Mechatronics 2023-2024

University of Groningen

Tutorial 1 (Solutions)

Self-Study Problems

Q 1: In a clean room facility of advanced manufacturing systems that build integrated circuits and microprocessors, an automated transport mechanism of the wafers has been developed and used which enables the transfer of both the unprocessed and processed silicon wafers from/to the chip production unit (such as the machine produced by ASML B.V, The Netherlands shown in fig. 1) to/from the pre/post-processing machine such as wafer-cutter or wire-bonding machine, with minimal contamination from dust or particle.

Figure 1: A machine from ASML B.V.

Suppose that pneumatic systems are used to carry the wafer safely to the clean compartment. For simplicity, assume that the dynamics of the wafer movement is given by

$$\dot{x} = -2x + F, \tag{1}$$

where F is the *applied force* from the pneumatic drive and x is the velocity of the wafer. If one of the functional requirements is to move the wafer from the velocity $x = 0$ to $x = 2$m/sec within 5 seconds, determine the minimal power requirement of the pneumatic drive. Note that $\frac{1}{1-e^{-10}} = 1.0000454$. (Explain your arguments.)

Solution:

First, we determine the solution of the differential equation $\dot{x} = -2x + F$. This can be done in 2 steps.

1. First, we solve the corresponding homogeneous solution, obtained by setting $F = 0$, giving us.

$$\dot{x} + 2x = 0.$$

To this end, we consider the homogeneous solution of the form $x_h(t) = e^{\lambda t}$. When substituting this into the differential equation above, we obtain

$$\lambda e^{\lambda t} + 2e^{\lambda t} = 0. \tag{2}$$

We can rewrite this as

$$e^{\lambda t}(\lambda + 2) = 0, \tag{3}$$

which yields $\lambda = -2$. Therefore, the homogeneous solution is of the form

$$x_h(t) = \alpha e^{-2t}. \tag{4}$$

According to the requirements, the solution must fulfill the following

$$x(0) = 0, \qquad x(5) = 2.$$

We can use the initial condition $x(0) = 0$ to find that

$$0 = x_h(0) = \alpha e^{-2*0}, \tag{5}$$

and hence $\alpha = 0$. Thus the homogeneous solution is $x_h(t) = 0$.

2. Secondly, we determine the forcing solution, which is of the form

$$x_f(t) = \int_0^t h(t-\tau) F d\tau \tag{6}$$

where $h(t)$ is given by

$$h(t) = \beta e^{-2t}. \tag{7}$$

To find the coefficient β, we need to apply the condition $h(0) = 1$, which yields

$$h(0) = \beta e^{-2*0} = 1. \tag{8}$$

Hence, we find that $\beta = 1$, and thus the forcing solution is given by

$$x_f(t) = \int_0^t e^{-2(t-\tau)} F d\tau. \tag{9}$$

When combining this, we see that the solution of the differential equation is given by $x(t) = x_h(t) + x_f(t)$, or more explicitely

$$x(t) = 0 + \int_0^t e^{-2(t-\tau)} F d\tau. \tag{10}$$

According to the requirements, we must fulfill

$$x(0) = 0, \qquad x(5) = 2. \tag{11}$$

Substituting we then have

$$2 = x(5) = \int_0^5 e^{-2(5-\tau)} F d\tau. \tag{12}$$

We can solve the integral as follows

$$2 = \int_0^5 e^{-2(5-\tau)} F d\tau = \frac{1}{2} F(e^{2r-10}) \Big|_0^5 = \frac{1}{2}\left[(e^{2*5-10}) - (e^{2*0-10})\right] = \frac{1}{2} F(1 - e^{-10}). \tag{13}$$

We can solve for F to get the force the system needs to move the wafer from velocity $x = 0$ to $x = 2\text{m/sec}$ in exactly 5 seconds. Forces higher than this will also be able to move the wafer within the allotted time, thus fulfilling the functional requirements. Therefore, what we are calculating is the minimal amount of force that the system needs to provide, which is given by.

$$F_{min} = \frac{4}{1 - e^{-10}} = 4(1.0000454) = 4.0001816\,\text{N}. \tag{14}$$

Next, the power is given by $P = F \cdot x$ (recall that x is in this case velocity). Minimal force will give us minimal power, from which it follows that:

$$P_{min} = F_{min} \cdot \max_{t \in [0,5]} x(t) = F_{min} x(5) = (4.0001816)(2) = 8.0003632\,\text{W}. \tag{15}$$

Q 2: Let the transfer function of a process from the input U to the measured output Y be given by

$$\frac{Y(s)}{U(s)} = \frac{b_0 s + b_1}{a_0 s^2 + a_1 s + a_2}, \tag{16}$$

where a_0, a_1, a_2, b_0, b_1 are the process parameters with positive constants but uncertain. The coefficients a_0, a_1, a_2, b_0, b_1 belong to the closed interval $[1, 2]$.

