Mechatronics

Week 7 Day 1

Components of a Mechatronics System

Electronic domain

Mechanical, electrical, hydraulic, or thermal domain
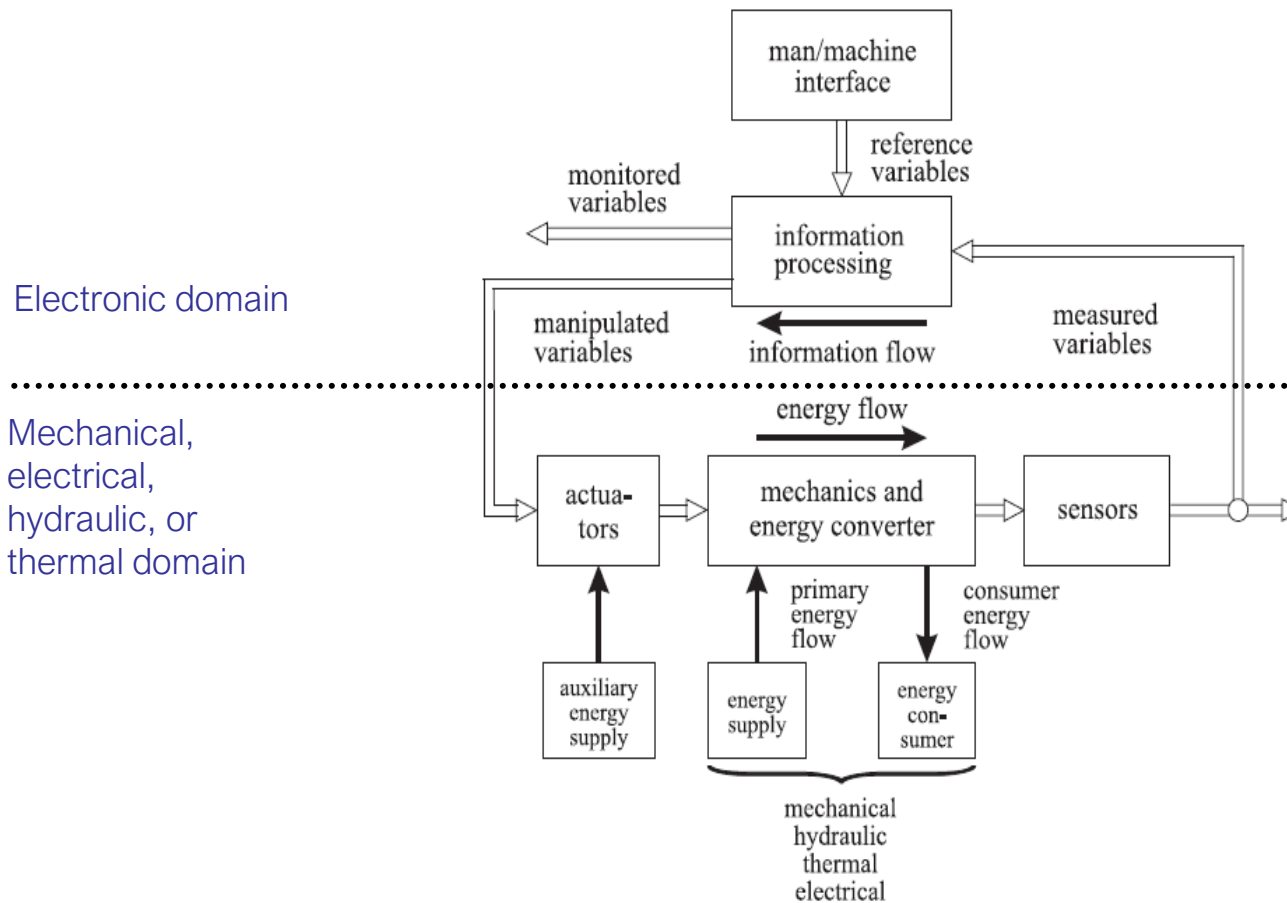
The figure is taken from (Isermann, 2008).

Today's lecture:
Delayed systems

Learning objectives

After today's lecture, you will be able to

- analyse the stability properties of a system through its Nyquist plot
- determine the stability properties of delayed systems
- calculate the phase margin and critical time delay of a system

Additionally, we will refresh-or introduce-some concepts pertaining to Nyquist's criterion

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Time-delays can occur due to different circumstances, eg.

- Signal communication between a terminal and another

- Heavy computational time in the microprocessor (or microcontroller)

- Signal conversion (eg. time consumed by DAC or ADC)

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Time-delays can occur due to different circumstances, eg.

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- Heavy computational time in the microprocessor (or microcontroller)
- Signal conversion (eg. time consumed by DAC or ADC)

Time-delays can affect the stability of the system.

Delays effect on stability

The delay operator can be analysed in the frequency domain. To this end, consider a delay operator of the form

$$y(t) = u(t - T),$$

where T is the time delay, the its transfer function is given by

$$\frac{Y(s)}{U(s)} = e^{-Ts}$$

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Our aim is to determine how time-delays affect the closed-loop system stability.

Delays Effect on Stability-Padé

To analyse the stability of the delayed system we can use the first-order Padé approximation

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Then, we can analyse the closed-loop system as usual:
- Calculate poles
- Check if they lie on the left half s-plane (negative real parts)

Example: Padé approximation

- Take system with transfer function

$$G(s) = \frac{1}{s + 2}$$

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- We add a 2 second delay

$$G_d(s) = e^{-2s} \frac{1}{s + 2}$$

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$$G_d(s) = e^{-2s} \frac{1}{s+2}$$

- Padé approximation of e^{-2s} can be computed

$$e^{-2s} = \frac{1 - \frac{2}{2}s}{1 + \frac{2}{2}s} = \frac{1-s}{1+s}$$

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- Then the delayed system is

$$G_d(s) = \frac{1-s}{1+s} \cdot \frac{1}{s+2} = \frac{1-s}{(1+s)(2+s)}$$

Example: Padé approximation

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$$G_d(s) = \frac{1-s}{1+s} \cdot \frac{1}{s+2} = \frac{1-s}{(1+s)(2+s)}$$

The system now has two poles at $s = -1$, $s = -2$, but it's still stable

Example: Padé approximation

Example: Stable System Becoming Unstable Due to Delay

Original System:

$$G(s) = \frac{1}{s+1}$$

- Stable: Pole at $s = -1$, which is in the left half of the s-plane.

With Delay:

Adding a time delay T, the system becomes:

$$G_{\text{delayed}}(s) = \frac{e^{-sT}}{s+1}$$

Using the **First-Order Pade Approximation:**

$$e^{-sT} \approx \frac{1 - \frac{sT}{2}}{1 + \frac{sT}{2}}$$

$$G_{\text{delayed}}(s) \approx \frac{1 - \frac{sT}{2}}{(s+1)\left(1 + \frac{sT}{2}\right)}$$

Denominator with Delay:

$$\text{Denominator: } \frac{s^2 T}{2} + \left(1 + \frac{T}{2}\right) s + 1$$

Stability Analysis:

- **Without Delay:** The pole is at $s = -1$, system is stable.
- **With Delay:** Poles are determined by the quadratic equation. For larger T, poles may cross to the right half-plane, causing instability.

Critical Delay:

For $T = 2$:

$$\text{Poles: } s = -1.0 \, (\text{stable})$$

If T increases further, instability may occur.

Delays Effect on Stability-Padé

To analyse the stability of the delayed system we can use the Padé approximation

$$e^{-Ts} = \frac{1 - \frac{T}{2}s}{1 + \frac{T}{2}s}$$

Then, we can analyse the closed-loop system as usual:
- Calculate poles
- Check if they lie on the left half s-plane (negative real parts)

⛔**Disadvantage:** For high order systems this task can be challenging

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- Calculate poles
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⬣ Disadvantage: For high order systems this task can be challenging

💡 Alternative? Nyquist stability criterion

It tells us the number of closed-loop poles on the right half plane without calculating them
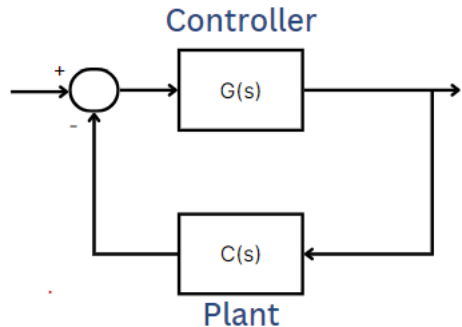
Nyquist's stability criterion

Nyquist Stability Criteria-Preliminaries
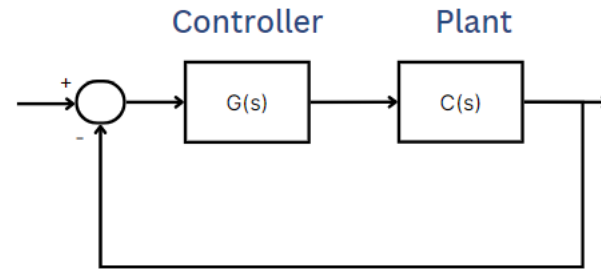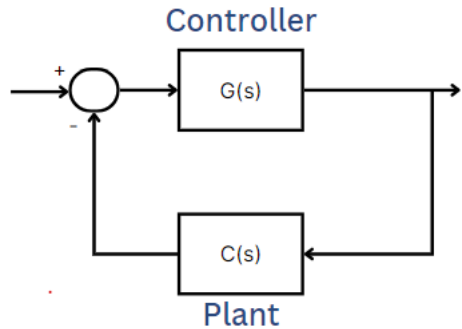
For the systems

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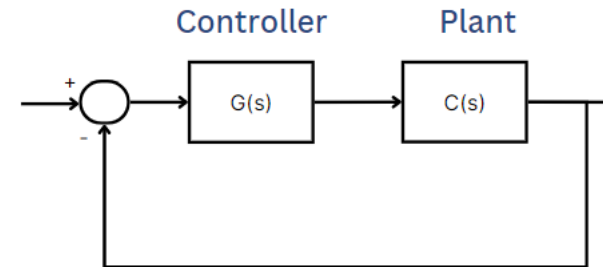
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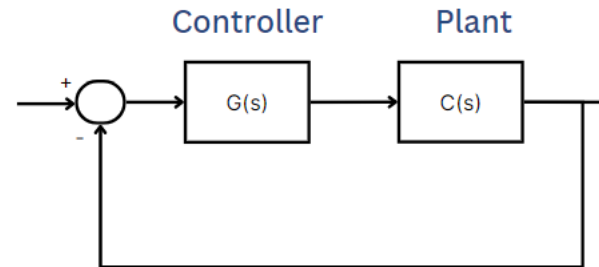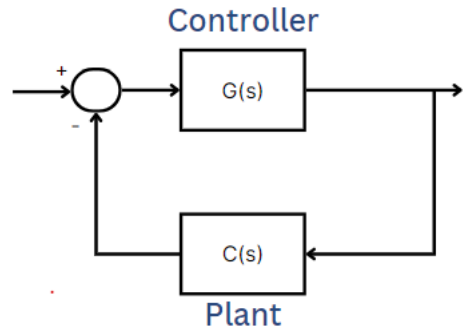we have closed loop transfer functions

$$H_{CL}(s) = \frac{G(s)}{1+C(s)G(s)}$$

$$H_{CL}(s) = \frac{C(s)G(s)}{1+C(s)G(s)}$$

Nyquist Stability Criteria-Preliminaries

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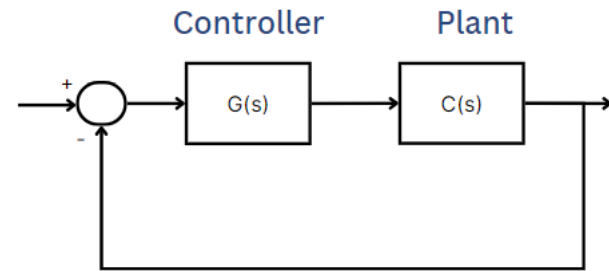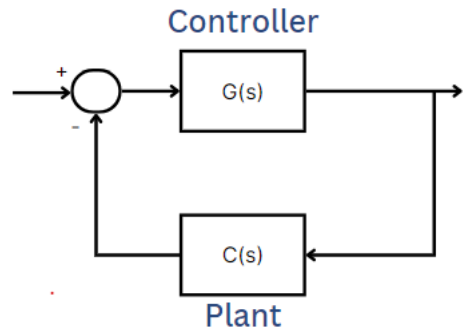
$$H_{CL}(s) = \frac{G(s)}{1+C(s)G(s)} \qquad\qquad H_{CL}(s) = \frac{C(s)G(s)}{1+C(s)G(s)}$$

