EXAM MECHATRONICS

Wednesday, January 22^{nd} 2020, 8:30 - 11:30 hrs.

Last Name: ______________________ First Name: ______________________ StudentID: ____________

INSTRUCTIONS:

- This exam consists of 23 pages with 6 open questions. Make sure you have all pages and questions.
- Read carefully each question and answer accordingly.
- The answers to each question (including motivation) have to be placed inside the answer boxes.
- Write neatly.
- Write your name and student number in all pages. The exercises will be collected separately.
- If you like, you can use and add additional paper, which needs to include your name and student number. Please provide separate papers for separate exercises. Hand in the exercises on separate piles.
- You can earn a maximum of 100 points at the exam. The amount of points spread over the exercises is 110 points, i.e., there are 10 bonus points to be earned.
- You will only get a grade if you have finalized the practical.
- This is a CLOSED book exam.

Preliminaries

Across and Through variable table

Table 1.2. Ideal system elements (linear)

System type	Mechanical translational	Mechancial rotational	Electrical	Fluid	Thermal
A-type variable	Velocity, v	Velocity, Ω	Voltage, e	Pressure, P	Temperature, T
A-type element	Mass, m	Mass moment of inertia, J	Capacitor, C	Fluid Capacitor, C_f	Thermal capacitor, C_h
Elemental equations	$F = m\dfrac{dv}{dt}$	$T = J\dfrac{d\Omega}{dt}$	$i = C\dfrac{de}{dt}$	$Q_f = C_f\dfrac{dP}{dt}$	$Q_h = C_h\dfrac{dT}{dt}$
Energy stored	Kinetic	Kinetic	Electric field	Potential	Thermal
Energy equations	$\mathscr{E}_k = \frac{1}{2}mv^2$	$\mathscr{E}_k = \frac{1}{2}J\Omega^2$	$\mathscr{E}_e = \frac{1}{2}Ce^2$	$\mathscr{E}_p = \frac{1}{2}C_f P^2$	$\mathscr{E}_t = \frac{1}{2}C_h T^2$
T-type variable	Force, F	Torque, T	Current, i	Fluid flow rate, Q_f	Heat flow rate, Q_h
T-type element	Compliance, $1/k$	Compliance, $1/K$	Inductor, L	Inertor, I	None
Elemental equations	$v = \dfrac{1}{k}\dfrac{dF}{dt}$	$\Omega = \dfrac{1}{K}\dfrac{dT}{dt}$	$e = L\dfrac{di}{dt}$	$P = I\dfrac{dQ_f}{dt}$	
Energy stored	Potential	Potential	Magnetic field	Kinetic	
Energy equations	$\mathscr{E}_P = \dfrac{1}{2k}F^2$	$\mathscr{E}_P = \dfrac{1}{2K}T^2$	$\mathscr{E}_m = \frac{1}{2}Li^2$	$\mathscr{E}_k = \frac{1}{2}IQ_f^2$	
D-type element	Damper, b	Rotational damper, B	Resistor, R	Fluid resistor, R_f	Thermal resistor, R_h
Elemental equations	$F = bv$	$T = B\Omega$	$i = \dfrac{1}{R}e$	$Q_f = \dfrac{1}{R_f}P$	$Q_h = \dfrac{1}{R_h}T$
Rate of energy dissipated	$\dfrac{dE_D}{dt} = Fv = \dfrac{1}{b}F^2 = bv^2$	$\dfrac{dE_D}{dt} = T\Omega = \dfrac{1}{B}T^2 = B\Omega^2$	$\dfrac{dE_D}{dt} = ie = Ri^2 = \dfrac{1}{R}e^2$	$\dfrac{dE_D}{dt} = Q_f P = R_f Q_f^2 = \dfrac{1}{R_f}P^2$	$\dfrac{dE_D}{dt} = Q_h$

Note: A-type variable represents a spatial difference across the element.

The other analogy for linear systems as was treated in Control Engineering, and is useful for Euler-Lagrange modeling.

	Kinetic coenergy	**Potential energy**	**Rayleigh dissipation function**
Translation	$T^*(\dot{x}) = \frac{1}{2} m \frac{dx}{dt}^2$	$V(x) = \frac{1}{2} k x^2$	$\mathcal{D}(\dot{x}) = \frac{1}{2} b \frac{dx}{dt}^2$
Rotation	$T^*(\dot{\theta}) = \frac{1}{2} J \frac{d\theta}{dt}^2$	$V(\theta) = \frac{1}{2} k \theta^2$	$\mathcal{D}(\dot{\theta}) = \frac{1}{2} b \frac{d\theta}{dt}^2$
Electric	$T^*(\dot{q}) = \frac{1}{2} L \frac{dq}{dt}^2$	$V(q) = \frac{1}{2C} q^2$	$\mathcal{D}(\dot{q}) = \frac{1}{2} R \frac{dq}{dt}^2$

Euler-Lagrange equations

The Euler-Lagrange equation, considering external forces and dissipation, is given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q} = F - \frac{\partial \mathcal{D}(\dot{q})}{\partial \dot{q}} \tag{1}$$

where $\mathcal{L}(q,\dot{q})$ denotes the Lagrangian of the system, q the position, and $\mathcal{D}(\dot{q})$ the Rayleigh dissipation function.

