Class notes

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Contents

1 Lecture 1

1.1 Control Theory

the science of modifying the physical environment. Is a mathematically oriented science. Alt: the use of algorithms and feedback in engineered systems.

1. **Dynamical system:** system whose behavior changes over time, often in response to external stimulation or forcing.
2. **Open/closed loop:** In a closed loop system, the output of system 1 is used as the input of system 2, and the output of system 2 becomes the input of system 1. If he interconnection between system 2 and system 1 is removed then the system is said to be open loop.
3. **Control Law:** The algorithm that computes the control action as a function of the sensor values.
4. **Feedforward:** measure a disturbance before it enters the system, and this information can then be used to take corrective action before the disturbance has influenced the system. The effect of the disturbance is thus reduced by measuring it and generating a control signal that counteracts it.
5. **Bistability:** use of positive feedback (more in section 1.2.1) is to create switching behavior, in which a system maintains a given state until some input crosses a threshold.
6. **Hysteresis:** the dependence of the state of a system on its history.
7. **Steady-state error:** difference between the desired value and the actual value of a system output in the limit as time goes to infinity (i.e. when the response of the control system has reached steady-state). In other words, nothing changes anymore and nothing will be a function of time.

The system can be influenced externally by an operator who introduces command signals to the system. Modeling in a control context allows the design of robust interconnections between subsystems, a feature that is crucial in the operation of all large engineered systems.

1.2 Feedback

Situation in which two (or more) dynamical systems are connected together such that each system influences the other and their dynamics are thus strongly coupled. Formal methods are then necessary to understand it. The key uses of feedback is to provide robustness to uncertainty and to change the dynamics of a system.

1.2.1 Negative/positive Feedback

Negative feedback: we attempt to regulate the system by reacting to disturbances in a way that decreases the effect of those disturbances.
Positive feedback: increase in some variable or signal leads to a situation in which that quantity is further increased through its dynamics. This has a destabilizing effect and is usually accompanied by a saturation that limits the growth of the quantity. Although often considered undesirable, this behavior is used in biological (and engineering) systems to obtain a very fast response to a condition or signal.

1.2.2 Drawbacks and Advantages

Advantages: By using feedback to create a system whose response matches a desired profile, we can hide the complexity and variability that may be present inside a subsystem

Drawbacks: **feedback instability**, system must not only to be stable under nominal conditions but also to remain stable under all possible perturbations of the dynamics. Also, feedback often injects measurement noise into the system and there's high complexity in the task of embedding a control system in a product

1.3 Feedback Control Systems: An Introduction

A basic feedback control system is the on-off control

$$u = \begin{cases} u_{max} & if \quad e > 0 \\ u_{min} & if \quad e < 0 \end{cases} \tag{1}$$

where the control error $e = r - y$ is the difference between the reference signal (or command signal) r and the output of the system y and u is the actuation command. It is common to make modifications by introducing either a dead zone or hysteresis (more in section 1.1), since there is no value assigned for the condition $e = 0$ (visible in Figure 1).

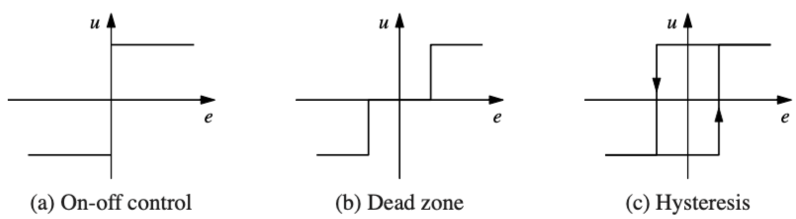

Figure 1: Input/output characteristics of on-off controllers. Each plot shows the input on the horizontal axis and the corresponding output on the vertical axis. Ideal on-off control is shown in (a), with modifications for a dead zone (b) or hysteresis (c). Note that for on-off control with hysteresis, the output depends on the value of past inputs.

All feedback control systems follow the block diagram visible in Figure 2. A block diagram is a type of schematic diagram used in control engineering (more on how to do it in textbook (Åström and Murray, 2021), 2.3, page 44).

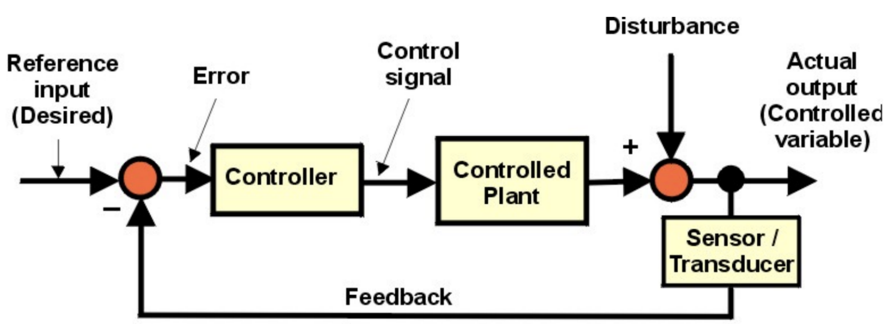

Figure 2: Feedback control systems diagram

where the controlled plant is whatever we have to control, the controllable inputs is what we call "control", uncontrollable input is the **disturbance**, sensor output is a physical variable that can be measured y and reference value y_r is compared to the output y to produce an error e. The controller will take this error e and output a control u to the controlled plant. We call this design the **control algorithm**.

1.3.1 Example: Cruise control

Textbook section 3.1, shows the control system depicted in figure 3, where u is the quantity proportional to the fuel injection and $F_d, u, v \in \mathbb{R}$. Here v is the state, and T the controlled input, according to the roles explained in feedback control figure 2 and its description.

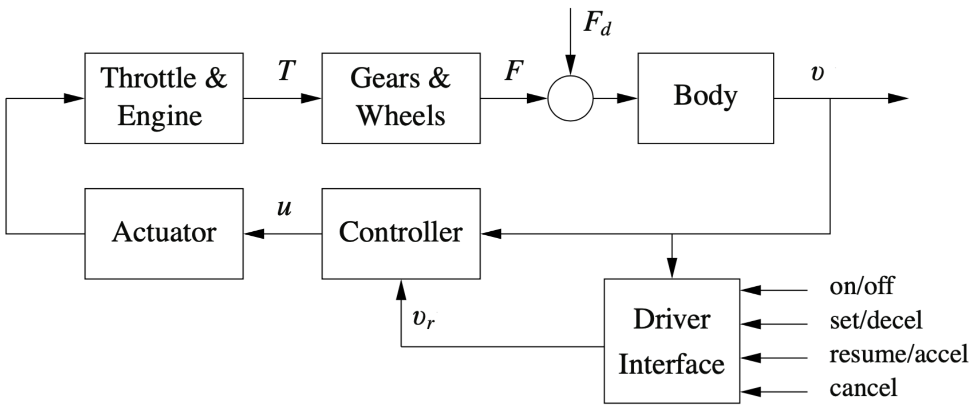

Figure 3: Block diagram of a cruise control system for an automobile. The throttle-controlled engine generates a torque T that is transmitted to the ground through the gearbox and wheels. Combined with the external forces from the environment F_d, such as aerodynamic drag and gravitational forces on hills, the net force F causes the car to move. The velocity of the car v is measured by a control system that adjusts the throttle through an actuation mechanism. A driver interface allows the system to be turned on and off and the reference speed v_r to be established.

However we can visualise this as the equation of motion $m \cdot a = F$

$$m\frac{d}{dt}v(t) = F(t) \qquad \text{(we add disturbances)}$$
$$m\frac{d}{dt}v(t) = F(t) - F_d(t) \qquad \text{(we assume } v(t), F(t), F_d(t) \in \mathbb{R}\text{)}$$

Let's relate the force with the input u

$$F(t) = \frac{n}{r}T(\alpha_n \cdot v(t)) \cdot u(t) = g(v(t)) \cdot u(t) \tag{2}$$

in which n is the gear ratio, r is the wheel radius and α is a coefficient related to the gear ratio, $\alpha \cdot v$ is the rotational velocity and the torque T is a quadratic function of its argument, which can be defined as

$$T(\omega) = \alpha + \beta \cdot v - \gamma \cdot v^2 \tag{3}$$

where $\alpha, \beta, \gamma > 0$ are all constant coefficients

We can write the force of disturbance $F_d(t)$ as

$$F_d(t) = \frac{1}{2}\rho C_d A v^2(t) + mgC_r sign(v(t)) \tag{4}$$

where ρ is the density of air, C_d is drag, mg is the mass multiplied by gravity,A is the front area of the car and C_r is rolling friction. $Sign(v(t))$ is given by the following

$$sign(v(t)) = \begin{cases} +1 & v(t) > 0 \\ 0 & v(t) = 0 \\ -1 & v(t) < 0 \end{cases} \tag{5}$$

They are all constant coefficients, so let's rename $\frac{1}{2}pC_dA$ as $a > 0$. More generally, we can state that for all

$$m\frac{dv(t)}{dt} = -av(t)^2 + g(v(t))u(t) + d \tag{6}$$

where d is disturbances, which is unknown, $-av(t)^2$ is aerodynamic drag and $g(v(t))u(t)$ is the driving force. The disturbance d in this example is given by the unknown rolling friction and the slope of the road θ, which we can say is the gravitational force $F_g = -mg \cdot \sin(\theta)$ affects the car as visible in figure 4

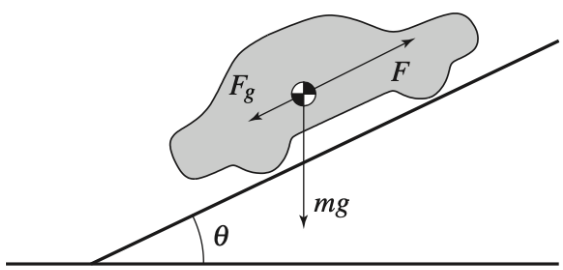

Figure 4: Effect of the gravitational force

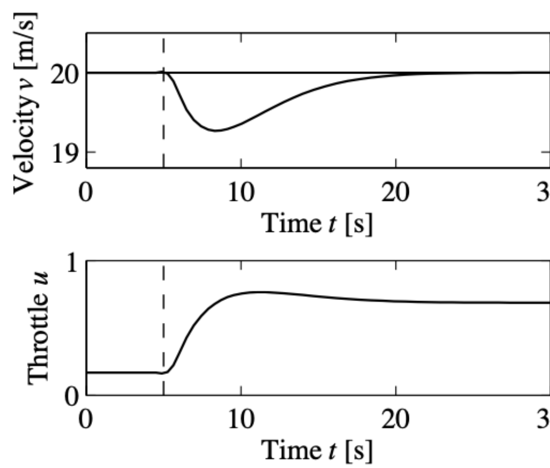

Figure 5: Closed loop response

Figure 6: Car with cruise control encountering a sloping road. A schematic diagram is shown in 4, and 5 shows the response in speed and throttle when a slope of 4° is encountered. The hill is modeled as a net change of 4° in hill angle θ, with a linear change in the angle between $t = 5$ and $t = 6$. The PI controller has proportional gain is $k_p = 0.5$, and the integral gain is $k_i = 0.1$.

1.4 Linearization at an equilibrium point

raw lecture

The following are non-linear because they are not linear equation.

$$m\dot{v} = -av^2 + g(v)u + d \tag{7}$$

We define the equilibrium point as the moment in which there is no disturbance. Also defined as the tuple of the two scalars $(\bar{v}, \bar{u})$ such that the following nonlinear eq is satisfied

$$-a\bar{v}^2 + g(\bar{v})\bar{u} = 0 \tag{8}$$

where $\bar{u}$ measures a constant signal and accounts for the force you need to apply in order to contrarrest the drag and keep the velocity constant, while $\bar{v}$ accounts for constant speed. If $g(\bar{v}) \neq 0$ then $\bar{u} = \frac{a\bar{v}^2}{g(\bar{v})}$

We do a Taylor Expansion at the equilibrium point

$$f(u, v) = f(\bar{u}, \bar{v}) + \frac{\partial f}{\partial v}|_{v=\bar{v}, u=\bar{u}}(v - \bar{v}) + \frac{\partial f}{\partial u}|_{v=\bar{v}, u=\bar{u}}(u - \bar{u}) + R_2(v - \bar{v}, u - \bar{u}) \tag{9}$$

We can calculate $R_2(v - \bar{v}, u - \bar{u})$, with the following equation

$$R_2(v - \bar{v}, u - \bar{u}) = \frac{1}{2}[v - \bar{v}, u - \bar{u}]\frac{\partial^2 f}{\partial (v, u)^2}|_{\hat{v}, \hat{u}} \begin{matrix} v - \bar{v} \\ u - \bar{u} \end{matrix} \tag{10}$$

where $\frac{\partial^2 f}{\partial (v,u)^2}$ is the Hessian, which is a 2x2 matrix, because we compute at the equilibrium, we know $f(\bar{u}, \bar{v}) = 0$.

Because we are around the equilibrium, we can approximate the left-hand side of the equation with a nonlinear approximation, such as

$$m\dot{v} = \frac{\partial f}{\partial v}|_{v=\bar{v},u=\bar{u}}(v - \bar{v}) + \frac{\partial f}{\partial u}|_{v=\bar{v},u=\bar{u}}(u - \bar{u}) + d \tag{11}$$

$$\frac{\partial f}{\partial v}|_{v=\bar{v},u=\bar{u}} = -2a\bar{v} + \frac{dg}{dv}|_{v=\bar{v}} \cdot \bar{u} = \alpha \tag{12}$$

$$\frac{\partial f}{\partial u}|_{v=\bar{v},u=\bar{u}} = g(\bar{v}) = \beta \qquad (\tilde{v} = v - \bar{v}, \tilde{u} = u - \bar{u})$$

$$m\dot{\tilde{v}} = -\alpha\tilde{v} + \beta\tilde{u} + d \tag{13}$$

This is allowed because this is representing derivatives, while the tilde only represents difference. This is now a linear equation where the squared has disappeared

1.5 Controller

Some of the more used controllers are Proportional-integral controllers (PI controllers).

1.5.1 Proportional control

On-off controllers (described in section 1.3) often gives rise to oscillations is that the system overreacts since a small change in the error makes the actuated variable change over the full range. This effect is avoided in proportional control, where the characteristic of the controller is proportional to the control error for small errors. This can be achieved with the control law

$$u = \begin{cases} u_{max} & e \geq e_{max} \\ k_p e & e_{min} < e < e_{max} \\ u_{min} & e \leq e_{min} \end{cases} \tag{14}$$

where k_p is the controller gain, $e_{min} = u_{min}/k_p$ and $e_{max} = u_{max}/k_p$

We define **proportional band** as the interval (e_{min}, e_{max}) because the behavior of the controller is linear when the error is in this interval $k_p e = k_p(r - y)$

1.5.2 Integral control

Integral control arises when we need some level of control signal is required for the system to maintain a desired value, then we must have $e \neq 0$ in order to generate the requisite input. Then we rely on the concept of Hysteresis (definition in section 1.1) to make the control action proportional to the integral of the error.

$$u(t) = k_i \int_0^t e(\tau) d\tau \tag{15}$$

where k_i is the integral gain. A controller with integral action has zero steady-state error (definition in section 1.1). The catch is that there may not always be a steady state because the system may be oscillating.

1.5.3 Derivative control

We can further provide the controller with an anticipating ability by using a prediction of the error. A simple prediction is given by the linear interpolation

$$e(t + T_d) \approx e(t) + T_d \frac{de(t)}{dt} \tag{16}$$

which predicts the error T_d time units ahead.

1.5.4 PI Controller

$$\tilde{u}(t) = K_p e(t) + K_i \int_0^t e(s)ds \tag{17}$$

where $e(t) = -\tilde{v}(t) = (v(t) - \bar{v})$ and K_p, K_i are gains to be tuned. It's called a proportional integral because the error is proportionally changed. The integration looks at the recent time and multiplies this accumulation. However, it's hard to handle an integral in a differential equation, so we'd rather be able to tune K_p, K_i such that $\lim_{t\to\infty} e(t) = 0$. Implementation of the PI controller

unsure of this dot

$$\xi(t) = \int_0^t e(s)ds \tag{18}$$

$$\dot{\xi(t)} = e(t) = -\tilde{v}(t) \tag{19}$$

$$\tilde{u}(t) = K_p e(t) + K_i \xi(t) = -K_p \tilde{v}(t) + K_i \xi(t) \tag{20}$$

1.5.5 PID Controller

Combining proportional, integral and derivative control, we obtain a controller whose control action is a combination of the past as represented by the integral of the error, the present as represented by the proportional term and the future as represented by a linear extrapolation of the error (the derivative term). This is visible in equation 21.

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau)d\tau + k_d \frac{de(t)}{dt} \tag{21}$$

1.6 Modelling

A model is a precise representation of a system's dynamics used to answer questions via analysis and simulation. We will introduce two specific methods commonly used in feedback and control systems: differential equations and difference equations and work in "state space" form (more in section 1.7.2) . However, models can be divided in multiple classifications, as seen in figure 7.

Deterministic exact relationship between measurable and derived variables, no stoch. uncertainty	**Stochastic** contains quantities described using stochastic variables or processes	
Dynamic variables changing with time, differential or difference equations	**Static** instantaneous links between variables, e.g., Ohm's law	
Continuous time $t \in \mathbb{R}$ relations with help of differential equations	**Discrete time** $k \in \mathbb{Z}$ relations with help of difference equations	**Hybrid** $(t, k) \in \mathbb{R} \times \mathbb{Z}$ mix of differential difference equations
Lumped Ordinary differential eqs. finite # of variables, discrete set of variables variables in space	**Distributed** Partial differential eq.s infinite # of variables, continuous set of variables variables in space	

Figure 7: Table of the characteristics defining different types of models

1.6.1 Normalisation and scaling

By adding dimension-free variables we may reduce the number of parameters and improve the numerical conditioning of the model to allow faster and more accurate simulations. This is often done by defining new variables by dividing the independent variables by the chosen normalization unit.

For example consider the spring mass equation 31 but without damping. where, to fix the ideas, t is measured in sec and q in m. Define the dimension-free independent variables (time and position) $z = \frac{q}{l}, \tau = \omega_0 t$ where $l(m)$ and $\omega_0(sec^{-1})$ are the chosen spatial and temporal length scale. In the chosen rescaled spatial and temporal variables, we have

$$z(\tau) := \frac{q(t)}{l}|_{t=\frac{\tau}{\omega_0}} \tag{22}$$

$$\frac{d}{d\tau}z(\tau) = \frac{\dot{q}(t)}{l}|_{t=\frac{\tau}{\omega_0}}\frac{1}{\omega_0} \tag{23}$$

$$\frac{d^2}{d\tau^2}z(\tau) = \frac{\ddot{q}(t)}{l}|_{t=\frac{\tau}{\omega_0}}\frac{1}{\omega_0^2} \tag{24}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{25}$$

$$u(\tau) = \frac{F(t)}{l}|_{t=\frac{\tau}{\omega_0}}\frac{1}{m\omega_0^2} \tag{26}$$

Then the original model can be rewritten as

$$m\omega_0^2\frac{\ddot{q}(t)}{l}|_{t=\frac{\tau}{\omega_0}}\frac{1}{\omega_0^2} + k\frac{q(t)}{l}|_{t=\frac{\tau}{\omega_0}} = \frac{F(t)}{l}|_{t=\frac{\tau}{\omega_0}} \tag{27}$$

$$m\omega_0^2\frac{d^2}{d\tau^2}z(\tau) + kz(\tau) = \frac{F(t)}{l}|_{t=\frac{\tau}{\omega_0}} \qquad \text{(divide by } m\omega_0^2\text{)}$$

$$\frac{d^2}{d\tau^2}z(\tau) + \frac{k}{m\omega_0^2}z(\tau) = \frac{F(t)}{l}|_{t=\frac{\tau}{\omega_0}}\frac{1}{m\omega_0^2} \tag{28}$$

$$\frac{d^2}{d\tau^2}z(\tau) + z(\tau) = u(\tau) \tag{29}$$

where $\tau, z(\tau), u(\tau)$ are dimension-free variables. Once the solution $z(\tau)$ is computed one can recover the physical position $q(t) = lz(t)|_{\tau=\omega_0 t}$. Similarly we can get velocity, acceleration and force.

1.7 Systems

1.7.1 Dynamical systems

We will use dynamical systems (definition in section 1.1). The **state** of a dynamical system is a collection of variables that completely characterizes the motion of a system for the purpose of predicting future motion, we call the set of all possible states the **state space** (more in section1.7.2).

The simplest model of a dynamical system is an Ordinary Differential Equation (ODE). In mechanics, one of the simplest such differential equations is that of a spring–mass system with damping, following figure 13:

$$m\ddot{q} + c(\dot{q}) + kq = 0 \tag{30}$$

where $c(\dot{q})$ is a nonlinear function defining the dampening, q is position, $\dot{q}$ is velocity, $\ddot{q}$ is acceleration and both q and its derivatives represent the instantaneous state of the system. We say that this system is a **second-order system** since the dynamics depend on the first two derivatives of q and an **autonomous system** because there are no external influences.

If there are external influences, then we get a Forced/Controlled Differential Equation

$$m\ddot{q} + c(\dot{q}) + kq = u(t) \tag{31}$$

where $u(t)$ represents the effect of external inputs, also written as $F(t)$.

Other characteristics of systems are linearity, time-invariance and causality.

- A system is **linear** if the superposition (addition) of two inputs yields an output that is the sum of the outputs that would correspond to individual inputs being applied separately.
- A system is **time-invariant** if the output response for a given input does not depend on when that input is applied.

Some interesting concepts are

1. **Step response**: describes the relationship between an input that changes from zero to a constant value abruptly (a step input) and the corresponding output.

