Geo Summary

Wednesday, 5 April 2023 8:44 PM

Week 1

Smt about how splitting geodesics are clear / vertices, edges, faces

Def E v.s. , $\mathcal{E}$ set and $f: \mathcal{E}\times\mathcal{E} \longrightarrow E$, $f(A,B) = \overrightarrow{AB}$ s.t.

① $\forall A \in \mathcal{E}$, $f(A,\cdot): \mathcal{E} \to E$ bijection

② $\forall A,B,C \in \mathcal{E}$, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\Rightarrow \mathcal{E}$ is an affine space.

Def Curve $\gamma: [\alpha,\beta] \to \mathbb{R}^n$ smooth if $\gamma = (x^1(t), \dots, x^n(t))$ smooth

→ smooth map is called regular if $\forall\, t \in [\alpha,\beta]$, $\gamma'(t) \neq 0$

Def Let D_t, $t \in [\alpha,\beta]$ family of lines in $\mathbb{R}^2$.

Envelope is a curve $\gamma = \gamma(\tau)$ s.t. $\forall \tau\ \exists\, t(\tau)$ with $\gamma(\tau) \in D_t(\tau)$ and $\dot\gamma(\tau) \parallel D_t(\tau)$

↳ $\gamma = \gamma(\tau)$ is tangent to D_t, $t \in [\alpha,\beta]$

→ To find an evolute.

$D_t := u(t)x + v(t)y + w(t) = 0$, solve:

$$\begin{pmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} w(t) \\ w'(t) \end{pmatrix} = 0$$

→ worked examples for families of cycloids

Week 2

Def Arc-length: $I: [\alpha,\beta] \to \mathbb{R}^n$ smooth & at least C^1.

↳ $s = s(t) := \int_\alpha^t \left|\frac{d\gamma}{dt}\right| dt$

Prop If γ regular curve $\Rightarrow$ $s(t)$ smooth & strictly monotone (& also inverse $t = t(s)$)

$\Rightarrow |\gamma'(s)| = 1$

Def Curvature: $\gamma: I \to \mathbb{R}^2$ smooth & regular. If γ param by arc-length $\gamma = \gamma(s)$

$\Rightarrow \gamma''(s) = k(s)\, n(s)$

$(\tau(s) = \gamma'(s), n(s))$ gives a +ve basis wrt some fixed orientation of $\mathbb{R}^2$.

Prop (Frenet formulas in dim=2)

$$\left.\begin{aligned} \tau'(s) &= k(s)n(s) \\ n'(s) &= -k(s)\tau(s) \end{aligned}\right\} \quad \begin{pmatrix} \tau'(s) \\ n'(s) \end{pmatrix} = \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix}\begin{pmatrix} \tau(s) \\ n(s) \end{pmatrix}$$

→ n-dim case is in lecture notes

→ $k_i = k_i(s)$ for $i \in \{2, \dots, m-1\}$ are called torsions

Prop for a regular curve $\gamma: I \to \mathbb{R}^3$

→ Curvature → $k(t) = \dfrac{\|\dot\gamma \times \ddot\gamma\|}{\|\dot\gamma\|}$

→ Torsion → $\varkappa(t) = \dfrac{\det(\dot\gamma, \ddot\gamma, \dddot\gamma)}{\|\dot\gamma \times \ddot\gamma\|^2}$

If γ plane curve → $k(t) = \dfrac{\dot x\ddot y - \ddot x \dot y}{\ldots}$

$\|\dot\gamma \times \ddot\gamma\|^2$

If γ plane curve $\to$ $k(t) = \dfrac{\dot x \ddot y - \ddot x \dot y}{(\dot x^2 + \dot y^2)^{3/2}}$

Def $\gamma : I \to \mathbb{R}^2$ smooth and $t_0 \in I$ s.t. $\gamma'(t_0) \neq 0 \Rightarrow s(t_0) = 1/k(t_0)$ curvature radius

$\to$ $c(t_0) = \gamma(t_0) + s(t_0)\vec n(t_0)$ $\leftarrow$ curvature center

$\to$ circle with center $c(t_0)$ and radius $|s(t_0)|$ is osculating circle.

Def $\gamma : I \to \mathbb{R}^2$ smooth & regular curve. The evolute of γ is the envelope of family D_t of normal lines D_t to γ: $\gamma(t) \in D_t$, $D_t \parallel \vec n(t)$

Prop Let $\gamma : I \to \mathbb{R}^2$ be a smooth curve of C^2 class at least for which the curvature never vanishes. The evoute of γ is the curve $C = C(s)$ of curvature centers of γ.

