Geometry - Solutions to Exercises

As the course progresses, solutions to more exercises will be added to this document.

Contents

1 Week 1

Exercise VII.1

Let $\mathcal{C}$ be a conic in $\mathbb{R}^2$ and let A be a point of $\mathcal{C}$. Let us perform a change of coordinates such that A is the origin of our new coordinate system, that is $A = (0,0)$. Let $\mathcal{D}_t$ be a line of slope t passing through the point A. It will in general intersect $\mathcal{C}$ at two points (unless it is a tangent line). Let the other point of intersection between $\mathcal{D}_t$ and $\mathcal{C}$ be $M_t = (x(t), y(t))$.

A conic is given by the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{1}$$

where $a, b, c, d, e, f \in \mathbb{R}$, and at least one of a, b, c are non-zero. Since $\mathcal{D}_t$ has slope t and it runs through the origin, it can be written as $y = tx$. Then the intersection points between the conic and the line is given by (1) with $y = tx$:

$$(a + bt + ct^2)x^2 + (d + et)x + f = 0. \tag{2}$$

Note that one of the intersection points, A, is by assumption at the origin. This implies that $f = 0$. Let us now write (2) as

$$x(g(t)x + h(t)) = 0,$$

where $g(t) = a+bt+ct^2$ and $h(t) = d+et$. Again, the solution $x = 0$ corresponds to the point A. Thus, the other point must be

$$x = -\frac{h(t)}{g(t)}, \quad y = tx = -\frac{th(t)}{g(t)}.$$

Note that $g(t)$ is indeed non-zero by the definition of the conic.

Let us now consider the circle $x^2 + y^2 = 1$ minus a point, say $(-1, 0)$. The circle corresponds to the conic (1) with $a = c = 1$, $b = d = e = 0$ and $f = -1$. This time A is not located at the origin. The equation of the line with slope t running from $A = (-1, 0)$ to $M_t = (x(t), y(t))$ is given by $y = t(x + 1)$. The intersection points are given by the solutions to

$$x^2 + t^2(x + 1)^2 - 1 = 0.$$

This equation can be written

$$(1 + t^2)x^2 + 2t^2x + t^2 - 1 = 0.$$

Using the *abc*-formula gives

$$x = \frac{-2t^2 \pm \sqrt{4t^4 - 4(1 + t^2)(t^2 - 1)}}{2(1 + t^2)} = \frac{-2t^2 \pm \sqrt{4t^4 - 4(1 + t^2)}}{2(1 + t^2)} = \frac{-t^2 \pm 1}{1 + t^2}.$$

Note that the choice of minus-sign gives us the x-value for A, namely $x = -1$. Hence, we should use the plus-sign, which gives $x(t) = \frac{1-t^2}{1+t^2}$. Then $y(t) = t(x + 1) = \frac{2t}{1+t^2}$. This defines a parametrization of the circle.

Exercise VII.3

Let $\mathcal{C}$ be the curve parametrized by

$$\begin{cases} x &= \frac{t}{1+t^4} \\ y &= \frac{t^3}{1+t^4}. \end{cases}$$

The curve is an inclined figure 8, as seen in Figure 1.

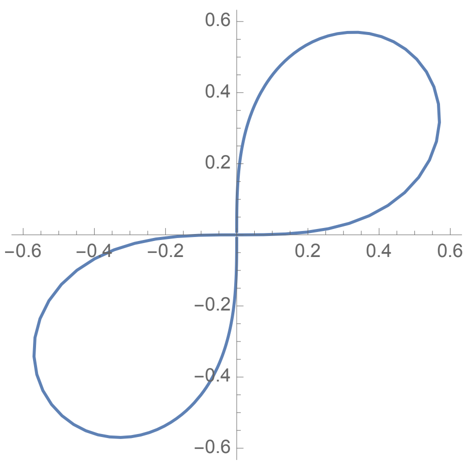


Figure 1: The curve from Exercise VII.3. The figure was created with Mathematica.