Canonical forms

The state-space representation for a given transfer function is not unique, i.e., there are infinite-number of possibilities to express a given transfer function in state-space form. However, there are several forms that can be helpful in the design of controller or observer. Let us consider the following general transfer function for single-input single-output system:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}. \tag{2}$$

For this transfer function, the state-space representation in **canonical observable form** is given by

$$\begin{gathered}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u \\
y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u.
\end{gathered} \tag{3}$$

On the other hand, the state-space representation in the **canonical controllable form** is given by

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \cdots & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u.
\end{aligned} \tag{4}$$

Z-transform.

Denote by $Z\{u(n)\}$ the Z-transform of discrete-time signal $u(n)$ where $n = 0, 1, \ldots$.

- Unit step signal $u(n)$: $Z\{u(n)\} = \dfrac{1}{1 - z^{-1}}$
- Time-shifting property: $Z\{u(n-k)\} = z^{-k}U(z)$

Transformation from s-domain to z-domain

- The bilinear transformation: $s \mapsto \dfrac{2}{T_s}\dfrac{1 - z^{-1}}{1 + z^{-1}} = \dfrac{2}{T_s}\dfrac{z - 1}{z + 1}$
- The Euler backward approximation: $s \mapsto \dfrac{1}{T_s}(1 - z^{-1}) = \dfrac{z - 1}{T_s z}$
- The Euler forward approximation: $s \mapsto \dfrac{z - 1}{T_s}$

Optimal state feedback control design(LQR)

The Riccati equation, that is related to the optimal state feedback, reads as

$$A^T P + PA - PBR^{-1}B^T P + Q = 0, \tag{5}$$

where $P = P^T > 0$, and where $Q = Q^T > 0$ and $R = R^T > 0$ are related to the cost function

$$J = \int_0^\infty \left(x^T(\tau)Qx(\tau) + u^T Ru(\tau)\right) d\tau. \tag{6}$$

The optimal state feedback controller is given by $u(t) = -R^{-1}B^T Px(t)$.

State observer design

For a state-space system described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{7}$$

where x is the actual state and y is the measured signal, a state observer for such system has the following structure

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x} + Du, \end{aligned} \tag{8}$$

where $\hat{x}$ is the estimated state and y is the corresponding estimated output.

Padé approximation

The first order Padé approximation of the delay transfer function e^{-Ts} is given by

$$e^{-Ts} \approx \frac{1 - \frac{T}{2}s}{1 + \frac{T}{2}s}. \tag{9}$$

Complex numbers

Consider a complex number

$$z = a + bj. \tag{10}$$

Then

$$\begin{aligned} |z| &= \sqrt{a^2 + b^2} \\ \angle z &= \tan^{-1}\left(\frac{b}{a}\right) \end{aligned} \tag{11}$$

Phase margin

The phase margin of a system $G(s)$ can be computed as

$$P_m = -\pi - \angle G(j\omega_*), \tag{12}$$

where the frequency ω_* verifies

$$|G(j\omega_*)| = 1. \tag{13}$$

The phase margin of a system with delay is computed as follows

$$P_{m_d} = P_m + T\omega_*, \tag{14}$$

where T is the time delay.

Absolute stability

Popov's criterion

Proposition 1. *If a linear system combines with a static nonlinearity in the feedback, and the following are fulfilled:*

- *A (the system's matrix) is asymptotically stable*
- *The nonlinearity belongs to a sector $[0, k]$*
- *There exists a constant $\alpha > 0$ such that for all $\omega \geq 0$*

$$\Re\left((1 + j\alpha\omega)G(j\omega)\right) + \frac{1}{k} \geq \epsilon \tag{15}$$

for arbitrarily small $\epsilon > 0$, then 0 is globally asymptotically stable.

Circle's criterion

Theorem 1. *If a linear system combines with a static nonlinearity in the feedback, and the following are fulfilled:*

- *A (the system's matrix) has no eigenvalues on the $j\omega$-axis and ρ eigenvalues in the RHP*
- *The nonlinearity belongs to the sector $[k_1, k_2]$*
- *One of the following holds*
  - *$0 < k_1 \leq k_2$, the Nyquist plot of $G(j\omega)$ does not enter the disk $D(k_1, k_2)$ and encircles it ρ times anti-clockwise*
  - *$0 = k_1 < k_2$, the Nyquist plot stays to the right of $\Re(s) > -\dfrac{1}{k_2}$*
  - *$k_1 < 0 < k_2$, the Nyquist plot of $G(j\omega)$ stays in $D(k_1, k_2)$*
  - *$k_1 < k_2 < 0$, the Nyquist plot of $-G(j\omega)$ does not enter $D(-k_1, -k_2)$ and encircles it ρ times anti-clockwise*

Then 0 is globally asymptotically stable.

1. (12 TOTAL points) Suppose that you are involved in the design of the new generation of the Ninebot Z10, like the one depicted in Fig. 1, which is a personal transport device. The main idea of the Ninebot is that the torque of the motor makes spin the wheel; this produces a translational force that allows the user to move forward or backward.

Figure 1: Ninebot by Segway Z10.

(a) (6 points) Identify two possible user demands, two possible functional requirements, and two possible design parameters.

