Control Engineering
Additional exercises

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Disclaimer I (October 29, 2018) *These additional exercises are for those students who want to practice more. They are not compulsory and I will not check whether or not you have solved them but I advise you to try to solve then to raise your chances to pass the exam. Furthermore, I have not written any solution manual, and due the tight time constraints, I will not be able to provide one during the course. Some of the exercises might be part of the mock exam. If this will be the case, then a solution will be provided.*

Disclaimer II (May 8, 2020) *I added a few sketchy solutions. Use them with caution, as they are in beta version and might contain errors. If time allows, I might add new exercises now and then. Thus check this document regularly. If I add a new exercise, I will update the date at the beginning of the document and highlight the addition in blue.*

(May 25, 2020) *I added 3 exercises on the contents of Lectures 3, 4 and 9.*

1. Lecture 1

   (a) Consider the equations of motion of the mass-spring damper system

   $$m\ddot{q}(t) + c\dot{q}(t) + kq(t) = u(t)$$

   Define the scaled variables[1] $z(\tau)$, τ, $v(\tau)$ and the parameter ζ such that the equations above can be written as

   $$\frac{d^2}{d\tau^2}z(\tau) + 2\zeta\frac{d}{d\tau}z(\tau) + z(\tau) = v(\tau).$$

   (b) Solve [1, Exercise 1.5, p. 26].

2. Lecture 2

   (a) Consider the cart-pendulum system in [1, Example 2.1] and the equations of motion [1, (2.9)] describing it and recalled below

   $$\begin{aligned} (M+m)\ddot{p} - ml(\cos\theta)\ddot{\theta} + c\dot{p} + ml(\sin\theta)\dot{\theta}^2 &= F \\ -ml(\cos\theta)\ddot{p} + (J+ml^2)\ddot{\theta} + \gamma\dot{\theta} - mgl\sin\theta &= 0. \end{aligned} \tag{1}$$

      i. Determine the translational kinetic co-energy of the cart with mass M, and the horizontal and vertical translational kinetic co-energy of the mass m to be balanced.

[1] The symbol v does *not* represent the velocity of the system.

ii. Determine the rotational kinetic co-energy of the pendulum with moment of inertia J.

iii. Denoted by c and γ the coefficients of viscous friction, determine the Rayleigh dissipation function.

iv. Denoted by g the acceleration due to gravity and by F the force applied at the cart, determine the nonconservative potential function $-u^\top q$, where $u^\top = [mg\ F]$, and q is the generalized displacement (to be determined).

v. Use the quantities determined before to show that [1, (2.9)] are the Euler-Lagrange equations of motion of the cart-pendulum system.

Solution

i. $\frac{1}{2}M\dot{p}^2$ (translational kinetic co-energy of the cart),
$\frac{1}{2}m(\dot{p} - l(\cos\theta)\dot{\theta})^2$ (horizontal translational kinetic co-energy of the mass m),
$\frac{1}{2}m(-l(\sin\theta)\dot{\theta})^2$ (vertical translational kinetic co-energy of the mass m).

ii. $\frac{1}{2}J\dot{\theta}^2$ (rotational kinetic co-energy of the pendulum with moment of inertia J).

iii. $\mathcal{D}(\dot{p}, \dot{\theta}) = \frac{1}{2}c\dot{p}^2 + \frac{1}{2}\gamma\dot{\theta}^2$.

iv. $-u^\top q = -mgl\cos\theta - Fp$, where $l\cos\theta$ is the vertical position of the mass m.
$f(q) = \begin{bmatrix} l\cos\theta & p \end{bmatrix}^\top$

v. The (non conservative) Lagrangian function (in what follows we use the symbol L instead of L_{NC}, for the sake of simplicity) can be written as

$$\begin{aligned} L(p, \theta, \dot{p}, \dot{\theta}) =& \frac{1}{2}M\dot{p}^2 + \frac{1}{2}m(\dot{p} - l(\cos\theta)\dot{\theta})^2 + \frac{1}{2}m(-l(\sin\theta)\dot{\theta})^2 + \frac{1}{2}J\dot{\theta}^2 \\ &+ mgl\cos\theta + Fp \\ &+ \int_0^t (\frac{1}{2}c\dot{p}^2 + \frac{1}{2}\gamma\dot{\theta}^2)dt. \end{aligned}$$

Note that the second and third term in L get simplified as:
$\frac{1}{2}m(\dot{p} - l(\cos\theta)\dot{\theta})^2 + \frac{1}{2}m(-l(\sin\theta)\dot{\theta})^2 = \frac{1}{2}m\dot{p}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - ml(\cos\theta)\dot{p}\dot{\theta}$.
The equations (1) are obtained from routine application of the Euler-Lagrange equations of motion.
The Euler-Lagrange equations of motion are:

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) - \frac{\partial L}{\partial q} = 0$$

with $q = \begin{bmatrix} p & \theta \end{bmatrix}^\top$.
First consider the p variable:

$$\begin{aligned} \frac{\partial L}{\partial p} &= F \\ \frac{\partial L}{\partial \dot{p}} &= (M+m)\dot{p} - ml(\cos\theta)\dot{\theta} + \int_0^t (c\dot{p})dt \\ \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{p}}\Big) &= (M+m)\ddot{p} + ml(\sin\theta)\dot{\theta}^2 - ml(\cos\theta)\ddot{\theta} + c\dot{p} \end{aligned}$$

The Euler-Lagrange equations of motion is:

$$(M+m)\ddot{p} + ml(\sin\theta)\dot{\theta}^2 - ml(\cos\theta)\ddot{\theta} + c\dot{p} = F$$

Next consider the θ variable:

$$\begin{aligned}
\frac{\partial L}{\partial \theta} &= ml(\sin\theta)\dot{p}\dot{\theta} - mgl\sin\theta \\
\frac{\partial L}{\partial \dot{\theta}} &= ml^2\dot{\theta} - ml(\cos\theta)\dot{p} + J\dot{\theta} + \int_0^t (\gamma\dot{\theta})dt \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= ml^2\ddot{\theta} + ml(\sin\theta)\dot{\theta}\dot{p} - ml(\cos\theta)\ddot{p} + J\ddot{\theta} + \gamma\dot{\theta}
\end{aligned}$$

The Euler-Lagrange equations of motion are:

$$\begin{aligned}
ml^2\ddot{\theta} + ml(\sin\theta)\dot{\theta}\dot{p} - ml(\cos\theta)\ddot{p} + J\ddot{\theta} + \gamma\dot{\theta} + ml(\sin\theta)\dot{\theta}\dot{p} - mgl\sin\theta &= 0 \\
ml^2\ddot{\theta} - ml(\cos\theta)\ddot{p} + J\ddot{\theta} + \gamma\dot{\theta} - mgl\sin\theta &= 0
\end{aligned}$$

