Lecture 15

Euclidean Isometries:

Thm: Every isometry $f: \mathbb{R}^n \to \mathbb{R}^n$ has the form $f(x) = Ax + B$ where A is an orthogonal matrix, i.e $AA^T = E$.

identity matrix (same as I)

Proof/

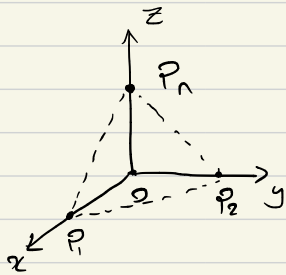
$$\langle f(x)-f(y), f(x)-f(y) \rangle_{eucl} = \langle x-y, x-y \rangle_{eucl}$$

Take an isometry $g: \mathbb{R}^n \to \mathbb{R}^n$

s.t. $g \circ f(o) = o$

$g \circ f(P_i) = P_i$,

where $o, P_1, \ldots, P_n$ form an affine Euclidean frame.



If $f(o) = B$ then

take $g_1 = x - \overrightarrow{OB}$

then $g \circ f(o) = g(B) = B - \overrightarrow{OB}$

$= B + \overrightarrow{BO} = o$

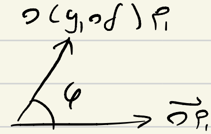
$g = g_{n+1} \circ g_n \cdots g_2 \circ g_1$

with such choice of g_i

$g \circ f \overset{?}{=} id \Rightarrow f = g^{-1}$

to prove

g_2 is a rotation that aligns the vector $\overrightarrow{o(g_1 \circ f)(P_1)}$ with $\overrightarrow{OP_1}$



To prove $\underline{g \circ f = id_n}$

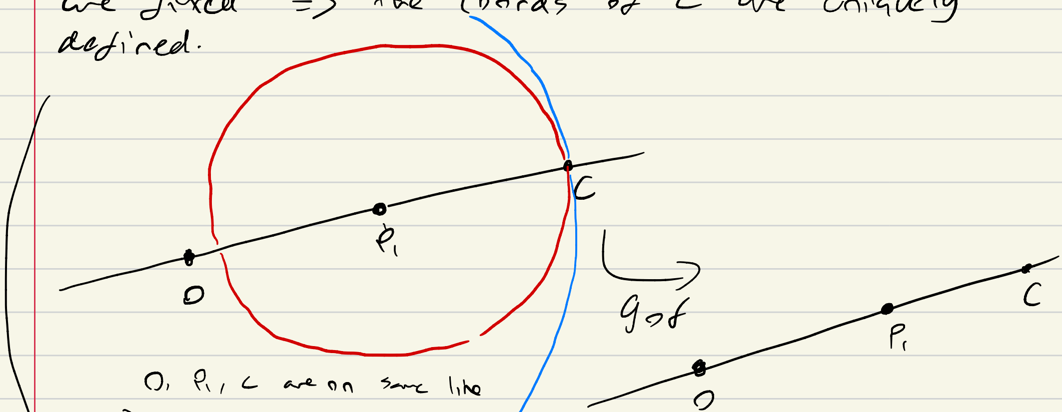
recall the fundamental theorem of affine geometry. Stating that if

$g \circ f$ maps 3 collinear pts. to collinear pts.

$\Rightarrow$ $g \circ f$ is affine, which would imply $g \circ f$ is linear since $g \circ f(0) = 0$. Then $g \circ f$ must be identity because it acts trivially on the affine frame.

It's left to show collinear points are mapped to collinear points.

The proof is by the property that a given pair of points O and P_1 and another pt. $C \in \mathbb{R}^n$, if O, C, P_1 are collinear and $d(C,O), d(C,P)$ are fixed $\Rightarrow$ the coords of C are uniquely defined.



O, P_1, C are on same line

$\Rightarrow \underset{\parallel}{g \circ f(O)}, \underset{\parallel}{g \circ f(P_1)}, \underset{\parallel}{g \circ f(C)}$

$\quad\;\; O \qquad\quad P_1 \qquad\quad C$

are on the same line also.

$\Rightarrow$ O, $g \circ f(C)$, P_1 are also collinear.

If $g_i \circ f(0) = 0$,

$\overrightarrow{0\, g_i \circ f(p_i)} = \vec{a}_i$

$$A = \begin{pmatrix} a^1_1 & \cdots & a^1_n \\ \vdots & & \vdots \\ a^n_1 & \cdots & a^n_n \end{pmatrix} \Rightarrow A(\overrightarrow{0p_i}) = \vec{a}_i$$

Reflections and Inversions:

The case of $\mathbb{R}^2 \approx \mathbb{C}$

$$\begin{array}{ccc} \mathbb{R}^2 & \approx & \mathbb{C} \\ \Psi & & \Psi \\ (u,v) & & z_1 + iz_2 \end{array}$$

$z_1 = u$
$z_2 = v$

Say you have a line in $\mathbb{R}^2$.

$$\underbrace{B\bar{z} + \bar{B}z}_{} + \underset{\mathbb{R}}{\underset{\cap}{C}} = 0$$

defines a real line in $\mathbb{C}$.

