

Introduction to Image Processing



Lecturer:

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Week 4 – Morphological Filters

December 13, 2024

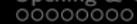
Morphological filtering



Dilation & Erosion



Opening & Closing



Grayscale morphology



Basics of morphological filters

Sets

Set-theory symbolics

union	\cup
intersection	\cap
set inclusion	\subseteq
element of	\in
universal set	E
empty set	\emptyset
set difference	$A \setminus B = \{a \in A : a \notin B\}$
complement	$A^c = E \setminus A$

Set Algebra

Recall the basic algebra rules for sets:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

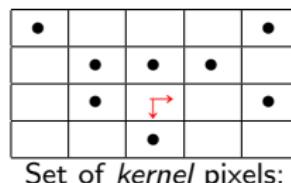
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

$$(A \subseteq B \wedge B \subseteq A) \Leftrightarrow A = B$$

Binary images as sets

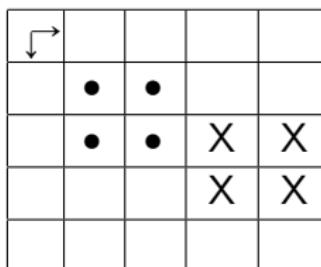


Set of *kernel* pixels:

$$\{(-2, -2), (2, -2), (-1, -1), (0, -1), (1, -1), (-1, 0), (2, 0), (0, 1)\}$$

Note: the origin $(0, 0)$ is denoted as $\downarrow\rightarrow$

Set translation



Set X (black dots) and its shift X_h (crosses) by $h = (2, 1)$.

$$X_h = \{x + h : x \in X\},$$

is the *translate* of X over the vector $h \in \mathbb{E}$. \mathbb{E} is the Euclidean space.

Notation

Let X, A be two sets, combined with a morphological operation op .

$$X \text{ op } A = \{x \text{ op } a : x \in X, a \in A\}$$

Then, the notation follows:

X : (binary) image

A : *structuring element (SE)*; kernel

op : morphological operation

Note: the size of the structuring element A can differ from the size of image X .

Morphological filtering
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Grayscale morphology
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Binary erosion- and dilation filters

Morphological filtering
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Dilation & Erosion
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Minkowski operations: addition

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$X \oplus A$ is the union of all translations of X by $a \in A$.

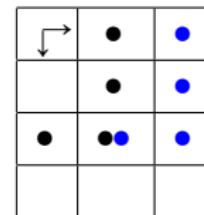
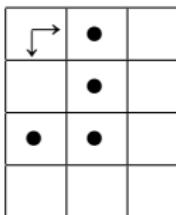
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(left) binary image $X = \{(0,2), (1,0), (1,1), (1,2)\}$

(middle) structuring element $A = \{(0,0), (1,0)\}$.

(right) $X \oplus A : x \in X, a \in A;$

black dots are $X + a_0$, blue dots are $X + a_1$.

Morphological filtering
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Grayscale morphology
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Minkowski operations: subtraction

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$X \ominus A$ is the intersection of all translations of X by $-a$, $a \in A$.

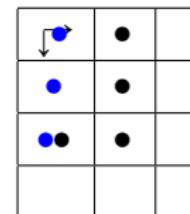
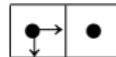
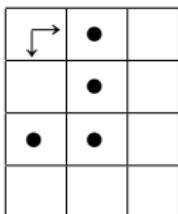
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(right) $X \ominus A : x \in X, a \in A;$

black dots are $X - a_0$, blue dots are $X - a_1$. The result includes only cells with *both* dots $\rightarrow X \ominus A = \{(0, 2)\}$.

Minkowski operations: dilation & erosion

- *Dilation* by structuring element A :

$$\delta_A(X) = X \oplus A$$

Minkowski operations: dilation & erosion

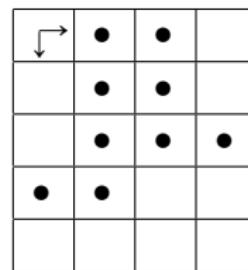
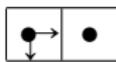
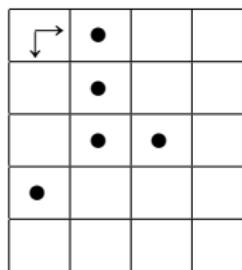
- *Dilation* by structuring element A :

$$\delta_A(X) = X \oplus A$$

- *Erosion* by structuring element A :

$$\varepsilon_A(X) = X \ominus A$$

Dilation: discrete case example



(left) binary image X .

(middle) structuring element A .

(right) dilation of X by A .

Image example: dilation

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.



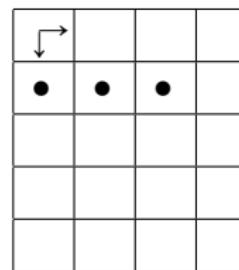
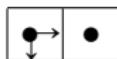
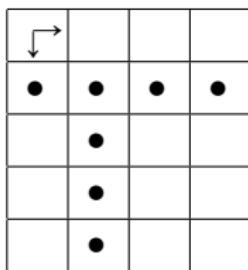
0	1	0
1	1	1
0	1	0

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.



(left) input. (right) dilation of input by 'cross' structuring element.

Erosion: discrete case example



(left) binary image X .

(middle) structuring element A .

(right) erosion of X by A .

Morphological filtering
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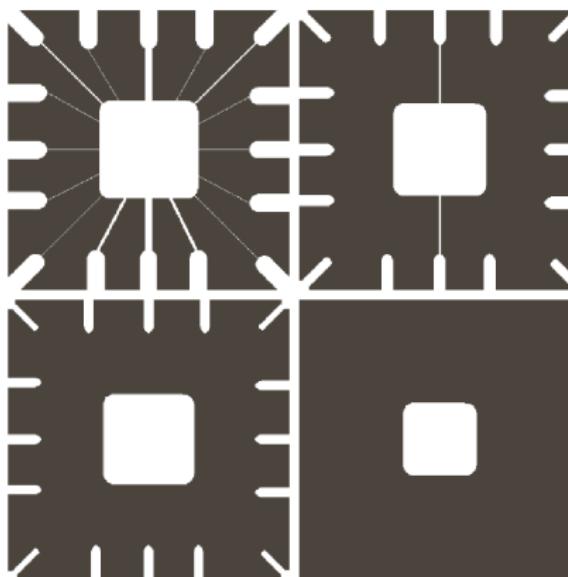
Dilation & Erosion
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Opening & Closing
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Grayscale morphology
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Image example: erosion

Erosion with *square* structuring element ($a \times b$ kernel, $x = 1 \forall x \in X$, center origin)



(top-left) Input. (top-right) erosion, kernel 11×11 .

