Mechatronics

Energy-based Modeling of Mechanical Systems via Euler-Langrage Equations

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Overview

From this lecture, you should be able to:

- Derive dynamics of mechanical systems using the Euler-Lagrange equations

Analytical mechanics — using energy

- Often an energy-based approach offers a simpler and more direct method for resolving the equations of motion. The approach utilises the energy of the system to derive the dynamic equations.

For conservative systems, we can express the total energy of the system as a function of state variables and show that is it invariant with time

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{T} + \mathcal{V}) = 0$$

where

T is the kinetic energy
V is the potential energy

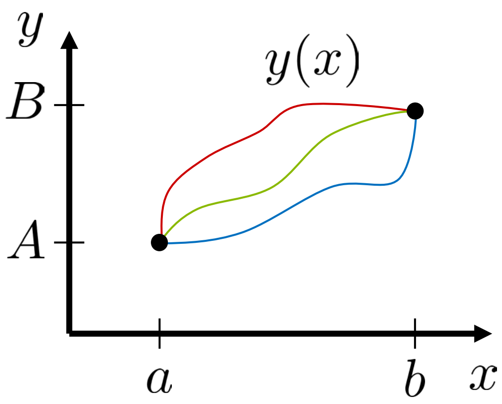
This poses the question, is there a systematic way to derive equations of motion from energy? To answer this, we need to take a detour through calculus of variations…

Calculus of variations

- Consider Y to be the set of differentiable trajectories $y(x)$ on the interval $[a, b]$ and let $F(x, y(x), y'(x))$ be a scalar function.

$$J(y) = \int_a^b F(x, y(x), y'(x))\ dx$$

is a functional on Y.



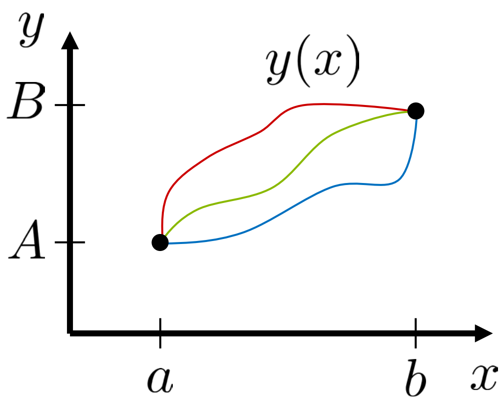
- A necessary condition for $y(x)$ to be a minimiser (optimal solution) of $J(y)$ is that it must satisfy

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Example

- Find the function $y(x)$ that minimises the integral

$$J(y) = \int_a^b y(x)^2 + y'(x)^2 \; dx$$



Example

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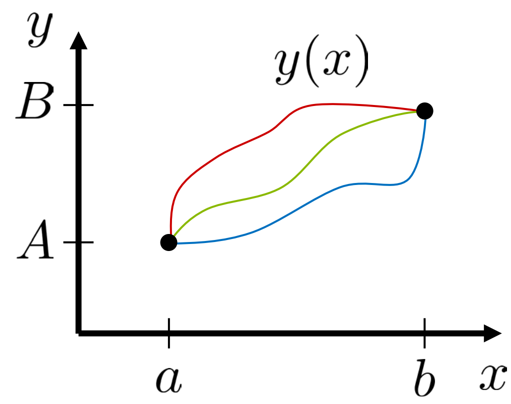


- Note that $F = y^2 + y'^2$:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$2y(x) - \frac{d}{dx}\left(2y'(x)\right) = 0$$

$$2y(x) - 2y''(x) = 0$$

$$y(x) = c_1 e^{x} + c_2 e^{-x}$$

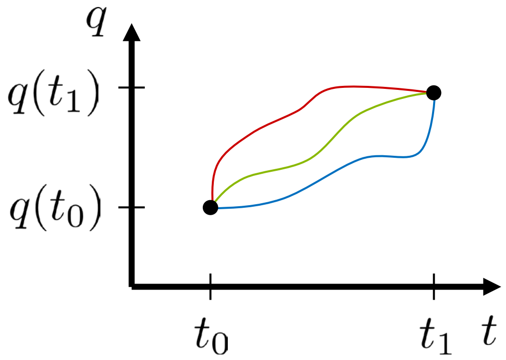
Euler-Lagrange Equations

- Lagrangian is the difference between the kinetic (co)energy and the potential energy.

