# Control Engineering
# Lecture 8
# ver. 2.3

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# Today

**Last lectures** (Sections 7.1-7.3)

- Observability
- Observable canonical form
- Asymptotic observer
- Output feedback control

**Today's lecture** (Sections 8.1-8.2)

- Laplace transform
- Transfer functions (Chapter 8)
- Transfer functions of linear differential equations
- Gains, poles and zeros
- Transfer functions (cont'd)
  - Block algebra
  - Control systems transfer functions

# Laplace transform

One-sided Laplace transform of the signal $f(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $f(t)$ grows no faster than $\mathrm{e}^{s_0 t}$, $s_0 \in \mathbb{R}$,

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st}\, dt, \quad \mathrm{Re}[s] > s_0$$

The growth condition on $f$ guarantees the Laplace transform is well-defined

$$\int_0^\infty f(t)e^{-st}\, dt \leq \int_0^\infty \mathrm{e}^{s_0 t} e^{-\mathrm{Re}[s]t - i\mathrm{Im}[s]t}\, dt = \int_0^\infty e^{-(\mathrm{Re}[s]-s_0)t}[\cos(\mathrm{Im}[s]t) - i\sin(\mathrm{Im}[s]t)]\, dt$$

Important properties:

- $\mathcal{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0)$
  $\mathcal{L}\left[\frac{d}{dt} f(t)\right] = \int_0^\infty \frac{d}{dt} f(t)e^{-st}\, dt = \int_0^\infty e^{-st} df = f(t)e^{-st}|_0^\infty - s\int_0^\infty f(t)e^{-st} dt$
- $\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{1}{s} F(s)$
- $\mathcal{L}[af_1(t) + bf_2(t)] = aF_1(s) + bF_2(s)$ (superposition)
- **Final Value Theorem (FVT)** If the limit $\lim_{t\to\infty} f(t)$ is finite, then

$$f_{steady} = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

  $\mathcal{L}\left[\frac{d}{dt} f(t)\right] = \int_0^\infty \dot{f}(t)e^{-st}\, dt = sF(s) - f(0)$. Take the limits as $s \to 0$ and move the limit inside the integral (it can be done in this case – see A. Lewis' textbook p. 627) to obtain $(\lim_{t\to\infty} f(t)) - f(0) = (\lim_{s\to 0} sF(s)) - f(0)$.

# Laplace transform

If the signal is vector-valued, i.e. $f(t) : \mathbb{R}_+ \to \mathbb{R}^n$, and

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \dots \\ f_n(t) \end{pmatrix},$$

then $F(s) = \mathcal{L}[f(t)]$ denotes the vector of the Laplace transforms of each component, that is

$$F(s) = \mathcal{L}[f(t)] = \begin{pmatrix} \mathcal{L}[f_1(t)] \\ \mathcal{L}[f_2(t)] \\ \dots \\ \mathcal{L}[f_n(t)] \end{pmatrix} = \begin{pmatrix} F_1(s) \\ F_2(s) \\ \dots \\ F_n(s) \end{pmatrix}.$$

# Laplace transform

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$

$$sX(s) - x(0) = AX(s) + BU(s)$$

Rearrangement gives $sX(s) - AX(s) = (sI - A)X(s) = x(0) + BU(s)$

$$X(s) = (sI_n - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

Laplace transform of the output

$$Y(s) = CX(s) + DU(s) = C(sI_n - A)^{-1}x(0) + \Big(C(sI - A)^{-1}B + D\Big)U(s)$$

# System transfer function

In the time domain, state and output response of a linear system are

$$
\begin{aligned}
x(t) = &\quad \mathrm{e}^{At}x(0) + \int_0^t \mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau \\
y(t) = &\quad C\mathrm{e}^{At}x(0) + \int_0^t C\mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau + Du(t)
\end{aligned}
$$

By comparison,

$$
\begin{aligned}
\mathcal{L}[\mathrm{e}^{At}] = &\quad (sI_n - A)^{-1} \\
\mathcal{L}[\textstyle\int_0^t \mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau] = &\quad (sI - A)^{-1}BU(s) = \mathcal{L}[\mathrm{e}^{At}B]\mathcal{L}[u(\tau)] \\
\mathcal{L}[C\mathrm{e}^{At}] = &\quad C(sI_n - A)^{-1} \\
\mathcal{L}[\textstyle\int_0^t C\mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau + Du(t)] = &\quad (C(sI - A)^{-1}B + D)U(s)
\end{aligned}
$$

# System transfer function

$$Y(s) = C(sI_n - A)^{-1}x(0) + \Big(C(sI_n - A)^{-1}B + D\Big)U(s)$$

The transfer function describes the input-output relation in the Laplace domain for **zero initial state** $x(0) = 0$

