# Mechatronics

University of Groningen

## Tutorial 3 (Solutions)

## Self-Study Problems

**Question 1:** Consider the problem of the dynamical modeling of an electro-hydraulic system for an assisted steering wheel in a car. It is a multi-domain system that consists of a mechanical system, fluid system and electrical system as shown in Figure 1. The voltage source, $V_s$, is used to rotate the valve through an electro-mechanical coupling device which is given by the relations

$$\omega = \alpha V_{coupling}, \qquad \text{and} \qquad i_L = \alpha T, \tag{1}$$

where $\alpha$ is the coupling constant, $\omega$ is the angular velocity of the valve, $i_L$ is the current through the inductor and $T$ is the torque applied to the valve. The moment of inertia of the valve is denoted by $J$. The angular position of the valve $\theta$ determines the flow rate $Q_m$ based on the following relation:

$$P_{12} = f(\theta) Q_m^2, \tag{2}$$

where $f$ is a nonlinear function that depends on the valve position $\theta$. The pressure source, $P_s$, provides a constant pressure. Based on the pressure across the hydraulic motor $P_{2r} = P_2 - P_r$ , the hydraulic motor generates a force $F$ that drives a mechanical system with mass $m$, and which is connected to a spring and a damper with constants $k$ and $b$, respectively. The displacement of the mechanical system is denoted by $x$ and the velocity is denoted by $v$. The fluid mechanical coupling device (or the hydraulic motor) satisfies the relations

$$F = DP_{2r}, \qquad \text{and} \qquad Dv = Q_m, \tag{3}$$

where $D$ is the coupling constant of the hydraulic motor.

Derive the state equations of the full systems with the voltage $V_s$ as the input and the displacement $x$ as the measured output.

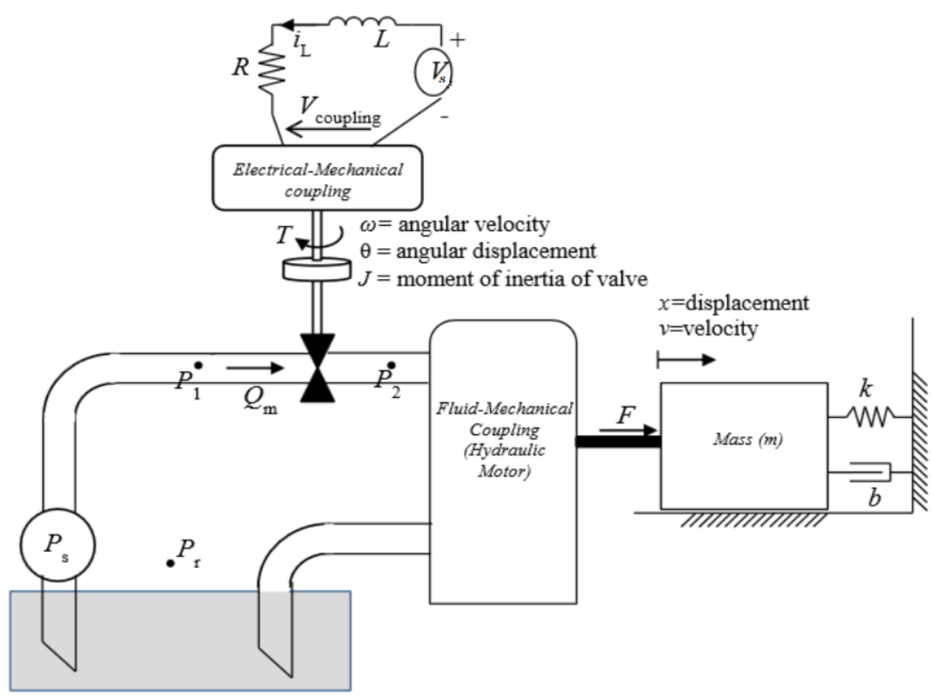



Figure 1: A simplified electro-hydro system of an assisted steering wheel in a car.

**Solution:** In order to derive the state equations of the full system, we first establish the elemental equations in each of the individual domains.

(i) In the electrical domain, the electrical inductor has the following dynamics

$$L\frac{di_L}{dt} = LV_L, \tag{4}$$

where $V_L$ is the voltage over the inductor.