(bottom-left) erosion, kernel 15×15 . (bottom-right) erosion, kernel 45×45 .

Erosion of a set by itself

$$X \ominus A = \{h \in E : A_h \subseteq X\}$$

As a consequence, the erosion of some set X using itself as structuring element equals the origin $(0, 0)$:

$$X \ominus X = \{h \in E : X_h \subseteq X\} = \{(0, 0)\}$$

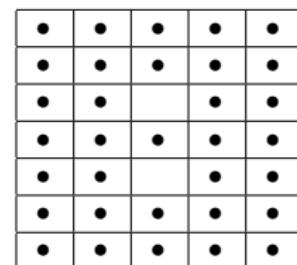
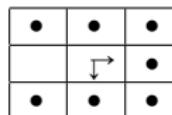
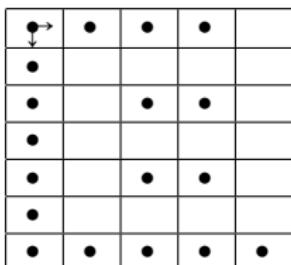
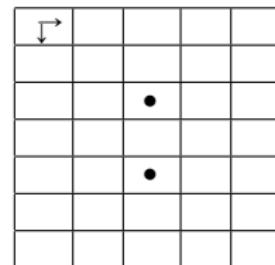
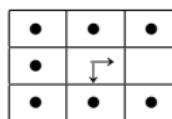
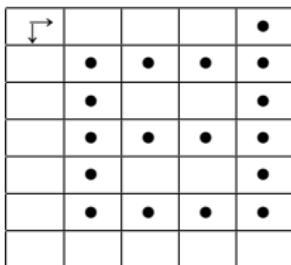
Duality w.r.t. set-complementation

Let X^c denote the complement of X , and $\overset{\vee}{A}$ the reflect (i.e. x-y axis mirror) of A . Then:

$$X \oplus A = (X^c \ominus \overset{\vee}{A})^c$$

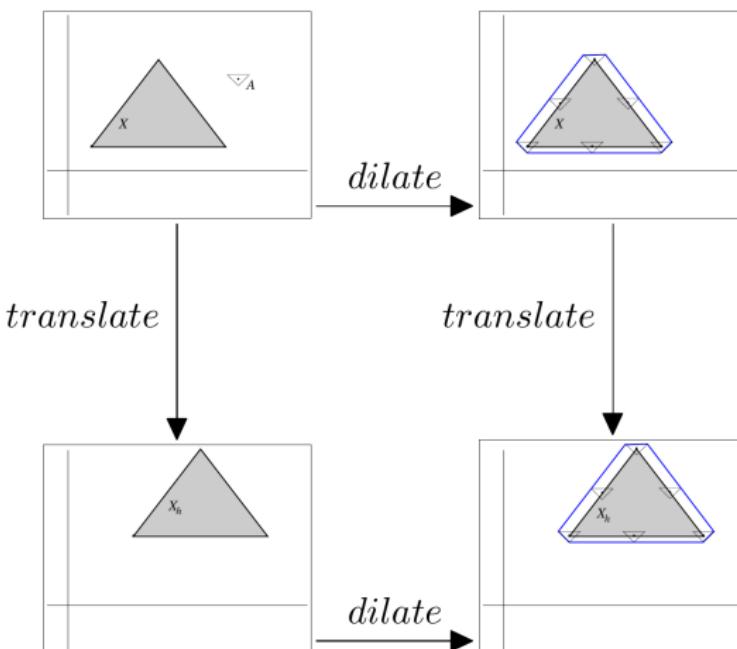
In words: *dilating* an image by A gives the same result as *eroding* the *background* by $\overset{\vee}{A}$ and taking the complement.

Duality: example for erosion



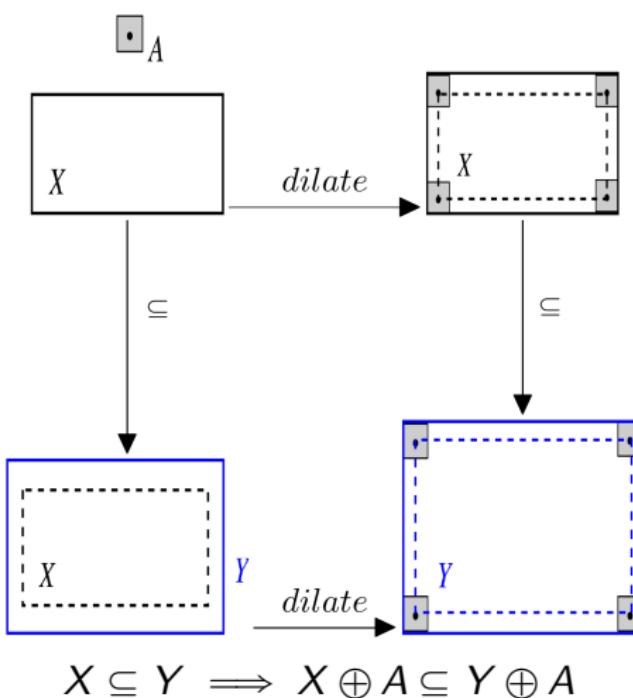
Top: (left) binary image X ; (middle) structuring element A ; (right) erosion of X by A .
 Bottom: (left) binary image X^c ; (middle) reflected structuring element $\overset{\vee}{A}$; (right)
 dilation of X^c by $\overset{\vee}{A}$.