2. **Frequency response:** way to describe a linear time-invariant system is to represent it by its response to sinusoidal input signals

1.7.2 State space models

In state space models we substitute the basic ODE or CDE models by the state space model

$$\frac{dx}{dt} = f(x,u), \quad y = h(x,u) \tag{32}$$

where $x \in \mathbb{R}^n$ is a vector of state variables, $u \in \mathbb{R}^m$ is a vector of control signals and $y \in \mathbb{R}^p$ is a vector of measurements. $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q$ are (possibly nonlinear) smooth mappings of their arguments to vectors of the appropriate dimension. The state vector dimension aka order of the system.

For example, it is natural to ask if possible states x can be reached with the proper choice of u (**reachability**) and if the measurement y contains enough information to reconstruct the state (**observability**).

This system can be modeled in state space as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u, \tag{33}$$

$$y = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} x + du. \tag{34}$$

we do so by converting our n-th order linear differential equation into a system of n first order linear differential equations (a Linear Time Invariant (LTI) system).

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n y = u \tag{35}$$

$$\dot{x_1} = \frac{d}{dt}\frac{d^{n-1}y}{dt^{n-1}} = \frac{d^n y}{dt^n} = -a_1 \frac{d^{n-1}y}{dtdt^{n-1}} - \cdots - a_n y + u = -a_1 x_1 - \cdots - a_n x_n + u \tag{36}$$

$$\dot{x_2} = \frac{d}{dt}\frac{d^{n-2}y}{dt^{n-2}} = \frac{d^{n-1}y}{dt^{n-1}} = x_1 \ldots \tag{37}$$

$$\dot{x_n} = \dot{y} = \frac{dy}{dt} = x_{n-1} \tag{38}$$

$$y = x_n \tag{39}$$

$$\dot{x} = \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -a_1 x_1 - \cdots - a_n x_n \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} + \begin{pmatrix} u \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, y = x_n \tag{40}$$

$$\tag{41}$$

which simplifies to what seen above The state space representation is not unique and different choices of coordinates lead to a different state space representation.

1.7.3 Examples of systems

A **balance system** is a class of mechanical system in which the center of mass is balanced above a pivot point. It's an example of a type of system that can be modeled using ordinary differential equations. Balance systems are a generalization of the spring–mass system we saw earlier. We can write the dynamics for a mechanical system in the general form

$$M(q)\ddot{q} + C(q, \dot{q}) + K(q) = B(q)u \tag{42}$$

, where $M(q)$ is the inertia matrix for the system, $C(q, \dot{q})$ represents the Coriolis forces as well as the damping, $K(q)$ gives the forces due to potential energy and $B(q)$ describes how the external applied forces couple into the dynamics.

1.7.4 Discrete-time Systems

This system describes the evolution of a system at discrete instants of time rather than continuously in time. If we refer to each of these times by an integer $k = 0, 1, 2, ...$, then we can ask how the state of the system changes for each k. The evolution of a discrete-time system can be written in the form

$$x[k+1] = f(x[k], u[k]), y[k] = h(x[k], u[k]) \tag{43}$$

where $x[k] \in \mathbb{R}^n$ is the state of the system at time k (an integer), $u[k] \in \mathbb{R}^p$ is the input and $y[k] \in \mathbb{R}^q$ is the output. An example of this are the Lotka–Volterra equations, also known as the predator–prey equations.

1.7.5 Linear Systems

The model consists of two functions: the function f gives the rate of change of the state vector as a function of state x and control u, and the function h gives the measured values as functions of state x and control u. A system is called a linear state space system if the functions f and h are linear in x and u. A linear state space system can thus be represented by

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{44}$$

where A, B, C and D are constant matrices.

Such a system is said to be linear time-invariant, or LTI for short. The matrix A is called the **dynamics matrix**, the matrix B is called the **control matrix**, the matrix C is called the **sensor matrix** and the matrix D is called the **direct term**.

For which x is the nx1 state $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, u is the mx1 control $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$ and y is the px1 ouptut $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$.
Therefore A will be a matrix size nxn, B nxm, C pxn and D pxm. Most of the time $D = 0$ so matrix D is a zero matrix.

SISO system single input single output: $m = 1, n = 1$, $x = \tilde{v}, u = \tilde{u}$, which causes $p = 1$.

1.7.6 Nonlinear Systems

Like the predator-prey equations, a nonlinear system will be defined by the following equations

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x, u) \end{cases} \tag{45}$$

Dealing with differential equations, for $t \in \mathbb{R}$ and

$$f(x, u) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}, h(x, u) = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_p \end{bmatrix} \tag{46}$$

Specially for discrete-time control systems (more in section 1.7.4)

$$\dot{x}(t) = f(x(t), u(t)) \tag{47}$$
$$u(t) = u(kh) \tag{48}$$
$$\dot{x}(t) = f(x(t), u(kh)) \tag{49}$$
$$x((k+1)h) \tag{50}$$

for $kh \leq t < (k+1)h, k = 0, 1, ...$

1.8 Euler's Method

Given an input signal $u(t), t = 0, 1, 2, \ldots$ and the state $x(0)$ at time 0, the state at each discrete time $t \geq 1$ can be recursively computed:

$$x(1) = Ax(0) + Bu(0) \tag{51}$$
$$x(2) = Ax(1) + Bu(1) = A^2x(0) + ABu(0) + Bu(1) \tag{52}$$
$$x(t) = A^t x(0) + \sum_{i=0}^{t-1} A^{t-1-i} Bu(i) \tag{53}$$
$$y(t) = CA^t x(0) + \sum_{i=0}^{t-1} CA^{t-1-i} Bu(i) + Du(t) \tag{54}$$

We can use Euler's Method to reduce difficulty in this recursive calculation. Euler's method, also known as Euler's Discretization, Euler's Integration or Euler's approximation is a first-order numerical procedure for solving ordinary differential equations (ODEs) with a given initial value.

$$y_{n+1} = y_n + hf(t_n, y_n) \qquad \text{(more specifically)}$$
$$x((k+1), h) = x(kh) + hf(x(kh), u(kh)) \tag{55}$$

for timestep $k = 0, 1$ and the step size h.

This is equivalent to the Taylor expansion of the function y around t_0:

$$y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{1}{2}h^2 y''(t_0) + O\left(h^3\right) \tag{56}$$

for which $O(h^3)$ is all Taylor expansion terms of degree three or higher in h. The differential equation states that $y' = f(t, y)$. If this is substituted in the Taylor expansion and the quadratic and higher-order terms are ignored, the Euler method arises.

The solutions to simulation following Euler's method with different constants of h is visible in Figure 8. We can see that as h gets smaller, the computed solution converges to the exact solution. The form of the solution is also worth noticing: after an initial transient, the system settles into a periodic motion. The portion of the response after the transient is called the steady-state response to the input.

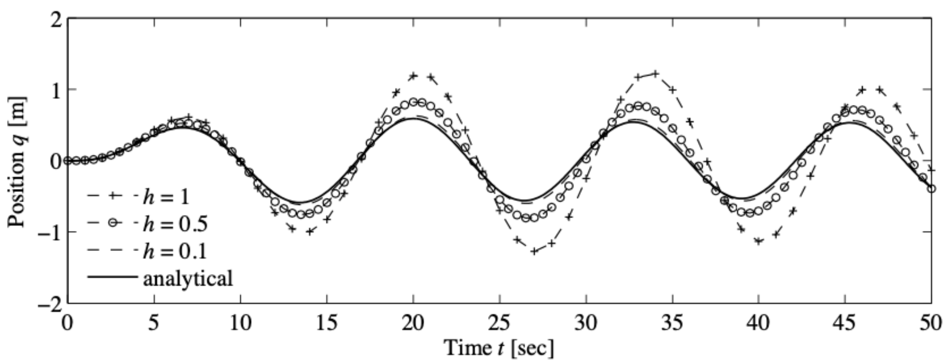

Figure 8: Simulation of the forced spring–mass system with different simulation time constants. The dashed line represents the analytical solution. The solid lines represent the approximate solution via the method of Euler integration, using decreasing step sizes.

In order to implement this method we follow these steps

1. Discretise the continuous time axis $\mathbb{T} = \mathbb{R}$ taking sampling times $\ldots, -h, 0, h, 2h, \ldots, k\Delta, \ldots, k \in \mathbb{Z}$ where $h > 0$ is the sampling interval

2. Aproximate the derivative $\dot{x}(t)$ at each sampling time as

$$\dot{x}(kh) = \lim_{h \to 0} \frac{x(kh + h) - x(kh)}{h} \approx \frac{x((k+1)h) - x(kh)}{h}$$

3. Obtain the discrete-time Euler approximation of the forced nonlinear differential equation

$$x((k+1)h) \approx x(kh) + h \cdot f(x(kh), u(kh))$$

4. To obtain the difference equation rename $\hat{x}(k) = x(kh), \hat{u}(k) = u(kh)$ so that you get $\hat{x}(k+1) = F(\hat{x}(k), \hat{u}(k))$

2 Lecture 2

2.1 Preliminary facts

1. A **particle** is a body whose dimensions may be neglected in describing its motion
2. The position of a particle in the space is defined by a vector $r \in \mathbb{R}^3$ given by its Cartesian coordinates $r = \begin{Bmatrix} x & y & z \end{Bmatrix}^T \in \mathbb{R}^3$. Hence to define the position of a system of N particles we need N vectors $r_1, r_2, \dots, r^N$.
3. The number n of independent quantities $q_1, q_2, \dots, q_n$ which must be specified in order to define uniquely the position of any particle is called the number of **degrees of freedom** and the quantities $q_1, q_2, \dots, q_n$ **generalized coordinates**
4. The generalized coordinates need not be the Cartesian coordinates of the particles and n need not be equal to N because coordinates might be constrained by relations of the form $f_j(r_1, r_2, \dots, r^N) = 0$, for which $j = 1, 2, \dots, m$ and $n = 3N - m$.
5. An $n \times n$ symmetric matrix M is positive semidefinite if and only if all its eigenvalues are non-negative and at least one is equal to zero.
6. An $n \times n$ symmetric matrix M is positive definite if and only if all its eigenvalues are positive.

Using the mass-spring damper example from section 2.7.3, the particle would be the mass m, the space is $\mathbb{R}$, the cartesian reference is the horizontal line pointing to the right with origin at the rest position of the spring, position is $r = q \in \mathbb{R}$ and generalised coordinate is q. Here $n = 1$ and there are no constraints.

2.2 Modelling Motion equations

Nature a system usually follows the path of the least energy usage (more in section 2.2.1), so there are general relationships that represent several of the different physical modeling domains. The variables used in this approach are the `generalized displacements` and `generalized momenta`.

	Effort	Flow	Generalized Displacement	Generalized Momentum
	e	f	q	p
Electric	voltage $u\,[V]$	current $i\,[A]$	charge $q\,[C]$	flux $\phi\,[Vs]$
Translation	force $F\,[N]$	velocity $v\,[m/s]$	displacement $x\,[m]$	momentum $p\,[Ns]$
Rotation	torque $M\,[Nm]$	angular velocity $\omega\,[rad/s]$	angular displacement $\theta\,[rad]$	rot. momentum b $[Nms]$
Hydraulic	pressure $p\,[N/m^2]$	vol. flow $Q\,[m^3/s]$	volume $V\,[m^3]$	momentum of flow tube $\Gamma\,[Ns/m^2]$
Thermodynamic	temp. $T\,[K]$	entropy flow $f_T\,[WK^{-1}]$	entropy $S\,[J/K]$	-

Figure 9: Domains and variables.

2.2.1 Hamilton's Principle

The principle of least action known as Hamilton's principle. In order to state Hamilton's principle we introduce the Lagrangian function L of a system, which is given by the difference between the total kinetic co-energy ($T*$) and the potential energy (V) of the system ($L = T* - V$). The motion of the system from time t1 to time t2 such that the line integral

$$I = \int_{t_1}^{t_2} L dt \tag{57}$$

has a stationary value for the correct path of the motion. The quantity I is referred to as the action or the action integral.

2.3 Euler-Lagrange formalism

Let us consider an electromechanical system with n degrees of freedom

The dynamical equations of such a system can be represented in terms of n so called generalized displacement coordinates, $q = (q_1, \ldots, q_n)$. These coordinates represent the actual position of the system under consideration, in the rotational domain they represent the angle, and in the electrical domain the charge (more in table 9).

Euler-Lagrange equation following from D'Alembert's principle, and reformulates it into a standard state-space form (more in section 1.7.2). From D'Alembert's principle from the force-balance of a system in equilibrium we can derive the following equations of motion

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{q}_i}(q, \dot{q})\right) - \frac{\partial T^*}{\partial \dot{q}_i}(q, \dot{q}) = F_i, i = 1, \ldots, n \tag{58}$$

Here $T*(q, \dot{q})$ denotes the total kinetic co-energy of the system while F_i are the rest of the forces acting on the system. Usually they can be divided in three parts: F_i^c the conservative forces, derived from potential energy solely depending on the generalised displacement coordinates and $F_i^d + F_i^e$ which are the dissipative and generalised external forces (like control). We denote the composition of forces as follows:

$$F_i = -\frac{\partial V}{\partial q_i}(q) + F_i^d + F_i^e, i = 1, \ldots, n \tag{59}$$

with $V(q)$ being the potential energy function.

Now we can define the Lagrangian function

$$L(q, \dot{q}) = T^*(\dot{q}) - E(q) \tag{60}$$

and the Euler-lagrangian equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \begin{cases} 0 \\ (Bu)_i + d_i \end{cases} \tag{61}$$

for which we choose 0 if the controls(inputs) are between i=m+1 and i= n and otherwise if it is m or smaller. A more specific algorithm and equations is visible in the summary. Overall, the Euler-lagrangian equations can be given by

$$m\ddot{q} + F^c = 0 \tag{62}$$

The advantage becomes more apparent in case of a nonlinear system. In the 1DOF pendulum, section 2.7.2, we have a external torque $F^c = u$ as input.

2.3.1 1st order Linear and nonlinear systems description

For linear systems we have an even number of state variables / number of storage elements.

The kinetic co-energy is independent of the generalised position q as seen in quadratic form where M is a $n \times n$ positive definite matrix. The potential energy has a positive semi-definite constant matrix and therefore the equations of motion become

$$\frac{d}{dt}\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}B \end{pmatrix} u \tag{63}$$

where B is the matrix multiplied by the external forces.

In nonlinear systems the kinetic co-energy and Coriolis and centrifugal forces rely on both $\dot{q}, q$ unlike linear form

$$T * (q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} \tag{64}$$

$$C_{li}(q, \dot{q}) = \frac{1}{2}\sum_{j=1}^{n}(\frac{\partial m_{il}}{\partial q_j}(q) + \frac{\partial m_{jl}}{\partial q_i}(q) + \frac{\partial m_{ij}}{\partial q_l}(q))\dot{q}_j \tag{65}$$

So the 2n-dimensional standard nonlinear state in which $x_1, x_2 = q, \dot{q}$ we can say that the initial motion equations leads us to

$$\frac{d}{dt}(x_1 x_2) = \left(x_2 - M^{-1}(x_1)(C(x_1, x_2)x_2 + k(s_1))\right) + \begin{pmatrix} 0 \\ M^{-1}(x_1)B \end{pmatrix} u \tag{66}$$

If this were a linear system, $M(x_1) = M, k(x_1))M, k(x_1) = Kx_1, C(x_1, x_2)x_2 = 0$ this last one is due to the form of kinetic energy independent from the generalised. The calculation of the lagrangian, M, k and B in the motion equations is present in the textbook.

2.3.2 Rayleigh Dissipation function

So far, we only have discussed conservative systems. Dissipation however, is often desirable to have present in your model, because its effects often can not be neglected if we do not like oscillatory behavior. Rayleigh dissipation function denoted by $\mathcal{D}$ is a function depending on the generalized velocity coordinates such that the associated dissipative forces F_i^d are given by

$$F_i^d = -\frac{\partial D}{\partial \dot{q}_i}(\dot{q}) \tag{67}$$

$$D(\dot{q}) = \frac{1}{2}c\dot{q}^2 \tag{68}$$

which we can replace in the equation 59

2.4 Electrical Networks

By using the same technique for both electrical and electrical systems we offer a unified framework for electromechanical systems. The selection of generalised coordinates are consistent with mechanical constraints. In electrical networks, the network topology defines these constraints while the displacement is the electric charge. Because the generalised coordinates have to be independent, charges cannot be associated to each individual dynamic element (inductor charge, capacitor charge) because they are constrained by Kirchoff laws. So we choose loops in the network.

Kirchoff's voltage law is applied to the set of independent network elements (n_R resistors , $_L$, inductors N_E voltage sources gives a system of k independent equations of the form

$$\begin{bmatrix}\psi_R & | & \psi_L & | & \psi_C\end{bmatrix}\begin{bmatrix}u_R\\u_L\\u_C\end{bmatrix} - \begin{bmatrix}\psi_R & | & \psi_L & | & \psi_C\end{bmatrix}u_E = 0 \tag{69}$$

$$\psi u - E = 0 \tag{70}$$

Where ψ is the network topology following from KVL separated into three sets, u are the vector with the component voltages and u_E is the vector of voltage sources.

We calculate the magnetic co-energy associated with the inductors

$$T*(i_L) = \int \phi_L^T di_L \tag{71}$$

Where $\psi_L = \psi_L(i_L)$ denotes the inductor flux-linkages vector. In the linear case $\psi_L = L(i_L), T*(i_L) = \frac{1}{2}I_L^T i_L$ what is d

The total electrical energy associated with the capacitors

$$V(q_C) = \int u_C^T dq_C \tag{72}$$

Which in the linear case is $u_C = C^{-1}q_C, V(q_C) = \frac{1}{2}q_C^T C^{-1} q_C$

For the Rayleigh dissipation

$$D(i_R) = \int u_R^T di_R \tag{73}$$

Where $u_R = Ri_R, D(i_R) = \frac{1}{2}i_R^T R i_R$.

We can therefore write the environment variables and the original equation 69

$$u_R = \frac{\partial D}{\partial i_R}(i_R) \tag{74}$$

$$u_L = \frac{d}{dt}(\frac{\partial T*}{\partial i_L}(i_L)) \tag{75}$$

$$u_C = \frac{\partial V}{\partial q_C}(q_C) \tag{76}$$

$$\psi_R\frac{\partial D}{\partial i_R}(i_R) + \psi_L\frac{d}{dt}(\frac{\partial T*}{\partial i_L}(i_L)) + \psi_C\frac{\partial V}{\partial q_C}(q_C) - E = 0 \tag{77}$$

We know the generalised coordinates are the loop charges defined by

$$q = \int_0^t i_{loop}(\tau)d\tau = q_{loop}(t) - q_{loop}(0) \tag{78}$$

Where i_{loop} is associated to the individual element currents i as $i = \begin{bmatrix}i_R\\i_L\\i_C\end{bmatrix} = \psi^T i_{loop}$

by substituting these in the i_R, i_L, i_C provided by the electrical version of the Euler-Lagrange (77) we obtain which with help of $\mathcal{L}(q,\dot{q}) = \bar{T}*(\dot{q}) - \bar{V}(q)$ the following formula

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = -\frac{\partial \bar{D}}{\partial \dot{q}}(\dot{q}) + E \tag{79}$$

RLC Networks with current sources asks to express the Lagrangian function also in terms of the magnetic energy and electric co-function. This means that the equations remain the same but $q_C \to \phi_L, i \leftrightarrow u, E \to J$. The final equation is referred to as the co-langragian equation since it forms precisely the dual from the previous case

$$\frac{d}{dt}\left(\frac{\partial L*}{\partial \dot{\phi}}(\phi,\dot{\phi})\right) - \frac{\partial L*}{\partial \phi}(\phi,\dot{\phi}) = -\frac{\partial \bar{D}*}{\partial \dot{\phi}}(\dot{\phi}) + J \tag{80}$$

2.5 Hamiltonian Systems

The Lagrangian of mechanical systems is given in terms of the generalized kinetic co-energy and the generalized potential energy results in the most natural physical states given by the generalized displacement and momenta coordinates. Let's introduce the representation of physical systems given by the Hamiltonian framework (no dissipative forces).

We start with generalized momenta

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}(q, \dot{q}) \tag{81}$$

and in general $n \times n$ matrix N

$$N_{ij} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \dot{q}_j} \tag{82}$$

will be non-singular, because all $p = (p_i, \ldots, p_n)$ are independent functions.

The Hamiltonian function $H(p, q)$ can be obtained with the Legendre transform of $\mathcal{L}(q, \dot{q})$

$$H(p, q) = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}(q, \dot{q}) = p_{\dot{q}}^T - \mathcal{L}(q, \dot{q}) \tag{83}$$

2.5.1 Linear and nonlinear systems

The generalised momentum of a linear system is $p = M\dot{q}$ and the hamiltonian is visible in equation 83. We substitute $\dot{q} = M^{-1}p$ and we can say that the kinetic energy now is $T(p) = \frac{1}{2}p^T M^{-1} p$ and the potential energy $V(q) = \frac{1}{2}q^T K q$. Therefore the first order dynamical equations are given by

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & M^{-1} \\ -K & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} u \tag{84}$$

this makes sense according to 2.3.1

check if correct

The Hamiltonian equations of motion relate to the Hamiltonian control system. They constitute a control system in standard state space form with most natural physical state variables (q,p), so $H(q, p)$ can be directly related to the energy of the system.

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(q, p) \tag{85}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}(q, p) + u_i, i = 1, \ldots, n \tag{86}$$

The internal energy of the system $H(q, p)$ results in the following control system

$$H(q, p) = \frac{1}{2}p^T M^{-1}(q) p + V(q) \tag{87}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(q, p) = M^{-1}(q) p \tag{88}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}(q, p) + u_i \tag{89}$$

$$= -\frac{1}{2}\frac{\partial}{\partial q_i}(p^T M^{-1}(q) p) - \frac{\partial V}{\partial q_i}(q) + u_i, i = 1, \ldots, n \tag{90}$$

A nice property about the time derivative of the Hamiltonian systems is that it shows the power-balance of the system, stating that the rate of power preserved inside the system ($\dot{H}$) is equal to the power supplied to the system from the outside ($\dot{q}^T u$).