$$\begin{aligned}\mathcal{L} &= \mathcal{T}^\star(q,\dot{q}) - \mathcal{V}(q) \\ &= \frac{1}{2}\dot{q}^\top M(q)\dot{q} - \mathcal{V}(q)\end{aligned}$$

- The functional that nature naturally minimises is the integral of the Lagrangian.

$$J(q) = \int_{t_0}^{t_1} \mathcal{L}(t, q(t), \dot{q}(t))\ dt$$



- Consequently, trajectories of (conservative) mechanical systems are solutions to the differential equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$

Higher dimensions and inputs

- For systems with multiple degrees of freedom, the EL equations is applied in each coordinate.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$

If there is no index, assume a vector value

- If there are external forces action on the system, they can be added to the right hand side of the Lagrangian dynamics

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = F_{i,ext}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = F_{ext}$$

- The external forces could be due to damping (friction, damper), a control input, or some other external force.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \underbrace{F_{i,control} - F_{i,damping}}_{F_{i,ext}}$$

EL-equations

$$\mathcal{L} = \mathcal{T}^\star(q,\dot{q}) - \mathcal{V}(q)$$
$$= \frac{1}{2}\dot{q}^\top M(q)\dot{q} - \mathcal{V}(q)$$

A general form for the dynamic equations can be constructed

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = F_{ext}$$

$$\frac{d}{dt}\left(M(q)\dot{q}\right) - \frac{\partial \mathcal{T}^\star}{\partial q} + \frac{\partial \mathcal{V}}{\partial q} = F_{control} - F_{damping}$$

$$M(q)\ddot{q} + \underbrace{\dot{M}(q)\dot{q} - \frac{\partial \mathcal{T}^\star}{\partial q}}_{C(q,\dot{q})\dot{q}} + \underbrace{\frac{\partial \mathcal{V}}{\partial q}}_{g(q)} = F_{control} - \underbrace{D(q,\dot{q})\dot{q}}_{F_{damping}}$$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D(q,\dot{q})\dot{q} + g(q) = F_{control}$$

Commonly called τ

where $M(q)$ is the mass matrix, $C(q,\dot{q})$ is the centripetal-Coriolis matrix, $D(q,\dot{q})$ is the damping matrix and $g(q)$ is a vector of conservative forces from a potential (e.g., gravity, springs).

Some useful properties

- The mass matrix is symmetric positive definite

$$M(q) = M^\top(q) > 0$$

- The damping matrix is symmetric positive semi-definite

$$D(q,\dot{q}) = D^\top(q,\dot{q}) \geq 0$$

- The matrix $N(q,\dot{q}) = \dot{M}(q) - 2C(q,\dot{q})$ is skew-symmetric

$$N(q,\dot{q}) = -N^\top(q,\dot{q})$$

- The total energy satisfies

$$\frac{d}{dt}(\mathcal{T}^\star + \mathcal{V}) = \dot{q}^\top \tau - \dot{q}^\top D(q,\dot{q})\dot{q} \leq \dot{q}^\top \tau$$