3. Lecture 3

(a) Consider the point mass satellite model

$$\begin{aligned}
\ddot{r} =& \ r\dot{\theta}^2 - \frac{k}{r^2} + u_1 \\
\ddot{\theta} =& \ -\frac{2\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2
\end{aligned}$$

i. Introduce the state variables $x_1 = r, x_2 = \dot{r}, x_3 = \theta, x_4 = \dot{\theta}$ and derive the vector field $f(x,u)$ in $\dot{x} = f(x,u)$.

ii. Compute *all* the control inputs $\bar{u}_1,\ \bar{u}_2$ such that $\bar{x}_1(t) = \rho, \bar{x}_2(t) = 0, \bar{x}_3(t) = \omega t, \bar{x}_4(t) = \omega$, with ρ, ω constant real numbers is a solution (circular orbit) of the system.

iii. Based on the answer to the previous question, show that if $\bar{u}_1 > \frac{k}{\rho^2}$ then the solution $\bar{x}(t)$ does not exist.

iv. Linearize the nonlinear system around $(\bar{x}(t), \bar{u}(t))$. You should obtain matrices A, B that depend on the parameters ρ, ω, k.

Solution

$$\text{i.} \quad f(x,u) = \begin{bmatrix} x_2 \\ x_1 x_4^2 - \frac{k}{x_1^2} + u_1 \\ x_4 \\ -\frac{2x_2x_4}{x_1} + \frac{1}{x_1}u_2 \end{bmatrix}$$

ii. $\bar{u}_1 = \frac{k}{\rho^2} - \rho\omega^2$, $\bar{u}_2 = 0$. iii. If $\bar{u}_1 > \frac{k}{\rho^2}$, there is no ω that satisfy the equation $\bar{u}_1 = \frac{k}{\rho^2} - \rho\omega^2$.

$$iv. \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega^2 + \frac{2k}{\rho^3} & 0 & 0 & 2\rho\omega \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega}{\rho} & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix}$$

(b) (Dynamic feedback linearization of a unicyle) The unicyle is a nonholonomic mobile robot whose kinematic equations are given by the nonlinear model

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega \end{aligned}$$

where $(x, y) \in \mathbb{R}^2$ is the Cartesian position of the unicyle in a reference frame and θ is the orientation with respect to the x axis. The control inputs w, ω represent the driving and steering velocity respectively. Note that the system has $n = 3$ state variables and $m = 2$ control inputs.

The dynamic feedback linearization is an algorithm to *exactly* transform nonlinear models into linear ones. Consider the dynamic controller[2]

$$\begin{aligned} \dot{\xi} &= u_1\cos\theta + u_2\sin\theta \\ v &= \xi \\ \omega &= \frac{-u_1\sin\theta + u_2\cos\theta}{\xi} \end{aligned}$$

where $\xi \in \mathbb{R}$ is the state variable of the controller, and $u_1 \in \mathbb{R}, u_2 \in \mathbb{R}$ are two new control inputs. Observe that the controller is well-defined provided that the system evolution is away from $\xi = 0$.

i. Write the closed-loop system.

ii. Define the nonlinear change of coordinates

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \xi\cos\theta \\ \xi\sin\theta \end{bmatrix}$$

and show that in the new coordinates the closed-loop system takes the form

$$\dot{z} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

where $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^\top$. Observe that the system is equivalent to two decoupled double integrators, namely

$$\ddot{z}_1 = u_1, \quad \ddot{z}_2 = u_2.$$

Answer i.

$$\begin{aligned} \dot{x} &= \xi\cos\theta \\ \dot{y} &= \xi\sin\theta \\ \dot{\theta} &= \frac{-u_1\sin\theta + u_2\cos\theta}{\xi} \\ \dot{\xi} &= u_1\cos\theta + u_2\sin\theta \end{aligned}$$

[2] De Luca, Oriolo, Vendittelli, "Stabilization of the unicycle via feedback linearization", IFAC Robot control, Vienna, Austria, 2000, pp. 687–692.

ii.

$$
\begin{aligned}
\dot{z}_1 &= \dot{x} = \xi \cos\theta = z_3 \\
\dot{z}_2 &= \dot{y} = \xi \sin\theta = z_4 \\
\dot{z}_3 &= \dot{\xi}\cos\theta - \xi\sin\theta\dot{\theta} = (u_1\cos\theta + u_2\sin\theta)\cos\theta - \xi\sin\theta\frac{-u_1\sin\theta + u_2\cos\theta}{\xi} \\
&= \ldots = u_1 \\
\dot{z}_4 &= \dot{\xi}\sin\theta + \xi\cos\theta\dot{\theta} = (u_1\cos\theta + u_2\sin\theta)\sin\theta + \xi\cos\theta\frac{-u_1\sin\theta + u_2\cos\theta}{\xi} \\
&= \ldots = u_2
\end{aligned}
$$

Moroever, $\ddot{z}_1 = \dot{z}_3 = u_1$ and $\ddot{z}_2 = \dot{z}_4 = u_2$.

4. Lecture 4

(a) Consider the linear system with constant input

$$\dot{x} = Ax + b$$

where $x \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ a constant vector and $A \in \mathbb{R}^{n\times n}$ a nonsingular matrix.

i. Compute the equilibrium $\bar{x}$ of the system.

ii. Define the new variable $\tilde{x} = x - \bar{x}$ and show that it satisfies

$$\dot{\tilde{x}} = \tilde{A}\tilde{x}.$$

What is the matrix $\tilde{A}$ equal to?

iii. Give necessary and sufficient conditions for the equilibrium $\bar{x}$ to be globally asymptotically stable.

(b) [1, Exercise 4.6, questions (a), (b)].
Hint In (a) neglect that a is a bifurcation parameter.

(c) [1, Exercise 4.12].

(d) [1, Exercise 4.13].
Hint Study the stability of each equilibrium using the linearized dynamics.

(e) [1, Exercise 4.16].

(f) Determine whether or not the linearized system computed in Lecture 3, Exercise (a) is asymptotically stable. If the circular orbit is perturbed by a sufficiently small perturbation, will the orbit of the point mass satellite converge again to the circular orbit?

Answer The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega^2 + \frac{2k}{\rho^3} & 0 & 0 & 2\rho\omega \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega}{\rho} & 0 & 0 \end{bmatrix}$$

is *easily* computed to give

$$\det(sI_4 - A) = s^4 - (-3\omega^2 + \frac{2k}{\rho^3})s^2$$

Hence, the eigenvalues are

$$\{0, 0, \pm\sqrt{-3\omega^2 + \frac{2k}{\rho^3}}\}$$

and the system is *not* asymptotically stable. Hence , if perturbed, the solution will not converge to the circular orbit.