Using a PI controller with transfer function $C(s) = K_p + \frac{K_i}{s}$, find the conditions on K_p and K_i such that the closed-loop system remains stable for any possible values of a_0, a_1, a_2, b_0, b_1.

Solution:

Let $G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s+b_1}{a_0 s^2+a_1 s+a_2}$. For the stability analysis we must use the characteristic polynomial which, for example, can be obtained from the sensitivity transfer function:

The sensitivity transfer function relates the reference with the error and is given by

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)C(s)} = \frac{a_0 s^3 + a_1 s^2 + a_2 s}{a_0 s^3 + (a_1 + K_p b_0)s^2 + (a_2 + K_i b_0 + K_p b_1)s + K_i b_1}. \tag{17}$$

So, the stability of the closed-loop system is checked by investigating the stability of the denominator of the above transfer function, given by

$$P(s) = a_0 s^3 + (a_1 + K_p b_0)s^2 + (a_2 + K_i b_0 + K_p b_1)s + K_i b_1. \tag{18}$$

By the Routh-Hurwitz criterion we need the following:

- All coefficients of $P(s)$ must be positive and non-zero. The PI controller gains have to follow $K_p > 0$ and $K_i > 0$, and a_0, a_1, a_2, b_0, b_1 belong to closed interval $[1, 2]$, and are therefore always positive. Hence by definition, this first argument is fulfilled. (note that the previous argument on K_p is not necessary!).
- The first column of the Routh-Hurwitz table must not change sign. For $P(s)$, this table is given by

a_0	$a_2 + K_i b_0 + K_p b_1$
$a_1 + K_p b_0$	$K_i b_1$
$\frac{(a_1+K_p b_0)(a_2+K_i b_0+K_p b_1)-K_i b_1 a_0}{a_1+K_p b_0}$	0
$K_i b_1$	0

The first element of the first column is a_0, which belongs to the closed interval $[1, 2]$, so will be positive.

The second element of the first column would require $K_p > -\frac{a_1}{b_0}$. Since a_1 and b_0 are both positive, $-\frac{a_1}{b_0} < 0$ Thus, with $K_p > 0$ this second element will always be positive as well.

The third element of the first column requires

$$\frac{(a_1 + K_p b_0)(a_2 + K_i b_0 + K_p b_1) - K_i b_1 a_0}{a_1 + K_p b_0} > 0. \tag{19}$$

The denominator is always positive, so the numerator will also have to be positive to make the element positive overall. Thus, a requirement will be

$$(a_1 + K_p b_0)(a_2 + K_i b_0 + K_p b_1) - K_i b_1 a_0 > 0. \tag{20}$$

The fourth element will be positive as long as $K_i > 0$, since b_1 is restricted to the interval $[1, 2]$.

In conclusion, the conditions on K_p and K_i that ensure that the closed-loop system remains stable are:

i) $K_i > 0$;
ii) $K_p > 0$;
iii) $(a_1 + K_p b_0)(a_2 + K_i b_0 + K_p b_1) - K_i b_1 a_0 > 0$.

Q 3: Write the state space equations of the following system described by second-order ordinary differential equations (ODEs).

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} d & e \\ f & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} + \begin{bmatrix} g & h \\ 0 & i \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad y = \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix},$$

where u_1 and u_2 are the external inputs and y is the vector of measurements containing $\dot{x}$ and $\dot{z}$, and assume that $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is invertible.