2.5.2 Examples

1. **One link robot arm**. Given the generalised momenta coordinate is $p = ml^2\dot{q}$ and the hamiltonian is $H(q,p) = \frac{1}{2ml^2}p^2 - mgl\cos q + mgl$, the hamiltonian control system is

$$\dot{q} = \frac{1}{ml^2}p \tag{91}$$
$$\dot{p} = -mgl\sin q + u \tag{92}$$

2. **Two links rigid robot arm**. Given that there is no dissipation and we know $q_i = \theta_i, \ddot{q_i} = \dot{\theta}_i, i = 1,2$ then the generalised momentum is $q_i = \frac{\partial L}{\partial \dot{q_i}}(q,\dot{q}), p = M(q)\dot{q}$. In general $M(q)$ is positive definite and we know from the Example 1.5 in the textbook (Åström and Murray, 2021) that

$$\mathcal{L}(\theta,\dot{\theta}) = \frac{m_1 l_1^2\dot{\theta}_1^2}{2} + \frac{m_2}{2}(l_1^2\dot{\theta}_1^2 + l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 + 2l_1l_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\cos\theta_2) \tag{93}$$
$$+ m_1gl_1\cos\theta_1 + m_2gl_1\cos\theta_1 + m_2gl_2\cos\theta_1 + \theta_2 \tag{94}$$
$$\mathcal{L}(\theta,\dot{\theta}) = \frac{\dot{\theta}_1^2}{2} + \frac{1}{2}(\dot{\theta}_1^2 + (\dot{\theta}_1+\dot{\theta}_2)^2 + 2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\cos\theta_2) \tag{95}$$
$$+ g\cos\theta_1 + g\cos\theta_1 + g\cos\theta_1 + \theta_2 \tag{96}$$
$$\mathcal{L}(\theta,\dot{\theta}) = \frac{\dot{\theta}_1^2}{2} + \frac{1}{2}(\dot{\theta}_2^2 + 2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)(1+\cos\theta_2)) \tag{97}$$
$$+ 2g\cos\theta_1 + g\cos\theta_1 + \theta_2 \tag{98}$$
$$\frac{\partial L}{\partial \dot{\theta}_1} = 3\dot{\theta}_1 + \dot{\theta}_2 + (2\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2 \tag{99}$$
$$\frac{\partial L}{\partial \dot{\theta}_2} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_1\cos\theta_2 \tag{100}$$
$$q_1 = (3 + 2\cos(\theta_2)\dot{\theta}_1 + (1+\cos(\theta_2)\dot{\theta}_2 \tag{101}$$
$$q_2 = (1+\cos(\theta_2)\dot{\theta}_1 + \dot{\theta}_2 \tag{102}$$

and therefore the hamiltonian $H(q,p)$ is given by

$$H(q,p,u) = (1+\sin{\theta_2}^2)^{-1}(\frac{1}{2}p_1^2 - (1+\cos\theta_2)p_1p_2 + \frac{1}{2}(3+2\cos\theta_2)p_2^2) - 2g\cos\theta_1 - g\cos\theta_1 + \theta_2 \tag{103}$$

and the hamiltonian control system can be calculated from there (more in textbook (Åström and Murray, 2021, section 1.4)

2.6 Electromechanical systems

One of the main advantage of the Lagrangian or Hamiltonian formalism is that mechanical and electrical systems are treated analogously. An electromechanical transducer is used to convert electrical energy into mechanical (or acoustical) energy, and vice versa.

MEMS example combination of mechanical and electrical

Let's look into the optical switch visible in figure 10. It has 3 parts: actuator (electric), suspension beam (mechanical) and reflection mirror with optical fibre grooves (optical).

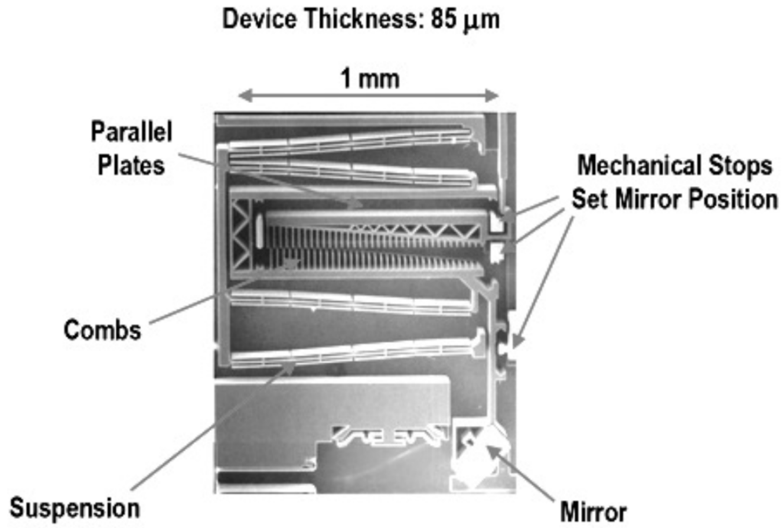

Figure 10: MEMS optical switch (courtesy of ioλon, Inc.).

Mechanical model: The mechanical part of the system is determined by the following three param- eters: (1) the effective mass m of the moving part; (2) the suspension stiffness k, which is generally a function of the mechanical displacement x such that $k(x) = k_1 x + k_2 x^3$ and (3) the damping coefficient b which will be assumed to be linear.

Taking as generalized coordinate and velocity pair the mechanical displacement $q_m = x$ and its time-derivative $\dot{q}_m = x$, it easy to see the mechanical kinetic co-energy, stored mechanical potential energy and Rayleigh dissipation function are

$$T_m * (\dot{q}_m) = \frac{m}{2}\dot{q}_m^2 \tag{104}$$

$$V_m(q_m) = \frac{k_1}{2}q_m^2 + \frac{k_2}{4}q_m^4 \tag{105}$$

$$D_m(\dot{q}_m) = \frac{b}{2}\dot{q}_m^2 \tag{106}$$

Electrical Model: The electrical part considers generation of electrostatic force by applying a voltage to terminals of the comb drive. This is done with a capacitor with a position dependent capacity

$$C(x) = \frac{\epsilon_0 A}{\alpha} = \frac{2\eta\epsilon_0\beta(x + x_0)}{\alpha}, (x > -x_0) \tag{107}$$

where ϵ_0 is the dielectric constant of vacuum, η is the number of movable comb fingers, α is the gap between the fingers, and β is the thickness of the structural layer. Knowing q_e is the electrical generalised coordinate, the electrical energy storage, parasitic Ohmnic loses, Lagrangian equation and Rayleigh dissipation are

$$V_e(q, q_e) = \frac{1}{2}C^{-1}(q_m)q_e^2 \tag{108}$$

$$D_e(\dot{q_e}) = \frac{1}{2}r\dot{q_e}^2 \tag{109}$$

$$\mathcal{L}(q_m, q_e, \dot{q_m}^2) = \frac{m}{2}\dot{q}_m^2 - (\frac{1}{2}k_1)q_m^2 + \frac{k_2}{4}q_m^4 + \frac{1}{2}C^{-1}(q_m)q_e^2 \tag{110}$$

$$D(\dot{q_m}, \dot{q_e}) = \frac{b}{2}\dot{q}_m^2 + \frac{1}{2}r\dot{q_e}^2 \tag{111}$$

We also know

$$q = \begin{bmatrix} q_m \\ q_e \end{bmatrix}, \dot{q} = \begin{bmatrix} \dot{q}_m \\ \dot{q}_e \end{bmatrix}, F^e = \begin{bmatrix} 0 \\ u_E \end{bmatrix} \tag{112}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \begin{bmatrix} m\dot{q_m} \\ 0 \end{bmatrix}, \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}}) = \begin{bmatrix} m\ddot{q_m} \\ 0 \end{bmatrix}, \frac{\partial D}{\partial \dot{q}} = \begin{bmatrix} b\dot{q_m} \\ r\dot{q_e} \end{bmatrix} \tag{113}$$

$$\frac{\partial \mathcal{L}}{\partial q} = \begin{bmatrix} -k_1 q_m - k_2 q_m^3 - \frac{1}{2}\frac{\partial C^{-1}(q_m)}{\partial q_m} q_e^2 \\ -\frac{q_e}{C(q_m)} \end{bmatrix} \tag{114}$$

where u_E is the actuator input voltage and the $\frac{1}{2}\frac{\partial C^{-1}(q_m)}{\partial q_m} q_e^2$ term represents the interaction force that links the mechanical and electrical subsystems

Hence applying Euler-lagrange equations gives the system

$$m\ddot{q_m} + k_1 q_m + k_2 q_m^3 + \frac{1}{2}\frac{\partial C^{-1}(q_m)}{\partial q_m} q_e^2 = -b\dot{q}_m \tag{115}$$

$$C^{-1}(q_m)q_e = -r\dot{q}_e + u_E \tag{116}$$

In this case, the generalised mass matrix M is singular, not invertible. We can still set $x_1 = q_m, x_2 = \dot{q_m}, x_3 = q_e$ and get three first-order differential equations.

Another way to obtain a set of first-order differential equations is using the Hamiltonian formalism. The way the Hamiltonian equations are defined here do not admit the inclusion of dissipation.

2.7 Examples

2.7.1 Double Pendulum example

We have two particles ($N = 2$) in $\mathbb{R}^2$ whose positions in the cartesian coordinates are given by $r_i = (x_i, y_i)$. Hence a total of 4 variables for two particles. Both particles are constrained to be a constant distance $d > 0, ||r_1 - r_2|| = d$. In this case there is a constraint $m = 1$ and $f_1(r_1, r_2) = ||r_1 - r_2|| - d$ according to what was said in section 2.1. The two particles are constrained to evolve on the surface $M = \{(r_1, r_2) \in \mathbb{R}^4 : ||r_1 - r_2|| = d\}$.

Each particle obeys Newton's equation of motion $m\ddot{r} = F + F^c$ where F^c are forces that constrain the two particles to be a constant distance d. These are unknown forces but we know that its direction vector F_1^{cT}, F_2^{cT} is orthogonal to the tangent space to M at each point. This phenomenon is visible in the figure 12 for the first mass.

We set $n = 3$ generalised coordinates that are the x, y position of the first particle and the angle of the second particle with respect to the x-axis. Position variables r_1, r_2 are related by a so-called immersion formula

not in textbook nor notes, wtf is this

$$r = \begin{bmatrix} \frac{r_1}{r_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + d \begin{bmatrix} \cos q_3 \\ \sin q_3 \end{bmatrix}} \end{bmatrix} = X(q) \tag{117}$$

For each $q \in \mathbb{R}^3$ the resulting vector $r = X(q)$ belongs to the surface M,

$$||r_1 - 2_2|| = ||d \begin{bmatrix} \cos q^3 \\ \sin q^3 \end{bmatrix}|| = d((\cos q_3)^2 + (\sin q_3)^2)^{\frac{1}{2}} = d \tag{118}$$

Now given $r = X(q)$ we can calculate $\dot{r} = X(\dot{q})$ and update the constraints

$$\frac{\partial X}{\partial q} = \begin{bmatrix} \frac{\partial X}{\partial q_1} & \frac{\partial X}{\partial q_2} & \frac{\partial X}{\partial q_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -d\sin q_3 \\ 0 & 1 & d\cos q_3 \end{bmatrix} \tag{119}$$

$$\begin{bmatrix} F_1^{cT} & F_2^{cT} \end{bmatrix} \begin{bmatrix} \frac{\partial X}{\partial q_1} & \frac{\partial X}{\partial q_2} & \frac{\partial X}{\partial q_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \tag{120}$$

$$F_1^{cT} = -F_2^{cT}, F_2^{cT} \perp \begin{bmatrix} -\sin q_3 \\ \cos q_3 \end{bmatrix} \tag{121}$$

Now let's get into calculating the Euler-Lagrange equation. We will consider newtons equations of motion for the two particles

$$\begin{bmatrix} m_1 \ddot{r_1} \\ m_1 \ddot{r_2} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \begin{bmatrix} F_1^c \\ F_2^c \end{bmatrix} \qquad \text{(multiply both sides by } \frac{\partial X}{\partial q_i}^T \text{ for } i = 1, 2, 3)$$

$$\frac{\partial X}{\partial q_i}^T \begin{bmatrix} m_1 \ddot{r_1} \\ m_1 \ddot{r_2} \end{bmatrix} = 0 \qquad \text{(it simplifies bc they're orthogonal)}$$

$$\frac{\partial X}{\partial q_i}^T \begin{bmatrix} m_1 \ddot{r_1} \\ m_1 \ddot{r_2} \end{bmatrix} = \frac{d}{dt}\frac{\partial T*}{\partial \dot{q}_i} - \frac{\partial T*}{\partial q_i} \tag{122}$$

where

$$T*(q, \dot{q}) = \frac{1}{2}(m_1 \dot{r_1}^T \dot{r_1} + m_2 \dot{r_2}^T \dot{r_2}) \dot{r} = \frac{\partial X}{\partial q}\dot{q} \tag{123}$$

Assume that the forces $F_i(r)$ are given by the gradient of a potential function $\hat{V}(r)$

$$F_i(r) = -\frac{\partial \hat{V}(r)}{\partial r_i} \tag{124}$$

. Then $\hat{V}(r)$ in the q-coordinates $V(q) = \hat{V}(r)|_{r=X(q)}$ satisfies (chain rule)

$$\frac{\partial V}{\partial q_i} = \frac{\partial \hat{V}}{\partial r}|_{r=X(q)}^T \frac{\partial X}{\partial q_i} = -\begin{bmatrix} F_1^T & F_2^T \end{bmatrix} \frac{\partial X}{\partial q_i} \tag{125}$$

We can conclude that we can define the Lagrangian function $L(q, \dot{q}) = T*(q, \dot{q}) - V(q)$ for $i = 1, 2, 3$ to derive the equations of motions of the 2 particles constrained to evolve at a constant distance and subject to conservative forces.

$$\frac{d}{dt}\frac{\partial T*}{\partial \dot{q}_i} - \frac{\partial T*}{\partial q_i} = 0 \tag{126}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{127}$$

2.7.2 Pendulum Example

make it follow algorithm

In order to model this we need the following: particle $p \in \mathbb{R}^3$ or $\mathbb{R}^2$ is a mass without physical dimension. We will use a generalised coordinate system $q \in \mathbb{R}^n$, for N particles, you have 3 dimensions, so we have a 3N matrix. The goal is to have $n < 3N$. There exists smooth constraints $f_j(r_i, \cdots, r_N) = 0, j = 1, 2, \cdots, m$ so that $n = 3N - m$. Here r are each of the coordinates in the coordinate vector.

An example is a 1 degree-of-freedom (actuated) pendulum, visible in figure 11

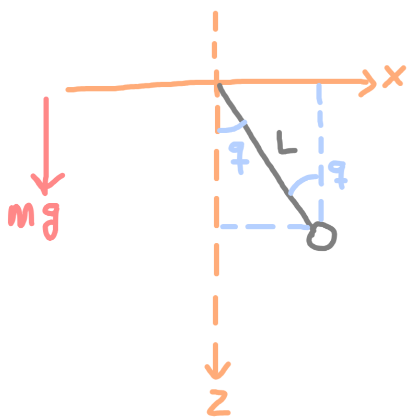

Figure 11: Gravity and angles in the pendulum

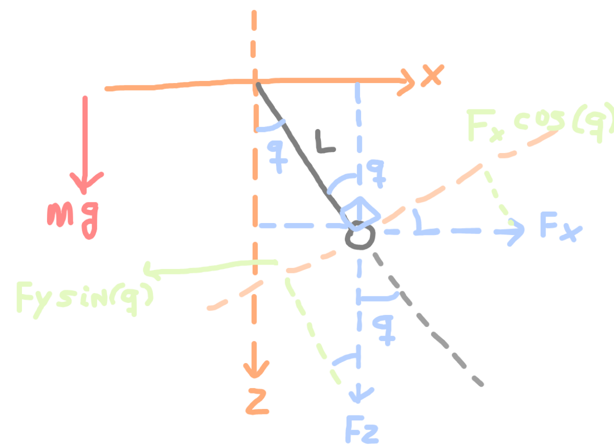

Figure 12: Additional forces

There is only one constraint: the position of the particle is limited by the radius of the circle the pendulum can make. Therefore its possible set of positions is limited to the coordinates in the set $\{r = \begin{bmatrix} z \\ x \end{bmatrix} | x^2 + z^2 = l^2\}$

An intuitive simplification of the coordinate system is the use of the angle $q = \theta$, so

$$r = \begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} l \cdot \cos q \\ l \cdot \sin q \end{bmatrix} = X(q) \tag{128}$$

where $q = \theta$. Consider that generally $\frac{d}{dt} r(t) = \frac{d}{dt} X(q(t)) = \frac{\partial X(q)}{\partial q}|_{q=q(t)} \dot{q}(t)$

Let's express the kinetic energy of this system

$$T(\dot{r}) = \frac{m}{2}(\dot{x}^2 + \dot{z}^2) \qquad \text{(where we use the speed on the axis)}$$

$$\dot{r} = \begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -l \cdot \sin q \cdot \dot{q} \\ l \cdot \cos q \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial q} \\ \frac{\partial x_2}{\partial q} \end{bmatrix} \frac{\partial X(q)}{\partial q} \cdot \dot{q} \tag{129}$$

$$= \frac{m}{2} \dot{r}^T \dot{r} = \frac{m}{2} \begin{bmatrix} \dot{z} & \dot{x} \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} \tag{130}$$

and the acceleration vector

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d}{dt} \begin{bmatrix} \frac{\partial x_1}{\partial q} \cdot \dot{q} \\ \frac{\partial x_2}{\partial q} \cdot \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 x_1}{\partial q^2} \\ \frac{\partial^2 x_2}{\partial q^2} \end{bmatrix} \dot{q}^2 + \begin{bmatrix} \frac{\partial x_1}{\partial q} \\ \frac{\partial x_2}{\partial q} \end{bmatrix} \ddot{q}^2$$

(the second order derivative is the angular acceleration)

$$\begin{bmatrix} \frac{\partial^2 x_1}{\partial q^2} \\ \frac{\partial^2 x_2}{\partial q^2} \end{bmatrix} = \begin{bmatrix} -l \cos q \\ -l \sin q \end{bmatrix}, \dot{r} = \begin{bmatrix} -l \cdot \sin q \cdot \dot{q} \\ l \cdot \cos q \dot{q} \end{bmatrix} \tag{131}$$

$$\ddot{r} = \begin{bmatrix} -l \cos q \\ -l \sin q \end{bmatrix} \dot{q}^2 + \begin{bmatrix} -l \cdot \sin q \cdot \dot{q} \\ l \cdot \cos q \dot{q} \end{bmatrix} \ddot{q}^2 \tag{132}$$

and equations of motion state $m \cdot a = F$

$$
\begin{aligned}
m \begin{bmatrix} \ddot{z} \\ \ddot{x} \end{bmatrix} &= \begin{bmatrix} mg \\ 0 \end{bmatrix} \left(\begin{bmatrix} F_z \\ F_x \end{bmatrix} \right) && (133) \\
\frac{\partial x^T}{\partial q} \cdot m\ddot{r} &= \begin{bmatrix} -l\sin q & l\cos q \end{bmatrix} \cdot m \left(\begin{bmatrix} -l\cos q \\ -l\sin q \end{bmatrix} \dot{q}^2 + \begin{bmatrix} -l \cdot \sin q \cdot \dot{q} \\ l \cdot \cos q \dot{q} \end{bmatrix} \ddot{q}^2 \right) && (134) \\
&= m((l^2 \sin q \cos q - l^2 \sin q \cos q)\dot{q}^2 + (l^2 \sin q^2 + l^2 \cos q^2)\ddot{q}) = ml^2\ddot{q} && (135)
\end{aligned}
$$

but we want to relate it to the kinetic energy, so let's rewrite $T(\dot{r})$ using our already calculated derivatives

$$
\begin{aligned}
T(\dot{r})|_{\dot{r}} &= \frac{\partial x}{\partial q}\dot{q} = \frac{m}{2}\frac{\partial x}{\partial q}^T \frac{\partial x}{\partial q}\dot{r} && (136) \\
&= \frac{m}{2}\dot{q}^2 \begin{bmatrix} -l\sin q & l\cos q \end{bmatrix} \begin{bmatrix} -l\sin q \\ l\cos q \end{bmatrix} = \frac{m}{2}\dot{q}^2 l^2 (\sin^2(q) + \cos^2(q)) = \frac{m}{2}\dot{q}^2 l^2 = T(\dot{q}) && (137)
\end{aligned}
$$

But where do Euler-Lagrange equations arise?

