Control Engineering
Lecture 5
ver. 2.1.3

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Last Lecture

- Stability of linear systems
- Stability for nonlinear systems via linear approximation
- Routh Hurwitz theorem

Today

- Solutions to linear state space equations
  (matrix exponential, convolution integral)
- Solutions to linear systems “transformable in diagonal form”
- Input/output response
  - Impulse, step response

State-space description

Solutions to linear state space equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Autonomous state equation:

$$\frac{dx(t)}{dt} = Ax(t)$$

Initial condition: $x(0) = x_0$.

To compute an explicit expression of this response, we need to introduce the

Matrix Exponential

Matrix Exponential

Consider

$$\frac{dx(t)}{dt} = Ax(t), \quad x(0) = x_0$$

By the fundamental theorem of calculus

$$\int_0^t \frac{dx(\tau_1)}{d\tau_1} d\tau_1 = \int_0^t Ax(\tau_1) d\tau_1 \Rightarrow x(t) = x_0 + \int_0^t Ax(\tau_1) d\tau_1$$

By the same reason, we have $x(\tau_1) = x_0 + \int_0^{\tau_1} Ax(\tau_2) d\tau_2$, which gives

$$\begin{aligned} x(t) = \quad & x_0 + \int_0^t Ax(\tau_1) d\tau_1 = x_0 + \int_0^t A[x_0 + \int_0^{\tau_1} Ax(\tau_2) d\tau_2] d\tau_1 \\ = \quad & x_0 + \int_0^t Ax_0 d\tau_1 + \int_0^t \int_0^{\tau_1} A^2 x(\tau_2) d\tau_2 d\tau_1 \\ = \quad & x_0 + Atx_0 + \int_0^t \int_0^{\tau_1} A^2 x(\tau_2) d\tau_2 d\tau_1 \end{aligned}$$

Matrix Exponential

$$\begin{aligned}
x(t) = & \quad x_0 + \int_0^t Ax(\tau_1)d\tau_1 = x_0 + \int_0^t A[x_0 + \int_0^{\tau_1} Ax(\tau_2)d\tau_2]d\tau_1 \\
= & \quad x_0 + \int_0^t Ax_0 d\tau_1 + \int_0^t \int_0^{\tau_1} A^2 x(\tau_2)d\tau_2 d\tau_1 \\
= & \quad x_0 + Atx_0 + \int_0^t \int_0^{\tau_1} A^2 x(\tau_2)d\tau_2 d\tau_1
\end{aligned}$$

This computation can be repeated infinite times (replace $x(\tau_2)$ etc) to obtain

$$x(t) = x_0 + Atx_0 + A^2 \tfrac{t^2}{2!} x_0 + A^3 \tfrac{t^3}{3!} x_0 + \ldots = [I + At + A^2 \tfrac{t^2}{2!} + A^3 \tfrac{t^3}{3!} + \ldots] x_0$$

The series $I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \ldots$ converges for every finite t and the resulting matrix is denoted by e^{At}, the exponential of the matrix At
Hence

$$x(t) = \mathrm{e}^{At} x_0$$

Matrix Exponential

Example Double integrator (approximation of nonholonomic mobile robots)

$$\ddot{q} = u, \quad y = q$$

State space form ($x = (q\ \dot{q})^T$)

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Notice that

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

therefore

$$\mathrm{e}^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots = I + At = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Hence the homogenous solution ($u = 0$) to the double integrator is

$$x_h(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x_0 = \begin{pmatrix} x_{01} + x_{02} t \\ x_{02} \end{pmatrix}, \quad y_h(t) = q(t) = x_1(t) = x_{01} + x_{02} t$$

Matrix Exponential

Example Harmonic oscillator (spring-mass system)

$$\ddot{q} + \omega_0^2 q = u, \quad y = q$$

State space form ($x = (\omega_0 q \ \dot{q})^T$)

$$\dot{x} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

The homogenous solution is

$$x_h(t) = \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix} x_0, \quad y(t) = \frac{1}{\omega_0} x_{h1}(t)$$

For instance, if $x(0) = (0 \ \omega_0)^T$, then

$$y(t) = \sin(\omega_0 t)$$

The displacement of a mass-spring system that starts from the rest position with an initial velocity ω_0 has a sinusoidal evolution.