(g) Consider the following linearized model of an inverted pendulum [Textbook, (2.10)]

$$\begin{aligned} \tfrac{d}{dt}\begin{bmatrix}\theta \\ \dot\theta\end{bmatrix} &= \begin{bmatrix} \dot\theta \\ \frac{mgl}{J_t}\theta - \frac{\gamma}{J_t}\dot\theta + \frac{l}{J_t}u \end{bmatrix} \\ y &= \theta \end{aligned}$$

Determine all the coefficients k_p, k_d, if any, of the so-called proportinal-derivative controller

$$u = k_p y + k_d \dot y$$

such that the closed-loop system is asymptotically stable.
Hint Use the Routh-Hurwitz criterion.
Solution The closed loop system is given by the equations

$$\tfrac{d}{dt}\begin{bmatrix}\theta \\ \dot\theta\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{l}{J_t}(mg + k_p) & \frac{1}{J_t}(-\gamma + lk_d) \end{bmatrix}\begin{bmatrix}\theta \\ \dot\theta\end{bmatrix}$$

whose characteristic polynomial is given by

$$s^2 + \frac{1}{J_t}(\gamma - lk_d)s - \frac{l}{J_t}(mg + k_p)$$

The associated Routh-Hurwitz table is

$$\begin{array}{c|ll} 2 & 1 & -\frac{l}{J_t}(mg + k_p) \\ 1 & \frac{1}{J_t}(\gamma - lk_d) & \\ 0 & -\frac{l}{J_t}(mg + k_p) & \end{array}$$

For the characteristic polynomial to have all the roots with strictly negative real parts it is necessary and sufficient that

$$\gamma - lk_d > 0, \quad mg + k_p < 0$$

or

$$k_d < \frac{\gamma}{l}, \quad k_p < -mg.$$

5. Lecture 5

(a) Show that the signal

$$x(t) = \mathrm{e}^{\sigma t}\begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} x_0$$

is *the* solution to system

$$\dot x = \Lambda x, \quad x(0) = x_0, \quad \text{with} \quad \Lambda = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$$

Hint By direct substitution. Compute $\dot x(t)$ and check that it is equal to $\Lambda x(t)$ for all t.

(b) [1, Exercise 5.4].

(c) [1, Exercise 5.7]

6. Lecture 6

(a) Consider the model of a forced mass-spring system (linear oscillator)

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Recall that the exponential of A in this case is given by

$$\mathrm{e}^{At} = \begin{bmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{bmatrix}$$

i. Compute the reachability Gramian $W(0,T) = \int_0^T \mathrm{e}^{A\theta} BB^T (\mathrm{e}^{A\theta})^T d\theta$.

ii. Give the expression of the control $u(\tau)$ for $\tau \in [0,T]$ that steers an initial state x_0 to a final state x_f in T units of time.

iii. Evaluate the expression of the control above in the case you want to control the mass-spring system from its rest position $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to the final position $x_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (unitary displacement and zero velocity) in $T = \frac{\pi}{\omega_0}$ units of time.

Hint Use the integrals

$$\int \sin^2(\omega_0\theta) d\theta = \frac{1}{2}\theta - \frac{\sin(2\omega_0\theta)}{4\omega_0} + c, \qquad \int \cos^2(\omega_0\theta) d\theta = \frac{1}{2}\theta + \frac{\sin(2\omega_0\theta)}{4\omega_0} + c$$

$$\int \sin(\omega_0\theta)\cos(\omega_0\theta) d\theta = -\frac{1}{4\omega_0}\cos(2\omega_0\theta) + c.$$

Solution i.

$$\begin{aligned} W(0,T) &= \int_0^T \begin{bmatrix} \sin^2(\omega_0\theta) & \sin(\omega_0\theta)\cos(\omega_0\theta) \\ \sin(\omega_0\theta)\cos(\omega_0\theta) & \cos^2(\omega_0\theta) \end{bmatrix} d\theta \\ &= \begin{bmatrix} \frac{1}{2}T - \frac{\sin(2\omega_0 T)}{4\omega_0} & -\frac{1}{4\omega_0}(\cos(2\omega_0 T) - 1) \\ -\frac{1}{4\omega_0}(\cos(2\omega_0 T) - 1) & \frac{1}{2}T + \frac{\sin(2\omega_0 T)}{4\omega_0} \end{bmatrix} \end{aligned}$$

ii. We have

$$\begin{aligned} u(\tau) = {} & B^T (\mathrm{e}^{A(T-\tau)})^T W(0,T)^{-1} \left(-\mathrm{e}^{AT} x_0 + x_f\right), \quad \tau \in [0,T] \\ = {} & \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\omega_0(T-\tau)) & -\sin(\omega_0(T-\tau)) \\ \sin(\omega_0(T-\tau)) & \cos(\omega_0(T-\tau)) \end{bmatrix} W(0,T)^{-1} \left(-\mathrm{e}^{AT} x_0 + x_f\right) \end{aligned}$$

where

$$\begin{aligned} W(0,T)^{-1} &= \frac{1}{\frac{T^2}{4} - \frac{1}{4\omega_0^2}\sin^2(\omega_0 T)} \begin{bmatrix} \frac{1}{2}T + \frac{\sin(2\omega_0 T)}{4\omega_0} & \frac{1}{4\omega_0}(\cos(2\omega_0 T) - 1) \\ \frac{1}{4\omega_0}(\cos(2\omega_0 T) - 1) & \frac{1}{2}T - \frac{\sin(2\omega_0 T)}{4\omega_0}, \end{bmatrix} \\ -\mathrm{e}^{AT} x_0 + x_f &= \begin{bmatrix} -\cos(\omega_0 T)x_{01} - \sin(\omega_0 T)x_{02} + x_{f1} \\ \sin(\omega_0 T)x_{01} - \cos(\omega_0 T)x_{02} + x_{f2} \end{bmatrix} \end{aligned}$$

6.1 (Double integrator) Consider the double integrator. Find a piecewise constant control strategy that drives the system from the origin to the state $x = (1, 1)$.

Figure 1: [1, Exercise 6.1]

Replacing these identities in the expression of $u(\tau)$ above returns the solution. iii. In this case

$$\begin{aligned} u(\tau) = &\; B^T(\mathrm{e}^{A(T-\tau)})^T W(0,T)^{-1}\left(-\mathrm{e}^{AT}x_0 + x_f\right) \\ = &\; \begin{bmatrix}\sin(\pi-\omega_0\tau) & \cos(\pi-\omega_0\tau))\end{bmatrix}\begin{bmatrix}\frac{2\omega_0}{\pi} & 0 \\ 0 & \frac{2\omega_0}{\pi}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = -\frac{2\omega_0}{\pi}\sin(\omega_0\tau), \quad \tau \in [0, \frac{\pi}{\omega_0}]. \end{aligned}$$

(b) Consider the linearized model of the point mass satellite obtained in Lecture 3, Exercise (a) and determine whether or not the system is controllable. Note that the system has two inputs.