(ii) In the mechanical rotational domain, the angular displacement and angular velocity are related by

$$\frac{d\theta}{dt} = \omega. \tag{5}$$

(iii) In the mechanical rotational domain, the differential equation of the angular velocity can be obtained using the rotational version of Newton's law, giving us

$$J\frac{d\omega}{dt} = T. \tag{6}$$

(iv) In the mechanical translational domain, the displacement and velocity are related by

$$\frac{dx}{dt} = v. \tag{7}$$

(v) In the mechanical translational domain, the differential equation of the velocity can be obtained using Newton's law, giving us

$$\frac{dv}{dt} = F - bv - kx. \tag{8}$$

The above elemental equations suggest to take $(i_L, \theta, \omega, x, v)$ as state variables. Thus, in order to obtain the state-space equations, all we need to do is write the differential equations above in terms of $(i_L, \theta, \omega, x, v)$ and the input $V_s$. This can be done in the following way.

(i) The voltage in the circuit satisfies

$$V_L = V_s - V_R - V_{coupling}.$$

Moreover, we have that $V_R = Ri_L$, while $V_{coupling}$ and $\omega$ are related through the following coupling equation

$$V_{coupling} = \frac{1}{\alpha}\omega.$$

We can combine these with eq. (4) to obtain that

$$\begin{aligned} L\frac{di_L}{dt} &= V_L \\ &= V_s - V_R - V_{coupling} \\ &= V_s - Ri_L - \frac{1}{\alpha}\omega. \end{aligned}$$

Thus, we have the following dynamical equation for $i_L$:

$$\frac{di_L}{dt} = \frac{1}{L}V_s - \frac{R}{L}i_L - \frac{1}{L\alpha}\omega.$$

(ii) From eq. (5) we directly obtain the differential equation of $\theta$, given by

$$\frac{d\theta}{dt} = \omega.$$

(iii) The torque $T$ and current $i_L$ are related through the following coupling equation

$$T = \frac{1}{\alpha}i_L.$$

We can use this in eq. (6) to obtain the differential equation of $\omega$, given by

$$\frac{d\omega}{dt} = \frac{1}{J\alpha}i_L.$$

(iv) From eq. (7) we directly obtain the differential equation of $x$, given by

$$\frac{dx}{dt} = v.$$

(v) The fluid pressure satisfies

$$P_{2r} = P_s - P_{12},$$

where the pressure $P_{12}$ is related to the fluid flow by

$$P_{12} = f(\theta)Q_m^2.$$

Moreover, we have the following relations between the force and pressure, and velocity and fluid flow, respectively

$$F = DP_{2r}, \qquad Dv = Q_m.$$

When combining this with eq. (8), we obtain the following

$$\begin{aligned} m\frac{dv}{dt} &= F - bv - kx \\ &= DP_{2r} - bv - kx \\ &= D(P_s - P_{12}) - bv - kx \\ &= D(P_s - f(\theta)Q_m^2) - bv - kx \\ &= D(P_s - f(\theta)(Dv)^2) - bv - kx. \end{aligned}$$

Therefore, we have the following differential equation for $v$

$$\frac{dv}{dt} = \frac{D}{m}(P_s - f(\theta)(Dv)^2) - \frac{b}{m}v - \frac{k}{m}x.$$

We have now obtained five differential equations, namely given by

$$\begin{aligned} \frac{di_L}{dt} &= \frac{1}{L}V_s - \frac{R}{L}i_L - \frac{1}{L\alpha}\omega, \\ \frac{d\theta}{dt} &= \omega, \\ \frac{d\omega}{dt} &= \frac{1}{J\alpha}i_L, \\ \frac{dx}{dt} &= v, \\ \frac{dv}{dt} &= \frac{D}{m}(P_s - f(\theta)(Dv)^2) - \frac{b}{m}v - \frac{k}{m}x. \end{aligned}$$

From these equations, together with the fact that the output is given by $x$, we can set up the state equations of the system as

$$\begin{bmatrix} \dot{i}_l \\ \dot{\theta} \\ \dot{\omega} \\ \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{1}{L\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{J\alpha} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -{}^k/_m & -{}^b/_m \end{bmatrix} \begin{bmatrix} i_L \\ \theta \\ \omega \\ x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{D}{m}(P_s - f(\theta)(Dv)^2) \end{bmatrix} + \begin{bmatrix} {}^1/_L \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_s,$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_L \\ \theta \\ \omega \\ x \\ v \end{bmatrix}.$$

**Question 2:** Consider the pendulum in Figure 2, where a solid mass with weight $m$ is connected to the ceiling by a massless spring where the spring constant is denoted by $k$. The length $z$ is the elongation of the spring from the rest position of $L$. In order to use the Euler-Lagrange method, find the corresponding kinetic and potential energy of the pendulum and derive the system equations via Euler-Lagrange. (Compare the answer with the one obtained from the Newton's method in Problem 1 of the self-study problems of Tutorial 2.)