Transfer function is the function

$$G(s) = \frac{Y(s)}{U(s)} = C(sI_n - A)^{-1}B + D$$

# Laplace transform table

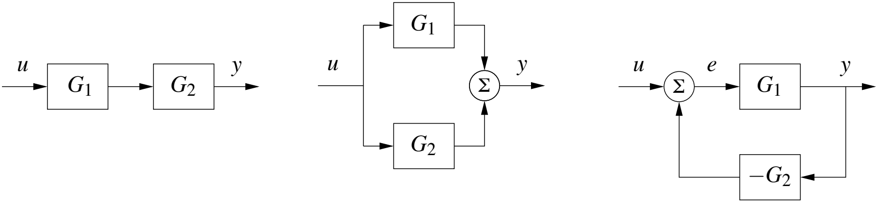
| $f(t) \quad (t \geq 0)$ | $\mathcal{L}[f(t)]$ | Region of Convergence |
|---|---|---|
| $1$ | $\frac{1}{s}$ | $\sigma > 0$ |
| $\delta_D(t)$ | $1$ | $\lvert\sigma\rvert < \infty$ |
| $t$ | $\frac{1}{s^2}$ | $\sigma > 0$ |
| $t^n \quad n \in \mathbb{Z}^+$ | $\frac{n!}{s^{n+1}}$ | $\sigma > 0$ |
| $e^{\alpha t} \quad \alpha \in \mathbb{C}$ | $\frac{1}{s-\alpha}$ | $\sigma > \Re\{\alpha\}$ |
| $te^{\alpha t} \quad \alpha \in \mathbb{C}$ | $\frac{1}{(s-\alpha)^2}$ | $\sigma > \Re\{\alpha\}$ |
| $\cos(\omega_o t)$ | $\frac{s}{s^2+\omega_o^2}$ | $\sigma > 0$ |
| $\sin(\omega_o t)$ | $\frac{\omega_o}{s^2+\omega_o^2}$ | $\sigma > 0$ |
| $e^{\alpha t}\sin(\omega_o t+\beta)$ | $\frac{(\sin\beta)s+\omega_o^2\cos\beta-\alpha\sin\beta}{(s-\alpha)^2+\omega_o^2}$ | $\sigma > \Re\{\alpha\}$ |
| $t\sin(\omega_o t)$ | $\frac{2\omega_o s}{(s^2+\omega_o^2)^2}$ | $\sigma > 0$ |
| $t\cos(\omega_o t)$ | $\frac{s^2-\omega_o^2}{(s^2+\omega_o^2)^2}$ | $\sigma > 0$ |
| $\mu(t)-\mu(t-\tau)$ | $\frac{1-e^{-s\tau}}{s}$ | $\lvert\sigma\rvert < \infty$ |

**Table 4.1.** Laplace transform table

The symbol $\sigma$ here specifies the region of convergence $\mathrm{Re}[s] > s_0$

# Block diagrams and transfer functions

$$\dot{x} = Ax + Bu, \; y = Cx + Du \qquad G(s) = C(sI - A)^{-1}B + D$$



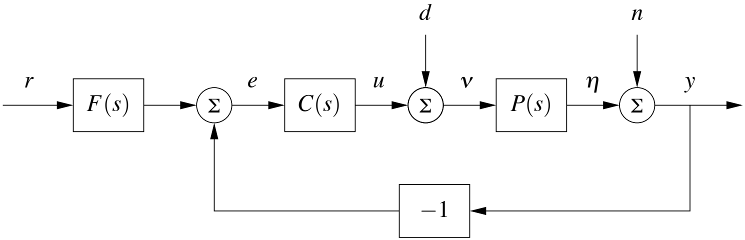

(a) **Series connection** $y = G_2(G_1 u) = G_2 G_1 u \Rightarrow \boldsymbol{G_{yu} = G_2 G_1}$

(b) **Parallel connection** $y = G_1 u + G_2 u = (G_1 + G_2)u$
$\Rightarrow \boldsymbol{G_{yu} = G_1 + G_2}$

(b) **Feedback connection** $y = G_1(u - G_2 y) = G_1 u - G_1 G_2 y$

$$\Rightarrow \boldsymbol{G_{yu} = (1 + G_1 G_2)^{-1} G_1}$$

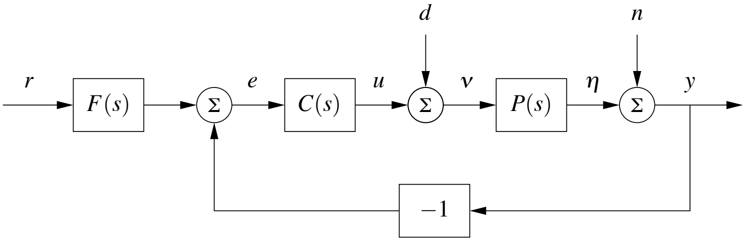
# Control system transfer functions




Negative feedback loop

- *P* **process**
- *C* **feedback controller**
- *F* **feedforward controller**

# Control system transfer functions



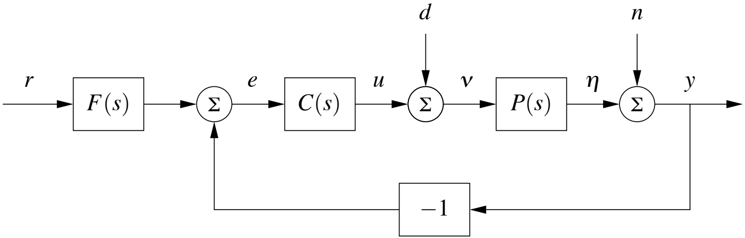

Find the transfer function relating the signals $r$ **reference**, $d$ **load disturbance**, $n$ **measurement noise**, $y$ **measured output**, $e$ **error**

$$e = Fr - y = Fr - (n + P(d + u)) = Fr - (n + P(d + Ce))$$

Hence

$$(1 + PC)e = Fr - n - Pd \Rightarrow e = \frac{F}{1 + PC}r - \frac{1}{1 + PC}n - \frac{P}{1 + PC}d$$