Matrix Exponential

Example Harmonic oscillator (spring-mass system)
If

$$A = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}$$

then

$$\begin{aligned} I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots = & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} t + \begin{pmatrix} -\omega_0^2 & 0 \\ 0 & -\omega_0^2 \end{pmatrix} \frac{t^2}{2!} \\ & + \begin{pmatrix} 0 & -\omega_0^3 \\ \omega_0^3 & 0 \end{pmatrix} \frac{t^3}{3!} + \dots \\ = & \begin{pmatrix} 1 - \omega_0^2 \frac{t^2}{2!} + \dots & \omega_0 t - \omega_0^3 \frac{t^3}{3!} + \dots \\ -\omega_0 t + \omega_0^3 \frac{t^3}{3!} - \dots & 1 - \omega_0^2 \frac{t^2}{2!} + \dots \end{pmatrix} \\ = & \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix} \end{aligned}$$

Hence

$$\mathrm{e}^{At} = \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix}$$

Zero input case

Given the initial state $x(t_0)$, then the autonomous state equation $\dot{x} = Ax$ has a **unique solution** (called the **homogeneous** solution)

$$x_h(t) = e^{A(t-t_0)}x(t_0) = e^{A(t-t_0)}x_0$$

with the **matrix exponential**

$$e^{At} = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots$$

- It generalizes the scalar case

$$\dot{x} = ax, \quad x(0) = x_0 \quad \Leftrightarrow \quad x(t) = \mathrm{e}^{at}x_0$$

- The series $\sum_{i=0}^{\infty} \frac{A^i t^i}{i!}$ converges for any t (ie., e^{At} is a well-defined matrix)
- $\frac{d}{dt}e^{At} = \quad Ae^{At} = e^{At}A$
- $e^{At+Bt} = e^{At}e^{Bt}$
- $e^{A\,0} = I$ (I is the identity matrix)

Zero input case

The property $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$ can be justified as follows:

$$\begin{aligned}
\frac{d}{dt}e^{At} &= \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \cdots\right) \\
&= \left(0 + A + \frac{A^2t}{1!} + \frac{A^3t^2}{2!} + \cdots\right) \\
&= A\left(I + At + \frac{A^2t^2}{2!} + \cdots\right) = Ae^{At} \\
&= \left(I + At + \frac{A^2t^2}{2!} + \cdots\right)A = e^{At}A
\end{aligned}$$

Homogeneous solution is linear in the initial condition

$$\left.\begin{array}{lcl}
\dot{x} = Ax, x(0) = x_{01} & \Rightarrow & x_{h1}(t) = e^{At}x_{01} \\
\dot{x} = Ax, x(0) = x_{02} & \Rightarrow & x_{h2}(t) = e^{At}x_{02}
\end{array}\right\} \Rightarrow
\begin{array}{c}
\dot{x} = Ax, x(0) = \alpha x_{01} + \beta x_{02} \\
\Downarrow \\
\begin{aligned}
x(t) &= e^{At}(\alpha x_{01} + \beta x_{02}) \\
&= \alpha e^{At}x_{01} + \beta e^{At}x_{02} \\
&= \alpha x_{h1}(t) + \beta x_{h2}(t)
\end{aligned}
\end{array}$$

Diagonal form I

We saw in Lecture 4 that **if** the dynamical matrix A ($n \times n$) has all the eigenvalues $s_1, \ldots, s_n$ *real and distinct* ($s_i \neq s_j$ for all $i \neq j$ and $s_i \in \mathbb{R}$), **then** such a matrix can be transformed into a **diagonal form**.[1]