(b) (6 points) Consider the Mechatronics block diagram shown in Fig. 2. Identify one element of the Ninebot Z10 for each of the following items.

- measured variables
- reference variables
- sensors
- actuators
- man/machine interface
- energy supply

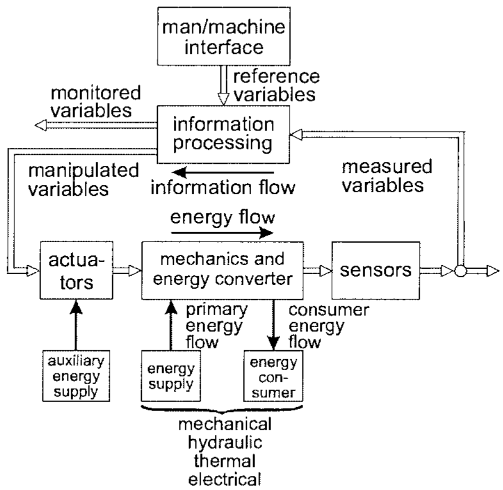

Figure 2: Mechatronics Block

2. (25 TOTAL points) Consider the fan represented by Fig. 3, where an RL circuit controls the motor that makes spin the propeller. The voltage across the motor terminals is v_a, the variables θ_m, ω_m, and T_m denote the angular position of the motor's shaft, the angular velocity of the motor's shaft, and the torque in the motor's shaft, respectively. The variable k_T represents the constant of the rotational spring that interconnects the motor and the propeller. The inertia of the propeller is J, B_θ is the damping coefficient due to the air, and θ denotes the angular position of the propeller. The coupling equations for this system are given by

$$\begin{aligned} v_a &= k_a \omega_m \\ T_m &= k_f i_m, \end{aligned} \tag{16}$$

where k_a and k_f are coupling constants.

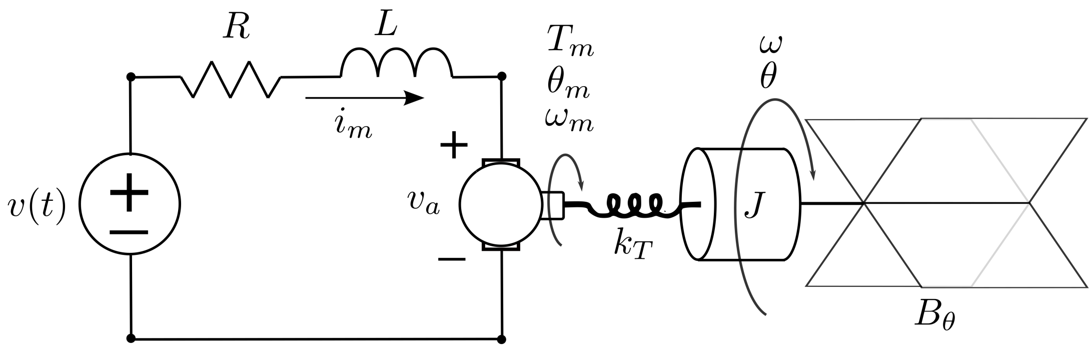

Figure 3: Fan.

(a) (4 points) Identify the A, T, and D variables of this system.

A-type variable:

- Velocity ω (or Ω).

T-type variables:

- Torque T_k (torque of the rotational spring)
- current $i_L = i_m$.

D-type variables:

- There are no "D-variables".

(b) (16 points) Derive the state-space model of this system. Consider $v(t)$ as the input, and the angular velocity ω as the output.

The behavior of the elements of this system is described by the following equations

$$\begin{aligned} v_L &= L i_L', & T_{k_T} &= k_T(\theta_m - \theta), \\ v_R &= R i_R, & T_J &= J\dot{\omega}, \\ & & T_{B_\theta} &= B_\theta \omega. \end{aligned} \tag{17}$$

The equations that describe the interaction among those elements are

$$\begin{aligned} i_L &= i_m = i_R, & T_m &= T_{k_T}, \\ v(t) &= v_R + v_L + v_a, & T_{k_T} &= T_J + T_{B_\theta}. \end{aligned} \tag{18}$$

Then, for the mechanical part, we get

$$\begin{aligned} T_m &= T_J + T_{B_\theta} \\ &= J\dot{\omega} + B_\theta \omega. \end{aligned} \tag{19}$$

From the coupling equations we have

$$k_f i_m = J\dot{\omega} + B_\theta\omega \iff \dot{\omega} = \frac{1}{J}\left(k_f i_L - B_\theta \omega\right). \tag{20}$$

For the electrical part, we get

$$v(t) = Ri_L + Li_L' + v_a \iff Li_L' = v(t) - Ri_L - v_a. \tag{21}$$

Hence, using the coupling equations, we obtain

$$Li_L' = v(t) - Ri_L - k_a\omega_m. \tag{22}$$

Notice that

$$T_m = T_{k_T} \implies \dot{T}_m = \dot{T}_{k_T}. \tag{23}$$

Therefore,

$$k_f i_L' = k_T(\omega_m - \omega) \iff \omega_m = \frac{k_f}{k_T} i_L' + \omega. \tag{24}$$