# Control system transfer functions




$$e = \underbrace{\frac{F}{1+PC}}_{G_{er}} r - \underbrace{\frac{1}{1+PC}}_{G_{en}} n - \underbrace{\frac{P}{1+PC}}_{G_{ed}} d$$

Reference-output transfer function

$$\begin{aligned} e = Fr - y \Rightarrow y = Fr - e = \quad & Fr - \frac{F}{1+PC} r + \frac{1}{1+PC} n + \frac{P}{1+PC} d \\ = \quad & \underbrace{\frac{FPC}{1+PC}}_{G_{yr}} r + \underbrace{\frac{1}{1+PC}}_{G_{yn}} n + \underbrace{\frac{P}{1+PC}}_{G_{yd}} d \end{aligned}$$

# Today

- Transfer functions (Chapter 8 of the textbook)
  - Block algebra
  - Control systems transfer functions
- Laplace transforms (Section 8.5)
- **Bode plots** (Section 8.4)
  - Harmonic response

# Frequency or harmonic response

## Harmonic response (Lecture 6)

The steady state output response of the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

with $A$ a Hurwitz matrix, to a sinusoidal input

$$u(t) = \cos \omega t$$

is a sinusoidal signal

$$y(t) = M(\omega) \cos(\omega t + \theta(\omega))$$

of the same frequency $\omega$ with amplitude

$$M(\omega) = \sqrt{\mathrm{Re}(W(i\omega))^2 + \mathrm{Im}(W(i\omega))^2}$$

and phase

$$\theta(\omega) = \arctan \frac{\mathrm{Im}(W(i\omega))}{\mathrm{Re}(W(i\omega))}$$

where

$$W(s) = C(sI - A)^{-1}B + D$$

# Properties of the frequency response

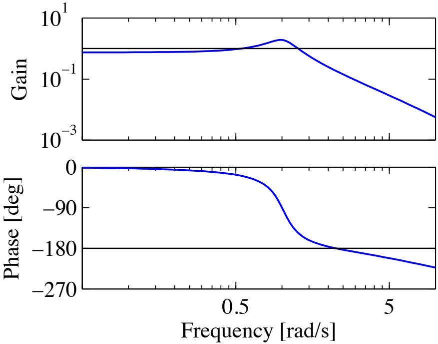
Recall

$$y_{st} \quad = \quad M(\omega)\cos(\omega t + \theta(\omega))$$

where

$$W(i\omega) = M(\omega)\mathrm{e}^{i\theta(\omega)} \quad \text{and} \quad W(s) = C(sI - A)^{-1}B + D$$

**Zero frequency or DC gain**
$W(i\omega)|_{\omega=0} = C(-A)^{-1}B + D$
(defined only if $A$ non-singular)




**Bandwidth** $\omega_b$ It is the frequency for which

$$\frac{M(\omega)}{M(0)} \geq \frac{1}{\sqrt{2}}, \quad \text{for all } \omega \in [0, \omega_b]$$

when $M(0) \neq 0$ ($A$ non-singular).
That is, $[0, \omega_b]$ is the range of frequencies $\omega$ over which the gain has decreased by not more than $\frac{1}{\sqrt{2}}$

# Properties of the frequency response

Recall

$$y_{st} = M(\omega)\cos(\omega t + \theta(\omega))$$

where

$$W(i\omega) = M(\omega)\mathrm{e}^{i\theta(\omega)} \quad \text{and} \quad W(s) = C(sI - A)^{-1}B + D$$

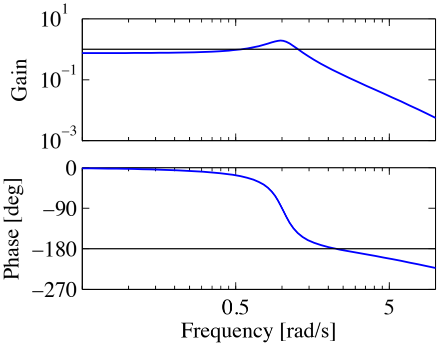
The **resonant peak** $M_r$ is the largest value of the frequency/harmonic response

$$M_r = \max_{\omega \geq 0} M(\omega)$$

The **peak frequency** $\omega_{mr}$ is the frequency at which the reasonant peak is attained

$$\omega_{mr} = \arg\max_{\omega \geq 0} M(\omega)$$

Hence $M(\omega_{mr}) = M_r$.