Answer. We compute the reachability matrix

$$W_r = \begin{bmatrix}B & AB & A^2B & A^3B\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0 & \dots \\ 1 & 0 & 0 & 2\omega & \dots \\ 0 & 0 & 0 & \frac{1}{\rho} & \dots \\ 0 & \frac{1}{\rho} & \frac{-2\omega}{\rho} & 0 & \dots\end{bmatrix}$$

Note that we only computed the first four columns of W_r corresponding to B, AB because such columns have already rank 4 hence it is guaranteed that $\operatorname{rank}(W_r)$ has also rank 4. We conclude that the reachability matrix is a full row rank matrix and the system is reachable.

7. Lecture 7

(a) [1, Exercise 6.1] (see Figure 1)
Solution The state solution of the system when $u(t) = \bar{u} = \text{constant}$ and $x(0) = (0\ 0)^\top$ is given by

$$x(t) = \int_0^t \mathrm{e}^{A\theta}Bd\theta\bar{u} = \int_0^t \begin{bmatrix}\theta\\1\end{bmatrix}d\theta\bar{u} = \begin{bmatrix}\frac{t^2}{2}\\t\end{bmatrix}\bar{u}.$$

We look for the time $\bar{t}$ and the input $\bar{u}$ such that

$$\begin{bmatrix}\frac{\bar{t}^2}{2}\\ \bar{t}\end{bmatrix}\bar{u} = \begin{bmatrix}1\\1\end{bmatrix}.$$

These equations return $\bar{t}\bar{u} = 1$ and $\frac{\bar{t}^2}{2}\bar{u} = 1$, whose solution is $\bar{t} = 2$ and $\bar{u} = 1/2$.

(b) [1, Exercise 6.8]

(c) [1, Exercise 6.9]

(d) Consider the linearized model of the point mass satellite obtained in Lecture 3, Exercise (a) and determine the feedback gains

$$\begin{aligned} u_1 = &\; \begin{bmatrix}k_{11} & k_{12} & 0 & -2\rho\omega\end{bmatrix}x \\ u_2 = &\; \begin{bmatrix}0 & \frac{2\omega}{\rho} & k_{23} & k_{24}\end{bmatrix}x \end{aligned}$$

that stabilize the system.

Solution We observe that the term $-2\rho\omega x_4$ in u_1 and the term $\frac{2\omega}{\rho}x_2$ in u_2 decouple the radial and the tangential dynamics turning the closed-loop system into two independent SISO systems for which the control u_1, u_2, specifically the gains $k_{11}, k_{12}, k_{23}, k_{24}$ can be computed with the usual formulas for SISO systems.

8. Lecture 9 (Observability and observer design)

   (a) Consider the linearized model of a point mass satellite

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega^2 + \frac{2k}{\rho^3} & 0 & 0 & 2\rho\omega \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega}{\rho} & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix}$$

   There are 4 sensors available, each one measuring one of the states of the system. Find the minimal number of sensors that you require to design an observer. Is this choice of sensor(s) unique?
   Hint A multi-output system (that is, $y \in \mathbb{R}^p$ with $p > 1$) is observable if the rank of the observability matrix W_o is full and equal to n, the dimension of the state space.

9. Lecture 10 (Transfer functions)

   (a) [1, Exercise 8.6] (see Figure 2)
   Hint Use the reachable canonical form and the invariance to coordinate changes of the transfer function.

   8.6 (Transfer function for state space system) Consider the linear state space system

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$

   Show that the transfer function is

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n},$$

   where

$$b_1 = CB, \quad b_2 = CAB + a_1 CB, \quad \ldots, \quad b_n = CA^{n-1}B + a_1 CA^{n-2}B + \cdots + a_{n-1}CB$$

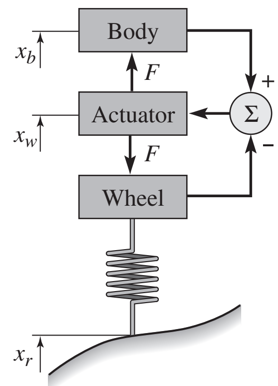
   and $\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_n$ is the characteristic polynomial for A.

   Figure 2: [1, Exercise 8.6]

   (b) [1, Exercise 8.14] (see Figure 3) No need to answer the final part of the exercise (numerical value of $G_{ax_r}(i\omega_t)$).
   Solution The transfer function relating the force F (output of the damper) and the input $x_w - x_b$ is $F = k(x_w - x_b) + cs(x_w - x_b)$. The transfer function of the body dynamics (double integrator, $m_b\ddot{x}_b = F$) is $m_b s^2 x_b = F$. The transfer function of the

8.14 (Vehicle suspension [HB90]) Active and passive damping are used in cars to give a smooth ride on a bumpy road. A schematic diagram of a car with a damping system in shown in the figure below.

(Porter Class I race car driven by Todd Cuffaro)



This model is called a *quarter car model*, and the car is approximated with two masses, one representing one fourth of the car body and the other a wheel. The actuator exerts a force F between the wheel and the body based on feedback from the distance between the body and the center of the wheel (the *rattle space*).

Let x_b, x_w and x_r represent the heights of body, wheel and road measured from their equilibria. A simple model of the system is given by Newton's equations for the body and the wheel,

$$m_b\ddot{x}_b = F, \qquad m_w\ddot{x}_w = -F + k_t(x_r - x_w),$$

where m_b is a quarter of the body mass, m_w is the effective mass of the wheel including brakes and part of the suspension system (the *unsprung mass*) and k_t is the tire stiffness. For a conventional damper consisting of a spring and a damper, we have $F = k(x_w - x_b) + c(\dot{x}_w - \dot{x}_b)$. For an active damper the force F can be more general and can also depend on riding conditions. Rider comfort can be characterized by the transfer function G_{ax_r} from road height x_r to body acceleration $a = \ddot{x}_b$. Show that this transfer function has the property $G_{ax_r}(i\omega_t) = k_t/m_b$, where $\omega_t = \sqrt{k_t/m_w}$ (the *tire hop frequency*). The equation implies that there are fundamental limitations to the comfort that can be achieved with any damper.