Note that $y = xt^2$, i.e. $t^2 = y/x$. Thus,

$$x = \frac{t}{1 + y^2/x^2} = \frac{x^2 t}{x^2 + y^2} \implies t = \frac{x^2 + y^2}{x}.$$

Plugging this back into $y = xt^2$ gives us the Cartesian equation:

$$xy = (x^2 + y^2)^2.$$

Exercise VII.5

We consider the curve parametrized by

$$\begin{cases} x = t + \frac{1}{2t^2} \\ y = t^2 + \frac{2}{t}. \end{cases}$$

We find the singularities:

$$\frac{dx}{dt} = 1 - \frac{1}{t^3}, \quad \frac{dy}{dt} = 2t - \frac{2}{t^2}.$$

If we set them both to be equal to zero, we find that $t = 1$ is the only possibility. That is, $t = 1$ is the only singularity of the curve.

We see that when $|t|$ is big, then $x \approx t$ and $y \approx t^2$. In other words, $y = x^2$. This means that, when $|t|$ is large, then the curve approximates the parabola given by $y = x^2$. Likewise, when $|t|$ is small, then $x \approx \frac{1}{2t^2}$ and $y \approx \frac{2}{t}$. In other words, $y^2 = 8x$. This means that, when $|t|$ is small, then the curve approximates the parabola given by $y^2 = 8x$.

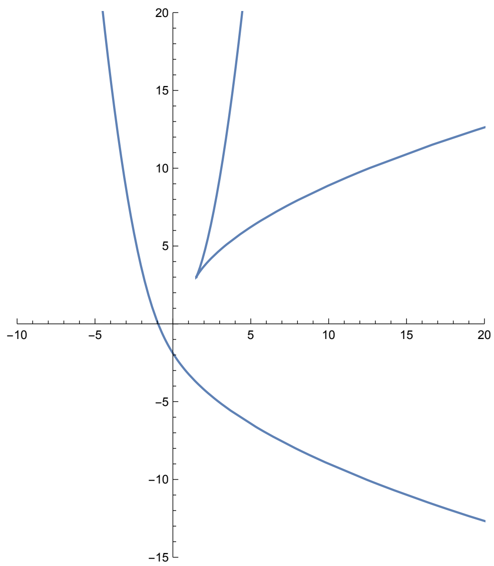


Figure 2: The curve from Exercise VII.5. The figure was created with Mathematica.

Exercise VII.15

The exercise asks us to first find the equation of the line D'. In Figure 3, the path for the line D as it is first reflected, and then refracted, and so becomes D' is drawn in red.

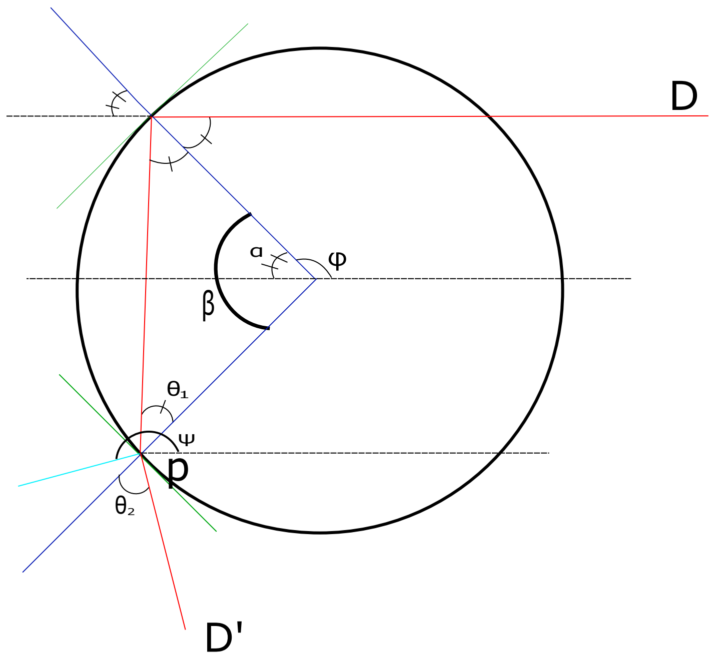


Figure 3: The figure is related to Exercise VII.15. Please see the problem solution for an explanation of the figure. The figure was created with Inkscape.

Let D be a line parallel to the x-axis given by $y = \sin\varphi$. Thus, the line is reflected by the circle at the point $(\cos\varphi, \sin\varphi)$. Observe that the angle α in the figure is equal to $\pi - \varphi$. The angles identified by two short lines are indeed identical, as they are located between parallel lines, and a straight line through the parallel lines. Furthermore, these angles must also be identical to the angles identified by only one short line, as opposite angles are equal. Note also that the two angles at the top are equal since this is a reflection. Finally, the bottom angle, θ_1, is equal to the top angles since we have an isosceles triangle. The blue lines extending to the border of the circle are both radii of the circle, i.e. of equal length. Thus, we have $\theta_1 = \alpha = \pi - \varphi$.