Translation invariance



$$(X \oplus A)_h = X_h \oplus A$$

Scaling subsets



Boundary extraction via morphology

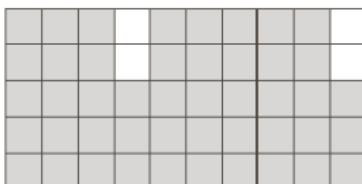
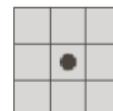
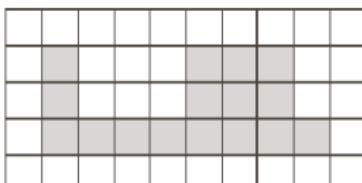
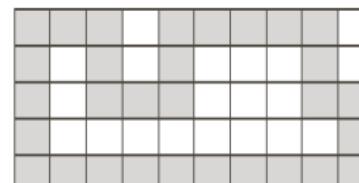
Let B be a structuring element containing the origin. The *morphological boundary* of A is defined by:

$$\beta(A) = A \setminus (A \ominus B)$$

Boundary extraction via morphology

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 A  B  $A \ominus B$  $\beta(A)$

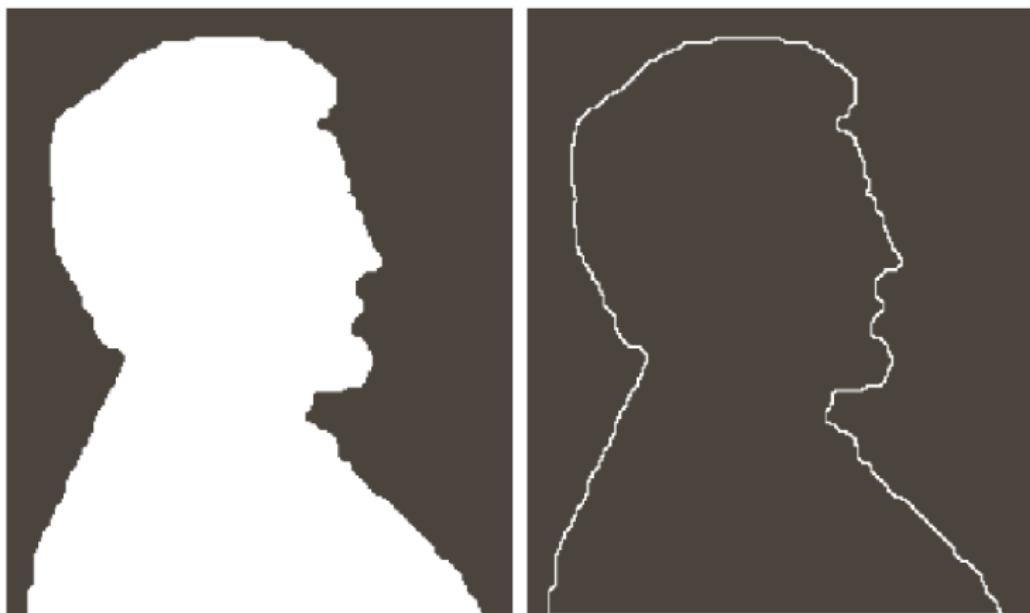
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Grayscale morphology
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Boundary extraction



Algebraic properties

$$X \oplus A = A \oplus X$$

$$(X \oplus A) \oplus B = X \oplus (A \oplus B)$$

commutativity

associativity

Algebraic properties

$$X \oplus A = A \oplus X$$

commutativity

$$(X \oplus A) \oplus B = X \oplus (A \oplus B)$$

associativity

$$(X \cup Y) \oplus A = (X \oplus A) \cup (Y \oplus A)$$

distributivity

$$(X \cap Y) \ominus A = (X \ominus A) \cap (Y \ominus A)$$

distributivity

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translation invariance

$$(X \oplus A)_h = X_h \oplus A$$

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$$(X \oplus A)_h = X_h \oplus A$$

translation invariance

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translation invariance

$$X \subseteq Y \implies X \oplus A \subseteq Y \oplus A$$

increasing

$$X \subseteq Y \implies X \ominus A \subseteq Y \ominus A$$

increasing

Algebraic properties

$$X \oplus A = A \oplus X$$

commutativity

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$$(X \oplus A)_h = X_h \oplus A$$

translation invariance

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translation invariance

$$X \subseteq Y \implies X \oplus A \subseteq Y \oplus A$$

increasing

$$X \subseteq Y \implies X \ominus A \subseteq Y \ominus A$$

increasing

$$(X \ominus A) \ominus B = X \ominus (A \oplus B)$$

iteration

Decomposition of a structuring element

$$X \oplus (A \oplus B) = (X \oplus A) \oplus B$$

Goal: decompose a structuring element C as a Minkowski sum of smaller sets.

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A structuring element B , in shape of a discrete *disc* (i.e. “cross”), consists of a center pixel and its 4-connected neighbors. Define the k -fold dilation of B with itself by

$$B_0 = B \quad \wedge \quad B_k = B \oplus B_{k-1}$$

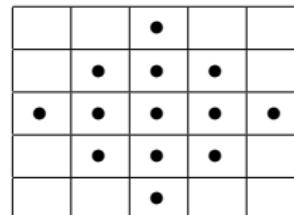
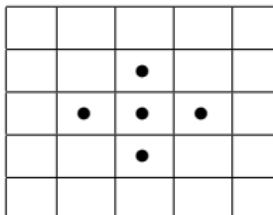
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Left: the set $B_0 = B$. Right: the set B_1 .

Decomposition of a structuring element

Let B_k be a *square* consisting of k rows by k columns.

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where L_r is a row of k 1-pixels,

$$L_r = \{(0, 0), (1, 0), (2, 0), \dots, (k - 1, 0)\}$$

and L_c is a column of k 1-pixels,

$$L_c = \{(0, 0), (0, 1), (0, 2), \dots, (0, k - 1)\}$$

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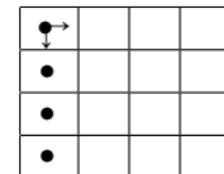
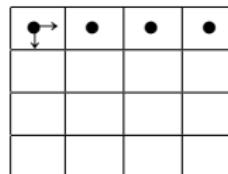
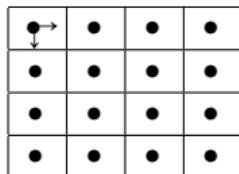
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Left: square B_1 . Middle: the set L_r . Right: the set L_c .