In general, we have

$$H_{CL}(s) = \frac{L(s)}{1 + C(s)G(s)}$$

where characteristic polynomial $\chi(s) := 1 + C(s)G(s)$ contains information about the closed loop system poles.
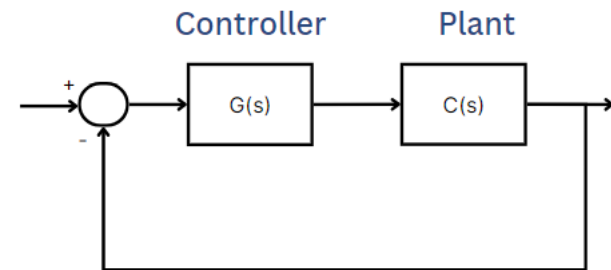
Nyquist Stability Criteria-Preliminaries

Let us establish three important concepts

- Relationship between the poles of $1 + C(s)G(s)$ and those of loop gain $C(s)G(s)$?
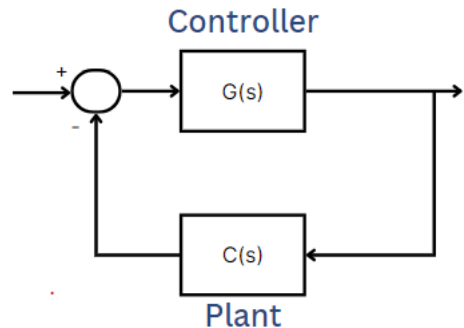
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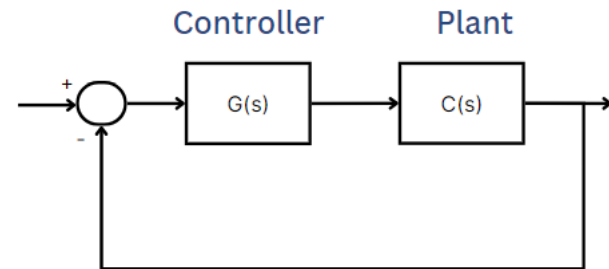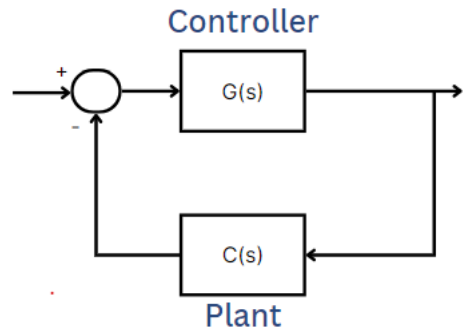
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- Cauchy's argument principle

Nyquist Stability Criteria-Preliminaries

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Let $C(s) := \frac{N_C(s)}{D_C(s)}$ and $G(s) := \frac{N_G(s)}{D_G(s)}$, then

$$C(s)G(s) = \frac{N_C(s)N_G(s)}{D_C(s)D_G(s)}$$

and

$$1 + C(s)G(s) = \frac{N_C(s)N_G(s) + D_C(s)D_G(s)}{D_C(s)D_G(s)}$$

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Since the denominators are the same

$$\boxed{\text{Poles of } 1 + C(s)G(s) = \text{Poles of } C(s)G(s)}$$

Nyquist Stability Criteria-Preliminaries

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$$H_{CL}(s) = \frac{L(s)}{1 + C(s)G(s)}$$

It can be written as

$$H_{CL}(s) = \frac{L(s)}{\dfrac{N_C(s)N_G(s) + D_C(s)D_G(s)}{D_C(s)D_G(s)}} = \frac{L(s)D_C(s)D_G(s)}{N_C(s)N_G(s) + D_C(s)D_G(s)}$$

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Comparing it with

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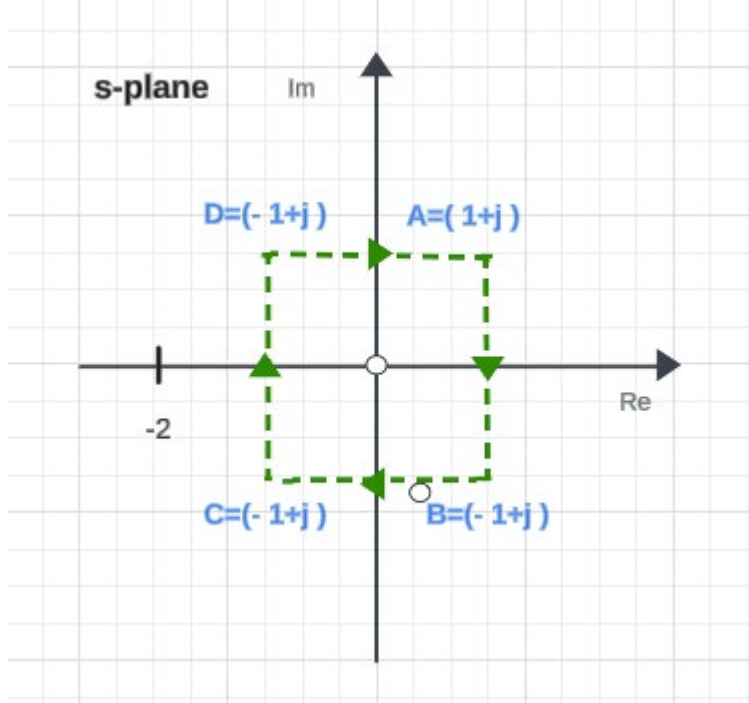we can see that the denominator and numerator are the same

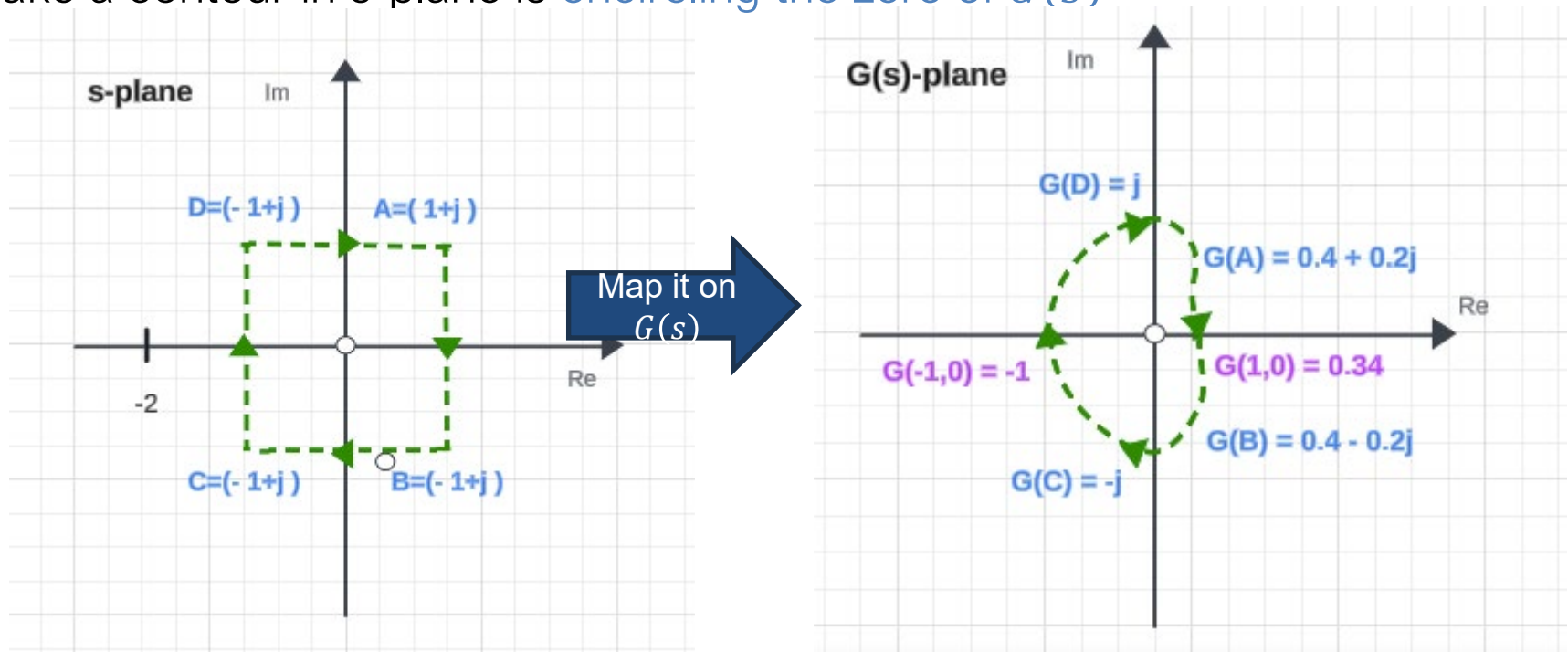$$\boxed{\text{Zeros of } 1 + C(s)G(s) = \text{Poles of } H_{CL}(s)}$$

Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

Consider $G(s) = \dfrac{s}{s+2}$. Zero at $s = 0$

Take a contour in s-plane is encircling the zero of $G(s)$

Nyquist Stability Criteria-Preliminaries

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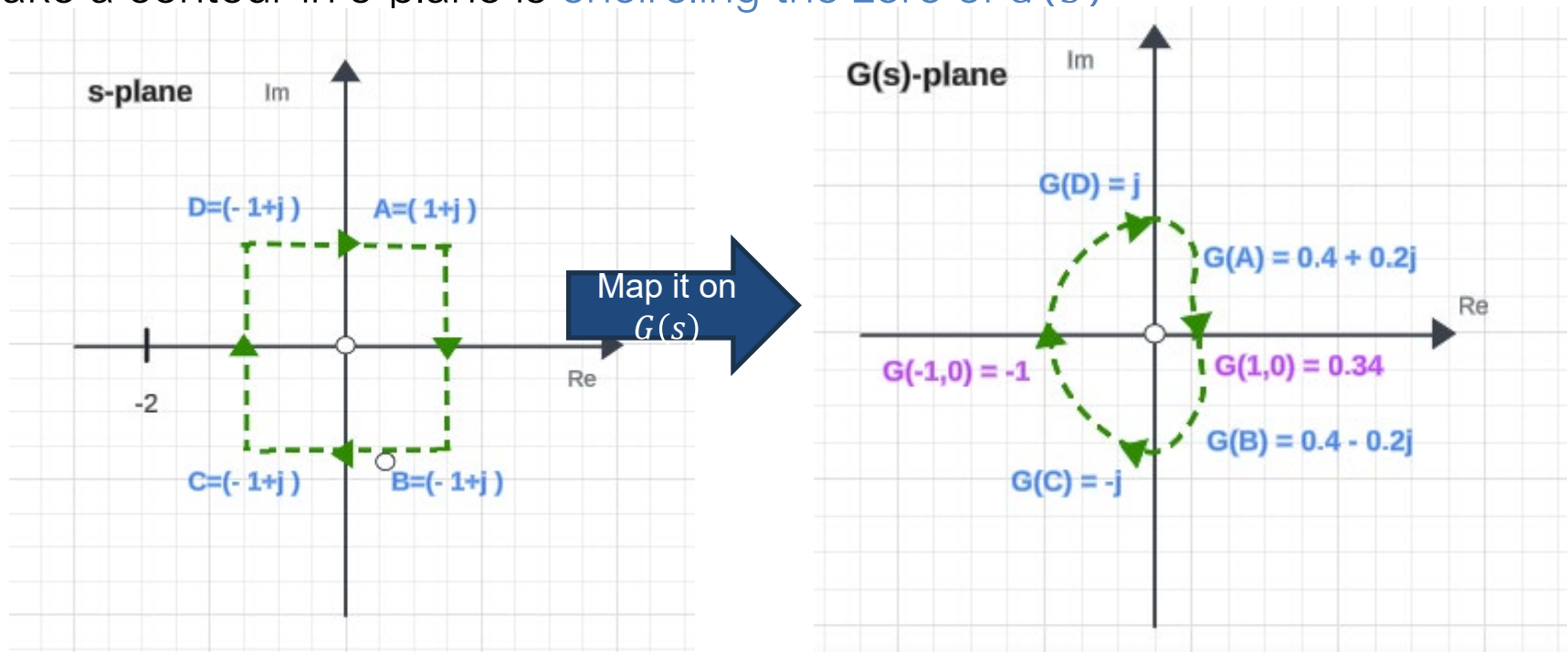# Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