Snell tells us now how to find θ_2:

$$\sin \theta_2 = n_{12} \sin \theta_1,$$

where n_{12} is the refraction index. Note that, as $|\sin \theta| \leq 1$ for any θ, we have, if $n_{12} > 1$ (between water and air $n_{12} = 4/3 > 1$), some restriction to what the angles φ, and henceθ_2, may be. Precisely, we find

$$\theta_2 = \arcsin(n_{12} \sin \theta_1) = \arcsin(n_{12} \sin(\pi - \varphi)) = \arcsin(n_{12} \sin(\varphi)).$$

Thus, we must have $|n_{12} \sin(\varphi)| \leq 1$ (the domain of the function arcsin is $[-1, 1]$). In the case of refraction from water to air, where $n_{12} = 4/3$, we then have $\arcsin(-3/4) \leq \varphi \leq \arcsin(3/4)$.

Now, to find the equation of the line D', we are going to use its normal vector, which has been coloured turquoise in Figure 3. Note that the x-value of the unit normal vector is given by $\cos \psi$ and the y-value is given by $\sin \psi$. Here ψ is in fact a function of φ, $\psi = \psi(\varphi)$. So, what is then ψ? Note that the inner angle of the isosceles triangle mentioned above is equal to $\beta := \pi - 2\theta_1 = \pi - 2(\pi - \varphi) = 2\varphi - \pi$. Then, the angle between the x-axis and D' is $\varphi + \beta + \theta_2$. To get to the outward normal vector (turquoise line) of D' we need to subtract $\pi/2$ from this angle:

$$\psi = \varphi + \beta + \theta_2 - \pi/2 = \varphi + 2\varphi - \pi + \theta_2 - \pi/2 = 3\varphi - \pi + \theta_2 - \pi/2.$$

Thus, we have been able to find an expression for the unit normal vector to D', given by $(\cos \psi, \sin \psi)$ (let us not fill in the expression we found for ψ, but only remember that it is indeed dependent on φ). The equation describing D' is then given by

$$\cos(\psi)x + \sin(\psi)y + \omega = 0, \tag{3}$$

where ω is the point of intersection. Let us find an expression for ω: Recall that the point p (see Figure 3) where the line is refracted is given by $\beta + \varphi = 3\varphi - \pi$. Thus, the x-value is $\cos(3\varphi - \pi)$ and the y-value is $\sin(3\varphi - \pi)$. Plugging this into (3) gives us

$$\omega = -\cos(\psi)\cos(3\varphi - \pi) - \sin(\psi)\sin(3\varphi - \pi) = \cos(\psi)\cos(3\varphi) + \sin(\psi)\sin(3\varphi).$$

To find the envelope, we will consider the line we just found, and the derivative of it,

$$\begin{cases} \cos(\psi)x + \sin(\psi)y + \omega = 0 \\ \frac{\mathrm{d}}{\mathrm{d}\varphi}(\cos(\psi))x + \frac{\mathrm{d}}{\mathrm{d}\varphi}(\sin(\psi))y + \frac{\mathrm{d}}{\mathrm{d}\varphi}(\omega) = 0. \end{cases}$$

Now, note that $\frac{\mathrm{d}}{\mathrm{d}\varphi}(\cos(\psi)) = -\sin(\psi)\psi'$ and $\frac{\mathrm{d}}{\mathrm{d}\varphi}(\sin(\psi)) = \cos(\psi)\psi'$, where $\psi' := \frac{\mathrm{d}\psi}{\mathrm{d}\varphi}$. We will also denote by $\omega' = \frac{\mathrm{d}\omega}{\mathrm{d}\varphi}$. Thus, the system of equations becomes

$$\begin{cases} \cos(\psi)x + \sin(\psi)y + \omega = 0 & \text{(I)} \\ -\sin(\psi)\psi' x + \cos(\psi)\psi' y + \omega' = 0 & \text{(II)}. \end{cases}$$

