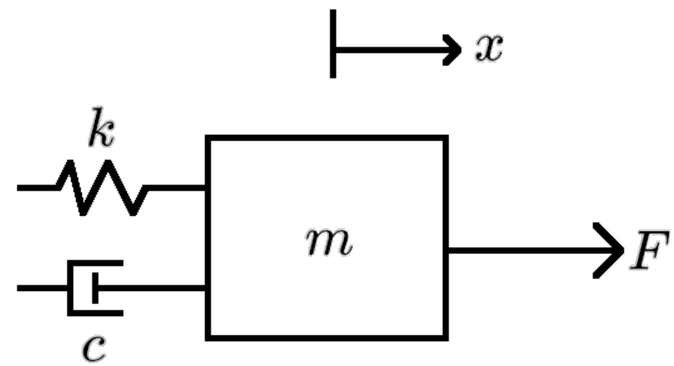
Electrical system



$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_s$$

≈

Translational Mechanical System



$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F$$

Voltage (V_s)
Inductance (L)
Resistance (R)
Reciprocal of Capacitance ($\frac{1}{C}$)
Charge (q)
Current (I$= \frac{dq}{dt}$)

≈

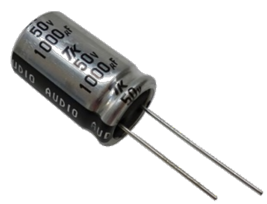
Force (F)
Mass (m)
Friction coefficient (c)
Spring Constant (k)
Displacement (x)
Velocity ($v = \frac{dx}{dt}$)

A- and T-type variables, and A, T, and D-type elements

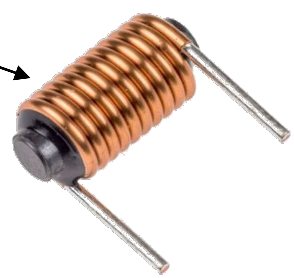
The internal behaviour of a physical system is governed by its energy

From an energy-based perspective, we can classify the elements of a physical system as:

- Energy storing elements
  - a capacitor



  - an inductor



  - a mass
  - a spring

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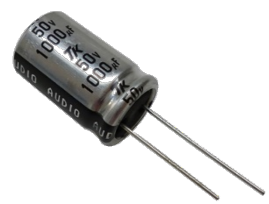
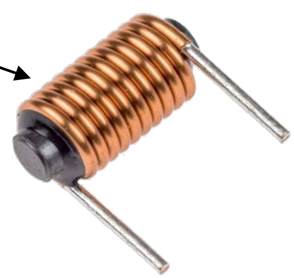
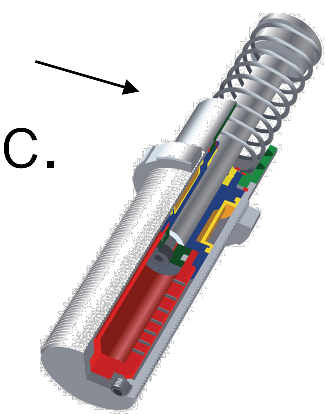




- Dissipative elements
  - a resistor
  - mechanical dampers, etc.