Morphological filtering
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Dilation & Erosion
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Opening & Closing
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Grayscale morphology
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Combined morphological filters - opening and closing

Opening

The *opening* γ_A is an erosion followed by a dilation.

$$\gamma_A(X) = X \circ A := (X \ominus A) \oplus A$$

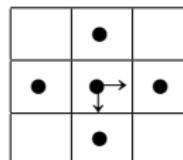
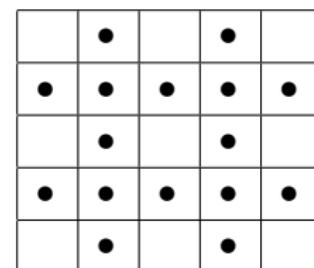
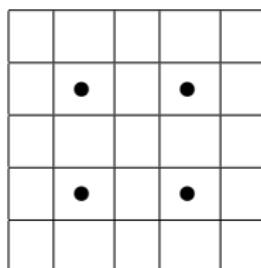
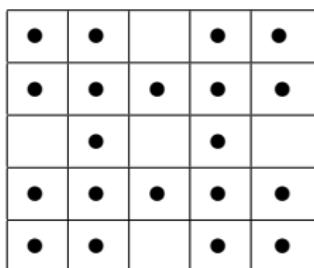
That is, $\gamma_A = \delta_A \varepsilon_A$.

Opening

The *opening* γ_A is an erosion followed by a dilation.

$$\gamma_A(X) = X \circ A := (X \ominus A) \oplus A$$

That is, $\gamma_A = \delta_A \varepsilon_A$.



- (a) image X . (b) $\delta_A = X \ominus A$. (c) $\gamma_A = \delta_A \oplus A$. (d) structuring element A .

Closing

The *closing* ϕ_A is a dilation followed by an erosion.

$$\phi_A(X) = X \bullet A := (X \oplus A) \ominus A$$

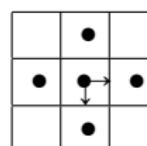
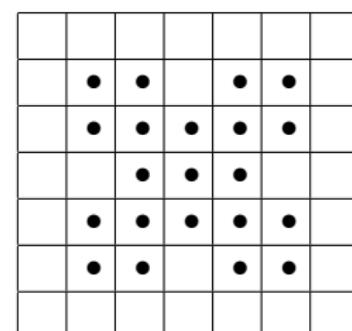
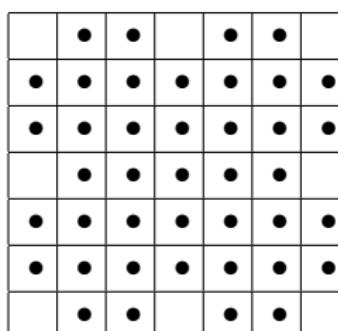
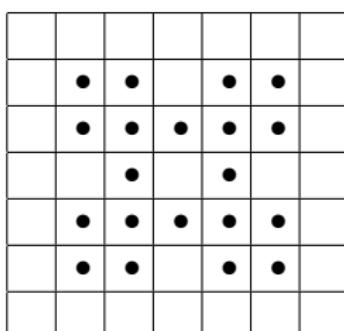
That is, $\phi_A = \varepsilon_A \delta_A$.

Closing

The *closing* ϕ_A is a dilation followed by an erosion.

$$\phi_A(X) = X \bullet A := (X \oplus A) \ominus A$$

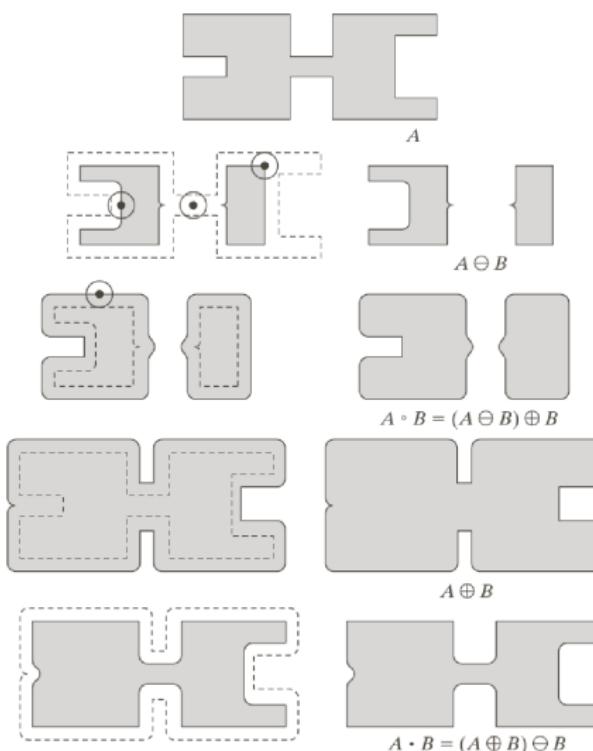
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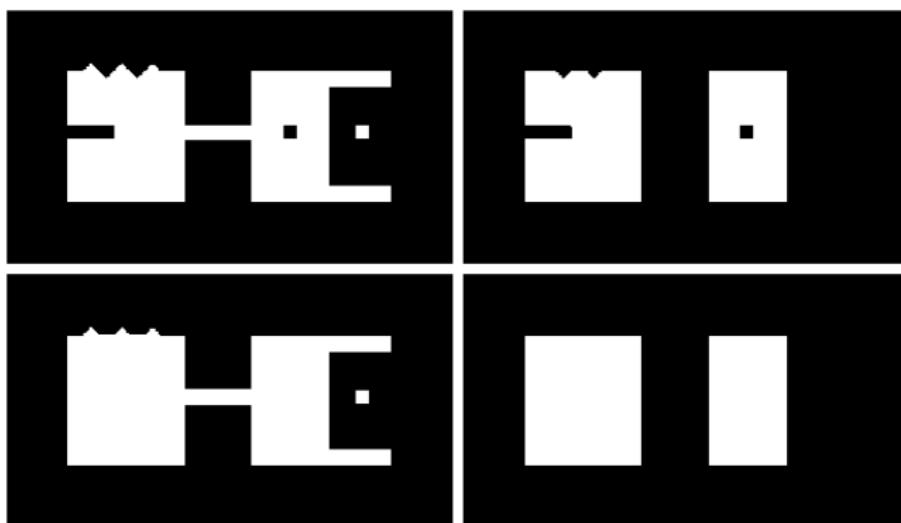
- (a) image X . (b) $\varepsilon_A = X \oplus A$. (c) $\phi_A = \varepsilon_A \ominus A$. (d) structuring element A .

