Control Engineering 2017-2018
Mock Exam 2017
Prof. C. De Persis

- You have **3 hours** to complete the exam.
- You **can** use books and notes but **not** smart phones, computers, tablets and the like.
- Please write your answers using a pen, **not a pencil**.
- There are questions/exercises labeled as **Bonus**. These questions/exercises are optional and give you **extra** points if answered correctly.
- Please write down your Surname, Name, Student ID on each sheet.
- You will be given 2 sheets. If you need more, please ask. Please hand in **all the sheets** that you have used and the **text of the exam**.
- If you return the sheets, then your exam will be graded, unless you explicitly write "do not grade" on the first page.
- If your exam is graded, then the grade will be registered, even if the grade is lower than the one you got at the previous exam(s).

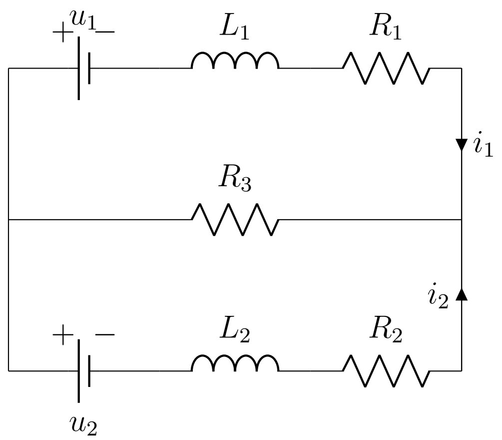
Good luck!

For the grader only

	Exercise 1	Exercise 2	Exercise 3	Exercise 4	Exercise 5
Points					
Bonus	×	×	×	×	

Exercise 1. A DC microgrid (10pt)

Consider the following electrical circuit, which needs to serve a resistive load R_3. The two voltage sources u_1, u_2 represent DC sources interconnected to the resistive load R_3 via the two RL lines with parameters R_1, R_2, L_1, L_2.



Take as the generalized displacement vector $q = [q_1\ q_2]^\top$ the charges $q_k = \int i_k dt$, $k = 1, 2$.

a. (2pt) Determine the kinetic co-energy $T^*(q, \dot{q})$ of the system.

b. (2pt) Determine the potential energy $V(q)$ of the system.

c. (1pt) Determine the Lagrangian of the system.

d. (3pt) Determine the dissipation function of the system.

e. (2pt) Determine the equations of motion of the system.

Solutions.

a. $T^*(\dot{q}) = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2\dot{q}_2^2$

b. $V(q) = 0$

c. $L(\dot{q}) = T^* - V = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2\dot{q}_2^2$

d. Dissipation function $\frac{1}{2}(R_1\dot{q}_1^2 + R_2\dot{q}_2^2 + R_3(\dot{q}_1 + \dot{q}_2)^2)$

e. One point for the dissipation function $\frac{1}{2}R_1(\dot{q}_1 - \dot{q}_2)^2$, one for each correct equation of motion.

$$L_1\ddot{q}_1 + R_1\dot{q}_1 + R_3(\dot{q}_1 + \dot{q}_2) = u_1 \qquad L_2\ddot{q}_2 + R_2\dot{q}_2 + R_3(\dot{q}_1 + \dot{q}_2) = u_2$$

Exercise 2. Stabilizing a nonlinear system (10pt)

Consider the following dynamics

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -x_1 + p\sin(x_2) \\ -pb\sin(x_1) + 2x_2 - e^{x_1}u \end{bmatrix} \\ y &= x_1 \end{aligned} \tag{1}$$

and its linearization around the equilibrium pair

$$(\bar{x}, \bar{u}) = (\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0), \tag{2}$$

given by

$$\begin{aligned} \Delta\dot{x} &= \underbrace{\begin{bmatrix} -1 & p \\ pb & 2 \end{bmatrix}}_{A} \Delta x + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{B} \Delta u \\ \Delta y &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \Delta x, \end{aligned} \tag{3}$$

where $\Delta x = x - \bar{x}$, $\Delta u = u - \bar{u}$, $\Delta y = y - \bar{y}$, and $\bar{y} = C\bar{x}$. Note that in this case $\Delta x = x$, $\Delta u = u$.

