university of groningen

Exam Numerical Methods

November 7th 2019   18.45-21.45

It is allowed to use a book (paper version only) and lecture notes, as well as a (graphical) pocket calculator. The use of electronic devices (tablet, laptop, mobile phone, etc.) is not allowed.

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: $\boxed{10}$

Practica: $\boxed{18}$ For the 6 computer practica a maximum of 6*3=18 points can be earned.

1. The equation $e^{-x} = 5x - 10$ has a solution near $x=2.0$.

   (a) $\boxed{5}$ (1) Compute one iteration with the Secant method, starting with $x_0 = 2.0$, $x_1 = 2.1$.
   (2) Determine the most accurate ('the best') error estimate for x_2.
   (3) What can you say regarding the number of iterations required for an accuracy of 1.0E-9 ?

   (b) $\boxed{4}$ Determine a $x_{n+1} = g(x_n)$ method with optimal linear convergence factor for this problem, by introduction and optimisation of a parameter α.

   (c) $\boxed{4}$ When $x_{n+1} = \frac{1}{5}e^{-x_n} + 2$ is used, again with $x_0=2$, the first iterations are given by

x_0	x_1	x_2	x_3	x_4
2.00000000	2.02706706	2.02634426	2.02636331	2.02636280

   (1) How fast does this method converge? Give order and factor of the error reduction.
   (2) Determine an error estimate for x_4.

2. To compute the value of π one could use a numerical method to compute $\int_0^1 \frac{4}{1+x^2}\,dx$

   (a) $\boxed{4}$ Suppose a grid contains 5 segments. Compute the sub-area for the middle segment:
   (1) If the Trapezoidal method is used.
   (2) If Simpson's method is used.

   With the Trapezoidal method the following results are obtained, with $I(n)$ the approximation of the integral on a grid with n segments.

n	$I(n)$
32	3.14142989
64	3.14155196
128	3.14158248
256	3.14159011

   (b) $\boxed{5}$ (1) Compute the q-factor and explain that extrapolation is allowed.
   (2) Compute improved solutions for $I(256)$ and $I(128)$ by means of extrapolation. Combine these extrapolations into a highly accurate approximation of π. Does this make sense in this case? Explain why (not).

   (c) $\boxed{4}$ (1) Give an error estimate for $I(256)$ based on $I(n)$ values.
   (2) How many segments (powers of 2) are probably required for an accuracy 1.0E-8?

3. Consider on $[0;5]$ the o.d.e. $y'(x) = \frac{1}{10}y^2(x) + \alpha$, with boundary condition $y(0) = 2$.

   (a) $\boxed{5}$ Take $\alpha = -\frac{2}{10}$. Use a grid with $\Delta x=0.5$.
   (1) Use explicit Euler to compute $y(x)$ at $x=1$ (these are 2 steps).
   (2) Use Heun's method (RK2) to compute $y(x)$ at $x=0.5$.

P.T.O.

(b) $\boxed{8}$ For $\alpha = -\frac{1}{10}$, Heun's method is used on a number of grids ($N = 10, 20, 40$ segments). The table below shows solutions at only a selection of x locations.

x_n	$N = 10$	$N = 20$	$N = 40$
2	0.39312040	0.36681244	0.35888496
3	-0.58166323	-0.61912718	-0.62994516
4	-1.44495774	-1.49233433	-1.50488101
5	-2.09054035	-2.14428082	-2.15717831

(1) Is there a stability limit visible? Explain.
(2) Compute the q-ratio for $x = 4$. What can you conclude?
(3) Give error estimates ϵ for the solutions at $x = 3$ and $x = 4$ on the finest grid. Which value is the larger one? Explain how come.
(4) Compute an improvement for the solution at $x = 4$ by means of extrapolation.
(5) Explain whether the stability restriction $|1 + ah| < 1$ is applicable in this case.

4. We will fit a (least-squares) straight line through the data

x_i	-1	3	7	11
f_i	4.8	5.9	7.6	8.0

(a) $\boxed{3}$ Apply a coordinate transformation, such that the $\hat{x}_i$ points are centered around $x = 0$ with: $\hat{x}_2 = -1$, $\hat{x}_3 = 1$ (determine $\hat{x}_1$ and $\hat{x}_4$ yourself).

(b) $\boxed{6}$ Determine the straight-line fit through the original data and use this to predict the value at $x = 10$.

5. Consider $A\vec{x} = \vec{b}$, and the initial vector $\vec{x}_0$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -3 & 1 & 0 \\ 0 & 1 & -3 & 2 \\ 1 & 0 & 0 & 2 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} -1 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad \vec{x}_0 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(a) $\boxed{6}$ (1) Matrix A is not tri-diagonal. Explain how one can modify the equation $A\vec{x} = \vec{b}$ easily into a system that is tri-diagonal.
(2) Discretisation of the diff.eqn. $y''(x) + \alpha\, y(x) = \frac{\beta}{x}$, with boundary conditions $y(0) = -1$, $y(1) = 1$, on an equidistantly spaced grid (2 internal points) leads to a system that is equivalent to the given $A\vec{x} = \vec{b}$. Determine α and β.

(b) $\boxed{6}$ (1) Compute $\|r^{(0)}\|_\infty$, i.e. the max.-norm of the initial residual, for the given $A\vec{x} = \vec{b}$.
(2) Will the Jacobi method converge? Explain.
(3) Compute $\vec{x}_1$, the result after 1 SOR iteration with $\omega = 2$ for the given $A\vec{x} = \vec{b}$.

6. Consider the diffusion eqn. $\partial\phi/\partial t = \kappa\, \partial^2\phi/\partial x^2$, with $k = 10^{-3}$. The initial and boundary conditions are $\phi(x, 0) = 100\sin(\pi x) + 20$ and $\phi(0, t) = \phi(1, t) = 20$.
For $\partial^2/\partial x^2$ the standard [1 -2 1]-formula is used, with $\Delta x = 1/200$ (constant).

(a) $\boxed{4}$ (1) Determine the maximum time step when the explicit Euler method is used.
(2) Is it wise to use this maximum time step? Explain why (not).

(b) $\boxed{3}$ Give both an advantage and a disadvantage of implicit methods.

(c) $\boxed{5}$ A term 2ϕ is added to the LHS: $$\frac{\partial\phi}{\partial t} + 2\phi = \kappa\frac{\partial^2\phi}{\partial x^2}$$

Describe exactly how the matrix-vector system changes because of this extra term in case of the implicit Euler method, where the extra term is taken half at the new and half at the old time level: $2\phi \rightarrow \phi^{n+1} + \phi^n$.

Total: $\boxed{100}$