There exists an *invertible* matrix T such that

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \lambda_n \end{pmatrix}$$

Observe that

$$\begin{aligned} \tilde{A}^2 = &\quad TAT^{-1}TAT^{-1} = TAIAT^{-1} = TA^2T^{-1} \\ \tilde{A}^3 = &\quad \tilde{A}^2\tilde{A} = TA^2T^{-1}TAT^{-1} = TA^3T^{-1} \\ &\vdots \\ \tilde{A}^k = &\quad \ldots = TA^kT^{-1} \end{aligned}$$

[1] See "Handout on the diagonalization of a matrix" on Brightspcae, which is the solution to [Textbook, Exercise 4.14]

Diagonal form I

Applying the definition $e^{\tilde{A}t} = \sum_{i=0}^{\infty} \frac{\tilde{A}^i t^i}{i!} = I + \tilde{A}t + \frac{\tilde{A}^2 t^2}{2!} + \frac{\tilde{A}^3 t^3}{3!} + \cdots$

$$\mathrm{e}^{\tilde{A}t} = \begin{pmatrix} 1 + \lambda_1 t + \lambda_1^2 \frac{t^2}{2!} + \ldots & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 + \lambda_n t + \lambda_n^2 \frac{t^2}{2!} + \ldots \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\lambda_1 t} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \mathrm{e}^{\lambda_n t} \end{pmatrix}$$

because

$$I = \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{pmatrix} \quad \tilde{A}t = \begin{pmatrix} \lambda_1 t & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n t \end{pmatrix} \quad \tilde{A}^2 \frac{t^2}{2!} = \begin{pmatrix} \lambda_1^2 \frac{t^2}{2!} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n^2 \frac{t^2}{2!} \end{pmatrix}$$

and the series $1 + \lambda t + \lambda^2 \frac{t^2}{2!} + \ldots$ is the MacLaurin series of $\mathrm{e}^{\lambda t}$, to which it converges for any finite t.

Diagonal form I

The matrix exponential $\mathrm{e}^{\tilde{A}t}$ can be used to derive the solution to the original system $\dot{x} = Ax$.
Consider the change of coordinates

$$z = Tx \quad \Leftrightarrow \quad T^{-1}z = x$$

In the new coordinates z the system $\dot{x} = Ax$ becomes

$$\dot{z} = T\dot{x} = TAx = TAT^{-1}z = \tilde{A}z$$

Hence

$$z(t) = \mathrm{e}^{\tilde{A}t}z(0) = \begin{pmatrix} \mathrm{e}^{\lambda_1 t} & 0 & \dots & 0 & 0 \\ 0 & \mathrm{e}^{\lambda_2 t} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \mathrm{e}^{\lambda_n t} \end{pmatrix} z(0)$$

In the original coordinates

$$x(t) = T^{-1}z(t) = T^{-1}\mathrm{e}^{\tilde{A}t}z(0) = \underbrace{T^{-1}\mathrm{e}^{\tilde{A}t}T}_{\mathrm{e}^{At}}x(0)$$

Diagonal form II

We saw in Lecture 4 that **if** the dynamical matrix A ($n \times n$) has the eigenvalues $s_1, \ldots, s_n$ distinct and some of them are complex conjugate, **then** such a matrix can be converted into a **block diagonal form**.[2]

There exists an *invertible* matrix T such that

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} \Lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \Lambda_k \end{pmatrix}, \quad \text{where} \quad \Lambda_i = \lambda_i \in \mathbb{R}, \quad \Lambda_i = \begin{pmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{pmatrix}$$

Then

$$\mathrm{e}^{\tilde{A}t} = \begin{pmatrix} \mathrm{e}^{\Lambda_1 t} & 0 & \ldots & 0 & 0 \\ 0 & \mathrm{e}^{\Lambda_2 t} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \mathrm{e}^{\Lambda_n t} \end{pmatrix}, \quad \text{where} \quad \begin{array}{l} \mathrm{e}^{\Lambda_i t} = \mathrm{e}^{\lambda_i t} \quad \text{or} \\ \mathrm{e}^{\Lambda_i t} = \mathrm{e}^{\sigma_i t} \begin{pmatrix} \cos(\omega_i t) & \sin(\omega_i t) \\ -\sin(\omega_i t) & \cos(\omega_i t) \end{pmatrix} \end{array}$$

