Control Engineering 2017-2018
Mock Exam Solutions
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- You have **3 hours** to complete the exam.
- You **can** use books and notes but **not** smartphones, computers, tablets and the like.
- Please write your answers using a pen, **not a pencil**.
- There are questions/exercises labeled as **Bonus**. These questions/exercises are optional and give you **extra** points if answered correctly.
- Please write down your Surname, Name, Student ID on each sheet.
- You will be given 2 sheets. If you need more, please ask. Please hand in **all the sheets** that you have used and the **text of the exam**.
- If you return the sheets, then your exam will be graded, unless you explicitly write "do not grade" on the first page.
- If your exam is graded, then the grade will be registered, even if the grade is lower than the one you got at the previous exam(s).

Good luck!

For the grader only

	Exercise 1	Exercise 2	Exercise 3	Exercise 4	Exercise 5
Points					
Bonus	×	×	×	×	

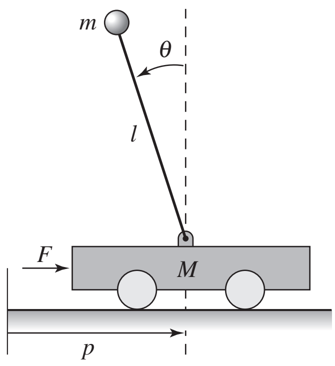

Figure 1: Inverted pendulum on a cart [Example 2.1, your textbook]

Exercise 1. Inverted pendulum on cart (10pt)

Consider the cart-pendulum system in Figure 1 and the equations of motion [Equation (2.9), your textbook] describing it and recalled below.

$$\begin{aligned} (M+m)\ddot{p} - ml(\cos\theta)\ddot{\theta} + c\dot{p} + ml(\sin\theta)\dot{\theta}^2 &= F \\ -ml(\cos\theta)\ddot{p} + (J+ml^2)\ddot{\theta} + \gamma\dot{\theta} - mgl\sin\theta &= 0. \end{aligned} \tag{1}$$

a. (3pt) Determine the translational kinetic co-energy of the cart with mass M, and the horizontal and vertical translational kinetic co-energy of the mass m to be balanced.

b. (1pt) Determine the rotational kinetic co-energy of the pendulum with moment of inertia J.

c. (2pt) Denoted by c and γ the coefficients of viscous friction, determine the Rayleigh dissipation function.

d. (2pt) Denoted by g the acceleration due to gravity and by F the force applied at the cart, determine the nonconservative potential function $-u^\top q$, where $u^\top = [mg\ F]$, and q is the generalized displacement (to be determined).

e. (2pt) Use the quantities determined before to show that (1) are the Euler-Lagrange equations of motion of the cart-pendulum system.

Solutions.

a. $\frac{1}{2}M\dot{p}^2$ (translational kinetic co-energy of the cart),

$\frac{1}{2}m(\dot{p} - l(\cos\theta)\dot{\theta})^2$ (horizontal translational kinetic co-energy of the mass m),

$\frac{1}{2}m(-l(\sin\theta)\dot{\theta})^2$ (vertical translational kinetic co-energy of the mass m).

b. $\frac{1}{2}J\dot{\theta}^2$ (rotational kinetic co-energy of the pendulum with moment of inertia J).

c. $\mathcal{D}(\dot{p},\dot{\theta}) = \frac{1}{2}c\dot{p}^2 + \frac{1}{2}\gamma\dot{\theta}^2$.

d. $-u^\top q = -mgl\cos\theta - Fp$,

where $l\cos\theta$ is the vertical position of the mass m and Fp is the work done by the external force F.

e. The Lagrangian function can be written as

$$\begin{aligned} L(p,\theta,\dot{p},\dot{\theta}) =& \frac{1}{2}M\dot{p}^2 + \frac{1}{2}m(\dot{p} - l(\cos\theta)\dot{\theta})^2 + \frac{1}{2}m(-l(\sin\theta)\dot{\theta})^2 + \frac{1}{2}J\dot{\theta}^2 \\ &+ mgl\cos\theta + Fp \\ &+ \int_0^t (\frac{1}{2}c\dot{p}^2 + \frac{1}{2}\gamma\dot{\theta}^2)dt. \end{aligned}$$

Note that the second and third term in L get simplified as:

$\frac{1}{2}m(\dot{p} - l(\cos\theta)\dot{\theta})^2 + \frac{1}{2}m(-l(\sin\theta)\dot{\theta})^2 = \frac{1}{2}m\dot{p}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - ml(\cos\theta)\dot{p}\dot{\theta}$.