$\to$ Formula (evolute)

$$C(t) = (x(t), y(t)) + \frac{1}{k(t)} \times \frac{(-\dot y(t), \dot x(t))}{\sqrt{\dot x(t)^2 + \dot y(t)^2}} = (x(t), y(t)) + \frac{\dot x^2 + \dot y^2}{\dot x \ddot y - \ddot x \dot y}(-\dot y(t), \dot x(t))$$

Thm Huygen's principle

- γ smooth C^2 curve in $\mathbb{R}^2$. Assume curvature never vanishes.

Wavefronts are the family of parallel curves s.t. $\gamma_a = \gamma(s) + a\vec n(s)$

Then the set $\{\gamma_a(s) \mid \gamma_a'(s) = 0, a \in \mathbb{R}\}$ of sg. pts. of wavefronts $\gamma_a(s)$ is given by the evolute of γ.

$\to$ **Week 3 + 4**

$\to$ Regular curves in $\mathbb{R}^2$

- Param equ: $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $t \in I$ s.t. $\|(\dot x, \dot y)\| \neq 0 \ \forall t \in I$
- Implicit equ: $S = \{(x,y) : f(x,y) = 0\}$ w/ $\operatorname{grad} f|_{(x,y)} \neq 0$
- Graph: $y = g(x)$

$\to$ Regular hypersurfaces in $\mathbb{R}^n$

- Param equ. $\begin{cases} x^1 = x^1(u^1, \dots, u^{n-1}) \\ \vdots \\ x^n = x^n(u^1, \dots, u^{n-1}) \end{cases}$ s.t. $\operatorname{rank}\left(\dfrac{\partial x^i}{\partial u^j}\right) = n-1$ $\leftarrow$ equ. that $\dfrac{\partial x^1}{\partial u^j}, \dots, \dfrac{\partial x^n}{\partial u^j}$ are lin. indepn. vectors.
- Implicit equ: $S = \{(x^1, \dots, x^n) : f(x^1 \dots x^n) = 0\}$ where $\operatorname{grad} f|_S \neq 0$
- Graph: $x^i = g(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n)$

Def A quartic in $\mathbb{R}^n$ is equ. class of polynom. of deg 2:

$P(x^1, \dots, x^n) = \sum_{i,j=1}^n a_{ij} x^i x^j + \sum_{j=1}^n b_j x^j + c$

$\to$ Polynom P, Q equiv $\Leftrightarrow$ $P = \lambda Q$ for $\lambda \neq 0$

Rem A quartic is the locus of pts $\{(x^1, \dots, x^n) : Q(x^1, \dots, x^n) = 0\}$

$\to$ Classifications of quadrics (up to isometries of affine Euclidean space $\mathbb{R}^n$.)

$\to$ dim $n = 2$ (regular curves)

- ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

- ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- Parabola $y = \frac{x^2}{a}$
- Cone $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ (singular set)

→ $n=3$

(regular surfaces)

- ellipsoid. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- 1-sheeted hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- 2-sheeted hyperboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- elliptic paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
- hyperbolic paraboloid $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$
- Elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- Hyperbolic cylinder $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- Parabolic cylinder $z = \frac{x^2}{a^2}$

(Singular surfaces)

- cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$
- Cylinder over 1-cone $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

for Christoffel symbols $g^{im} \in g^{-1}$ & summed up till m.

Def **Tangent spaces**

Let $M \subset \mathbb{R}^n$ be a regular hypersurface in $\mathbb{R}^n$

T_pM is a point $p \in M$ s.t. :

① T_pM is the hyperplane of $\mathbb{R}^n$ orthogonal to $\vec{n}(p)$ & passing through p.