$$
\begin{aligned}
&\frac{\partial T}{\partial q} = ml^2\dot{q}, \frac{d}{dt}\frac{\partial T}{\partial q} = ml^2\ddot{q} && (138) \\
&\frac{\partial T}{\partial q} = 0, \frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = ml^2\ddot{q} = \frac{\partial X^T}{\partial q} m\ddot{r} && (139) \\
&F(r) = \begin{bmatrix} mg \\ 0 \end{bmatrix} = -\frac{\partial}{\partial r}\hat{v}(r) = -\begin{bmatrix} \frac{\partial \hat{v}(r)}{\partial z} \\ \frac{\partial \hat{v}(r)}{\partial x} \end{bmatrix} = -\begin{bmatrix} -mg \\ 0 \end{bmatrix} && \text{(this is the form of a conservative force)} \\
&\hat{v}(r) = -mgz, \hat{v}(r)|_{r=X(q)} = -mgl\cos q = V(q) && (140) \\
&\frac{\partial v}{\partial q} = \frac{\partial}{\partial q}\hat{v}(r)|_{r=X(q)} = \frac{\partial \hat{v}}{\partial r}^T|_{r=X(q)} \frac{\partial X(q)}{\partial q} = \begin{bmatrix} -mg & 0 \end{bmatrix} \begin{bmatrix} -l\sin q \\ l\cos q \end{bmatrix} = mgl\sin q && (141)
\end{aligned}
$$

v(r) is the potential energy and is equivalent to the effect of the displacement of the particle in the gravitational force, same in both coordinate systems

and we go back to the equations of motion, were we want to project over a surface

$$
\begin{aligned}
\frac{\partial x^T}{\partial q} \cdot m\ddot{r} = \frac{\partial x^T}{\partial q} \begin{bmatrix} mg \\ 0 \end{bmatrix} &= \begin{bmatrix} -l\sin q & l\cos q \end{bmatrix} \begin{bmatrix} mg \\ 0 \end{bmatrix} = -mgl\sin q, \frac{d}{dt}\frac{\partial T}{\partial q} - \frac{\partial T}{\partial q} = -\frac{\partial v}{\partial q} && (142) \\
L(q, \dot{q}) &= T(\dot{q}) - V(q) && \text{(this is the langrangian function)} \\
\frac{\partial L}{\partial \dot{q}} &= \frac{\partial T}{\partial \dot{q}} && (143) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} &= 0 && \text{(Euler-lagrange eq of motion)}
\end{aligned}
$$

This enforces naturally a constraint in a more numerical manner

$$
\frac{\partial x^T}{\partial q} \begin{bmatrix} F_z \\ F_x \end{bmatrix} = -lF_z \sin q + lF_x \cos q = l(F_x \cos q - F_z \sin q) = u \tag{144}
$$

projection of the force onto the tangent of the rod with length L. The force that makes the particle move are actually just these projections. These projections we call the torque and is not a force that can be expressed in terms of gradients, so it's non-conservative. We'll have to separately add it to the Lagrange equation. Friction is also unaccounted for, so we use the Rayleigh dissipation function $\mathcal{D}$ for all the velocities

define Rayleigh

$$L_{NC}(q,\dot q) = L(q,\dot q) + uq + \int_0^t \mathcal{D}(\dot q(s))ds \tag{145}$$

$$\dot q^T \frac{\partial}{\partial q} \geq 0, \forall \dot q, \mathcal{D}(\dot q) = c\dot q^2 \tag{146}$$

$$\frac{d}{dt}\frac{\partial L_{NC}}{\partial \dot q} - \frac{\partial L_{NC}}{\partial q} = 0 \tag{147}$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot q}\left(L(q,\dot q) + uq + \frac{1}{2}\int_0^t c(\dot q(s))^2 ds\right) - \frac{\partial}{\partial \dot q}(L(q,\dot q) + uq + \frac{1}{2}\int_0^t c(\dot q(s))^2 ds) = 0 \tag{148}$$

$$= \frac{d}{dt}\frac{\partial L(\dot q)}{\partial \dot q} + c\dot q - \frac{\partial L}{\partial q} - u = 0 \tag{149}$$

$$L(q,q) = \frac{m}{2} l^2 \dot q^2 - (-mgl\cos q) \tag{150}$$

$$\frac{d}{dt}\frac{\partial L(\dot q)}{\partial \dot q} = \frac{d}{dt} ml^2 \dot q = ml^2 \ddot q, \frac{\partial L}{\partial q} = -mgl\sin q \tag{151}$$

$$ml^2\ddot q + c\dot q - -mgl\sin q - u = 0 \Rightarrow ml^2\ddot q + c\dot q - -mgl\sin q = u \tag{152}$$

which in state space form we can rewrite as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{mgl}{ml^2}\sin x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix} u = f(x,u) \tag{153}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = h(x) \tag{154}$$

since we know $x_1 = q, x_2 = \dot q, u = F, y = q$.

2.7.3 Damped Spring-mass Example

Some definitions:
Asymptotically stable: if the initial state of the system is away from the rest position, the system will return to the rest position eventually. Proof in textbook (Åström and Murray, 2021) page 42 using Lyapunov stability analysis

We will use the modelling equations 31, and the equations mentioned in the previous sections. We know the Lagrangian equation is $L(q,\dot q) = \frac{1}{2}m\dot q^2 - \frac{1}{2}kq^2$, and the derivatives are $\frac{\partial L}{\partial \dot q} = m\ddot q$, $\frac{\partial L}{\partial q} = -kq$. These are the conservative forces, derived from q.

Once we account for external non-conservative forces, we get

$$L_{NC}(q,\dot q) = L(q,\dot q) + (Bu+d)^T q + \int_0^t \mathcal{D}(\dot q(s))ds = \frac{1}{2}m\dot q^2 - \frac{1}{2}kq^2 + F_q + \int_0^t \frac{c}{2}\dot q^2 ds \tag{155}$$

where $Bu + d$ is an n-dimensional vector obtained by the projection of the nonconservative forces onto the tangent space, $\mathcal{D}(\dot q(s))$ is the Rayleigh dissipation function, which in most of the cases is linear friction or constant resistances ($\frac{c}{2}\dot q^2$).

Formulas are:

$$(Bu+d)_i = F_{NC}^T \frac{\partial X}{\partial q_i} \tag{156}$$

$$\dot q^T \frac{\partial \mathcal{D}}{\partial \dot q} = \sum_{i=1}^n \dot q_i, \frac{\partial \mathcal{D}}{\partial \dot q} \geq 0 \tag{157}$$

$$\tag{158}$$

And the partials become $\frac{d}{dt}\frac{\partial L_{NC}}{\partial \dot{q}} = m\ddot{q} + c\dot{q}$, $\frac{\partial L}{\partial q} = -kq + F$. Hence $m\ddot{q} + c\dot{q} + kq = F$ according to the Euler-lagrange equations.

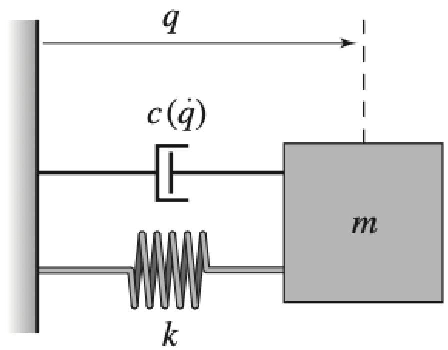

Figure 13: Spring–mass system with nonlinear damping. The position of the mass is denoted by q, with $q = 0$ corresponding to the rest position of the spring. The forces on the mass are generated by a linear spring with spring constant k and a damper with force dependent on the velocity $\dot{q}$.

We define output as $y(t) = q(t)$, input $u(t) = F(t)$ and states $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}$. Then the state space representation (more in section 1.6) can be defined with the following system of equations

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\frac{k}{m}x_1(t) - \frac{c}{m}x_2(t) + \frac{1}{m}u(t) \end{bmatrix} = f(x(t), u(t)) \tag{159}$$

$$y(t) = x_1(t) = h(x(t)) \tag{160}$$

this is obviously a linear system (more about linear systems in section 1.7.5)

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t) = Ax(t) + Bu(t) \tag{161}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) = Cu(t) \tag{162}$$

2.7.4 Keynes' economy model Example

An example of discrete-time system (Textbook, Exercise 2.4, p. 62) y(t): gross national product (GNP), year t [output] c(t): total consumption, year t [state] i(t): total investments, year t [state] g(t): expenses of government, year t [input] y(t) = c(t) + i(t) + g(t)

Assumptions:

1. c(t+1)=ay(t),a>0 (consumption increases with GNP with 1-year delay)
2. i(t+1)=b(c(t+1)-c(t)),b>0 (investment proportional to rate of change of consumption)

Then with variables

$$c(t+1) = ac(t) + ai(t) + ag(t) \tag{163}$$

$$i(t+1) = b(ac(t) + ai(t) + ag(t) - c(t))(ba - b)c(t) + bai(t) + bag(t) \tag{164}$$

$$y(t) = c(t) + i(t) + g(t), t = 0, 1, 2, ... \tag{165}$$

$$x := \begin{bmatrix} c \\ i \end{bmatrix}, u \qquad := g \tag{166}$$

$$\tag{167}$$

we obtain the linear state space model of the system

$$x(t+1) = \begin{bmatrix} a & a \\ ba-b & ba \end{bmatrix} x(t) + \begin{bmatrix} a \\ ba \end{bmatrix} u(t) = Ax(t) + Bu(t) \tag{168}$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) + u(t) = Cx(t) + Du(t) \tag{169}$$

2.7.5 RLC Circuit Example

A particle is the electron, the position in space si $q \in \mathbb{R}$ is the electric charge transported by the current $i = \dot{q}$ in the circuit in a period of time).

1. Magnetic energy of the inductor
$$T*(\dot{q}) = \frac{1}{2}L\dot{q}^2 \tag{170}$$
2. Electrical energy of the capacitor
$$E(q) = \frac{1}{2C}q^2 \tag{171}$$
3. Rayleigh function associated with the resistor
$$\dot{\mathcal{D}}(\dot{q}) = \frac{1}{2}R\dot{q}^2 \tag{172}$$
4. Energy required to move charge q through potential V: qV
5. Lagrangian non-conservative
$$L_{NC}(q,\dot{q}) = \frac{1}{2}L\dot{q}^2 - \frac{1}{2C}q^2 + qV + \int_0^t \frac{1}{2}R\dot{q}^2 ds \tag{173}$$
6. Euler-Lagrangian equation
$$0 = \frac{d}{dt}\left(L\dot{q}^2 - \int_0^t \frac{1}{2}R\dot{q}^2 ds\right) - \left(-\frac{1}{C}q + V\right) \Rightarrow L\ddot{q} + R\dot{q} + \frac{1}{C}q = V \tag{174}$$

3 Lectures 3 and 4

3.1 Linearization of nonlinear systems

Nonlinear systems are important because nonlinear behaviour is dairly common in physical systems. Linear models are often only valid in neighborhood of the operating point, based on linearization. For outside this neighbourhood more advanced tools are necessary.

Same thing we have already seen in the previous week look where was this

$$\dot{x} = f(x,u), x \in \mathbb{R}^n, u \in \mathbb{R}^m y = h(x,u), y \in \mathbb{R}^p \tag{175}$$

Let's get started with the 1DOF frictionless robot manipulator

$$\dot{x} = \begin{pmatrix} x_2 \\ \frac{-mgl}{ml^2}\sin(x_1) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix} u = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin x_1 + \frac{1}{ml^2}u \end{pmatrix} y = \begin{pmatrix} 1 & 0 \end{pmatrix} x \tag{176}$$

We define the state variables x as $x_1 = q$, the angular velocity will be $x_2 = \dot{q}$ and $\dot{x_1} = x_2$ and the angular acceleration will be $\dot{x}_2 = \ddot{q} = -\frac{mgl}{ml^2}\sin x_1 + \frac{1}{ml^2}u$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 + \frac{1}{ml^2}u \end{bmatrix} \equiv f(x,u) \tag{177}$$

while $y = x = h(x,u)$.

So a linear system is $\dot{x} = x + u = Ax + B$ where A=1,B=1 and solution is $x(t) = l^t x(0) + \int_0^t e^{t-s}u(s)ds$.

3.1.1 Nonlinearity phenomena

Finite escape time: Let's try to integrate a nonlinear system $\dot{x} = x^2 + u$ with no input $u(t) = 0, \forall t \geq 0$ and initial condition $x(0) = 1$

$$\dot{x} = x^2, \frac{dx}{x^2} = dt \tag{178}$$

$$-\frac{1}{x} = t + k, -\frac{1}{t+k} = x \tag{179}$$

$$-\frac{1}{x(0)} = 0 + k, k = -\frac{1}{1} = -1 \tag{180}$$

$$x(t) = \frac{1}{1-t} \tag{181}$$

so the system leads to infinity ($x(t) \to +\infty$ as $t \to 1^-$), this is called finite escape time, and only happens in nonlinear systems because solutions always exist for linear systems. We don't like this behavior and is hard to handle.

Non-unique solution: Let's try to integrate a system $\dot{x} = 2\sqrt{x} + u$ with no input $u(t) = 0, \forall t \geq 0$ and initial condition $x(0) = 0$. There are two possible solutions $x(t) = 0$ or $x(t) = t^2$, in which $x(t)$ can either remain at 0 constantly or exponentially increase. This is a phenomenon occurring in nonlinear systems.

$$x(t) = t^2, \dot{x} = 2t, 2\sqrt{x(t)} = 2\sqrt{t^2} = 2t \tag{182}$$

$$x(t) = 0, \dot{x} = 0, 2\sqrt{x(t)} = 2\sqrt{0} = 0 \tag{183}$$

To prevent this to happen, one should ask the vector field $f(x)$ to be locally Lipschitz: for each $\bar{x} \in \mathbb{R}^n$ there exists a neighborhood $I_{\bar{x}}$ of x and a constant $L > 0$ such that $\forall x, y \in I_{\bar{x}}$.

$$||f(x) - f(y)|| < L||x - y|| \tag{184}$$

where $||.||$ denotes the Euclidean norm of a vector

Other nonlinear phenomena include multiple isolated equilibria, limit cycles, bifurcations and chaos, though overall we'll want to focus in linear systems in the scope of this course. To make nonlinear systems linear, we learn about linearization both in section 3.1.3 even though we already got a preview in section 1.4.

3.1.2 Equilibrium

We say the equilibrium point is $\dot{x} = f(x,u)$ for an x that is a nx1 matrix and u is a mx1 matrix. The pair of vectors n-dimensional and m-dimensional $(\bar{x}, \bar{u})$ is an equilibrium for the environment if $f(\bar{x}, \bar{u}) = 0_{nx1}$.

For example, for a

$$f(x,u) = \begin{bmatrix} \bar{x_2} \\ -\frac{g}{l}\sin \bar{x_1} + \frac{1}{ml^2}\bar{u} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{185}$$

$$\begin{cases} \bar{x_2} = 0 \\ \bar{u} = -gml\sin \bar{x_1} \end{cases} \tag{186}$$

let's do it for a satellite problem. We have mass $M = 1$ in an inverse square force field $|F_g| = \frac{k}{r^2}$ for $r = \sqrt{x^2 + y^2}$ where k is a constant of proportionality indicating the strength of the force field. Here the radius is not a constraint, but we can calculate angular acceleration as $\dot{\theta} = -\frac{2\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2$ and $\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1$. The generalised coordinates are $q = \begin{bmatrix} r \\ \theta \end{bmatrix}$ and therefore $X(q) = r\begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$ the claim is that if $\bar{u_1} = 0$ and $\bar{u_2} = 0$, then we would have constant radius and angle $\bar{r}(t) = \bar{r}, \bar{\theta}(t) = \bar{\theta}$ This is a solution we're interested on so we will want to substitute and check if it works

$$r\ddot{(t)} = 0 \tag{187}$$
$$0 = \bar{r}\bar{\theta}^2 - \frac{k}{\bar{r}^2} \tag{188}$$
$$\bar{r}\bar{\theta}^2 = 0 \tag{189}$$
$$\frac{k}{\bar{r}^2} = 0 \tag{190}$$
$$\bar{r}^3\bar{\theta}^2 = k \qquad \text{(if this holds then it is a valid solution)}$$

3.1.3 Linearization

How to linearize:

- Determine a solution (like a point)
- Approximate functions around this solution via truncated Taylor's expansion
- Convert the system to local coordinates around this solution
- Obtain an approximate linear model

Considering any system $\dot{x} = f(x, u)$, a solution to the system from the initial condition $x_0 \in \mathbb{R}^n$ is any function of time $\bar{x}(t)$ that solves the Cauchy problem , so

make a section about the cauchy problem

$$\dot{\bar{x}} = f(\bar{x}(t), \bar{u}(t)), \bar{x}(0) = x_0, \forall t \geq 0 \tag{191}$$

where the pair is defined. This is oftentimes called the operating solution. The interpretation is that if the system is initially at the $\bar{x}(0)$ and an input $\bar{u}(t) : \mathbb{R} \to \mathbb{R}^m$ is applied then the system will evolve as $\bar{x}(t)$ for all time t.

Example 1: Satellite problem
Point mass in an inverse square law force field, for which the motion equations are the following, with u_1 radial thrust and u_2 tangential thrust

$$\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1 \tag{192}$$
$$\ddot{\theta} = -\frac{2\dot{\theta}\dot{r}}{r} + \frac{u_2}{r} \tag{193}$$

a solution is $\bar{u_1}(t) = 0, \bar{u_2}(t) = 0, \bar{r}(t) = const, \bar{\theta}(t) = \bar{\theta}(t)$ which when applied to the equations of motion leads to

$$0 = \bar{r}\bar{\theta}(t)^2 - \frac{k}{\bar{r}^2} \tag{194}$$
$$0 = -\frac{2\bar{\theta} \cdot 0}{\bar{r}} + \frac{1}{\bar{r}} \cdot 0 \tag{195}$$
$$\bar{r}^3\bar{\theta}^2 = k \tag{196}$$

This is a case in which the constant state-input pair $(\bar{x}, \bar{u})$ is a solution of the system. The interpretation is that if the system is initially at the $\bar{x}(0)$ and an input $\bar{u}(t) : \mathbb{R} \to \mathbb{R}^m$ is applied then the system will remain in $\bar{x}(t)$ for all time t.

Example 2: Inverted pendulum

Determining an operating point $\bar{x}, \bar{u}$ for the general nonlinear system $\dot{x} = f(x, u), y = h(x, u)$ can also be done as follows

$$f(x,u) = f(\bar{x},\bar{u}) + \frac{\partial f}{\partial x}|_{x=\bar{x},u=\bar{u}}(x-\bar{x})\frac{\partial f}{\partial u}|_{x=\bar{x},u=\bar{u}}(u-\bar{u}) + R_2(x-\bar{x},u-\bar{u})$$

(bc its an equilibrium the first term disappears)

$$f = \begin{bmatrix} f_1(x,u) \\ f_n(x,u) \end{bmatrix} = \begin{bmatrix} f_1(x-1,x_2,u_1,u_2) \\ f_2(x-1,x_2,u_1,u_2) \end{bmatrix} \quad \text{(if n=2)}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}, \frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix} \quad \text{(the first is nxn and second nxm)}$$

$$R_2(x-\bar{x},u-\bar{u}) = \begin{bmatrix} R_21(x-\bar{x},u-\bar{u}) \\ R_22(x-\bar{x},u-\bar{u}) \end{bmatrix}, R_{2i} = \begin{bmatrix} x-\bar{x} \\ u-\bar{u} \end{bmatrix}^T \begin{bmatrix} \frac{\partial^2 f_i}{\partial x^2} & \frac{\partial^2 f_i}{\partial u \partial x} \\ \frac{\partial^2 f_i}{\partial u \partial x} & \frac{\partial^2 f_i}{\partial u^2} \end{bmatrix} \begin{bmatrix} x-\bar{x} \\ u-\bar{u} \end{bmatrix} \cdot \frac{1}{2} \tag{197}$$

$$\begin{cases} \dot{\Delta x} = A\Delta x + B\Delta u \\ \Delta y = C\Delta x + D\Delta u \end{cases} \tag{198}$$

We want to reduce 2nd order differential equations like the one in the satellite problem to first order system of equations. Let $(\hat{x}, \hat{u})$ be a point on the segment connecting (x, u) with $(\bar{x}, \bar{u})$.

We define the new variable (small perturbation around the equilibrium)

$$\Delta x = x - \bar{x}, \Delta u = u - \bar{u} \tag{199}$$

$$\dot{x} = \dot{\bar{x}} + \Delta\dot{x} = f(\hat{x},\hat{u}) + \frac{\partial f}{\partial x}|_{x=\bar{x}} + \frac{\partial f}{\partial u}|_{x=\bar{x}} + R_2(x-\bar{x},u-\bar{u}) \quad \text{(same to h)}$$

$$\Delta x, \Delta y \approx 0, \Delta\dot{x} \approx \frac{\partial f}{\partial x}|_{x=\bar{x}} + \frac{\partial f}{\partial u}|_{x=\bar{x}} \tag{200}$$

$$A = \frac{\partial f}{\partial x}|_{\hat{x},\hat{u}}, B = \frac{\partial f}{\partial u}|_{\hat{x},\hat{u}}, C = \frac{\partial h}{\partial x}|_{\hat{x},\hat{u}}, D = \frac{\partial h}{\partial u}|_{\hat{x},\hat{u}} \tag{201}$$

As far as $\Delta x, \Delta u$ are small, the function $\bar{x} + \Delta x$ approximates the solution x to the original nonlinear system when the input $u = \bar{u} + \Delta u$ is applied.

Example Inverted pendulum let's calculate the equilibrium

$$\dot{x} = \begin{bmatrix} x_2 \\ \sin x_1 - cx_2 + u\cos x_1 \end{bmatrix}, c > 0, \begin{cases} \bar{x_2} = 0 \\ \sin\bar{x_1} = 0, \bar{x_1} = k\pi \end{cases} \quad \text{(compute A and B)}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ \cos x_1 - u\sin x_1 & -c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix}$$

(simplification is possible because they are at an equilibrium point)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 \\ \cos x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{(variation of the actual angular velocity from the equilibrium)}$$

$$\begin{cases} \dot{\Delta x} = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases} \tag{202}$$

rate in which the angular displacement changes over time.