Replacing the latter in (22), we have

$$Li_L' = v(t) - Ri_L - k_a\omega - \frac{k_a k_f}{k_T} i_L' \iff i_L' = \frac{k_T}{Lk_T + k_a k_f}\left(v(t) - Ri_L - k_a\omega\right). \tag{25}$$

Then, we can write the dynamics of the system as follows

$$\begin{aligned} \begin{bmatrix} \dot{\omega} \\ i_L' \end{bmatrix} &= \underbrace{\begin{bmatrix} -\frac{B_\theta}{J} & \frac{k_f}{J} \\ -\frac{k_a k_T}{Lk_T + k_a k_f} & -\frac{Rk_T}{Lk_T + k_a k_f} \end{bmatrix}}_{A} \begin{bmatrix} \omega \\ i_L \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{k_T}{Lk_T + k_a k_f} \end{bmatrix}}_{B} v(t) \\ y &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} \omega \\ i_L \end{bmatrix} \end{aligned} \tag{26}$$

(c) (5 points) Based on your answer to the previous item, obtain the transfer function $G(s) = \dfrac{Y(s)}{U(s)}$

Consider the matrix

$$(sI_2 - A) \begin{bmatrix} s + \frac{B_\theta}{J} & -\frac{k_f}{J} \\ \frac{k_a k_T}{Lk_T + k_a k_f} & s + \frac{Rk_T}{Lk_T + k_a k_f} \end{bmatrix} \tag{27}$$

whose inverse is

$$\frac{1}{\mathtt{det}} \begin{bmatrix} s + \frac{Rk_T}{Lk_T + k_a k_f} & \frac{k_f}{J} \\ -\frac{k_a k_T}{Lk_T + k_a k_f} & s + \frac{B_\theta}{J} \end{bmatrix} \tag{28}$$

with

$$\texttt{det} := s^2 + \frac{B_\theta \left(Lk_T + k_a k_f\right) + JRk_T}{J\left(Lk_T + k_a k_f\right)} s + \frac{B_\theta R k_T + k_a k_T k_f}{J\left(Lk_T + k_a k_f\right)}. \tag{29}$$

Thus $G(s) = C(sI - A)^{-1}B$ takes the form

$$\begin{aligned} G(s) &= \frac{\frac{k_f k_T}{J(Lk_T + k_a k_f)}}{\texttt{det}} \\ &= \frac{\frac{k_f k_T}{J(Lk_T + k_a k_f)}}{s^2 + \frac{B_\theta(Lk_T + k_a k_f) + JRk_T}{J(Lk_T + k_a k_f)} s + \frac{B_\theta R k_T + k_a k_T k_f}{J(Lk_T + k_a k_f)}} \end{aligned} \tag{30}$$

3. (30 TOTAL points) Consider a chemical process, where the relation between the input and the output of the system is given by

$$5\frac{d^2y}{dt^2}(t) - 8\frac{dy}{dt}(t) + 10y(t) = 10\frac{du}{dt}(t) + 4u(t). \tag{31}$$

(a) (4 points) Obtain the transfer function $G(s) = \dfrac{Y(s)}{U(s)}$. Then, compute the zero(s) and the pole(s) of the transfer function. Is this system stable? Justify your answer.

Consider the Laplace transformation of (31) with initial conditions equal to zero, that is,

$$\left(5s^2 - 8s + 10\right) Y(s) = (10s + 4).$$

The equation above can be rewritten as

$$G(s) = \frac{10s + 4}{5s^2 - 8s + 10}.$$

Hence, the transfer function has a zero at $s = -0.4$, and poles at $s = 0.8 \pm 1.166j$. Since the poles are located at the right of the imaginary axis, the system is **unstable**.

(b) (4 points) Obtain the state-space representation of this system in the canonical controllable form.

To use the formula (4) we need that the denominator of $G(s)$ has the form $s^2 + a_1 s + a_2$. Hence, we compute

$$G(s) \cdot \frac{\frac{1}{5}}{\frac{1}{5}} = \frac{2s + 0.8}{s^2 - 1.6s + 2}. \tag{32}$$

Therefore,

$$\begin{aligned} a_1 &= -1.6, \qquad & b_0 &= 0 \\ a_2 &= 2, & b_1 &= 2 \\ & & b_2 &= 0.8. \end{aligned} \tag{33}$$

Then, the matrices A, B, C, and D are given by

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 1.6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.8 & 2 \end{bmatrix}, \quad D = b_0 = 0. \tag{34}$$

(c) (9 points) Consider the state-space model that you obtained in the previous item, and the cost function

$$J = \int_0^\infty \left(2x_1^2(t) - 4x_1(t)x_2(t) + 10x_2^2(t) + 20u^2(t)\right) dt. \tag{35}$$

Design an LQR that stabilizes the system and minimizes (35). **Note:** If you could not obtain the state-space representation, or you are not sure of your result, consider

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0. \tag{36}$$

However, if you do this you will be penalized with two points in this exercise.