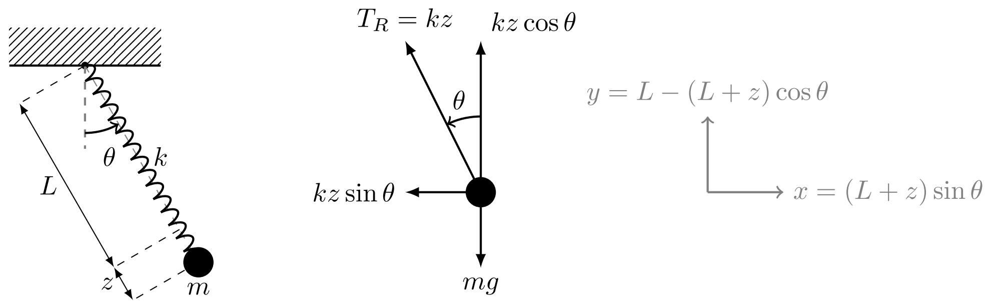



Figure 2: Modeling a pendulum connected to a spring

**Solution:**

The kinetic energy of the system is given by the sum of the decomposed horizontal and vertical kinetic energy of the mass $m$, that is

$$E_k = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2.$$

From Figure 2, we see that we can write the coordinates of the mass in terms of the generalized coordinates $z$ and $L$ in the following way

$$x = (L+z)\sin\theta, \qquad \text{and} \qquad y = L - (L+z)\cos\theta.$$

Substituting this into the above, and working out the derivatives, we find that the kinetic energy is given by

$$\begin{aligned}
E_k &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \\
&= \frac{1}{2}m\left(\frac{d}{dt}((L+z)\sin\theta)\right)^2 + \frac{1}{2}m\left(\frac{d}{dt}(L-(L+z)\cos\theta)\right)^2 \\
&= \frac{1}{2}m\left(\dot{z}\sin\theta + (L+z)\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}m\left(-\dot{z}\cos\theta + (L+z)\dot{\theta}\sin\theta\right)^2.
\end{aligned}$$

When working out the squares, and using that $\sin^2\theta + \cos^2\theta = 1$, we find that we can simplify this to

$$\begin{aligned}
E_k &= \frac{1}{2}m\left(\dot{z}\sin\theta + (L+z)\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}m\left(-\dot{z}\cos\theta + (L+z)\dot{\theta}\sin\theta\right)^2 \\
&= \frac{1}{2}m\left(\dot{z}^2\sin^2\theta + 2(L+z)\dot{\theta}\dot{z}\sin\theta\cos\theta + (L+z)^2\dot{\theta}^2\cos^2\theta\right) \\
&\quad + \frac{1}{2}m\left(\dot{z}^2\cos^2\theta - 2(L+z)\dot{\theta}\dot{z}\sin\theta\cos\theta + (L+z)^2\dot{\theta}^2\sin^2\theta\right) \\
&= \frac{1}{2}m\left(\dot{z}^2(\sin^2\theta+\cos^2\theta) + (L+z)^2\dot{\theta}^2(\sin^2\theta+\cos^2\theta)\right) \\
&= \frac{1}{2}m\left(\dot{z}^2 + (L+z)^2\dot{\theta}^2\right).
\end{aligned}$$

The potential energy in the system is given by the sum of the gravitational energy of the mass, and the potential energy stored in the spring, and is hence given by

$$E_p = mgy + \frac{1}{2}kz^2.$$

Using that $y = L - (L+z)\cos\theta$, this becomes

$$E_p = mg(L - (L+z)\cos\theta) + \frac{1}{2}kz^2.$$

Combining the above, the Lagrangian of the system is given by

$$\begin{aligned} L &= E_k - E_p \\ &= \frac{1}{2}m(\dot{z}^2 + (L+z)^2\dot{\theta}^2) - mg(L - (L+z)\cos\theta) - \frac{1}{2}kz^2. \end{aligned}$$

Since we are not considering any external forces are acting on the system, we see that the Euler-Lagrange equations are then defined by the following equations

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}}\right] - \left[\frac{\partial L}{\partial z}\right] &= 0, \\ \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] - \left[\frac{\partial L}{\partial \theta}\right] &= 0. \end{aligned}$$

The first of these gives us

$$\begin{aligned} 0 &= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}}\right] - \left[\frac{\partial L}{\partial z}\right] \\ &= \frac{d}{dt}\left[m\dot{z}\right] - \left[m(L+z)\dot{\theta}^2 + mg\cos\theta - kz\right] \\ &= m\ddot{z} - m(L+z)\dot{\theta}^2 - mg\cos\theta + kz. \end{aligned}$$