Solution:

Let us define

$$\begin{aligned} \zeta_1 &= x, \\ \zeta_2 &= z, \\ \zeta_3 &= \dot{x} \quad \Rightarrow \quad \dot{\zeta}_3 = \ddot{x}, \\ \zeta_4 &= \dot{z} \quad \Rightarrow \quad \dot{\zeta}_4 = \ddot{z}, \end{aligned}$$

such that we can write

$$\begin{aligned} \zeta &= (\zeta_1, \zeta_2, \zeta_3, \zeta_4)^T = (x, z, \dot{x}, \dot{z})^T \\ u &= (u_1, u_2)^T. \end{aligned} \tag{21}$$

We can use this to rewrite the ODE as

$$\underbrace{\begin{bmatrix} a & b \\ b & c \end{bmatrix}}_{M_1} \begin{bmatrix} \dot{\zeta}_3 \\ \dot{\zeta}_4 \end{bmatrix} + \underbrace{\begin{bmatrix} d & e \\ f & 0 \end{bmatrix}}_{M_2} \begin{bmatrix} \zeta_3 \\ \zeta_4 \end{bmatrix} + \underbrace{\begin{bmatrix} g & h \\ 0 & i \end{bmatrix}}_{M_3} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad y = \begin{bmatrix} \zeta_3 \\ \zeta_4 \end{bmatrix} \tag{22}$$

To obtain the state space representation, we can rewrite these equations as

$$M_1 \begin{bmatrix} \dot{\zeta}_3 \\ \dot{\zeta}_4 \end{bmatrix} = -M_2 \begin{bmatrix} \zeta_3 \\ \zeta_4 \end{bmatrix} - M_3 \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

As matrix M_1 is invertible, we can write

$$\begin{bmatrix} \dot{\zeta}_3 \\ \dot{\zeta}_4 \end{bmatrix} = -M_1^{-1} M_2 \begin{bmatrix} \zeta_3 \\ \zeta_4 \end{bmatrix} - M_1^{-1} M_3 \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} + M_1^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Using this, we can now write the state space equations as

$$\begin{aligned} \dot{\zeta} &= \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline \multicolumn{2}{c|}{-M_1^{-1} M_3} & \multicolumn{2}{c}{-M_1^{-1} M_2} \end{array}\right] \zeta + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline \multicolumn{2}{c}{M_1^{-1}} \end{array}\right] u, \\ y &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \zeta, \end{aligned}$$

where $M_1 = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $M_2 = \begin{bmatrix} d & e \\ f & 0 \end{bmatrix}$ and $M_3 = \begin{bmatrix} g & h \\ 0 & i \end{bmatrix}$.

Q 4: Consider the simple pendulum, where a solid mass with weight m is connected to the ceiling by a rigid, massless string (fig. 2). T_R is the reaction force from the string.

(a) Using the Cartesian coordinate of the ball ($x = L\sin\theta, y = L - L\cos\theta$) and using the Newtonian approach, compute the horizontal and the vertical Newton's laws of the ball.

(b) Based on the result in a), show that the system equations of motion can be simplified to:

$$mL\ddot{\theta} + mg\sin\theta = 0. \tag{23}$$

Figure 2: Modeling of a simple pendulum system

Solution:

a) By looking at the second figure in Figure 2, we can see that on the horizontal axis the only force acting on the ball is $T_R\sin\theta$. This force is acting in the opposite direction to that of the positive direction on the x-axis. Therefore, the horizontal Newton's law will be given by

$$\begin{aligned}
-T_R\sin\theta &= m\frac{d^2x}{dt^2}\\
-T_R\sin\theta &= m\frac{d^2}{dt^2}(L\sin\theta)\\
-T_R\sin\theta &= m\frac{d}{dt}(L\dot{\theta}\cos\theta)\\
-T_R\sin\theta &= mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta.
\end{aligned} \tag{24}$$

When looking in the vertical direction, there are two forces acting on the ball: $T_R\cos(\theta)$ in the positive direction, and mg in the negative direction. Therefore, the vertical Newton's law is given by

$$\begin{aligned}
T_R\cos\theta - mg &= m\frac{d^2y}{dt}\\
T_R\cos\theta - mg &= m\frac{d^2}{dt}(L - L\cos\theta)\\
T_R\cos\theta - mg &= m\frac{d}{dt}(L\dot{\theta}\sin\theta)\\
T_R\cos\theta - mg &= mL\ddot{\theta}\sin\theta + mL\dot{\theta}^2\cos\theta.
\end{aligned} \tag{25}$$

b) From eq. (24) we have

$$T_R = \frac{-mL\ddot{\theta}\cos\theta + mL\dot{\theta}^2\sin\theta}{\sin\theta}.$$

