Q1 cont

A) Compute evolute of Nephroid: $\varphi \mapsto (3\cos(\varphi) + \cos(3\varphi),\ 3\sin(\varphi) + \sin(3\varphi)),\ \varphi \in [0, 2\pi]$
Determine singular ~~pairs of~~ points of the evolute

$x(\varphi) = 3\cos(\varphi) + \cos(3\varphi)$ $\qquad$ $x''(\varphi) = -3\cos\cancel{-9}(\varphi) - 9\cos(3\varphi)$
$y(\varphi) = 3\sin(\varphi) + \sin(3\varphi)$ $\qquad$ $y''(\varphi) = -3\sin(\varphi) - 9\sin(3\varphi)$
$x'(\varphi) = -3\sin(\varphi) - 3\sin(3\varphi)$
$y'(\varphi) = 3\cos(\varphi) + 3\cos(3\varphi)$

For evolute function $E(\varphi) = (x(\varphi), y(\varphi)) + \dfrac{x'(\varphi)^2 + y'(\varphi)^2}{x'(\varphi)y''(\varphi) - x''(\varphi)y'(\varphi)}\,(-y'(\varphi), x'(\varphi))$

$$p(\varphi) = \frac{x'(\varphi)^2 + y'(\varphi)^2}{x'(\varphi)y''(\varphi) - x''(\varphi)y'(\varphi)} = \frac{(-3\sin(\varphi) - 3\sin(3\varphi))^2 + (3\cos(\varphi) + 3\cos(3\varphi))^2}{(-3\sin\varphi - 3\sin(3\varphi))(-3\sin(\varphi) - 9\sin(3\varphi)) - (3\cos(\varphi) + 3\cos(3\varphi))(-3\cos(\varphi) - 9\cos(3\varphi))}$$

$\cancel{= + ((9\sin^2(\varphi) + 9\sin^2(3\varphi) + 9\sin(\varphi)\sin(3\varphi)) + 9\cos^2(\varphi) + 9\cos^2(3\varphi) +}$

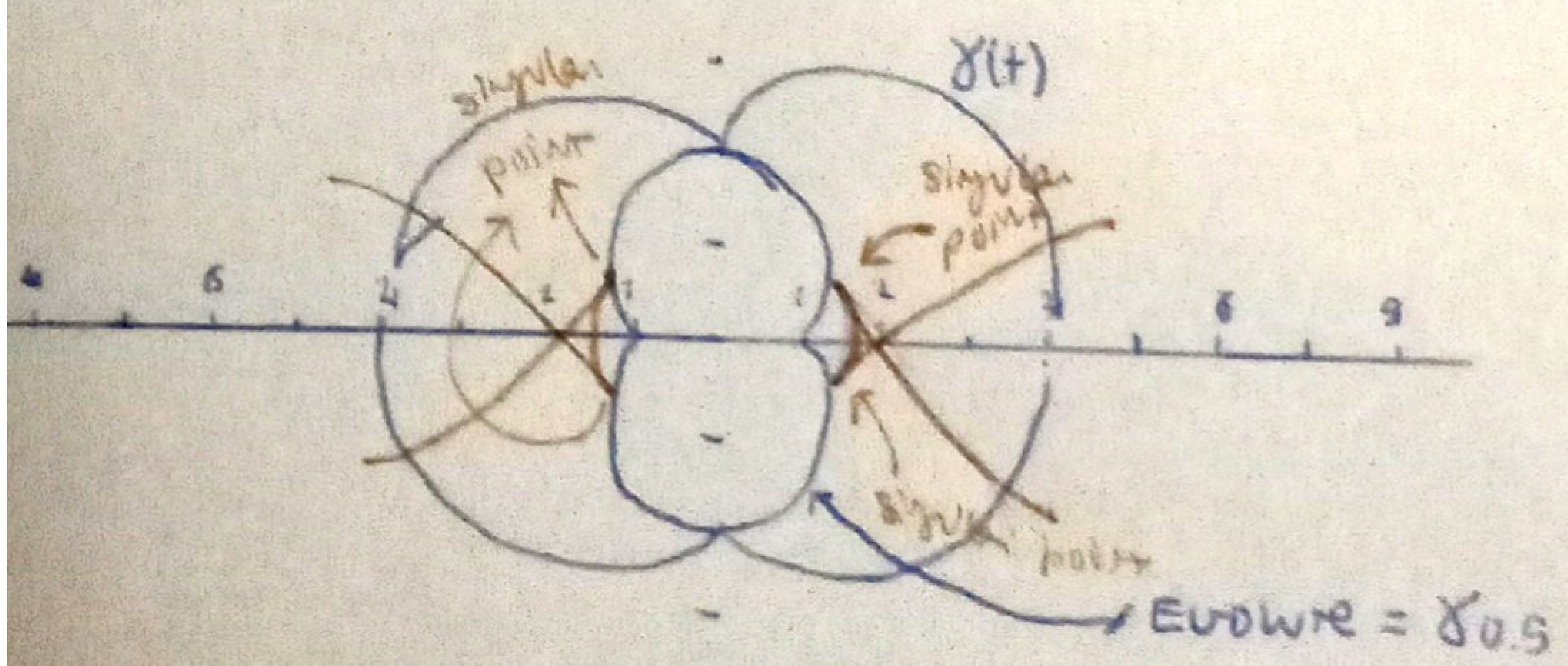
$a = 9\sin^2(\varphi) + 9\sin^2(3\varphi) + 18\sin(\varphi)\sin(3\varphi) + 9\cos^2(\varphi) + 9\cos^2(3\varphi) + 18\cos(\varphi)\cos(3\varphi)$
$\quad = 36\cos^2(\varphi)$
$b = 72\cos^2(\varphi)$