Example: cart-pole system

Generalised coordinates:

$$q = \begin{bmatrix} s \\ \theta \end{bmatrix}$$

$$x_c = s$$

$$x_p = s + \ell \sin\theta$$

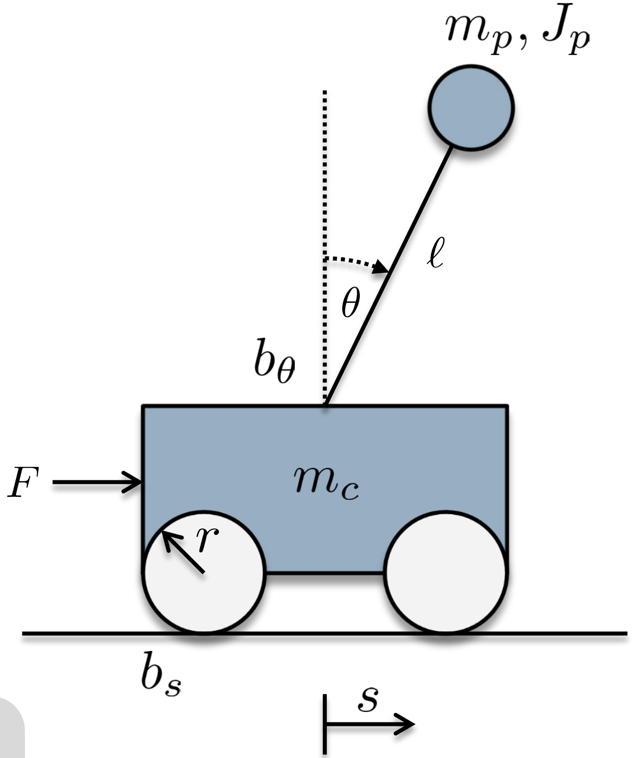
$$y_p = \ell \cos\theta$$

Velocities:

$$\dot{x}_c = \dot{s}$$

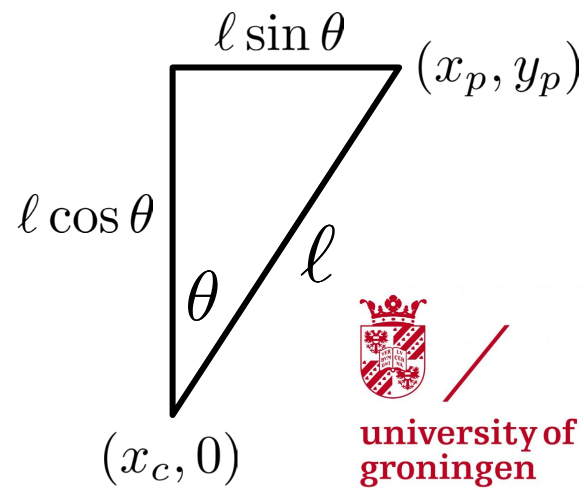
$$\dot{x}_p = \dot{s} + \ell\dot{\theta}\cos\theta$$

$$\dot{y}_p = -\ell\dot{\theta}\sin\theta$$



Kinetic energy:

$$\begin{aligned}
\mathcal{T}^\star &= \frac{1}{2}m_c\dot{x}_c^2 + \frac{1}{2}m_p\left(\dot{x}_p^2 + \dot{y}_p^2\right) + \frac{1}{2}J_p\dot{\theta}^2 \\
&= \frac{1}{2}m_c\dot{s}^2 + \frac{1}{2}m_p\left(\dot{s}^2 + 2\ell\dot{s}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2\right) + \frac{1}{2}J_p\dot{\theta}^2 \\
&= \frac{1}{2}\begin{bmatrix}\dot{s} \\ \dot{\theta}\end{bmatrix}^\top \underbrace{\begin{bmatrix} m_c + m_p & m_p\ell\cos\theta \\ m_p\ell\cos\theta & m_p\ell^2 + J_p \end{bmatrix}}_{M(q)} \begin{bmatrix}\dot{s} \\ \dot{\theta}\end{bmatrix}
\end{aligned}$$



