EXAM MECHATRONICS
TUESDAY 23 JANUARY 2018, 9.00 - 12.00 h.

Last Name: ______________________ First Name: ______________________ StudentID: ____________

INSTRUCTIONS:

- This exam consists of 30 pages with 6 open questions. Make sure you have all pages and questions.
- Read carefully each question and answer accordingly.
- The answers to each question (including motivation) have to be placed inside the answer boxes.
- Write neatly.
- Write your name and student number in all pages. The exercises will be collected separately.
- If you like, you can use and add additional paper, which needs to include your name and student number. Please provide separate papers for separate exercises. Hand in the exercises on separate piles.
- You can earn a maximum of 100 points at the exam. The amount of points spread over the exercises is 110 points, i.e., there are 10 bonus points to be earned.
- You will only get a grade if you have finalized the practical.
- This is a CLOSED book exam.

Preliminaries

Across and Through variable table

Table 1.2. Ideal system elements (linear)

System type	Mechanical translational	Mechancial rotational	Electrical	Fluid	Thermal
A-type variable	Velocity, v	Velocity, Ω	Voltage, e	Pressure, P	Temperature, T
A-type element	Mass, m	Mass moment of inertia, J	Capacitor, C	Fluid Capacitor, C_f	Thermal capacitor, C_h
Elemental equations	$F = m\dfrac{dv}{dt}$	$T = J\dfrac{d\Omega}{dt}$	$i = C\dfrac{de}{dt}$	$Q_f = C_f\dfrac{dP}{dt}$	$Q_h = C_h\dfrac{dT}{dt}$
Energy stored	Kinetic	Kinetic	Electric field	Potential	Thermal
Energy equations	$\mathscr{E}_k = \frac{1}{2}mv^2$	$\mathscr{E}_k = \frac{1}{2}J\Omega^2$	$\mathscr{E}_e = \frac{1}{2}Ce^2$	$\mathscr{E}_p = \frac{1}{2}C_f P^2$	$\mathscr{E}_t = \frac{1}{2}C_h T^2$
T-type variable	Force, F	Torque, T	Current, i	Fluid flow rate, Q_f	Heat flow rate, Q_h
T-type element	Compliance, $1/k$	Compliance, $1/K$	Inductor, L	Inertor, I	None
Elemental equations	$v = \dfrac{1}{k}\dfrac{dF}{dt}$	$\Omega = \dfrac{1}{K}\dfrac{dT}{dt}$	$e = L\dfrac{di}{dt}$	$P = I\dfrac{dQ_f}{dt}$	
Energy stored	Potential	Potential	Magnetic field	Kinetic	
Energy equations	$\mathscr{E}_P = \dfrac{1}{2k}F^2$	$\mathscr{E}_P = \dfrac{1}{2K}T^2$	$\mathscr{E}_m = \frac{1}{2}Li^2$	$\mathscr{E}_k = \frac{1}{2}IQ_f^2$	
D-type element	Damper, b	Rotational damper, B	Resistor, R	Fluid resistor, R_f	Thermal resistor, R_h
Elemental equations	$F = bv$	$T = B\Omega$	$i = \dfrac{1}{R}e$	$Q_f = \dfrac{1}{R_f}P$	$Q_h = \dfrac{1}{R_h}T$
Rate of energy dissipated	$\dfrac{dE_D}{dt} = Fv = \dfrac{1}{b}F^2 = bv^2$	$\dfrac{dE_D}{dt} = T\Omega = \dfrac{1}{B}T^2 = B\Omega^2$	$\dfrac{dE_D}{dt} = ie = Ri^2 = \dfrac{1}{R}e^2$	$\dfrac{dE_D}{dt} = Q_f P = R_f Q_f^2 = \dfrac{1}{R_f}P^2$	$\dfrac{dE_D}{dt} = Q_h$

Note: A-type variable represents a spatial difference across the element.

The other analogy for linear systems as was treated in Control Engineering, and is useful for Euler-Lagrange modeling.