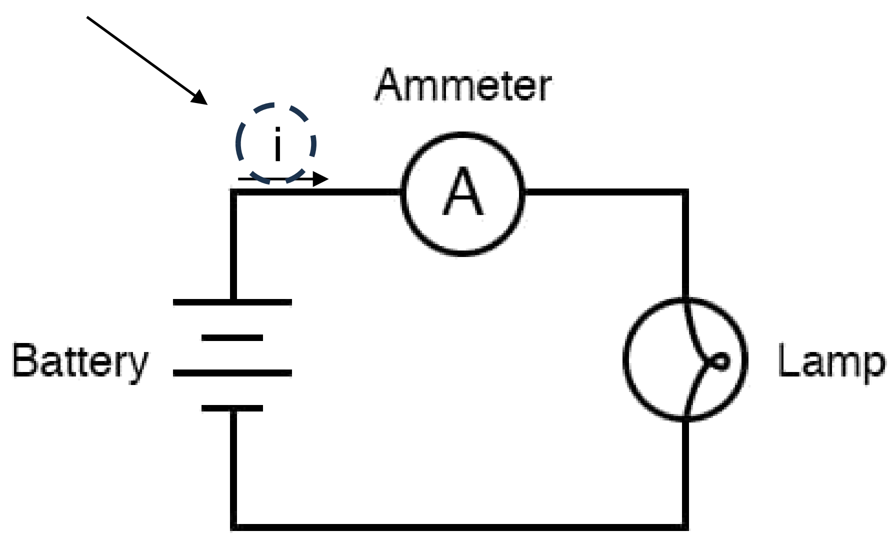


T- and A-type variables

T- and A-type variables

The dual variables related to energy storing elements can be categorized as

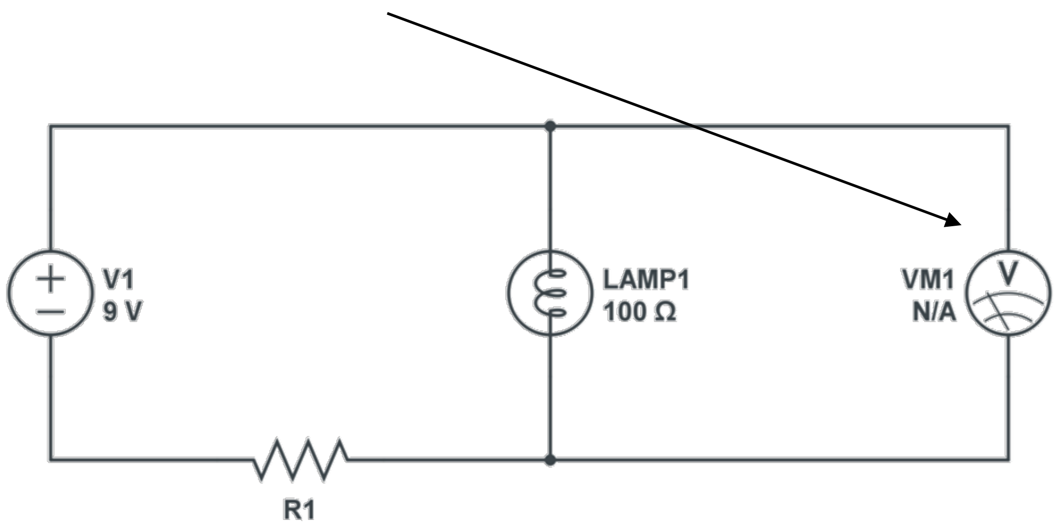
- T-type variables
  - The physical quantity goes <u>through</u> the element, and it is measured by connecting an instrument <u>in series</u> with the corresponding element
  - Examples: current, fluid flow, force etc.



T- and A-type variables

The dual variables related to energy storing elements can be categorized as

- A-type variables
  - These variables act across the element and are measured by connecting an instrument in parallel to the corresponding element
  - Examples: voltage, pressure, velocity etc.



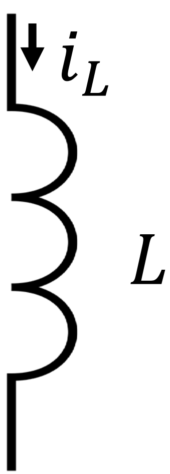
A, T, D-type elements

A, T, D-type elements

T-type elements

These energy storage elements are related to T-type variables

- e,g. inductors, inertors, springs etc.



$$\varepsilon_L = \frac{1}{2} L i_L^2$$

A, T, D-type elements

T-type elements

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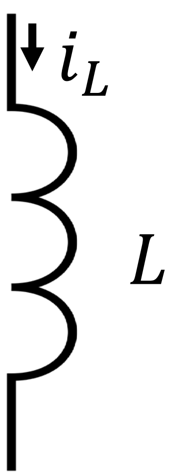
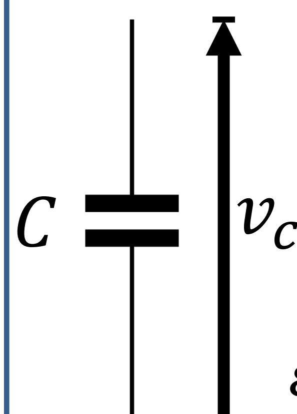


$$\varepsilon_L = \frac{1}{2} L i_L^2$$

A-type elements

These energy storage elements are related to A-type variables

- e.g. capacitors, fluid capacitors, mass etc.



$$\varepsilon_C = \frac{1}{2} C v_C^2$$

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A, T, D-type elements

T-type elements

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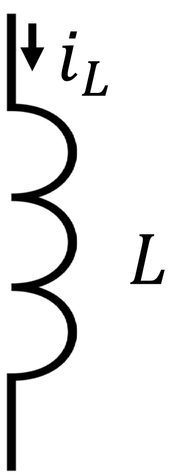
- e,g. inductors, inertors, springs etc.



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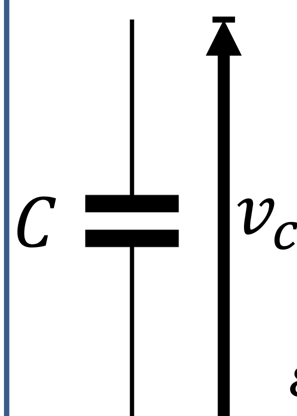
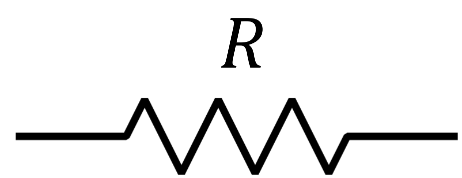


$$\varepsilon_C = \frac{1}{2} C v_C^2$$

D-type elements

Dissipative elements are referred to as D-type elements

- e.g. resistors, fluid resistor, mechanical dampers, etc.



No D-type variable!

Point to remember

- The elemental equations of A-type and T-type elements are differential equations
  - E.g., $I_C = C\frac{dv_c}{dt}$, $V_L = L\frac{di_L}{dt}$

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- The elemental equations of A-type and T-type elements are differential equations
  - E.g., $I_C = C\frac{dv_c}{dt}, \; V_L = L\frac{di_L}{dt}$
- The elemental equations of D-type elements are static relationships
  - E.g., $V_R = Ri_R$