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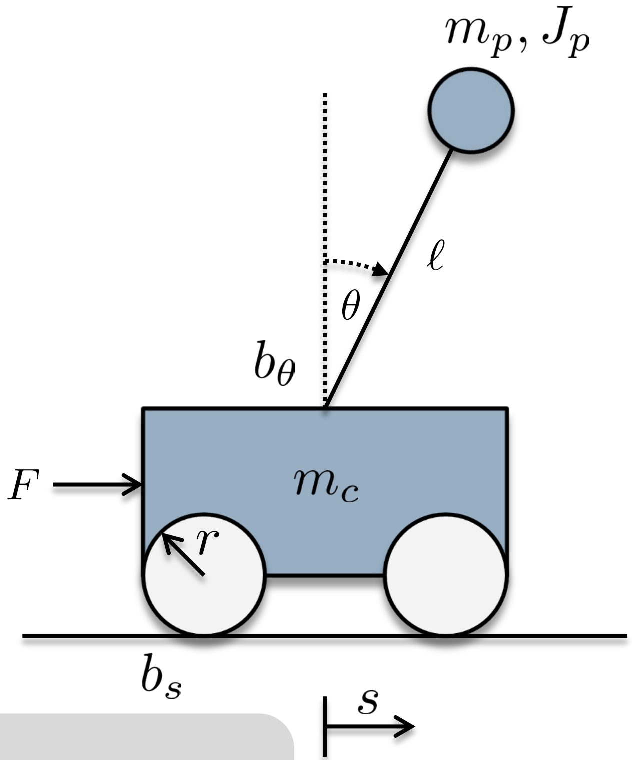
Example: cart-pole system



Generalised coordinates:

$$q = \begin{bmatrix} s \\ \theta \end{bmatrix}$$

$$x_c = s$$

$$x_p = s + \ell \sin\theta$$

$$y_p = \ell \cos\theta$$

Potential energy:

$$\begin{aligned}\mathcal{V} &= m_p g y_p \\ &= m_p g \ell \cos\theta\end{aligned}$$

Lagrangian:

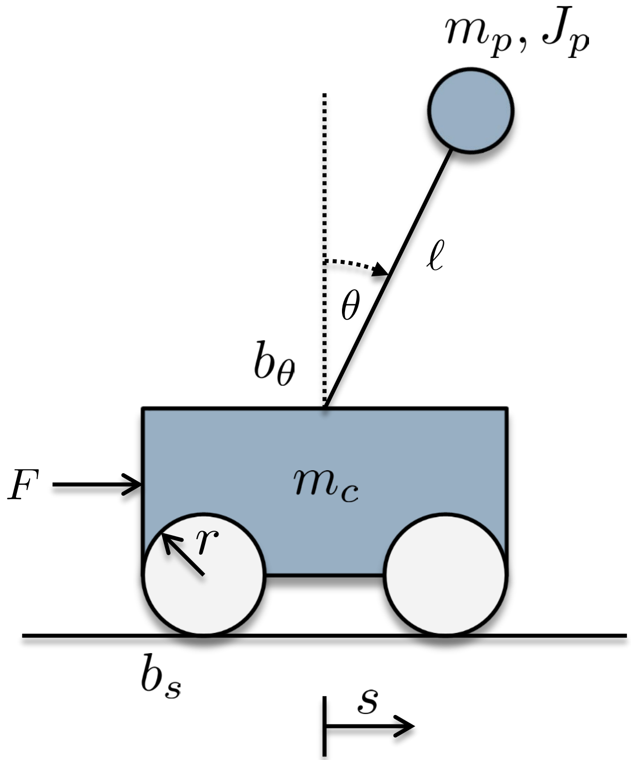
$$\begin{aligned}\mathcal{L} &= \mathcal{T}^\star - \mathcal{V} \\ &= \underbrace{\frac{1}{2} m_c \dot{s}^2 + \frac{1}{2} m_p \left(\dot{s}^2 + 2\ell \dot{s}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2\right) + \frac{1}{2} J_p \dot{\theta}^2}_{\mathcal{T}^\star} - m_p g \ell \cos\theta \\ &= \frac{1}{2}\begin{bmatrix}\dot{s} \\ \dot{\theta}\end{bmatrix}^\top \underbrace{\begin{bmatrix} m_c + m_p & m_p \ell \cos\theta \\ m_p \ell \cos\theta & m_p \ell^2 + J_p \end{bmatrix}}_{M(q)} \begin{bmatrix}\dot{s} \\ \dot{\theta}\end{bmatrix} - m_p g \ell \cos\theta\end{aligned}$$