The equations (1) are obtained from routine application of the Euler-Lagrange equations of motion.

The Euler-Lagrange equations of motion is:

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} = 0 \tag{2}$$

First consider the p direction and velocity:

$$\begin{aligned} \frac{\partial L}{\partial p} &= F \\ \frac{\partial L}{\partial \dot{p}} &= (M+m)\dot{p} - ml(\cos\theta)\dot{\theta} + \int_0^t (c\dot{p})dt \\ \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{p}}\Big) &= (M+m)\ddot{p} + ml(\sin\theta)\dot{\theta}^2 - ml(\cos\theta)\ddot{\theta} + c\dot{p} \end{aligned}$$

The Euler-Lagrange equation of motion (2) is:

$$(M+m)\ddot{p} + ml(\sin\theta)\dot{\theta}^2 - ml(\cos\theta)\ddot{\theta} + c\dot{p} = F \tag{3}$$

Next consider the θ direction and velocity:

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= ml(\sin\theta)\dot{p}\dot{\theta} - mgl\sin\theta \\ \frac{\partial L}{\partial \dot{\theta}} &= ml^2\dot{\theta} - ml(\cos\theta)\dot{p} + J\dot{\theta} + \int_0^t (\gamma\dot{\theta})dt \\ \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\theta}}\Big) &= ml^2\ddot{\theta} + ml(\sin\theta)\dot{\theta}\dot{p} - ml(\cos\theta)\ddot{p} + J\ddot{\theta} + \gamma\dot{\theta} \end{aligned}$$

The Euler-Lagrange equation of motion (2) is:

$$\begin{aligned} & ml^2\ddot{\theta} + ml(\sin\theta)\dot{\theta}\dot{p} - ml(\cos\theta)\ddot{p} + J\ddot{\theta} + \gamma\dot{\theta} - ml(\sin\theta)\dot{p}\dot{\theta} + mgl\sin\theta = 0 \\ & (ml^2 + J)\ddot{\theta} - ml(cos\theta)\ddot{p} + \gamma\dot{\theta} + mgl\sin\theta = 0 \end{aligned} \tag{4}$$

Exercise 2. Stabilizing an inverted pendulum (10pt)

Consider the model of an inverted pendulum [Example 2.2, your textbook]

$$\begin{aligned} \dot{x} &= \begin{bmatrix} x_2 \\ \frac{mgl}{J_t}\sin x_1 - \frac{\gamma}{J_t}x_2 + \frac{l}{J_t}(\cos x_1)u \end{bmatrix} \\ y &= x_1 \end{aligned} \tag{5}$$

and its linearization around the upright position [Exercise 2, Tutorial 5], i.e. around the equilibrium pair

$$(\bar{x}, \bar{u}) = (\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0), \tag{6}$$

given by

$$\begin{aligned} \Delta\dot{x} &= \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{mgl}{J_t} & -\frac{\gamma}{J_t} \end{bmatrix}}_{A} \Delta x + \underbrace{\begin{bmatrix} 0 \\ \frac{l}{J_t} \end{bmatrix}}_{B} \Delta u \\ \Delta y &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \Delta x, \end{aligned} \tag{7}$$

where $\Delta x = x - \bar{x}$, $\Delta u = u - \bar{u}$, $\Delta y = y - \bar{y}$, and $\bar{y} = C\bar{x}$. Note that in this case $\Delta x = x$, $\Delta u = u$.

