Control Engineering

Instruction Lecture 4: Solutions

Linear systems (state space analysis), State feedback (state space design) | Chapter 5,6

Exercise 1. Step response

Consider a mass-spring-damper system modeled as

$$m\frac{d^2q}{dt^2} + b\frac{dq}{dt} + kq = u, \tag{1}$$

with q the position of mass m [kg], b [kg/s] the damping coefficient, and k [kg/s^2] the spring constant. The step response of (1) for four sets of parameters is shown in Figure 1 below.

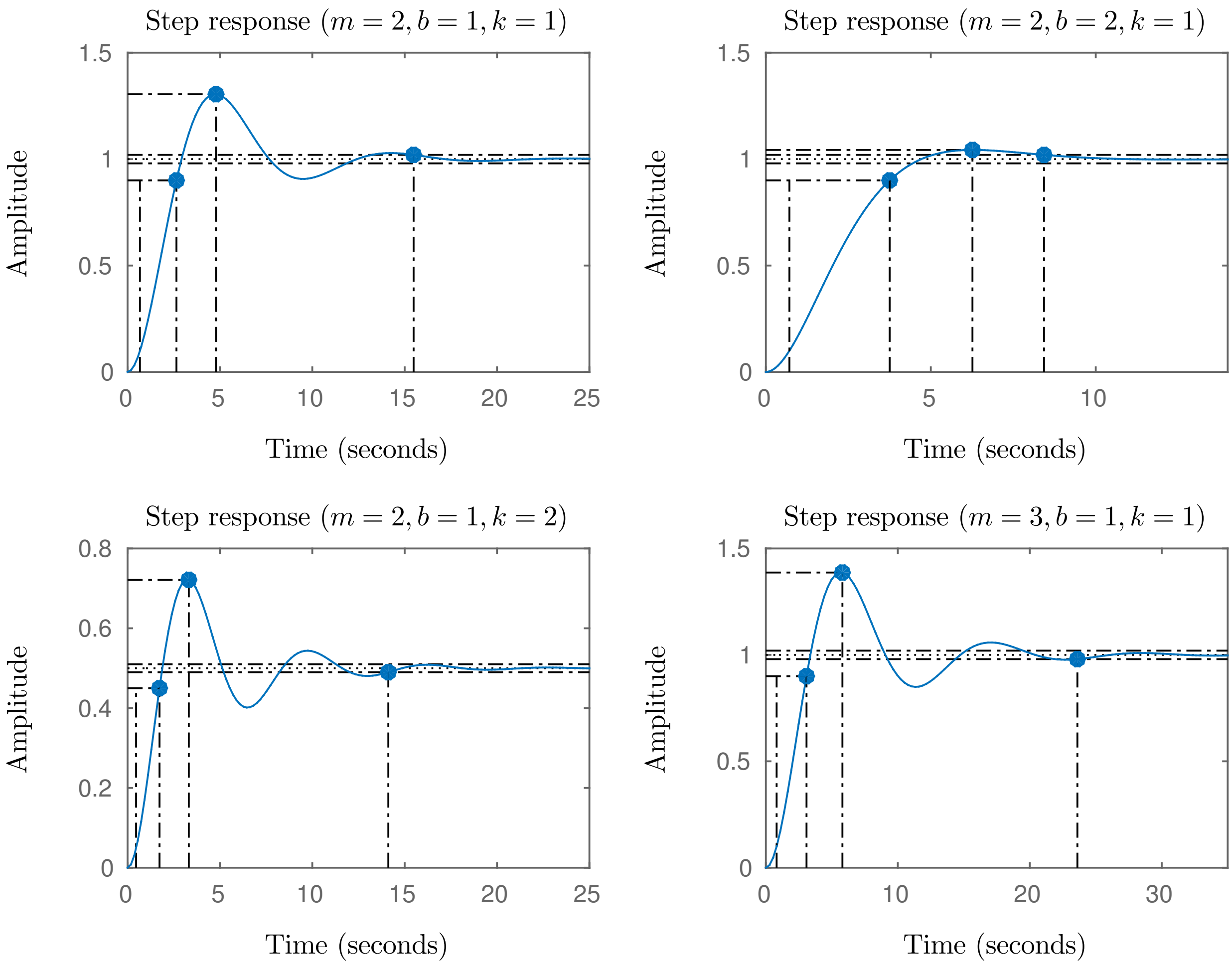

Figure 1: Step responses of the system (1) for different values of m, b, k.