$(b_1 + ib_2)(z_1 - iz_2)$
$+ (b_1 - ib_2)(z_1 + iz_2)$
$= 2(b_1 z_1 + b_2 z_2)$

$$\underbrace{2\vec{B}\cdot\vec{z} + C}_{} = 0$$

$b_1 z_1 + b_2 z_2 + C = 0$

$\hookrightarrow$ looks like $ax + bx + c = 0$ $\therefore$ line.

An equation for a circle is similar:

$A|z|^2 + B\bar{z} + C = 0$

$\underline{A, C \in \mathbb{R}}$

$$-\frac{C}{A} + \frac{|B|^2}{A^2} = \frac{|B|^2 - CA}{A^2} = r^2$$

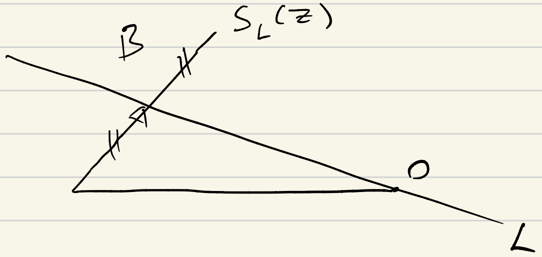
$$\Rightarrow \text{radius } r = \frac{\sqrt{|B|^2 - CA}}{|A|} > 0$$

when radius $\to \infty$ becomes equation of a line.

$\|z - a\|^2 = r^2$
$\langle z-a, z-a\rangle = r^2$
$(z-a)(\bar{z}-\bar{a}) = r^2$
$\vdots$
$A|z|^2 + \bar{B}z + B\bar{z}$
$+ A(|a|^2 + r^2)$
$= 0$

Reflection:

$B \perp L$



$$2\vec{B}\cdot\vec{z} = B\bar{z} + \bar{B}z$$

$$S_L(z) = z - \frac{2\cdot\vec{B}\cdot\vec{z}}{|\vec{B}|}\cdot\frac{\vec{B}}{|\vec{B}|}$$

$$= \cancel{z} - \frac{B\bar{z} + \cancel{\bar{B}z}}{|B|^2}B$$

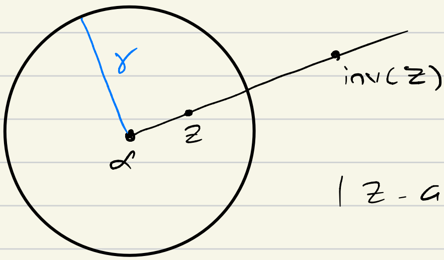
$$= -\frac{B}{\bar{B}}\bar{z}$$

In general (where O may not lay on L)

$$\boxed{S_L(z) = -\frac{B}{\bar{B}}\bar{z} - \frac{C}{\bar{B}},\quad C \in \mathbb{R}}$$

Reflection in a line

Inversions: (reflection in a circle)



$$|z - a| \cdot |inv(z) - a| = \gamma^2$$

$$inv(z) = a + \frac{z-a}{|z-a|} \cdot |inv(z) - a|$$

$$= a + \frac{(z-a)\gamma^2}{|z-a|}$$

$$= \frac{a\cancel{(z-a)}(\bar{z}-\bar{a}) + \cancel{(z-a)}\gamma^2}{\cancel{(z-a)}(\bar{z}-a)}$$

$$= \frac{A(a(\bar{z}-\bar{a}) + z^2)}{A(\bar{z}-\bar{a})}$$

$$= \frac{Aa\bar{z} + A(z^2 - |a|^2)}{A\bar{z} - A\bar{a}}$$

$$inv(z) \boxed{= \frac{B\bar{z} + C}{A\bar{z} - \bar{B}}}$$ inversion on a circle

If $a = 0 \Rightarrow inv(z) = \frac{\gamma^2}{\bar{z}} = \frac{C}{A\bar{z}}$

If we set $A = 0$ we get general equation of reflection for a line.

Theorem: An isometry of $\mathbb{C} \approx \mathbb{R}^2$ can be written as composition of ≤ 3 reflections in lines.

More generally an isometry of $\mathbb{R}^n$ (Eucl. metric) is a composition of $\leq n+1$ reflections in hyperplanes.