(There are other def. in the notes)

Def **Riemannian metric**

$G(p) : T_pM \times T_pM$

$$\underset{\vec{r}(u^1,\dots,u^{n-1})}{\overset{G(p)}{\parallel}} = \left(g_{ij}(u^1,\dots,u^{n-1})\right)$$

where $g_{ij}(u^1,\dots u^{n-1}) = \langle \vec{r}_i , \vec{r}_j \rangle$

where $\vec{r}_i := \frac{d\vec{r}}{du^i}$

Def $\vec{n} : M^{n-1} \longrightarrow S^{n-1}$ is a **Gauss map**

regular hypersurface

$\det(d|_p \vec{n}) =: K(p)$ is **Gaussian curvature** of M^{n-1} at p

Def $\vec{n}: \overset{\text{regular}}{M^{n-1}} \longrightarrow S^{n-1}$ is a Gauss map

$\det(d_p \vec{n}) =: K(p)$ is Gaussian curvature of M^{n-1} at p

Def 2nd fundamental form $\mathbb{I} := \mathbb{I}(p)(\underbrace{X,Y}_{\in T_pM^{n-1}}) = -\langle d_p\vec{n}(X), Y\rangle$

$$Q_{ij}(p) := \mathbb{I}\left(\vec{r}_i(p), \vec{r}_j(p)\right) = \left\langle \frac{\partial^2 \vec{r}}{\partial u_i u_j}\Big|_p, \vec{n}(p)\right\rangle \leftarrow \text{symmetric}$$

→ Gaussian curvature: $K(p) = (-1)^{n-1} \dfrac{\det(Q_{ij}(p))}{\det(G_{ij}(p))}$

→ for dim $n=3$: let $G = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$, $Q = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$, $L = \dfrac{\partial^2 \vec{r}}{(\partial u^1)^2}$, $M = \dfrac{\partial^2 \vec{r}}{\partial u^1 \partial u^2}$, $N = \dfrac{\partial^2 \vec{r}}{(\partial u^2)^2}$

Thus we get: $K(p) = \dfrac{NN - M^2}{EG - F^2}$

Def Mean curvature $\dfrac{1}{n-1} \times$ trace of $d_p\vec{n} = -G^{-1}\,\mathbb{I}(p)$. Denoted by $H(p)$

→ 0 mean curvature = surface locally minimizes area/volume

→ 0 Gaussian curvature = locally flat

Def $X \in T_pM^{n-1}$ and π the 2-plane in $\mathbb{R}^n$ spanned by vectors X and $\vec{n}(p)$.

Curvature K_X of the plain curve $\gamma = \pi \cap M^{n-1}$ is the normal curvature in direction X

→ $K_X = \mathbb{I}(p)\left(X/\|X\|, X/\|X\|\right)$

Def $(e_1, \dots, e_{n-1})$ ONB for T_pM^{n-1} s.t. $Q_{ij}(p)$ is diagonal: $(Q_{ij}(p)) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$K_{e_i} = \lambda_i$ are called principal curvature

Rem Gaussian curvature $K(p) = \lambda_1 \cdot \ldots \cdot \lambda_{n-1}$ product of principal curvatures $\lambda_k, k = 1, \dots, n-1$

and mean curvature $H(p) = \frac{1}{n-1}(\lambda_1 + \dots + \lambda_{n-1})$ is $(H(p) + K(p))/(n-1)$

Thm Theorema Egregium: Gaussian curvature of regular surface $M^2 \subset \mathbb{R}^3$ is an intrinsic invar of M^2

⟶ Gaussian curvature depends only on Riemannian metric

Cor Gaussian curvature invar. under isometries

→ If $f: M^2 \to N^2$ smooth s.t. $f^*(G_N) = G_M \Leftrightarrow G_M(X,Y) = G_N(df(X), df(Y))$, where G_M = Riem. metric on M^2, G_N = Riem. metric on N^2

$\Rightarrow K_N \circ f = K_M$

Def Let $p \in M^2$. By applying an isometry of $\mathbb{R}^3$ if necessary, we can assume that $\underbrace{T_pM^2 \perp \vec{n}(p) = (0,0,1)}$.

↳ same as $\dfrac{\partial z}{\partial x}\Big|_p = \dfrac{\partial z}{\partial y}\Big|_p = 0$ (p singular for $z = z(x,y)$). p is:

- elliptic if $\det(D^2_p z) > 0$
- hyperbolic if $\det(D^2_p z) < 0$
- Parabolic if $\det(D^2_p z) = 0$ but $D^2_p z \neq 0$

Prop
- If Gaussian curvature $K(p) > 0 \Rightarrow p$ elliptic
- If $K(p) < 0 \Rightarrow p$ hyperbolic

Def $M^{n-1} \subset \mathbb{R}^n$ regular surface with Riemannian metric. A geodesic on M^{n-1} is a smooth curve $\gamma: I \to M^{n-1}$ parametrised by a constant multiple of arc length that locally realizes distance function $d(p,q) := \inf_{\alpha} \int_{t_1}^{t_2} \sqrt{g_{ij}(\gamma(t))\,\dot\gamma^i(t)\,\dot\gamma^j(t)}\,dt$