For this exercise, take $p = 0.3$, $b = 4$.

a. (2pt) Compute the reachability matrix of system (3) and discuss whether or not the system is rechable.

b. (4pt) If the system is reachable, determine the matrix $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ in the state feedback control $\Delta u = -K\Delta x$ such that the closed-loop matrix $A - BK$ has its eigenvalues equal to $-3, -2$.

c. (4pt) Consider the original *nonlinear* system (1), in closed-loop with the control $u = \bar{u} - K(x + \bar{x}) = -Kx$:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -x_1 + p\sin(x_2) \\ -pb\sin(x_1) + 2x_2 - e^{x_1}(-k_1x_1 - k_2x_2) \end{bmatrix} \\ y &= x_1 \end{aligned} \tag{4}$$

Determine the system linearized around the equilibrium

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and check that the dynamic matrix is Hurwitz, i.e. it has all its eigenvalues with strictly negative real parts. What can you conclude about the stability of the equilibrium $\bar{x}$ of the closed-loop nonlinear system (4)? Explain in one sentence.

Solutions.

a. The reachability matrix is $\begin{bmatrix} 0 & -0.3 \\ -1 & -2 \end{bmatrix}$. This matrix has full rank. The system is therefore reachable.

b. The characteristic polynomial of A has coefficient list 1, $a_1 = -1$, $a_2 = -2.36$. The desired polynomial has coefficient list 1, $p_1 = 5$, $p_2 = 6$. Therefore,

$$\begin{aligned} K &= \begin{bmatrix} p_1 - a_1 & p_2 - a_2 \end{bmatrix} \tilde{W}_r W_r^{-1} \\ &= \begin{bmatrix} 6 & 8.36 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6.67 & -1 \\ -3.33 & 0 \end{bmatrix} = \begin{bmatrix} -7.87 & -6 \end{bmatrix} \end{aligned}$$

c. The new linearized A is $\begin{bmatrix} -1 & 0.3 \\ -6.67 & -4 \end{bmatrix}$, with eigenvalues -3 and -2, so it is indeed Hurwitz. The equilibrium of the closed-loop system is therefore asymptotically stable.

Exercise 3. Observer and output feedback control for a linear system (10pt)

Consider again the linearized system (2), rewritten here as

$$\begin{aligned}
\dot{\Delta x} &= \underbrace{\begin{bmatrix} -1 & p \\ pb & 2 \end{bmatrix}}_{A} \Delta x + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{B} \Delta u \\
\Delta y &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \Delta x,
\end{aligned} \tag{5}$$

a. (2pt) Determine the observability matrix W_o and discuss whether the system is observable or not.

b. (2pt) Determine the observable canonical form of the system (matrices $\tilde{A}$ and $\tilde{C}$).

c. (3pt) Determine the gain matrix L such that the eigenvalues of $A - LC$ are equal to $-2, -1$. Write explicitly the observer for system (5).

d. (3pt) Using the matrix K of the stabilizing state feedback obtained in Exercise 2, point b. determine a *dynamic output* feedback controller that solves the output regulation problem, that is, (i) the closed-loop system is asymptotically stable and (ii) the output y asymptotically converges to the constant reference signal r.

Solutions.

a. $W_o = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0.3 \end{smallmatrix}\right]$. The system is observable.

b. We use the familiar standard pattern.

$$\tilde{A} = \begin{bmatrix} 1 & 1 \\ 2.36 & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

c. The desired polynomial has coefficient list 1, $q_1 = 3$, $q_2 = 2$. Therefore,