Figure 3: [1, Exercise 8.14]

wheel dynamics ($m_w\ddot{x}_w = -F + k_t(x_r - x_w)$) is given by $m_w s^2 x_w = -F + k_t(x_r - x_w)$. We then have

$$F = m_b s^2 x_b = k(x_w - x_b) + cs(x_w - x_b)$$

From which

$$x_w = \frac{m_b s^2 + cs + k}{k + cs} x_b$$

Moreover, from $m_w s^2 x_w = -F + k_t(x_r - x_w)$, we obtain

$$k_t x_r = (m_w s^2 + k_t)x_w + F = \frac{(m_w s^2 + k_t)(m_b s^2 + cs + k)}{k + cs} x_b + m_b s^2 x_b$$

Solving for x_b we arrive at

$$x_b = \frac{k_t(k + cs)}{(m_b s^2 + cs + k)(m_w s^2 + k_t) + m_b s^2(cs + k)} x_r$$

Finally, bearing in mind that $a = s^2 x_b$ and multiplying both sides by s^2 give the required transfer function

$$a = \frac{k_t s^2(k + cs)}{(m_b s^2 + cs + k)(m_w s^2 + k_t) + m_b s^2(cs + k)} x_r$$

(c) [1, Exercise 8.15] (see Figure 4)

Hint Assume that the transfer function corresponds to a reachable and an observable state space model, thus its poles are the eigenvalues of the dynamic matrix of the state space model. State conditions under which the poles have all strictly negative real parts using Routh-Hurwitz theorem, thus guaranteeing the existence of the harmonic response. Finally, bear in mind that the Laplace transform of a sinusoidal signal is given by

$$\mathcal{L}[\sin(\omega_0 t)] = \frac{\omega_0}{s^2 + \omega_0^2}$$

If you do not want to use the Laplace transform, then simply use the frequency response of a linear system.

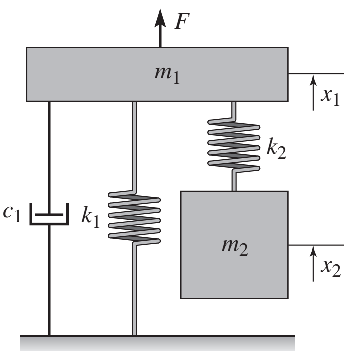
Solution The transfer function can be derived from the Euler-Lagrange equations of motion, converting them into a system of first order linear differential equations and then applying the usual formula $C(sI - A)^{-1}B + D$ to obtain $G_{x_1 F}$. Since $G_{x_1 F}$ is the transfer function of a reachable and observable state-space system, its poles coincide with the eigenvalues of the dynamic matrix of the system. Hence the system is asymptotically stable if and only if the poles have all strictly negative real parts. We apply Routh-Hurwitz theorem. Let us first construct the table

$$\begin{array}{c||ccc}
4 & m_1 m_2 & m_1 k_2 + m_2(k_1 + k_2) & k_1 k_2 \\
3 & m_2 c_1 & k_2 c_1 & \\
2 & \star & k_1 k_2 & \\
1 & \circ & & \\
0 & k_1 k_2 & &
\end{array}$$

where

$$\begin{aligned}
\star &= -\tfrac{1}{m_2 c_1}\left(m_1 m_2 k_2 c_1 - (m_1 k_2 + m_2(k_1 + k_2))m_2 c_1\right) \\
&= -\left(m_1 k_2 - (m_1 k_2 + m_2(k_1 + k_2))\right) \\
&= m_2(k_1 + k_2)
\end{aligned}$$

8.15 (Vibration absorber) Damping vibrations is a common engineering problem. A schematic diagram of a damper is shown below:



The disturbing vibration is a sinusoidal force acting on mass m_1, and the damper consists of the mass m_2 and the spring k_2. Show that the transfer function from disturbance force to height x_1 of the mass m_1 is

$$G_{x_1F} = \frac{m_2s^2 + k_2}{m_1m_2s^4 + m_2c_1s^3 + (m_1k_2 + m_2(k_1 + k_2))s^2 + k_2c_1s + k_1k_2}.$$

How should the mass m_2 and the stiffness k_2 be chosen to eliminate a sinusoidal oscillation with frequency ω_0. (More details are vibration absorbers is given in the classic text by Den Hartog [DH85, pp. 87–93].)

Figure 4: [1, Exercise 8.15]

and

$$\circ = \frac{c_1(k_2(k_1+k_2)-k_1k_2)}{k_1+k_2} = \frac{c_1k_2^2}{k_1+k_2}$$

Bearing in mind that all the parameters are positive, the systems is always asymptotically stable. The frequency response to a sinusoidal oscillation (input) of frequency ω_0 is $|G_{x_1F}(i\omega_0)|\sin(\omega_0 t + \angle G_{x_1F}(i\omega_0))$. Hence, if $|G_{x_1F}(i\omega_0)| = 0$, then the oscillation is eliminated. $|G_{x_1F}(i\omega_0)| = 0$ if and only if $-m_2\omega_0^2 + k_2 = 0$.

10. Lecture 12 (Frequency domain analysis)

   (a) [1, Exercise 9.6] (see Figure 5)

   Hint To solve the exercise, first draw the Nyquist plot of $L(s)$ using the Bode diagrams given in Figure 6.
   Solution The loop transfer function is $L(s) = \frac{k_d s + k_p}{s^2}$, whose Nyquist plot is given in Figure 7. The closed-loop system is asymptotically stable. The gain margin is $+\infty$, since there is no value of the open-loop gain that can lead the system from stability to instability (Figure 7). To compute the phase margin, first we compute the gain cross-over frequency ω_{gc}, which is obtained solving the equation

$$|L(i\omega_{gc})| = 1.$$

   We note that

$$|L(i\omega_{gc})| = \frac{\sqrt{k_d^2\omega_{gc}^2 + k_p^2}}{\omega_{gc}^2}.$$

   Hence, ω_{gc} is the solution of the quartic equation

$$\omega_{gc}^4 - k_d^2\omega_{gc}^2 - k_p^2 = 0.$$

   The unique *real and positive* solution is

$$\omega_{gc} = \frac{\sqrt{k_d^2 + \sqrt{k_d^4 + 4k_p^2}}}{\sqrt{2}}$$

   Replacing the values of k_p, k_d given in the text of the exercise, we have

$$\omega_{gc} = \frac{\sqrt{4\zeta^2\omega_0^2 + \sqrt{16\zeta^4\omega_0^4 + 4\omega_0^4}}}{\sqrt{2}} = \omega_0\sqrt{2\zeta^2 + \sqrt{4\zeta^4 + 1}}$$

   Correspondingly, we have that

$$\angle L(i\omega_{gc}) = -\pi + \arctan\frac{2\zeta\omega_{gc}}{\omega_0} = -\pi + \arctan\left(2\zeta\sqrt{2\zeta^2 + \sqrt{4\zeta^4+1}}\right)$$

   The phase margin φ_m is then equal to $\arctan\left(2\zeta\sqrt{2\zeta^2 + \sqrt{4\zeta^4+1}}\right)$.

   (b) i. Draw the Nyquist plot of the transfer function e^{-s} representing a unitary delay.
   ii. Draw the Nyquist plot of the transfer function $\frac{\mathrm{e}^{-s}}{s}$ representing a delayed integrator.

10.6 (Stability margins for second-order systems) A process whose dynamics is described by a double integrator is controlled by an ideal PD controller with the transfer function $C(s) = k_d s + k_p$, where the gains are $k_d = 2\zeta\omega_0$ and $k_p = \omega_0^2$. Calculate and plot the gain, phase and stability margins as a function ζ.