From the second one, we obtain

$$\begin{aligned} 0 &= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] - \left[\frac{\partial L}{\partial \theta}\right] \\ &= \frac{d}{dt}\left[m(L+z)^2\dot{\theta}\right] - \left[-mg(L+z)\sin\theta\right] \\ &= m(L+z)^2\ddot{\theta} + 2m(L+z)\dot{z}\dot{\theta} + mg(L+z)\sin\theta. \end{aligned}$$

Thus, the system equations are given by

$$\begin{aligned} m\ddot{z} - m(L+z)\dot{\theta}^2 - mg\cos\theta + kz &= 0, \\ m(L+z)^2\ddot{\theta} + 2m(L+z)\dot{z}\dot{\theta} + mg(L+z)\sin\theta &= 0. \end{aligned}$$

**Question 3:** Consider the Segway system shown in Figure 3. We can distinguish this system to consist of two parts: a rolling base with mass $M$, and an inverted pendulum with mass $m$. Motion of this Segway is created by applying a torque on the rotational base. We assume the radius of the base to be $R$, and we represent the moment of inertia of the base as $J$. Moreover, we denote the length of the inverted pendulum to be $L$.

We want to derive the equations of motions via Euler-Lagrange formalism. Therefore, let us use the generalized coordinate $q = (q_1, q_2)^T$ where $q_1$ is the angle of the inverted pendulum $\theta$, and $q_2$ is the horizontal coordinate of the base $x$.

a) Show that the corresponding kinetic and potential energy of the Segway system is given by

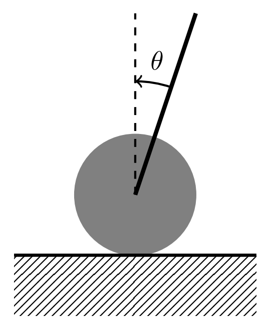
$$\begin{aligned} E_k &= \frac{1}{2}\begin{bmatrix}\dot{q}_1 & \dot{q}_2\end{bmatrix}\begin{bmatrix} J + mL^2 & mL\cos q_1 \\ mL\cos q_1 & m + M\end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix} \\ E_p &= mgL\cos q_1 \end{aligned} \tag{9}$$

b) Show that the external forces of the Segway system are given by

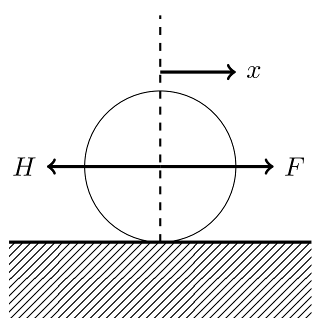
$$F_{ext} = \begin{bmatrix} -T \\ \frac{T}{R} - \frac{T}{L}\cos(q_1) \end{bmatrix} \tag{10}$$

c) Derive the equations of motion using the Euler-Lagrange equations.

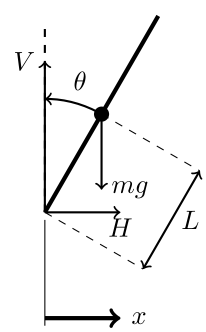
(a) The Segway system



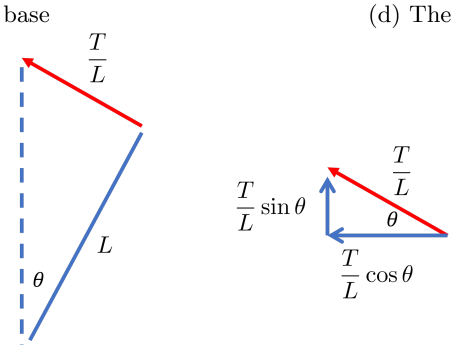

(b) Schematic of the Segway system




(c) The rolling base




(d) The inverted pendulum




(e) Force decomposition of the reaction torque acting on the pendulum.

Figure 3: Modeling the Segway transportation system

**Solution:**

a) The kinetic energy in the system is given by the sum of the decomposed horizontal and vertical kinetic energy of the inverted pendulum, the horizontal kinetic energy of the base, and the rotational kinetic energy of the base, that is

$$E_k = \underbrace{\frac{1}{2}m\left(\frac{d}{dt}(x_{pend})\right)^2}_{\text{Horizontal K.E. pendulum}} + \underbrace{\frac{1}{2}m\left(\frac{d}{dt}(y_{pend})\right)^2}_{\text{vertical K.E. pendulum}} + \underbrace{\frac{1}{2}M\left(\frac{d}{dt}(x)\right)^2}_{\text{horizontal K.E. base}} + \underbrace{\frac{1}{2}J\left(\frac{d}{dt}(\theta)\right)^2}_{\text{rotational K.E. base}}.$$