Note that $R = 20$, and

$$Q = \begin{bmatrix} 2 & -2 \\ -2 & 10 \end{bmatrix}. \tag{37}$$

Then we compute

$$PA = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 1.6 \end{bmatrix} = \begin{bmatrix} -2P_2 & P_1 + 1.6P_2 \\ -2P_3 & P_2 + 1.6P_3 \end{bmatrix}. \tag{38}$$

Therefore,

$$A^\top P + PA = \begin{bmatrix} -4P_2 & P_1 + 1.6P_2 - 2P_3 \\ P_1 + 1.6P_2 - 2P_3 & 2P_2 + 3.2P_3 \end{bmatrix}. \tag{39}$$

Now,

$$PB = \begin{bmatrix} P_2 \\ P_3 \end{bmatrix}. \tag{40}$$

Hence,

$$PBR^{-1}B^\top P = \frac{1}{20} \begin{bmatrix} P_2^2 & P_2P_3 \\ P_2P_3 & P_3^2 \end{bmatrix}. \tag{41}$$

Accordingly, (5) takes the form

$$\begin{bmatrix} -\frac{1}{20}P_2^2 - 4P_2 + 2 & P_1 + 1.6P_2 - 2P_3 - \frac{1}{20}P_2P_3 - 2 \\ P_1 + 1.6P_2 - 2P_3 - \frac{1}{20}P_2P_3 - 2 & 2P_2 + 3.2P_3 - \frac{1}{20}P_3^2 + 10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{42}$$

Then, we have that

$$P_2^2 + 80P_2 - 40 = 0 \tag{43}$$

with solutions $P_2 = -80.4969$ (this one leads to a complex number for P_3), $P_2 = 0.4969$. Thus,

$$\begin{aligned} P_3^2 - 3.2(20)P_3 - 40P_2 - 200 &= 0 \\ P_3^2 - 3.2(20)P_3 - 40(0.4969) - 200 &= 0 \end{aligned} \tag{44}$$

with solutions $P_3 = -3.2686$ (P_3 can't be negative), $P_3 = 67.2686$. Then, we compute P_1 as follows

$$\begin{aligned} P_1 &= -1.6P_2 + 2P_3 + \frac{1}{20}P_2P_3 + 2 \\ P_1 &= -1.6(0.4969) + 2(67.2686) + \frac{1}{20}(0.4969)(67.2686) + 2 \\ P_1 &= 137.4135 \end{aligned} \tag{45}$$

Finally, we compute the control law as follows

$$u(t) = -kx(t), \qquad \begin{aligned} k &= R^{-1}B^\top P \\ &= \begin{bmatrix} 0.0248 & 3.3634 \end{bmatrix}. \end{aligned} \tag{46}$$

If the student considered the system (36), then Q and R reamin the same, but the rest of the answer is:

$$PA = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} P_2 & -P_1 - 5P_2 \\ P_3 & -P_2 - 5P_3 \end{bmatrix}. \tag{47}$$

Therefore,

$$A^\top P + PA = \begin{bmatrix} 2P_2 & -P_1 - 5P_2 + P_3 \\ -P_1 - 5P_2 + P_3 & -2P_2 - 10P_3 \end{bmatrix}. \tag{48}$$

Since B is the same, we get that (5) takes the form

$$\begin{bmatrix} -\frac{1}{20}P_2^2 + 2P_2 + 2 & -P_1 - 5P_2 + P_3 - \frac{1}{20}P_2P_3 - 2 \\ -P_1 - 5P_2 + P_3 - \frac{1}{20}P_2P_3 - 2 & -2P_2 - 10P_3 - \frac{1}{20}P_3^2 + 10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{49}$$

Then, we have that
$$P_2^2 - 40P_2 - 40 = 0 \tag{50}$$
with solutions $P_2 = 40.9762$ (this one leads to negative P_3), $P_2 = -0.9762$. Thus,
$$\begin{aligned} P_3^2 + 200P_3 + 40P_2 - 200 &= 0 \\ P_3^2 + 200P_3 - 40(0.9762) - 200 &= 0 \end{aligned} \tag{51}$$
with solutions $P_3 = -201.1882$ (P_3 can't be negative), $P_3 = 1.1882$. Then, we compute P_1 as follows
$$\begin{aligned} P_1 &= -5P_2 + P_3 - \frac{1}{20}P_2P_3 - 2 \\ P_1 &= 5(0.9762) + 1.1882 + \frac{1}{20}(0.9762)(1.1882) - 2 \\ P_1 &= 4.1272 \end{aligned} \tag{52}$$
Finally, we compute the control law as follows
$$u(t) = -kx(t), \qquad \begin{aligned} k &= R^{-1}B^\top P \\ &= \begin{bmatrix} -0.0488 & 0.0594 \end{bmatrix}. \end{aligned} \tag{53}$$

(d) (8 points) Suppose that the independent measurement of the states is risky. Hence, we want to implement an observer for the state-space system obtained in item **b)**. Determine if it is possible to design an observer that satisfies the following constraints
$$L = \begin{bmatrix} L_1 \\ 0 \end{bmatrix}, \quad \lambda_1\{\mathcal{A}_o\} = -0.75, \quad \Re\{\lambda_2\{\mathcal{A}_o\}\} < 0, \tag{54}$$
with $\mathcal{A}_o := A - LC$. **Note:** If you could not obtain the state-space representation, or you are not sure of your result, consider the state-space system (36). However, if you do this you will be penalized with three points in this exercise.