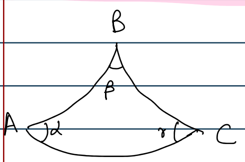
↖ C^1 smooth curves on M^{n-1} s.t. $\gamma(t_1) = p$, $\gamma(t_2) = q$

Thm Let M^{n-1} param by $(u^1, \dots, u^{n-1})$. If $\gamma: I \to M^{n-1}$ geodesic, it satisfies:

$$\ddot u^i + \Gamma^i_{jk}\dot u^j \dot u^k = 0 \quad ; \quad \Gamma^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\partial g_{js}}{\partial u^k} + \frac{\partial g_{ks}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^s}\right)$$ ← Christoffel symbols

→ Week 5

Thm (Local) Gauss Bonnet: M^2 regular surface. $K(p)$ Gaussian curvature. Geodesic Δ:



Then $\int K\,dS = \alpha + \beta + \gamma - \pi$ & $dS = \sqrt{\det G}\,du^1 \wedge du^2$

↑ area element of M^2

Cor Angle-sum of a geodesic ΔABC:
- for $\mathbb{H}^2$ $< \pi$
- for $\mathbb{R}^2$ $= 0$
- for S^2 $> \pi$

Def Affine subspaces. A subset $\mathcal{F}$ of an affine space $\mathcal{E}$ is an affine subspace if for some point $A \in \mathcal{F}$ the space $F = \{\vec{AB}, B \in \mathcal{F}\}$ is a vector subspace of E. $\mathcal{F}$ is directed by F. (Remember $\vec{f}: \mathcal{E} \times \mathcal{E} \to E$)

Prop Let $A \in \mathcal{E}$ and F be a VS of E. Then there is a unique affine subspace of $\mathcal{E}$ directed by F & passing through A

→ Two lines in affine space are parallel if they have the same direction

↳ $\mathcal{F}_i = A_i + F$ for $i = 1, 2$

Def $\mathcal{E}$ & $\mathcal{F}$ affine spaces directed by E & F. $f: \mathcal{E} \to \mathcal{F}$ affine mapping if $\exists A \in \mathcal{E}$ s.t. $\vec{AB} \mapsto \vec{f}(\vec{AB}) := \overrightarrow{f(A)f(B)}$ is a linear mapping from E to F.

Prop Every affine mapping $f: \mathcal{E} \to \mathcal{F}$ has the form $f(x) = Ax + B$

normal subgroup in affine group

Cor Group of invertible affine transformations of an n-dimensional affine space $\mathcal{E}$ is isomorphic to $GL(n, \mathbb{K}) \ltimes \mathbb{K}^n$

Thm Fundamental theorem of affine geometry
- Let $\mathcal{E}$ and $\mathcal{F}$ be affine spaces of dim ≥ 2. If $\varphi: \mathcal{E} \to \mathcal{F}$ bijectively, s.t. triples $\varphi(A), \varphi(B)$ & $\varphi(C)$ colinear (on same line) $\Rightarrow \varphi$ is an affine map (and also an isomorphism).

→ Week 6

Def $A_0, \dots, A_k \in \mathcal{E}$ affinely independent if minimal affine subspace $\mathcal{F} \subset \mathcal{E}$ containing these points ($\mathcal{F} = \text{span}\langle \vec{A_0}, \dots, \vec{A_k} \rangle$) has dim $= k$

Def $A_0, \dots, A_k \in \mathcal{E}$ affinely independent if minimal affine subspace $\mathcal{F} \subset \mathcal{E}$ containing these points
($\mathcal{F} = \text{span}\langle A_0, \dots, A_k \rangle$) has dim $= k$
↳ eqv to saying vectors $\overrightarrow{A_0A_1}, \dots, \overrightarrow{A_0A_k}$ are lin. indep (and span v.s. F)
If span $\langle A_0, \dots, A_n \rangle = \mathcal{E}$ & $A_0, \dots, A_n$ affinely indep $\Leftrightarrow$ $A_0, \dots, A_n$ form an affine frame of $\mathcal{E}$

→ Affine subspaces $\mathcal{F} \subset \mathcal{E}$ of dim $= k$ are specified by $k+1$ affinely indep. points