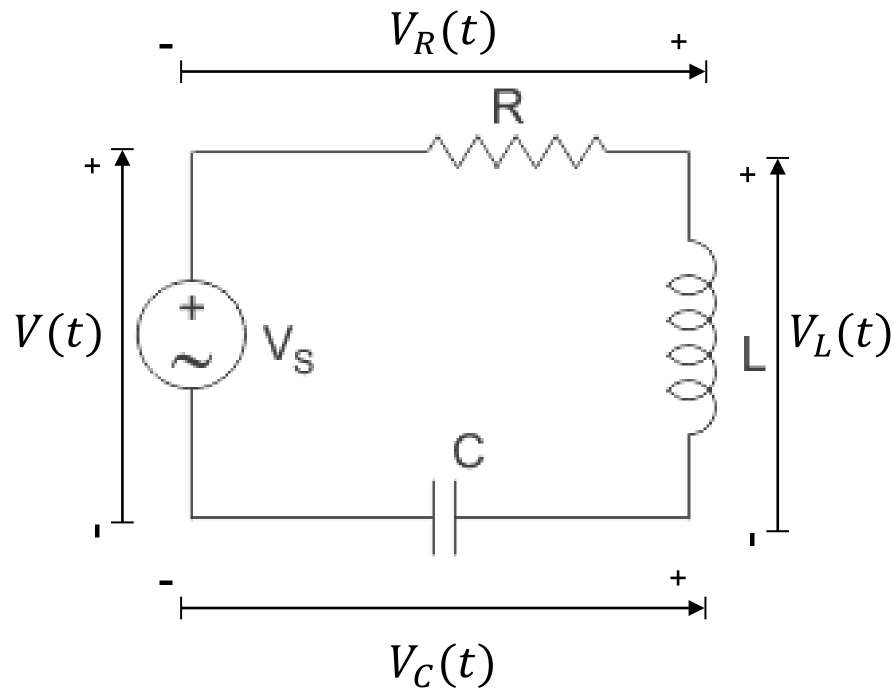
Example: Series RLC circuit

Input= *V(t)*

Output= *I(t)*

I(0)=0

$$\ddot{I}(t) + \frac{R}{L}\dot{I}(t) + \frac{1}{LC}I(t) = \frac{1}{L}\dot{v}(t)$$



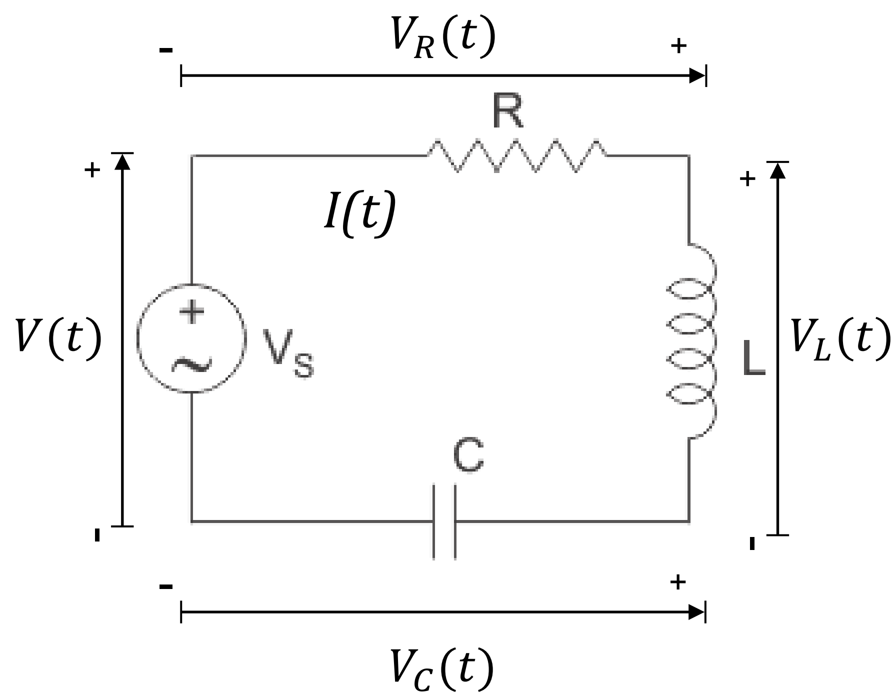
Example: Series RLC circuit

Input= *V(t)*

Output= I(t), i.e., current in the circuit

I(0)=0

$$\ddot{I}(t) + \frac{R}{L}\dot{I}(t) + \frac{1}{LC}I(t) = \frac{1}{L}\dot{V}(t)$$



- Second-order system because
  - Output variable (*I(t)*) *has the highest degree of two*

  → *There are two energy storing elements (inductor and capacitor)*

  → *You need two state variables to represent the system*

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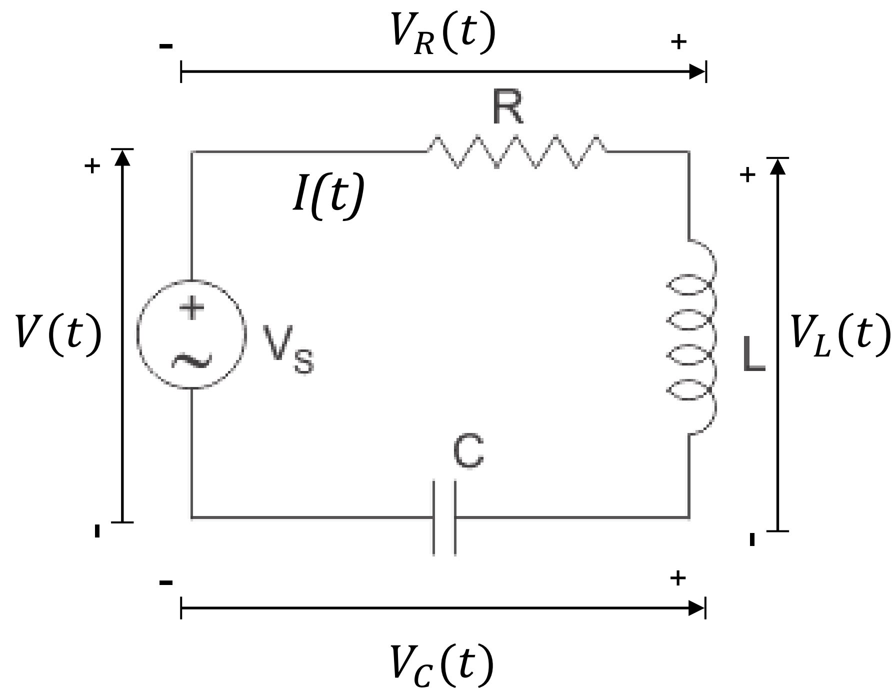