Using the generalized coordinates $(q_1, q_2) = (\theta, x)$, we can express the coordinates of the inverted pendulum in terms of $q_1$ and $q_2$ in the following way

$$x_{pend} = x + L\sin\theta = q_2 + L\sin(q_1), \qquad \text{and} \qquad y_{pend} = L\cos\theta = L\cos(q_1).$$

We can use this in the expression above to obtain

$$\begin{aligned}
E_k &= \frac{1}{2}m\left(\frac{d}{dt}(q_2 + L\sin(q_1))\right)^2 + \frac{1}{2}m\left(\frac{d}{dt}(L\cos(q_1))\right)^2 + \frac{1}{2}M\left(\frac{d}{dt}(q_2)\right)^2 + \frac{1}{2}J\left(\frac{d}{dt}(q_1)\right)^2, \\
&= \frac{1}{2}m\left(\dot{q}_2 + L\dot{q}_1\cos(q_1)\right)^2 + \frac{1}{2}m\left(-L\dot{q}_1\sin(q_1)\right)^2 + \frac{1}{2}M\dot{q}_2^2 + \frac{1}{2}J\dot{q}_1^2 \\
&= \frac{1}{2}m\dot{q}_2^2 + mL\dot{q}_1\dot{q}_2\cos(q_1) + \frac{1}{2}mL^2\dot{q}_1^2\cos^2(q_1) + \frac{1}{2}mL^2\dot{q}_1^2\sin^2(q_1) + \frac{1}{2}M\dot{q}_2^2 + \frac{1}{2}J\dot{q}_1^2 \\
&= \frac{1}{2}(m+M)\dot{q}_2^2 + \frac{1}{2}\left(J + mL^2\left(\sin^2(q_1) + \cos^2(q_1)\right)\right)\dot{q}_1^2 + mL\dot{q}_1\dot{q}_2\cos(q_1).
\end{aligned}$$

Using that $\sin^2(q_1) + \cos^2(q_1) = 1$, we can simplify this to

$$E_k = \frac{1}{2}(m+M)\dot{q}_2^2 + \frac{1}{2}\left(J + mL^2\right)\dot{q}_1^2 + mL\dot{q}_1\dot{q}_2\cos(q_1).$$

Note that this is a quadratic form, and can be written more compactly as

$$E_k = \frac{1}{2}\begin{bmatrix}\dot{q}_1 & \dot{q}_2\end{bmatrix}\begin{bmatrix}J + mL^2 & mL\cos q_1 \\ mL\cos q_1 & m + M\end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}.$$

The potential energy in the system is given by the gravitational energy of the inverted pendulum, and can be directly computed as

$$E_p = mgy_{pend} = mgL\cos(q_1).$$

b) Let $(x_{base}, y_{base})$ denote the coordinates of the base, and let $(x_{pend}, y_{pend})$ denote the coordinates of the inverted pendulum. Then, we can collect these in

$$z = \begin{bmatrix}z_1 \\ z_2 \\ z_3 \\ z_4\end{bmatrix} := \begin{bmatrix}x_{base} \\ y_{base} \\ x_{pend} \\ y_{pend}\end{bmatrix},$$

which represents the spatial coordinates of the system. Using that

$$x_{pend} = q_2 + L\sin(q_1), \qquad \text{and} \qquad y_{pend} = L\cos(q_1),$$

we can rewrite these spatial coordinates in terms of the generalized coordinates $(q_1, q_2) = (\theta, x)$ in the following way

$$z = \begin{bmatrix}x_{base} \\ y_{base} \\ x_{pend} \\ y_{pend}\end{bmatrix} = \begin{bmatrix}q_2 \\ 0 \\ q_2 + L\sin(q_1) \\ L\cos(q_1)\end{bmatrix}.$$

Now, note that the derivative of the spatial coordinates satisfies

$$\begin{aligned}
\dot{z}_1 &= \dot{q}_2 \\
\dot{z}_2 &= 0 \\
\dot{z}_3 &= \dot{q}_2 + L\dot{q}_1 \cos(q_1) \\
\dot{z}_4 &= -\dot{q}_1 L \sin(q_1),
\end{aligned}$$

or, more compactly

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ L\cos q_1 & 1 \\ -L\sin q_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}. \tag{11}$$