For this exercise, take $\gamma = 0.2$, $J_t = 0.1$, $l = 0.5$, $m = 2$. g is the acceleration due to gravity.

a. (2pt) Compute the reachability matrix of system (7) and discuss whether or not the system is reachable.

b. (4pt) If the system is reachable, determine the matrix $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ in the state feedback control $\Delta u = -K\Delta x$ such that the closed-loop matrix $A - BK$ has its eigenvalues equal to $-1, -2$.

c. (4pt) Consider the original model of the inverted pendulum, i.e. the *nonlinear* system (5), in closed-loop with the control $u = \bar{u} - K(x + \bar{x}) = -Kx$:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} x_2 \\ \frac{mgl}{J_t}\sin x_1 - \frac{\gamma}{J_t}x_2 - \frac{l}{J_t}(\cos x_1)k_1 x_1 - \frac{l}{J_t}(\cos x_1)k_2 x_2 \end{bmatrix} \\ y &= x_1 \end{aligned} \tag{8}$$

Determine the system linearized around the equilibrium

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and check that the dynamic matrix is Hurwitz, i.e. it has all its eigenvalues with strictly negative real parts. What can you conclude about the stability of the equilibrium $\bar{x}$ of the closed-loop nonlinear system (8)? Explain in one sentence.

Solutions.

a. The reachability matrix is $\begin{bmatrix} 0 & 5 \\ 5 & -10 \end{bmatrix}$. This matrix has full rank. The system is therefore reachable.

b. The characteristic polynomial of A has coefficient list 1, $a_1 = 2$, $a_2 = -10g$. The desired polynomial has coefficient list 1, $p_1 = 3$, $p_2 = 2$. Therefore,

$$\begin{aligned} K &= \begin{bmatrix} p_1 - a_1 & p_2 - a_2 \end{bmatrix} \tilde{W}_r W_r^{-1} \\ &= \frac{1}{5} \begin{bmatrix} 1 & 10g + 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2g + 0.4 & 0.2 \end{bmatrix} \end{aligned}$$

c. The new linearized A is $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, with eigenvalues -1 and -2, so it is indeed Hurwitz. The equilibrium of the closed-loop system is therefore asymptotically stable.

Exercise 3. Observer and output feedback control for an inverted pendulum (10pt)

Consider again the linearized system (6), rewritten here as

$$\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{mgl}{J_t} & -\frac{\gamma}{J_t} \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ \frac{l}{J_t} \end{bmatrix}}_{B} u \\ y &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x, \end{aligned} \tag{9}$$

For this exercise, take $\gamma = 0.2$, $J_t = 0.1$, $l = 0.5$, $m = 2$. g is the acceleration due to gravity.

a. (2pt) Determine the observability matrix W_o and discuss whether the system is observable or not.

b. (2pt) Determine the observable canonical form of the system (matrices $\tilde{A}$ and $\tilde{C}$).

c. (3pt) Determine the gain matrix L such that the eigenvalues of $A - LC$ are equal to $-1, -1$. Write explicitly the observer for system (9).

d. (3pt) Using the matrix K of the stabilizing state feedback obtained in Exercise 2, point b. determine a dynamic output feedback controller that solves the output regulation problem, that is, (i) the closed-loop system is asymptotically stable and (ii) the output y asymptotically converges to the constant reference signal r.

   Hint If you did not find K and L, then use $K = \begin{bmatrix} 2g + 0.4 & 0.2 \end{bmatrix}, L = \begin{bmatrix} 0 & 10g + 1 \end{bmatrix}^\top$.

Solutions.

a. $W_o = I$. The system is observable.

b. Recall from the previous exercise that the characteristic polynomial of A has coefficient list 1, $a_1 = 2$, $a_2 = -10g$ (this is not worth new points). For $\tilde{A}$ and $\tilde{C}$, we use the familiar standard pattern.

$$\tilde{A} = \begin{bmatrix} 0 & 10g \\ 1 & -2 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

c. The desired polynomial has coefficient, $p_1 = 2$, $p_2 = 1$. Therefore,

$$L = W_o^{-1}\tilde{W}_o \begin{bmatrix} p_1 - a_1 & p_2 - a_2 \end{bmatrix}^\top = I \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 10g + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 10g + 1 \end{bmatrix}$$

   The observer is therefore

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ &= \begin{bmatrix} 0 & 1 \\ 10g & -2 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u + \begin{bmatrix} 0 \\ 10g + 1 \end{bmatrix} (y - \hat{y}) \\ \hat{y} &= C\hat{x} = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x} \end{aligned}$$

d. Set $u = -K\hat{x} + k_r r$ and substitute $\hat{y}$ with its definition.