Consider $G(s) = \frac{s}{s+2}$. Zero at $s = 0$

Take a contour in s-plane is encircling the zero of $G(s)$

When you encircle a zero in the s-plane, you encircle the origin in G(s)-plane in clockwise direction
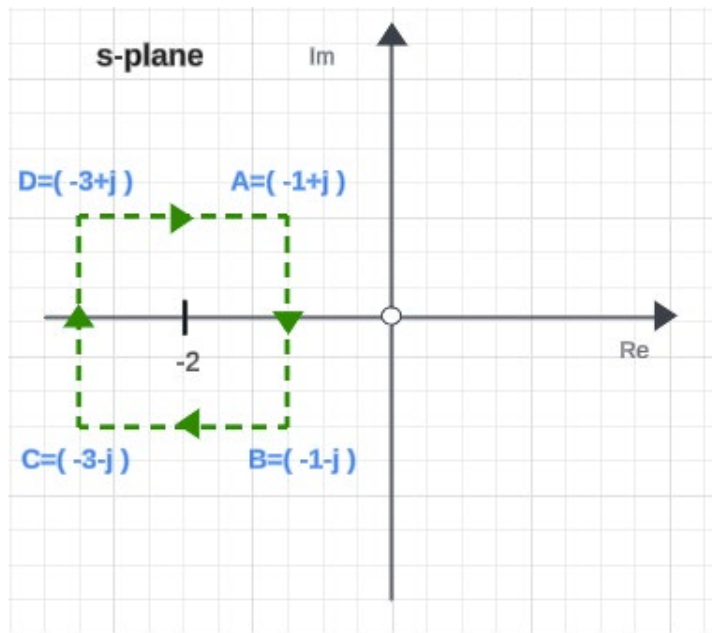
Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

Consider $G(s) = \frac{s}{s+2}$. Pole at $s = -2$

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Nyquist Stability Criteria-Preliminaries

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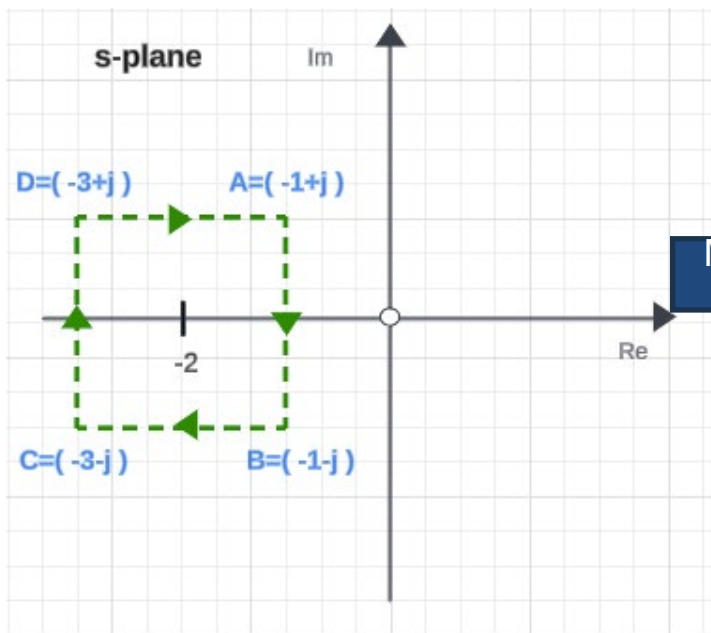Take a contour in s-plane is encircling the pole of $G(s)$

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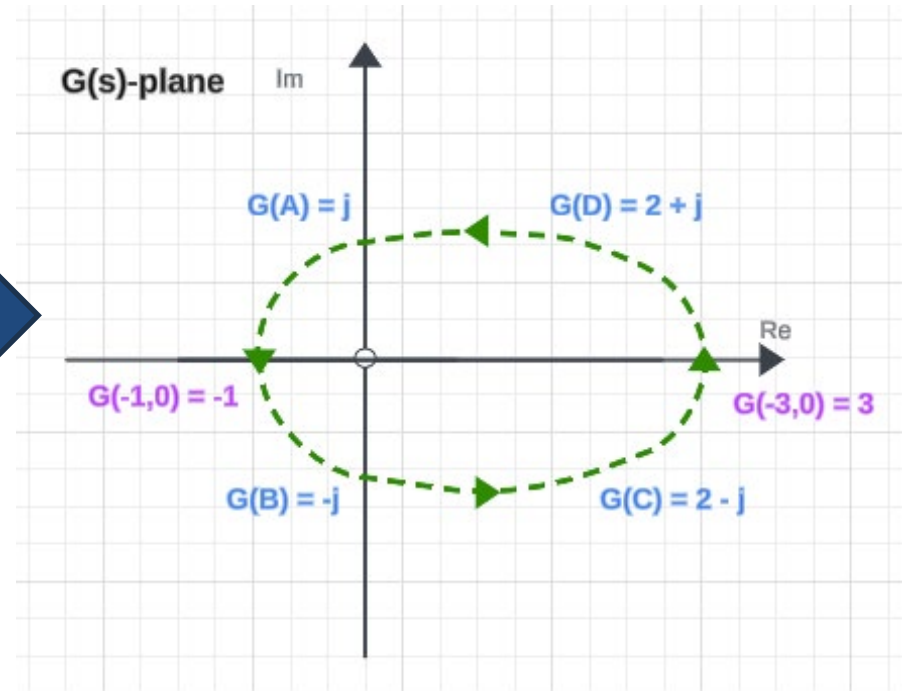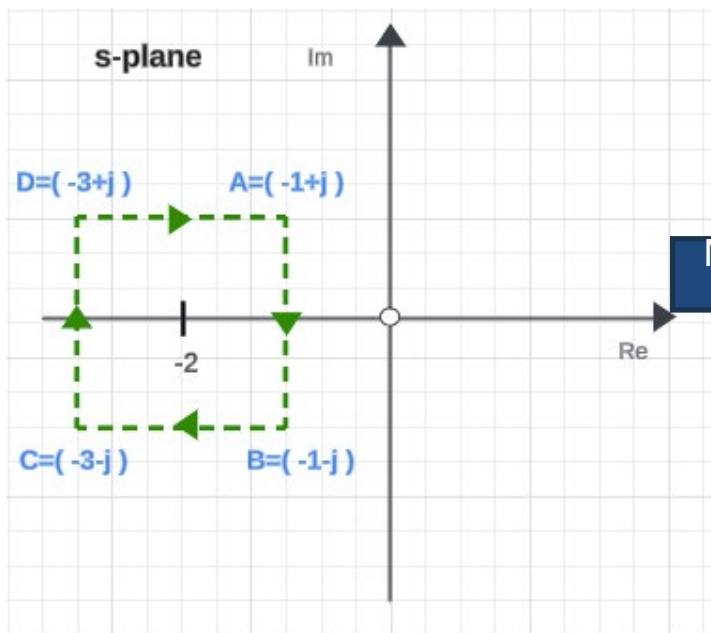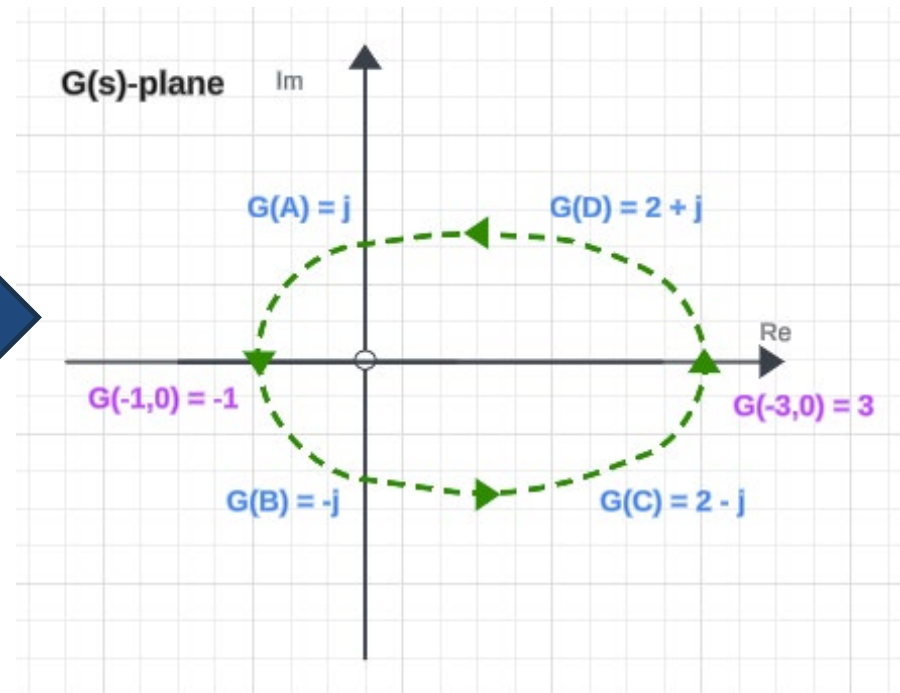
Concept 3. Cauchy's argument principle

Consider $G(s) = \frac{s}{s+2}$. Pole at $s = -2$

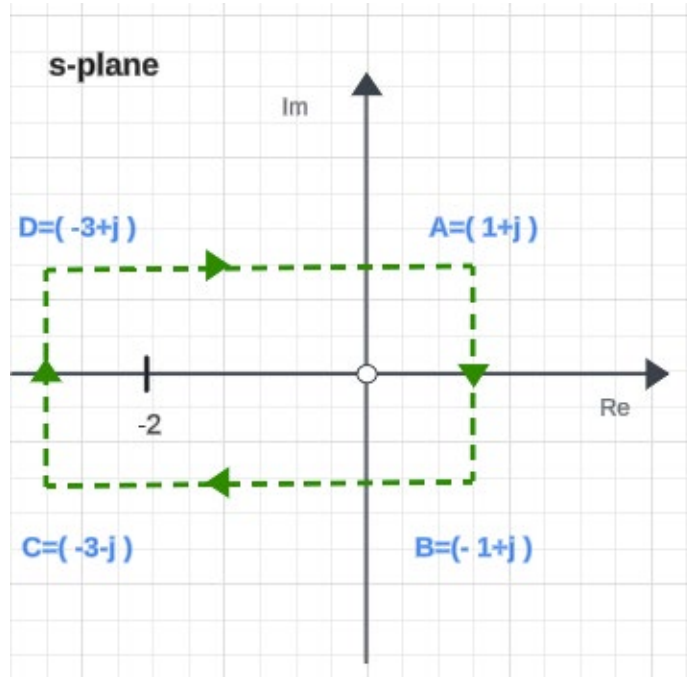Take a contour in s-plane is encircling the pole of $G(s)$

When you encircle a **pole** in the **s-plane**, you **encircle the origin** in **G(s)-plane** in **counter clockwise** direction

Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

Consider $G(s) = \frac{s}{s+2}$. Zero $s = 0$, pole $s = -2$

Take a contour in s-plane is encircling both zero and pole of $G(s)$

Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

Consider $G(s) = \dfrac{s}{s+2}$. Zero $s = 0$, pole $s = -2$

Take a contour in s-plane is encircling both zero and pole of $G(s)$

s-plane

Im

D=(-3+j)   A=(1+j)

-2

C=(-3-j)   B=(- 1+j)