Read off the following quantities from Figure 1 and discuss the influence of changing the system parameters b, k, m:

1. Steady state value $q_{ss}[m]$
2. Rise time $T_r[s]$
3. Overshoot $M_p[\%]$

4. Settling time $T_s[s]$ (2% of the final value)

SOLUTION

	steady state $q_{ss}[m]$	rise time $T_r[s]$	overshoot $M_p[\%]$	settling time $T_s[s]$
$m=2, b=1, k=1$	1	1.98	30.5	15.5
$m=2, b=2, k=1$	1	3.04	4.32	8.43
$m=2, b=1, k=2$	0.5	1.27	44.3	14.1
$m=3, b=1, k=1$	1	2.27	38.7	23.6

Increasing the damping coefficient b has the following effect:

- No effect on steady state value
- Increasing rise time
- Decreasing overshoot
- Decreases settling time

Increasing the spring constant k has the following effect:

- Decreasing steady state value
- Decreasing rise time
- Increasing overshoot
- Settling time is almost the same.

Increasing the mass m has the following effect:

- No effect on steady state value
- Minor increase in rise time
- Increasing overshoot
- Increasing settling time

Exercise 2.

Consider a linear system that integrates the input one more time compared to the double integrator of Example 5.2:

$$\frac{d^3q}{dt^3} = u, \qquad y = q. \tag{2}$$

This system can be called a *triple integrator*, because the input is now integrated three times to determine the output y.

1. Derive a state space model from (2) with the position q as the output.
2. Calculate the matrix exponential e^{At} and use Proposition 5.1 to calculate the homogenuous solution (i.e., set $u = 0$) to (2) (i.e., calculate $y(t)$).
   Note: for (2) the matrix exponential can be calculated quite easily. In general this is quite a tedious task and numerical software (e.g. MATLAB) can be used to approximate it.

SOLUTION

1. Define $x_1 = q, x_2 = \dot{q}, x_3 = \ddot{q}$ to obtain the state space model below:

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

2. $$e^{At} = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

   Using Proposition 5.1 we obtain

$$\begin{aligned} x(t) &= e^{At}x(0) = \begin{pmatrix} x_1(0) + tx_2(0) + \frac{1}{2}t^2x_3(0) \\ x_2(0) + tx_3(0) \\ x_3(0) \end{pmatrix} \\ &= x_1(0) + tx_2(0) + \frac{1}{2}t^2x_3(0). \end{aligned}$$

Exercise 3. Pole placement

Consider again the mass-spring-damper system (1) modeled as

$$m\frac{d^2q}{dt^2} + b\frac{dq}{dt} + kq = u, \tag{3}$$

with q the position of the mass m, b the damping coeffient, and k the spring constant.

1. Rewrite (3) into state-space form, with output $y = q$.
2. Verify that the system (3) is reachable for all $k, b, m > 0$ using the reachability matrix W_r.
3. Determine the open loop characteristic polynomial $\lambda(s) = s^2 + a_1 s + a_2$ of (3) and calculate the eigenvalues of the system matrix A.

Let the parameters of the system (3) by given by $m = 2\,\text{kg}, b = 1\,\text{kg/s}, k = 2\,\text{kg/s}^2$.

4. Check whether the system is stable.
5. Suppose that we would like to place the eigenvalues at $-4, -4$ using state feedback. Calculate the closed loop characteristic polynomial $p(s) = s^2 + p_1 s + p_2$.
6. Using Theorem 6.3, design a state feedback that places the eigenvalues at the desired position. Verify that the eigenvalues of the closed loop system are at $-4, -4$.
7. Below a plot of the open and closed loop step response is show. Exp the differences between the two systems.