We have that
$$\begin{aligned} \mathcal{A}_o &= \begin{bmatrix} 0 & 1 \\ -2 & 1.6 \end{bmatrix} - \begin{bmatrix} L_1 \\ 0 \end{bmatrix} \begin{bmatrix} 0.8 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -0.8L_1 & 1 - 2L_1 \\ -2 & 1.6 \end{bmatrix}. \end{aligned} \tag{55}$$
Thus, the characteristic polynomial of $\mathcal{A}_o$ is given by
$$\begin{aligned} p(\mathcal{A}_o) &= \det(I_2\lambda - \mathcal{A}_o) \\ &= \det\left(\begin{bmatrix} \lambda + 0.8L_1 & -1 + 2L_1 \\ 2 & \lambda - 1.6 \end{bmatrix}\right) \\ &= (\lambda + 0.8L_1)(\lambda - 1.6) - 2(-1 + 2L_1) \\ &= \lambda^2 + (0.8L_1 - 1.6)\lambda + 2 - 5.28L_1. \end{aligned} \tag{56}$$
Now, we equate the polynomial above with
$$\begin{aligned} p_d(\mathcal{A}_o) &= (\lambda + 0.75)(\lambda + a) \\ &= \lambda^2 + (0.75 + a)\lambda + 0.75a, \end{aligned} \tag{57}$$
where the constant a must be positive to fulfill the requirement $\Re\{\lambda_2\{\mathcal{A}_o\}\} < 0$. Hence, we have the following system of equations
$$\begin{aligned} 0.8L_1 - 1.6 &= 0.75 + a \\ 2 - 5.28L_1 &= 0.75a \end{aligned} \tag{58}$$

The value of a that solves the system of equations above is -1.8381. Hence, the conclusion is that **it's not possible to design an observer that fulfills all the requirements.**

If the student used the system (36), we have the following result:

We have that

$$\begin{aligned}\mathcal{A}_o &= \begin{bmatrix}0 & -1\\ 1 & -5\end{bmatrix} - \begin{bmatrix}L_1\\ 0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\\ &= \begin{bmatrix}-L_1 & -1\\ 1 & -5\end{bmatrix}.\end{aligned} \tag{59}$$

Thus, the characteristic polynomial of $\mathcal{A}_o$ is given by

$$\begin{aligned}p(\mathcal{A}_o) &= \det(I_2\lambda - \mathcal{A}_o)\\ &= \det\left(\begin{bmatrix}\lambda + L_1 & 1\\ -1 & \lambda+5\end{bmatrix}\right)\\ &= (\lambda+L_1)(\lambda+5)+1\\ &= \lambda^2 + (5+L_1)\lambda + 1 + 5L_1.\end{aligned} \tag{60}$$

If we equate the polynomial above with (57) we obtain the following system of equations

$$\begin{aligned}5 + L_1 &= 0.75 + a\\ 1 + 5L_1 &= 0.75a.\end{aligned} \tag{61}$$

The solutions to the systems of equations above are

$$L_1 = 0.5147, \qquad a = 4.7647. \tag{62}$$

Since $a > 0$, the matrix $\mathcal{A}_o$ is Hurwitz. Accordingly, we conclude that in this case, **it's possible to design an observer that complies with the requirements.**

(e) (5 points) Consider the gain k obtained for the LQR of the item **c)**, and the matrix L obtained in **d)**. Now consider the augmented state

$$\xi = \begin{bmatrix}x\\ \tilde{x}\end{bmatrix}, \tag{63}$$

and the control law $u = -k\hat{x}$. Express the dynamics of ξ as

$$\dot{\xi} = \mathcal{A}_T\xi \tag{64}$$

and provide the eigenvalues of the matrix $\mathcal{A}_T$.

Define the matrix

$$\mathcal{A}_c := A - Bk = \begin{bmatrix}0 & 1\\ -2.0248 & -1.7634\end{bmatrix}. \tag{65}$$

Then,

$$\dot{\xi} = \begin{bmatrix}\dot{x}\\ \dot{\tilde{x}}\end{bmatrix} = \underbrace{\begin{bmatrix}\mathcal{A}_c & Bk\\ \mathbf{0}_{2\times 2} & \mathcal{A}_o\end{bmatrix}}_{\mathcal{A}_T}\begin{bmatrix}x\\ \tilde{x}\end{bmatrix}. \tag{66}$$

Moreover,

$$\texttt{eig}(\mathcal{A}_T) = \begin{bmatrix}\texttt{eig}(\mathcal{A}_c)\\ \texttt{eig}(\mathcal{A}_o)\end{bmatrix} = \begin{bmatrix}-0.8817 + 1.1169j\\ -0.8817 - 1.1169j\\ -0.75\\ 1.8381\end{bmatrix}. \tag{67}$$

If the student uses (36), we get

$$\mathcal{A}_c := A - Bk = \begin{bmatrix} 0 & -1 \\ 1.0488 & -5.0594 \end{bmatrix}, \tag{68}$$

and

$$\texttt{eig}(\mathcal{A}_T) = \begin{bmatrix} \texttt{eig}(\mathcal{A}_c) \\ \texttt{eig}(\mathcal{A}_o) \end{bmatrix} = \begin{bmatrix} -0.2166 \\ -4.8428 \\ -0.75 \\ -4.7647 \end{bmatrix}. \tag{69}$$

4. (19 TOTAL points) Suppose that transfer function that describes an industrial process is given by

$$G(s) = \frac{2}{s-4}. \tag{70}$$

(a) (4 points) We want to design a controller that ensures the poles of the closed-loop system are located at -2 and -5. Is it possible to fulfill this requirement with the implementation of a PD? Justify your answer.