Now we can find an expression for x and y:

$$\begin{aligned} 0 &= \sin(\psi)\psi' \cdot \text{(I)} + \cos(\psi) \cdot \text{(II)} \\ &= (\sin^2(\psi) + \cos^2(\psi))\psi' y + \sin(\psi)\psi'\omega + \cos(\psi)\omega' \\ &= \psi' y + \sin(\psi)\psi'\omega + \cos(\psi)\omega'. \end{aligned}$$

Thus, after a similar computation for x,

$$\begin{aligned} x &= -\cos(\psi)\omega + \sin(\psi)\frac{\omega'}{\psi'} \\ y &= -\sin(\psi)\omega - \cos(\psi)\frac{\omega'}{\psi'}. \end{aligned}$$

This defines the envelope depicted in Figure 4, when the refraction index is 4/3 (between water and air).

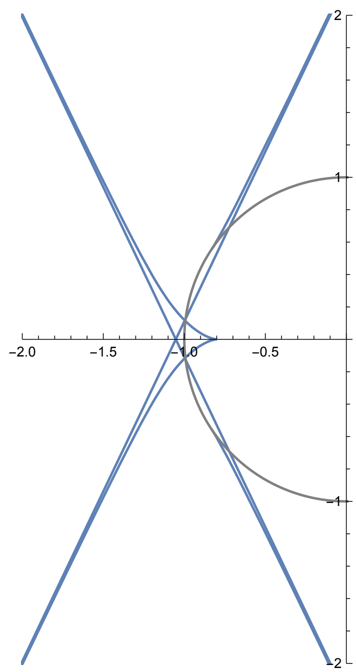


Figure 4: The envelope found in Exercise VII.15. The figure was created with Mathematica.

We could also find the equation for the line D' in another way, which is useful for finding the asymptotes. That is, we could instead consider the angle between the $x - axis$ and the line D'; its largest value correspond to the asymptote. One can show that this angle is $\xi = \pi - \theta_2 - (\beta - \alpha) = \pi - 3\varphi - \theta_2$. Then the direction of D' is given by $y = \tan \xi$.

One finds the maximum value for ξ by computing the derivative with respect to φ, solving for φ, and plugging back into ξ. This yields the asymptotes to the envelope curve. The extrema of ξ are at

$$\varphi_{\pm} = \pm \arccos \left(-\frac{3\sqrt{n^2 - 1}}{2\sqrt{2}n} \right), \tag{4}$$

and the asymptotes are parallel to $y = x \tan \varphi_{\pm}$.

For more on this, see http://people.uncw.edu/hermanr/Documents/Talks/Rainbow_History.pdf

2 Week 2

Exercise VII.9

We are given a circle of radius r, which we assume is centred at the origin in the plane. Thus, it is given by $x^2 + y^2 = r^2$, and parametrised by $x = r\cos t$, $y = r\sin t$. Furthermore, we have a point $F = (x_F, y_F)$, we which assume lies outside the circle, which implies $x_F^2 + y_F^2 > r^2$. We are interested in finding the envelope of the perpendicular bisector MF. The perpendicular bisector is the line perpendicular to MF, which runs through the midpoint of MF, which we will denote by A, see Figure 5. Note that the point M is given by $(r\cos t, r\sin t)$, and so we find A:

$$(r\cos t, y\sin t) + \frac{1}{2}(x_F - r\cos t, y_F - r\sin t) = \frac{1}{2}(x_F + r\cos t, y_F + r\sin t).$$

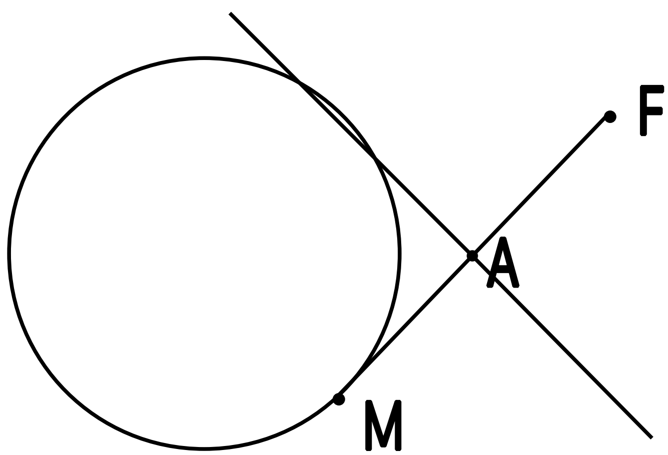


Figure 5: The perpendicular bisector is the line running through A, perpendicular to the line MF.