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The number of independent energy storing elements in a system is equal to the order of the system and to the number of state variables in the system model

Compliance= Spring
Voltage (V) is denoted by e

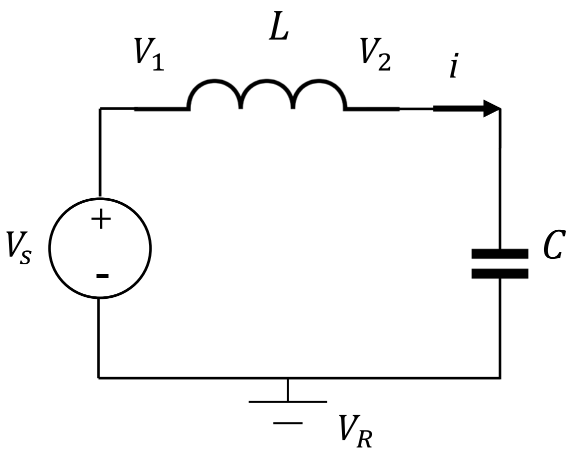
Table 1.2. Ideal system elements (linear)

System type	Mechanical translational	Mechancial rotational	Electrical	Fluid	Thermal
A-type variable	Velocity, v	Velocity, Ω	Voltage, e	Pressure, P	Temperature, T
A-type element	Mass, m	Mass moment of inertia, J	Capacitor, C	Fluid Capacitor, C_f	Thermal capacitor, C_h
Elemental equations	$F = m\dfrac{dv}{dt}$	$T = J\dfrac{d\Omega}{dt}$	$i = C\dfrac{de}{dt}$	$Q_f = C_f\dfrac{dP}{dt}$	$Q_h = C_h\dfrac{dT}{dt}$
Energy stored	Kinetic	Kinetic	Electric field	Potential	Thermal
Energy equations	$\mathscr{E}_k = \dfrac{1}{2}mv^2$	$\mathscr{E}_k = \dfrac{1}{2}J\Omega^2$	$\mathscr{E}_e = \dfrac{1}{2}Ce^2$	$\mathscr{E}_p = \dfrac{1}{2}C_f P^2$	$\mathscr{E}_t = \dfrac{1}{2}C_h T^2$
T-type variable	Force, F	Torque, T	Current, i	Fluid flow rate, Q_f	Heat flow rate, Q_h
T-type element	Compliance, $1/k$	Compliance, $1/K$	Inductor, L	Inertor, I	None
Elemental equations	$v = \dfrac{1}{k}\dfrac{dF}{dt}$	$\Omega = \dfrac{1}{K}\dfrac{dT}{dt}$	$e = L\dfrac{di}{dt}$	$P = I\dfrac{dQ_f}{dt}$	
Energy stored	Potential	Potential	Magnetic field	Kinetic	
Energy equations	$\mathscr{E}_p = \dfrac{1}{2k}F^2$	$\mathscr{E}_p = \dfrac{1}{2K}T^2$	$\mathscr{E}_m = \dfrac{1}{2}Li^2$	$\mathscr{E}_k = \dfrac{1}{2}IQ_f^2$	
D-type element	Damper, b	Rotational damper, B	Resistor, R	Fluid resistor, R_f	Thermal resistor, R_h
Elemental equations	$F = bv$	$T = B\Omega$	$i = \dfrac{1}{R}e$	$Q_f = \dfrac{1}{R_f}P$	$Q_h = \dfrac{1}{R_h}T$
Rate of energy dissipated	$\dfrac{dE_D}{dt} = Fv$	$\dfrac{dE_D}{dt} = T\Omega$	$\dfrac{dE_D}{dt} = ie$	$\dfrac{dE_D}{dt} = Q_f P$	$\dfrac{dE_D}{dt} = Q_h$
	$= \dfrac{1}{b}F^2$	$= \dfrac{1}{B}T^2$	$= Ri^2$	$= R_f Q_f^2$	
	$= bv^2$	$= B\Omega^2$	$= \dfrac{1}{R}e^2$	$= \dfrac{1}{R_f}P^2$	

Note: A-type variable represents a spatial difference across the element.