Now let's say i want to know $f(x, u)$

$$R_2 1(x, u) = 0 \tag{203}$$

$$\begin{bmatrix} \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \cos x_1 - u \sin x_1 \\ -c \end{bmatrix} \tag{204}$$

$$\begin{bmatrix} \frac{\partial f_2}{\partial x_1^2} & \frac{\partial f_2}{\partial x_1 x_2} & \frac{\partial f_2}{\partial u \partial x_1} \\ \frac{\partial f_2}{\partial x_1 x_2} & \frac{\partial f_2}{\partial x_2^2} & \frac{\partial f_2}{\partial u \partial x_2} \end{bmatrix} = \begin{bmatrix} -u \sin x_1 - \cos x_1 & 0 & -\sin x_1 \\ 0 & 0 & 0 \\ -\sin x_1 & 0 & 0 \end{bmatrix} \tag{205}$$

3.2 Stability

$\bar{x}$ is a **stable solution** if $\forall \epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if the initial condition x_0 differs from solution $\bar{x}$ more than δ_ϵ ($|x_0 - \bar{x}| < \delta_\epsilon$), then $|x(t, x_0) - \bar{x}| < \epsilon, \forall t \geq 0$.

An equilibrium which is not stable it is said to be **unstable**.

$\bar{x}$ is an **asymptotically stable equilibrium (AS)** if it's stable and there exists $\delta > 0$ such that it $|x_0 - \bar{x}| < \delta_\epsilon$ then $\lim_{t \to t_0} |x(t, x_0) - \bar{x}| = 0$. notes says $t \to +\infty$

In the slides we refer to $\bar{x}$ as $x(t, a)$ and the **globally asymptotically stable (GAS)** solution happens if it's stable and for all $x_0 \in \mathbb{R}^n$, $\lim_{t \to t_0} |x(t, x_0) - \bar{x}| = 0$.

Some remarks: Stability notions also apply to equilibrium points (special case). Planar systems

- Asymptotically stable (AS) equilibrium is a sink
- Unstable equilibrium is a source (all trajectories move away from equilibrium)
- Unstable equilibrium is a saddle (some trajectories move away, some converge to equilibrium)
- Equilibrium point that is stable but not AS is a center

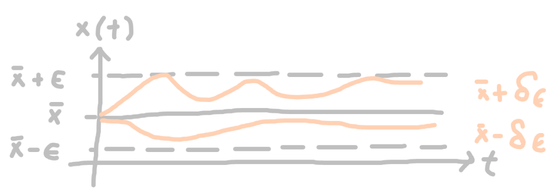

Figure 14: Hand-drawn

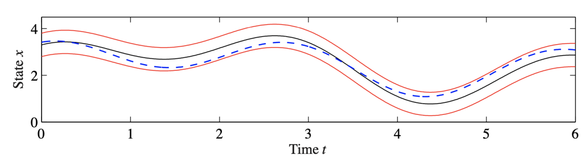

Figure 15: Slides

Figure 16: Plot showing the distance equilibrium point $\bar{x}$ and its variance

3.2.1 Stability in linear systems

For linear systems, there is input $u = 0$ and the stability of $\bar{x} = 0$ (always an equilibrium) depends on the eigenvalues of A, $\lambda(A) = \{s \in \mathbb{C} : \det(sl - A) = 0\}$. For $\bar{x} = 0, A\bar{x} = 0, \dot{x} = Ax$

1. A diagonal (n=2)

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{cases} \dot{x_1} = \lambda_1 x_1 \\ \dot{x_2} = \lambda_2 x_2 \end{cases} \tag{206}$$

$$\begin{cases} x_1(t) = e^{\lambda_1 t} x_1(0) \\ x_2(t) = e^{\lambda_2 t} x_2(0) \end{cases} \tag{207}$$

$$\det(SI_2 - A) = \det \begin{bmatrix} S - \lambda_1 & 0 \\ 0 & S - \lambda_2 \end{bmatrix} = (S - \lambda_1)(S - \lambda_2) = 0 \tag{208}$$

So the eigenvalues of A are λ_1, λ_2.

- $\lambda_1 < 0, \lambda_2 < 0$ $\bar{x} = 0$ is a globally asymptotically stable equilibrium.
- $\lambda_1 > 0, \lambda_2 < 0$ $x_1(0) \neq 0, x_1(t) \to \infty$ as $t \to \infty$. Because we can find at least one solution in which the system is unstable, we can argue that it's unstable
- $\lambda_1 \leq 0, \lambda_2 \leq 0 \begin{cases} x_1(t) = x_1(0) \\ x_2(t) = e^{\lambda_2 t} x_2(0) \end{cases}$ then is asymptotically stable

Both real and positive, then globally asymptotically stable, At least one real eigenvalue positive, then unstable, All of them smaller or equal to 0, asymptotically stable.

2. A block diagonal n=3

$$A = \begin{bmatrix} \lambda_1 & \theta_{2\times 1} \\ \theta_{1\times 2} & \lambda_2 \end{bmatrix}, \lambda_1 = \lambda \in \mathbb{R}, \lambda_2 = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}, \sigma, \omega \in \mathbb{R} \tag{209}$$

$$\det(SI_2 - A) = \det \begin{bmatrix} s - \lambda_1 & 0 & 0 \\ 0 & s - \theta & -\omega \\ 0 & \omega & s - \delta \end{bmatrix} = (s - \lambda_1)[(s - \sigma)^2 + \omega^2] = 0 \tag{210}$$

$$s_1 = \lambda_1, s_{2,3} = \sigma \pm \sqrt{\sigma^2 - \sigma^2 - \omega^2} = \sigma \pm i\omega, s^2 = 2\theta s + \sigma^2 + \omega^2 = 0 \tag{211}$$

$$\begin{cases} x_1(t) = \lambda_1 x_1(t) \\ \begin{bmatrix} x_2^\circ(t) \\ x_3^\circ(t) \end{bmatrix} = \lambda_2 \begin{bmatrix} x_2(t) \\ x_3(t) \end{bmatrix} \end{cases} \Rightarrow \begin{cases} x_1(t) = e^{\lambda_1 t} x_1(0) \\ \begin{bmatrix} x_2(t) \\ x_3(t) \end{bmatrix} = e^{\sigma t} \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} x_2(0) \\ x_3(0) \end{bmatrix} \end{cases} \tag{212}$$

The solutions to the system are

- is globally asymptotically stable GAS if λ_1 under 0 and $\mathrm{Re}[\lambda_2] = \mathrm{Re}[\lambda_3] < 0$
- is an unstable equilibrium if it exists a j for which $Re[\lambda_j] > 0$
- is an stable equilibrium if $\mathrm{Re}(\lambda_j) \leq 0, \forall j$

3.2.2 Diagonalisation

A is diagonalisable if there exists an nxn matrix T that is non singular such that $T^{-1}AT = \tilde{A}$ is a diagonal matrix, where λ_1 is either $\lambda_1 \in \mathbb{R}$ or $\lambda_1 = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}$. The stability properties of the system $z^\circ = \tilde{A}z$, tell me the stability properties of the system $x^\circ = Ax$ as seen below.

$$z^\circ = T^{-1}x^\circ = T^{-1}Ax = T^{-1}ATz = \tilde{A}z \tag{213}$$

The main result is that if the matrix A has all the eigenvalues with strictly negative real part then $\bar{x} = 0$ is a globally asymptotically stable equilibrium, if there exists an eigenvalue where real part is positive then $\bar{x} = 0$ is unstable.

Remark: if A si block diagonalisable and all the eigenvalues are such that their real parts are nonpositive (with at least one =0) then $\bar{x} = 0$ is a stable equilibrium.

Example: Double integrator

$$\begin{bmatrix} x_1^\circ \\ x_2^\circ \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{214}$$

for any $x_2 \neq 0$ it will go to infty. i can't reduce it to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3.2.3 Block-diagonalisation

v are vectors and each of them are vectors. eigenvalues are distinct. by computing eigenvectors and putting them on T it works if the eigenvalues are real we're fine, but what otherwise?

Let's call eigenvalues λ_i and its eigenvector v_i for $i = 1, 2$ in A. For eigenvectors $v_i \neq 0, Av_i = \lambda_i v_i$. We know eigenvectors are of shape nx1, so a vector of eigenvectors is of shape nxn. Because we know the determinant is not 0, it's nonsingular. Now let's expand on the $Av_i = \lambda_i v_i$ claim we said

$$Av_i = \lambda_i v_i \tag{215}$$

$$A[v_1, v_2, \ldots, v_n] = [v_1, v_2, \ldots, v_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \tag{216}$$

$$AT = T\tilde{A} \tag{217}$$

$$T^{-1}AT = \tilde{A} \tag{218}$$

A has distinct eigenvalues then we can compute T such that $\tilde{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix} = T^{-1}AT$.

Now let's say $n = 2$ and $\lambda_{1,2} = \sigma \pm i\omega$, and then $\tilde{A} = \begin{bmatrix} \sigma \pm i\omega & 0 \\ 0 & \sigma \pm i\omega \end{bmatrix} = T^{-1}AT$.

Let's also add nonsingular matrix $S = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$S^{-1}\tilde{A}S = S^{-1}T^{-1}ATS = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sigma \pm i\omega & 0 \\ 0 & \sigma \pm i\omega \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \tag{219}$$

$$S^{-1} = 1 \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{220}$$

Example for a linear system (more in handout)

$$\dot{x} = Ax = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x \tag{221}$$

$$\tag{222}$$

has eigenvalues $\lambda(A)$ and eigenvectors v_1, v_2 that make transformation matrix T and diagonal matrix $\tilde{A}$

$$\lambda(A) = \{s \in \mathbb{C} : s^2 + s + 1 = 0\} = \{-\frac{1}{2} + i\frac{\sqrt{3}}{2},\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}\} \tag{223}$$

$$v_1 = \begin{pmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix} \tag{224}$$

$$T = \begin{pmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 & 1 \end{pmatrix}, T^{-1} = \begin{pmatrix} \frac{i}{\sqrt{3}} & \frac{i+\sqrt{3}}{2\sqrt{3}} \\ -\frac{i}{\sqrt{3}} & \frac{-i+\sqrt{3}}{2\sqrt{3}} \end{pmatrix} \tag{225}$$

$$\tilde{A} = T^{-1}AT = \begin{pmatrix} \frac{-i-\sqrt{3}}{2\sqrt{3}} & 0 \\ 0 & \frac{-i+\sqrt{3}}{2\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \sigma + i\omega & 0 \\ 0 & \sigma - i\omega \end{pmatrix} \tag{226}$$

Now what if we were to make new coordinates $z = T^{-1}z$, then the system $\dot{x} = Ax$ becomes $\dot{z} = \tilde{A}z$ where $\tilde{A} = T^{-1}AT$. We have seen that when $z = 0$ is asymptotically stable, then $\lambda_i < 0$ and $\sigma_i < 0$. Because we know the following relationship

$$||T|| = sup_{z\neq 0}\frac{||T_z||}{||z||} \tag{227}$$

$$||x(t)|| = ||Tz(t) \leq ||T||||z(t)|| \tag{228}$$

we can say that $x = 0$ is also asymptotically stable and

$$\lim_{t\to+\infty} ||z(t)|| = 0, \lim_{t\to+\infty} ||x(t)|| = 0 \tag{229}$$

3.2.4 Routh-Hurwitz criteria (RHC)

Imagine that the determinant of a matrix A is a polynomial of the type $p(s) = s_n + a_1 s^{n-1} + a_2 s^{n-1} + \cdots + a_{n-1}s + a_n$. The root will not give you much info, you want to further analysis. By RHC if and only if the roots of $p(s) = 0$ have all strictly negative real parts there are no sign changes in the first column.

n even (similarly for *n* odd)

n	1	a_2	a_4	...	a_{n-2}	a_n
n-1	a_1	a_3	a_5	...	a_{n-1}	
n-2	b_1	b_2	b_3	...		
⋮						
	h_1	h_2	h_3	...		
	k_1	k_2	k_3	...		
	ℓ_1	ℓ_2	ℓ_3	...		
⋮						
2						
1						
0						

Figure 17: Odd root number n

n odd

n	1	a_2	a_4	...	a_{n-1}
n-1	a_1	a_3	a_5	...	a_n
n-2					
⋮					
2					
1					
0					

Figure 18: Even root number n

Figure 19: Tables for the calculation of the RHC

The first column is filled in with the equation

$$l_i = -\frac{1}{k_1}\begin{bmatrix} [|cc|]h_1 & h_{i+1} \\ k_1 & k_{i+1} \end{bmatrix}, i = 1, 2, \ldots \tag{230}$$

3.2.5 Stability of NL systems

$\ddot{x} = Ax, \bar{x} = 0$, stability depends on the eigenvalues of A $\ddot{x} = f(x), f(\bar{x}) = 0, \Delta\dot{x} = A\Delta x, \Delta x = x - \bar{x}, A = \frac{\partial f}{\partial x}|_{x=\bar{x}}$ If A has all eigenvalues with strictly negative real part then bar x is an AS equilibrium of $\ddot{x} = f(x)$. If there exists an eigenvalue of A with strictly positive dear part then x is unstable for $\ddot{x} = f(x)$

Example inverted pendulum

$$\dot{x} = f(x, u) = \begin{pmatrix} x_2 \\ \sin x_1 - cx_2 + u\cos x_1 \end{pmatrix}, c > 0 \tag{231}$$

we set $u = 0, f(x) = \begin{pmatrix} x_2 \\ \sin x_1 - cx_2 \end{pmatrix}$, then for an equilibrium $\bar{x} = 0$ we calculate the Jacobian

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ \cos x_1 & -c \end{bmatrix} \tag{232}$$

$$A = \frac{\partial f}{\partial x}|_{x=\bar{x}} = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix} \tag{233}$$

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 \\ -1 & \lambda + c \end{bmatrix} \tag{234}$$

we solve $\lambda^2 + c\lambda - 1 = 0$ so we get $\lambda_{1,2} = \frac{-c \pm \sqrt{c^2+4}}{2}$

3.3 Qualitative Analysis of Nonlinear Systems

3.3.1 Vector field

For the system

$$\dot{x} = f(x), x \in \mathbb{R}^n, \tag{235}$$

for each $x \in \mathbb{R}^n$, $f(x)$ is a vector representing the velocity of the system at that point.

3.3.2 Phase Portrait

A phase portrait is a graphical representation of qualitative behavior of planar dynamical system

$$\dot{x} = f(x), x \in \mathbb{R}^2 \tag{236}$$

For any $x \in \mathbb{R}^2$ draw an arrow with the tail at x, length is given by $||f(x)||$ and direction of the arrow provided y $f(x)$. Example $\dot{x} = x_2$ superposition $-\sin(x_1)$ angular velocity $\hat{x} = \begin{bmatrix} \pi/2 \\ -1 \end{bmatrix}, f(\hat{x}) = \begin{bmatrix} -1 \\ -\sin(\pi/2) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

Similarly can be done with other points such that we fill in the space to obtain a result like the image visible in 20

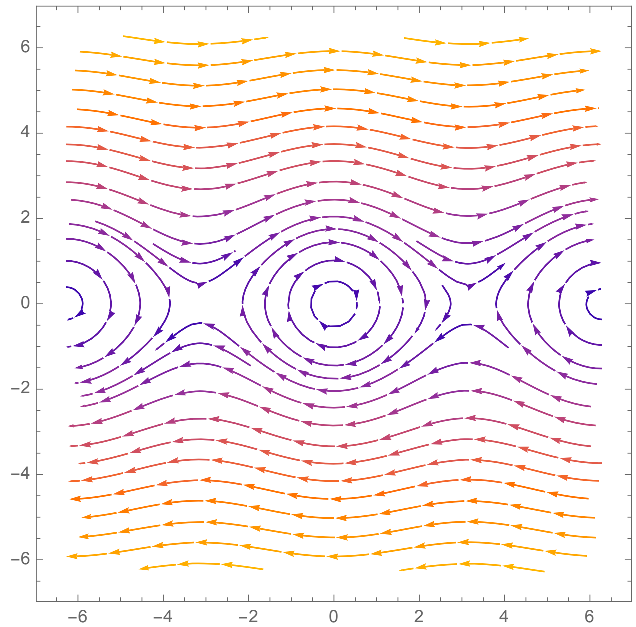

Figure 20: Phase portrait of $f(\hat{x})$

4 Lecture 5 and 6

4.1 Linearity

The linearity of a system

$$\dot{x} = Ax + Bu \tag{237}$$
$$y = Cx + Du \tag{238}$$

is $y(t; x_0, u)$, be the output response $y(t)$ corresponding to the initial condition x_0 and input u. The input/output response has linear properties with respect to the initial conditions and inputs

- $y(t; \alpha x_1 + \beta x_2, 0) = \alpha y(t; x_1, 0) + \beta y(t; x_2, 0)$
- $y(t; \alpha x_0, \delta_u) = \alpha y(t; x_0, 0) + \delta y(t; 0, u)$
- superposition principle $y(t; \alpha x_1 + \beta x_2, 0) = \alpha y(t; x_1, 0) + \beta y(t; x_2, 0)$

Example: RLC circuit

This is an example on the diagonalisation of a matrix A and the computation of the corresponding matrix exponential and convolution integral. This example is left as a reading assignment.

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{239}$$
$$y(t) = \begin{bmatrix} 0 & \frac{1}{L} \end{bmatrix} x(t) \tag{240}$$

With initial condition $x(0) = 0, u(t) = 1, L = 1, R = 2, C = \frac{4}{3}$

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 \\ \frac{3}{4} & \lambda + 2 \end{bmatrix}\right) = \lambda^2 + 2\lambda + \frac{3}{4} = 0 \tag{241}$$

$$\lambda_{1,2} = -1 \pm \frac{\sqrt{4-3}}{2} = -1 \pm \frac{1}{2} \tag{242}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{3}{4} & -2 \end{bmatrix} = T^{-1}\lambda T, T = \frac{1}{\lambda_2 - \lambda_1}\begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}, T^{-1} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \tag{243}$$

$$\lambda_1 = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 1 \\ -3/2 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \tag{244}$$

$$e^{At} = T^{-1}e^{\lambda t}T = \frac{1}{\lambda_2 - \lambda_1}\begin{pmatrix} (\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}) & (e^{\lambda_2 t} - e^{\lambda_1 t}) \\ \lambda_1\lambda_2(e^{\lambda_1 t} - e^{\lambda_2 t}) & (\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}) \end{pmatrix} \tag{245}$$

$$y(t) = \int_0^t Ce^{A(t-\tau)}Bu(t)d\tau = \int_0^t \begin{pmatrix} 0 & 1 \end{pmatrix} e^{A(t-\tau)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot d\tau = \int_0^t -(\lambda_2 e^{\lambda_2(t-\tau)} - \lambda_1 e^{\lambda_1(t-\tau)})d\tau \tag{246}$$

$$= -(-e^{\lambda_2(t-\tau)} + e^{\lambda_1(t-\tau)})|_{\tau=0} = -(e^{\lambda_1 t} - e^{\lambda_2 t}) = -e^{-\frac{1}{2}t} + e^{\frac{3}{2}t} \tag{247}$$

4.2 Matrix Exponential

To compute an elicit expression of the response to a linear state space equation 237 we can use the autonomous state equation $\frac{dx(t)}{dt} = Ax(t)$ with initial condition $x(0) = x_0$, but then we need to introduce the matrix exponential. By the fundamental theorem of calculus

$$\int_0^t \frac{dx(\tau_1)}{d\tau_1}d\tau_1 = \int_0^t Ax(\tau_1)d\tau_1 \Rightarrow x(t) = x_0 + \int_0^t Ax(\tau_1)d\tau_1 \tag{248}$$

$$x(t) = x_0 + \int_0^t Ax(\tau_1)d\tau_1 = x_0 + \int_0^t A[x_0 + \int_0^t Ax(\tau_2)d\tau_2]d\tau_1 \tag{249}$$

$$= x_0 + \int_0^t Ax_0\tau_1 + \int_0^t \int_0^{\tau_1} A^2 x(\tau_2)d\tau_2 d\tau_1 \tag{250}$$

$$= x_0 + Atx_0 + \int_0^t \int_0^{\tau_1} A^2 x(\tau_2)d\tau_2 d\tau_1 \tag{251}$$

$$= e^{At}x_0 \tag{252}$$

Example 1: Double integrator This is the state space form of $x = (q\dot{q})^T$ and it has a homogeneous solution $u = 0$

$$\ddot{q} = u, y = q \tag{253}$$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{254}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{255}$$

$$e^{At} = \begin{pmatrix} 1 & t \\ 1 & 0 \end{pmatrix} = x_h(t) \tag{256}$$

$$x_0 = \begin{pmatrix} x_{01} + x_{02}t \\ x_{02} \end{pmatrix}, y_h(t) = q(t) = x_1(t) = x_{01} + x_{02}t \tag{257}$$

Example 2: Spring-mass system

$$\ddot{q} + \omega_0^2 q = u, y = q \tag{258}$$

$$\dot{x} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{259}$$

$$x_h(t) = \begin{bmatrix} \cos(\omega_0 t) & \sin \omega_0 t \\ -\sin \omega_0 t & \cos(\omega_0 t) \end{bmatrix} x_0, y(t) = \frac{1}{\omega_0} x_{h1}(t) \tag{260}$$

$$e^{At} = \begin{pmatrix} \cos(\omega_0 t) & \sin \omega_0 t \\ -\sin \omega_0 t & \cos(\omega_0 t) \end{pmatrix} \tag{261}$$

4.2.1 Solutions to linear systems

The solutions of a linear system converges to a matrix exponential as seen in section 4.2

Examples

In robots we use a lot $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and the answer should be given by $e^{At} = I_2 + At + \frac{A^2 t^2}{2!} + \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} + 0$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \tag{262}$$

but this is a little too simple since the matrix is omnipotent, so let's do one which is a little more difficult based off of the spring mass system. We make a choice of variables for which $x_2 = \dot{q}$

This is just a Taylor expansion around the point x=0 to the sin and cos, which allows the simplification.

Properties of e^{At} include

$$\frac{d}{dt} e^{At} = Ae^{At} \tag{263}$$

$$e^{AT+Bt} = e^{At} e^{Bt} \tag{264}$$

$$e^{At}|_{t=0} = I_n \tag{265}$$

. Laplace transform is not necessary but can be used instead of the folllowing: .