Opening and closing - implications

- An opening smooths contours, cuts narrow bridges, removes small islands and sharp corners.
- A closing fills narrow channels and small holes.



Opening and closing: example

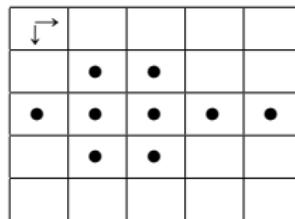
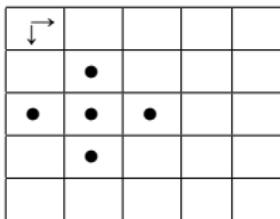
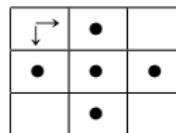
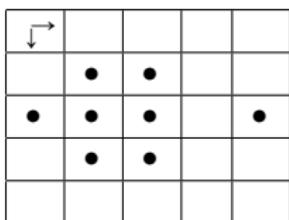


Upper left: Original. Upper right: Opening.

Lower left: Closing. Lower right: Closing of the opening.

The structuring element is a square.

Opening & Closing: discrete-case examples



Upper left: binary image X . Upper right: structuring element A .
Lower left: *opening* of X by A . Lower right: *closing* of X by A .

Morphological filtering
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Dilation & Erosion
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Opening & Closing
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Grayscale morphology
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Example



Properties of Openings and Closings

- The *opening* γ_A is:

Properties of Openings and Closings

- The *opening* γ_A is:
 - ① *increasing*: $X \subseteq Y \implies \gamma_A(X) \subseteq \gamma_A(Y)$

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Morphological filtering
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Dilation & Erosion
oooooooooooooooooooo

Opening & Closing
oooooooo

Grayscale morphology
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Morphological operations on scalar functions (1D) and images (2D)

Morphological filtering
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Dilation & Erosion
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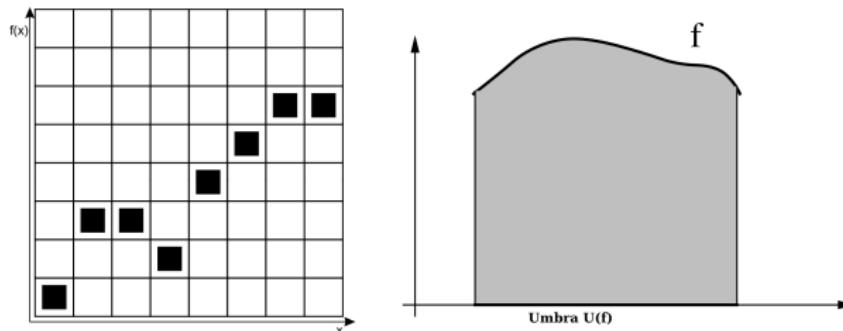
Opening & Closing
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Grayscale morphology
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Grayscale morphology - 1D

We start the explanation by looking at morphological filtering of *digitized 1D functions $f(x)$* .

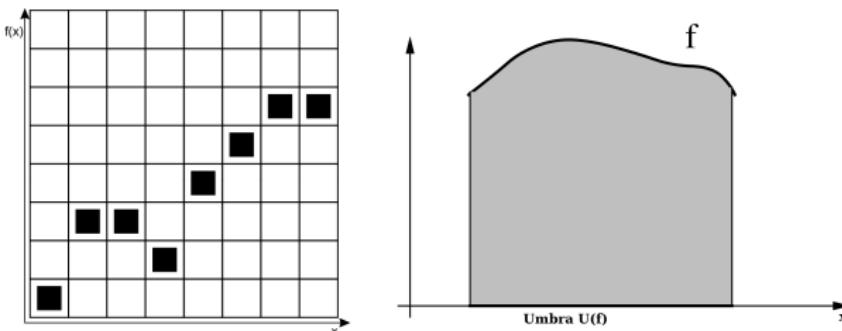
$$f = f(x) : x \in X$$



Grayscale morphology - 1D

We start the explanation by looking at morphological filtering of *digitized 1D functions $f(x)$* .

$$f = f(x) : x \in X$$



In order to dilate or erode the *curve*, we require the *integral* under the curve. This is what we call *the umbra*.

$$U(f) = \int f(x)$$

Morphological filtering
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Dilation & Erosion
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Opening & Closing
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Grayscale morphology
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The Umbra function

Umbra means shadow.



The Umbra function

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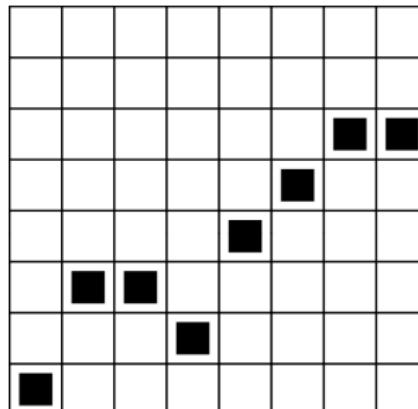
Umbra of a **set** X includes both X and the volume of units in its “shadow”.