# The Bode plot

Recall that the transfer function $G(s)$ describes the harmonic (frequency) response of a linear system

$$u(t) = \sin(\omega t) \Rightarrow y(t) = M(\omega)\sin(\omega t + \varphi(\omega))$$

where

$$G(i\omega) = M(\omega)\mathrm{e}^{i\varphi(\omega)}$$

$$M(\omega) = |G(i\omega)| = \sqrt{(\mathrm{Re}\, G(i\omega))^2 + (\mathrm{Im}\, G(i\omega))^2}$$

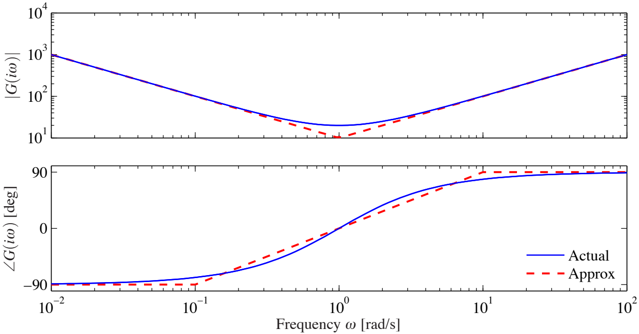
$$\varphi(\omega) = \arctan \frac{\mathrm{Im}\, G(i\omega)}{\mathrm{Re}\, G(i\omega)}$$

**Convention**

- $-\pi < \varphi(\omega) \leq \pi$
- If $\varphi(\omega)$ represented in **degrees**, $\varphi(\omega) = \angle G(i\omega)$
- If $\varphi(\omega)$ represented in **radians**, $\varphi(\omega) = \arg G(i\omega)$

# The Bode plot

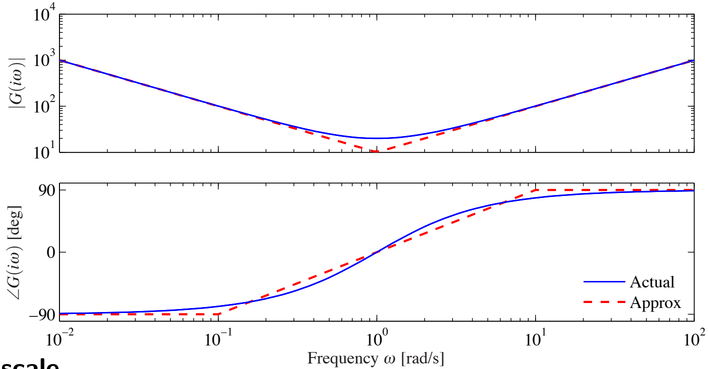
Te Bode plot is a representation of the **gain** curve $|G(i\omega)|$ and of the **phase** curve $\angle G(i\omega)$ as a function of $\omega$




Bode plot of the transfer function $G(s) = 20 + \frac{10}{s} + 10s$ (PID controller)
**log-log** representation (gain curve)
**log-linear** representation (phase curve)

# The Bode plot

The Bode plot is a representation of the **gain** curve $|G(i\omega)|$ and of the **phase** curve $\angle G(i\omega)$ as a function of $\omega$



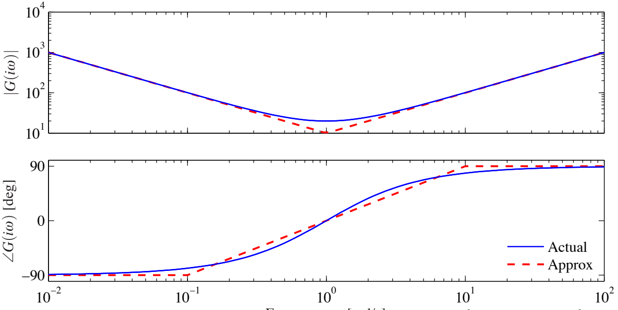

**Log scale**

A **decade** is the frequency band from $\omega_1$ to $10\omega_1$, where $\omega_1$ is any frequency

The tick mark representing a frequency $\omega$ is at a distance $c \log \omega$, with $c > 0$ the actual length of the decade (e.g. 2 cm), from the tick mark representing frequency $10^0 = 1$

# The Bode plot

The Bode plot is a representation of the **gain** curve $|G(i\omega)|$ and of the **phase** curve $\angle G(i\omega)$ as a function of $\omega$




Bode plots can be obtained from experiments without $(A, B, C, D)$

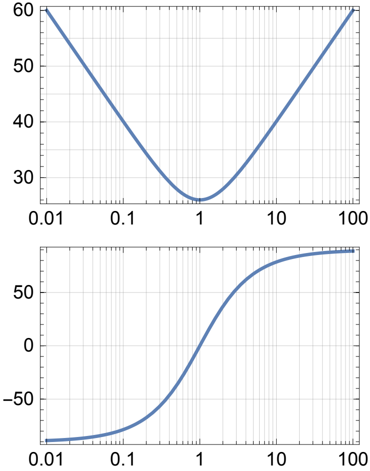
- Select a range of frequencies $\omega_i$, $i = 1, 2, \ldots, N$
- For every $i = 1, 2, \ldots, N$ run experiment $i$: apply the input $u(t) = \sin(\omega_i t)$ and measure the output response $y(t) = M(\omega_i) \sin(\omega_i t + \varphi(\omega_i))$
- Set $|G(i\omega_i)| := M(\omega_i)$ and $\angle G(i\omega) := \varphi(\omega_i)$ and draw the points $(\omega_i, |G(i\omega_i)|)$, $(\omega_i, \angle G(i\omega))$
- Interpolate