	Kinetic coenergy	Potential energy	Rayleigh dissipation function
Translation	$T^*(\dot{x}) = \frac{1}{2}m\frac{dx}{dt}^2$	$V(x) = \frac{1}{2}kx^2$	$\mathcal{D}(\dot{x}) = \frac{1}{2}b\frac{dx}{dt}^2$
Rotation	$T^*(\dot{\theta}) = \frac{1}{2}J\frac{d\theta}{dt}^2$	$V(\theta) = \frac{1}{2}k\theta^2$	$\mathcal{D}(\dot{\theta}) = \frac{1}{2}b\frac{d\theta}{dt}^2$
Electric	$T^*(\dot{q}) = \frac{1}{2}L\frac{dq}{dt}^2$	$V(q) = \frac{1}{2C}q^2$	$\mathcal{D}(\dot{q}) = \frac{1}{2}R\frac{dq}{dt}^2$

Canonical forms

The state-space representation for a given transfer function is not unique, i.e., there are infinite-number of possibilities to express a given transfer function in state-space form. However, there are several forms that can be helpful in the design of controller or observer. Let us consider the following general transfer function for single-input single-output system:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}. \tag{1}$$

For this transfer function, the state-space representation in **canonical observable form** is given by

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} &= \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u.
\end{aligned} \tag{2}$$

On the other hand, the state-space representation in the **canonical controllable form** is given by

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \cdots & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u.
\end{aligned} \tag{3}$$

Z-transform.

Denote by $Z\{u(n)\}$ the Z-transform of discrete-time signal $u(n)$ where $n = 0, 1, \ldots$.

- Unit step signal $u(n)$: $Z\{u(n)\} = \dfrac{1}{1 - z^{-1}}$
- Time-shifting property: $Z\{u(n-k)\} = z^{-k}U(z)$

Transformation from s-domain to z-domain

- The bilinear transformation: $s \mapsto \dfrac{2}{T}\dfrac{1 - z^{-1}}{1 + z^{-1}}$
- The backward-Euler transformation: $s \mapsto \dfrac{1}{T}(1 - z^{-1})$

Optimal state feedback control design(LQR)

The Riccati equation, that is related to the optimal state feedback, reads as

$$A^T P + PA - PBR^{-1}B^T P + Q = 0, \tag{4}$$

where $P = P^T > 0$, and where $Q = Q^T > 0$ and $R = R^T > 0$ are related to the cost function

$$J = \int_0^\infty \left(x^T(\tau)Qx(\tau) + u^T Ru(\tau)\right) d\tau. \tag{5}$$

The optimal state feedback controller is given by $u(t) = -R^{-1}B^T Px(t)$.

State observer design

For a state-space system described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{6}$$

where x is the actual state and y is the measured signal, a state observer for such system has the following structure

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x}, \end{aligned} \tag{7}$$

where $\hat{x}$ is the estimated state and y is the corresponding estimated output.

Transfer function of time delay

For a time delay operator

$$y(t) = u(t - T) \tag{8}$$

where T is the delay time, its Laplace transform is given by

$$\frac{Y(s)}{U(s)} = e^{-sT}. \tag{9}$$

From this transfer function, the corresponding Bode plot is given by unity amplitude for all frequencies and the phase plot is linear with respect to the frequency, i.e.,

$$\phi(\omega) = -\omega T, \tag{10}$$

for all frequencies ω.

The first order Padé approximation of the delay transfer function $e^{-\omega T}$ is given by

$$e^{-\omega T} \approx \frac{1 - \frac{T}{2}s}{1 + \frac{T}{2}s}. \tag{11}$$

Absolute stability

Popov's criterion

Proposition 1. *If a linear system combines with a static nonlinearity in the feedback, and the following are fulfilled:*

- *A (the system's matrix) is asymptotically stable*
- *The nonlinearity belongs to a sector $[0, k]$*
- *There exists a constant $\alpha > 0$ such that for all $\omega \geq 0$*

$$\Re\left((1 + j\alpha\omega)G(j\omega)\right) + \frac{1}{k} \geq \epsilon \tag{12}$$

for arbitrarily small $\epsilon > 0$, then 0 is globally asymptotically stable.