[2] See "Handout on the diagonalization of a matrix" on Nestor, which is the solution to [Textbook, Exercise 4.14]

Diagonal form II

To show that

$$\mathrm{e}^{\Lambda_i t} = \mathrm{e}^{\sigma_i t} \begin{pmatrix} \cos(\omega_i t) & \sin(\omega_i t) \\ -\sin(\omega_i t) & \cos(\omega_i t) \end{pmatrix} \quad \text{if} \quad \Lambda_i = \begin{pmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{pmatrix}$$

observe that

$$\Lambda_i = \begin{pmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{pmatrix} = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix} + \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = \Lambda_i^{(1)} + \Lambda_i^{(2)}$$

and recall the property $\mathrm{e}^{At+Bt} = \mathrm{e}^{At}\mathrm{e}^{Bt}$.
It follows that

$$\mathrm{e}^{\Lambda_i t} = \mathrm{e}^{\Lambda_i^{(1)} t}\mathrm{e}^{\Lambda_i^{(2)} t} = \begin{pmatrix} \cos(\omega_i t) & \sin(\omega_i t) \\ -\sin(\omega_i t) & \cos(\omega_i t) \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\sigma_i t} & 0 \\ 0 & \mathrm{e}^{\sigma_i t} \end{pmatrix} = \mathrm{e}^{\sigma_i t} \begin{pmatrix} \cos(\omega_i t) & \sin(\omega_i t) \\ -\sin(\omega_i t) & \cos(\omega_i t) \end{pmatrix}$$

Jordan form

And when A can **not** be converted into a (block) diagonal form?

Jordan form

One can convert any matrix into the Jordan form J and use it to compute

$$\mathrm{e}^{At} = T\mathrm{e}^{Jt}T^{-1}$$

The Jordan from is discussed at pp. 139–141 of the textbook but it is not part of the program of the course.

Example I

Compute the state response of the linear autonomous system

$$\dot{x} = Ax, \quad \text{with} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Eigenvalues of A: $\{1, 1 \pm i1\}$

Consider the transformation matrix[3]

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{with inverse} \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

It can be checked that

$$\tilde{A} = TAT^{-1} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & -1 & 1 \end{array}\right] = \left[\begin{array}{c|c} \Lambda_1 & 0_{1\times 2} \\ \hline 0_{2\times 1} & \Lambda_2 \end{array}\right]$$

which is the desired block diagonal form

[3] This is obtained following the "Handout on the diagonalization of a matrix" available on Nestor, which is the solution to [Textbook, Exercise 4.14]

Example II

State response

$$x(t) = \mathrm{e}^{At}x(0) = T^{-1}\mathrm{e}^{\tilde{A}t}Tx(0)$$

The matrix exponential can be computed as

$$\begin{aligned}
\mathrm{e}^{At} = \quad & T^{-1}\mathrm{e}^{\tilde{A}t}T = T^{-1}\left[\begin{array}{c|c} \mathrm{e}^{\Lambda_1 t} & 0_{1\times 2} \\ \hline 0_{2\times 1} & \mathrm{e}^{\Lambda_2 t} \end{array}\right] T \\
= \quad & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{e}^t & 0 & 0 \\ 0 & \mathrm{e}^t\cos t & \mathrm{e}^t\sin t \\ 0 & -\mathrm{e}^t\sin t & \mathrm{e}^t\cos t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \\
= & \begin{bmatrix} \mathrm{e}^t & 0 & 0 \\ \mathrm{e}^t - \mathrm{e}^t\cos t & \mathrm{e}^t\cos t & \mathrm{e}^t\sin t \\ \mathrm{e}^t\sin t & -\mathrm{e}^t\sin t & \mathrm{e}^t\cos t \end{bmatrix}
\end{aligned}$$