Def Barycentric coordinates
$A_0, \dots, A_k \in \mathcal{E}$ affinely indep. pts & $d_0, \dots, d_k \in \mathbb{K}$ weights s.t. $\sum_{i=0}^{d} d_i \neq 0$, the barycenter is $G := O + \frac{\sum_{i=0}^k \lambda_i \overrightarrow{OA_i}}{\underbrace{\sum_{i=0}^k \lambda_i}_{\lambda}}$ w/ $O \in \mathcal{E}$ arbitrary
→ $1/\lambda \sum_{i=0}^k \lambda_i \overrightarrow{GA_i} = 0$

Prop For every pt. $C \in \text{span}\langle A_0, \dots, A_k \rangle$ $\exists$ weights $\lambda_0, \dots, \lambda_k \in \mathbb{K}$, $\sum_{i=0}^k \lambda_i \neq 0$ s.t. C is the barycenter of weighted points $(A_0, d_0), \dots, (A_k, d_k)$
Weights $(\lambda_0, \dots, \lambda_k)$ are unique up to simultaneous scaling by $\beta \in \mathbb{K}$: $(\lambda_0, \dots, \lambda_k)$ and $(\overline{\lambda}_0, \dots, \overline{\lambda}_k)$ given same barycenter $\Leftrightarrow$
$(\overline{\lambda}_0, \dots, \overline{\lambda}_k) = (\beta\lambda_0, \dots, \beta\lambda_k)$

Def $(\lambda_0 : \dots : \lambda_k)$, w/ $(\lambda_0 : \dots : \lambda_k) = (\beta\lambda_0 : \dots : \beta\lambda_k)$ are barycentric/homogeneous coordinates

→ Classical thms: Thales', Pappus', Desargues'

→ Week 6

- Euclid's postulates

→ Euclidean distance $d(u,v) = \|u-v\|$ coincides w/ $\inf_{\gamma} \int_{t_1}^{t_2} \sqrt{\langle \dot\gamma, \dot\gamma \rangle}\, dt$ w/ infimum taken among all C^1-smooth curves
$\gamma : I \to \mathbb{R}^n$ s.t. $\gamma(t_1) = u$, $\gamma(t_2) = v$

- By defn. of isometry as bijection $f^* G = G$. In other words:
↳ * $\langle f(u) - f(v), f(u) - f(v) \rangle = \langle u-v, u-v \rangle$. Such maps always smooth & affine.

Thm Let $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfy *. Then it can be written as $f = Ax + B$ w/ A orthogonal ($AA^T = A^TA = I$)
← orthogonal transformations of $\mathbb{R}^n$.

Cor Group of isometries of $\mathbb{R}^n$, $\text{Iso}(\mathbb{R}^n) \cong O(n) \ltimes \mathbb{R}^n$

- rigid-motions, reflections & inversions

→ Week 7 + 8

Def V is v.s. over $\mathbb{K}$. Projective space $P(V)$ is the set of all lines in V.
If $\dim V = n$,
→ then $P_{n-1}(\mathbb{K})$ or $\mathbb{K}P^{n-1}$
→ $(\lambda_1 : \dots : \lambda_n) \in P_{n-1}(\mathbb{K})$ w/ $(\lambda_1, \dots, \lambda_n)$ is a vector in V taken up to proportionality

Prop $P_n(\mathbb{K}) = \mathbb{K}^n \cup P_{n-1}(\mathbb{K}) = \mathbb{K}^n \cup \mathbb{K}^{n-1} \cup \dots \cup \mathbb{K} \cup \{pt\}$
→ projective completion of affine space $\mathbb{K}^n$ by adding a hyperplane $P(\{\sum_{i=0}^n x^i = 0\})$ at infinity

Def $f : \mathbb{K}^n \to \mathbb{K}^n$ isomorphism of $\mathbb{K}^n$. As f preserves lines, it induces a bijection $f : P_{n-1}(\mathbb{K}) \to P_{n-1}(\mathbb{K})$ is called a projective transform
i.e., $PSL(n, \mathbb{K})$

Prop $PSL(n, \mathbb{K}) \cong SL(n, \mathbb{K}) / \{\pm I\}$

Ex Every ~~proj.~~ transformation of $P_1(\mathbb{K})$ is of the form $f = \frac{ax+b}{cx+d}$ w/ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{K})$

Thm $PSL(2, \mathbb{C})$ generated by an even number of inversions and reflections in arbitrary circles/lines in $\mathbb{C}$.

Thm $PSL(2, \mathbb{R})$ is the group of orientation-preserving isometries of upper-half plane.