Circle's criterion

Theorem 1. *If a linear system combines with a static nonlinearity in the feedback, and the following are fulfilled:*

- *A (the system's matrix) has no eigenvalues on the $j\omega$-axis and ρ eigenvalues in the RHP*
- *The nonlinearity belongs to the sector $[k_1, k_2]$*
- *One of the following holds*
  - *$0 < k_1 \leq k_2$, the Nyquist plot of $G(j\omega)$ does not enter the disk $D(k_1, k_2)$ and encircles it ρ times anti-clockwise*
  - *$0 = k_1 < k_2$, the Nyquist plot stays to the right of $\Re(s) > -\frac{1}{k_2}$*
  - *$k_1 < 0 < k_2$, the Nyquist plot of $G(j\omega)$ stays in $D(k_1, k_2)$*
  - *$k_1 < k_2 < 0$, the Nyquist plot of $-G(j\omega)$ does not enter $D(-k_1, -k_2)$ and encircles it ρ times anti-clockwise*

Then 0 is globally asymptotically stable.

1. (10 TOTAL points) Consider the robotic exoskeleton shown in figure 1

Figure 1: A robotic exoskeleton.

This device can, for example, aid paralyzed people to re-gain motion.

(a) (5 points) Identify three possible user demands, three possible functional requirements, and three possible design parameters.

(b) (5 points) Referring to the mechatronic block diagram shown in Figure 2 identify at least 2 elements of the exoskeleton for the each of the following items.

- measured variables
- manipulated variables
- reference variables
- sensors
- actuators
- man/machine interface
- energy supply

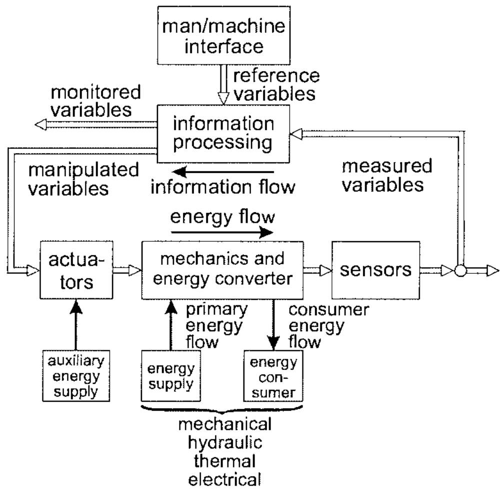


Figure 2: Mechatronics Block

2. (35 TOTAL points) Suppose that a transfer function of an industrial process from the input u to the measurement output y is given by

$$\frac{Y(s)}{U(s)} = \frac{s}{s^2 + 2s + 3}. \tag{13}$$

(a) (1 point) Is the system stable or unstable? Why?

(b) (1 point) Obtain the state-space representation of the system in the controllable canonical form.

(c) (15 points) Suppose that the production cost of the industrial process is given by

$$J = \int_0^{\infty} \left(5x_1^2(\tau) + 2x_1(\tau)x_2(\tau) + x_2^2(\tau) + 2u^2(\tau)\right) d\tau. \tag{14}$$

Using the state-space representation obtained in part (b), show that the solution to the Riccati equation is given by

$$P = \begin{bmatrix} 2.58 & 0.78 \\ 0.78 & 0.59 \end{bmatrix} \tag{15}$$

and then obtain the optimal feedback controller that stabilizes the system and minimizes the cost J.

(d) (15 points) Suppose now that the industrial process is hazardous. Therefore, it is not possible to have measurements of all the states. Design a state observer to estimate the internal states. It is required that the observer has an exponential convergence rate of -10 and -20, in other words, the closed-loop characteristic polynomial of the observer error dynamics must be identified with $p(s) = (s + 10)(s + 20)$.

(e) (3 points) Finally, assume that you want to implement the optimal state feedback controller using the estimated states. Is this possible?

If yes, why and how would you do it?

If not, why and what would you propose as a solution?

Motivate your answer and provide enough justification.

3. (25 TOTAL points) Consider the problem of dynamical modeling of a flexible conveyor belt as shown in Figure 3.