Figure 5: [1, Exercise 9.6]

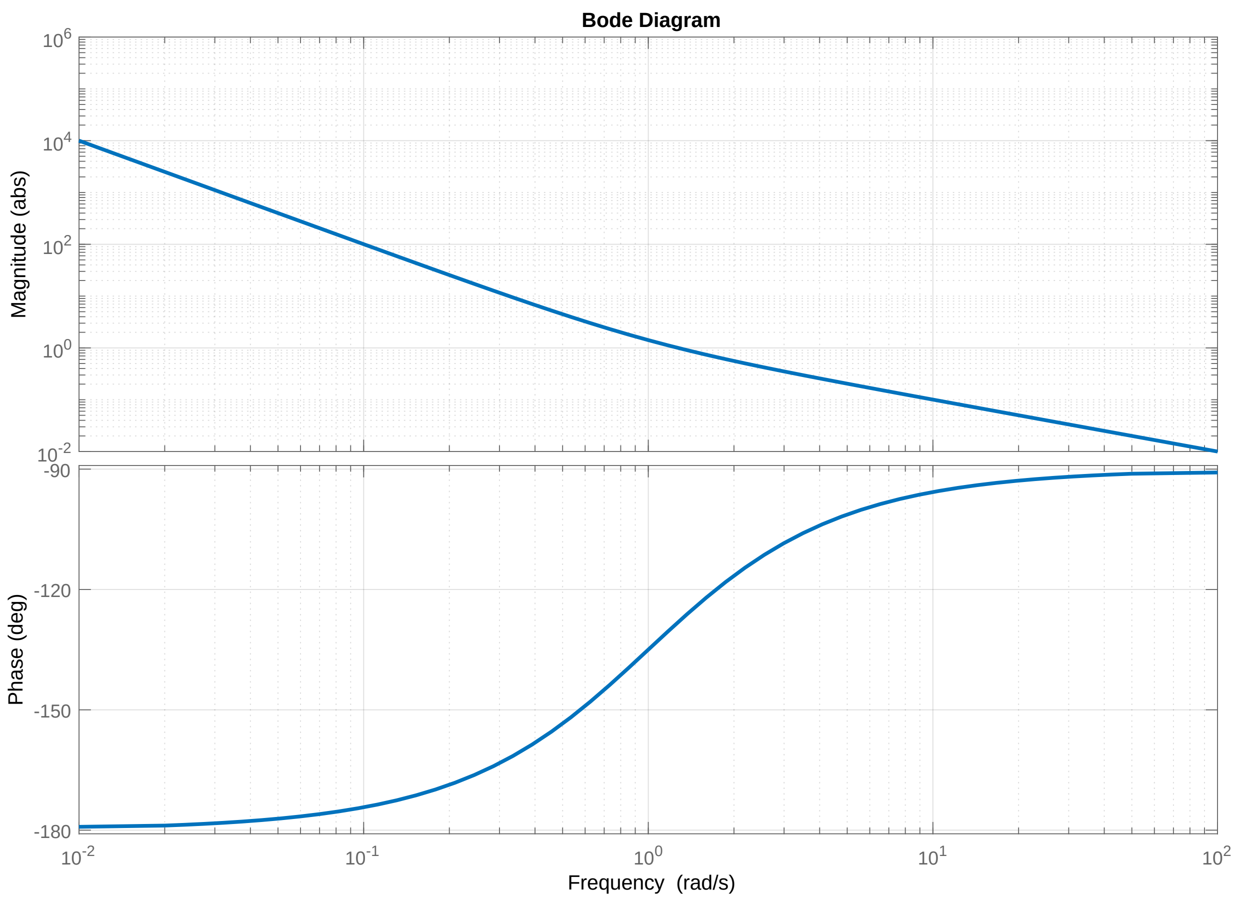


Figure 6: [1, Exercise 9.6] Bode diagrams of the loop transfer function $L(s) = \frac{k_d s + k_p}{s^2}$.

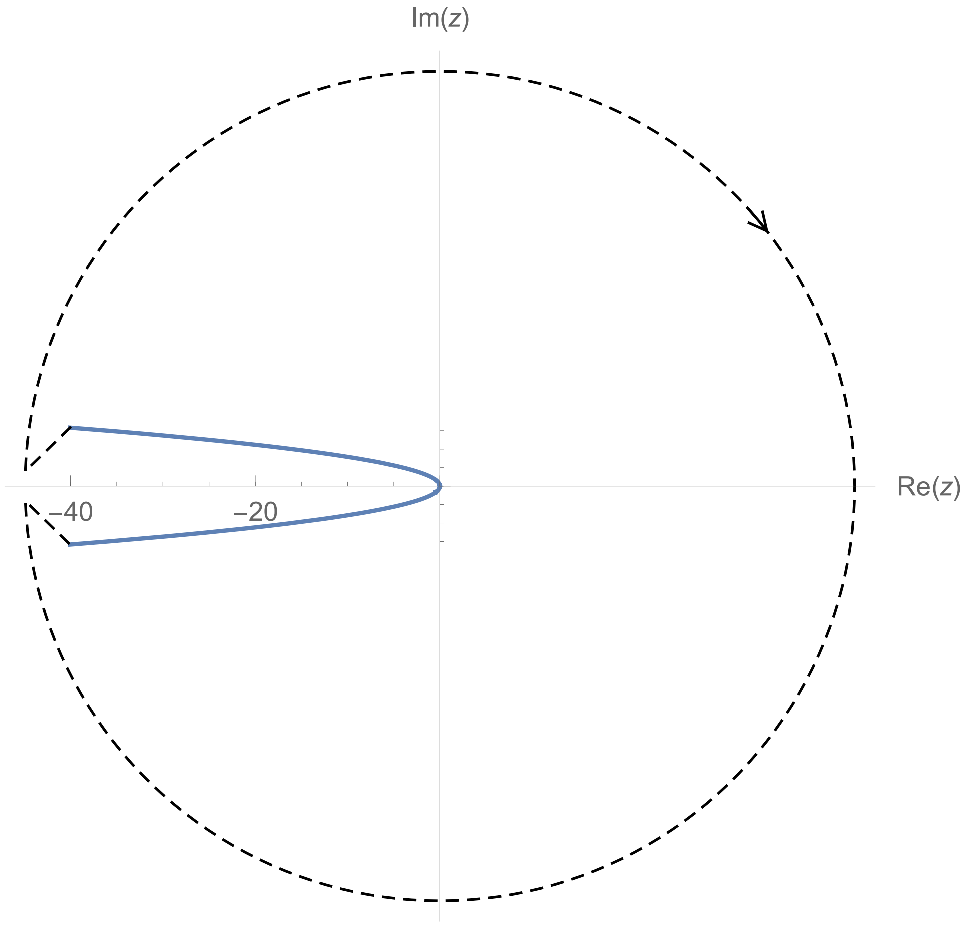


Figure 7: [1, Exercise 9.6] Nyquist diagrams of the loop transfer function $L(s) = \frac{k_d s + k_p}{s^2}$ (with $k_d = k_p = 1$).