Calculate

$$k_r = \frac{-1}{(C(A - BK)^{-1}B)} = \frac{-1}{-2.5} = 0.4 \tag{10}$$

This yields the controller

$$\begin{aligned}
\dot{\hat{x}} &= (A - LC - BK)\hat{x} + Bk_r r + Ly \\
&= \begin{bmatrix} 0 & 1 \\ -10g - 3 & -3 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 10g + 1 \end{bmatrix} y + \begin{bmatrix} 0 \\ 2 \end{bmatrix} r \\
u &= -K\hat{x} + k_r r.
\end{aligned}$$

Exercise 4. Loop shaping (10pt)

Consider a process whose dynamics is modeled as a single integrator, i.e.,

$$P(s) = \frac{1}{s},$$

which is controlled by a lead controller of the form

$$C(s) = k\frac{s+1}{s+p}.$$

The Bode diagram (magnitude and phase) of the loop transfer function $L(s) = C(s)P(s)$ with $k = 1$ is given in Figure 2.

a. (2pt) Determine the value p of the pole from the Bode diagram.

b. (2pt) Sketch the Nyquist plot of $L(s)$.

c. (2pt) Determine the phase crossover frequency and the gain margin.
Hint If you did not answer b., use the Nyquist plot in Fig. 3.

d. (2pt) Determine the gain crossover frequency and the phase margin.

e. (2pt) Discuss the closed-loop system stability or instability.
Hint If you did not answer b., use the Nyquist plot in Fig. 3.

f. **(Bonus)** (3pt) Determine the phase margin (in degrees) and gain crossover frequency analytically.
Hint Recall that the solution to a quartic equation of the form $ax^4 + bx^2 + c = 0$ can be found solving the second-order equation $ay^2 + by + c = 0$.

Solutions.