Umbra function: 1D

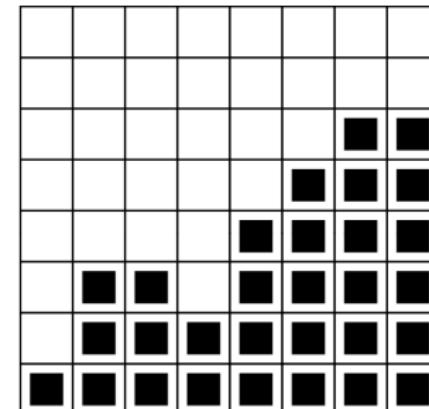
Formally, the umbra is a mapping $\mathbb{Z}^1 \rightarrow \mathbb{Z}^2$ of a 1D function $f(x)$ where

$$y_{max} = f(x)$$

$$U(f(x)) = \{(x, y) : y \leq y_{max}\}$$



f



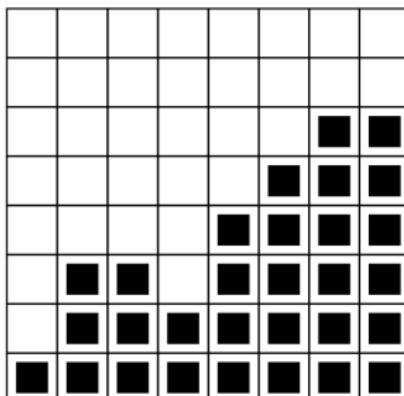
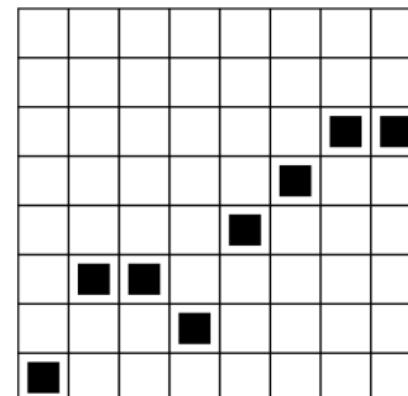
$U(f)$

Top function: 1D

Let R be set of samples in \mathbb{Z}^2 . Then the top function is a mapping $\mathbb{Z}^2 \rightarrow \mathbb{Z}^1$, where

$$R = f(x, y)$$

$$T(R) = \{y : f(x, y) > f(x, y + 1)\}$$

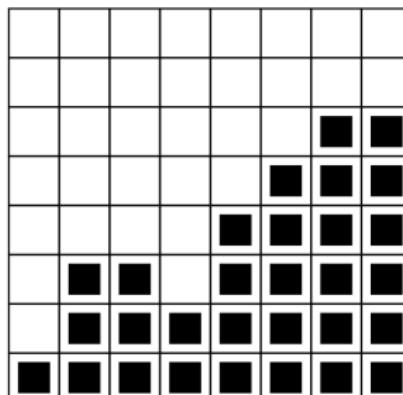
 R  $T(R)$

Top function: 1D

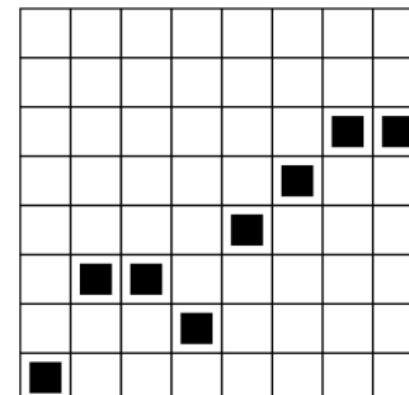
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R

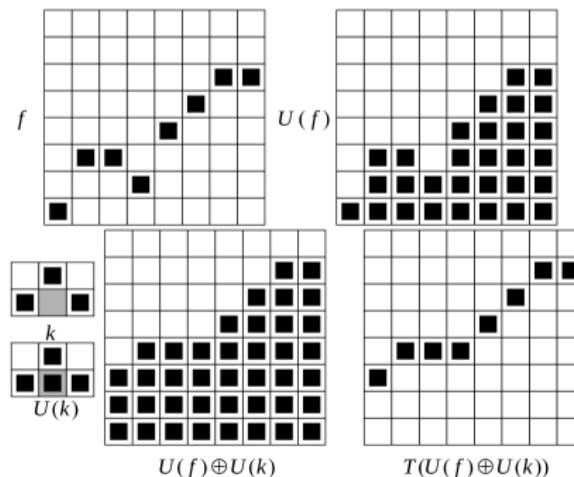


$T(R)$

So, we have $T(U(f)) = f$.

Gray level dilation: 1D

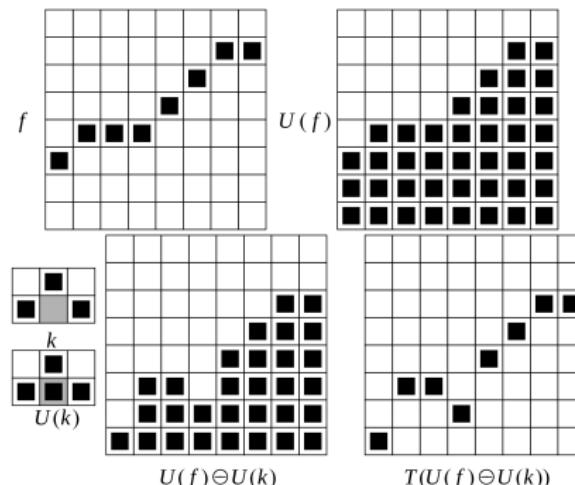
- grayscale dilation: $f \oplus_{gray} k = T(U(f) \oplus U(k))$



Note: kernel k above is meant as an illustration, not as a value equivalent.

Gray level erosion: 1D

- grayscale dilation: $f \oplus_{gray} k = T(U(f) \oplus U(k))$
- grayscale erosion: $f \ominus_{gray} k = T(U(f) \ominus U(k))$



Note: kernel k above is meant as an illustration, not as a value equivalent.

Gray level opening & closing: 1D

- grayscale dilation: $f \oplus_{gray} k = T(U(f) \oplus U(k))$
- grayscale erosion: $f \ominus_{gray} k = T(U(f) \ominus U(k))$
- grayscale opening/closing: as before using \oplus_{gray} and \ominus_{gray} .

From now on we simply write \oplus instead of \oplus_{gray} . The meaning of \oplus and \ominus depend on the context.

Gray level dilation/erosion: 1D

Let K be the domain of the function k .