# Bode plots

- Sometimes, instead of representing $\log|G(i\omega)|$ it is represented a version scaled by a factor 20, $|G(i\omega)|_{dB} := 20\log|G(i\omega)|$

  $$|G(i\omega)| = 10^{|G(i\omega)|_{dB}/20}$$

  E.g. $\overline{\omega} = 0.1\mathrm{rad}/\mathrm{sec}$
  $|G(i\overline{\omega})|_{dB} = 40\mathrm{dB}$
  $\Rightarrow |G(i\overline{\omega})| = 100$
- The adopted unit is the **decibel** (dB)
- A log-linear representation of the gain curve is used



# Sketching Bode plots

The transfer function can be factorized as

$$G(s) = \frac{k\Pi_i(s + a_i)\Pi_i(s^2 + 2\xi_i\alpha_i s + \alpha_i^2)}{s^g\Pi_i(s + b_i)\Pi_i(s^2 + 2\zeta_i\omega_{0i}s + \omega_{0i}^2)}$$

where $\underline{s + a_i, s + b_i}$ corresponds to <u>real</u> zeros, poles and $\underline{s^2 + 2\xi_i\alpha_i s + \alpha_i^2, s^2 + 2\zeta_i\omega_{0i}s + \omega_{0i}^2}$ to <u>complex</u> zeros, poles $(0 < \xi_i < 1,\ 0 < \zeta_i < 1)$.

The factorization can be rewritten as

$$G(s) = k\frac{\Pi_{i=1}a_i(\frac{s}{a_i} + 1)\Pi_{i=1}\alpha_i^2(\frac{s^2}{\alpha_i^2} + 2\frac{\xi_i}{\alpha_i}s + 1)}{s^g\Pi_{i=1}b_i(\frac{s}{b_i} + 1)\Pi_{i=1}\omega_{0i}^2(\frac{s^2}{\omega_{0i}^2} + 2\frac{\zeta_i}{\omega_{0i}}s + 1)}$$

or

$$G(s) = \frac{k\Pi_i a_i\Pi_i\alpha_i^2}{\Pi_i b_i\Pi_i\omega_{0i}^2}\frac{\Pi_{i=1}(\frac{s}{a_i} + 1)\Pi_{i=1}(\frac{s^2}{\alpha_i^2} + 2\frac{\xi_i}{\alpha_i}s + 1)}{s^g\Pi_{i=1}(\frac{s}{b_i} + 1)\Pi_{i=1}(\frac{s^2}{\omega_{0i}^2} + 2\frac{\zeta_i}{\omega_{0i}}s + 1)}$$

# Sketching Bode plots

$$G(s) = \frac{b_1(s)b_2(s)}{a_1(s)a_2(s)}$$

By the property of the magnitude of a complex number and of the log function[1]

$$\begin{aligned} \log|G(s)| &= \log\left|\frac{b_1(s)b_2(s)}{a_1(s)a_2(s)}\right| = \log\frac{|b_1(s)b_2(s)|}{|a_1(s)a_2(s)|} = \log\frac{|b_1(s)||b_2(s)|}{|a_1(s)||a_2(s)|} \\ &= \log|b_1(s)| + \log|b_2(s)| - \log|a_1(s)| - \log|a_2(s)| \end{aligned}$$

## Gain curve

The gain curve is computed by adding and subtracting gains corresponding to terms in the numerator and the denominator

---

[1] $M_1 e^{i\varphi_1} \cdot M_2 e^{i\varphi_2} = M_1 M_2 e^{i(\varphi_1+\varphi_2)}$. Hence $|M_1 e^{i\varphi_1} \cdot M_2 e^{i\varphi_2}| = |M_1 M_2 e^{i(\varphi_1+\varphi_2)}| = |M_1 M_2| \cdot |e^{i(\varphi_1+\varphi_2)}| = M_1 M_2$.