Re

Map it on $G(s)$

G(s)-plane

Im

G(D) = 2 + j

G(A) = 0.4+0.2j

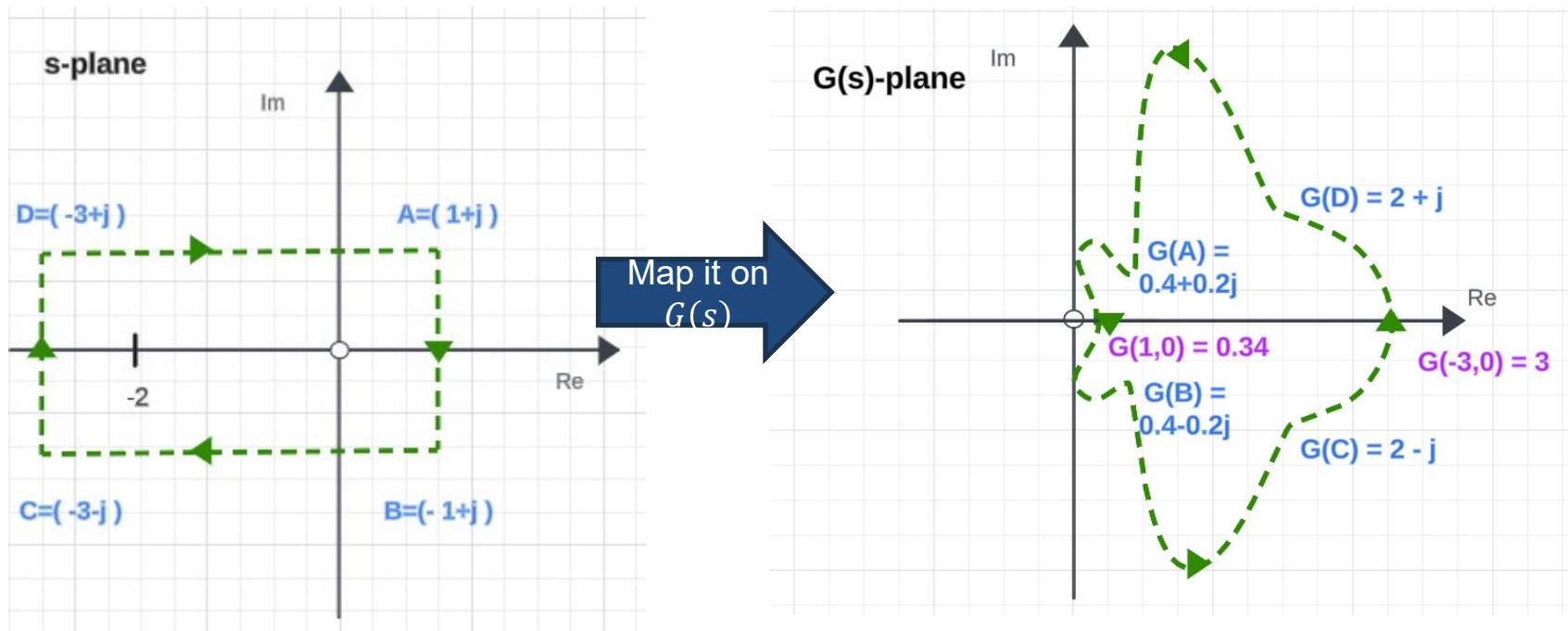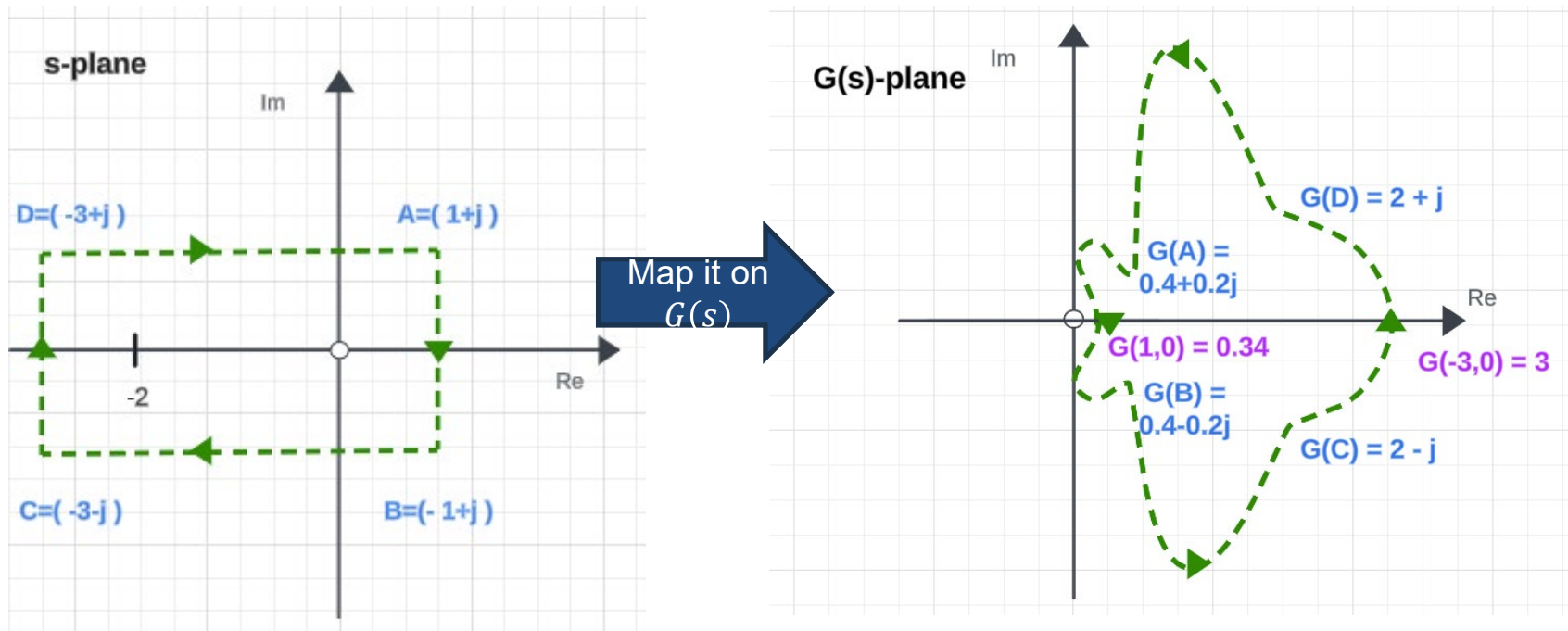G(1,0) = 0.34

G(-3,0) = 3

G(B) = 0.4-0.2j

G(C) = 2 - j

Re

Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

Consider $G(s) = \frac{s}{s+2}$. Zero $s = 0$, pole $s = -2$

Take a contour in s-plane is encircling both zero and pole of $G(s)$

When you encircle both a pole and zero in the s-plane, you do not encircle the origin in G(s)-plane

Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

- The encirclement (in clockwise direction) of the poles and zeros in a contour in the s-plane is related to the encirclement of the origin in the $G(s)$-plane
- The contour must not touch any pole or zero

Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

- The encirclement (in clockwise direction) of the poles and zeros in a contour in the s-plane is related to the encirclement of the origin in the $G(s)$-plane
- The contour must not touch any pole or zero

Generally we have
$$N = Z - P,$$
where
- N is the number of encirclements of origin in the $G(s)$-plane
- Z is the number of encircled zeros of $G(s)$ in the s-plane
- P is the number of encircled poles of $G(s)$ in the s-plane

Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

Consider $G(s) = \dfrac{s}{s+2}$
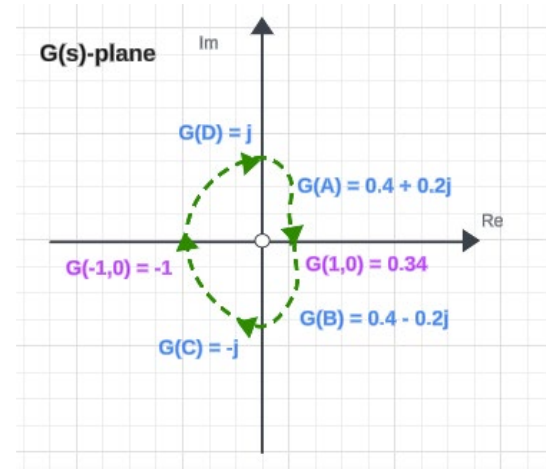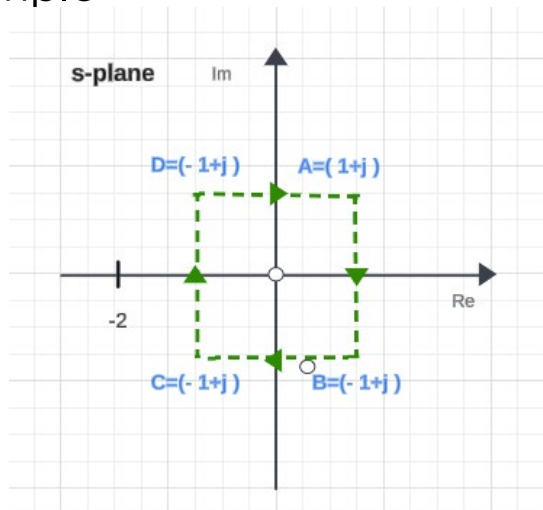
In the example

we have $Z = 1, P = 0$, then
$$N = Z - P = 1$$
which implies one encirclement of origin in the G(s)-plane in clock-wise direction

Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

Consider $G(s) = \dfrac{s}{s+2}$

In the example

we have $Z = 0, P = 1$, then
$$N = Z - P = -1$$
which implies one encirclement of origin in the G(s)-plane in counterclock-wise direction

Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

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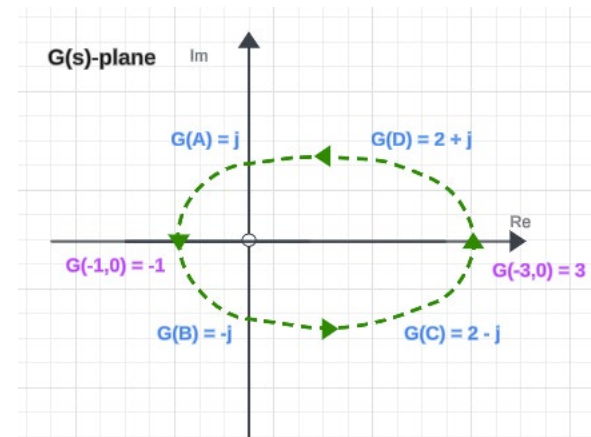In the example

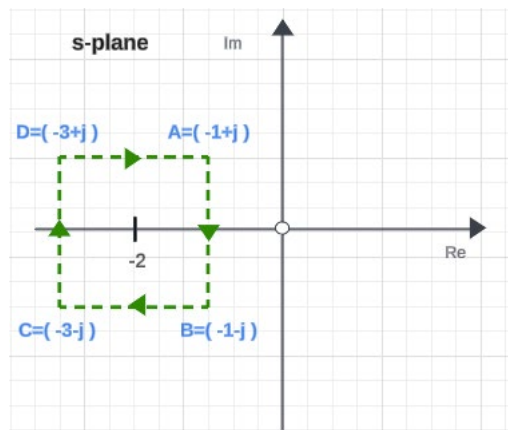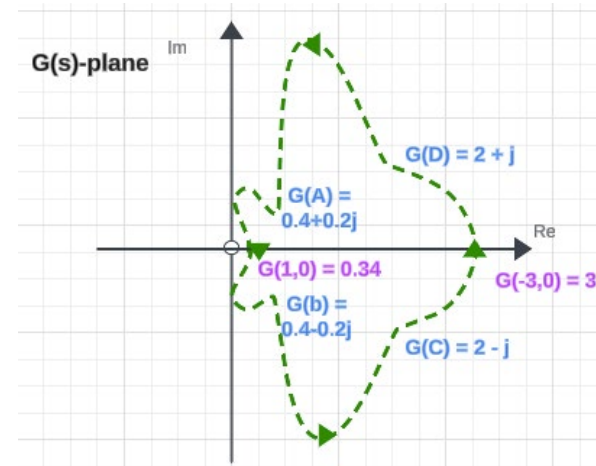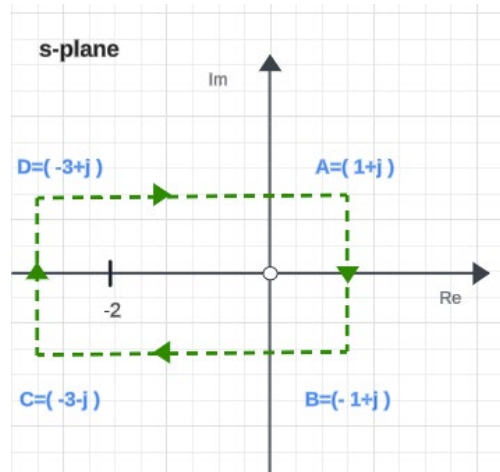we have $Z = 1, P = 1$, then
$$N = Z - P = 0$$
which implies no encirclement of origin in the G(s)-plane

Nyquist Stability Criterion

How do we use Nyquist Stability Criterion?

- We can construct a contour that covers the entire right-half plane
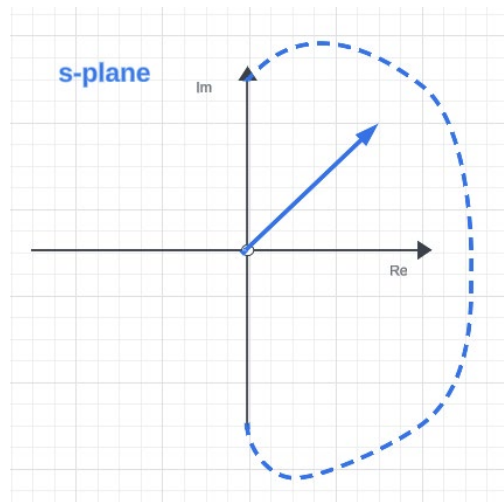
Nyquist Stability Criterion

How do we use Nyquist Stability Criterion?