- grayscale dilation:

$$(f \oplus_{gray} k)(x) = (T(U(f) \oplus U(k)))(x)$$

Gray level dilation/erosion: 1D

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$$(f \oplus_{gray} k)(x) = (T(U(f) \oplus U(k)))(x) = \max_{z \in K} (f(x + z) + k(z))$$

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Gray level dilation/erosion: 1D

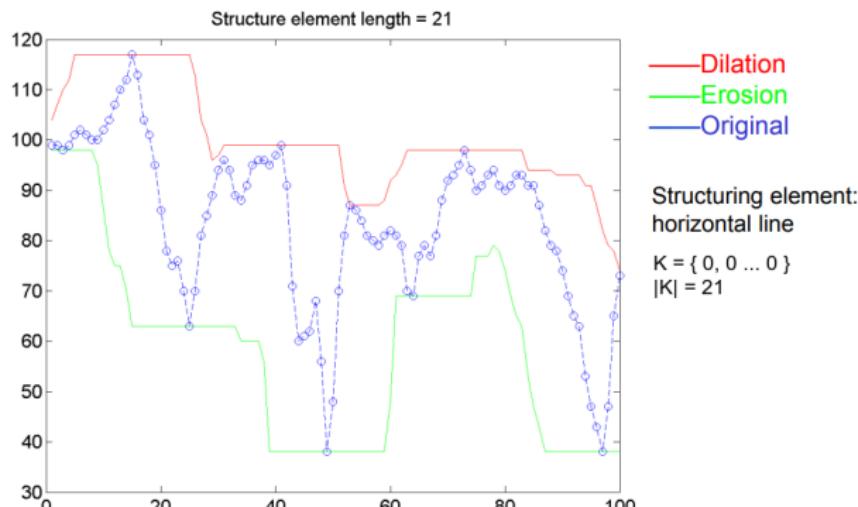
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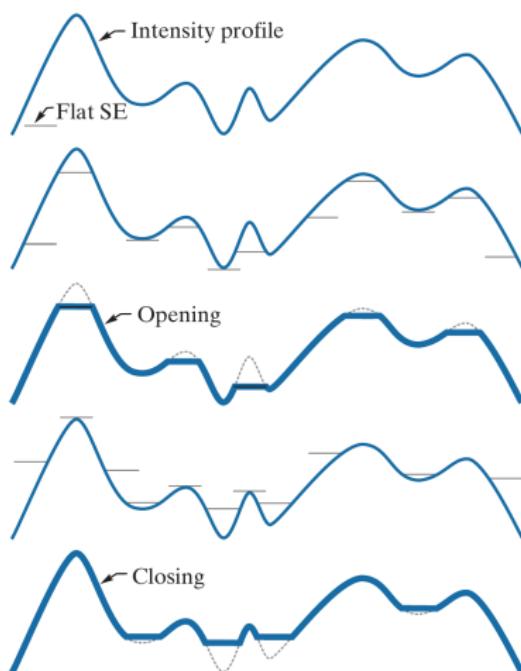
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Flat 1-D opening and closing



Scalar image dilation & erosion: 2D

Let $f : F \rightarrow G$ be a gray value image with domain F and $b : B \rightarrow G$ a structuring element with domain B . The gray value range is G .

Scalar image dilation & erosion: 2D

Let $f : F \rightarrow G$ be a gray value image with domain F and $b : B \rightarrow G$ a structuring element with domain B . The gray value range is G .

- The *gray value dilation* of f by b is defined as

$$(f \oplus b)(x, y) = \max_{(s,t) \in B} [f(x - s, y - t) + b(s, t)]$$

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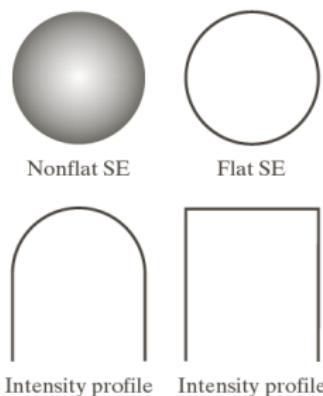
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- The *gray value erosion* of f by b is defined as

$$(f \ominus b)(x, y) = \min_{(s,t) \in B} [f(x + s, y + t) - b(s, t)]$$

Structuring functions: 2D



- For processing a gray value image $f(x, y)$ one can use another gray value image $b(x, y)$, called a **nonflat** structuring function (or element).
- If $b(x, y) = \text{constant}$ we call it a **flat** structuring function (or element).

Flat scalar image dilation & erosion: 2D

In practice, the function b is replaced by zero-constant $b : B \rightarrow \{0\}$, i.e. a flat structuring element with value 0. This results in:

- The *flat gray value dilation* of f by b is defined as

$$(f \oplus b)(x, y) = \max_{(s,t) \in B} [f(x - s, y - t)]$$

(*local maximum filter*)

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- associated with *gray value smoothing*, defined as

$$(\tilde{f}, b)(x, y) = \text{median}_{(s,t) \in B} [f(x + s, y + t)]$$

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- those three operations compose the *rank-order filters*

Morphological filtering
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Dilation & Erosion
oooooooooooooooooooo

Opening & Closing
oooooooo

Grayscale morphology
oooooooooooooooooooo●oooooooo

Flat Gray value dilation and erosion



Original



Dilation



Erosion

Gray value opening and closing

Now that we have defined grayscale dilation and erosion, operators like grayscale opening and closing can be defined as before:

Gray value opening and closing

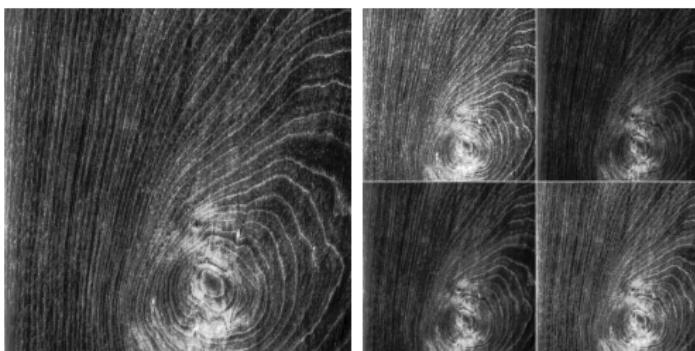
Now that we have defined grayscale dilation and erosion, operators like grayscale opening and closing can be defined as before:

- The *gray value opening* of f by b is the function $f \circ b$ defined by $f \circ b = (f \ominus b) \oplus b$
- The *gray value closing* of f by b is the function $f \bullet b$ defined by $f \bullet b = (f \oplus b) \ominus b$

Gray value opening and closing

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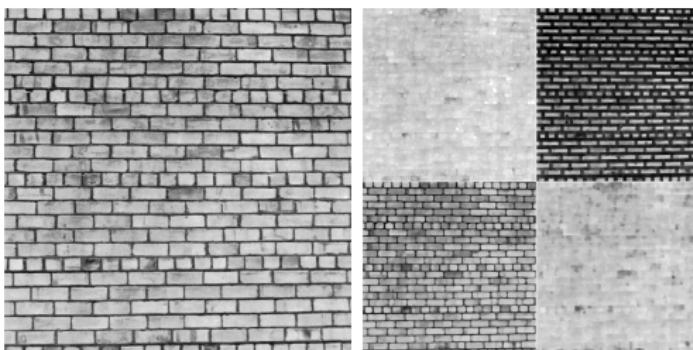


Left: a gray value image. Right: upper left: gray value dilation; upper right: gray value erosion; lower left: gray value opening; lower right: gray value closing. (*flat structuring element*)

Gray value opening and closing

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Left: a gray value image. Right: upper left: gray value dilation; upper right: gray value erosion; lower left: gray value opening; lower right: gray value closing. (*flat structuring element*)

Morphological filtering
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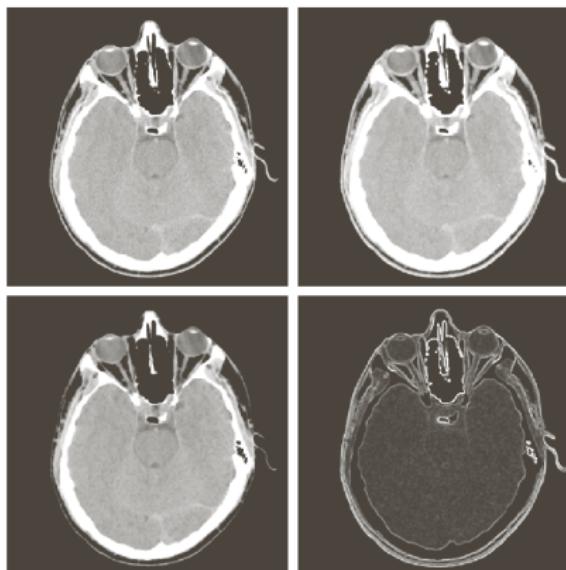
Dilation & Erosion
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Opening & Closing
oooooooo

Grayscale morphology
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Morphological gradient

$$g = (f \oplus b) - (f \ominus b)$$



(a) CT image. (b): Dilation. (c) Erosion. (d) Morphological gradient.

Morphological filtering
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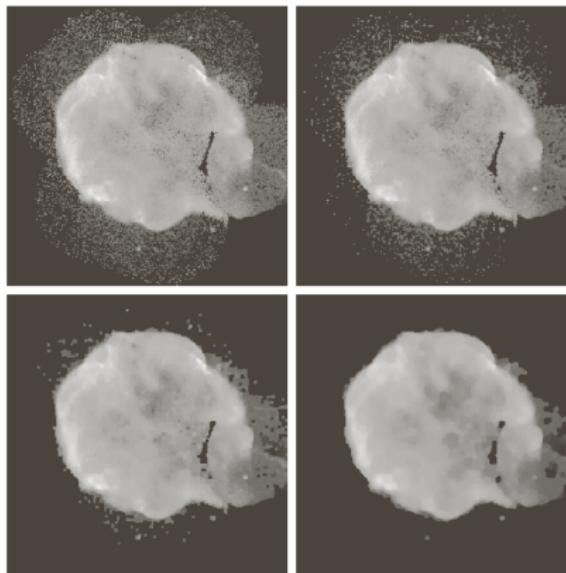
Dilation & Erosion
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Opening & Closing
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Grayscale morphology
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Morphological smoothing

$$g = (f \circ b) \bullet b$$



(a) Image of supernova.

(b)-(d): opening-closing of (a) with flat disk of radius 2, 3, 5.

Top/bottom-hat filter

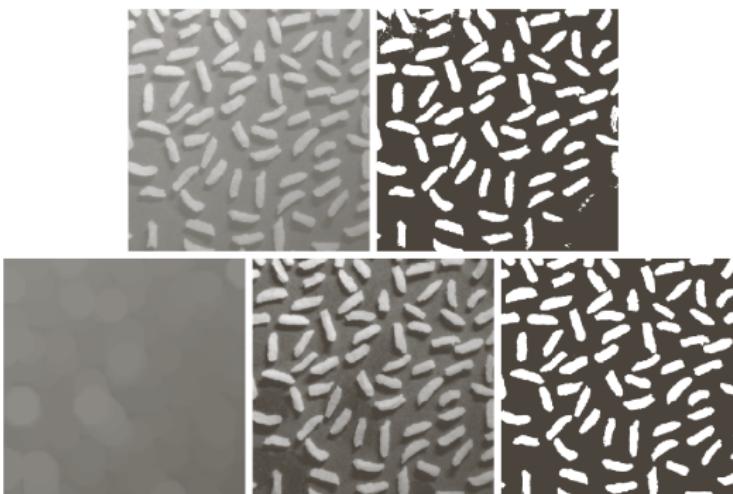
- *Top-hat filter.* Subtract the opening of f from f : $T_{\text{hat}} = f - (f \circ b)$

Top/bottom-hat filter

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Top/bottom-hat filter

- *Top-hat filter.* Subtract the opening of f from f : $T_{\text{hat}} = f - (f \circ b)$
- *Bottom-hat filter.* Subtract f from the closing of f : $B_{\text{hat}} = (f \bullet b) - f$
- Useful for correcting uneven illumination.



(a) Input. (b) Thresholded image. (c) Opening of (a) by disk of radius 40.
(d) Top-hat of (a). (e) Thresholded top-hat.

Morphological filtering
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Dilation & Erosion
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Opening & Closing
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Grayscale morphology
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That's it for this week!