The answer is no. Below, a detailed explanation.
The transfer function of a PD controller is given by

$$C(s) = k_d s + k_p. \tag{71}$$

Moreover, the poles of the closed-loop transfer function are the zeros of

$$1 + C(s)G(s) = 1 + \frac{2(k_d s + k_p)}{s-4} = \frac{(1+2k_d)s + 2k_p - 4}{s-4}. \tag{72}$$

Since the polynomial in the numerator has only one root, the closed-loop system will have only **one** pole. Consequently, we can't assign **two** poles.

(b) (8 points) Consider the closed-loop system requirement of **a)**, that is, poles at -2 and -5. Is it possible to complete the control task with a PID controller, with proportional gain $K_p = 7$? If the answer is yes, then provide K_d and K_i.

The transfer function of a PID controller is given by

$$C(s) = \frac{k_d s^2 + k_p s + k_i}{s}. \tag{73}$$

Moreover, the poles of the closed-loop transfer function are the zeros of

$$1 + C(s)G(s) = 1 + \frac{2(k_d s^2 + k_p s + k_i)}{s^2 - 4s} = \frac{(1+2k_d)s^2 + (2k_p - 4)s + 2k_i}{s^2 - 4s}. \tag{74}$$

Since $k_p = 7$, we need to determine if there exist values of k_d and k_i such that the solution of

$$(1+2k_d)s^2 + 10s + 2k_i = 0 \tag{75}$$

are $s = -2$ and $s = -5$. To simplify the computations, we can rewrite the ploynomial above as follows

$$s^2 + \frac{10}{1+2k_d}s + \frac{2k_i}{1+2k_d} = 0 \tag{76}$$

and we can equate

$$(s+5)(s+2) = s^2 + 7s + 10. \tag{77}$$

Hence, we get

$$\begin{aligned} \frac{10}{1+2k_d} = 7 \quad &\Longleftrightarrow \quad 10 = 7 + 14k_d \\ &\Longleftrightarrow \quad k_d = \frac{3}{14} \approx 0.2143, \end{aligned} \tag{78}$$

and

$$\frac{2k_i}{1+2(0.2143)} = 10 \quad \Longleftrightarrow \quad k_i = \frac{100}{14} \approx 7.1429 \tag{79}$$

(c) (5 points) Consider that the transfer function of the closed-loop system is given by

$$H_{cl}(s) = \frac{G(s)C(s)}{1+G(s)C(s)}. \tag{80}$$

Using the gains of the PID controller obtained in **b)**, discretize the system using the Euler forward approximation and determine the maximum sampling time before the discretized closed-loop system becomes unstable.

The closed-loop transfer function is given by

$$H_{cl}(s) = \frac{\frac{2(k_d s^2 + k_p s + k_i)}{s^2-4s}}{\frac{(1+2k_d)s^2+(2k_p-4)s+2k_i}{s^2-4s}} = \frac{\frac{2}{14}\left(3s^2+98s+100\right)}{\frac{1}{14}\left(20s^2+140s+200\right)} = \frac{3s^2+98s+100}{10s^2+70s+100}. \tag{81}$$

Then, using the approximation $s \approx \dfrac{z-1}{T_s}$, we get

$$\begin{aligned} H_{cl}(z) &= \frac{\frac{1}{T_s^2}\left[3\left(z-1\right)^2 + 98T_s(z-1) + 100T_s^2\right]}{\frac{1}{T_s^2}\left[10\left(z-1\right)^2 + 70T_s(z-1) + 100T_s^2\right]} \\ &= \frac{3z^2 + (98T_s - 6)z + 100T_s^2 - 98T_s + 3}{10\left[z^2 + (7T_s-2)z + 10T_s^2 - 7T_s + 1\right]}, \end{aligned} \tag{82}$$

where the stability is determined by the values of z such that

$$z^2 + (7T_s - 2)z + 10T_s^2 - 7T_s + 1 = 0, \tag{83}$$

which are given by

$$\begin{aligned} z_{1,2} &= \frac{2 - 7T_s \pm \sqrt{(7T_s-2)^2 - 4(10T_s^2 - 7T_s + 1)}}{2} \\ &= 1 - \frac{7T_s \pm \sqrt{49T_s^2 - 28T_s + 4 - 40T_s^2 + 28T_s - 4}}{2} \\ &= 1 - \frac{7T_s \pm 3T_s}{2} \\ &= \begin{cases} 1 - 2T_s \\ 1 - 5T_s. \end{cases} \end{aligned} \tag{84}$$

Since $T_s > 0$, we conclude that any sampling time greater than 0.4 seconds will place one of the poles out of the unit circle making the system unstable.
Note: a much simpler way to obtain the result is to check the closed-loop characteristic polynomial $(s+2)(s+5)$.