Substituting this in eq. (25) we get

$$\frac{-mL\ddot{\theta}\cos\theta + mL\dot{\theta}^2\sin\theta}{\sin\theta}\cos\theta - mg = mL\ddot{\theta}\sin\theta + mL\dot{\theta}^2\cos\theta.$$

Multiplying both sides by $\sin\theta$ yields

$$-mL\ddot{\theta}\cos^2\theta + mL\dot{\theta}^2\sin\theta\cos\theta - mg\sin\theta = mL\sin^2\theta\ddot{\theta} + mL\cos\theta\sin\theta\dot{\theta}^2.$$

By canceling out the equal terms we obtain

$$-mL\ddot{\theta}\cos^2\theta - mg\sin\theta = mL\sin^2\theta\ddot{\theta}$$

By moving everything to the left side of the equation we obtain

$$-mL\ddot{\theta}\cos^2\theta - mg\sin\theta - mL\sin^2\theta\ddot{\theta} = 0,$$

which we can rewrite as

$$-mL\ddot{\theta}(\cos^2\theta + \sin^2\theta) - mg\sin\theta = 0.$$

We know that $\sin^2\theta + \cos^2\theta = 1$, and therefore we obtain

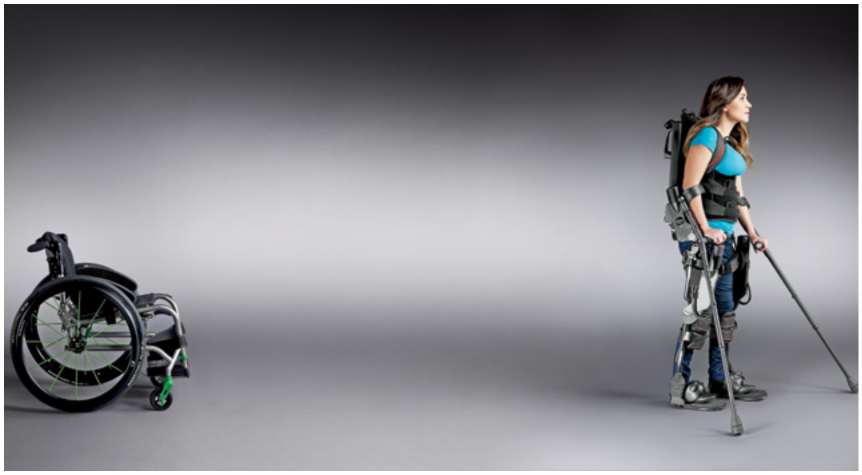
$$mL\ddot{\theta} + mg\sin\theta = 0.$$

Hence, we see that the system equations of motion can be simplified to

$$mL\ddot{\theta} + mg\sin\theta = 0.$$

In-tutorial Problems

E 1: You are the leader of a group of engineers in charge of designing an exoskeleton to aid people who have lost leg-motion capabilities.



As a leader, you are asked to provide a plan of the different steps to follow in the design process. Following the V-diagram, identify at least two elements of each component of the Mechatronic Design Cycle. In particular:

(a) Which fields of engineering would be involved in the design process?

(b) Identify some elements of the electrical domain and of the mechanical domain.

(c) Identify some (natural) functional requirements of the exoskeleton.

(d) What are some essential sensors and actuators?

(e) Where does a mathematical model comes into play?

(f) What kind of control problems will your team face?

(g) What type of testing (simulations/prototypes) would you propose?

(h) Identify some difficulties that the integration of distinct systems may present.

*For this question, you can join in groups and discuss with your fellow students.

Solution:

(a) Mechanical Engineering (structure of the exoskeleton, mechanical systems that support movement and provide support), Biomedical Engineering (interactions between the exoskeleton and the human body, simulations and studies to maximize the performance and minimize the injuries of the exoskeleton), Electrical Engineering (electrical systems, sensors, and actuators incorporated into the exoskeleton).

(b) Electrical domain: sensors, actuators, power supply, wiring, and connectors. Mechanical domain: frame and structure, joints and linkages, limb attachments.

(c) Range of motion, support the body weight, adaptability, adjustability, safety.

(d) Sensors: force sensors, joint angle sensors, accelerometers, and gyroscopes. Actuators: Motors, pneumatic and hydraulic systems, clutches, and brakes.