Example: cart-pole system

Lagrangian:

$$\mathcal{L} = \frac{1}{2}\begin{bmatrix}\dot{s}\\ \dot{\theta}\end{bmatrix}^\top \underbrace{\begin{bmatrix} m_c + m_p & m_p\ell\cos\theta \\ m_p\ell\cos\theta & m_p\ell^2 + J_p\end{bmatrix}}_{M(q)} \begin{bmatrix}\dot{s}\\ \dot{\theta}\end{bmatrix} - m_p g\ell\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) - \frac{\partial\mathcal{L}}{\partial q} = F_{ext}$$



$$\frac{\partial\mathcal{L}}{\partial\dot{q}} = \begin{bmatrix} m_c + m_p & m_p\ell\cos\theta \\ m_p\ell\cos\theta & m_p\ell^2 + J_p\end{bmatrix}\begin{bmatrix}\dot{s}\\ \dot{\theta}\end{bmatrix}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) = \begin{bmatrix} 0 & -m_p\ell\dot{\theta}\sin\theta \\ -m_p\ell\dot{\theta}\sin\theta & 0\end{bmatrix}\begin{bmatrix}\dot{s}\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix} m_c + m_p & m_p\ell\cos\theta \\ m_p\ell\cos\theta & m_p\ell^2 + J_p\end{bmatrix}\begin{bmatrix}\ddot{s}\\ \ddot{\theta}\end{bmatrix}$$

$$\frac{\partial\mathcal{L}}{\partial q} = \begin{bmatrix} 0 \\ -m_p\ell\dot{s}\dot{\theta}\sin\theta + m_p g\ell\sin\theta\end{bmatrix}$$

$$F_{ext} = \begin{bmatrix} F - b_s\dot{s} \\ -b_\theta\dot{\theta}\end{bmatrix}$$

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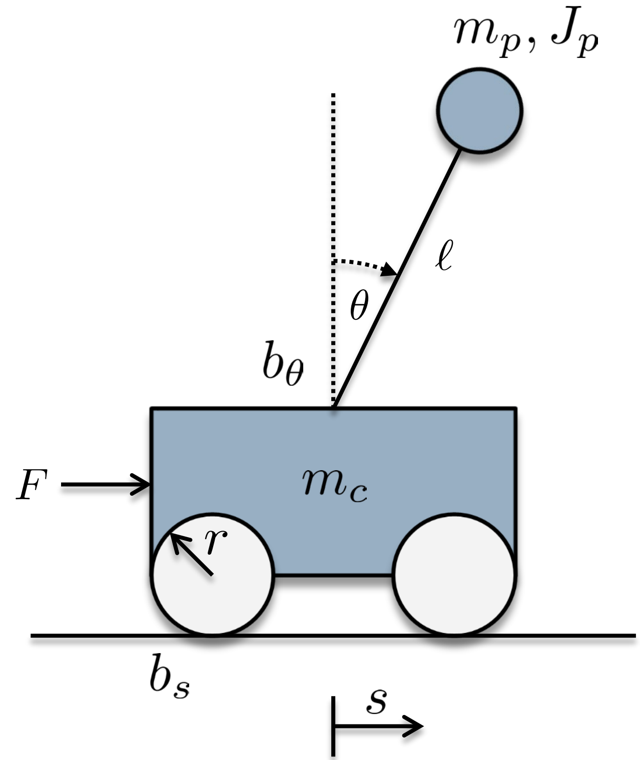
Example: cart-pole system