Hence, the state response of the linear autonomous system is

$$\begin{aligned}
x(t) = \mathrm{e}^{At}x(0) \quad &= \begin{bmatrix} \mathrm{e}^t & 0 & 0 \\ \mathrm{e}^t(1-\cos t) & \mathrm{e}^t\cos t & \mathrm{e}^t\sin t \\ \mathrm{e}^t\sin t & -\mathrm{e}^t\sin t & \mathrm{e}^t\cos t \end{bmatrix} x(0) \\
&= \mathrm{e}^t \begin{bmatrix} x_1(0) \\ (1-\cos t)x_1(0) + \cos(t)x_2(0) + \sin(t)x_3(0) \\ \sin t x_1(0) - \sin(t)x_2(0) + \cos(t)x_3(0) \end{bmatrix}
\end{aligned}$$

Input/output response

If $u \neq 0$, the solution to

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

becomes

$$x(t) = e^{A(t-t_0)}x(t_0) + \boldsymbol{\int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau}, \quad t \geq t_0$$

The integral term is called the **convolution** integral

Output response

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ &= Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\, d\tau + Du(t) \\ & \qquad t \geq t_0 \end{aligned}$$

How to prove this?

Input/output response

The solution to

$$\dot{x} = Ax + Bu$$

$$x(t) = \mathrm{e}^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} Bu(\tau)\, d\tau, \quad t \geq t_0$$

Write $(e^{A(t-\tau)} = \mathrm{e}^{At-A\tau} = \mathrm{e}^{At}\mathrm{e}^{-A\tau})$

$$x(t) = e^{A(t-t_0)}x(t_0) + e^{At}\int_{t_0}^{t} e^{-A\tau} Bu(\tau)\, d\tau, \quad t \geq t_0$$

Differentiating

$$\begin{aligned}
\dot{x}(t) &= Ae^{A(t-t_0)}x(t_0) + Ae^{At}\int_{t_0}^{t} e^{-A\tau} Bu(\tau)\, d\tau + e^{At}e^{-At}Bu(t) \\
&= A\left(e^{A(t-t_0)}x(t_0) + e^{At}\int_{t_0}^{t} e^{-A\tau} Bu(\tau)\, d\tau\right) + e^{A0}Bu(t) \\
&= Ax(t) + Bu(t)
\end{aligned}$$

Linear state space system

We consider Single Input Single Output (SISO) linear state space systems, i.e.,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

with $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$, A $n \times n$ matrix, B, C^T n dim. vector.
Solution:

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ &= Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} \underbrace{Ce^{A(t-\tau)}Bu(\tau)}_{1\times 1} \, d\tau + Du(t) \\ & \quad t \geq t_0 \end{aligned}$$

The integrand reduces to a <u>scalar</u> function, and computing the integral is computationally less expensive.

Linearity

The linearity of the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

is reflected on the linearity of the input/output response.
Let

$$y(t; x_0, u)$$

be the output response $y(t)$ corresponding to the initial condition x_0 and input u.
The **input/output response** has linear properties with respect to the initial conditions and the inputs:

$$\begin{aligned} &\text{(i)} \quad y(t;\alpha x_1 + \beta x_2, 0) = \alpha y(t;x_1,0) + \beta y(t;x_2,0), \\ &\text{(ii)} \quad y(t;\alpha x_0, \delta u) = \alpha y(t;x_0,0) + \delta y(t;0,u), \\ &\text{(iii)} \quad y(t;0,\delta u_1 + \gamma u_2) = \delta y(t;0,u_1) + \gamma y(t;0,u_2). \end{aligned}$$

(iii) is the so-called **superposition principle**

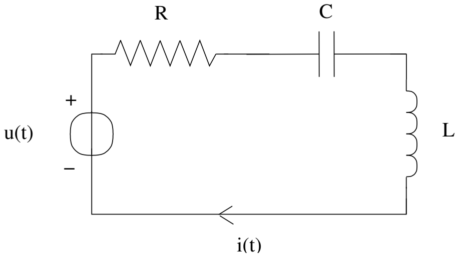
Example: RLC circuit (mass-spring-damper system)

This is an example on the diagonalization of a matrix A and the computation of the corresponding matrix exponential and convolution integral. This example is left as a reading assignment.

