Geometry 2024 – Course programme

Week 1: $\mathbb{R}^n$ and $\mathbb{C}^n$ as affine spaces. Real and complex + smooth and regular curves. Characterisations of regular curves in $\mathbb{R}^2$ and $\mathbb{C}^2$: Implicit equations, graphs, parametric curves.

Examples: The lemniscate of Bernoulli, semi-cubical parabola, hypo- and epicycloids. Conic sections.

Reading material: [1, Chapter 7].

Week 2: Arc-length. Frenet-Serret formulas (Euclidean case). Curvature and torsions of a regular curve and their explicit formulas in $\mathbb{R}^3$. Envelopes and Evolutes. Wave-fronts and Huygens's principle.

Examples: Envelopes of Simson and Steiner lines. Evolutes of specific curves.

Reading material: [1, Chapter 7].

Week 3: Hypersurfaces in $\mathbb{R}^n$ and $\mathbb{C}^n$. Quadrics in $\mathbb{R}^n$ and $\mathbb{C}^n$. Tangent spaces and first fundamental form (aka Riemannian metric). Normal and principal curvatures.

Examples: Spherical, Euclidean, and hyperbolic geometry.

Reading material: [1, Chapters 6.1 and 8] and [3, Chapters 16.1 and 16.2].

Week 4: Euler's theorem. The second fundamental form. Gaussian curvature. Gauss's Theorema Egregium; Gaussian curvature as an isometry invariant for 2-surfaces. Hyperbolic, elliptic and parabolic points.

Examples: planes, spheres, ellipsoids, cylinders, cones, helicoids and catenoids, pseudosphere, Dini's surface, and Möbius band.

Reading material: [1, Chapter 8] and [3, Chapters 19.1-4].

Week 5: Geodesics as local minimisers and geodesic equations. The local Gauss-Bonnet theorem.

Examples: Geodesic triangles in spherical, Euclidean, and hyperbolic geometry.

Reading material: [3, Chapters 21.1-3]; see also [4, Chapter 3].

Week 5/6: General affine geometry: Affine subspaces and mappings. The affine group. Parallelism. Barycentric and homogeneous coordinates. Real projective space $\mathbb{R}P^1$. Thales's, Pappus's and Desargues's theorems. The fundamental theorem of affine geometry.

Reading material: [1, Chapter 1].

Week 6/7: Euclidean geometry. Euclid's postulates and their connection to the different geometries treated in this course. Isometry group of the Euclidean space $\mathbb{R}^n$ and the sphere S^{n-1}. Reflections and inversions, connection to the upper-half plane.

Reading material: [1, Chapters 2 and 3.4].

Week 7/8: Projective geometry. Projective spaces, homogeneous coordinates. Projective transformations and projective linear group. Perspective, perspectivities, and hyperplanes at infinity. Projective completion. Projective duality and general Desargues's theorem.

Examples: Real and complex projective spaces, $\mathrm{PSL}(2, \mathbb{R})$, $\mathrm{PSL}(2, \mathbb{C})$ and the circular group, general Pappus's theorem, projective frames, the-cross-ratio, and perspective.

Reading material: [1, Chapter 5].

Week $8^\star$: Differential invariants: Group actions (on the space of derivatives) and invariantisation. (Special) affine arc-length and affine Frenet-Serret formulas.

Examples: Euclidean curvature as a differential invariant for isometry group $\mathrm{SO}(2) \ltimes \mathbb{R}^2$. Special affine curvature as a differential invariant for the special affine group $\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$.

Reading material: [2].

Information about the exam/resit:

- The topics of week $8^\star$ **will not** be part of the exam, resp. resit.
- You will be allowed to take **one A4 sheet of paper** (2-sided, written by hand or typeset) with you to the exam and resit.
  Tip: Don't overdo it, but rather write down important definitions with examples, e.g. of Gaussian curvature, geodesic equations, homogeneous/barycentric coordinates, etc.
- The exam dates are on **April 9, 8:30-10:30** (exam) and **June 27, 11:45-13:40** (resit); please check the location here.

Literature (on which the lecture notes are based)

[1] M. Audin, *Geometry*, Springer, Berlin, Heidelberg, 2003, **The coursebook**.

[2] B. Brongers and T.V. Henriksen, *Differential invariants*, Lecture notes, RUG, 2023.

[3] H.S.M. Coxeter, *Introduction to geometry*, Wiley, 2d edition, 1991.

[4] R. van der Veen, *Geometry*, Lecture notes, RUG, 2022.