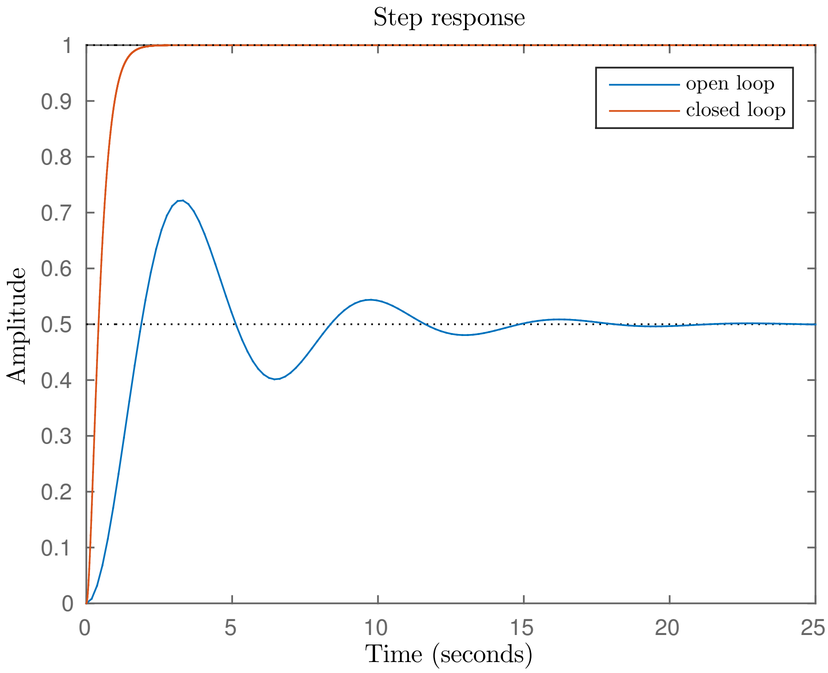

Figure 2: Step responses of the open- and closed-loop systems.

SOLUTION

1. Define $x = (x_1, x_2) = (q, \dot{q})$ to obtain

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x. \end{aligned}$$

2. For arbitrary $k, b, m > 0$ the reachability matrix W_r is calculated as

$$W_r = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & -\frac{b}{m^2} \end{pmatrix}.$$

   Since we have that $\det(W_r) = -\frac{1}{m^2} \neq 0$, it follows that W_r is invertible and therefore the system is reachable.

3. The open loop characteristic polynomial λ is given by

$$\lambda(s) = s^2 + \frac{b}{m}s + \frac{k}{m}.$$

   Hence, the eigenvalues are given by

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

4. For $m = 2, b = 1, k = 2$ the eigenvalues are given by

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1 - 16}}{4} = -\frac{1}{4} \pm \frac{1}{4}\sqrt{15}i.$$

   Since both eigenvalues are in the open left half plane, the system is asymptotically stable. Furthermore, since we have a nonzero complex part, there will be oscillations, see also Figure 1 in Exercise 1.

5. The closed loop characteristic polynomial $p(s)$ is given by

$$p(s) = s^2 + 8s + 16.$$

6. From (6.21) it follows that

$$K = \begin{pmatrix} 30 & 15 \end{pmatrix}$$

The reference gain k_r is given by

$$k_r = 32.$$

Now substitute $u = -Kx + k_r r$, with r the new reference signal. The closed loop dynamics then follows as

$$\dot{x} = A_{cl}x + B_{cl}r, \ y = C_{cl}x + D_{cl}r,$$

with

$$A_{cl} = A - BK, \ \ B_{cl} = k_r B, \ \ C_{cl} = C, \ \ D_{cl} = D.$$

The characteristic polynomial of the closed loop system $\lambda_{cl}(s)$ is given by

$$\lambda_{cl}(s) = \det(sI - (A - BK)) = (s+4)^2.$$

Hence, the eigenvalues of the closed loop system are at the desired location.

7. Differences between the open- and closed-loop systems:
   - The closed-loop system has a zero steady state error
   - No oscillations/overshoot in the closed-loop system
   - Faster rise and settling time of the closed-loop system

Exercise 4. (Book exercise 6.9 altered)

Consider the system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \tag{4}$$

with control law

$$u = -k_1 x_1 - k_2 x_2. \tag{5}$$

1. Show that the system is not reachable.
2. Show that the eigenvalues of the system cannot be assigned to arbitrary values.
3. Can the system (4) be stabilized using feedback law (5)? I.e., can you find k_1, k_2 such that the closed-loop system is asymptotically stable?

SOLUTION

1. The reachability matrix is given by

$$W_r = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

which has a zero determinant. Hence, the system is unreachable.