For the original spatial coordinates, we can express the effect of an input torque $T$ between the base and the pendulum in the following way. In the $x_{base}$ direction, the torque generates a force of

$$F = \frac{T}{R}$$

in the positive direction. The same torque creates a reaction torque on the pendulum, generating a force of $\frac{T}{L}$ through its center of mass. This reaction force can be decomposed into the $x_{pend}, y_{pend}$ coordinates as can be seen in Figure 3e. Using this decomposition, the total forces acting on the spatial coordinates are described by

$$F_{spatial} = \begin{bmatrix} \frac{T}{R} \\ 0 \\ -\frac{T}{L}\cos(q_1) \\ \frac{T}{L}\sin(q_1) \end{bmatrix}$$

Now, recall that the power supplied by the torque input should be the same in any chosen coordinate system. In the spatial coordinates, this power is given by the force/torque multiplied by the velocity, and is hence given by

$$\dot{z}^\top F_{spatial}.$$

In the generalized coordinates, this power is given by

$$\begin{bmatrix} \dot{q}_1 & \dot{q}_2 \end{bmatrix} F_{ext},$$

where $F_{ext}$ is what we would like to determine. Using the relation between $\dot{z}$ and $(\dot{q}_1, \dot{q}_2)$ as given in eq. (11), we thus find that

$$\dot{z}^\top F_{spatial} = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & L\cos(q_1) & -L\sin(q_1) \\ 1 & 0 & 1 & 0 \end{bmatrix} F_{spatial}.$$

Therefore, we see that

$$\begin{aligned}
F_{ext} &= \begin{bmatrix} 0 & 0 & L\cos(q_1) & -L\sin(q_1) \\ 1 & 0 & 1 & 0 \end{bmatrix} F_{spatial} \\
&= \begin{bmatrix} 0 & 0 & L\cos(q_1) & -L\sin(q_1) \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{T}{R} \\ 0 \\ -\frac{T}{L}\cos(q_1) \\ \frac{T}{L}\sin(q_1) \end{bmatrix} \\
&= \begin{bmatrix} -T\cos^2(q_1) - T\sin^2(q_1) \\ \frac{T}{R} - \frac{T}{L}\cos(q_1) \end{bmatrix}.
\end{aligned}$$

Using that $\sin^2(q_1) + \cos^2(q_1) = 1$, we thus find that the external forces in generalized coordinates are given by

$$F_{ext} = \begin{bmatrix} -T \\ \frac{T}{R} - \frac{T}{L}\cos(q_1) \end{bmatrix}.$$

c) From the answer of part a), we obtain that the Lagrangian is given by

$$\begin{aligned} L &= E_k - E_p \\ &= \frac{1}{2}(M+m)\dot{q}_2^2 + \frac{1}{2}(J+mL^2)\dot{q}_1^2 + mL\dot{q}_1\dot{q}_2\cos q_1 - mgL\cos q_1. \end{aligned}$$

Using the external forces obtained in part b), the Euler-Lagrange equations for this system are given by

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_1}\right] - \left[\frac{\partial L}{\partial q_1}\right] &= -T \\ \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_2}\right] - \left[\frac{\partial L}{\partial q_2}\right] &= \frac{T}{R} - \frac{T}{L}\cos(q_1). \end{aligned}$$

From the first of these equations, we obtain

$$\begin{aligned} -T &= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_1}\right] - \left[\frac{\partial L}{\partial q_1}\right] \\ &= \frac{d}{dt}\left[(J+mL^2)\dot{q}_1 + mL\dot{q}_2\cos(q_1)\right] - \left[-mL\dot{q}_1\dot{q}_2\sin(q_1) + mgL\sin(q_1)\right] \\ &= (J+mL^2)\ddot{q}_1 + mL\ddot{q}_2\cos(q_1) - mL\dot{q}_2\dot{q}_1\sin(q_1) + mL\dot{q}_1\dot{q}_2\sin(q_1) - mgL\sin(q_1) \\ &= (J+mL^2)\ddot{q}_1 + mL\ddot{q}_2\cos(q_1) - mgL\sin(q_1). \end{aligned}$$

When working out the second equation, we obtain

$$\begin{aligned} \frac{T}{R} - \frac{T}{L}\cos(q_1) &= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_2}\right] - \left[\frac{\partial L}{\partial q_2}\right] \\ &= \frac{d}{dt}\left[(M+m)\dot{q}_2 + mL\dot{q}_1\cos(q_1)\right] \\ &= (M+m)\ddot{q}_2 + mL\ddot{q}_1\cos(q_1) - mL\dot{q}_1^2\sin(q_1). \end{aligned}$$