$p(\varphi) = 1/2$ $\qquad$ $n(\varphi) = (-3\cos(\varphi) - 3\cos(3\varphi),\ -3\sin(\varphi) - 3\sin(3\varphi))$

$$E(\varphi) = \left(\frac{(3\cos(\varphi) + \cos(3\varphi))(-3\cos(\varphi) - 3\cos(3\varphi))}{2},\ \frac{(3\sin(\varphi) + \sin(3\varphi))(-3\sin(\varphi) - 3\sin(3\varphi))}{2}\right)$$

$E(\varphi) = (-\cos(\varphi)(-2 + \cos(2\varphi)),\ 2\sin^3(\varphi))$
Singular points can be found by checking which value of φ makes the gradient 0.
This is accomplished by $\varphi = \pi/2,\ 3\pi/2$, knowing $\varphi = [0, 2\pi]$
~~$-3\sin(\varphi) - 3\sin$~~
$x'(\pi/2) = -3\sin(\pi/2) - 3\sin(3\pi/2) = -3 - -3 = 0$
$x'(3\pi/2) = -3\sin(3\pi/2) - 3\sin(9\pi/2) = --3 - 3 = 0$
$y'(\pi/2) = 3\cos(\pi/2) + 3\cos(3\pi/2) = 0 + 0 = 0$
$y'(3\pi/2) = 3\cos(3\pi/2) + 3\cos(9\pi/2) = 0 + 0 = 0$

B) The evolute of a curve consists of singular points of the wavefronts $\gamma_\alpha(\varphi) = (x(\varphi), y(\varphi)) + \alpha n(\varphi) = ((3\cos(\varphi) + \cos(3\varphi)) + \alpha(-3\cos(\varphi) - 3\cos(3\varphi)),$
$(3\sin(\varphi) + \sin(3\varphi)) + \alpha(-3\sin(\varphi) - 3\sin(3\varphi)))$



According to Huygen's principle, and the set of singular points of γ_α is $\{\gamma_\alpha(s) \mid \gamma'_\alpha(s) = 0,\ \alpha \in \mathbb{R}\}$ given by the evolute. In the drawing, this relationship can be seen, as if $\alpha = 0.5$, the wavefront is the envelope and we can see the singular points given.

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Q2 Consider surface in $\mathbb{R}^2$ given by implicit eq $M^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^4 + y^4 - z^4 = 1\}$

A) Prove M^2 is a regular, orientable surface

Let $f(x,y) = \sqrt[4]{x^4+y^4-1}$: $\mathbb{R}^2 \to \mathbb{R}$ and observe the gradient is not 0, this follows that M^2 is a regular surface

$\partial f/\partial x = x^3 / \sqrt[4]{(x^4+y^4-1)^3}$ $\quad \partial f/\partial y = y^3 / 4\sqrt[4]{(x^4+y^4-1)^3}$

The surface is a particular type of elliptic hyperboloid $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1\right)$. This surface is homeomorphic to the standard sphere $S^2 \in \mathbb{R}^3$. Therefore is diffeomorphic to the standard sphere, which is orientable. Due to this diffeomorphism and the fact that the surface doesn't contain a Möbius band, we can say it's orientable.