Lagrangian:

$$\mathcal{L} = \frac{1}{2}\begin{bmatrix}\dot{s}\\ \dot{\theta}\end{bmatrix}^\top \underbrace{\begin{bmatrix} m_c + m_p & m_p\ell\cos\theta \\ m_p\ell\cos\theta & m_p\ell^2 + J_p\end{bmatrix}}_{M(q)} \begin{bmatrix}\dot{s}\\ \dot{\theta}\end{bmatrix} - m_p g\ell\cos\theta$$

Euler-Lagrange equations:

$$\underbrace{\begin{bmatrix} m_c + m_p & m_p\ell\cos\theta \\ m_p\ell\cos\theta & m_p\ell^2 + J_p\end{bmatrix}}_{M(q)} \begin{bmatrix}\ddot{s}\\ \ddot{\theta}\end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -m_p\ell\dot{\theta}\sin\theta \\ 0 & 0\end{bmatrix}}_{C(q,\dot{q})} \begin{bmatrix}\dot{s}\\ \dot{\theta}\end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} b_s & 0 \\ 0 & b_\theta\end{bmatrix}}_{D(q,\dot{q})} \begin{bmatrix}\dot{s}\\ \dot{\theta}\end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -m_p g\ell\sin\theta\end{bmatrix}}_{g(q)} = \underbrace{\begin{bmatrix} F \\ 0\end{bmatrix}}_{\tau}$$

Some useful properties

- The mass matrix is symmetric positive definite

$$M(q) = M^\top(q) > 0 \longrightarrow M(q) = \begin{bmatrix} m_c + m_p & m_p \ell \cos\theta \\ m_p \ell \cos\theta & m_p \ell^2 + J_p \end{bmatrix}$$

- The damping matrix is symmetric positive semi-definite

$$D(q,\dot{q}) = D^\top(q,\dot{q}) \geq 0 \longrightarrow D(q,\dot{q}) = \begin{bmatrix} b_s & 0 \\ 0 & b_\theta \end{bmatrix}$$

- The matrix $N(q,\dot{q}) = \dot{M}(q) - 2C(q,\dot{q})$ is skew-symmetric

$$N(q,\dot{q}) = -N^\top(q,\dot{q})$$

$$N(q,\dot{q}) = \underbrace{\begin{bmatrix} 0 & -m_p \ell \dot{\theta} \sin\theta \\ -m_p \ell \dot{\theta} \sin\theta & 0 \end{bmatrix}}_{\dot{M}(q)} - 2 \underbrace{\begin{bmatrix} 0 & -m_p \ell \dot{\theta} \sin\theta \\ 0 & 0 \end{bmatrix}}_{C(q,\dot{q})}$$

$$= \begin{bmatrix} 0 & m_p \ell \dot{\theta} \sin\theta \\ -m_p \ell \dot{\theta} \sin\theta & 0 \end{bmatrix}$$

State space model

- The Euler-Lagrange equations derived so far describe systems dynamics as n second-order differential equations. To construct a state-space model, we need first-order equations.

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D(q,\dot{q})\dot{q} + g(q) = \tau \quad \longrightarrow \quad \dot{x} = f(x,u)$$

- The EL equations can be rearranged to make $\ddot{q}$ the subject.

$$\ddot{q} = M(q)^{-1}\left\{\tau - \left[C(q,\dot{q}) + D(q,\dot{q})\right]\dot{q} - g(q)\right\}$$

- Defining the generalized velocity as $v = \dot{q}$, $2n$ first-order equations can be constructed.

$$\begin{aligned}\dot{v} &= M(q)^{-1}\left\{\tau - \left[C(q,v) + D(q,v)\right]v - g(q)\right\}\\ \dot{q} &= v\end{aligned}$$

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Summary

From this lecture, you should be able to:

- Derive dynamics of mechanical systems using the Euler-Lagrange equations