Therefore, the equations of motion of the system are given by

$$\begin{aligned} (J+mL^2)\ddot{q}_1 + mL\ddot{q}_2\cos(q_1) - mgL\sin(q_1) &= -T, \\ (M+m)\ddot{q}_2 + mL\ddot{q}_1\cos(q_1) - mL\dot{q}_1^2\sin(q_1) &= \frac{T}{R} - \frac{T}{L}\cos(q_1). \end{aligned}$$

# In-tutorial Problems

**E 1:** Consider the problem of the dynamical modeling of a hydraulic motor connected to a DC generator for supplying electrical power to the load, as shown in Figure 4. The fluid system contains a fluid inertor with inertance $I$, and a fluid capacitor with capacitance $C_f$. The mechanical component consists of two inertias ($J_t$ and $J_g$) connected with a rigid shaft where the angular velocity is given by $\Omega_1$, and the friction components are represented by the damping constants $B_m$ and $B_g$. The electrical system contains an electrical resistor with resistance $R$, and an electrical inductor with inductance $L$.

Recall that the fluid-mechanical coupling device satisfies

$$T_t = D_r P_{2r}, \qquad \text{and} \qquad \Omega_1 = \frac{1}{D_r} Q_m,$$

where $P_{2r} = P_2 - P_r$, and that the mechanical-electrical coupling device satisfies

$$T_g = \frac{1}{\alpha_T} i_L, \qquad \text{and} \qquad \Omega_1 = \alpha_T V_{12},$$

where $V_{12} = V_1 - V_2$.

Derive the state equations of the full systems with the pressure $P_s$ as the input and the electrical current $i_L$ as the measured output.

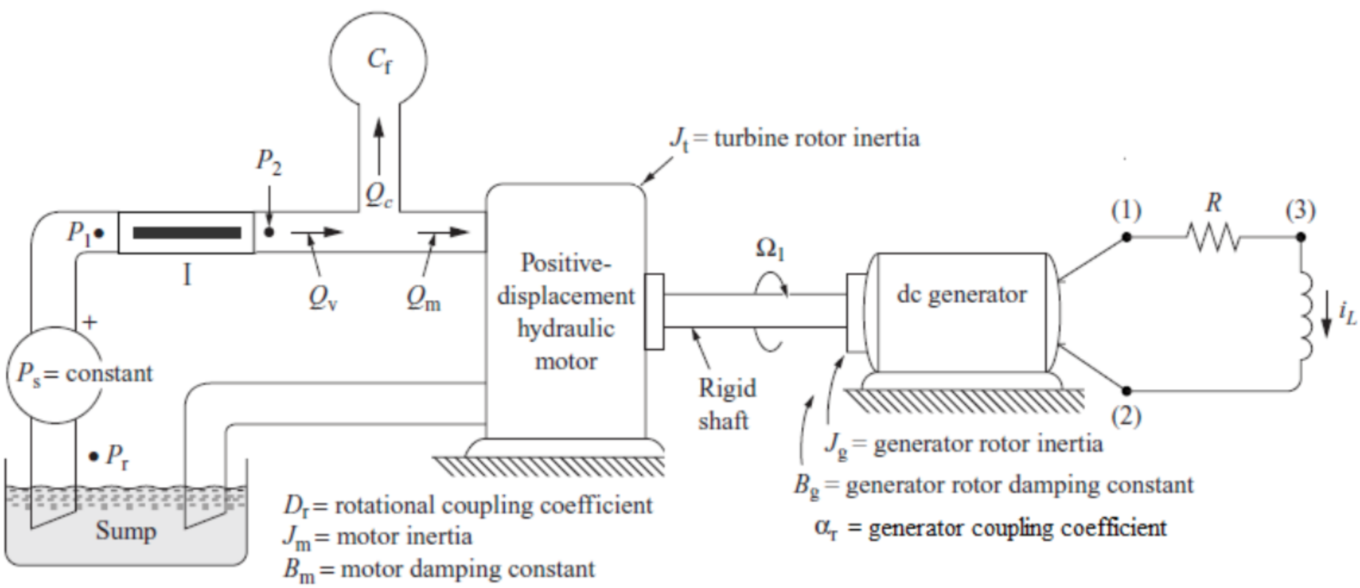



Figure 4: A hydraulic motor connected to a DC generator for supplying power to a load.