$$L = \begin{bmatrix} 4 \\ 41.2 \end{bmatrix}$$

The observer is therefore

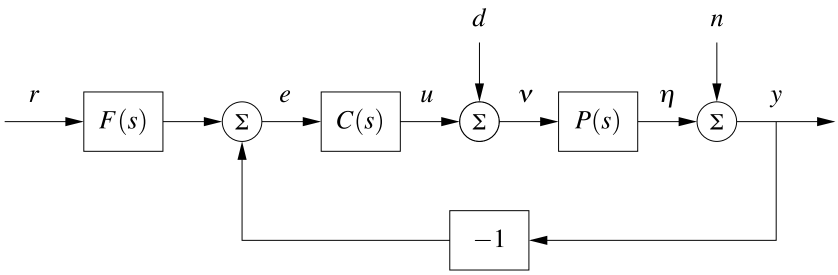
$$\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\
&= \begin{bmatrix} -1 & 0.3 \\ 1.2 & 2 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u + \begin{bmatrix} 4 \\ 41.2 \end{bmatrix} (y - \hat{y}) \\
\hat{y} &= C\hat{x} = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}
\end{aligned}$$

d. Set $u = -K\hat{x} + k_r r$ and substitute $\hat{y}$ with its definition. Calculate $k_r = -1/(C(A - BK)^{-1}B) = -20$. This yields the controller

$$\begin{aligned}
\dot{\hat{x}} &= (A - LC - BK)\hat{x} + Bk_r r + Ly \\
&= \begin{bmatrix} -5 & 0.3 \\ -47.87 & -4 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 20 \end{bmatrix} y + \begin{bmatrix} 4 \\ 41.2 \end{bmatrix} r \\
u &= -K\hat{x} + k_r r \\
&= \begin{bmatrix} 7.87 & 6 \end{bmatrix} \hat{x} - 20r.
\end{aligned}$$

Exercise 4. A case study (10pt)

Consider the following feedback system:



The process $P(s)$ is given by the transfer function

$$P(s) = \frac{20}{s^2 + 4.5s + 2}.$$

The contoller is a PI controller $C(s) = \frac{k_i}{s} + k_p$, and $F(s) = 1$. In this exercise $d = 0$.

a. (2pt) Determine the transfer function H_{yn} from the measurement error n to the output y.

b. (2pt) Tune the PI controller $C(s)$ to make sure the system is/remains able to reject a constant measurement error (i.e. a bias). (If you can do without the P or I part, leave it out). Additionally, make sure that in case the measurement error is a unit ramp, the steady state output is not larger than 10.

c. (3pt) Your colleague proposes to use the controller $C(s) = \frac{100}{s}$ just to be sure. Your boss opines that this might cause excessively large input u, even for a step reference $r(t) = \mathbb{1}(t)$ and no measurement errors. Investigate the stability properties in this situation. If the closed-loop system is not stable, recommend a controller that stabilizes the closed-loop system. Then determine the steady state input to the step reference $r(t) = \mathbb{1}(t)$.

   Hint. Use the Routh-Hurwitz criterion.

d. (3pt) Your boss based on your report implements an integral controller with $k_i \approx 0.0117$. This has the effect of placing the system's poles at $-a, -b, -b$ with $a \approx 4.017$ and $b \approx 0.242$. The closed-loop transfer function from r to y is now

$$H_{yr} = \frac{PC}{1 + PC} = \frac{20k_i}{(s + a)(s + b)^2}.$$

   Determine the step response $y(t)$ of the system in terms of k_i, a, b.

Solutions.

a. $y = n + PC(-y)$, so $H_{yn} = \frac{1}{1+PC}$. Filling in our P and C gives

$$H_{yn} = \frac{s(s^2 + 4.5s + 2)}{s(s^2 + 4.5s + 2) + 20(k_i + k_p s)} = \frac{s^3 + 4.5s^2 + 2s}{s^3 + 4.5s^2 + (2 + 20k_p)s + 20k_i}$$

b. In this case, the closed loop poles are stable, so we use the FVT.

$$y_{ss} = \lim_{s\to 0} s\frac{\bar{n}}{s}H_{yn} = \frac{0}{20k_i}.$$

Hence, we reject the step disturbance for any positive k_i. (The integrator part is needed.) For a noise of slope $\bar{n} = 1$,

$$y_{ss} = \lim_{s\to 0} s\frac{1}{s^2}H_{yn} = \frac{1}{10k_i}.$$

Requiring $\frac{1}{10k_i} \leq 10$, we need $k_i \geq 0.01$. As there is no requirement for k_p we leave the proportional control out and set $k_p = 0$.

c. We need the TF H_{ur} to see the effect of r on u. Noting $u = C(r - Pu)$, $H_{ur} = \frac{C}{1+PC}$. With the new information, this is

$$H_{ur} = \frac{100(s^2 + 4.5s + 2)}{s^3 + 4.5s^2 + 2s + 2000}$$

(Just finding the denominator or copying it from before is also okay, since the denominators are all the same for these TFs).