- We can construct a contour that covers the entire right-half plane

- We map the contour on right half s-plane to the $1 + C(s)G(s) -$plane. We call this Nyquist plot
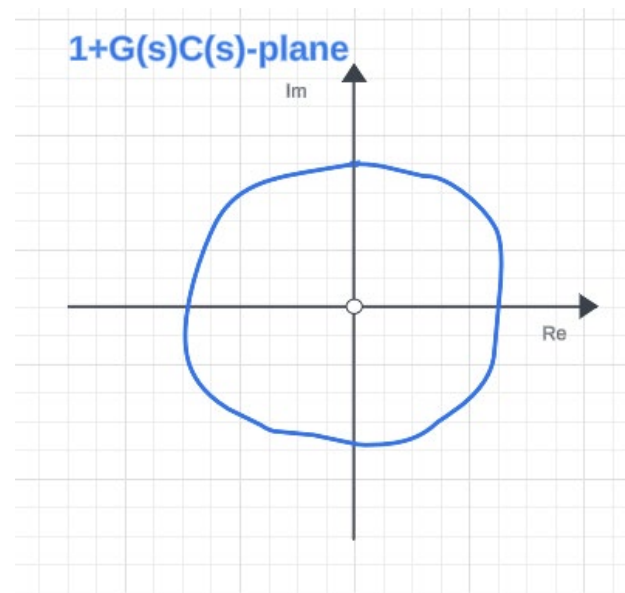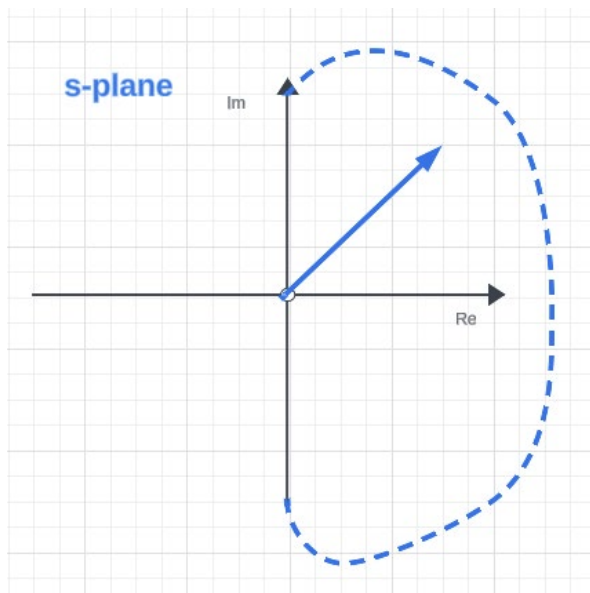
Nyquist Stability Criterion

- Then, we use Cauchy's argument principle (**Concept** 3)

$$N = Z - P,$$

where

- N is the number of encirclements of origin in the $1 + G(s)C(s)$-plane, which is given by the Nyquist plot

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  where

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    Concept 2. Zeros of $1 + C(s)G(s) =$ Poles of $H_{CL}(s)$
    which we can use to analyse closed loop stability

- Note that for stability of the closed loop system we want no closed-loop poles on the right half s-plane. So we require no zeros of $1 + C(s)G(s)$ encircled, which means $Z = 0$. Then the stability criteria becomes
$$\boxed{0 = N + P}$$

Nyquist Stability Criterion

- Instead of mapping $1 + C(s)G(s)$ we can directly map $C(s)G(s)$, and look for encirclements of the point (-1,0)
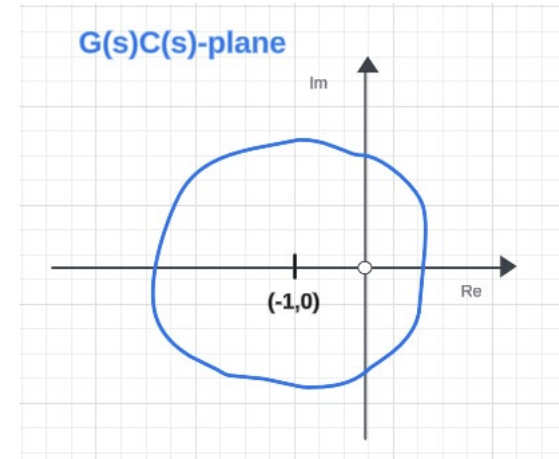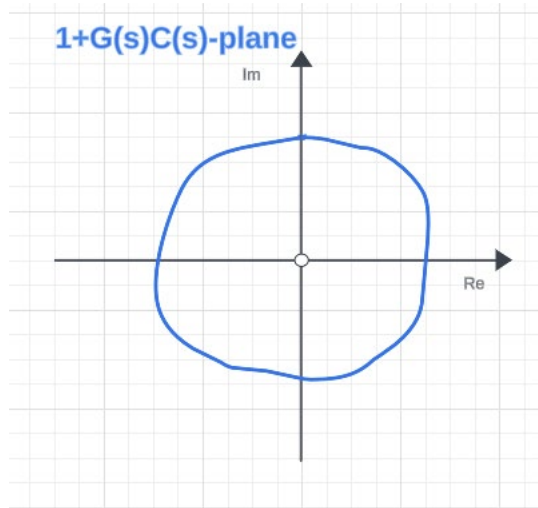
$$1 + C(s)G(s) \Leftrightarrow C(s)G(s) = -1$$

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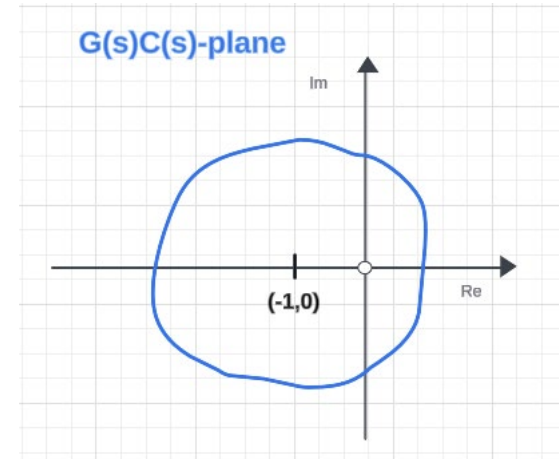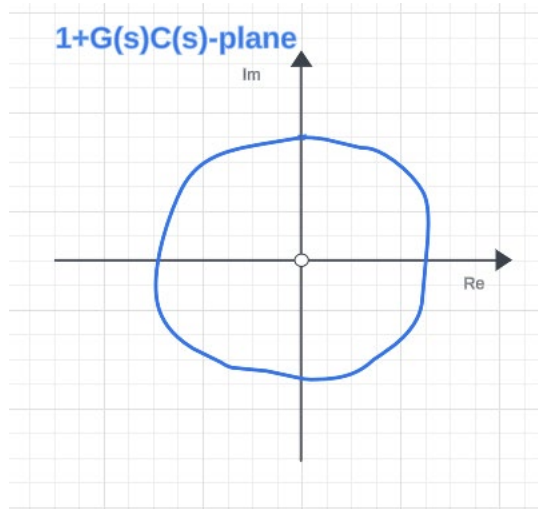
Then we can get $\boldsymbol{N}$ by simply looking at encirclements of map $\boldsymbol{C(s)G(s)}$ around point $(\boldsymbol{-1, 0})$

Nyquist Stability Criterion

In conclusion,
- We can get N by simply looking at encirclements of map $C(s)G(s)$ around point $(-1, 0)$
- We can get P by calculating poles of $C(s)G(s)$
- We can compute $Z = N + P$
- We have stability if $Z = 0$

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- We can get N by calculating poles of $C(s)G(s)$
- We can compute $Z = N + P$
- We have stability if $Z = 0$

Nyquist criterion:
A feedback control system is stable if and only if the number of counterclockwise encirclements of the point (-1,0) is equal to the number of poles of $G(s)C(s)$ on the right-half of the complex plane

Nyquist Stability Criterion

- Given the contour on the entire right half s-plane, its maping to the $C(s)G(s)$-plane (of $C(s)G(s)$) is called a **Nyquist plot**

- The Nyquist plot is the plot of $C(s)G(s)$ with $s = j\omega$ where ω goes from $-\infty$ to ∞

- From the Nyquist plot, we can check the number of counterclockwise encirclements of the point $(-1,0)$. Thus, we can determine the **stability** of the closed-loop system.

Example: Construction of Nyquist plot

Consider

$$C(s)G(s) = \frac{s+3}{s^2+2s+1} \text{ with } G(s) = \frac{1}{s^2+2s+1}, \qquad C(s) = s+3$$

Replacing $s = j\omega$, we obtain

$$C(j\omega)G(j\omega) = \frac{3+j\omega}{1-\omega^2+j2\omega}$$

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*Note that at $\omega = 0, C(j\omega)G(j\omega) = 3$ and at $\omega \to \pm\infty, C(j\omega)G(j\omega) \to 0$

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We can plot this points

C(s)G(s)-plane   Im

Re

ω → ∞
C(jω)G(jω)=0

ω=0
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$$C(j\omega)G(j\omega) = \frac{3 + j\omega}{1 - \omega^2 + j2\omega}$$

We can find the real and imaginary parts of $C(j\omega)$G$(j\omega)$

$$C(j\omega)G(j\omega) = \frac{3 + j\omega}{(1 - \omega^2) + j2\omega} \cdot \frac{(1 - \omega^2) - j2\omega}{(1 - \omega^2) - j2\omega} = \frac{[3 - \omega^2] - j[w(5 + \omega^2)]}{1 + 2\omega^2 + \omega^4}$$

Example: Construction of Nyquist plot

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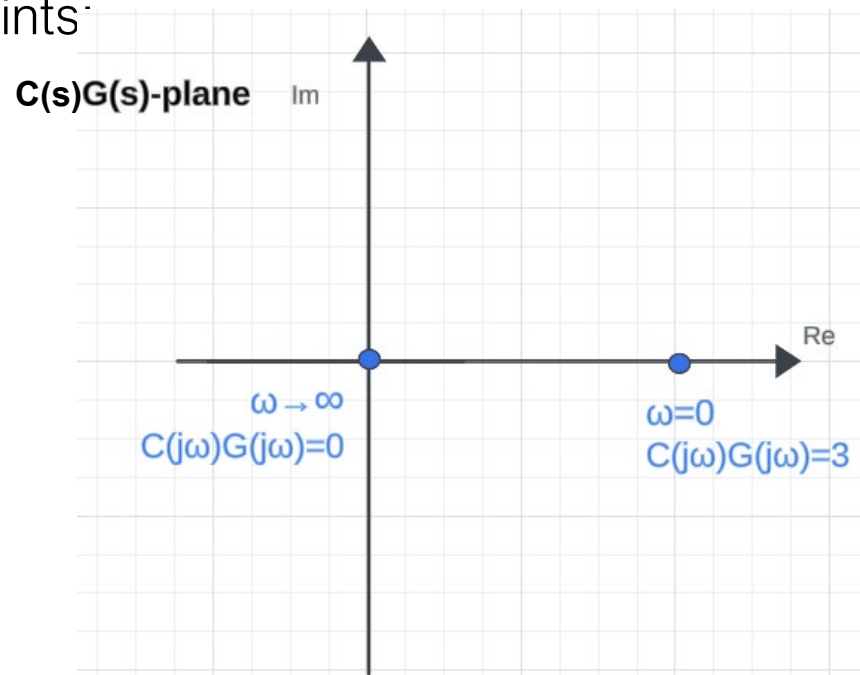$$C(s)G(s) = \frac{s+3}{s^2 + 2s + 1} \text{ with } G(s) = \frac{1}{s^2 + 2s + 1}, \qquad C(s) = s + 3$$

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which gives

$$\Re\{C(j\omega)G(j\omega)\} = \frac{3 - \omega^2}{1 + 2\omega^2 + \omega^4}, \Im\{C(j\omega)G(j\omega)\} = \frac{-\omega(5 + \omega^2)}{1 + 2\omega^2 + \omega^4}$$

Example: Construction of Nyquist plot

Real and imaginary parts of $C(j\omega)G(j\omega)$

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*Note that $\Re\{C(j\omega)G(j\omega)\} = 0$ when $\omega \to \infty$ (already plotted) and when $= \pm\sqrt{3}$;
We can calculate value of $\Im\{C(j\omega)G(j\omega)\}$ at which $\Re\{C(j\omega)G(j\omega)\} = 0$

$$\Im\{C(j\sqrt{3})G(j\sqrt{3})\} = \Im\{C(-j\sqrt{3})G(-j\sqrt{3})\} = -\frac{\sqrt{3}}{2}$$

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Real and imaginary parts of $C(j\omega)G(j\omega)$

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We can plot the crossings of real and imaginary lines:
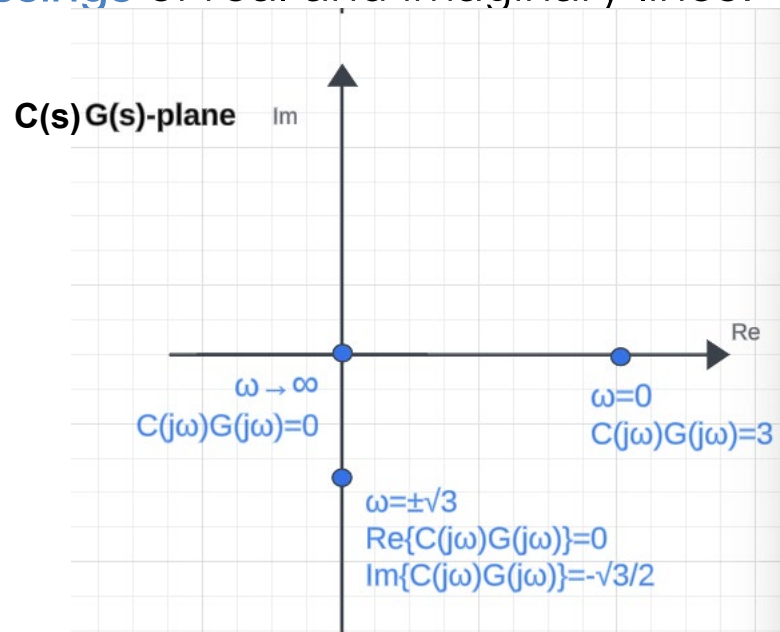
Example: Construction of Nyquist plot

We have plotted $C(j\omega)G(j\omega)$ for $\omega = 0$ and $\omega \to \pm\infty$, and real and imaginary line crossings.