(e) A mathematical model can be used to describe the kinematics (movement) and dynamics (forces and torques) of the exoskeleton and its interaction with the user. It helps in understanding the relationship between joint angles, joint velocities, and joint torques, allowing for the prediction and analysis of the exoskeleton's motion and forces

(f) Control of the joints and coordination: control team should control and actuate multiple joints for a smooth and natural joint movement. Control of balance and stability: to prevent falls.

(g) Simulation-Based Testing:

- Kinematic and Dynamic Simulation

- Human-Exoskeleton Interaction Simulation

Prototype Testing

- Functional Prototype
- Performance and Ergonomics Testing
- Safety and Reliability Testing

E 2: In recent years, autonomous systems (robots) have been deployed in the warehouse. These systems are able to coordinate their tasks with each other in order to optimize the queuing time, space and energy usage. An example of the autonomous system is shown in fig. 3 where the robot can navigate on the floor and bring items on the shelf to any position in the warehouse.

Figure 3: KIVA system in an automated logistic system

Suppose that the movement of the systems can be modeled as a first-order system

$$\dot{x} = -x + F, \tag{26}$$

where the state x is the velocity and F is the force from a DC motor. If one of the functional requirements is to be able to move from 0 m/s to 5 m/s within 2 seconds, determine whether a 20 Watt DC motor can meet the requirement. Note that $\frac{5}{1-e^{-2}} = 5.7826$. (Explain your arguments.)

Solution:

First, we determine the solution of the differential equation $\dot{x} = -x + F$. This can be done in 2 steps.

1. First, we solve the corresponding homogeneous solution, obtained by setting $F = 0$, giving us.

$$\dot{x} + x = 0.$$

To this end, we consider the homogeneous solution of the form $x_h(t) = e^{\lambda t}$. When substituting this into the differential equation above, we obtain

$$\lambda e^{\lambda t} + e^{\lambda t} = 0. \tag{27}$$

We can rewrite this as

$$e^{\lambda t}(\lambda + 1) = 0, \tag{28}$$

which yields $\lambda = -1$. Therefore, the homogeneous solution is of the form

$$x_h(t) = \alpha e^{-t}. \tag{29}$$

According to the requirements, the solution must fulfill the following

$$x(0) = 0, \qquad x(2) = 5.$$

We can use the initial condition $x(0) = 0$ to find that

$$0 = x_h(0) = \alpha e^{-1*0}, \tag{30}$$

and hence $\alpha = 0$. Thus the homogeneous solution is $x_h(t) = 0$.

2. Secondly, we determine the forcing solution, which is of the form

$$x_f(t) = \int_0^t h(t-\tau) F d\tau \tag{31}$$

where $h(t)$ is given by

$$h(t) = \beta e^{-t}. \tag{32}$$

To find the coefficient β, we need to apply the condition $h(0) = 1$, which yields

$$h(0) = \beta e^{-1*0} = 1. \tag{33}$$

Hence, we find that $\beta = 1$, and thus the forcing solution is given by

$$x_f(t) = \int_0^t e^{-(t-\tau)} F d\tau. \tag{34}$$

When combining this, we see that the solution of the differential equation is given by $x(t) = x_h(t) + x_f(t)$, or more explicitly

$$x(t) = 0 + \int_0^t e^{-(t-\tau)} F d\tau. \tag{35}$$

According to the requirements, we must fulfill

$$x(0) = 0, \qquad x(2) = 5. \tag{36}$$

Substituting we then have

$$5 = x(2) = \int_0^2 e^{-(2-\tau)} F d\tau. \tag{37}$$

We can solve the integral as follows

$$5 = \int_0^2 e^{-(2-\tau)} F d\tau = F e^{\tau-2} \Big|_0^2 = F\left[e^{2-2} - e^{0-2}\right] = F(1 - e^{-2}). \tag{38}$$

We can solve for F to get the force the system needs to move the wafer from velocity $x = 0$ to $x = 5$m/sec in exactly 2 seconds. Forces higher than this will also be able to move the wafer within the allotted time, thus fulfilling the functional requirements. Therefore, what we are calculating is the minimal amount of force that the system needs to provide, which is given by.

$$F_{min} = \frac{5}{1 - e^{-2}} = 5.782\,\text{N}. \tag{39}$$

Next, the power is given by $P = F \cdot x$ (recall that x is in this case velocity). The minimal force will give us minimal power. Then it follows that:

$$P_{min} = F_{min} \cdot \max_{t\in[0,2]} x(t) = F_{min} x(2) = (5.7826)(5) = 28.913\,\text{W}. \tag{40}$$

Therefore, a 20 Watt DC Motor will not meet the requirement.