Example 3

Electrical system



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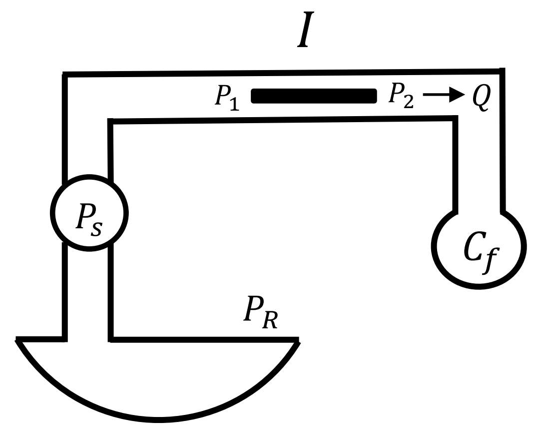
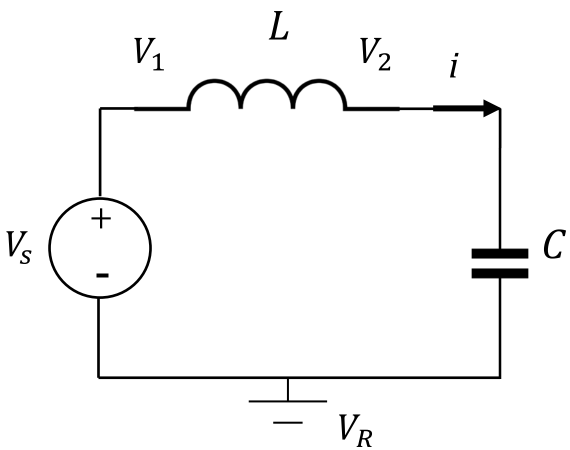
Fluid system



1 → T-type element
2 → A-type element

Example 3

Electrical system



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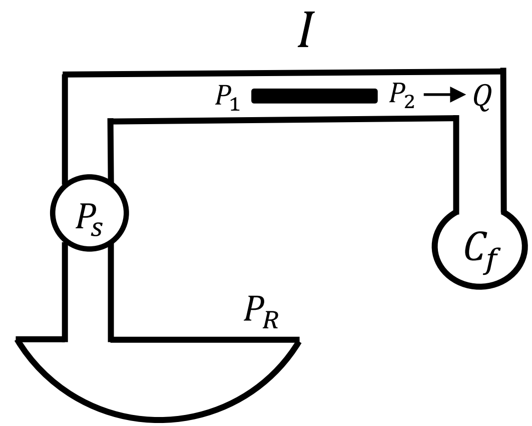
Fluid system



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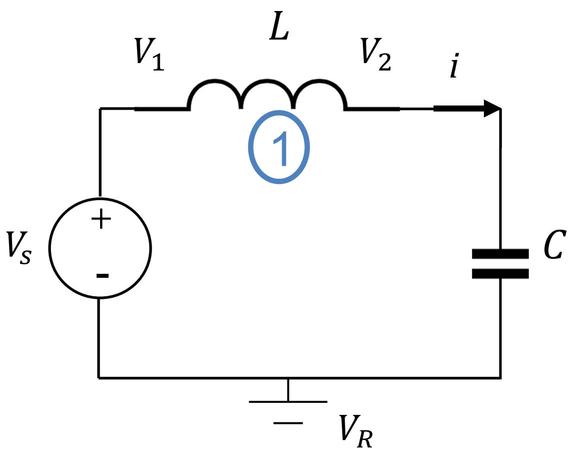
Example 3

Electrical system



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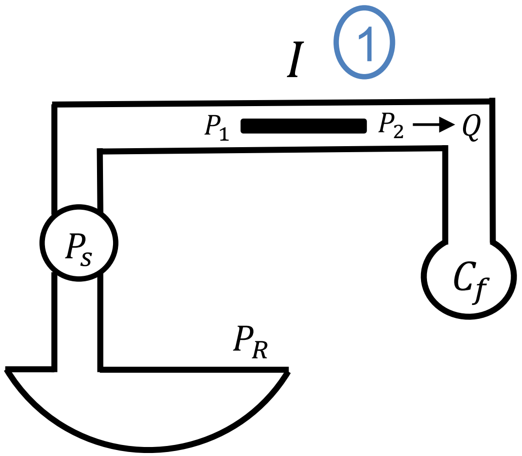
Fluid system



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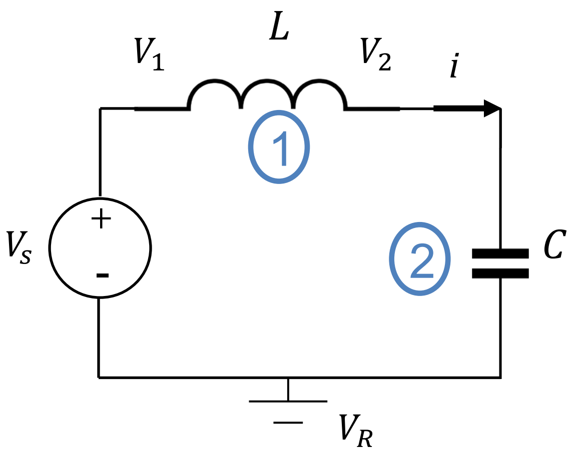
Example 3

Electrical system



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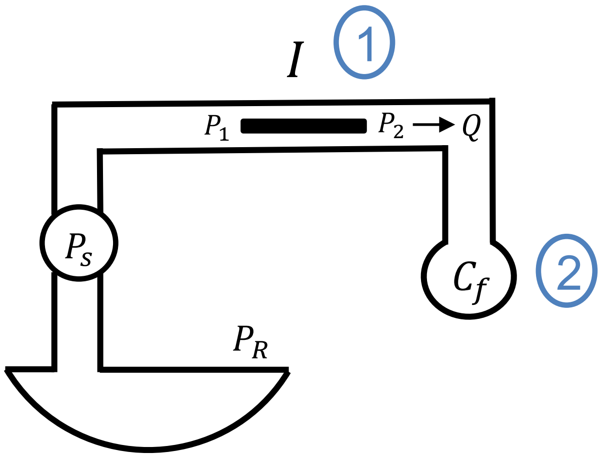
Fluid system