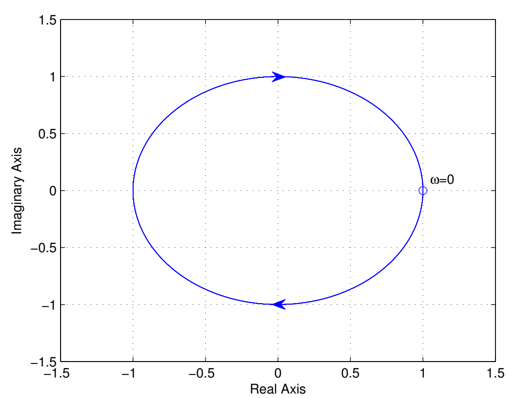


Figure 8: Nyquist plot for e^{-s}

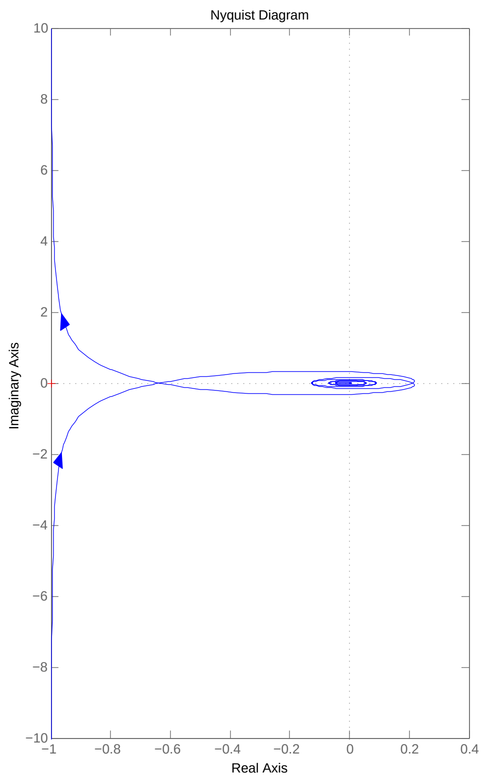


Figure 9: Nyquist plot for e^{-s}/s

iii. Find analytically all the phase-crossover frequencies of the transfer function $\frac{e^{-s}}{s}$, that is all the frequencies for which the Nyquist plot crosses the negative real axis.

Hint The Nyquist plot in ii. can help you answering the question.

iv. Solve questions i., ii., iii. for the transfer function $\frac{e^{-\tau s}}{s}$ with delay $\tau > 0$.

v. Find values of τ for which the closed-loop system is asymptotically stable. For these values of τ, compute the gain stability margin. For those values of τ for which the closed-loop system is not asymptotically stable, can you find a simple controller that stabilizes the closed-loop system?

Solution i. See Figure 8. ii. See Figure 9. iii. All the values of ω_{gc} are found solving the equation $\angle L(i\omega_{gc}) = -(2k+1)\pi$, $k = 0, 1, 2, \ldots$, which in this case becomes $-\frac{\pi}{2} - \omega = -(2k+1)\pi$, $k = 0, 1, 2, \ldots$, which gives the solution $\omega_{gc} = 2k\pi + \frac{\pi}{2}$, $k = 0, 1, 2, \ldots$. iv. The Nyquist plots of e^{-s} and $\frac{e^{-s}}{s}$ are drawn similarly to the ones in Figure 8 and Figure 9. With similar calculations as in iii., we find that $\omega_{gc} = 2k\frac{\pi}{\tau} + \frac{\pi}{2\tau}$, $k = 0, 1, 2, \ldots$. v. The smallest ω_{gc} is equal to $\frac{\pi}{2\tau}$, at which the loop transfer function magnitude is given by $|L(i\omega_{gc})| = \frac{1}{\omega_{gc}} = \frac{2\tau}{\pi}$. Hence, if $\frac{2\tau}{\pi} < 1$, then there are no net encirclements of the point -1 and the system is asymptotically stable. We conclude that the system is asymptotically stable for sufficiently small values of the delay ($\tau < \frac{\pi}{2}$). For these values, the gain margin is given by $\frac{\pi}{2\tau}$. For values of the delay that do not satisfy the condition $\tau < \frac{\pi}{2}$, one can guarantee stability using a proportional control with gain k_p such that $\frac{2k_p\tau}{\pi} < 1$, or, equivalently, $k_p < \frac{\pi}{2\tau}$.

(c) Lecture 13 (PID control)

i. [1, Exercise 10.6]

Solution We first note that

$$\frac{b}{s(s+a_1)(s+a_2)} = \frac{\alpha}{s} + \frac{\beta}{s+a_1} + \frac{\gamma}{s+a_2}$$

where

$$\alpha = \frac{b}{a_1 a_2}, \quad \beta = \frac{b}{a_1(a_1-a_2)}, \quad \gamma = \frac{b}{a_2(a_1-a_2)},$$

from which we can compute the step output response of the system with transfer function $G_{qp}(s)$, given by

$$y(t) = \begin{cases} 0 & 0 \le t < \tau_e \\ \alpha \mathbb{1}(t-\tau_e) + \beta e^{-a_1(t-\tau_e)} + \gamma e^{-a_2(t-\tau_e)} & t \ge \tau_e, \end{cases}$$

where the function $\mathbb{1}(t)$ denotes the step input.

Replacing the numerical values of the parameters we obtain:

$$a_1 = 400, \quad a_2 = 6.66, \quad \alpha = 3.91, \quad \beta = 0.066, \quad \gamma = -3.97,$$

which shows that the term associated to the pole a_1 can be neglected, since it vanishes very fast due to the high value of a_1 and the small value of β. This implies that, to the purpose of tuning the PID controller via the Ziegler-Nichols improved step method, we can approximate the transfer function as

$$G_{qp}(s) \approx \frac{b}{s+a_2} e^{-s\tau_e}$$

and the step response as

$$y(t) \approx \begin{cases} 0 & 0 \le t < \tau_e \\ \alpha \mathbb{1}(t-\tau_e) + \gamma e^{-a_2(t-\tau_e)} & t \ge \tau_e, \end{cases}$$

We observe that the steepest tangent is obtained at $t = \tau_e$ and given by

$$\dot{y}(\tau_e) = -a_2\gamma = 26.44.$$

From this we deduce that this tangent intercepts the vertical axis at $-a = a_2\gamma\tau_e = -3.96$ (see [1, Figure 10.7(a)] and the horizontal axis at $\tau = \tau_e = 0.15$. Using [1, Table 10.1], one can propose a PI controller with transfer function

$$G_{ue}(s) = k_p + \frac{k_i}{s} = k_p(1 + \frac{1}{T_i s}) = \frac{0.9}{3.96}(1 + \frac{1}{3 \cdot 0.15 s}) = 0.227(1 + \frac{1}{0.45 s})$$

(d) Consider again [1, Exercise 10.4] solved in Lecture 13 (see the slides). Do the proportional controllers tuned according to Ziegler-Nichols' step and frequency response method stabilize the closed-loop system?
Hint Use the results of Exercise (b), Lecture 12 in this document.