E 3: Let the transfer function of a process from the input U to the measured output Y be given by

$$\frac{Y(s)}{U(s)} = \frac{b_0}{a_0 s^2 - a_1 s + a_2}, \tag{41}$$

where a_0, a_1, a_2, b_0, b_1 are the process parameters with positive constants but uncertain.

(a) Suppose that we will assign a proportional controller with transfer function $C(s) = K_p$ with the gain $K_p > 0$. Can the proportional controller stabilize the closed-loop system?

(b) Suppose that we use a PD (Proportional + Derivative) controller with transfer function $C(s) = K_p + K_d s$ with the gains K_p and K_d. Find the conditions on the gains K_p and K_d such that the closed-loop systems is stable.

Hint: you can, for example, use Routh-Hurwitz test on the closed-loop system.

Solution:
Let in general, $C(s) = K_p + K_d s$. We start by computing the sensitivity function as follows

$$\begin{aligned} \frac{E(s)}{R(s)} = \frac{1}{1 + G(s)C(s)} &= \frac{1}{1 + \frac{(K_p + K_d s)b_0}{a_0 s^2 - a_1 s + a_2}} \\ &= \frac{1}{\frac{a_0 s^2 - a_1 s + a_2 + (K_p + K_d s)b_0}{a_0 s^2 - a_1 s + a_2}} \\ &= \frac{a_0 s^2 - a_1 s + a_2}{a_0 s^2 + (K_d b_0 - a_1)s + a_2 + K_p b_0}. \end{aligned} \tag{42}$$

Recall that a closed-loop linear system is stable if and only if the characteristic polynomial (denominator of (42)) has roots with strictly negative real part.

(a) For a proportional controller we set $K_d = 0$, which yields the sensitivity function

$$\frac{E(s)}{R(s)} = \frac{a_0 s^2 - a_1 s + a_2}{a_0 s^2 - a_1 s + a_2 + K_p b_0}.$$

Then, the characteristic polynomial is

$$p(s) = a_0 s^2 - a_1 s + a_2 + b_0 K_p. \tag{43}$$

The Routh-Hurwitz table of this polynomial reads

a_0	$a_2 + b_0 K_p$
$-a_1$	0
$a_2 + b_0 K_p$	0

Since $a_0 > 0$ and $-a_1 < 0$, and K_p does not have an influence on either of these values, the system is unstable for any value of K_p. Hence, the closed-loop system cannot be stabilized by a purely proportional controller.

(b) When $C(s) = K_p + K_d s$, we find that the sensitivity function is given by

$$\frac{E(s)}{R(s)} = \frac{a_0 s^2 - a_1 s + a_2}{a_0 s^2 + (K_d b_0 - a_1)s + a_2 + K_p b_0}.$$

Then, the characteristic polynomial is

$$p(s) = a_0 s^2 + (b_0 K_d - a_1)s + a_2 + b_0 K_p. \tag{44}$$

The Routh-Hurwitz table reads as:

a_0	$a_2 + b_0 K_p$
$b_0 K_d - a_1$	0
$a_2 + b_0 K_p$	0

Recall that we need to ensure that there are no sign changes along the first column of the Routh-Hurwitz table. So, since $a_0 > 0$, we have to ensure that

i) $b_0 K_d - a_1 > 0$,

ii) $a_2 + b_0 K_p > 0$.

From the conditions, we then know that the closed-loop system is stable when

$$K_p > -\frac{a_2}{b_0} \qquad \text{and} \qquad K_d > \frac{a_1}{b_0}.$$

E 4: Write the state space equations of the following systems described by second-order ordinary differential equations (ODEs).

(a) $\begin{bmatrix} \ddot{x} + \dot{x} + x + \dot{z} + z \\ \ddot{z} + \dot{z} + z + \dot{x} + x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\qquad y = \begin{bmatrix} x \\ z \end{bmatrix}$, where y is the vector of measurements containing x and z.