Let us now find the equation of the perpendicular bisectors. We first find its direction, using the fact that it is perpendicular to MF, and so their dot-product vanishes:

$$0 = (x_F - r\cos t, y_F - r\sin t) \cdot (x, y) = (x_F - r\cos t)x + (y_F - r\sin t)y.$$

The equation for the line should also include the intersection term c, i.e. the line is given by

$$(x_F - r\cos t)x + (y_F - r\sin t)y + c = 0.$$

We find c by noting that the point A lies on the line, and so

$$c = -\frac{1}{2}\left((x_F - r\cos t)^2 + (y_F - r\sin t)^2\right).$$

Now the envelope is given by the solution to the following system:

$$\begin{cases} (x_F - r\cos t)x + (y_F - r\sin t)y - \frac{1}{2}\left((x_F - r\cos t)^2 + (y_F - r\sin t)^2\right) = 0 \\ r\sin tx - r\cos ty - x_F r\sin t + y_F r\cos t = 0. \end{cases}$$

The solution is

$$\begin{cases} x = \frac{2y_F^2\cos t - r(1+x_F^2+y_F^2)\cos t + 2x_F(r - y_F\sin t)}{2r(r - x_F\cos t - y_F\sin t)} \\ y = \frac{2ty_F - 2x_F y_F\cos t + 2x_F^2\sin t - r(1+x_F^2+y_F^2)\sin t}{2r(r - x_F\cos t - y_F\sin t)}. \end{cases}$$

An example of the envelope is then given in Figure 6.

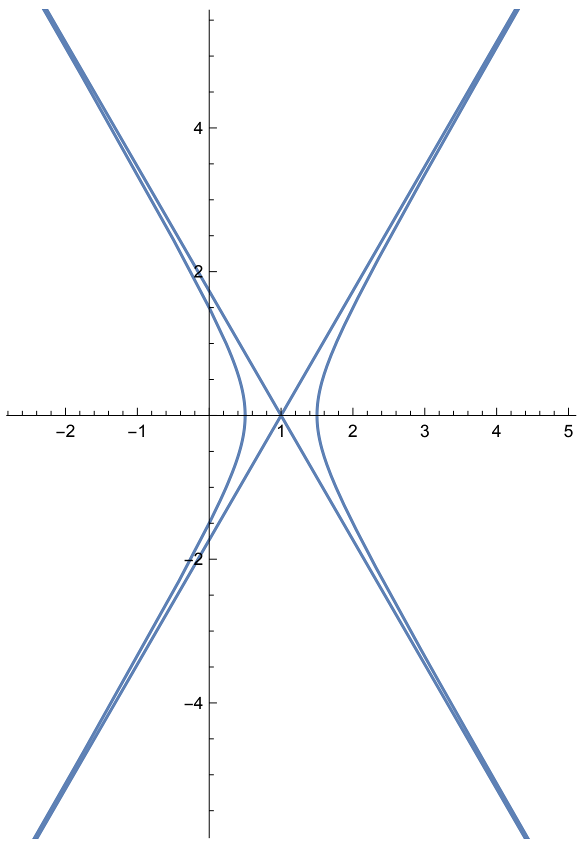


Figure 6: Here the circle is centred at $(0,0)$ and with radius 1. Furthermore, $(x_F, y_F) = (2,0)$.

Exercise VII.23

Note that a singularity here means a point where the derivative vanishes. Sometimes this is also called a critical point.