B) Prove that M^2 has a closed geodesic

First we calculate the geodesics of M^2. By the symmetry of variables, it suffices to consider the intersection of M^2 with the plane $x=0$. Consider the reflection S_x of $\mathbb{R}^3$ in this plane $S_x(x,y,z) = (-x,y,z)$. Then S_x is an isometry of M^2 and hence maps geodesics to geodesics.

Consider geodesic γ passing through $p = (0,0,1) \in M^2$ in the $v = (0,1)$ since $S_x(p) = p$ and $dS_x(v) = v$, it follows that $S_x(\gamma) = \gamma$.

We can consider a closed geodesic that curve that returns to its starting point with same tangent direction. By definition, tangent planes that in their intersection create a curve over a elliptic paraboloid satisfy these requirements.

C) Calculate the Gaussian curvature of M^2 verify that everywhere is non-positive

$K = \det(II) / \det(I) = LN - M^2 / EG - F^2$

$$\cancel{G = \begin{pmatrix} \frac{\partial f}{\partial u^2} & \frac{\partial f}{\partial u \partial v} \\ \frac{\partial f}{\partial u \partial v} & \frac{\partial f}{\partial v^2} \end{pmatrix} = \frac{1}{(-1+x^4+y^4)^{3/2}} \begin{pmatrix} x^6 & x^3y^3 \\ x^3y^3 & y^6 \end{pmatrix}} \quad \det(G) = \frac{\cancel{x^6y^6 - x^6y^6}}{(-1+x^4+y^4)^3} = 0$$

$$G = \begin{pmatrix} \left(\frac{\partial f}{\partial u}\right)^2 & \left(\frac{\partial f}{\partial u}\cdot\frac{\partial f}{\partial v}\right) \\ \left(\frac{\partial f}{\partial u}\cdot\frac{\partial f}{\partial v}\right) & \left(\frac{\partial f}{\partial v}\right)^2 \end{pmatrix} = \begin{pmatrix} x^6 & x^3y^3 \\ x^3y^3 & y^6 \end{pmatrix} \cdot \frac{1}{(-1+x^4+y^4)^{3/2}}$$

$$\det(G) = \left(\frac{x^6}{(-1+x^4+y^4)^{3/2}} \cdot \frac{y^6}{(-1+x^4+y^4)^{3/2}} - \frac{\cancel{x^3y^3}\, x^6y^6}{(-1+x^4+y^4)^3}\right)^{-1} = 0$$

d) Determine the type (elliptic, hyperbolic or parabolic) of points of $M^2 \subset \mathbb{R}^3$

Because the entirety of M^2 has negative curvature, $K_p < 0$, $\forall p \in M^2$ therefore they're hyperbolic points

e) Use C to show that the angle sum of every geodesic triangle $ABC \subset M^2$ is strictly less than π.

Because the curvature is negative in the entirety of M^2, the local Gauss-Bonnet theorem $\int_{ABC} K \cdot dS = \alpha + \beta + \gamma - \pi$ is negative, since $K<0$ and dS is the area, which will be positive. This means that the integral result will be negative and for the equivalence to hold, the angle sum for every geodesic triangle ABC will be strictly ~~tha~~ less than ~~a~~ π.

Q3: Consider an $(n+1)$-dimensional affine space K^{n+1} over a field $\mathbb{K}$. Let P_1 and P_2 be two distinct hyperplanes in this space and let $f: P_1 \to P_2$ be a perspectivity with center $O \in \mathbb{K}^{n+1} \setminus (P_1 \cup P_2)$. Show that f is an affine map iff P_1 and P_2 are parallel.

Let's define the perspectivity f for hyperplanes. For point $x \in P_1$, $f(x)$ is defined as the intersection point of the line OX with P_2 if these hyperplanes intersect. If the line OX is parallel to P_2, then $f(x)$ is defined to be the point at infinity of P_2.
First let's stablish that P_1 and P_2 must be parallel. Without loss of generality, we can assume O is the origin in $\mathbb{K}^{n+1}$ and P_i is given by $\{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{K}^{n+1} \mid x_{n+1} = c_i\}$ where $c_i \neq$ for $i = 1, 2$.
If P_1 and P_2 are not parallel, then f sends a point from P_1 to a point at infinity in P_2, which violates the definition of perspectivity. Now let's consider f from P_1 to P_2 with center O, $f(x)$ is defined as the intersection of line OX and P_2. If OX is parallel to P_2 then $f(x)$ is a point at infinity of P_2.
f is an affine map iff the linear part of f is a translation, and this happens only if P_1 and P_2 are parallel.

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