The Routh-Hurwitz criterion tells us this system has stable poles if 4.5 and 2000 are positive (they are) and if $4.5 \cdot 2 > 2000$ (that's not true). Hence the resulting closed loop system is not stable.

In the first part, we found a denominator of $s^3+4.5s^2+2s+20k_i$ (having set $k_d = 0$). Therefore, stability is ensured if $9 > 20k_i$, or $k_i < 9/20 = 0.45$. This means that k_i should be between 0.01 and 0.45.

The steady state input is then obtained as

$$u_{steady} = \lim_{s\to 0} s\frac{k_i(s^2 + 4.5s + 2)}{s^3 + 4.5s^2 + 2s + 20k_i}\frac{1}{s} = \frac{2k_i}{20k_i} = 0.1.$$

d. The output $Y(s) = R(s)H_{yr}(s) = \frac{20k_i}{s(s+a)(s+b)^2}$. Splitting the fraction yields

$$\frac{20k_i}{ab^2s} - \frac{20k_i}{a(a-b)^2(s+a)} - \frac{20k_i(a-2b)}{b^2(a-b)^2(s+b)} - \frac{20k_i}{b(a-b)(s+b)^2}$$

with inverse Laplace transform

$$y(t) = 20k_i\left(\frac{(2b-a)e^{-bt}}{b^2(a-b)^2} + \frac{1}{ab^2} - \frac{e^{-at}}{a(a-b)^2} - \frac{te^{-bt}}{b(a-b)}\right).$$

Exercise 5. Loop shaping (10pt)

Let's consider a process modeled as

$$P(s) = \frac{(s+z)}{(s+1)(s+p)}.$$

a. (2pt) On a separate page you will find a Bode diagram of $C(s)P(s)$, with $C(s)$ a pure integral controller. Find the values of p and z.

b. (2pt) Sketch the Nyquist plot belonging to $P(s)C(s)$.

c. (2pt) Determine the gain margin and phase crossover frequency.

d. (2pt) Determine the phase margin and gain crossover frequency.

e. (2pt) Is the closed-loop system stable?

f. **(Bonus)** (3pt) Sketch the Nyquist plot of $P(s)$.

Solutions.