*Also note that $\Re\{C(j\omega)G(j\omega))\} < 0$ for large values of ω, so negative values of $\boldsymbol{\Re\{C(j\omega)G(j\omega))\}}$ are expected before approaching $\omega \to \pm\infty$

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- Knowing this we can approximately connect the points

C(s)G(s)-plane   Im

Re

$\omega \to \infty$
C(jω)G(jω)=0

ω=0
C(jω)G(jω)=3

ω=±√3
Re{C(jω)G(jω)}=0
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Example: Construction of Nyquist plot

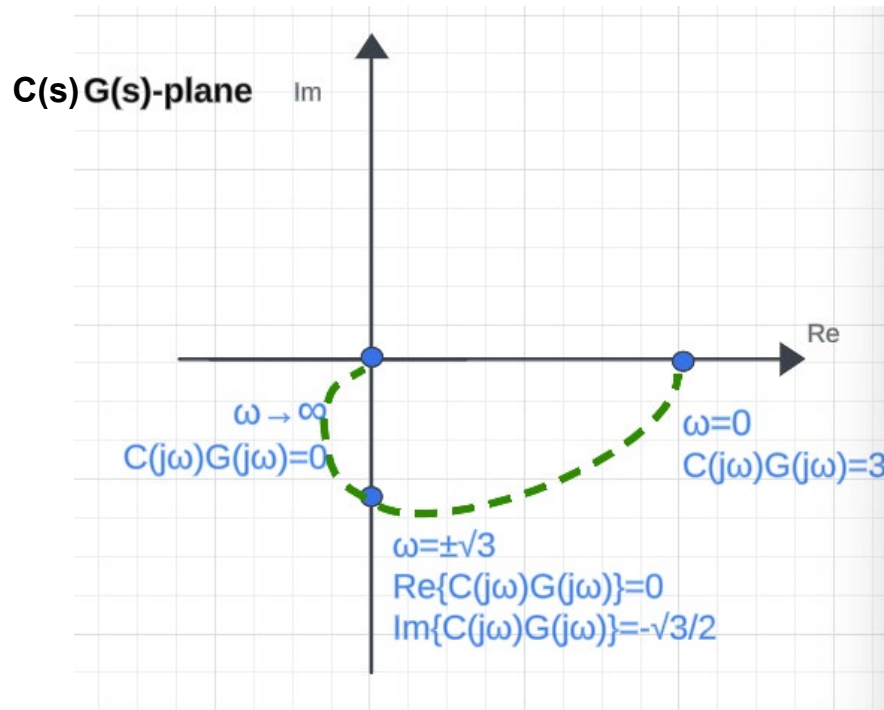We have plotted $C(j\omega)G(j\omega)$ for $\omega = 0$ and $\omega \to \pm\infty$, and real and imaginary line crossings.

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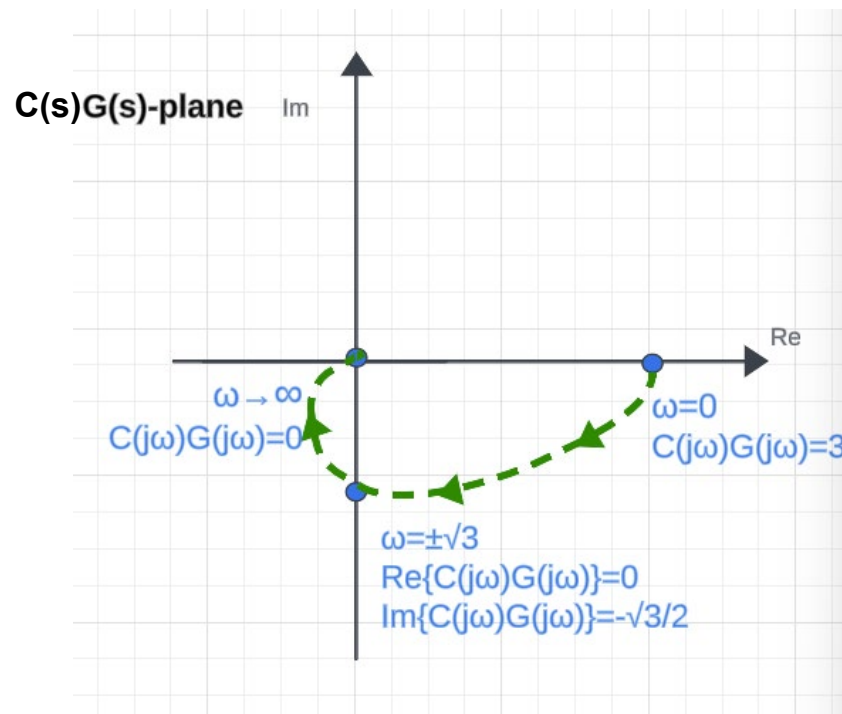
- Knowing this we can approximately connect the points
- The direction is plotted starting from $\omega = 0$ towards $\omega \to \infty$

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Re

$\omega \to \infty$
C(jω)G(jω)=0

ω=0
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ω=±√3
Re{C(jω)G(jω)}=0
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Example: Construction of Nyquist plot

The Nyquist plot is symmetric around the Real line, so we can get an estimation of the full plot:
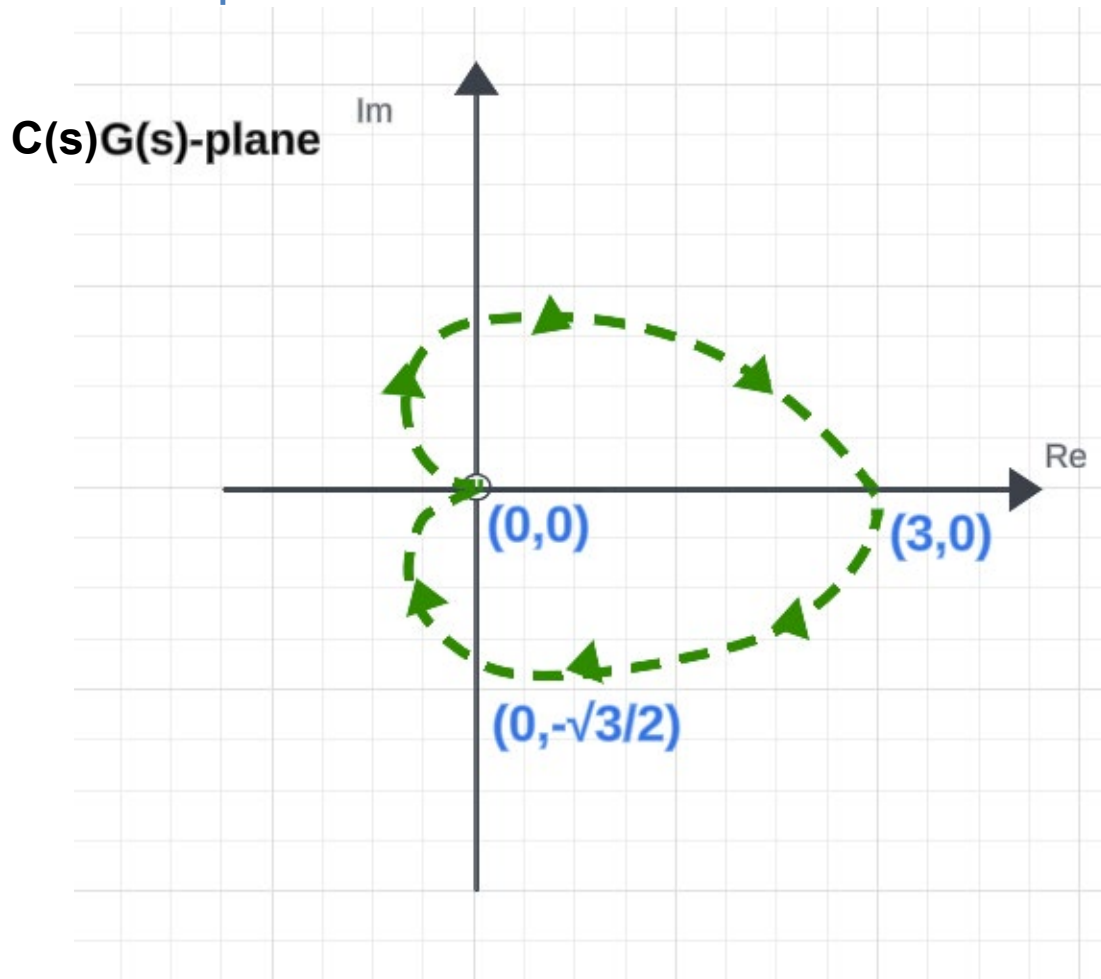
Example: Construction of Nyquist plot

In MATLAB

Create the system: sys=tf([0 1 3], [1 2 1])
Use function to obtain plot: nyquist(sys)

Example: Stability using Nyquist

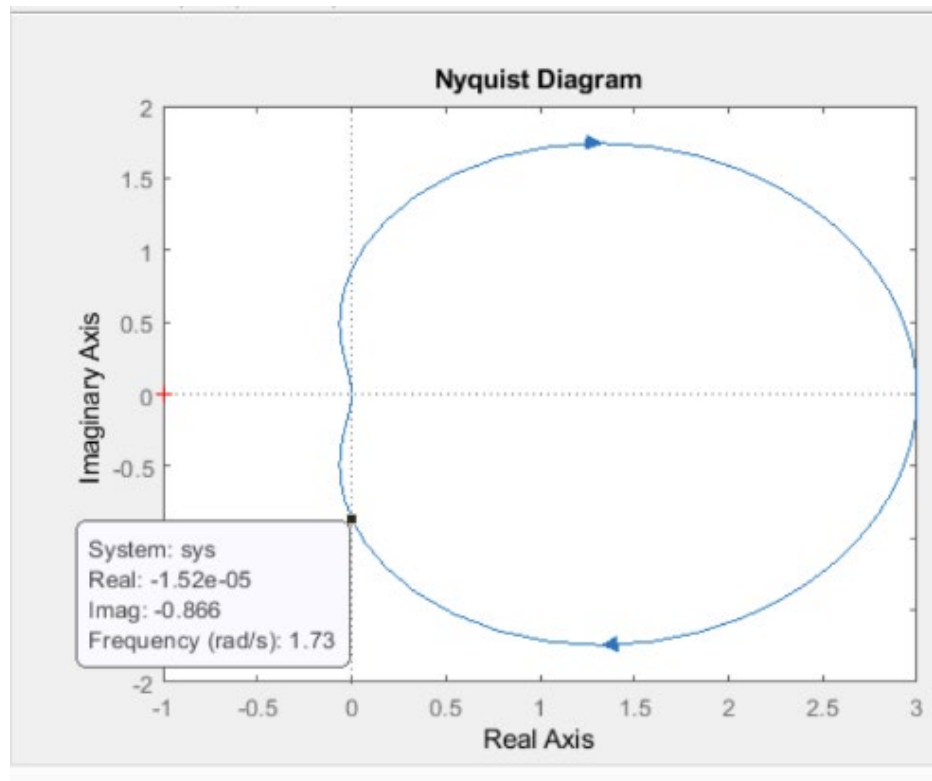Take a system with transfer function $G(s)$ and controller with transfer function $C(s)$.
The loop gain is defined as $L(s) := G(s)C(s)$ and is stable
The Nyquist plot looks as follows:

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Nyquist Diagram

Since $L(s)$ is stable we have no poles in the right-half plane.