a. $p = 10$

b. See figure 4

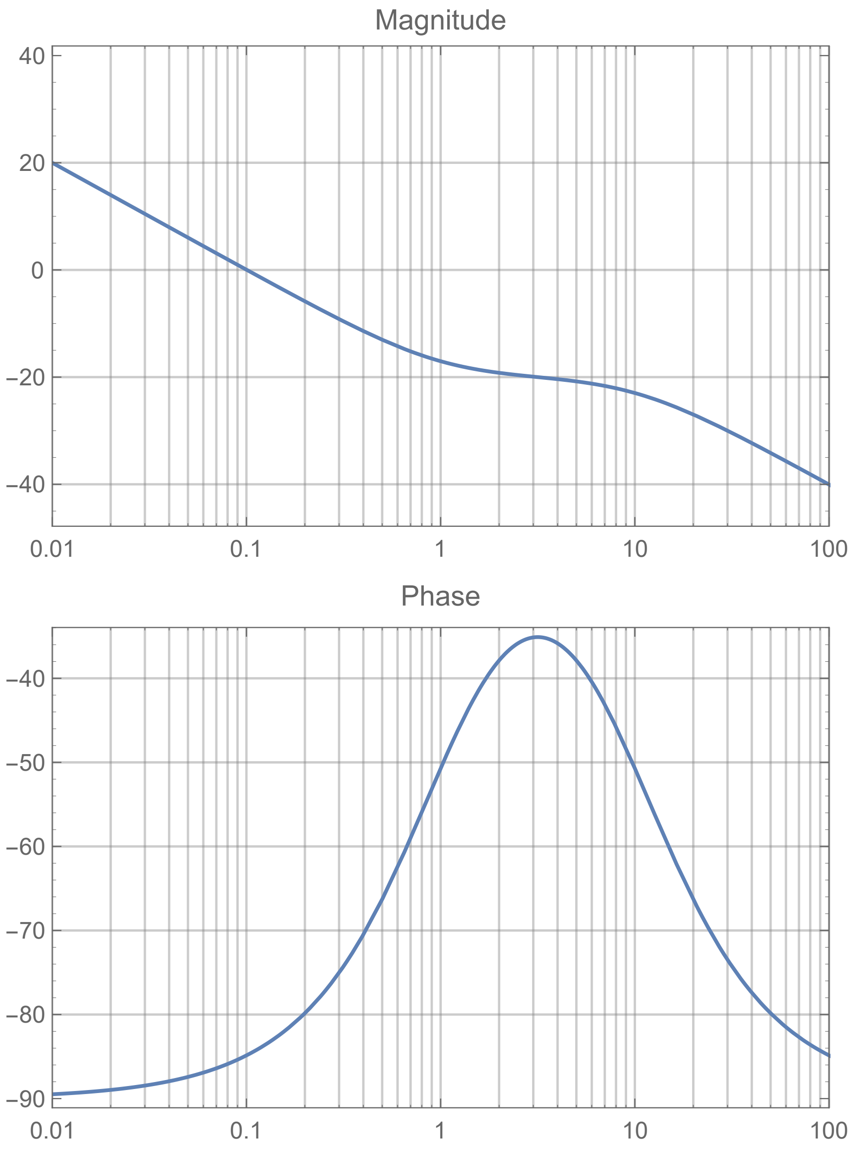

Figure 2: Bode diagram.

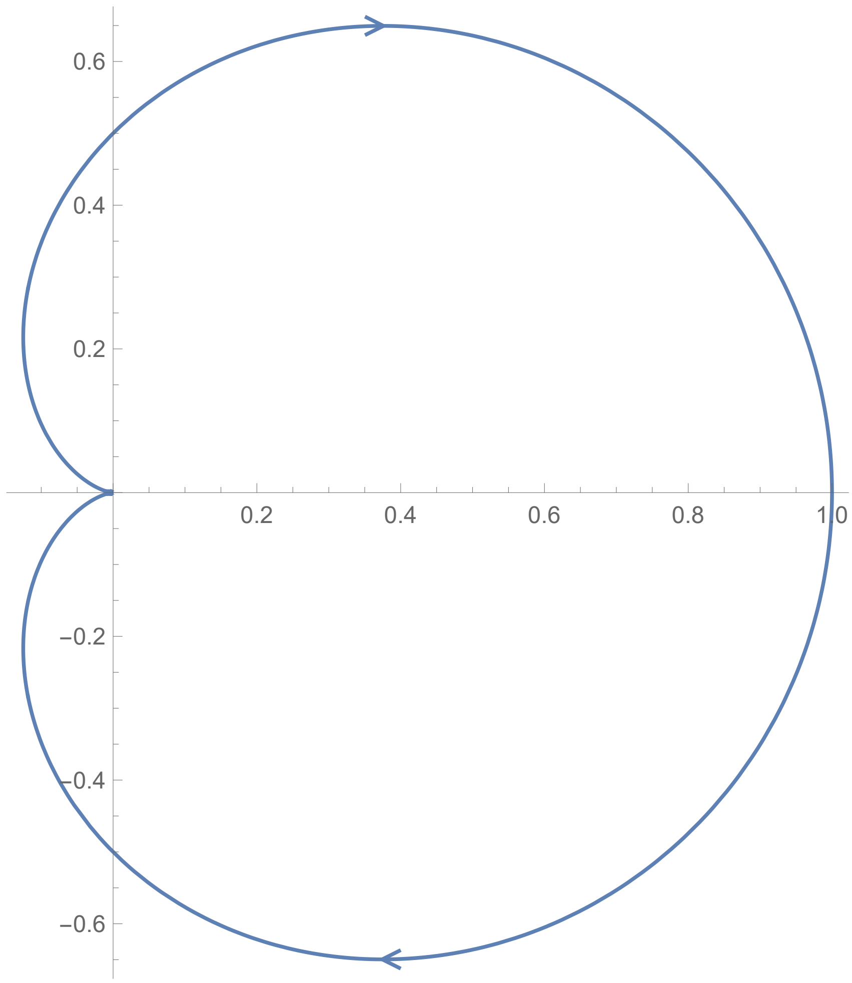

Figure 3: Alternative Nyquist plot of the incorrect loop transfer function $\frac{1}{(s+1)^2}$