For a more general exercise on diagonalization see [Textbook, Exercise 4.14], solved in "Handout on the diagonalization of a matrix" on Nestor.

Input:
$u(t)$

Output:
$y(t) = i(t)$.



$$\begin{aligned} \frac{dx(t)}{dt} &= \begin{bmatrix} 0 & 1/L \\ -1/C & -R/L \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1/L \end{bmatrix} x(t) \end{aligned}$$

Example: RLC circuit

Initial condition: $x(0) == 0$

Input signal: $u(t) = 1$

$$y(t) = \underbrace{Ce^{At}x(0)}_{=0} + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + \underbrace{Du(t)}_{=0}$$

Example: RLC circuit

Take $L = 1$, $R = 2$ and $C = \frac{4}{3}$, then

$$\begin{aligned} \det(\lambda I - A) &= \det\left(\begin{bmatrix} \lambda & -1 \\ 3/4 & \lambda + 2 \end{bmatrix}\right) \\ &= \lambda^2 + 2\lambda + \frac{3}{4} = 0 \end{aligned}$$

$$\lambda_{1,2} = -1 \pm \frac{\sqrt{4-3}}{2} = -1 \pm \frac{1}{2} = -\frac{1}{2}, -\frac{3}{2}$$

$$A = \begin{bmatrix} 0 & 1 \\ -3/4 & -2 \end{bmatrix} = T^{-1}\Lambda T, \qquad e^{At} = T^{-1} e^{\Lambda t} T$$

Example: RLC circuit

An eigenvector $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ satisfies by definition

$$A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3/4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

First row: $v_2 = \lambda v_1$,
Take $v_1 = 1$

$$\lambda_1 \text{ eigenvector: } \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \qquad \lambda_2 \text{ eigenvector: } \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

Example: RLC circuit

The eigenvalues determine the diagonal form and the eigenvectors determine the change of coordinates

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \qquad T = \underbrace{\frac{1}{(\lambda_2 - \lambda_1)}}_{=-1} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$$

$$\begin{aligned} e^{At} &= T^{-1} e^{\Lambda t} T \\ &= \underbrace{\frac{1}{(\lambda_2 - \lambda_1)}}_{=-1} \begin{bmatrix} (\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}) & e^{\lambda_2 t} - e^{\lambda_1 t} \\ \lambda_1 \lambda_2 (e^{\lambda_1 t} - e^{\lambda_2 t}) & (\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}) \end{bmatrix} \end{aligned}$$

Example: RLC circuit

$$
\begin{aligned}
y(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau \\
&= \int_0^t \begin{bmatrix} 0 & 1 \end{bmatrix} e^{A(t-\tau)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 1\,d\tau \\
&= \int_0^t -(\lambda_2 e^{\lambda_2(t-\tau)} - \lambda_1 e^{\lambda_1(t-\tau)})\,d\tau \\
&= -\left(-e^{\lambda_2(t-\tau)} + e^{\lambda_1(t-\tau)}\right)\Big|_{\tau=0} \\
&= -(e^{\lambda_1 t} - e^{\lambda_2 t}) = -e^{-\frac{1}{2}t} + e^{-\frac{3}{2}t}
\end{aligned}
$$

Coordinate invariance

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

and the same system in the new coordinates $z = Tx$, with T nonsingular

$$\begin{aligned} \dot{z} &= T(Ax + Bu) = TAx + TBu \\ &= \overbrace{TAT^{-1}}^{\tilde{A}} z + \overbrace{TB}^{\tilde{B}} u \\ y &= \underbrace{CT^{-1}}_{\tilde{C}} z + Du \end{aligned}$$