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Since $L(s)$ is stable we have no poles in the right-half plane.
We see no encirclements of point $(1, 0)$ in the Nyquist plot.
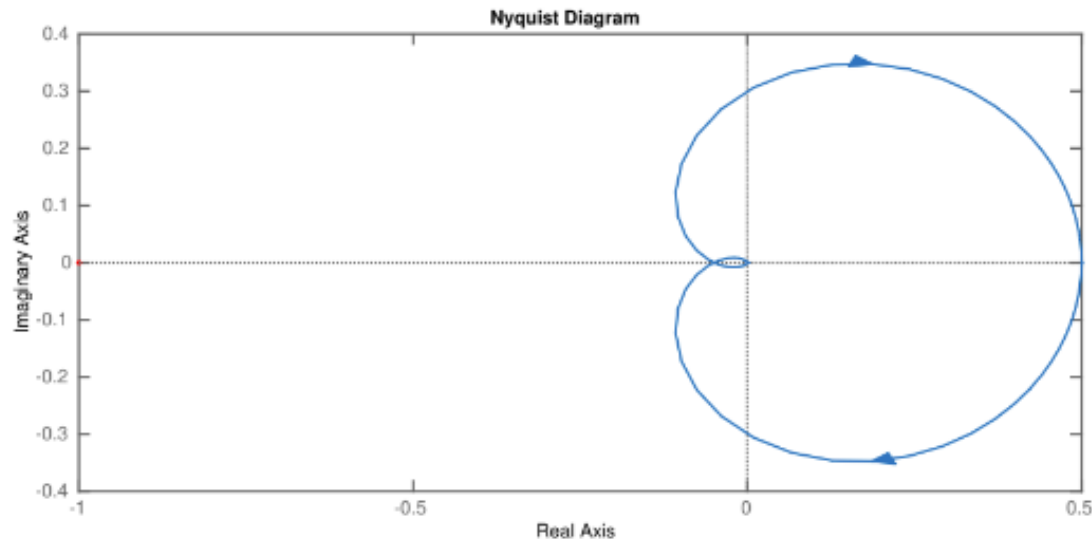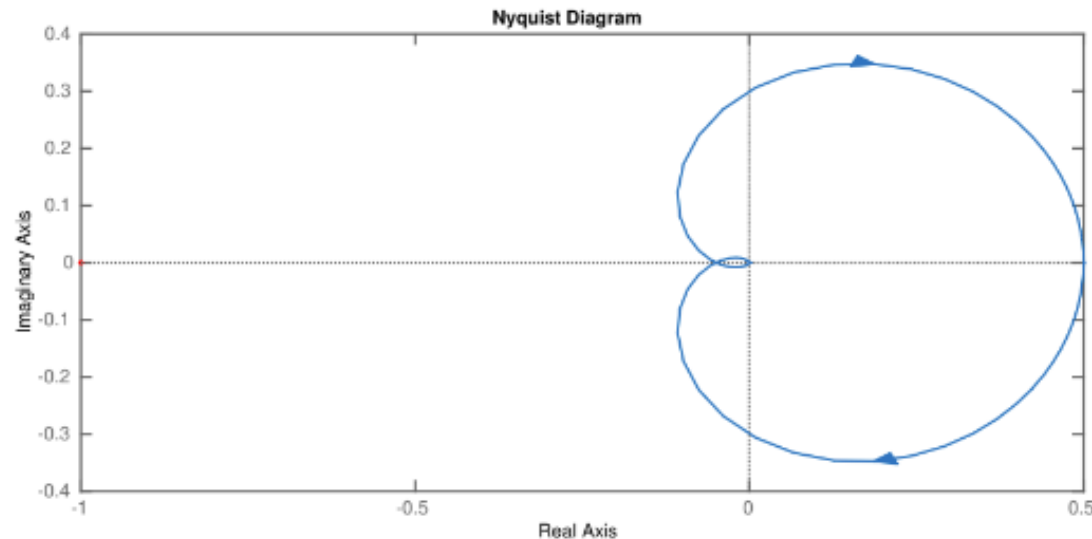
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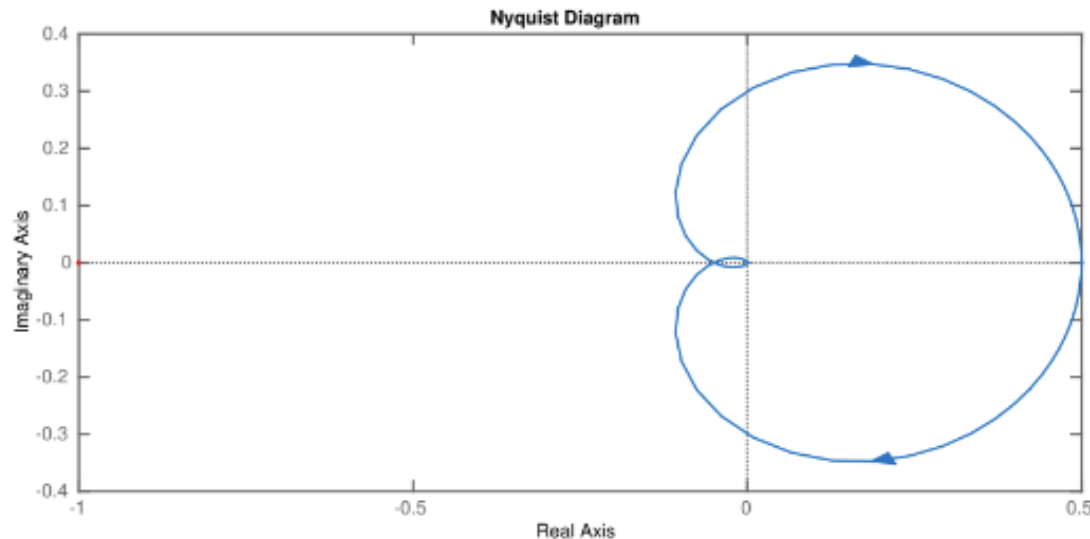The Nyquist plot looks as follows:

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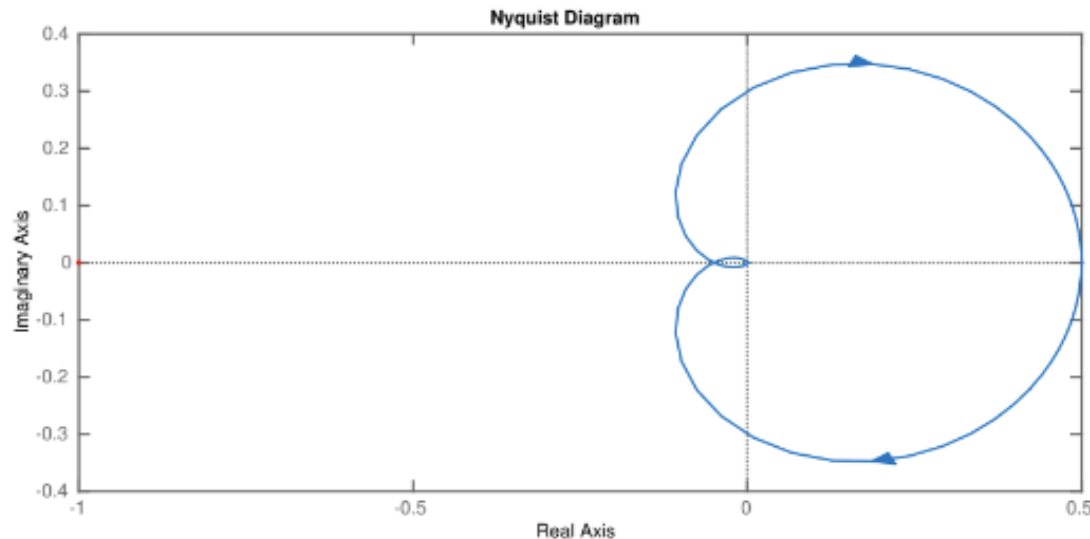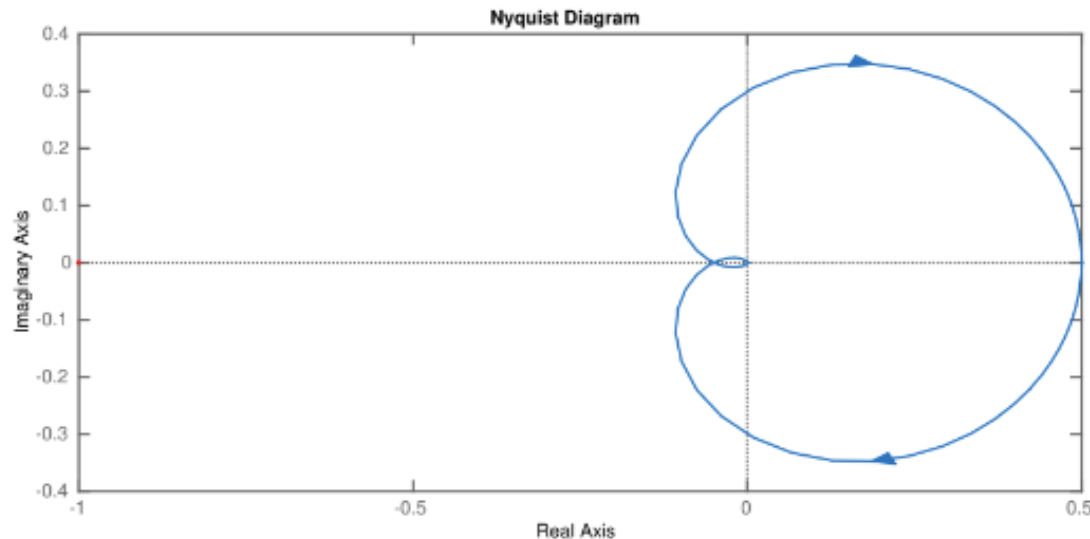Since $L(s)$ is stable we have no poles in the right-half plane.
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⚠ How do we take into account delays? Phase and gain margin

Gain margin and phase margin

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- They determine how much the gain and phase of the system can shift before they reach instability

- They can be **computed** for a system

- They can be **obtained** from **Nyquist plot**

- The **critical time delay** the system can undergo until reaching instability can be computed from phase margin. This shows us how robust the system is to delays.

Gain and phase margin from Nyquist

Take a Nyquist plot

    *Due to symmetry only one half is enough

    *We care about at intersection with unit circle and real axis so for simplicity we only
    draw relevant segment

C(s)G(s)-plane

Re
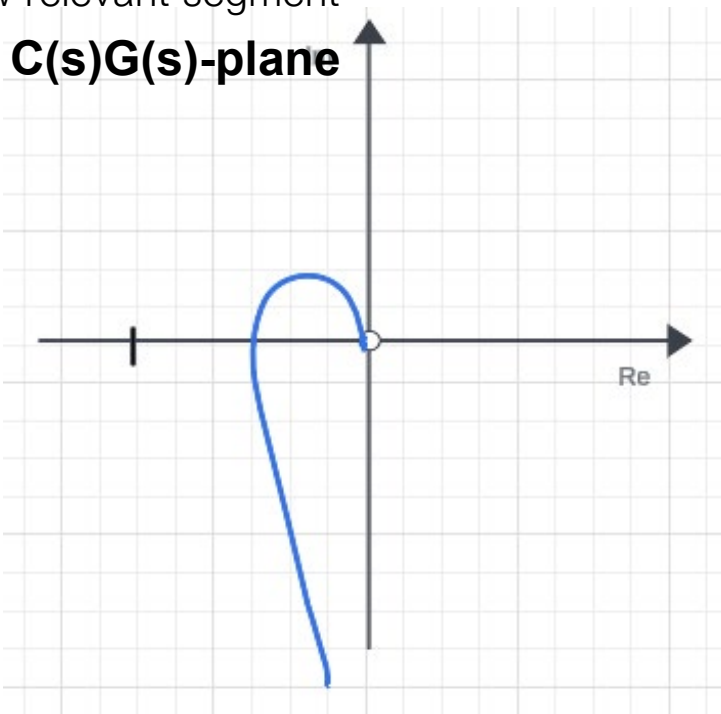
Gain and phase margin from Nyquist

We find crossing of Nyquist plot and unitary circle

C(s)G(s)-plane

(-1,0)