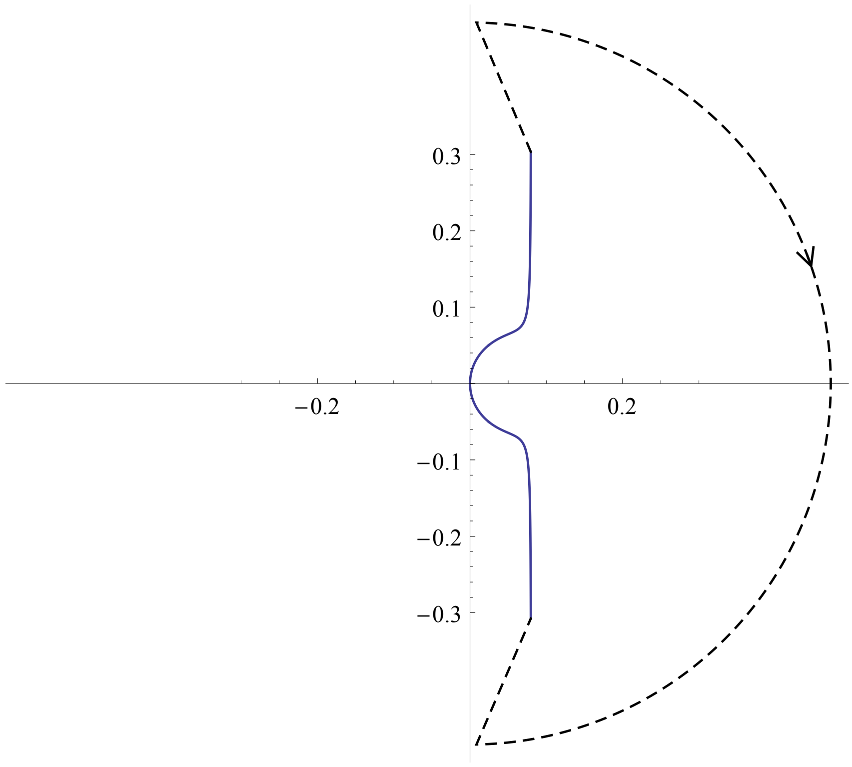

Figure 4:

c. It is seen that there is no phase crossover frequency and that the gain margin is infinite (the gain k can be increased indefinitely without leading the system to instability).

d. The phase cross-over frequency is $\omega_{pc} = 0.1$ rad/sec at which $\arg L(i\omega_{pc}) = -84°$, whence $\varphi_m = 96°$.

e. The system has no open-loop unstable poles in the Nyquist contour and 0 net encirclements of the point -1, hence the system is asymptotically stable.

f.
$$|L(i\omega)| = \frac{\sqrt{\omega^2+1}}{\omega\sqrt{\omega^2+100}} \text{ (1pt)}$$

$|L(i\omega_{gc})| = 1$ if

$$\omega_{gc}^2 = \frac{\omega_{gc}^2+1}{\omega_{gc}^2+100} \Leftrightarrow \omega_{gc}^4 + 99\omega_{gc}^2 - 1 = 0. \text{ (0.5pt)}$$

Hence,

$$\omega_{gc} = \sqrt{\frac{-99+\sqrt{99^2+4}}{2}} = 0.1 \text{ (0.5pt)}$$

It follows that

$$\varphi_m = 180° + \angle L(i\omega_{gc}) = 180° + \frac{180°}{\pi}(\arctan\omega_{gc} - \arctan\frac{\omega_{gc}}{10}) - 90° = 95.13°. \text{ (1pt)}$$

Exercise 5. Cruise control under periodic disturbance (10pt)

Consider the feedback loop represented in Fig. 5 where

$$P(s) = \frac{b}{s+a}$$

is the transfer function of the linearized dynamics of a car, where a, b are positive real numbers representing physical parameters. Set $a = 3$ and $b = 2$. Assume that $F(s) = 1$, and $n = 0$.

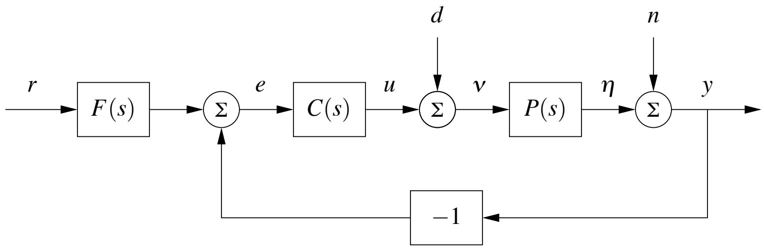

Figure 5: Feedback control system.

