Geometry 2024
Take home exam, Tuesday 9

- Below you can find your exam questions. There are 3 questions summing up to 85 points; you get extra 15 points for a clear writing of solutions.
- This is a take home exam and you may consult the materials that we used during the course such as the course book, the BrightSpace page of the course and the two A4 pages that you brought with you today. **Other materials are not allowed!**
- Please prepare handwritten solutions, scan and upload them to BrightSpace on **Wednesday April 10** before 23:59. The deadline is strict. Good luck!

QUESTIONS

1. $\boxed{10+10 = 20 \text{ pts}}$

   a) Compute the evolute of the nephroid

   $$\varphi \mapsto (3\cos\varphi + \cos(3\varphi), 3\sin\varphi + \sin(3\varphi)), \quad \varphi \in [0, 2\pi].$$

   Determine the singular points of the evolute.

   b) Draw (schematically) the evolute constructed in part a) together with the wavefronts of the nephroid. Explain how your drawing illustrates Huygens's principle.

2. $\boxed{10+5+10+10+10 = 45 \text{ pts}}$ Consider the following surface in $\mathbb{R}^3$ given by the implicit equation

   $$M^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^4 + z^4 = 1 + y^4\}.$$

   a) Prove that M^2 is a regular and orientable surface in $\mathbb{R}^3$;

   b) Prove that M^2 has a closed geodesic;

   c) Compute the Gaussian curvature of M^2. Verify that it is everywhere non-positive;

   d) Determine the type (elliptic, hyperbolic or parabolic) of points of $M^2 \subset \mathbb{R}^3$;

   e) Use part c) to show that the anglesum of every geodesic triangle $ABC \subset M^2$ is strictly less than π.

see next page

3. $\boxed{10+10 = 20\text{pts}}$

a) Consider an $(n+1)$-dimensional affine space $\mathbb{K}^{n+1}$ over a field $\mathbb{K}$. Let P_1 and P_2 be two distinct hyperplanes in this space and let $f\colon P_1 \to P_2$ be a perspectivity with center $O \in \mathbb{K}^{n+1} \setminus (P_1 \cup P_2)$. Show that f is an affine map if and only if P_1 and P_2 are parallel. Prove that the linear part of f is then a scaling (i.e., has the form $x \mapsto \lambda x$ for some constant $\lambda \in \mathbb{K} \setminus \{0\}$).

b) Derive the general form of a projective transformation f of $P(\mathbb{K}^{n+1})$, where $\mathbb{K}^{n+1}$ is an $(n+1)$-dimensional vector space over $\mathbb{K}$. Show that there exist projective transformations that are not induced from a (single) perspectivity.

Hint for part b): First endow $P(\mathbb{K}^{n+1})$ with homogeneous coordinates $(x_1 : \ldots : x_{n+1})$. Then $P(\mathbb{K}^{n+1})$ can be viewed as the projective completion of the affine hyperplane $x_{n+1} = 1$ given by the set of points $\{(u_1 \ldots, u_n, 1) \mid (u_1, \ldots, u_n) \in \mathbb{K}^n\}$. Write the projective transformation f in these $(u_1, \ldots, u_n)$ coordinates.

End of exam