If A is not block-diagonalizable then Jordan form.

example

4.3 IO Response

If there is a control input $u \neq 0$, then the solution to 237 can be rewritten with the convolution integral

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} Bu(s) ds \tag{266}$$

$$y(t) = ce^{At} x_0 + c \int_0^t e^{A(t-s)} Bu(s) ds + Du(t) \tag{267}$$

$$\tag{268}$$

We can prove that the differentiation of 266 is equal to $Ax(t) + Bu(t)$.

In a linear state space system, we consider a Single Input Single Output linear state space system 237, with $x \in \mathbb{R}^n, u, y \in \mathbb{R}, A, B, C^T$. The solution can be written as

$$y(t) = Cx(t) + Du(t), t \geq t_0 = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \tag{269}$$

We set a coordinate invariance $z = TX$ where T is a nonsingular matrix with det $\neq 0$ and we make a coordinate change

$$\dot{z} = TAx + TBu = TAT^{-1}z + TBu \tag{270}$$

$$y = CT^{-1}z + Du \tag{271}$$

$$y(t) = CT^{-1}Te^{At}T^{-1}Tx_0 + \int_0^t CT^{-1}Te^{A(t-s)}T^{-1}TBu(s)ds + Du(t) \tag{272}$$

$$= \tilde{C}e^{\tilde{A}t}z_0 + \int_0^t \tilde{C}e^{\tilde{A}(t-s)}\tilde{B}u(s)ds + Du(t) \tag{273}$$

Since $e^{\tilde{A}t} = Te^{At}T^{-1}$, we have $y_z(t) = y_x(t)$ for all $t \geq 0$, that is the IO response of a linear system is invariant with respect to the choice of the coordinates.

4.3.1 Impulse Response

A **pulse** is a signal of duration ϵ and amplitude $\frac{1}{\epsilon}$

$$u(t) = P_\epsilon(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{\epsilon} & 0 \leq t \leq \epsilon \\ 0 & t > \frac{1}{\epsilon} \end{cases} \tag{274}$$

where the impulse is the limit of the pulse as $\epsilon \to 0$

$$\delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t) \tag{275}$$

. This function has some properties,

$$\delta(t) = 0, t \neq 0 \tag{276}$$

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1 \tag{277}$$

$$f(t)\delta(t-\tau) = f(\tau)\delta(t-\tau) \qquad \text{(for any continuous function } f(t) \text{ in } t = \tau)$$

$$\int_{-\infty}^{+\infty} f(t)\delta(t-\tau)dt = f(\tau)\int_{-\infty}^{+\infty} \delta(t-\tau)dt = f(\tau) \tag{278}$$

The impulse response $h(t)$ is the output response of a system that is initially at rest $x_0 = 0$ (don't take into account the initial condition), to an input that equals the impulse $\delta(t)$. In general the output and impulse response follow the structure visible below, given that $h(t) = y(t), x(t_0) = 0, u(t) = \delta(t)$

$$y(t) = Cx(t) + Du(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t), t \geq t_0 \tag{279}$$

$$h(t) = Ce^{At}B + D\delta(t) = \int_{t_0}^{t} Ce^{A(t-\tau)}B\delta(\tau)d\tau = Ce^{At}B \tag{280}$$

For simplicity and without loss of generality we can take $t_0 = 0, D = 0$

$$h_\epsilon(t) = \int_0^t ce^{A(t-s)} BP_\epsilon(s)ds = \int_0^\epsilon ce^{At}e^{-As}B\frac{1}{\epsilon}ds = ce^{At}\int_0^\epsilon e^{-As}B\frac{1}{\epsilon}ds \tag{281}$$

$$= ce^{At}(I - A\frac{\epsilon}{2} + A^2\frac{\epsilon^2}{6} - \dots)B \tag{282}$$

4.3.2 Step response

For a function f,

$$x(t) = e^{At}x_0 + \int_0^t e^{t-s}Bu(s)ds \tag{283}$$

We have already seen a pulse response $p_\epsilon(t)$

$$\lim_{\epsilon\to 0} p_\epsilon(t) = Ce^{At}B + D\delta(t) \tag{284}$$

The step response $r(t)$ is the output response of a system which is initially $x_0 = 0$ to an input that equals a unit step function

$$1(t) = \begin{cases} 0, t < 0 \\ 1, t \geq 0 \end{cases} \tag{285}$$

Let's assume A is nonsingular, which we can expect given that is also asymptotically stable. We know its asymptotically stable because the first section will go to 0 while the steady state response will remain. In this case we also count that $D \neq 0$.

$$\begin{aligned}
&\int_0^t e^{A(t-s)}Bds && (\theta = t - s)\\
&= \int_0^t e^{A\theta}Bd\theta && \text{(A is nonsingular)}\\
&= \int_0^t A^{-1}Ae^{A\theta}Bd\theta && (Ae^{A\theta} = \tfrac{d}{d\theta}e^{A\theta})\\
&= A^{-1}e^{A\theta}|_{\theta=0}^{\theta=t}B = A^{-1}(e^{At} - I)B = A^{-1}e^{At}B - A^{-1}B && (286)\\
r(t) &= CA^{-1}e^{At}B - CA^{-1}B + D\mathbb{I}(t) && \text{(transient - steady state response)}\\
r(t) &= -CA^{-1}B + D && (\text{if } t \to \infty)
\end{aligned}$$

4.3.3 Frequency or Harmonic response

The harmonic response has a sinusoidal input $u(t) = \cos\omega t$. To compute the harmonic response we can express this value differently using Euler's formula

$$\frac{e^{i\omega t} + e^{-i\omega t}}{2} = \frac{\cos\omega t + i\sin\omega t + \cos\omega t - i\sin\omega t}{2} = \cos\omega t \tag{287}$$

We use the basic eq to rewrite with $u(t) = \cos \omega t$ and now the previous s is now τ and $s = i\omega$

$$x(t) = \int_0^t Ce^{A(t-\tau)}Be^{s\tau}d\tau = Ce^{At}\int_0^t e^{-A\tau}Be^{s\tau}d\tau \tag{288}$$

$$= Ce^{At}\int_0^t e^{(SI-A)\tau}Bd\tau = Ce^{At}\int_0^t (SI-A)^{-1}(SI-A)e^{(SI-A)\tau}Bd\tau \tag{289}$$

$$= Ce^{At}(SI-A)^{-1}\int_0^t \frac{d}{dt}e^{(SI-A)\tau}Bd\tau = Ce^{At}(SI-A)^{-1}[e^{(SI-A)\tau} - I]B \tag{290}$$

$$y(t) = C(SI-A)^{-1}e^{At}B + C(SI-A)^{-1}Be^{st} = W(s)u(t) = W(s)e^{st} \tag{291}$$

$$W(s) = C(SI-A)^{-1}B = |W(s)|e^{i\theta(s)} \tag{292}$$

The response will be an attenuation or increase of the magnitude according to the transfer function $W(s)$

$$y(t) = \frac{|W(is)|e^{i\theta(\omega)}}{2}e^{i\omega t} + \frac{|W(-is)|e^{-i\theta(\omega)}}{2}e^{-i\omega t} = |W(i\omega)| \cdot \cos(\omega t + \theta(\omega)) \tag{293}$$

This can be rewritten (check slides) to a sum of two matrices, first half for the transient part and second half for the steady-state,

$$y(t) = Ce^{At}(x(0) - (sI-A)^{-1}B) + (C(sI-A)^{-1}B + D)e^{st} \tag{294}$$

where we can say that if A is asymptotically stable, then the steady state response to $u(t) = e^{st}$ is $y_{st} = W(s)e^{st}$ with $W(s) = C(sI-A)^{-1}B + D, s \in \mathbb{C}$

In summary, for the output response of a linear system 237, to a sinusoidal input $u(t) = \cos(\omega t)$ where ω is the frequency, is a sinusoidal signal $y(t)$ with magnitude $M(\omega)$ and phase θ visible in the equations below and graph 21.

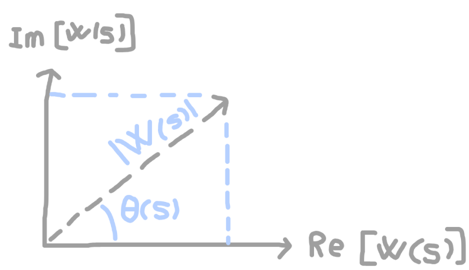

Figure 21: Imaginary and real parts of the magnitude and phase of a harmonic response

$$y(t) = M(\omega)\cos(\omega t + \theta(\omega)) \tag{295}$$

$$M(\omega) = \sqrt{\mathrm{Re}(W(i\omega))^2 + \mathrm{Im}(W(i\omega))^2} \tag{296}$$

$$\theta(\omega) = \arctan\frac{\mathrm{Im}(W(i\omega))}{\mathrm{Re}(W(i\omega))} \tag{297}$$

$$W(s) = C(sI-A)^{-1}B + D \tag{298}$$

We can get this attenuation also from the computation of the phase $\theta(\omega)$ and magnitude $M(\omega)$ in an RLC circuit. Example $L = 1, C = 1, R = 1$ RLC circuit has

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{299}$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \tag{300}$$

$$y_{st} = M(\omega)\cos(\omega t + \theta(\omega)) \tag{301}$$

$$W(i\omega) = M(\omega)e^{i\theta(\omega)} \tag{302}$$

$$W(s) = C(sI - A)^{-1}B + D = \frac{s}{s^2 + s + 1} \tag{303}$$

For $\omega = \frac{1}{\sqrt{2}}, \theta(\omega) = \arctan\frac{1}{\sqrt{2}}, M(\omega) = \sqrt{\frac{2}{3}}$ hence

$$u(t) = \cos(\frac{1}{\sqrt{2}}t) \tag{304}$$

$$y(t) = \sqrt{\frac{2}{3}}\cos(\frac{1}{\sqrt{2}}t + \arctan\frac{1}{\sqrt{2}}) \tag{305}$$

Now let's observe this plot 22 in which we can see the difference in time and amplitude of the input and output signals.

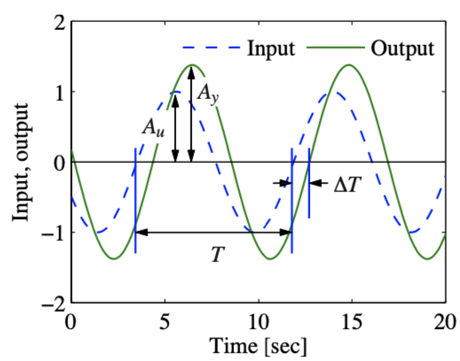

Figure 22: Input and output signals

Here we can estimate gain and phase graphically, since $M(\omega) = \frac{A_y}{A_u}$ and $\theta(\omega) = \phi_y - \phi_u = -\frac{2\pi}{T}\Delta T$.

Some properties of the frequency response are:

- **Zero frequency or DC gain** is defined only if A is non-singular and states that $W(i\omega)|_{\omega=0} = C(-A)^{-1}B + D$.
- **Bandwidth** ω_b is the frequency for which, when $M(0) \neq 0$ (A is non-singular),

$$\frac{M(\omega)}{M(0)} \geq \frac{1}{\sqrt{2}}, \forall \omega \in [0, \omega_b] \tag{306}$$

  in which $[0, \omega_b]$ is the range of frequencies ω over which the gain has decreased by not more than $\frac{1}{\sqrt{2}}$.
- **Resonant peak** M_r is the largest value of the frequency/harmonic response

$$M_r = \max_{\omega \geq 0} M(\omega) \tag{307}$$

- **Peak frequency** ω_{mr} is the frequency at which the resonant peak is attained $\omega_{mr} = \arg\max_{\omega \geq 0} M(\omega)$, so $M(\omega_{mr}) = M_r$

4.4 Reachability and Controllability

A system (*) is reachable if for any pair of states x_0, x_f there exists $T > 0$ and $u|_{[0,T]}$ such that $x(T) = x_f$

1. Let's say you have an unitary maze affected by action of force u(t), given a solution position, figure out the velocity with respect to a T you also calculate.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{308}$$

$$x(T) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds \tag{309}$$

$$= e^{At}x_0 + W(0,T) \tag{attempt 1}$$

$$= x_f \tag{310}$$

Since $e^{A(t-s)}B$ is a nx1 matrix we want u(s) to be 1xn that makes the product invertible. We chose $u(s) = B^{Tr}(e^{A(t-s)})^{Tr}$ where tr is transpose. However, we also want to scale it to $u(s) = B^{Tr}(e^{A(t-s)})^{Tr} \cdot W(0,T)^{-1} \cdot (-e^{AT}x_0 + x_f)$ to simplify. This will lead us to $x(t) = x_f$ while keeping the product invertible.

Example: is the double integrator reachable? assume $\theta = T - t$

$$W(0,T) = \int_0^T e^{A(\theta)}BB^T(e^{A(\theta)})^T d\theta = \int_0^T \begin{pmatrix} \theta \\ 1 \end{pmatrix} \begin{pmatrix} \theta & 1 \end{pmatrix} d\theta = \int_0^T \begin{bmatrix} \theta^2 & \theta \\ \theta & 1 \end{bmatrix} d\theta = \begin{bmatrix} \frac{\theta^3}{3} & \frac{\theta^2}{2} \\ \frac{\theta^2}{2} & \theta \end{bmatrix}_{\theta=0}^{\theta=T} \tag{311}$$

$$W_r = [BAB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{312}$$

The double integrator is a reachable system. $W(0,T)$ nonsingular is independent of T, hence the reachability property is independent of T (if the system is reachable for some T, then it is reachable for any T).

An alternative definition is the Reachability Gramian condition, by which a linear system like equations 237 is reachable if and only if

$$W(0,T) = \int_0^T e^{A\theta}BB^T e^{A^T\theta} d\theta \tag{313}$$

is invertible.

However, a linear system is reachable if and only if the reachability matrix is invertible.

$$W_r = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{314}$$

, which also works to prove the reachability of the double integrator. There is proof for this relation in slides 6. If the number of inputs $m > 1$, then we have a $n \times mm$ matrix and then the linear system is only reachable if and only if the rank of W_r is n.

4.4.1 Reachable canonical form

Let's take advantage of the concept of coordinate invariance $z = Tx$ introduced in lecture 5, equations 270. The reachable canonical form is a special form of the reachable linear system equations 237 that is useful for feedback design. Based on the characteristical polynomial, we have the canonical form $\dot{z}$

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \tag{315}$$

$$\dot{z} = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u \tag{316}$$

so the new coordinate system is $\dot{z} = T(Ax + Bu) = TAT^{-1}z + TBu = \tilde{A}z + \tilde{B}u$ and computability matrix is $T = \tilde{W}_r W_r^{-1}$. More examples are available in the slides

5 Lecture 7

5.1 Output regulation problem

In the output regulation problem we consider the addition of a controller, which outputs $u = -k_x + k_r r$. In the system visible in 23, $u, y, r \in \mathbb{R}$ and $r(t) = r \forall t$

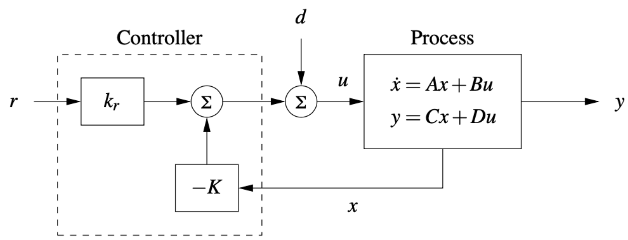

Figure 23: Output regulation problem with a controller

Example 1:

$$\begin{cases} \dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y = \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{cases} \qquad \text{(in a closed loop system)}$$

$$\begin{cases} \dot{x} = Ax + B(-K_x + K_r r) = (A - BK)x + BK_r r \\ y = Cx \end{cases} \tag{317}$$

We know A,B,C but we don't know k, which is a 1xn vector of gains nor k_F, which is a scalar. We can regard this as the step response of a system driven by step input always =r, and we can therefore know what will be the solution to this problem. Under the condition that $A - BK$ is asymptotically stable, then the step response as $t \to \infty$

$$y_{steadystate} = -C(A - BK)^{-1} BK_r r \tag{318}$$

Design K_r such that $y_{steadystate} = r$, so $K_r = \frac{-1}{C(A-BK)^{-1}B}$.

5.1.1 Eigenvalue assignment problem

Give (A,B) find K such that A-BK has all its eigenvalues at the desired locations. Remark: If all the desired eigenvalues are in strictly negative real parts then A-BK is asymptotically stable.

A sufficient condition is that (A,B) is reachable. This just means that the rank of $[B, AB, \ldots A^{n-1}B]$ is n, the dimension of the state space, in the following case $n = 3$. Is more convenient to work with z set of coordinates.

$$z = Tx \qquad \text{(such that)}$$

$$\dot{z} = TAT^{-1}z + TBu \tag{319}$$

$$\dot{z} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \tag{320}$$

$$\det(SI_3 - A) = s^3 + a_1 s^2 + a_2 s + a_3 \tag{321}$$

$$\dot{z} = (\tilde{A} - \tilde{B}\tilde{K})z + \tilde{B}K_r r \tag{322}$$

$$u = -K_x + K_r r = -KT^{-1}z + K_r r \tag{323}$$

$$\begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{K_1} & \tilde{K_2} & \tilde{K_3} \end{bmatrix} = \begin{bmatrix} -a_1 - \tilde{K_1} & -a_2 - \tilde{K_2} & -a_3 - \tilde{K_3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{324}$$

$$\tag{325}$$

The characteristic polynomial of this matrix $\det\left(SI_3 - \tilde{A} - \tilde{B}\tilde{K}\right) = s^3 + (a_1 + \tilde{K_1})s^2 + (a_2 + \tilde{K_2})s + a_3 + \tilde{K_3}$.

Example: all eigenvalues are -1. Then $p(s) = (s+1)^3$ and $\tilde{k_1} = -a_1 + 3, \tilde{k_2} = -a_2 + 3, \tilde{k_3} = -a_3 + 1$.

Desired eigenvalues are given by the roots of $0 = p(s) = s^3 + p_1 s^2 + p_1 s + p_0$. To assign such eigenvalues choose $\tilde{K_1} = -a_1 + p_1, \tilde{K_2} = -a_2 + p_2, \ldots, \tilde{K_n} = -a_n + p_n$. But we don't really want $\tilde{K}$ vectors but K vectors. The relationship is given by a formula that already appeared before $KT^{-1} = \tilde{K}$ this is Ackermman's formula such that $K = \tilde{K}T$ where $T = \tilde{W_r}W_r^{-1}, \tilde{W_r} = [\tilde{B}, \tilde{A}\tilde{B}, \ldots A^{\tilde{n}-1}\tilde{B}$ were W_r is the reachability matrix

Back to the example:

$$W_r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tilde{A} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tilde{W_r} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{326}$$

$$\det(SI_3 - A) = \det \begin{pmatrix} s & -1 \\ 0 & s \end{pmatrix} = s^2 \tag{327}$$

Assign the eigenvalues to the roots of $p(s) = s^2$

$$\tilde{K} = \begin{pmatrix} 0 + 2s_c\omega_c & 0 + \omega_c^2 \end{pmatrix} \tag{328}$$

$$k = \tilde{K}T = \begin{pmatrix} 0 + 2s_c w_c & 0 + w_c^2 \end{pmatrix} I_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{pmatrix} w_c^2 & 2s_c w_c \end{pmatrix} \tag{329}$$

$$A - BK = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} w_c^2 & 2s_c w_c \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -w_c^2 & -2s_c w_c \end{bmatrix} \tag{330}$$

$$K_r = \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -w_c^2 & -2s_c w_c \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{-1} \right)^{-1} \tag{331}$$

$$u = -\begin{bmatrix} w_c^2 & 2s_c w_c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + k_r r \tag{332}$$

5.1.2 Design $p_{des}(s)$

Let's do it for a mass spring system

$$m\ddot{q} + c\dot{q} + kq = F \tag{333}$$
$$\ddot{q} + \frac{c}{m}\dot{q} + \frac{k}{m}q = \frac{F}{m} \tag{334}$$
$$\omega_0 = \sqrt{\frac{k}{m}}, \frac{c}{m} = 2s\sqrt{\frac{k}{m}}, \frac{c^2}{m} = 4s^2k \tag{335}$$
$$J^2 = \frac{c^2}{4km}, s = \frac{c}{2\sqrt{km}} \tag{336}$$
$$k\omega_0 u = \frac{1}{m}F \tag{337}$$
$$u = \frac{F}{k^2} \tag{338}$$

First let's select our state values, $x_1 = q, x_2 = \frac{\dot{q}}{\omega_0}$ so position and normalised velocity. Therefore

$$\dot{x_1} = \omega_0 x_2 \tag{339}$$
$$\dot{x_2} = \frac{1}{\omega_0}(-2s\omega_0\omega_0 x_2 + \omega_0^2 x_1 + k\omega_0^2 u) = -2s\omega_0 x_2 - \omega_0 x_1 + k\omega_0 u \tag{340}$$
$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & -2s\omega_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ k\omega_0 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases} \tag{341}$$

5.1.3 Step response

$$x(0) = 0 \tag{342}$$
$$u(t) = 1(t) = \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases} \tag{343}$$

If A is Asymptotically stable, then $y(t) = CA^{-1}e^{At}B - CA^{-1}B$ where we have the transient response minus the steady state response.

$$\det(ST_2 - A) = \det \begin{bmatrix} s & -\omega_0 \\ \omega_0 & s + 2s\omega_0 \end{bmatrix} = s^2 + 2s\omega_0 s + \omega_0^2 \tag{344}$$
$$s1, 2 = -s\omega_0 \pm \sqrt{s^2\omega_0^2 - \omega_0^2} = -s\omega_0 \pm \omega_0^2\sqrt{s^2 - 1} \tag{345}$$