The car is riding over a series of bumps in the road, whose effect on the car is modelled as a periodic disturbance

$$d(t) = \bar{d}\sin(t+\phi),$$

with the amplitude $\bar{d}$ and the phase ϕ unknown.

Design a feedback controller with proper (i.e., number of zeros not larger than the number of poles) transfer function $C(s)$ such that both the following specifications are guaranteed:

a. (2pt) **(Disturbance rejection)** When $r = 0$ and $d(t) = \bar{d}\sin(t+\phi)$, $\lim_{t\to+\infty} y(t) = 0$.

b. (4pt) **(Asymptotic stability)** The closed-loop system is asymptotically stable and its characteristic polynomial, i.e., the polynomial $n_L(s) + d_L(s)$, with $L(s) = C(s)P(s) = \frac{n_L(s)}{d_L(s)}$, is given by

$$s^3 + 3s^2 + 3s + 1.$$

Furthermore, answer the following questions:

c. (2pt) **(Tracking)** When $r = \bar{r} \neq 0$ and $d(t) = \bar{d}\sin(t+\phi)$, compute the steady state $\lim_{t\to+\infty} y(t)$, using Final Value Theorem or the step response.

d. (2pt) **(A new controller)** Give the expression of a new controller

$$C(s) = \frac{n_c(s)}{d_c(s)}$$

with $d_c(s)$ a polynomial to be determined exactly, and $n_c(s)$ a polynomial whose degree only must be determined, such that, (i) the closed-loop system is asymptotically stable, (ii) and $\lim_{t\to+\infty} y(t) = 0$ when $r = \bar{r} \neq 0$ and $d(t) = \bar{d}\sin(t+\phi)$.

Solutions.

a. For the first requirement the open-loop transfer function $L(s)$ must have a pair of complex conjugate poles on the imaginary axis with imaginary part equal to $\pm i1$ (1pt). This is achieved by considering the controller

$$C(s) = \frac{1}{s^2+1}\tilde{C}(s) \text{ (1pt)}$$

with $\tilde{C}(s)$ designed so as to fulfil the second specification (asymptotic stability).

b. By Sylvester's theorem (Lecture 13 and Tutorial 7), arbitrary pole assignment is possible (under co-primeness assumption) by considering the controller

$$\tilde{C}(s) = d_2 s^2 + d_1 s + d_0 \text{ (1pt)}$$

The characteristic polynomial $n_L(s) + d_L(s)$ of the closed-loop system is then given by

$$\begin{aligned} & n_L(s) + d_L(s) \\ = \; & (s^2+1)(s+a) + b(d_2 s^2 + d_1 s + d_0) \\ = \; & s^3 + (a + bd_2)s^2 + (bd_1 + 1)s + a + bd_0. \text{ (1pt)} \end{aligned}$$

When it is set equal to $s^3 + 3s^2 + 3s + 1$ gives

$$\begin{aligned} d_2 &= \tfrac{3-a}{b} \\ d_1 &= \tfrac{2}{b} \quad \text{(1pt)} \\ d_0 &= -\tfrac{2}{b} \end{aligned}$$

For $a = 3$ and $b = 2$,

$$\begin{aligned} d_2 &= 0 \\ d_1 &= 1 \quad \text{(1pt)} \\ d_0 &= -1 \end{aligned}$$

c. The closed-loop transfer function $G_{yr}(s)$ is given by

$$G_{yr}(s) = \tfrac{L(s)}{1+L(s)} = \tfrac{n_L(s)}{d_L(s)+n_L(s)} = \tfrac{b(s-1)}{s^3+3s^2+3s+1} \quad \text{(1pt)}$$

and hence

$$\lim_{t\to+\infty} y(t) = -b\bar{r} = -2\bar{r}. \text{ (1pt)}$$

d. The new controller should have the form

$$C(s) = \frac{1}{s(s^2+1)}\hat{C}(s) \text{ (1pt)}$$

with $\hat{C}(s)$ given by (in view of Sylvester's theorem and Tutorial 7)

$$\tilde{C}(s) = d_3 s^3 + d_2 s^2 + d_1 s + d_0 \text{ (1pt)}$$

whose coefficients have to be determined in such a way that the closed-loop system is asymptotically stable.