**E 2:** Consider the problem of the dynamical modeling of a flexible conveyor belt as shown in Figure 5. Using the Euler-Lagrange equations, derive the dynamical model of the system with generalized coordinates $\theta_1$, $\theta_2$ and generalized forces $T_{ext,1}$, $T_{ext,2}$

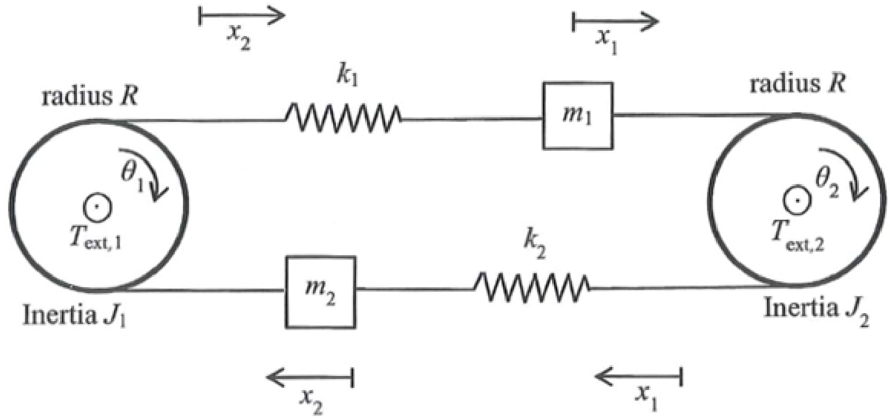



Figure 5: Modeling of a flexible conveyor belt with two masses, $m_1$ and $m_2$.

**E 3:** Consider the crane system shown in Figure 6. A force $F$ is acting on the upper body. The upper body with weight $M$ is connected to a lifted object with weight $m$ through a massless and solid string. The length of the string is $L$. The moment of inertia of the lifted body is assumed to be $J$. The surface on which the upper body lies, exerts a friction force $F_{friction} = c\dot{x}$ where $c$ is the friction constant and $x$ is the horizontal coordinate of the upper body. In order to use the Euler-Lagrange method, find the corresponding kinetic and potential energy of the pendulum, the dissipation energy $D$ (due to friction) and derive the system equations via Euler-Lagrange equations. Note that the external force would be $F_{ext} = \begin{bmatrix} F \\ 0 \end{bmatrix}$ with the Euler-Lagrange equation with dissipation be given by

$$\begin{bmatrix} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \\ \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} \end{bmatrix} + \begin{bmatrix} \frac{\partial D}{\partial \dot{x}} \\ \\ \frac{\partial D}{\partial \dot{\theta}} \end{bmatrix} = \begin{bmatrix} F \\ \\ 0 \end{bmatrix}. \tag{12}$$

**Remark:** The bob is not considered a point-mass anymore (like in the previous exercise). In other words, to compute the kinetic energy of the system you must consider also the inertial contribution (due to $J$).

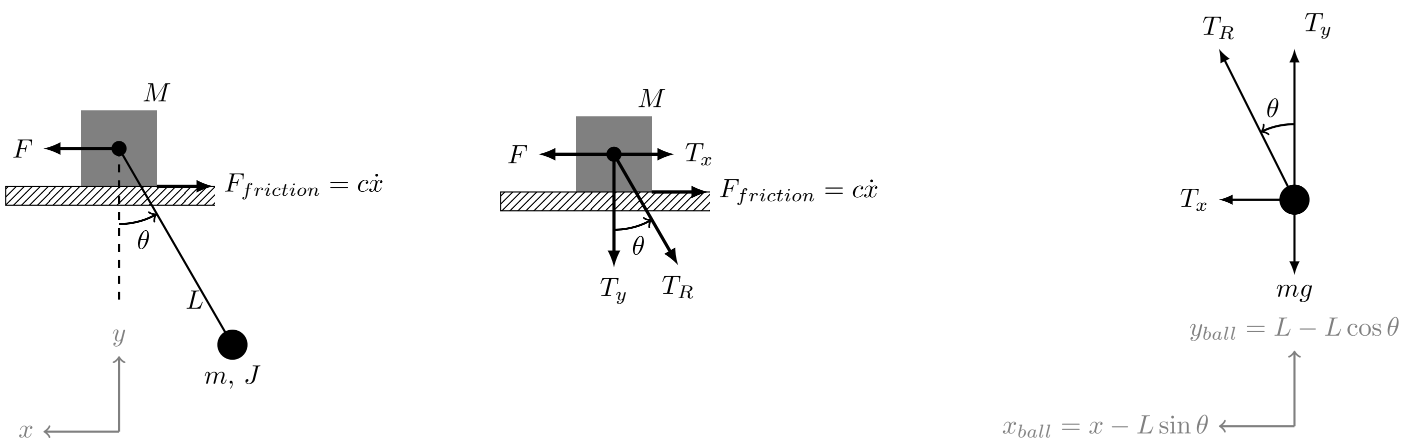



Figure 6: Modeling of a crane system