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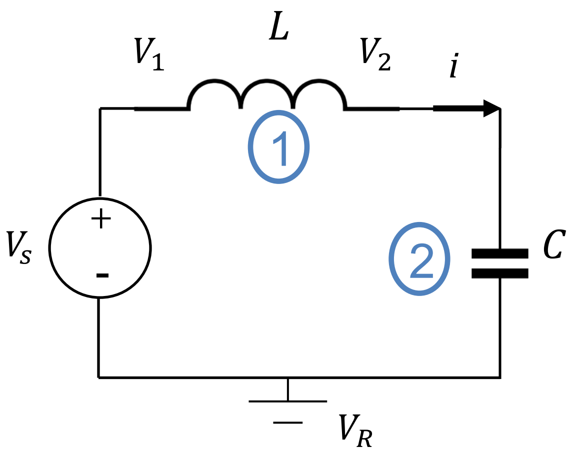
Example 3

Electrical system



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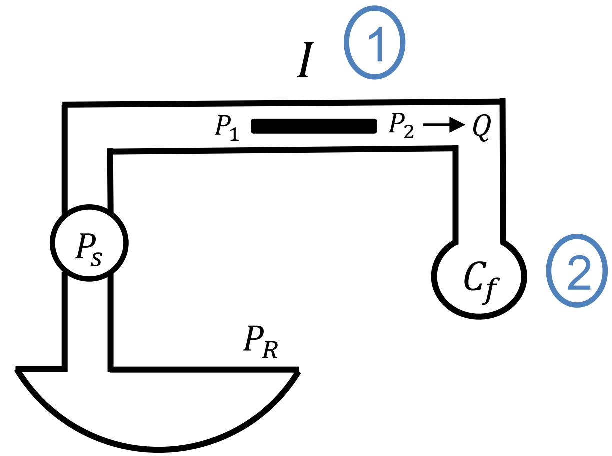
Fluid system



$$V_s = V_{12} + V_{2R} \quad (1)$$

$$C\frac{dV_{2R}}{dt} = \mathrm{i} \quad (2)$$

$$V_{12} = L\frac{di}{dt} \quad (3)$$

$$P_s = P_{12} + P_{2R} \quad (1')$$

$$C_f\frac{dP_{2R}}{dt} = \mathrm{Q} \quad (2')$$

$$P_{12} = I\frac{dQ}{dt} \quad (3')$$

1 → T-type element
2 → A-type element

Example 3

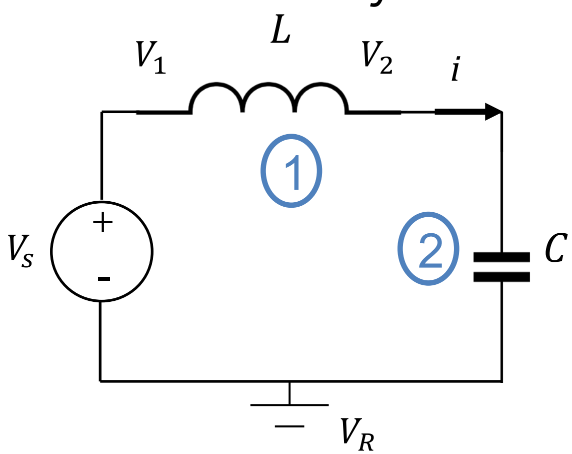
$$V_s = V_{12} + V_{2R} \qquad (1)$$

$$V_{12} = L\frac{di}{dt} \qquad (3)$$

$$P_s = P_{12} + P_{2R} \qquad (1')$$

$$P_{12} = I\frac{dQ}{dt} \qquad (3')$$

Electrical system



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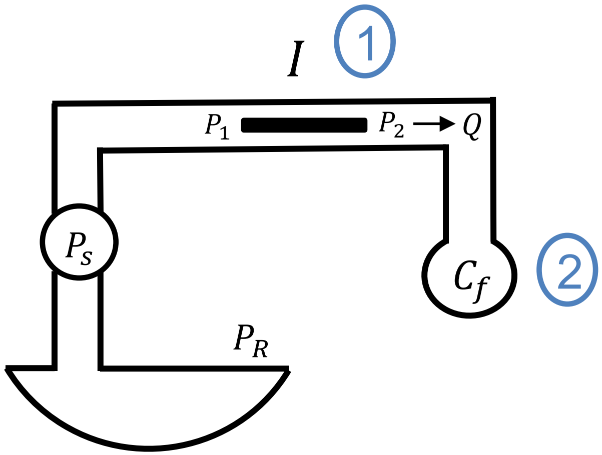
Fluid system