(d) (2 points) Suppose that you discretize the closed-loop transfer function obtained in **b)** using the bilinear approximation with a sampling time $T = 0.5$ [sec]. Will the discretized closed-loop transfer function be stable? Justify your answer.

The answer is yes. The simplest way to justify this answer is using the argument that this approximation maps every stable pole (in continuous time) inside the unit circle (in discrete time). Since, the controller in **b)** was designed to have a stable closed-loop system, it follows the stability of the discretized system. However, if the student decides to compute the poles, the correct values are 0.3333 and -0.1111, which satisfy the condition $|z| < 1$.

5. (18 points) Consider a stable linear system with transfer function $G(s)$, and a proportional controller with transfer function $C(s) = 5$. Fig. 4 shows the Nyquist plot of the open-loop transfer function $C(s)G(s)$.

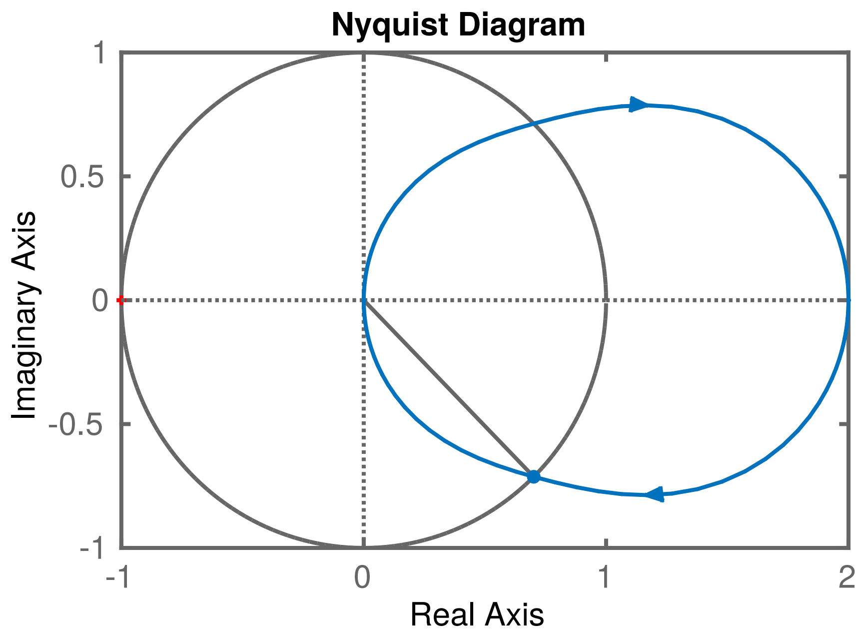

Figure 4: Nyquist plot of $C(s)G(s)$

(a) (3 points) What can be said about the stability of the closed-loop system? Justify your answer.

We can use the Nyquist criterion to claim stability because the plant is stable, which implies that $C(s)G(s)$ has no unstable poles, and the Nyquist plot is not encircling the point $(-1, 0)$.

(b) (5 points) Consider $\omega_* = 2.86$ [rad/sec], and a delay of 0.5 seconds. Hence, the open-loop transfer function with delay is given by $C(s)G(s)\mathrm{e}^{-0.5s}$. Is the closed-loop system (with delay) stable? Justify your answer. **Hint:** the phase margin can be estimated from the Nyquist plot.

From the Nyquist plot it's evident that $P_m < -\frac{\pi}{2}$ (considering the criterion that $P_m > 0$ implies stability) for the system without delay. Then, if we take into account the delay we have that

$$P_{m_d} = P_m + 0.5(2.86) < -\frac{\pi}{2} + 1.43 = -0.1408. \tag{85}$$

Accordingly, even if we don't know the exact value of P_m, we can determine that $P_{m_d} < 0$ which implies that the delayed system is stable.

(c) (5 points) Now, consider that the linear system $C(s)G(s)$ is interconnected with a nonlinearity $\varphi(y)$ that belongs to the sector $[k_1, k_2]$ as is shown in Figure 5.

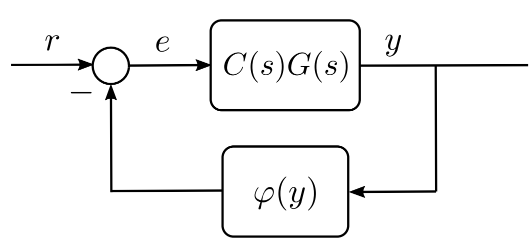

Figure 5: Linear system in closed-loop with a nonlinear feedback.

What can be said about the stability of the closed-loop system if $k_1 = 0$ and $k_2 = 10$? Justify your answer.

The closed-loop system will be stable since the Nyquist plot is located to the right of the point $(-0.1, 0)$. We obtain this result from the Circle criterion.

(d) (5 points) What can be said about the stability of the closed-loop system if $k_1 = -\frac{1}{3}$, $k_2 = 1$, and $C(s) = 10$? Justify your answer.

In this case, we can't say anything about the stability of the system. Note that the gain is twice bigger than in the previous items. As a consequence, the Nyquist plot will cross the real axis at $(4, 0)$. Since the new Nyquist plot doesn't remain inside the disk (k_1, k_2), the Circle criterion is not suitable to determine the stability properties of the closed-loop system.