# Sketching Bode plots

$$G(s) = \frac{b_1(s)b_2(s)}{a_1(s)a_2(s)}$$

By the property of the phase of a complex number[2]

$$\begin{aligned} \angle G(s) &= \angle \frac{b_1(s)b_2(s)}{a_1(s)a_2(s)} \\ &= \angle b_1(s) + \angle b_2(s) - \angle a_1(s) - \angle a_2(s) \end{aligned}$$

## Phase curve

The phase curve is computed by adding and subtracting phases corresponding to terms in the numerator and the denominator

[2] $M_1 e^{i\varphi_1} \cdot M_2 e^{i\varphi_2} = M_1 M_2 e^{i(\varphi_1+\varphi_2)}$

# Sketching Bode plots

Since a transfer function $G(s)$ can be written as the product of terms such as

$$k, \quad s, \quad s+a, \quad s^2 + 2\zeta\omega_0 s + \omega_0^2 \; (0 < \zeta < 1)$$

the Bode plots of $G(s)$ can be obtained summing the gain and phase curves of these simple terms

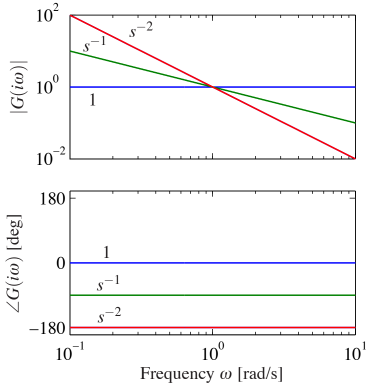
# Sketching Bode plots

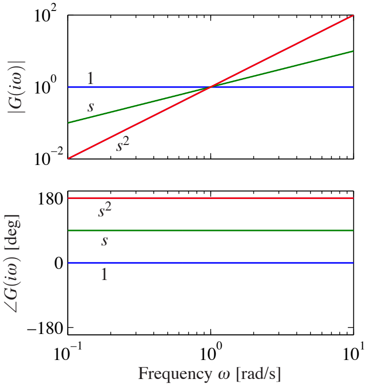
Bode plot of $\boldsymbol{G(s) = k}$

$$\log|G(i\omega)| = \log|k|, \quad \angle G(i\omega) = \angle k = \begin{cases} 0^\circ & if \quad k > 0 \\ 180^\circ & if \quad k < 0 \end{cases}$$

Bode plot of $\boldsymbol{G(s) = s^k}$

$$\log|G(i\omega)| = \log|(i\omega)^k| = \log\omega^k = k\log\omega, \quad \angle G(i\omega) = \angle(i\omega)^k = 90^\circ k$$



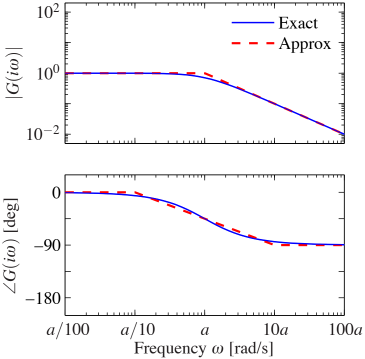



# Sketching Bode plots

Bode plot of $\boldsymbol{G(s) = \frac{a}{s+a} = \frac{1}{1+a^{-1}s}}$ $(a > 0)$

$$\log|G(i\omega)| = \log\frac{|a|}{|i\omega+a|} = \log\frac{|a|}{\sqrt{\omega^2+a^2}} \approx \begin{cases} 0 & \omega << a \\ \log a - \log\omega & \omega >> a \end{cases}$$

$$\angle G(i\omega) = -\angle(i\omega + a) = -\frac{180^\circ}{\pi}\arctan\frac{\omega}{a}\ (\mathrm{rad}) \approx \begin{cases} 0 & \omega << a \\ -45^\circ & \omega = a \\ -90^\circ & \omega >> a \end{cases}$$




$\omega = a$

**breakpoint** or **corner frequency**

$$|G(i\omega)|_{\omega=a} = \left.\frac{|a|}{\sqrt{\omega^2+a^2}}\right|_{\omega=a} = \frac{|a|}{\sqrt{2}|a|} = \frac{1}{\sqrt{2}}$$

# Sketching Bode plots

Bode plot of $\boldsymbol{G(s) = \frac{a}{s+a} = \frac{1}{1+a^{-1}s}}$ $(\mathbf{a < 0})$

$$\log|G(i\omega)| = \log\frac{|a|}{|i\omega + a|} = \log\frac{|a|}{\sqrt{\omega^2 + a^2}} \approx \begin{cases} 0 & \omega << |a| \\ \log|a| - \log\omega & \omega >> |a| \end{cases}$$

$$\angle G(i\omega) = -\angle(-\tfrac{i\omega}{|a|} + 1) = +\frac{180}{\pi}\arctan\frac{\omega}{|a|}\ (\mathrm{rad}) \approx \begin{cases} 0 & \omega << |a| \\ 45^\circ & \omega = |a| \\ 90^\circ & \omega >> |a| \end{cases}$$

If $a < 0$

- the gain diagram is the same as for the case $a > 0$;
- the phase diagram is symmetric with respect to the horizontal axis.