In an overdamped response $s > 1$, two distinct eigenvalues are both real $y(t) = CA^{-1}$. In $A = T^{-1}AT$, A has

$$\tilde{A} = \begin{bmatrix} -s\omega_0 + \omega_0^2\sqrt{s^2-1} & 0 \\ 0 & -s\omega_0 - \omega_0^2\sqrt{s^2-1} \end{bmatrix} \tag{346}$$

$$e^{\tilde{A}t} = \begin{bmatrix} e^{(-s\omega_0+\omega_0^2\sqrt{s^2-1})t} & 0 \\ 0 & e^{(-s\omega_0-\omega_0^2\sqrt{s^2-1})t} \end{bmatrix} \tag{347}$$

$$A = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & -2s\omega_0 \end{bmatrix}, A^{-1} = \frac{1}{\omega_0^2}\begin{bmatrix} -2s\omega_0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} = \frac{1}{\omega_0^2}\begin{bmatrix} -2s & -1 \\ 1 & 0 \end{bmatrix} \tag{348}$$

$$CA^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix}\frac{1}{\omega_0^2}\begin{bmatrix} -2s & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ k\omega_0 \end{bmatrix} = k \tag{349}$$

$$y(t) = k(1 - C_1 e^{(-s+\sqrt{s^2-1})\omega_0 t} + C_2 e^{(-s-\sqrt{s^2-1})\omega_0 t}) \tag{$t \to \infty$ and $y(t) \to k$}$$

If a critically damped response $s = 1$, $y(t) = k(1 - e^{-s\omega_0 t}(1 + \omega_0 t)$.

$$\tilde{A} = \begin{bmatrix} \pm s\omega_0 & 1 \\ 0 & \pm\omega_0 \end{bmatrix} \tag{350}$$

$$e^{\tilde{A}t} = \begin{bmatrix} e^{-s\omega_0 t} & te^{-s\omega_0 t} \\ 0 & e^{-s\omega_0 t} \end{bmatrix} \tag{351}$$

If a underdamped response $0 < s < 1$, you have some pseudo-oscillations

$$y(t) = k(1 - e^{s\omega_0 t}\cos\omega_d t - \frac{s}{\sqrt{1-s^2}}e^{-s\omega_0 t}\sin\omega_0 t \tag{352}$$

$$\omega_d = \omega_0\sqrt{1-s^2} \tag{damped frequency}$$

$$\frac{s}{\sqrt{1-s^2}} = \frac{\cos\phi}{\sin\phi} \tag{353}$$

$$y(t) = k(1 - e^{s\omega_0 t}\sin(\omega_d t + \phi) \tag{354}$$

5.1.4 Overshooting

The step response is characterised by the following elements present in figure 24:

- steady state value y_{st} is the final value where the output converges
- rise time T_r amount of time required for the output to go from 10% to 90% of its final value
- overshoot M_p percentage of the final value by which the signal rises above the final value (peak= 0.72 final= 0.5, overshoot = 0.72/0.5 *100 -100)
- settling time T_s amount of time required for the output to stay within 2% of its final value

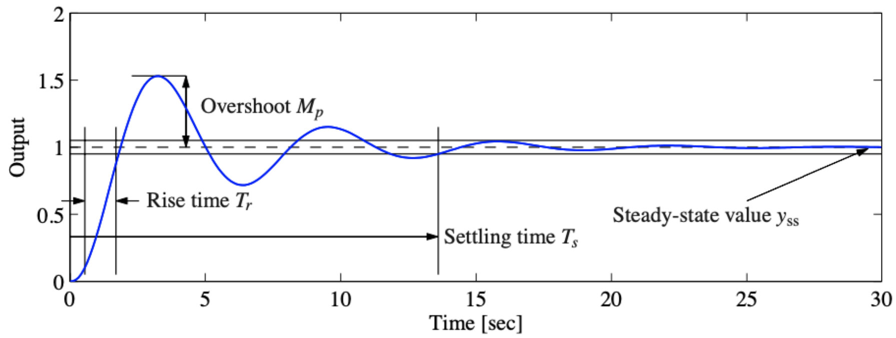

Figure 24: Step response characterisation and its elements.

For each percentage of overshooting, we may want to change the eigenvalues so that it follows table 25

Property	Value	$\zeta = 0.5$	$\zeta = 1/\sqrt{2}$	$\zeta = 1$
Steady-state value	k	k	k	k
Rise time	$T_r = 1/\omega_0 \cdot e^{\varphi/\tan\varphi}$	$1.8/\omega_0$	$2.2/\omega_0$	$2.7/\omega_0$
Overshoot	$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$	16%	4%	0%
Settling time (2%)	$T_s \approx 4/\zeta\omega_0$	$8.0/\omega_0$	$5.9/\omega_0$	$5.8/\omega_0$

Figure 25: Properties of the step response for a second-order system with $0 < \xi < 1$.

$$\lambda_1, \lambda_2 = \frac{-b \pm b^2 - 4km}{2m} \tag{355}$$

We know that our system is always stable, so when it's underdamped we have a sqrt of neg, so we have a complex number in the solution and therefore we have oscillations. An overdamped system will take longer to converge.

6 Lecture 8

6.1 State feedback controller

As already introduced in section 5.1, the idea of a state feedback controller is divided in 3 steps

- construct a device called observer that estimates state x. An observer with inputs $u(t)$ and $y(t)$ that outputs $\hat{x}(t)$ and $\hat{x} - x \to 0$, when $t \to +\infty$
- $u(t) = -k\hat{x}(t) + K_r r$
- fingers crossed (we still need to prove that it works)

Now the concept of observability in this controller is very important. For every linear system like equations 237, for any $t > 0$, and $u(.)|_{[0,T]}, y(.)|_{[0,T]}$, we can reconstruct the state $x(t)$. More numerically,

$$y(t) = Cx(t), \tag{356}$$
$$\dot{y}(t) = CAx(t), \tag{357}$$
$$y^{n-1}(t) = CA^{n-1}x(t) \tag{358}$$
$$y(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \\ y^{n-1}(t) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^{n-1} \end{bmatrix} x(t) = W_o x(t) \tag{359}$$

where W_o is the observability matrix.

The main assumption is that the observability matrix has rank n, in other words, the matrix is nonsingular. Since this is nonsingular we can use inverse to solve the equation. If the matrix has rank n then the system is observable and viceversa.

6.2 Observable Canonical Form

For a regular linear system like 237, we can say that if $n = 3$, then $\det(sI_3 - A) = s^3 + a_1 s^2 + a_2 s + a_3$, and when T is to be determined and nonsingular we introduce new coordinate $z = Tx$. We know that linear system is observable and this coordinate change should keep this property.

In the state space form $\dot{z} = \tilde{A}z + \tilde{B}u, y = \tilde{C}z$ where $\tilde{A} = TAT^{-1}, \tilde{B} = TB, \tilde{C} = CT^{-1}$ We'll see how this affects observability

$$\tilde{W_o} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CAT^{-1} \\ CA^{n-1}T^{-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^{n-1} \end{bmatrix} T^{-1} = W_o T^{-1} \tag{360}$$
$$T = \tilde{W_o}^{-1} W_o \tag{361}$$
$$\tag{362}$$

because T is non-singular and so is the observability matrix we can guarantee that the system is still observable.

6.3 Example

Let's make an **example** system in which valves v_1 and v_2 release drug concentration c_1 and c_2 proportional to coefficients k_1 and k_2. v_1 also has the input u and b and output proportional to coefficient k_o We can further define this environment as

$$\begin{cases} \dot{c_1} = -k_o c_1 + bu + k_1(c_2 - c_1) \\ \dot{c_2} = k_2(c_1 - c_2) \\ y = c_1 \end{cases} \tag{363}$$

And let's make that a state space form

$$\begin{bmatrix} \dot{c_1} \\ \dot{c_2} \end{bmatrix} = \begin{bmatrix} -k_o - k_1 & k_1 \\ k_2 & -k_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u \tag{364}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{365}$$

Because our observability matrix is $\begin{bmatrix} 1 & 0 \\ -k_o - k_1 & k_1 \end{bmatrix}$, the determinant is k_1 and therefore for the system to be observable it cannot lose its only observable component $k_1 = 0$.

If we wanted to make the observable canonical form, then we would get the $\det(sI - A_2) = s^2 + s(k_0 + k_1 + k_2) + k_0 k_2$

$$\dot{z} = \begin{bmatrix} -(k_0 + k_1 + k_2) & 1 \\ -k_0 k_2 & 0 \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \tag{366}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} z \tag{367}$$

$$\tilde{W_o}^{-1} = \begin{bmatrix} 1 & 0 \\ k_1 + k_2 + k_0 & 1 \end{bmatrix} \tag{368}$$

$$T = \begin{bmatrix} 1 & 0 \\ k_2 & k_1 \end{bmatrix} \tag{369}$$

6.4 Observer design

We know that the observer computes the error $e(t) = x(t) - \hat{x}(t)$, where $\hat{x}(t)$ has a different initial condition than $x(t)$. This error will converge to 0 as the t approaches infinity because the A is asymptotically stable. The optimal version of this is

$$\dot{\hat{x}} = A\hat{x} + Bu(t) + L(y(t) - C\hat{x}(t)) \tag{370}$$

where L is a nx1 matrix to be designed. However we know $y(t) = Cx(t)$

$$\dot{e}(t) = Ax(t) + Bu(t) - A\hat{x}(t) - Bu(t) - LC(x(t) - \hat{x}(t)) \tag{371}$$

$$= (A - LC)(x(t) - \hat{x}(t)) = (A - LC)e(t) \tag{372}$$

How do we design L when we are in observable canonical form such that A-LC is AS?

$$\tilde{A} - \tilde{L}\tilde{C} = T(A - LC)T^{-1} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{l_1} \\ \tilde{l_2} \\ \tilde{l_3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -a_1 - \tilde{L_1} & 1 & 0 \\ -a_2 - \tilde{l_2} & 0 & 1 \\ -a_3 - \tilde{l_3} & 0 & 0 \end{bmatrix} \tag{373}$$

$$\det\Big(sI_3 - (\tilde{A} - \tilde{L}\tilde{C})\Big) = s^3 + s^2(a_1 + \tilde{L_1}) + s(a_2 + \tilde{l_2}) + a_3 + \tilde{l_3} \tag{374}$$

$$p_{des}(s) = s^3 + p_1 s^2 + p_2 s + p_3 \tag{375}$$

$$\tilde{L} = \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \\ p_3 - a_3 \end{bmatrix}, L = T^{-1}\tilde{L} \tag{376}$$

By this point we have successfuly completed point 1 and 2 of the algorithm introduced on how to make a state feedback controller. Third and last is "fingers crossed".

We make the closed loop system

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + B(-K\hat{x} + K_r r) + L(y - C\hat{x}) \\ u = -K\hat{x} + K_r k \end{cases} \tag{377}$$

$$\begin{cases} \ddot{\hat{x}} = (A - BK - LC)\hat{x} + Ly + BK_r r \\ u = -K\hat{x} + K_r k \end{cases} \tag{378}$$

$$\begin{cases} \dot{x} = Ax + B(-K\hat{x} + K_r r) = Ax - BK\hat{x} + BK_r r \\ \ddot{\hat{x}} = (A - BK - LC)\hat{x} + Ly + BK_r r y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \end{cases} \tag{379}$$

By studying independently the observer and the -, this is the separation principle, we obtain a dynamic matrix. We make a change od coordinates $x = x, e = x - \hat{x}$ and we arrive at

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} BK_R \\ 0 \end{bmatrix} r \tag{380}$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{381}$$

$$y_{steady\ state} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}^{-1} \begin{bmatrix} BK_R \\ 0 \end{bmatrix} r = r \tag{382}$$

7 Lectures 9, 10 and 11

7.1 Laplace Transform

$$f(.) \cdot \mathbb{R} \geq 0 \to \mathbb{R} \tag{383}$$

$$t \to f(t) \tag{384}$$

$$\int_0^{+\infty} |e^{-st} f(t)| dt < \infty \tag{385}$$

$$\mathcal{L}[f] = F(s) = \int_0^{+\infty} e^{-st} f(t) dt \tag{386}$$

$$\int_0^{+\infty} e^{-st} f(t) dt \leq \int_0^{+\infty} |e^{-st} f(t)| dt < \infty \tag{387}$$

where $s \in \mathbb{C}$.

If there exists $s_0 \in \mathbb{C}$ such that $\int_0^{+\infty} |e^{-s_0 t} f(t)| dt$ then the same holds true for $\int_0^{+\infty} |e^{-st} f(t)| dt$ for all s such that $\mathrm{Re}\{s\} > \mathrm{Re}\{s_0\}$. $\mathcal{L}[f]$ is always defined on a domain of convergence which is $\mathrm{Re}\{s\} > \delta, \delta \in \mathbb{R}$.

Example: for an identity equation which is 1 when $t \geq 0$, then

$$\int_0^{+\infty} |e^{-st} \mathbb{I}(t)| dt < \infty? \tag{388}$$

we can consider two coefficients c_1 and c_2 real numbers such that $c_1 f_1 + c_2 f_2$, we can prove that the domain of convergence exists, since if we linearly combine signals, we can linearly combine its laplace transforms $\mathcal{L}[c_1 f_1 + c_2 f_2] = c_1 \mathcal{L}[f_1] + c_2 \mathcal{L}[f_2]$. Other properties are the derivative $\mathcal{L}[\ddot{f(t)} = s\mathcal{L}[f] - f(0)$ and the final value theorem $\lim_{t \to \infty} f(t) < \infty$ then $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

For the linear systems equations 237 we always be guaranteed a domain of convergence. After the laplace transform we get

$$sX(s) - x_0 = AX(s) + BU(s) \tag{389}$$

$$Y(s) = CX(s) + DU(s) \tag{390}$$

$$X(s) = \mathcal{L}[e^{At}]x_0 + \mathcal{L}[\int_0^t e^{A(t-\tau)} Bu(\tau) d\tau] \tag{391}$$

$$Y(s) = \mathcal{L}[Ce^{At}]x_0 + \mathcal{L}[\int_0^t Ce^{A(t-\tau)} Bu(\tau) d\tau] + DU(s) \tag{392}$$

$$\mathcal{L}[e^{At}] = (sI_n - A)^{-1}, \mathcal{L}[\int_0^t e^{A(t-\tau)} Bu(\tau) d\tau] = (sI_n - A)^{-1} BU(s) \tag{393}$$

$$\mathcal{L}[Ce^{At}] = C(sI - A)^{-1}, \mathcal{L}[\int_0^t Ce^{A(t-\tau)} Bu(\tau) d\tau + DU(s)] = [C(sI - A)^{-1} B + D] U(s) \tag{394}$$

For $x_0 = 0$ the IO relation is given by the rational function or transfer function $[C(sI-A)^{-1}B+D]$ multiplied by $U(s)$ where we can say that

$$C(sI-A)^{-1}B + D = C\frac{(sI-A)^a B}{\det(sI-A)} + D \tag{395}$$

where $(sI-A)^a$ is a polynomial of degree at most $n-1$.

If there are no cancellations they're exactly the eigenvalues of the matrix A.

7.2 Kalman decomposition

$$z = Tx, z = \begin{bmatrix} x_{r0} \\ x_{r\tilde{o}} \\ x_{\tilde{r}o} \\ x_{r\tilde{o}} \end{bmatrix} \tag{396}$$

$$\dot{z} = \begin{bmatrix} A_{ro} & 0 & * & 0 \\ * & Ax_{r\tilde{o}} & * & * \\ 0 & 0 & Ax_{\tilde{r}o} & 0 \\ 0 & 0 & * & Ax_{r\tilde{o}}z \end{bmatrix} + \begin{bmatrix} B_{ro} \\ B_{r\tilde{o}} \\ 0 \\ 0 \end{bmatrix} u \tag{397}$$

$$y = \begin{bmatrix} C_{ro} & 0 & C_{\tilde{r}o} & 0 \end{bmatrix} z \tag{398}$$

where $\tilde{o}$ means it's unobservable, $\tilde{r}$ means its unreachable and they're reachable and observable otherwise. Which leads to the following relation between components visible in figure 26.

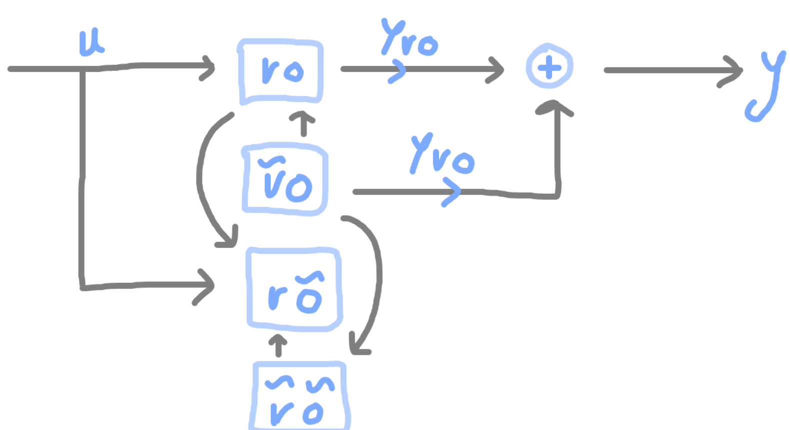

Figure 26: Block components of the kalman decomposition

Invariant to change of coordinates $\tilde{A} = TAT^{-1}, \tilde{B} = TB, \tilde{C} = CT^{-1}, \tilde{D} = D$ we can make the following change in the transfer function $G(s)$

$$G(s) = C(sI-A)^{-1}B + D \tag{399}$$

$$G(S) = \tilde{C}(sI_n - \tilde{A})^{-1}\tilde{B} + \tilde{D} = C_{ro}(sI-A)^{-1}B_{ro} \tag{400}$$

because what actually contributes to the transfer function is the subsystem with matrix A reachable observable, C reachable observable and B reachable observable, and therefore can be equal to A eigenvalues.

7.2.1 Negative feedback interconnection

We'll want to know if $G(s)$ has an unstable pole or zero and if $C(s)$ cancels the unstable pole or zero because then that causes unstable unobservable/unreachable eigeinvalue of the state space representation of $C(s)G(s)$

7.3 Bode diagrams

For linear system equations 237, A has all eigenvalues with strictly negative real parts and $u(t) = \cos\omega t$, then the steady state response

$$y(t) = |G(i\omega)| \cdot \cos(\omega t + LG(i\omega)) \tag{401}$$

$$G(s) = C(sI - A)^{-1}B = \frac{num_G(s)}{den_G(s)} \tag{402}$$

$$G(i\omega) = G(s)|_{s=i\omega} = |G(i\omega)|e^{i\angle G(i\omega)} \tag{403}$$

$$|G(i\omega)| = (\mathrm{Re}\{G\}^2 + \mathrm{Im}\{G\}^2)^{1/2} \tag{404}$$

$$L(G(i\omega)) = \arctan\frac{\mathrm{Im}\{G\}}{\mathrm{Re}\{G\}} \tag{405}$$

where the imaginary and real parts follow the same properties as explained in section 4.3.3. Bode diagram are pictorial representations of $|G(i\omega)|$ and $\angle G(i\omega)$ as function of ω

Decade: frequency band from ω_1 and $10\omega_1$. The tick mark representing the frequency $10^0 = 1$, where $c > 0$ is the actual length (i.e. 2cm) of the actual length of the decade

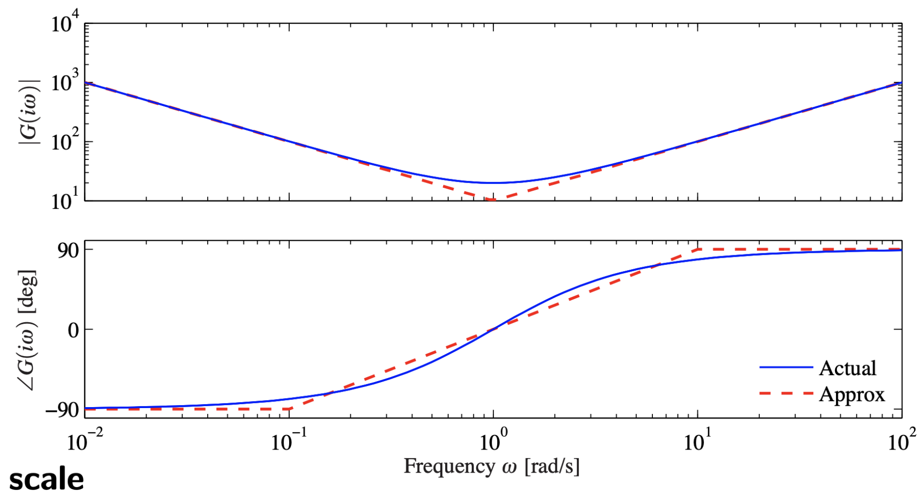

Figure 27: Caption

Example: PID $G(s)$

$$G(s) = 20 + \frac{10}{s} + 10s = \frac{10(1+s)^2}{s} = 20\log_{10}|G(i\omega)| = |G(i\omega)|_{dB} \tag{406}$$

$$|G(i\omega)| = 20\log_{10} 10\frac{|1+i\omega|^2}{|i\omega|} = 20 + 40\log_{10}|1+i\omega| - 20\log_{10}|i\omega| \tag{407}$$

$$\angle G(i\omega) = 0 + 2\angle 1 + i\omega - 90^\circ \tag{408}$$

$$\tag{409}$$

remark $20\log_{10}\sqrt{2} = 3dB$ and $\angle 1 + i\omega = \arctan\omega = \begin{cases} 90°, \omega >> 1 \\ 45°, \omega = 1 \\ 0°, \omega << 1 \end{cases}$ This results in the bode diagram seen in figure 29.