Putting (3) in (1), we obtain

Putting (3') in (1'), we obtain

1 → T-type element
2 → A-type element

Example 3

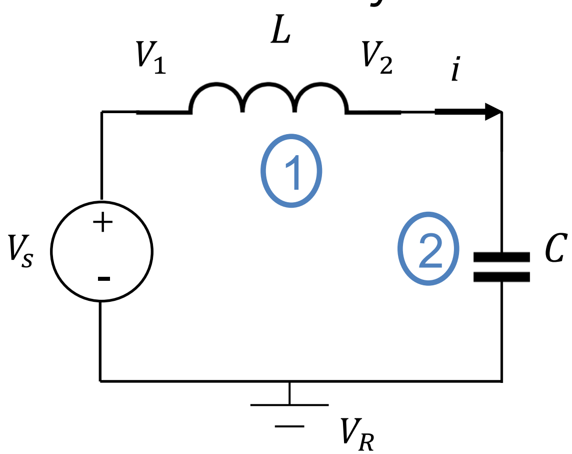
$$V_s = V_{12} + V_{2R} \qquad (1)$$

$$V_{12} = L\frac{di}{dt} \qquad (3)$$

$$P_s = P_{12} + P_{2R} \qquad (1')$$

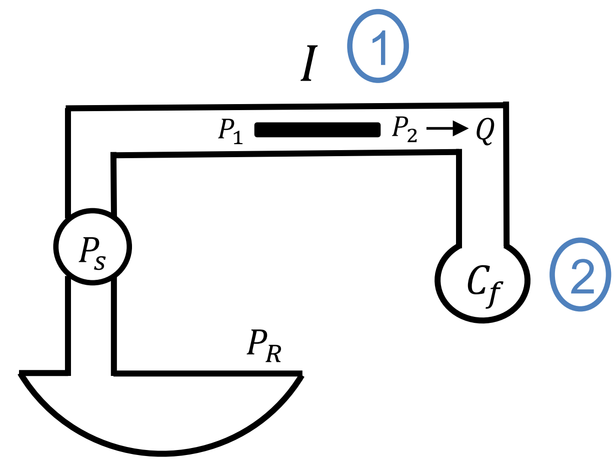
$$P_{12} = I\frac{dQ}{dt} \qquad (3')$$

Electrical system



$\approx$

Fluid system



Putting (3) in (1), we obtain

$$V_s = L\frac{di}{dt} + V_{2R}$$

$$\downarrow$$

$$\frac{di}{dt} = -\frac{1}{L}V_{2R} + \frac{1}{L}V_s \quad (\#)$$

Putting (3') in (1'), we obtain

$$P_s = I\frac{dQ}{dt} + P_{2R}$$

$$\downarrow$$

$$\frac{dQ}{dt} = -\frac{1}{I}P_{2R} + \frac{1}{I}P_s \quad (\#')$$

1 → T-type element
2 → A-type element

Example 3

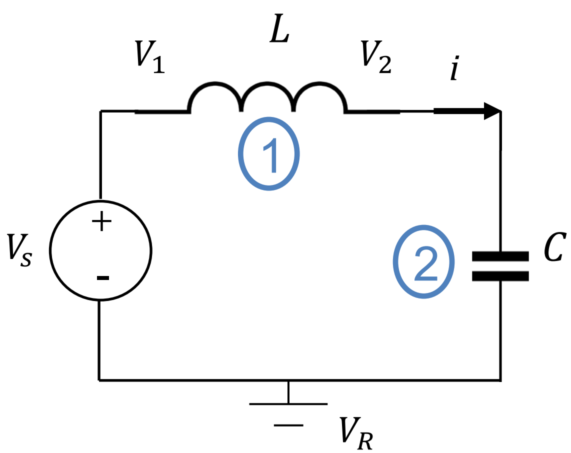
$$C\frac{dV_{2R}}{dt} = \text{i} \qquad (2)$$

$$\frac{di}{dt} = -\frac{1}{L}V_{2R} + \frac{1}{L}V_s \qquad (\#)$$

$$C_f\frac{dP_{2R}}{dt} = \text{Q} \qquad (2')$$

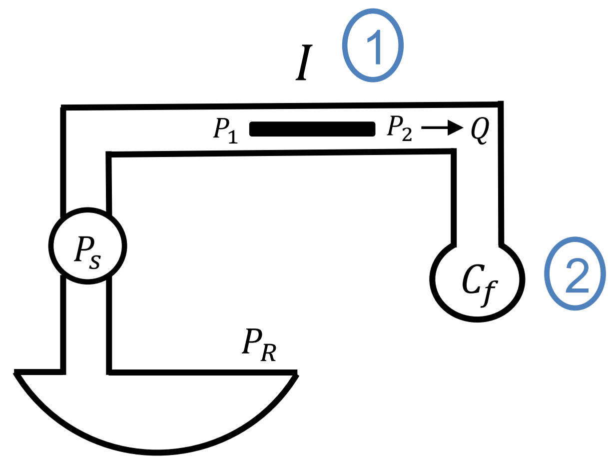
$$\frac{dQ}{dt} = -\frac{1}{I}P_{2R} + \frac{1}{I}P_s \qquad (\#')$$

Electrical system



≈

Fluid system



From (2), we have

$$\frac{dV_{2R}}{dt} = \frac{1}{C}i \qquad (\$)$$

↓ (\$) and (#)

From (2'), we have

$$\frac{dP_{2R}}{dt} = \frac{1}{C_f}Q \qquad (\$')$$

↓ (\$') and (#')