2. The closed-loop system takes the form

$$\dot{x} = \begin{pmatrix} -k_1 & 1-k_2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The characteristic polynomial is given by

$$\lambda(s) == (s+1)(s+k_1).$$

implying that the eigenvalues are given by $\lambda_1 = -1, \lambda_2 = -k_1$. Hence, one of the eigenvalues cannot be assigned to an arbitrary value.

3. By step 2 and choosing $k_1 > 0$ both eigenvalues are located in the open left half plane so the closed-loop system is asymptotically stable. Thus the system (4) can be stabilized.

Exercise 5. (Book exercise 6.3)

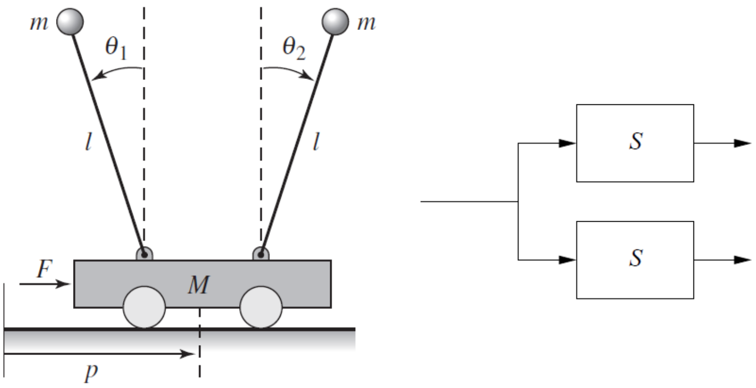

Figure 6.3: An unreachable system. The cart–pendulum system shown on the left has a single input that affects two pendula of equal length and mass. Since the forces affecting the two pendula are the same and their dynamics are identical, it is not possible to arbitrarily control the state of the system. The figure on the right is a block diagram representation of this situation.

Consider the system shown in Figure 6.3. The equations of motion are given in equation (2.10) in Example 2.2 of the book:

$$\ddot{\theta}_1 = \frac{mgl}{J_t}\sin\theta_1 - \frac{\gamma}{J_t}\dot{\theta}_1 + \frac{l}{J_t}\cos(\theta_1)u$$
$$\ddot{\theta}_2 = \frac{mgl}{J_t}\sin\theta_2 - \frac{\gamma}{J_t}\dot{\theta}_2 + \frac{l}{J_t}\cos(\theta_2)u.$$

1. Linearize each system around $\theta_i = 0,\ i = 1, 2$, and write the dynamics of the two systems as

$$\dot{x}_L = Ax_L + Bu, \qquad \dot{x}_R = Ax_R + Bu. \tag{6}$$

If x_L and x_R have the same initial condition, they will always have the same state regardless of the input that is applied.

2. Explain why this violates the definition of reachability (Definition 6.1) and further show that the reachability matrix W_r is not full rank.

Hint: first rewrite (6) into the form $\dot{x} = \tilde{A}x + \tilde{B}x$, where $x = \begin{pmatrix} x_L \\ x_R \end{pmatrix}$.

SOLUTION

1. Define $x_L = (\theta_1, \dot{\theta}_1)^T,\ x_R = (\theta_2, \dot{\theta}_2)^T$. The linearization of the two systems around $\theta_1 = \theta_2 = 0$ is given by (6) with

$$A = \begin{pmatrix} 0 & 1 \\ \frac{mgl}{J_t} & -\frac{\gamma}{J_t} \end{pmatrix},\ B = \begin{pmatrix} 0 \\ \frac{l}{J_t} \end{pmatrix}.$$

2. The system is not reachable since, if x_L, x_R have the same initial condition, it is impossible to drive the states to where x_L and x_R are different. This can also be proven from the reachability matrix W_r. First rewrite (6) in the form

$$\dot{x} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} x + \begin{pmatrix} B \\ B \end{pmatrix} u$$

so that its reachability matrix amounts to

$$W_r = \begin{pmatrix} B & AB & A^2B & A^3B \\ B & AB & A^2B & A^3B \end{pmatrix}. \tag{7}$$

It is easily seen, that this matrix can have at most 2 independent rows (not full row rank), so $\det W_r = 0$ and therefore W_r is not invertible. It follows that the system is not reachable.