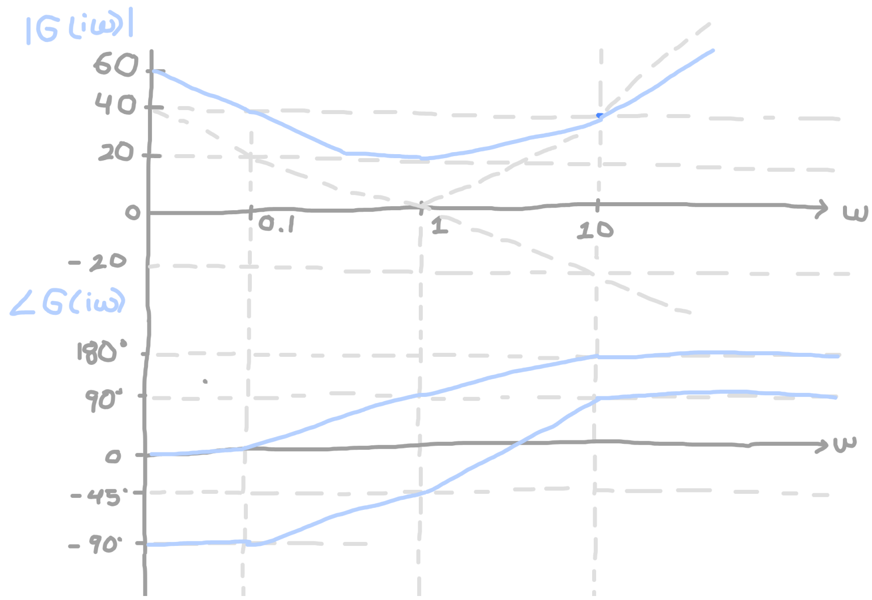

Figure 28: Bode diagram of PID controller

7.4 Nyquist plots

A nyquist diagram is defined by the $L(s) = \frac{1}{(s+1)^3} = C(s)G(s)$ in which $1 + L(s) = 0$ and $G(s) = \frac{L(s)}{1+L(s)}$ from the negative feedback controller explained in section 7.2.1

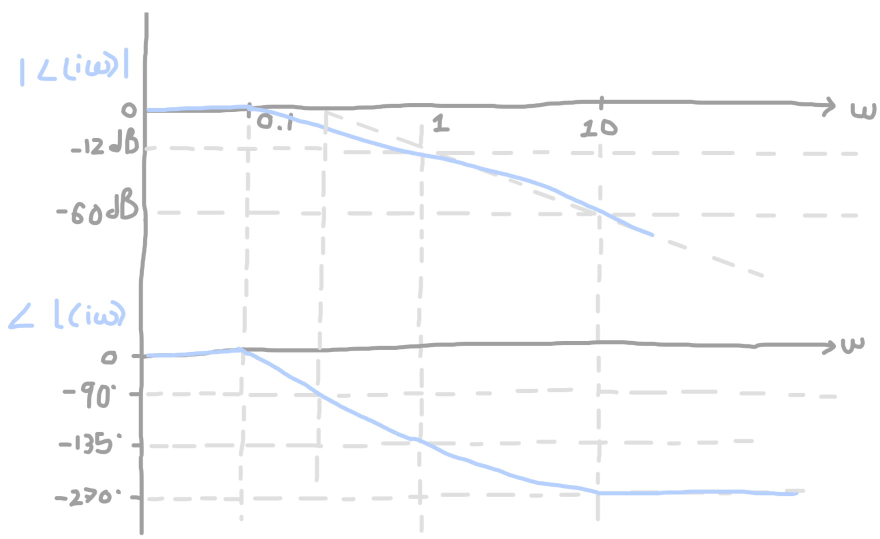

Figure 29: Bode diagram of $L(i\omega)$

Simplified nyquist criterion: $L(s)$ has no poles in $\mathbb{C}^F$ except for simple poles in the imaginary axis. The closed loop system is asymptotically stable iff the nyquist plot has no net encirclements of the point $-1+i0$, in other words if the number of closed-loop poles with positive part Z=N+P=0. Here, N number of net encirclements of $-1+i0$ by the nyquist plots of $L(i\omega)$.

7.5 Example

7.5.1 Bode diagram

$$\angle L(i\omega) = 2\arctan\left(\frac{\omega}{6}\right) - 2\arctan(\omega) - 90^\circ \tag{410}$$

$$L(s) = \frac{3(s+6)^2}{s(s+1)^2} = \frac{3 \cdot 36(\frac{5}{6}+1)^2}{s(s+1)^2} \tag{411}$$

7.5.2 Nyquist plot

$$L(i\omega) = \frac{3(i\omega+6)^2}{i\omega(i\omega+1)^2} \tag{412}$$

$$\angle L(i\omega) = -180^\circ \Leftrightarrow 2\arctan\left(\frac{\omega}{6}\right) - 2\arctan(\omega) = 90^\circ \tag{413}$$

$$|\angle L(i\omega)| = 3\frac{\omega^2+36}{\omega(\omega^2+1)} \tag{414}$$

$$|\angle L(i2)| = 12 \tag{415}$$

$$|\angle L(i3)| = 4.5 \tag{416}$$

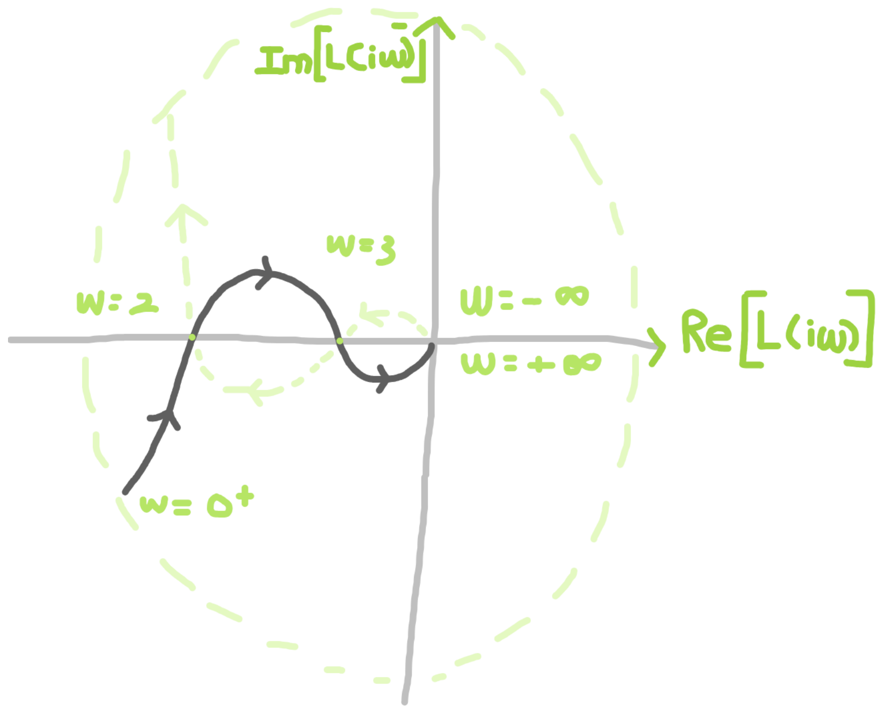

Figure 30: Caption

There is a critical point (-1,0) which appears that helps us calculate the number of encirclements of this problem. The number of encirclements is 0 and because the other thing sth, then the system is asymptotically stable.

Two ways to get the k' that tells the transition between stability states is the Routh-Hortwitz criteria and these calculations.

$$|L(i\omega)|_{\omega=3}| = \frac{3k'(i\omega+6)^2}{i\omega(i\omega+1)^2} \tag{417}$$

$$= |k'| \cdot \frac{45}{10} \tag{418}$$

$$|k'| = \frac{2}{9} \tag{419}$$

$$|L(i\omega)|_{\omega=2}| = \frac{3k'(i\omega+6)^2}{i\omega(i\omega+1)^2} \tag{420}$$

$$= |k'| \cdot 12 \tag{421}$$

$$|k'| = \frac{1}{12} \tag{422}$$

$$\tag{423}$$

For $|k'| > \frac{2}{9}$ the system is stable, for $\frac{1}{12} < |k'| < \frac{2}{9}$ unstable because we have one encirclement and $0 < |k'| < \frac{1}{12}$ stable

7.5.3 Closed loop system and its poles

The closed loop system is defined by

$$G_{y,r}(s) = \frac{L(s)}{1+L(s)} = \frac{\lambda_L(s)}{d_L(s)+n_L(s)} \tag{424}$$

and the poles are given by this equations and the Routh-Hurwitz criteria

$$s(s+1)^2 + 3k'(s+6)^2 = 0 \tag{425}$$
$$s^3 + (3k'+2)s^2 + (36k'+1)s + 108k' = 0 \tag{426}$$
$$\begin{cases} 3k'+2 > 0 \\ * > 0 \\ k' > 0 \end{cases} \tag{427}$$

7.5.4 Margin gain and phase

Calculate the stability margins and gain margin phase crossover frequency ω_{pc} is $-180°$

$$\angle L(i\omega) = -180° \tag{428}$$
$$g_m \cdot |L(i\omega)| = 1 \tag{429}$$
$$g_m = \frac{1}{|L(i\omega)|} \tag{430}$$
$$\angle L(i\omega) = 3\arctan\omega = -180° \tag{431}$$
$$\tag{432}$$

8 Lecture 12

8.1 PID control

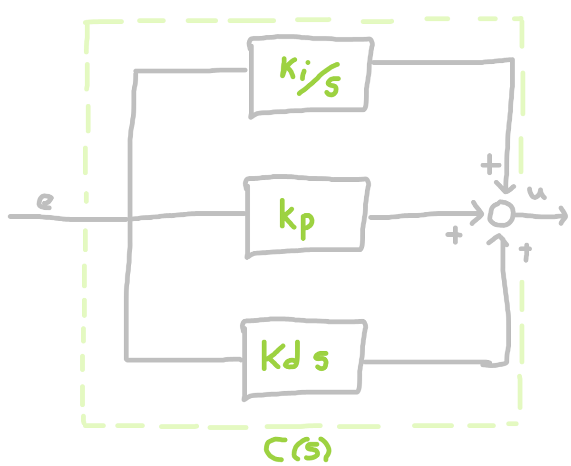

Figure 31: Caption

In PID control, K_i, K_p, K_d are design parameters

$$U(s) = \frac{K_i}{s} E(s) \tag{433}$$

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8.2 PD control

Board visible closer half

Board blurry:

$$E(s) = G_{er}(s)R(s) \tag{434}$$
$$R(s) = \frac{r}{s} \tag{435}$$
$$E(s) = R(s) - K_p P(s) E(s) \tag{436}$$
$$E(s) = \frac{R(s)}{1 + K_p P(s)} = \frac{d_p(s) \cdot r}{d_p(s) + K_p n_p(s) \cdot s} = \frac{A_0}{S} + \frac{A_1}{(s - \lambda_1} + \cdots + \frac{A_r}{s - \lambda_r} \tag{437}$$
$$+ \frac{B_1 S + C_1}{(s - \sigma_1)^2 + \omega_1^2} + \cdots + \frac{B_e s + C_e}{(s - \sigma_e)^2 + \omega_e^2} \tag{438}$$

We can do the laplace transform and get

$$e(t) = A_0 \mathbb{1}(t) + A_1 e^{\lambda t} \mathbb{1}(t) + \cdots + A_r e^{\lambda r t} \mathbb{1}(t) \tag{439}$$
$$+ \tilde{B}_{11} e^{\sigma_1 t} \cos(\omega t) + \tilde{B}_{12} e^{\sigma_1 t} \sin(\omega t) + \cdots + \tilde{B}_{e1} e^{\sigma_e t} \cos(\omega t) + \tilde{B}_{e2} e^{\sigma_e t} \sin(\omega t) \tag{440}$$

Given that the steady state response is a limit that converges to a value $\lim_{t \to +\infty} e(t) = A_0$ we apply finite limit theorem

$$\lim_{t \to +\infty} e(t) = A_0 = \lim_{s \to s} sE(s) = \lim s \to 0 s \frac{d_p(s) \cdot r}{d_p(s) + K_p n_p(s) \cdot s} = \frac{d_p(0) \cdot r}{d_p(0) + K_p n_p(0)} \tag{441}$$

If $P(s)$ has a 0 at $s = 0$ then $\lim_{t \to +\infty} e(t) = r$. Example: get the denominator of the closed loop

$$P(s) = \frac{1}{(s+1)^3} \tag{442}$$
$$C(s) = K_p \tag{443}$$
$$(s+1)^3 + K_p = s^3 + 3s^2 + 3s + K_p + 1 \tag{444}$$
$$\begin{cases} k_p + 1 > 0 \\ k_p - 8 < 0 \end{cases} \tag{445}$$
$$\frac{1}{-\frac{1}{3}(K_p - 8)} \begin{bmatrix} 3 & K_p + 1 \\ -\frac{1}{3}(K_p - 8) & 0 \end{bmatrix} = \frac{K_p + 1}{-\frac{1}{3}(K_p - 8)} \tag{we use this for the Routh-Hurwitz criteria}$$

8.3 PI control

PI Control board

8.4 Disturbance rejection problem

When we have an additional disturbance the control will adapt to reject it

$$U(s) = \frac{\frac{K_p s + K_i}{s} P(s)}{1 + \frac{K_p s + K_i}{s} P(s)} \cdot D(s) = -\frac{(K_p s + K_i) n_p(s)}{s d_p(s) + (K_p s + K_i) n_p(s)} \cdot \frac{d}{s} \tag{446}$$

$$\lim_{t \to +\infty} u(t) = -\lim s \to 0 s \cdot \frac{d}{s} \cdot \frac{(K_p s + K_i) n_p(s)}{s d_p(s) + (K_p s + K_i) n_p(s)} = -\frac{K_i n_p(0)}{0 + K_i n_p(0)} \cdot d = -d \tag{447}$$

$$\lim_{t \to +\infty} y(t) = 0 \tag{448}$$

8.5 Stabilization

With a $P(s) = \frac{b}{s+a}$ and $C(s) = \frac{K_p s + K_i}{s}$ we do

$$P_{des}(s) = s^2 + 2s\omega + \omega^2 \tag{449}$$

$$\begin{cases} a + bK_p = 2s\omega \\ K_i b = \omega^2 \end{cases} \tag{450}$$

PI control assigns completely the closed-loop dynamics to 1st order plants, PID control does exactly the same for 2nd order closed-loop dynamics.

8.5.1 Poles assignment *pbm*

pole assignment board

9 Lecture 13

9.1 Ziegler-nichols PID tuning

Table 1: Controller Parameters

Type	K_p	T_i	T_d
PI	$\frac{0.9}{a}$	3τ	
P	$\frac{K}{a}$		
PID	$\frac{1.2}{a}$	2τ	0.5τ

9.1.1 Improved Step response method

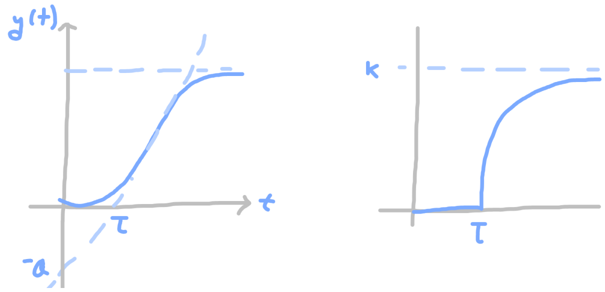

Figure 32: Caption

$$
\begin{aligned}
P(s) &= \frac{k}{1+ST}e^{-s\tau} \\
Y(s) &= \frac{k}{1+ST}e^{-s\tau}\frac{1}{s} \\
Y'(s) &= \frac{K}{s(1+sT)} = k(\frac{A}{s} + \frac{B}{1+sT}) = k\frac{(AT+B)s+A}{s(1+sT)} \\
&\begin{cases} AT+B=0 \\ A=1 \\ B=-T \end{cases} \\
Y'(s) &= k(\frac{1}{s} - \frac{T}{1+sT}) = K(\frac{1}{s} - \frac{1}{s+\frac{1}{T}}) \\
y'(t) &= k - ke^{-\frac{t}{T}} \\
y(t) &= \begin{cases} 0 & 0 \le t < \tau \\ k - ke^{\frac{t-\tau}{T}} & t > \tau \end{cases} \\
\dot{y}(t) &= \begin{cases} 0 & 0 \le t < \tau \\ ke^{\frac{t-\tau}{T}} + \frac{1}{T} & t \ge \tau \end{cases} \\
\max_{t\ge 0}\dot{y}(t) &= \frac{k}{T}, T = \frac{\tau k}{a}
\end{aligned}
$$

9.1.2 Frequency response method

Where T_c is the period of the oscillations

$$L(s) = K_p P(s) \tag{451}$$

$$T(s) = G_{yr}(s) = \frac{L(s)}{1+L(s)}, 1 + L(s) = 0, 1 + kP(i\omega) = 0, s = \pm i\omega \qquad \text{(are poles of T(s))}$$

In an example

$$P(s) = \frac{e^{-s}}{s} \tag{452}$$

$$K_p P(s) = K_p \frac{e^{-s}}{s} = L(s) \tag{453}$$

$$L(i\omega) = -1 = K_p \frac{e^{-i\omega}}{i\omega} \tag{454}$$

$$K_p = \frac{\cos(\omega) - i\sin(i\omega)}{i\omega} = \begin{cases} K_p \frac{\sin(\omega)}{-\omega} = -1, K_p = -\frac{\pi}{2} \\ K_p \frac{\cos(\omega)}{-\omega} = 0, \omega = \frac{\pi}{2} \end{cases} \tag{455}$$

Table 2: Controller Parameters

Type	K_p	T_i	T_d
PI	$0.5t_c$		
P	$\frac{K}{a}$		
PID	$\frac{1.2}{a}$		

9.2 Type k systems

Type k signals

$$r(t) = \frac{t^k}{k!} \tag{456}$$

$$\begin{cases} k = 0 & \text{step ineference} \\ k = 1 & r(t) = t \text{ ramp} \\ k = 2 & r(t) = \frac{t^2}{2}\text{parabola} \end{cases} \tag{457}$$

Type k systems

1. $T(s) = \frac{L(s)}{1+L(s)}$ all poles have negative real part
2. $\lim_{t\to t_o}[r(t) - y(t)] = const \neq 0$ if

To test if the system is of type K

1. $k > 0 \Leftrightarrow \lim_{s\to 0} s^k L(s) = const \neq 0$

2. $k = 0 \Leftrightarrow \lim_{s\to 0} L(s) = const \neq 0$

$$const \neq 0 = \lim_{s\to 0}[r(t) - y(t)] = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+L(s)}\frac{1}{s^{k+1}} \tag{458}$$

$$= \lim_{s\to 0}\frac{1}{s^k + s^k L(s)} \tag{459}$$

$$s^K L(s) = \frac{n_L(0)}{d_L(0)} \tag{460}$$

$$\lim_{t\to\infty}[r(t) - y(t)] = lim_{s\to 0}\frac{s}{s^k + s^k L(s)} = 0 \qquad \text{(laplace transform of error signal)}$$

$$E(s) = \frac{1}{1+L(s)}\frac{1}{s^{k+1}} = \frac{d_i(s)}{d_L(s)+n_L(s)}\cdot\frac{1}{s^{k+1}} = \frac{A_1}{s} + \frac{A_2}{s} + \cdots + \frac{A_{k+1}}{s^{k+1}} + \frac{n(s)}{d_L(s)+n_L(s)} \tag{461}$$

$$d_L(s) + n_L(s) = \frac{1}{(s+1)(s+2)}, \frac{n(s)}{d_L(s)+n_L(s)} = \frac{B_1}{s+1} + \frac{B_2}{s+2} \tag{462}$$

$$e(t) = R^{-1}[E(s)] = A_1 + B_1 e^{-t} + B_2 e^{-2t} \to_{t\to\infty} A_1 \tag{463}$$

Let's make an example with a type 1 reference signal and system of type 2 in which Y(s) is an angular position and U(s) is the torque

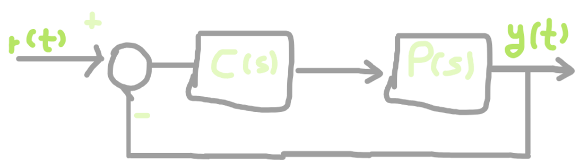

Figure 33: Caption

$$P(s) = \frac{a_0}{s^2} \tag{464}$$

$$r(t) = \omega t \tag{465}$$

$$C(s) = K_p, L(s) = a - \frac{K_p}{s^2} \tag{466}$$

$$T(s) = \frac{L(s)}{1+L(s)} = \frac{a_0 K_p}{s^2 + a_0 K_p} \tag{467}$$

$$C(s) = \frac{c_1 s + c_0}{d_1 s + d_0} \tag{468}$$

$$L(s) = \frac{(c_1 s + c_0)a_0}{(d_1 s + d_0)s^2} \tag{469}$$

$$P_{des} = (s+1)^3 \tag{470}$$

$$s^2(d_1 s + d_0) + a_0(c_1 s + c_0) = (s+1)^3 \qquad \text{((}s+1)^3\text{ is arbitrary choice)}$$

Sysem of 4 equations in 4 unknown, which is solvable $C(s) = \frac{3s+1}{s+3}$. By designing C(s) as stated earlier one can truely assign the denominator (poles) of the transform function.

$$r(t) = \frac{t^2}{2!}, y(t) = \frac{t^2}{2!} - e_\infty \tag{471}$$

$$e_\infty = \lim_{t\to\infty} e(t) = lim_{s\to 0} sE(s) = \frac{s+3}{s^2(s+3)+a_0(s+1)} = \frac{3}{a_0} \tag{472}$$

$$y(t) \to_{t\to\infty} \frac{t^2}{2!} - \frac{3}{a_0} \tag{473}$$

9.3 Performance specification: tracking a disturbance rejection

9.4 Internal model

References

Åström, K. J., & Murray, R. (2021). *Feedback systems: An introduction for scientists and engineers*. Princeton university press.