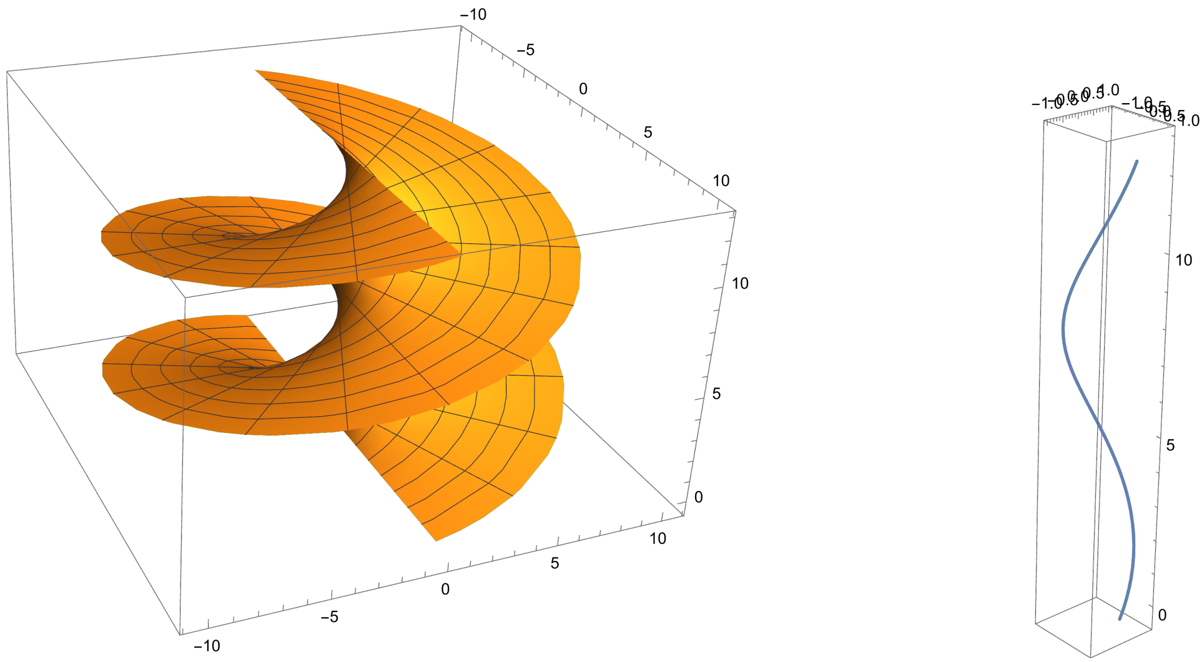
The evolute of a curve $\gamma(t)$ is given by $c(t) = \gamma(t) + n(t)/K(t)$, where $K(t)$ is the curvature, and $n(t)$ is the normal vector to $\gamma(t)$. Consider the differential,

$$\begin{aligned} c'(t) &= \gamma'(t) + \frac{n'(t)K(t) + n(t)K'(t)}{K(t)^2} \\ &= \tau(t) + \frac{-K(t)^2\tau(t) + n(t)K'(t)}{K(t)^2} = \frac{n(t)K'(t)}{K(t)^2} \end{aligned}$$

where $\tau(t) = \gamma'(t)$ is the tangent to $\gamma(t)$, and $n'(t) = -K(t)\tau(t)$. Thus, if $c'(t) = 0$, then so is $K'(t) = 0$.

3 Week 3

Exercise VIII.2



(a) The helicoid with $a = 2$, $t \in \{-10, 10\}$ and $\theta \in \{0, 2\pi\}$.

(b) The helicoid with $a = 2$, $t = 1$ and $\theta \in \{0, 2\pi\}$.

Figure 7: The figures related to Exercise VIII.2. Both figures were created with Mathematica.

The surface (curve) parametrized by

$$(\theta, t) \mapsto (t\cos\theta, t\sin\theta, a\theta)$$

for $t \in (-10, 10)$ $(t = 1)$ is drawn in Figure 7a (Figure 7b). We notice that the curves $t = \mathit{constant}$ are circular helices.

Let us consider the curve parametrized by $\theta \mapsto (0, 0, a\theta)$, i.e. the z-axis. At any point $p = (0, 0, a\theta_0)$ on this curve, we associate a vector $w(p) = (\cos\theta_0, \sin\theta_0, a\theta_0)$. Then $w(p) - p$ defines a straight line given by

$$y\sin\theta_0 + x\cos\theta_0 = 0.$$

We can extend this line in the x- and y-directions by considering $v(p) = (t\cos\theta_0, t\sin\theta_0, a\theta_0)$ for $t \in \mathbb{R}$. Thus, the lines extending from p draws the surface, it is a ruled surface.

Exercise VIII.10

The Whitney umbrella parametrized by $(u, v) \overset{f}{\mapsto} (uv, v, u^2)$ is drawn in Figure 8.