Re
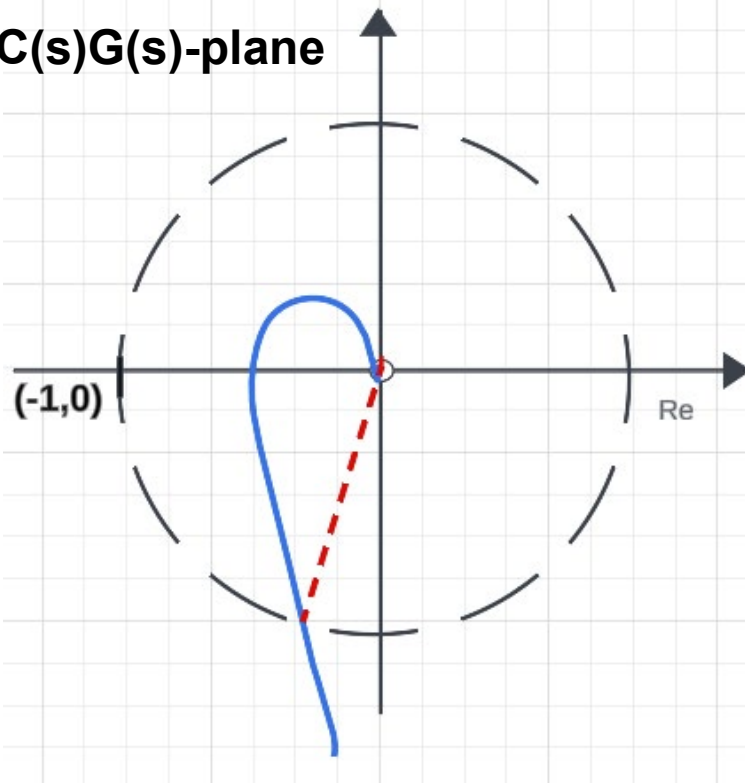
Gain and phase margin from Nyquist

We find crossing of Nyquist plot and unitary circle
The angle between positive real line and said crossing is phase margin

C(s)G(s)-plane
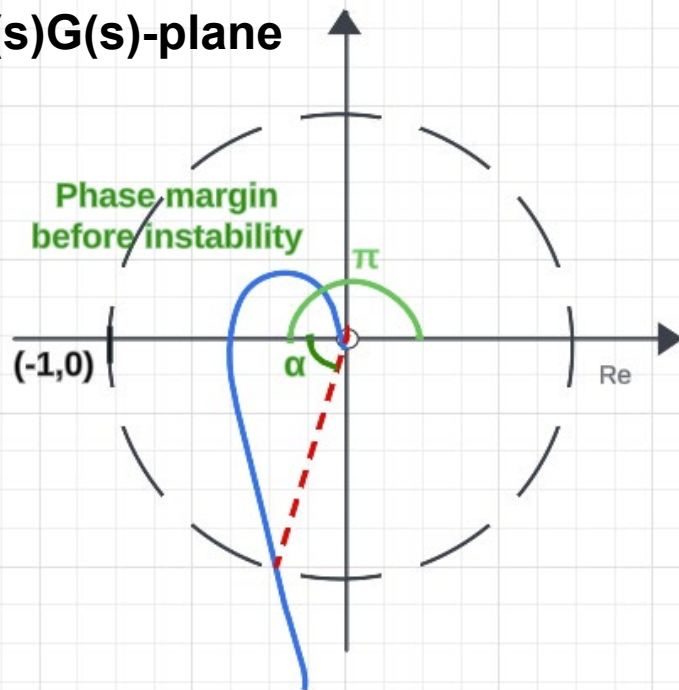
$$P_m = -\pi - \alpha$$

Gain and phase margin from Nyquist

Crossing of Nyquist plot and real line gives point (-1/a, 0)

C(s)G(s)-plane

(-1,0)   -1/a   α   Re
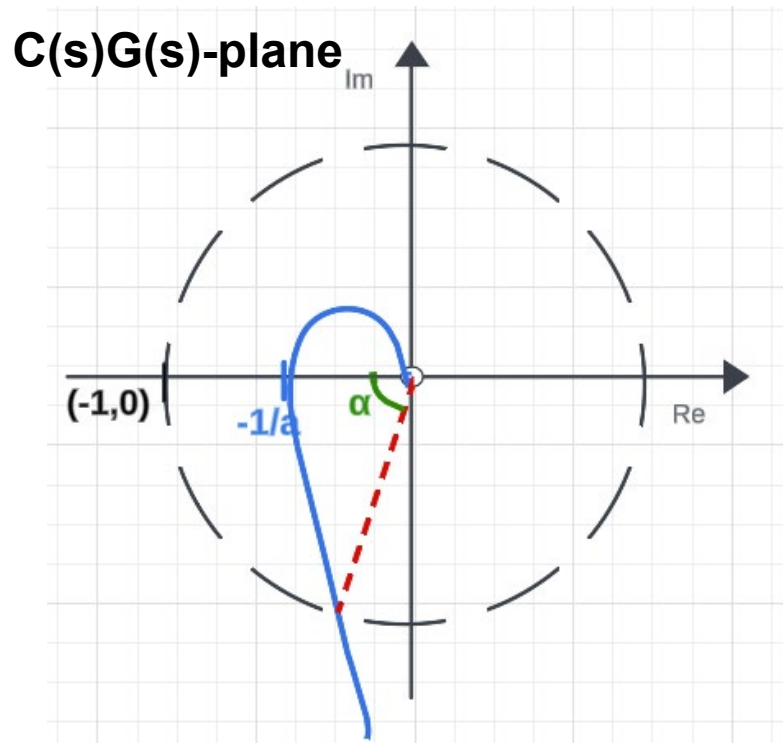
$$G_m = 20 \log a$$

Gain and phase margin from Nyquist

Crossing of Nyquist plot and real line gives point (-1/a, 0)
Gain margin is difference until point (-1/a,0) reaches (-1,0)

$$G_m = 20 log a$$
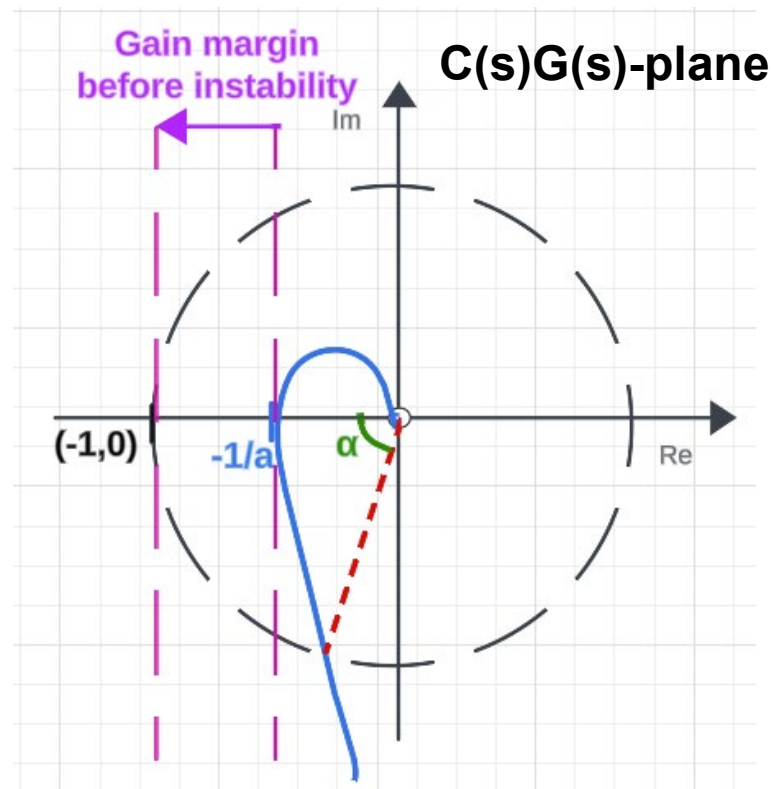
Gain and phase margin from Nyquist
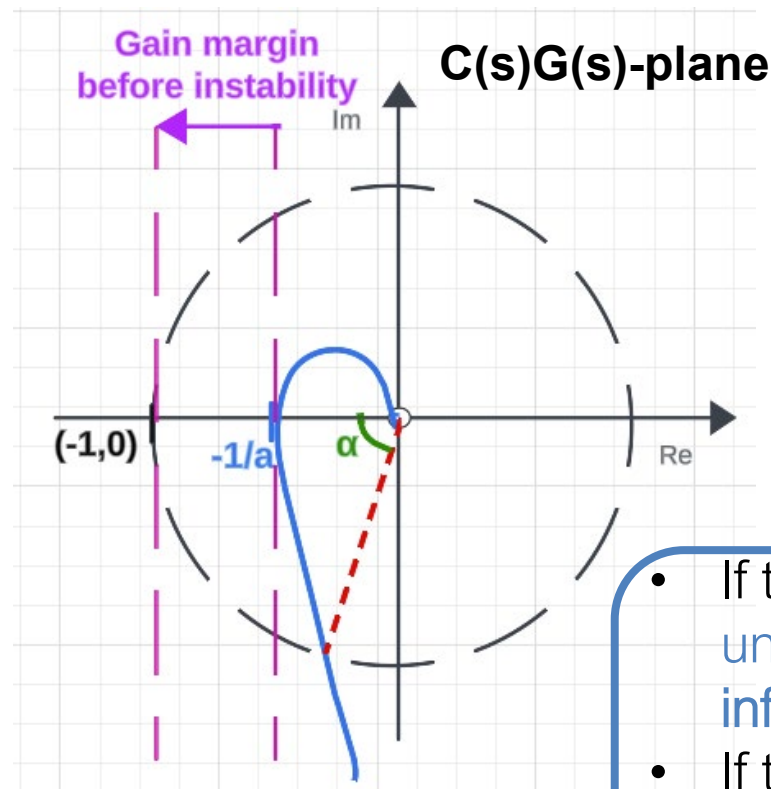
$$P_m = -\pi - \alpha$$

$$G_m = 20 \log a$$

IMPORTANT Remark

- If the Nyquist plot does not cross the unitary circle, then then phase margin is infinite
- If the Nyquist plot does not cross the real line in the left half plane, then the gain margin is infinite

Gain margin

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- It is expressed in dBs of phase shift required at $\pi (180°)$ to make the closed-loop system unstable.

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- For this purpose, we find ω^* such that $\angle G(j\omega^*)C(j\omega^*) = -\pi$. Then G_m is given by

$$G_m = \frac{1}{|G(j\omega^*)C(j\omega^*)|}$$

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Stability of delayed systems

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-magnitude is always one.

-there is a contribution $-T\omega$ to the phase shift (of the closed-loop system)

Consider ω^* such that $|G(j\omega^*)C(j\omega^*)| = 1$. If there exists a time delay, the closed-loop system is stable if the shift phase ω^* does not exceed P_m, that is,

$$P_m + (T\omega^*) < 0$$

Therefore, the critical time delay is given by

$$T = \frac{P_m}{\omega^*}$$

Example: computation of critical time delay

Consider
$$C(s)G(s) = \frac{s+3}{s^2+2s+1} \text{ with } G(j\omega) = \frac{1}{1-\omega^2+j2\omega} \text{ and } C(j\omega) = 3 + j\omega$$
We previously calculated real and imaginary parts:
$$\Re\{C(j\omega)G(j\omega)\} = \frac{3-\omega^2}{1+2\omega^2+\omega^4}, \Im\{C(j\omega)G(j\omega)\} = \frac{-\omega(5+\omega^2)}{1+2\omega^2+\omega^4}$$

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Step 1. Find ω^* such that $|C\{j\omega^*\}G\{j\omega^*\}| = 1$
This implies that
$$\left|\frac{3+j\omega^*}{(1-\omega^{*2})+j2\omega^*}\right| = 1 \Rightarrow \frac{\sqrt{9+(\omega^*)^2}}{\sqrt{(1-(\omega^*)^2)^2+4(\omega^*)^2}} = 1 \Rightarrow (\omega^*)^4 + (\omega^*)^2 - 8 = 0$$

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Letting $\alpha := (\omega^*)^2$, we have

$$\alpha^2 + \alpha - 8 = 0$$

which yields $\alpha = 2.3723$ or $\alpha = -3.3723$. Therefore

$$\boxed{\omega^* = \sqrt{\alpha} = \sqrt{2.3723} = 1.5402 \text{rad/sec}}$$

*We choose only positive value of α because fequency ω^* should be real number

Example: computation of critical time delay

Step 2. Compute $\angle C(j\omega^*)G(j\omega^*)$

This implies that

$$\angle C(j\omega^*)G(j\omega^*) = \frac{\Im\{C(j\omega^*)G(j\omega^*)\}}{\Re\{C(j\omega^*)G(j\omega^*)\}} = \frac{-\omega^*(5 + (\omega^*)^2)}{3 - \omega^{*2}} = \frac{-1.5402(5 + 2.3723)}{3 - 2.3723}$$

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$$\boxed{\angle C(j\omega^*)G(j\omega^*) \approx -1.5156\text{rad}}$$

Example: computation of critical time delay

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Step 5: The critical time delay is given by

$$T = -\frac{P_m}{\omega^*} = \frac{1.16260}{1.5402} = 1.0557 \; sec$$

This is the time delay that the system can handle before instability

Summary

- Delays may modify the stability properties of a system

- The Nyquist's criterion offers an alternative to check the stability of a system without computing its poles

- The phase margin of a system is helpful to determine whether the system is robust or not in the presence of delays

Next week:

Absolute stability.
Stability of nonlinear system as per
Circle criterion and Popov's criterion