11. Lecture 15-16 (Loop shaping)

(a) On the slides of Lecture 15, it is stated that the inverse function of

$$\phi_{max} = \frac{\pi}{2} - 2\arctan\sqrt{\alpha}$$

is given by

$$\alpha = \frac{1 - \sin\phi_{max}}{1 + \sin\phi_{max}}$$

Show that this is actually true.
Solution We have $\arctan\sqrt{\alpha} = \frac{\pi}{4} - \frac{\phi_{max}}{2}$. This in turn gives

$$\sqrt{\alpha} = \tan\left(\frac{\pi}{4} - \frac{\phi_{max}}{2}\right),$$

which, in view of the formula $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$, return the identity

$$\sqrt{\alpha} = \tan\left(\frac{\pi}{4} - \frac{\phi_{max}}{2}\right) = \frac{1 - \tan\frac{\phi_{max}}{2}}{1 + \tan\frac{\phi_{max}}{2}}.$$

Hence

$$\sqrt{\alpha} = \frac{\cos\frac{\phi_{max}}{2} - \sin\frac{\phi_{max}}{2}}{\cos\frac{\phi_{max}}{2} + \sin\frac{\phi_{max}}{2}}.$$

Taking the squares of both sides, and using elementary trigonometric identities ($\sin^2\alpha + \cos^2\alpha = 1$, $\sin(2\alpha) = 2\sin\alpha\cos\alpha$) we obtain

$$\alpha = \frac{1 - \sin\phi_{\max}}{1 + \sin\phi_{\max}}.$$

(b) [1, Example 12.8] Consider the problem of controlling the roll of a vectored thrust aircraft modelled as

$$P(s) = \frac{r}{Js^2}$$

by a lead controller

$$C(s) = k\frac{s+a}{s+b}.$$

In this exercise we set $r/J = 1$. Using the guidelines to design a lead controller provided in the slides of Lecture 15, justify the values $a = 2$, $b = 50$ proposed in the textbook and discuss a possible value of the gain k. The performance specifications are (i) $|L(i\omega)| \geq 11$ for $\omega \in [0, 10]$ and (ii) $m_\varphi \geq 60°$.

Solution Note that $C(s)$ can be rewritten as

$$k\frac{a}{b}\frac{Ts+1}{\alpha Ts+1}$$

where $T = \frac{1}{a} = \frac{1}{2}$ and $\alpha T = \frac{1}{b} = \frac{1}{50}$. From the last equation, we obtain $\alpha = \frac{1}{25}$. Using the formula for the maximal phase lead provided by the lead controller, namely $\phi_{max} = 90° - \frac{360°}{\pi}\arctan(\sqrt{\alpha}) = 67°$, we realize that the choice of α aims at guaranteeing a phase margin equal to $60° + 7°$, which is slightly larger than the prescribed one. On the other hand the choice of T guarantees that the maximal phase lead of the controller will be provided at frequency $\omega_{\max} = \frac{1}{T\sqrt{\alpha}} = \frac{2}{\frac{1}{5}} = 10$rad/s. To asses the value of k, we evaluate $|L(i\,10)| = k\frac{1}{100}\sqrt{\frac{104}{2600}} = k0.002$. To have $|L(i\,10)| > 11$, we must have $k > 5500$. Overall

$$C(s) = 5500\frac{s+2}{s+50}.$$

Drawing the Bode diagrams of $L(s)$ with this choice of the controller, we see that, due to the increase of the gain to fulfil the specification (i), the gain crossover frequency is increased to 70rad/s, and a phase margin of about 37°, thus much lower than the required one. From the inspection of the Bode diagrams, we see that lowering the gain crossover frequency to 20rad/s (which is the choice of the textbook, see Figure 12.5(b) therein), would indeed give a phase margin of 62rad/s, as requested. To have this gain crossover frequency, the gain k should be lowered to 1000. With this choice of k, however, we would have $|L(i\,10)| = 2$, which is lower than the quantity requested and leads to a worse tracking property of the closed-loop system (in fact, it can be computed that $1/|1 + L(i10)| \approx 0.5$, which is larger than the prescribed 0.1). We conclude that, with the choice of the controller suggested by the textbook, it is not possible to achieve simultaneously the two specifications (i) and (ii) simultaneously.

One can take other approaches and design different controllers. If $C(s) = k = 1$ (unitary proportional controller), at $\omega = 10$, $|L(i\omega)| = 10^{-2}$, hence, to guarantee specification (i) we should raise k to the value 1000. With this value of the gain, the new crossover frequency becomes $\omega_{gc,new} = 10^{\frac{3}{2}} \approx 30$rad/s. The proportional gain does not guarantee any stability (in fact, the closed-loop system would have a pair of poles at $\pm i\sqrt{k}$). To achieve stability with a guaranteed phase margin of $m_\varphi \geq 60°$, we use a lead controller that adds a phase $\phi_{max} = 60° + 7° = 67°$, to have some extra degree of freedom. Using the formula

$$\alpha = \frac{1 - \sin\phi_{max}}{1 + \sin\phi_{max}}.$$

after converting ϕ_{max} in radians, we obtain $\alpha = 0.04$. We now set $\frac{1}{T\sqrt{\alpha}}$ equal to a slightly larger value of the expected new gain crossover frequency, say $\omega_{gc,new} =$ 40rad/s, to take into account the additional gain provided by the lead controller. We then obtain the value of $T = \frac{1}{40 \cdot 0.2} = \frac{1}{8}$. Hence

$$C(s) = 1000\frac{\frac{1}{8}s+1}{\frac{1}{8}\frac{1}{15}s+1} = 1000 \cdot 15\frac{s+8}{s+120}$$

From the Bode diagram of the loop transfer function we see that the phase margin is equal to 45° (thus an improvement with respect to the previous solution) and that $|L(i\,10)| \approx 13$. This improvement comes at the price of having a larger crossover frequency, hence more sensitivity to noise. A controller for the fullfillment of both (i) and (ii) could be achieved by further trial-and-error tuning.

References

[1] Åstrom, Murray. Feedback Systems: An Introduction for Scientists and Engineers. First Edition, version 2.11b
`http://www.cds.caltech.edu/~murray/amwiki/index.php?title=First_Edition`