1 → T-type element
2 → A-type element

Example 3

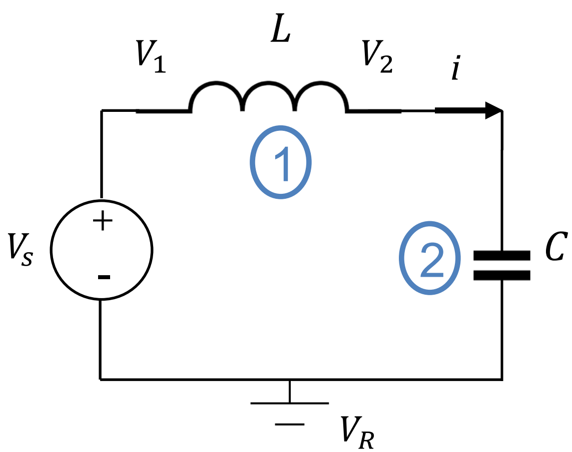
$$\frac{dV_{2R}}{dt} = \frac{1}{C}\mathrm{i} \qquad (\$)$$

$$\frac{di}{dt} = -\frac{1}{L}V_{2R} + \frac{1}{L}V_s \qquad (\#)$$

$$\frac{dP_{2R}}{dt} = \frac{1}{C_f}\mathrm{Q} \qquad (\$')$$

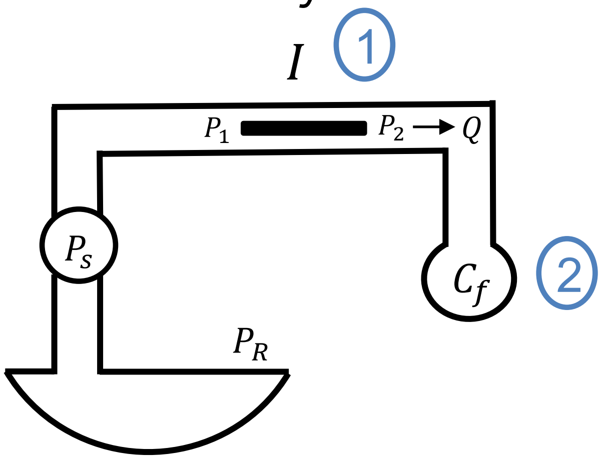
$$\frac{dQ}{dt} = -\frac{1}{I}P_{2R} + \frac{1}{I}P_s \qquad (\#')$$

Electrical system



$\approx$

Fluid system



↓ ($) and (#)

$$\begin{bmatrix} \frac{dV_{2R}}{dt} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} V_{2R} \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V_s$$

↓ ($') and (#')

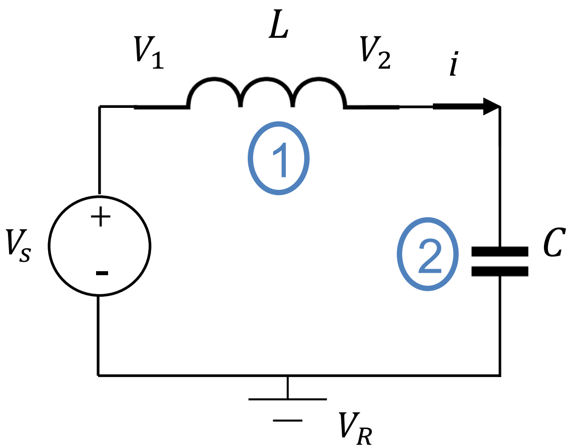
$$\begin{bmatrix} \frac{dP_{2R}}{dt} \\ \frac{dQ}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C_f} \\ -\frac{1}{I} & 0 \end{bmatrix} \begin{bmatrix} P_{2R} \\ Q \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} P_s$$

1 → T-type element

2 → A-type element

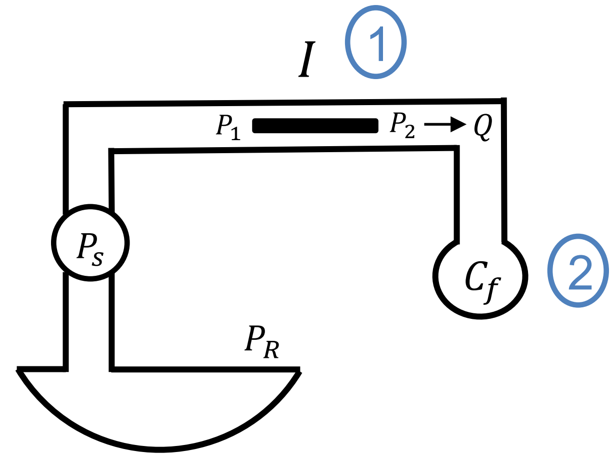
Example 3

Electrical system



$\approx$

Fluid system



↓

$$\begin{bmatrix} \dfrac{dP_{2R}}{dt} \\ \dfrac{di}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{C} \\ -\dfrac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} V_{2R} \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} V_s$$

↓

$$\begin{bmatrix} \dfrac{dP_{2R}}{dt} \\ \dfrac{dQ}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{C_f} \\ -\dfrac{1}{I} & 0 \end{bmatrix} \begin{bmatrix} P_{2R} \\ Q \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{I} \end{bmatrix} P_s$$

The A-type and T-type variables are suitable choices for state variables!

Summary of the lecture

- It is possible to establish analogies between two different physical domains
- The elements of a system can be classified as energy storing elements and dissipative elements
- The behaviour of the energy-storing elements is governed by A-type and T-type elements in terms of differential equations
- The behaviour of the dissipative elements is governed by static relationship between A-type and T-type variables
- The A-type and T-type variables are suitable state variable choices

Next lecture:

Modeling of interconnected multidomain systems