Coordinate invariance

The input-output response of the system in the x-coordinates is ($t_0 = 0$)

$$y_x(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

The input-output response of the system in the z-coordinates is ($t_0 = 0$)

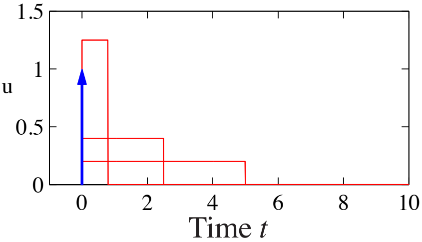
$$\begin{aligned}
y_z(t) &= \tilde{C}e^{\tilde{A}t}z(0) + \int_0^t \tilde{C}e^{\tilde{A}(t-\tau)}\tilde{B}u(\tau)\,d\tau + Du(t) \\
&= CT^{-1}Te^{At}T^{-1}z(0) + \int_0^t CT^{-1}Te^{A(t-\tau)}T^{-1}TBu(\tau)\,d\tau + Du(t) \\
&= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)
\end{aligned}$$

Since $e^{\tilde{A}t} = Te^{At}T^{-1}$, we have $y_z(t) = y_x(t)$ for all $t \geq 0$, that is, **the input-output response of a linear system is invariant with respect to the choice of the coordinates**.

Impulse response

Pulse Signal of duration ε and amplitude $\dfrac{1}{\varepsilon}$

$$u(t) = p_\varepsilon(t) = \begin{cases} 0 & t < 0 \\ 1/\varepsilon & 0 \le t < \varepsilon \\ 0 & t \ge \varepsilon. \end{cases}$$



The impulse is the limit of the pulse as $\varepsilon \to 0$

$$\delta(t) = \lim_{\varepsilon \to 0} p_\varepsilon(t)$$

Impulse response

The **impulse response** $h(t)$ is the output response of a system that is initially at rest, i.e., $x(0) = 0$, to an input that equals the impulse $\delta(t)$.

Impulse response

Impulse properties[4]

$$\delta(t) = 0, \quad t \neq 0 \tag{1}$$

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1 \tag{2}$$

For any continuous function $f(t)$ in $t = \tau$, by (1)

$$f(t)\delta(t-\tau) = f(\tau)\delta(t-\tau)$$

and by (2)

$$\int_{-\infty}^{+\infty} f(t)\delta(t-\tau)dt = f(\tau)\int_{-\infty}^{+\infty} \delta(t-\tau)dt = f(\tau) \tag{3}$$

[4] The analytical treatment of the properties of the impulse is out of the scope of this course.

Impulse response

Output response

$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)
\end{aligned}
$$

$$t \geq t_0$$

Impulse response: $h(t) = y(t)$ for $x(t_0) = 0$ and $u(t) = \delta(t)$:

$$h(t) = Ce^{At}B + D\delta(t)$$

since by (3)

$$\int_{t_0}^{t} Ce^{A(t-\tau)}B\delta(\tau)\,d\tau = Ce^{At}B$$

Question What is the impulse response of the RLC circuit studied before?

Impulse response

Impulse response - an intuitive explanation of the formula:

$$h(t) = \int_{t_0}^{t} Ce^{A(t-\tau)}B\delta(\tau)\,d\tau$$

For simplicity and without loss of generality, we take $t_0 = 0$ and $D = 0$.

$$\begin{aligned}
h_\varepsilon(t) = &\quad Ce^{At}\int_0^t e^{-A\tau}p_\varepsilon(\tau)\,d\tau B = Ce^{At}\int_0^\varepsilon e^{-A\tau}\frac{1}{\varepsilon}\,d\tau B \\
= &\quad Ce^{At}\int_0^\varepsilon \left(I - A\tau + A^2\frac{\tau^2}{2!} - \dots\right)\frac{1}{\varepsilon}\,d\tau B \\
= &\quad Ce^{At}\left(\varepsilon I - A\frac{\varepsilon^2}{2} + A^2\frac{\varepsilon^3}{6} - \dots\right)\frac{1}{\varepsilon}\,B \\
= &\quad Ce^{At}\left(I - A\frac{\varepsilon}{2} + A^2\frac{\varepsilon^2}{6} - \dots\right)B
\end{aligned}$$

Taking the limit as $\varepsilon \to 0$ returns the formula of the impulse response.