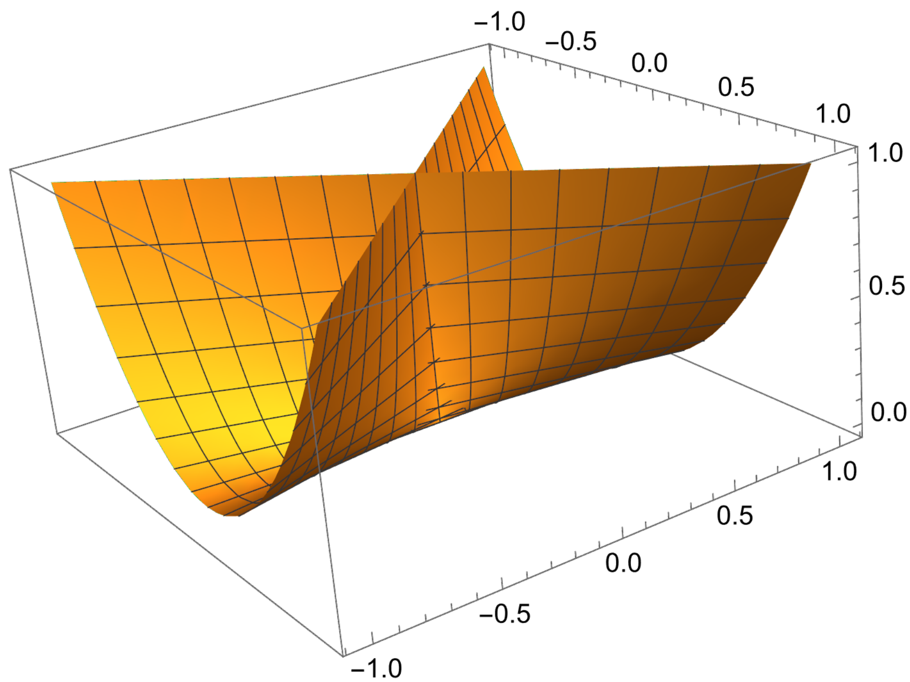


Figure 8: The Whitney umbrella. Figure created with Mathematica.

Note that

$$df|_{u,v} = \begin{pmatrix} v & u \\ 0 & 1 \\ 2u & 0 \end{pmatrix}.$$

This matrix has rank less than 2 only when $u = v = 0$, the origin in $\mathbb{R}^3$. Thus, the origin is a singular point (of rank 1), and it is the only one.

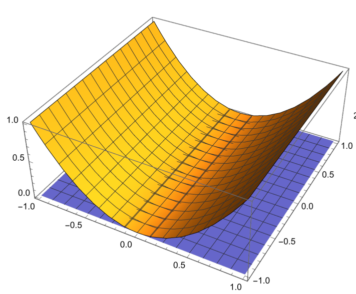
Furthermore, we see that the half-line $(x = y = 0, z > 0)$ is given by $u \neq 0$ and $v = 0$. Thus, there is always two choices giving the same point $(0, 0, u^2)$ on the half-line, namely $\pm u$. Outside the origin, everything is smooth, and so the two values $\pm u$ has to be approached by two different "branches" of the surface. Thus, the half-line consists of only self-intersecting points, in other words, of only double-points. Furthermore, Note that $N(u,v)$ is normal to each direction of $df|_{u,v}$:

$$(v,0,2u)\left(-2\frac{u}{v},2\frac{u^2}{v},1\right)^T=0,$$
$$(u,1,0)\left(-2\frac{u}{v},2\frac{u^2}{v},1\right)^T=0.$$

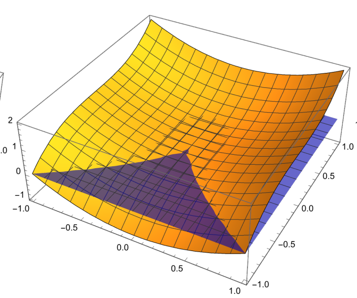
(We could of course also have computed the vector product $(v,0,2u)\times(u,1,0)$.) Thus, $N(u,v)$ is normal to the surface for all $v\neq 0$. As (u,v) tends to 0, then the first column of $df|_{0,0}$ vanishes so all lines are normal to that direction, whilst all lines of the form $(x,0,z)$ are normal to the direction given by the second column. But the normal vector cannot go in multiple directions, only two (also the "negative" direction).