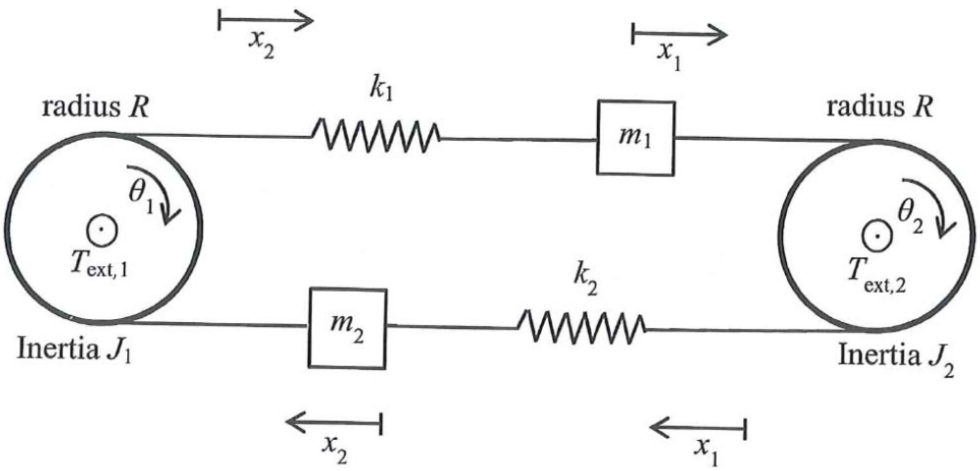


Figure 3: Modeling of a flexible conveyor belt with two masses, m_1 and m_2.

(a) (5 points) Identify the A-type and T-type elements and their respective variables.

(b) (20 points) Derive the dynamical model of the system with generalized coordinates θ_1, θ_2 and generalized forces $T_{ext,1}$, $T_{ext,2}$.

Remark: you are free to choose *any* modeling method: e.g. Newton, Euler-Lagrange, elemental equations of A,T type elements.

Hint: recall that the length of arc can be written as $x = R\theta$, where θ is the angle of rotation, R is the radius, and x is the length of the arc.

4. (10 TOTAL points) Consider the problem of dynamical modeling of an electro-hydraulic system for an assisted steering wheel in a car. It is a multi-domain system which consists of mechanical system, fluid system and electrical system as shown in figure 4. The Voltage source V_s is used to rotate the valve through an electro-mechanical coupling device which has the relation of

$$\omega = \alpha V_{coupling},\ i_L = \alpha T, \tag{16}$$

where α is the coupling constant, ω is the angular velocity of the valve, i_L is the current through the inductor and T is the torque applied to the valve. The moment of inertia of the valve is denoted by J. The angular position of the valve θ determines the flow rate Q_m based of the following relation:

$$P_{12} = \theta Q_m, \tag{17}$$

The pressure source P_s provides a constant pressure. Based on the pressure across the hydraulic motor P_{2r}, the hydraulic motor generates a force F that drives a mechanical system with mass m and which is connected to a spring and a damper with constants k and b , respectively. The displacement of the mechanical system is denoted by x and the velocity is denoted by v. The fluid mechanical coupling device (or the hydraulic motor) satisfies

$$F = DP_{2r},\ Dv = Q_m, \tag{18}$$

where D is the coupling constant of the hydraulic motor. Derive the state equations of the full system with the pressure P_s as the input and the spring force (F_k) as the measured output.