Step response

The **step response** $r(t)$ is the output response of a system which is initially at rest, i.e., $x(0) = 0$, to an input that equals a **unit step function**

$$1(t) = \begin{cases} 0 & \text{for } \; t < 0 \\ 1 & \text{for } \; t \geq 0 \end{cases}$$

Let $D = 0$ ($h(t) = C\mathrm{e}^{At}B$)

$$\begin{aligned} y(t) &= r(t) \\ &= \int_0^t h(t-\tau)\,d\tau \overset{\eta=t-\tau}{=} \int_t^0 h(\eta)\,(-d\eta) \\ &= \int_0^t h(\eta)\,d\eta = \int_0^t C\mathrm{e}^{A\eta}B\,d\eta \end{aligned}$$

$$r(t) = \int_0^t h(\eta)\,d\eta, \quad \text{thus} \quad \frac{dr(t)}{dt} = h(t)$$

If $D \neq 0$ then $r(t) = \int_0^t C\mathrm{e}^{A\eta}B\,d\eta + D1(t)$.

Step response

$$
\begin{aligned}
y(t) &= r(t) \\
&= \int_0^t h(\eta)\, d\eta \\
&= \int_0^t C\mathrm{e}^{A\eta} B\, d\eta + D1(t)
\end{aligned}
$$

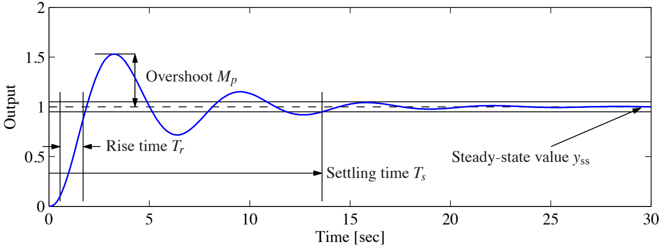
The convolution integral can be further manipulated (A nonsingular)

$$
\begin{aligned}
\int_0^t C\mathrm{e}^{A\eta} B\, d\eta = \int_0^t C\, \overbrace{A^{-1}A}^{I} \mathrm{e}^{A\eta} B\, d\eta &= \int_0^t CA^{-1} \frac{d\mathrm{e}^{A\eta}}{d\eta} B\, d\eta \\
&= CA^{-1} \int_0^t \frac{d\mathrm{e}^{A\eta}}{d\eta}\, d\eta B = CA^{-1} \int_0^t d\mathrm{e}^{A\eta} B \\
&= CA^{-1}(\mathrm{e}^{At} - I)B
\end{aligned}
$$

If A is asymptotically stable, then

$$
r(t) = \underbrace{CA^{-1}\mathrm{e}^{At}B}_{\text{transient}} \underbrace{-CA^{-1}B + D}_{\text{steady state}} \stackrel{t\to\infty}{\longrightarrow} -CA^{-1}B + D
$$

Step response characterization



- **steady state value** y_{st} final value where the output converges
- **rise time** T_r amount of time required for the output to go from 10% to 90% of its final value
- **overshoot** M_p percentage of the final value by which the signal rises above the final value
- **settling time** T_s amount of time required for the output to stay within 2% of its final value

Next Lecture(s)

Harmonic response

Reachability

Output regulation problem Given the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

find a state feedback control of the form

$$u = -Kx + k_r r$$

such that the output response of the closed-loop system converges to r, i.e.

$$y(t) \to r \quad \text{as} \quad t \to +\infty.$$