Exam Numerical Methods

November 5th 2020 18:45-21:45 (**deadline for handing in: 22:00**)

It is allowed to use a book (paper version only) and lecture notes, as well as a (graphical) pocket calculator. The use of electronic devices (tablet, laptop, mobile phone, etc.) is not allowed.

For handing in the exam, the use of electronic devices is of course allowed. The student is fully responsible for handing in his/her complete work before the deadline.
The student must upload the signed declaration before the start of the exam. After grading, short discussions with (a selection of) students will be held to check for possible fraud.

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: $\boxed{10}$

Practica: $\boxed{18}$ For the 6 computer practica a maximum of 6*3=18 points can be earned.

1. Consider the equation $\sin(\pi x) + 2\,x = 3$, with solution $x \approx 1.8$.

   (a) $\boxed{3}$ Someone uses the iterative method $x_{n+1} = \frac{3}{2} - \frac{1}{2}\sin(\pi x_n)$, with $x_0 = 2$ (so $x_0 \neq 1.8$). Will this method converge or diverge? Explain.

   (b) $\boxed{3}$ (1) Give the iteration formula when Newton's method is used for this problem.
   (2) Use this method to compute x_1, starting with $x_0 = 1.8$.

   (c) $\boxed{3}$ Determine a $x_{n+1} = g(x_n)$ method with optimal linear(!) convergence factor for this problem, by optimization of a parameter α, starting with $x_0 = 1.8$.

   (d) $\boxed{3}$ For the iterative method $x_{n+1} = \frac{1}{2}(\frac{3}{2} - \frac{1}{2}\sin(\pi x_n)) + \frac{1}{2}x_n$, with $x_0 = 1.8$, the first iterations are given by

n	1	2	3	4	5
x_n	1.7969 4631	1.7973 5299	1.7972 9962	1.7973 0664	1.7973 0571

   (1) Determine an error estimate for x_5.
   (2) The convergence is linear. Determine an improved value (for x_5) via extrapolation, using a technique similar to Euler's method for o.d.e.'s.

2. To compute the value of ln(3) one could use a numerical method to compute $\int_1^3 \frac{1}{x}\,dx$.

   (a) $\boxed{5}$ (1) For a grid with 5 segments, compute the sub-area for the middle segment:
   (i) If the Trapezoidal method is used.
   (ii) If Simpson's method is used.
   (2) Give an error bound for the Trapezoidal value for the middle segment in (1)(i).
   Hint: the min/max of the derivatives are 'easy' to find.

   With the Midpoint method the following results are obtained, with $I(n)$ the approximation of the integral on a grid with n segments.

n	8	16	32	64	128	256
$I(n)$	1.0963 2472	1.0980 3533	1.0984 6772	1.0985 7613	1.0986 0325	1.0986 1003

   (b) $\boxed{5}$ (1) Explain that extrapolations for $I(16)$ and $I(32)$ are allowed.
   (2) Compute improved solutions for $I(16)$ and $I(32)$ by means of extrapolation.
   (3) Combine these extrapolations into a highly accurate approximation of ln(3).
   (4) Compute the exact(!) error for this approximation.

   (c) $\boxed{4}$ (1) Give an error estimate for $I(256)$ based on $I(n)$ values.
   (2) How many segments (use powers of 2) are required for an accuracy of 1.0E-8?
   (3) How much is the gain in number of segments when using the extrapolation (b)(3)?

P.T.O.

3. Consider the differential equation $y'(x) = -x\,y^2$, with boundary condition $y(0) = 1$.

   (a) $\boxed{7}$ Compute the solution on a grid with $\Delta x = 0.5$
   (1) at $x = 1$, with the explicit Euler method (two steps).
   (2) at $x = 0.5$ (so not at $x = 1$!), with the Heun method (RK2).
   (3) at $x = 0.5$ (so not at $x = 1$!), with the implicit Euler method.

   (b) $\boxed{6}$ With a 3rd (!) order RK method the solution is determined on 4 grids with $\Delta x = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ and $\frac{1}{32}$. The result at a selection of x locations is as follows

x_n	$\Delta x = \frac{1}{4}$	$\Delta x = \frac{1}{8}$	$\Delta x = \frac{1}{16}$	$\Delta x = \frac{1}{32}$
0.5	0.8894 0021	0.8889 4901	0.8888 9614	0.8888 8978
1.0	0.6670 4851	0.6667 0839	0.6666 7153	0.6666 6725
1.5	0.4706 4742	0.4705 9393	0.4705 8885	0.4705 8831

   (1) Compute the q-factor at $x = 1.0$. What is your conclusion?
   (2) Give an improved solution (extrapolation) for the solution at $x = 1.0$.
   (3) Give error estimates for the solutions at $x = 1.0$ and $x = 1.5$ on the fine grid. Which of these error estimates is bigger? Explain why.
   (4) Is there an instability visible? Explain.

4. Consider the coordinate points $f(1) = 1$, $f(3) = 5$, $f(5) = 25$. To approximate $f(0)$, we will use an exponential fit $f(x) = a\,e^{bx}$ through the given points.

   (a) $\boxed{7}$ (1) Determine a coordinate transformation such that the new x-points are -1,0,1.
   (2) Set up a least-squares exponential fit through the original data.

   (b) $\boxed{2}$ Use the curve-fit to "approximate" $f(0)$. Why is approximate between quotes?

5. Consider the equation $A\vec{x} = \vec{b}$, with
$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -5 & 2 & 0 \\ 1 & 1 & -5 & -2 \\ -1 & 0 & 0 & 5 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

   (a) $\boxed{6}$ (1) Take $\vec{x}_0 = (1\ 1\ 1\ 1)^T$. Compute $\vec{x}_1$, the result after 1 Jacobi iteration.
   (2) Take $\vec{x}_0 = (1\ 1\ 1\ 1)^T$. Compute the result after 1 SOR iteration with $\omega = 5$.

   (b) $\boxed{4}$ (1) Compute $\|r^{(0)}\|_\infty$, i.e. the max.-norm of the initial residual for $\vec{x}_0 = (1\ 1\ 1\ 1)^T$.
   (2) How many Jacobi iterations are needed to reduce this error with a factor 10?

   (c) $\boxed{2}$ Explain how the system should be modified such that TDMA can be used.

6. Consider the diff. eqn. $\dfrac{\partial \phi}{\partial t} + \alpha\,\phi = \Gamma\,\dfrac{\partial^2 \phi}{\partial x^2}$, with bound. cond. $\phi(0,t) = \phi(1,t) = 20$ and initial condition $\phi(x,0) = A\sin(\pi x) + 20$. For $\partial^2/\partial x^2$ the usual [1 -2 1]-formula is used.

   (a) $\boxed{6}$ Take $\Gamma = 1.0\text{E-}4$, $\alpha = 0.5$, $A = 100$ and use a grid size $\Delta x = 0.01$.
   (1) Which restriction follows for the time step Δt in case of the explicit Euler method? Hint: include the effect of the term $\alpha\,\phi$ on the stability; $R \leq \frac{1}{2}$ does not apply!
   (2) Does this change when $A = 0$? Explain.

   (b) $\boxed{6}$ For the implicit Euler method, each time step a system $A\vec{x} = \vec{b}$ has to be solved.
   (1) Give the i-th row of the system, i.e. $A(i,:)$ and $b(i)$, for general Δt and Δx.
   (2) Is this method unconditionally stable, when $\Gamma = 1.0\text{E-}4$, $\alpha = 0.5$, $A = 100$?

Total: $\boxed{100}$