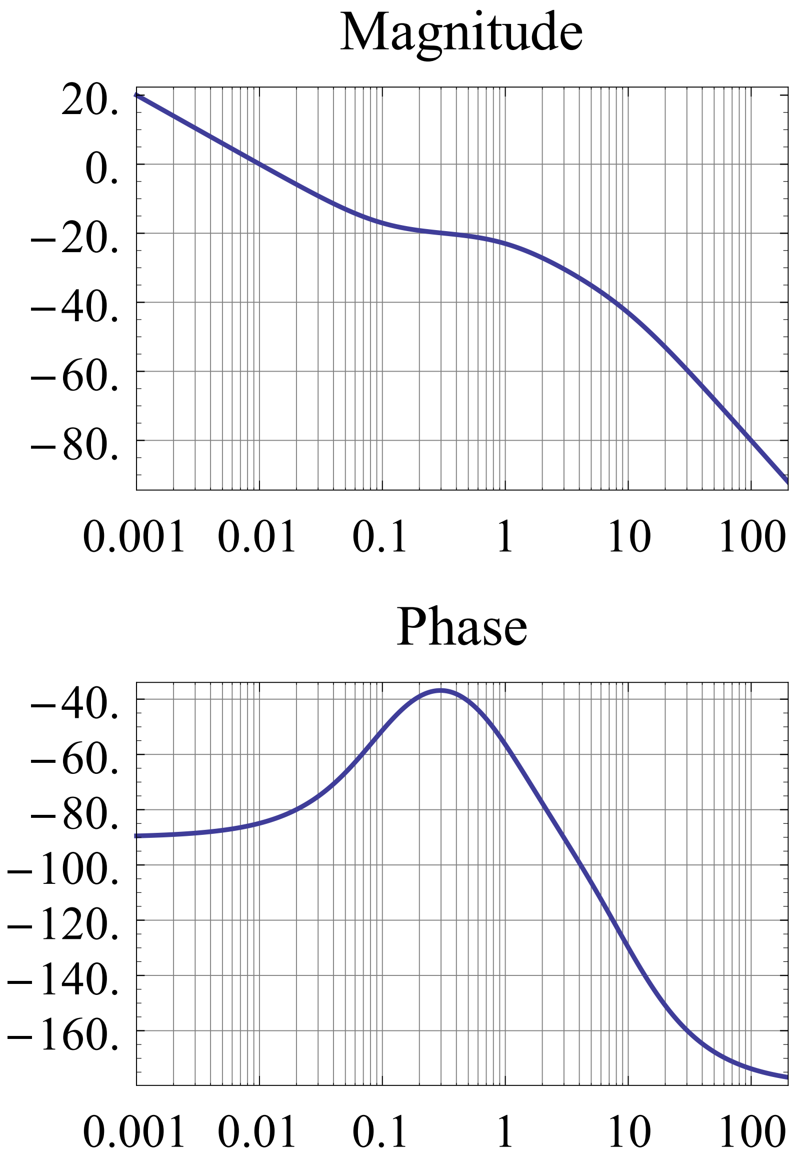
a. $p = 10, z = 0.1$.

b. See figure.

c. There is no phase crossover frequency and the gain margin is infinite.

d. The phase margin is 95 degrees and the gain crossover frequency is $\omega_{gc} = 0.01$.

e. $P = 0, N = 0, Z = 0$. The closed-loop system is stable.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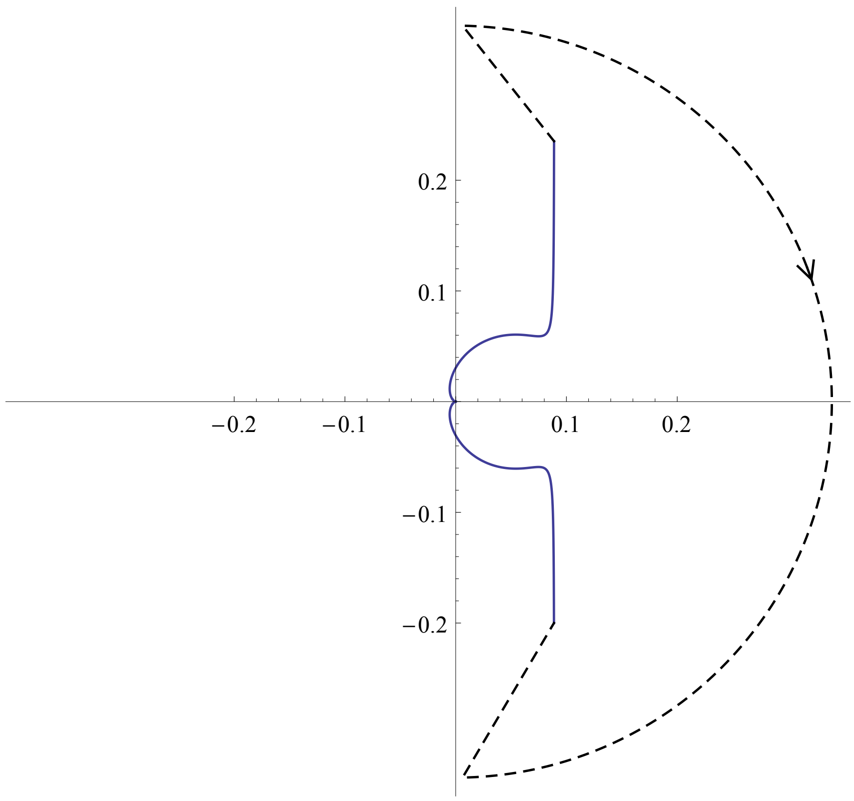

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