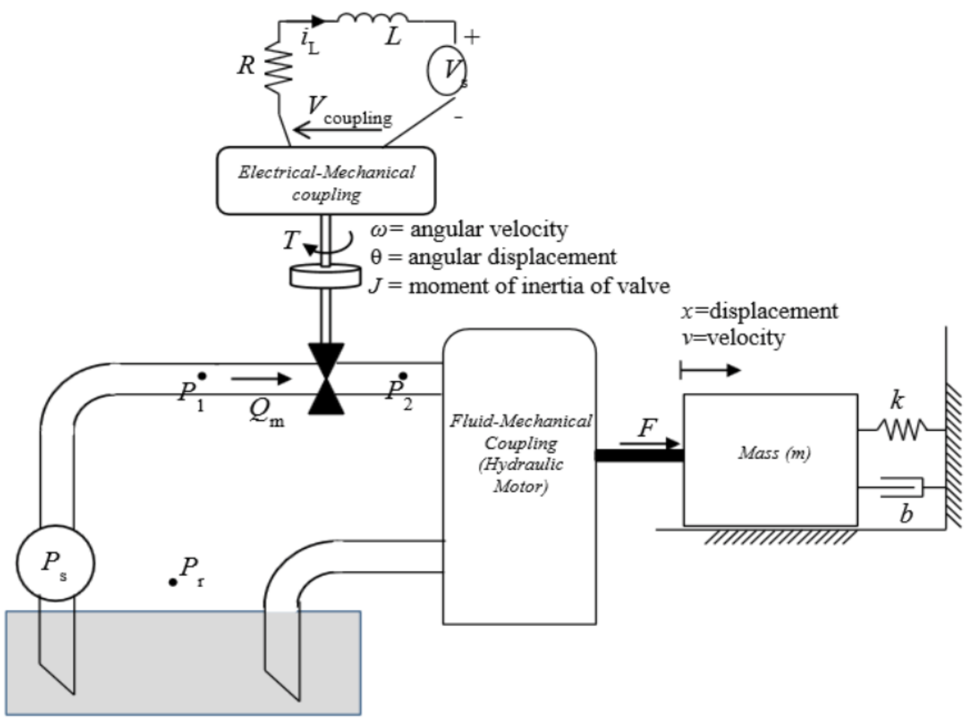


Figure 4: A simplified electro-hydro system of an assisted steering wheel in a car.

5. (20 TOTAL points) A simplified linear model of the delivery of a drug into the bloodstream is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -k_1 & 0 \\ k_1 & -k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \end{aligned} \tag{19}$$

where x_1 is the mass of the available drug, x_2 is the mass of the drug in the blood stream being absorbed, and u is the rate at which the drug is delivered. The values of k_1 and k_2 can be tuned in the drug delivery machine for desired purposes.

(a) (2.5 points) What is the equilibrium of the system given a constant input $u = u^* \geq 0$? Interpret your result.

(b) (2.5 points) Choose values of parameters k_1 and k_2 that ensures an equilibrium ratio of $\dfrac{x_2^*}{x_1^*} = 0.5$. Is this a realistic equilibrium point? Interpret your result.

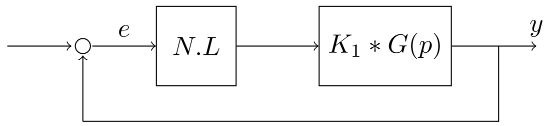
(c) (5 points) Using the values of k_1 and k_2 previously obtained, discretize the plant with a sampling period T_s by approximating the derivative with the forward Euler method.

Hint: recall that in the forward Euler method the time derivative $\dot{x}$ is approximated by $\dot{x} \approx \dfrac{x(k+1) - x(k)}{T_s}$

(d) (5 points) Obtain the corresponding discrete transfer function.

(e) (5 points) Choose an appropriate sampling period T_s that ensures that the discretized plant remains stable.

6. (10 bonus points) In the non-linear system below



the plant is asymptotically stable and the Nyquist plot of the loop gain $L(s) = K_1 G(s)$ is shown in figure 5

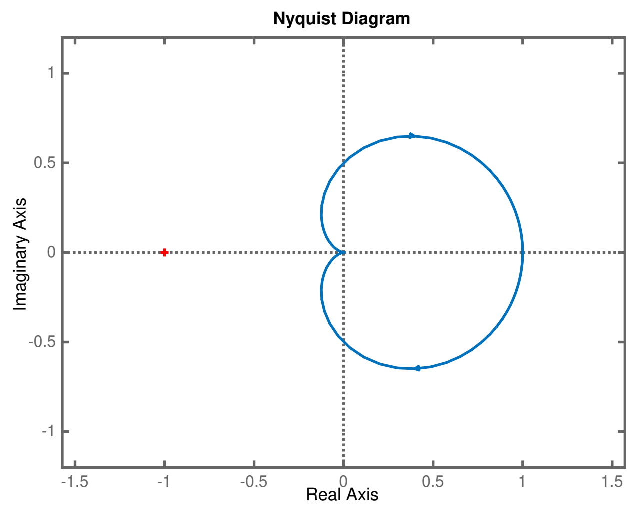


Figure 5: Nyquist plot of $K_1 G(s)$

The nonlinearity NL belongs to a sector with boundaries $[k_1, k_2]$.

(a) (5 points) It is known that due to the behavior of the nonlinearity, the boundaries satisfy $k_1 = 0$ and $k_2 > 0$. Choose appropriate bound k_2 such that the origin of the closed loop system is globally asymptotically stable. Motivate your answer.

(b) (5 points) It is known that due to the behavior of the nonlinearity, the boundaries satisfy $0 < k_1 < k_2$. Choose appropriate bounds k_1, k_2 such that the origin of the closed loop system is globally asymptotically stable. Motivate your answer.