# Sketching Bode plots

Bode plot of $\boldsymbol{G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = \frac{1}{\frac{s^2}{\omega_0^2} + 2\zeta\frac{s}{\omega_0} + 1}}$ ($\omega_0 > 0$, $0 < \zeta < 1$)

$$
\begin{aligned}
\log|G(i\omega)| &= \log\frac{\omega_0^2}{|-\omega^2 + \omega_0^2 + i2\zeta\omega_0\omega|} \\
&= 2\log|\omega_0| - \log\sqrt{(-\omega^2 + \omega_0^2)^2 + (2\zeta\omega_0\omega)^2}
\end{aligned}
$$

$$
\angle G(i\omega) = -\angle(-\omega^2 + \omega_0^2 + i2\zeta\omega_0\omega) = -\frac{180}{\pi}\arctan\frac{2\zeta\omega_0\omega}{-\omega^2 + \omega_0^2}
$$

$$
\log|G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ 2\log\omega_0 - 2\log\omega & \text{if } \omega \gg \omega_0, \end{cases}
$$

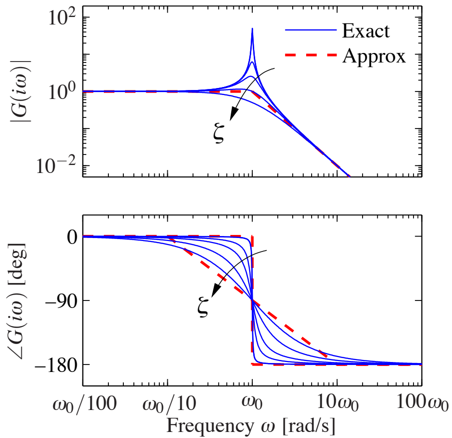
$$
\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ -180 & \text{if } \omega \gg \omega_0. \end{cases}
$$

# Sketching Bode plots

Bode plot of $G(s) = \dfrac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$ $(\omega_0 > 0,\ 0 < \zeta < 1)$

$$\log|G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ 2\log\omega_0 - 2\log\omega & \text{if } \omega \gg \omega_0, \end{cases}$$

$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ -180 & \text{if } \omega \gg \omega_0. \end{cases}$$



# Sketching Bode plots

Bode plot of $\boldsymbol{G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}}$ $(\omega_0 > 0,\ 0 < \zeta < 1)$

$$\frac{d}{d\omega} \log |G(i\omega)| = -\frac{2\omega(\omega^2 + \omega_0^2(2\zeta^2 - 1))}{(-\omega^2 + \omega_0^2)^2 + (2\zeta\omega_0\omega)^2}$$

$$\tfrac{d}{d\omega} \log |G(i\omega)| = 0 \quad \Leftrightarrow \quad \omega = 0, \quad \omega_r = \omega_0\sqrt{1 - 2\zeta^2} \quad \text{resonant frequency}$$

$$\log |G(i\omega_r)| = \log \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \quad \text{resonant peak}$$

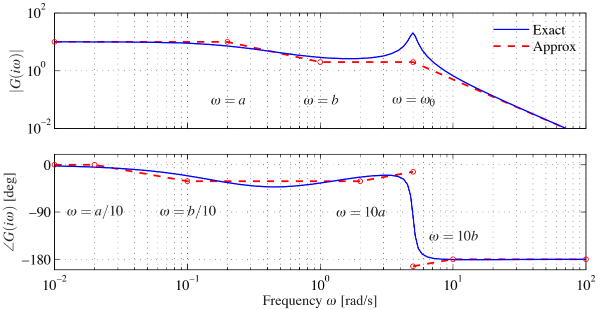
As $\zeta \to 0$, $\log |G(i\omega_r)| \to +\infty$

As $\zeta \to 1$, the transfer function $G(s)$ becomes $\frac{\omega_0^2}{(s+\omega_0)^2}$, that is the product of two first order transfer functions $G(s) = \frac{\omega_0}{s+\omega_0} \cdot \frac{\omega_0}{s+\omega_0}$. Its Bode diagrams is then equal to the one obtained by summing up the Bode diagrams of two identical first order transfer functions.

# Sketching Bode plots

**[Textbook, Example 8.8]**

$$G(s) = \frac{k(s+b)}{(s+a)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}, \quad 0 < a << b << \omega_0$$



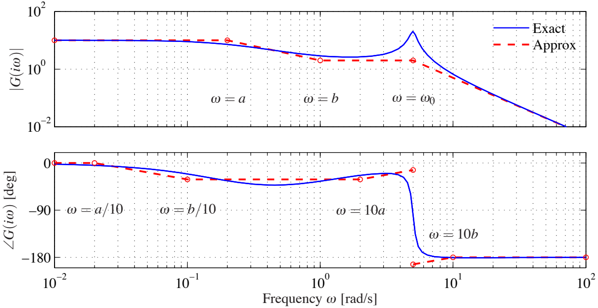

Note that it can be factorized in the terms depicted before

$$G(s) = \frac{kb}{a\omega_0^2}\frac{s+b}{b}\frac{a}{s+a}\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = \frac{kb}{a\omega_0^2}\frac{\frac{s}{b}+1}{(\frac{s}{a}+1)(\frac{s^2}{\omega_0^2} + 2\zeta\frac{s}{\omega_0} + 1)}$$

# Sketching Bode plots

**[Textbook, Example 8.8]**

$$G(s) = \frac{k(s+b)}{(s+a)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}, \quad 0 < a << b << \omega_0$$