Exercise VIII.11

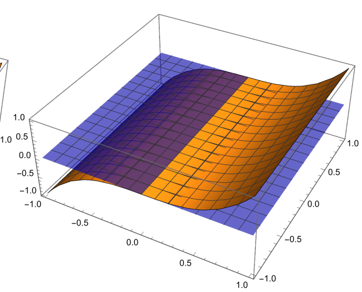
Here, we can find the tangent planes by parametrizing the surfaces by s and t. Set $x=s$, $y=t$, and $z=z(s,t)$, i.e. the surface is parametrized by $f(s,t)=(s,t,z(s,t))$. In **(a)**, we then have $z=s^2$. Computing the derivatives with respect to s and t, we find that the tangent plane is spanned by $\frac{\partial f}{\partial s}=(1,0,2s)$ and $\frac{\partial f}{\partial t}=(0,1,0)$. We are interested in the origin, i.e. $(s,t)=(0,0)$, and so the tangent plane is the xy-plane. The calculation is the same in all cases.



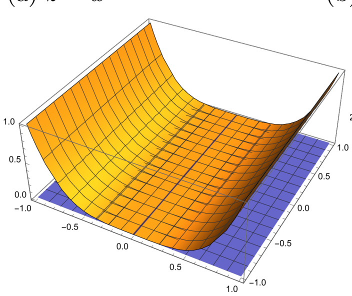
(a) $z=x^2$



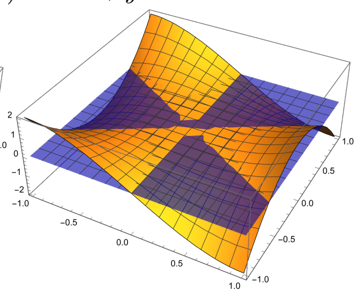
(b) $z=x^2+y^3$



(c) $z=x^3$



(d) $z=x^4$



(e) $z=x^3-3xy^2$

Exercise VIII.15

Consider the cone in Figure 10.

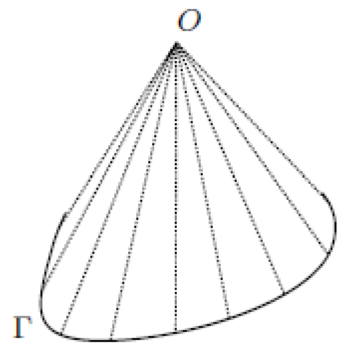


Figure 10: The cone from Audin, VIII Figure 3

This is a ruled surface. Clearly, the normal vector along the lines between Γ and O does not change, so the curvature along that line does not change either. More precisely, let us parametrize the curve Γ by $t \mapsto \gamma(t)$, where $\gamma : \mathbb{R} \to \mathbb{R}^3$. If we assume O is at the origin, then $\gamma(t)$ also gives us the direction from the curve to O, which we denote by $w(t) = \gamma(t)$. If O is not at the origin, then one simply needs to translate $w(t)$, so that $w(t) = O - \gamma(t)$. This will make no difference in the following, so we assume $O = (0, 0, 0)$. Then the cone is parameterized by $(t, u) \mapsto \gamma(t) + uw(t) = (1 + u)\gamma(t) =: f(t, u)$, where u is positive (since Γ is the end of the cone). Then

$$\frac{\partial f}{\partial t} = (1+u)\gamma'(t), \quad \frac{\partial f}{\partial u} = \gamma(t).$$

The normal vector is normal to both of these, so we may consider the cross/vector-product:

$$n(t,u) := \frac{\frac{\partial f}{\partial t} \times \frac{\partial f}{\partial u}}{\left|\left|\frac{\partial f}{\partial t} \times \frac{\partial f}{\partial u}\right|\right|} = \frac{(1+u)\gamma'(t) \times \gamma(t)}{||(1+u)\gamma'(t) \times \gamma(t)||} = \frac{\gamma'(t) \times \gamma(t)}{||\gamma'(t) \times \gamma(t)||},$$

where the $1 + u$ part cancels since it is just a positive constant. Thus, $n(t, u)$ does not depend on u, and so $\frac{\partial n}{\partial u} = 0$. Now, the Gaussian curvature is defined as $K(p) = \det d_p n(t, u)$. Since the column given by $\frac{\partial n}{\partial u}$ of $d_p n(t, u)$ is zero, the determinant is zero, so the curvature is zero. In fact, this is true for all ruled surfaces.