(b) $\begin{bmatrix} \ddot{x} + a\dot{x} + bx + c\dot{z} + dz \\ \ddot{z} + e\dot{z} + fz + g\dot{x} + hx \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\qquad y = \begin{bmatrix} \dot{x} \\ z \end{bmatrix}$, where u_1 and u_2 are the external inputs and y is the vector of measurement containing $\dot{x}$ and z.

Solution:

(a) From the differential equations, we see that

$$\begin{aligned} \ddot{x} &= -\dot{x} - x - \dot{z} - z \\ \ddot{z} &= -\dot{z} - z - \dot{x} - x. \end{aligned}$$

Let us define

$$\zeta = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \\ z \\ \dot{z} \end{pmatrix}.$$

Then

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_2 - x_1 - x_4 - x_3, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -x_4 - x_3 - x_2 - x_1. \end{aligned} \tag{45}$$

For the output we have

$$\begin{aligned} y_1 &= x = x_1, \\ y_2 &= z = x_3. \end{aligned} \tag{46}$$

When combining this, we have that the state space equations are given by

$$\begin{aligned} \dot{\zeta} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \zeta, \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \zeta. \end{aligned} \tag{47}$$

(b) From the differential equations, we see that

$$\begin{aligned} \ddot{x} &= u_1 - a\dot{x} - bx - c\dot{z} - dz, \\ \ddot{z} &= u_2 - e\dot{z} - fz - g\dot{x} - hx. \end{aligned}$$

Let us define

$$\zeta = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \\ z \\ \dot{z} \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Then

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -ax_2 - bx_1 - cx_4 - dx_3 + u_1, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -ex_4 - fx_3 - gx_2 - hx_1 + u_2. \end{aligned} \tag{48}$$

For the output, we have

$$\begin{aligned} y_1 &= \dot{x} = x_2, \\ y_2 &= z = x_3. \end{aligned} \tag{49}$$

When combining this, we have that the state space equations are given by Then the state-space equation reads as

$$\begin{aligned} \dot{\zeta} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -b & -a & -d & -c \\ 0 & 0 & 0 & 1 \\ -h & -g & -f & -e \end{bmatrix} \zeta + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \zeta. \end{aligned} \tag{50}$$

Recall that state-space representations are not unique!

E 5: A simplified drug-delivery model is given by

$$\frac{dc}{dt} = -kc + u, \tag{51}$$

where $c(t)$ is the drug concentration at time t in seconds, and $k > 0$ is a rate constant. Design a simple proportional controller $u(t) = K_p c(t)$ such that for any initial drug concentration $c(0) = c_0$ and any rate constant k, the drug concentration reaches $0.5c_0$ (50% of the initial concentration) in 2 hours.

Solution:
The closed-loop system reads as

$$\frac{dc}{dt} = -kc + K_p c, \tag{52}$$

which we can rewrite as

$$\dot{c} = (-k + K_p)c. \tag{53}$$

The solution to this differential equation is given by a homogeneous solution only. To this end, we consider the homogeneous solution of the form

$$c_h(t) = e^{\lambda t}.$$

When substituting this into the differential equation above, we obtain

$$\lambda e^{\lambda t} - (-k + K_p)e^{\lambda t}.$$

We can rewrite this as

$$e^{\lambda t}(\lambda + k - K_p) = 0,$$

which yields

$$\lambda = -k + K_p.$$

Therefore, the solution to the differential equation is of the form

$$c_h(t) = \alpha e^{(-k+K_p)t}.$$

We can determine the value of α from the initial condition $c_h(0) = c_0$, yielding

$$c_0 = c_h(0) = \alpha,$$

and hence $\alpha = c_0$. Therefore, the solution to the differential equation is given by

$$c(t) = c_0 e^{(-k+K_p)t}.$$

Now, from the requirements, we know that we want $c(t) = 0.5c_0$ after 2 hours. By converting this into seconds, we obtain the following condition

$$c(7200) = 0.5c_0.$$

Substituting this yields the equation:

$$0.5c_0 = c_0 e^{7200(-k+K_p)},$$

We need to solve for K_p, which we can do using the following steps. First we divide both sides by c_0,

$$\frac{0.5c_0}{c_0} = e^{7200(-k+K_p)},$$

after which we take the logarithm on both sides

$$\ln(0.5) = 7200(-k + K_p).$$

And thus, we find that the value of K_p should be

$$K_p = \frac{\ln(0.5)}{7200} + k.$$