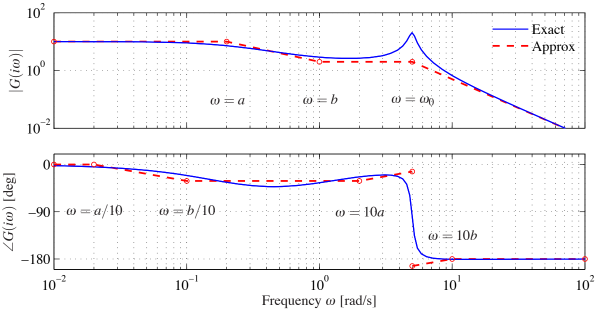



Zero-frequency gain $\dfrac{kb}{a\omega_0^2}$

# Sketching Bode plots

**[Textbook, Example 8.8]**

$$G(s) = \frac{k(s+b)}{(s+a)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}, \quad 0 < a << b << \omega_0$$

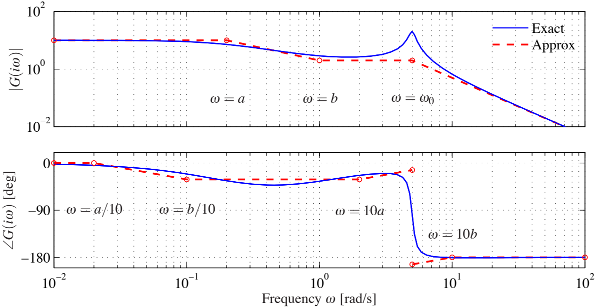



Low-frequency (the pole $\frac{a}{s+a}$ dominates)

# Sketching Bode plots

**[Textbook, Example 8.8]**

$$G(s) = \frac{k(s+b)}{(s+a)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}, \quad 0 < a << b << \omega_0$$

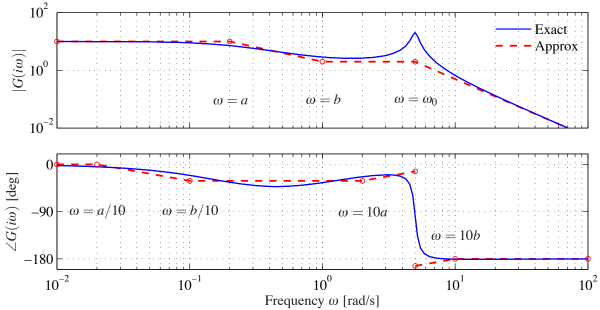



At $\omega \approx b$ the zero $\frac{s+b}{b}$ kicks in

# Sketching Bode plots

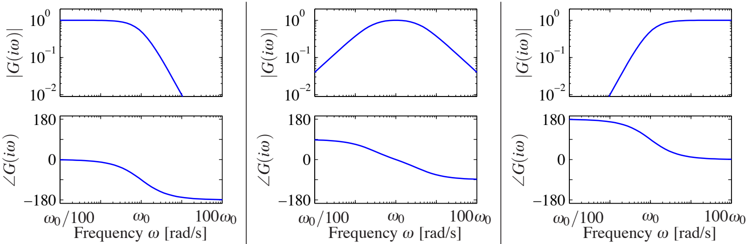
**[Textbook, Example 8.8]**

$$G(s) = \frac{k(s+b)}{(s+a)(s^2+2\zeta\omega_0 s+\omega_0^2)}, \quad 0 < a << b << \omega_0$$




At $\omega \approx \omega_0$ the complex conjugate poles $\dfrac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$ kicks in

# Low-, band-, high-pass filters




$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(a) Low-pass filter

$$G(s) = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

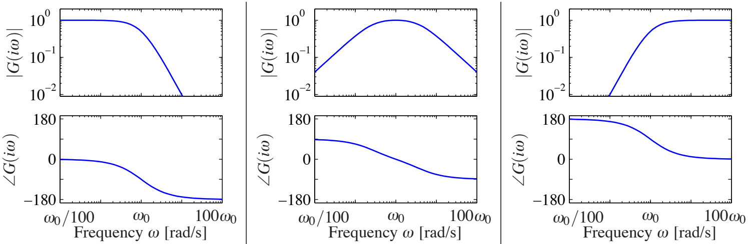
(b) Band-pass filter

$$G(s) = \frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(c) High-pass filter

| **Bandwidth** | frequency where gain has decreased by $\frac{1}{\sqrt{2}}$ from |
|---|---|
| Low-pass filter | DC gain |
| Band-pass filter | center of the band |
| High-pass filter | high-frequency gain |

# Low-, band-, high-pass filters




$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(a) Low-pass filter

$$G(s) = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(b) Band-pass filter

$$G(s) = \frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(c) High-pass filter

**Band pass filter**

Signals with frequencies around $\omega_0$ are passed through unchanged

Signals with frequencies around $\omega_0/100$ phase lead of 90 ($\approx$ differentiator)

Signals with frequencies around $100\omega_0$ phase lag of $-90$ ($\approx$ integrator)

# Next lecture(s)

- Frequency domain analysis (Chapter 9)
- Nyquist plot
- Stability margins