university of groningen

Exam Numerical Methods

November 9th 2021 18.45-21.45

It is allowed to use a book (paper version only) and lecture notes, as well as a (graphical) pocket calculator. The use of electronic devices (tablet, laptop, mobile phone, etc.) is not allowed.

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: $\boxed{10}$

Practica: $\boxed{18}$ For the 6 computer practica a maximum of 6*3=18 points can be earned.

1. Consider the equation $x\,e^{x} = 2,$ with solution $x \approx 0.8$.

   (a) $\boxed{3}$ Someone uses the iterative method $x_{n+1} = \ln(2/x_n)$, with $x_0 = 0.8$.
   Will this method converge or diverge? Explain.

   (b) $\boxed{3}$ (1) Give the iteration formula when Newton's method is used for this problem.
   (2) Use this method to compute x_1, starting with $x_0 = 0.8$, and give an error estimate.

   (c) $\boxed{7}$ For the iterative method $x_{n+1} = 2\,e^{-x_n}$, with $x_0 = 0.8$,
   the first iterations are given by

n	1	2	3	4	5
x_n	0.898658	0.814231	0.885959	0.824637	0.876788

   (1) Will this method give fast convergence? Explain.
   (2) Determine an error estimate for x_5.
   (3) How many more iterations are required for an accuracy of 1.0E-4?
   (4) The convergence is linear. Determine an improved value (for x_5) via extrapolation, using a technique similar to Euler's method for o.d.e.'s.
   Does this enhance the accuracy in this case? Explain.

2. Consider the integral $$\int_{-1}^{1} \frac{1}{x+2}\,dx = \ln(3) \approx 1.0986.$$

   (a) $\boxed{7}$ (1) For a grid with 5 segments, compute the sub-area for the middle segment:
   (i) If the Midpoint method is used.
   (ii) If the Trapezoidal method is used.
   (iii) If Simpson's method is used.
   (2) Give an error bound for the Trapezoidal value for the middle segment in (1)(ii).
   Hint: the min/max of the derivatives are 'easy' to find.

   With the Trapezoidal method the following results are obtained, with $I(n)$ the approximation of the integral on a grid with n segments.

n	16	32	64	128	256
$I(n)$	1.0980 3532	1.0984 6772	1.0985 7612	1.0986 0324	1.0986 1002

   (b) $\boxed{6}$ (1) Explain that an extrapolation for $I(32)$ is allowed.
   (2) Compute the improved solution for $I(32)$ by means of extrapolation.
   Compute the exact(!) error for this extrapolation.
   (3) Give an error estimate for $I(256)$ based on $I(n)$ values.
   (4) How many segments (use powers of 2) are required for an accuracy as in b(2)?

P.T.O.

3. Consider the differential equation $y'(x) = \alpha\, x\, y$, with boundary condition $y(0) = 1$. The parameter α will be determined below.

   (a) $\boxed{4}$ (1) Take $\alpha = 4$. Use explicit Euler, with steps of $\Delta x{=}0.5$, to approximate $y(1.0)$.
   (2) Take $\alpha = 4$. Use explicit Euler, with steps of $\Delta x{=}0.25$, to approximate $y(1.0)$.
   (3) Show that the exact(!) error at $x{=}0.5$ approximately decreases linearly.
   Hint: you may use that the exact value is $y(0.5) = \sqrt{e} \approx 1.649$.

   (b) $\boxed{4}$ Take $\alpha = 2$. Compute the solution at $x = 0.5$ on a grid with $\Delta x = 0.5$
   (1) using Heun's method (RK2).
   (2) using the implicit Euler method.

   (c) $\boxed{5}$ For $\alpha = -4$, the solution is determined using a 3rd (!) order RK method. The result at a selection of x locations, obtained on 5 grids, is as follows

x_n	$\Delta x = 0.4$	$\Delta x = 0.2$	$\Delta x = 0.1$	$\Delta x = 0.05$	$\Delta x = 0.025$
0.4	7.482667 E-1	7.281545 E-1	7.263596 E-1	7.261732 E-1	7.261519 E-1
0.8	2.894256 E-1	2.780319 E-1	2.780040 E-1	2.780321 E-1	2.780366 E-1
1.2	1.207098 E-2	5.266943 E-2	5.579896 E-2	5.609823 E-2	5.613051 E-2
1.6	-6.628671 E-3	3.894170 E-3	5.785295 E-3	5.956451 E-3	5.973816 E-3
2.0	1.092334 E-2	3.032294 E-5	2.989395 E-4	3.318199 E-4	3.350623 E-4

   (1) Is there a stability limit visible? Explain.
   (2) For $\Delta x = 0.2$ the computation is unstable. Why is that not visible?
   (3) Compute the q-factor at $x = 1.2$. What is your conclusion?
   (4) Give an improved value for the solution at $x = 1.2$ on the fine grid.

4. Consider the coordinate points $f(2) = 1$, $f(3) = 5$, $f(4) = 12$. To approximate $f(1)$, we will use a straight-line fit $y = a + bx$ through the given points.

   (a) $\boxed{7}$ (1) Determine a coordinate transformation such that the new x-points are -1,0,1.
   (2) Set up a least-squares fit through the original data.

   (b) $\boxed{2}$ (1) Use the curve-fit to approximate $f(1)$.
   (2) Determine the error for $x = 2$.

5. Consider the equation $A\vec{x} = \vec{b}$, with
$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -4 & 2 \\ 2 & 1 & -5 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix}$$

   (a) $\boxed{4}$ (1) Compute $\|r^{(0)}\|_\infty$, i.e. the max.-norm of the initial residual for $\vec{x}_0 = (1\ 1\ 1)^T$.
   (2) How many Jacobi iterations are needed to reduce this error with a factor 100?

   (b) $\boxed{6}$ (1) Take $\vec{x}_0 = (1\ 1\ 1)^T$. Compute $\vec{x}_1$, the result after 1 Jacobi iteration.
   (2) Take $\vec{x}_0 = (1\ 1\ 1)^T$. Compute the result after 1 SOR iteration with $\omega = 2$.

   (c) $\boxed{2}$ Explain how the system should be modified such that TDMA can be used and give the new matrix and vector.

6. Consider for x in [0,1] the partial diff. eqn. $\phi_t = k\,\phi_{xx} + H(x)$, with $H(x){=}400(x - x^2)$. Hence, the source term $H(x)$ is not depending on time. The boundary and initial conditions are $\phi(0,t){=}\phi(1,t){=}\phi(x,0){=}20$. For $\partial^2/\partial x^2$ the usual [1 -2 1]-formula is used.

   (a) $\boxed{7}$ Take $k = 0.01$ and use a grid size $\Delta x = 0.5$ (i.e. 2 segments, 3 grid points in total).
   (1) For which time step Δt will the explicit Euler method be stable?
   (2) Take $\Delta t = 0.25$ and give the solution after 1 time step using explicit Euler.

   (b) $\boxed{5}$ For the implicit Euler method, each time step a system $A\vec{x} = \vec{b}$ has to be solved. Give the i-th row of the system, i.e. $A(i,:)$ and $b(i)$, for general Δt and Δx.

Total: $\boxed{100}$