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Answers

1. $\boxed{\text{5 points}}$ In a small town, there is a yearly tradition where three contestants are chosen to participate in a series of challenges. Each contestant is given a unique challenge, and either succeeds or fails. The challenges vary in difficulty, and are not necessarily identical for each contestant. The success of one contestant might influence the success of the others. The town's historian has kept records of the event for many years, and has found that on average, 1.8 of the three contestants succeed in their challenges each year. This year, three new contestants - Alice, Bob, and Charlie - are stepping up to the plate. Let X_1, X_2, X_3 be random variables that represent the success of the trials of Alice, Bob and Charlie (taking value 1 if success, 0 otherwise). Let $X = X_1 + X_2 + X_3$ denote the number of successes among Alice, Bob, and Charlie. Given the historical average, we know that $\mathbb{E}[X] = 1.8$, but we do not know the joint distribution of $(X1, X2, X3)$.

   - What is the largest possible value of $\mathbb{P}(X = 3)$?

     First let's analyse the expected value, following the Bernoulli distribution characteristic describing expectation ($\mathbb{E}[X] = p$)

     $$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] = p_1 + p_2 + p_3 = 1.8 \tag{1}$$

     We know this is a Bernoulli distribution because is an experiment which results in success with probability p and failure with probability $(1 - p)$.
     We know that the results of one contestant might influence other's success. There are three possible scenarios of correlation: perfect correlation ($p_1 = p_2 = p_3 = 0.6$), partial correlation and perfect independence ($\mathbb{P}(X = 3) = p_1 * p_2 * p_3$).
     Let's work with the case of perfect correlation. That is when the success of the three participants is totally synchronous. The probability of all of them succeeding is the same probability as one of them doing so. Therefore $\mathbb{P}(X = 3) = 0.6$.
     If we have independent probabilities, we know that due to the existence of constraint in equation 1, their $\mathbb{P}(X = 3$ won't be higher than $\mathbb{P}(X = 3) = 0.6^3 \leq 0.6$. We take into consideration that the highest result of multiplication of three numbers which sum has to equal the same number will come from all three being the same. With regards to the partial correlation, the probability $\mathbb{P}(X = 3)$ can be rewritten as

     $$\mathbb{P}(X = 3) = p_1 * \mathbb{P}(X_2 = 1|X_1 = 1) * \mathbb{P}(X_3|X_2 = 1, X_1 = 1) \tag{2}$$

     which will follow the same principle as the previous. Therefore, the largest possible value for $\mathbb{P}(X = 3)$ is 0.6.

   - What is the smallest possible value of $\mathbb{P}(X = 3)$?

     In the previous subsection we discussed how the partial correlation and total independence cases will always lead to lower $\mathbb{P}(X = 3)$ than the perfect correlation due to the multiplication of these individually correlated probabilities. In the total independence case, any of these values could be approximating 0, and the individual probabilities could compensate to satisfy constraint in equation 1. However then $\mathbb{P}(X = 3) \approx 0$ or even $\mathbb{P}(X = 3) = 0$ which would be the lowest possible value.

   - What is the answer to these questions if the chances of success of the contestants are independent and identically distributed?

     If they're independent, then the lowest possible value is 0, following the reasoning of the previous subsection, while the highest possible value is $\mathbb{P}(X = 3) = 0.6^3 = 0.216$, as it follows the function

     $$\mathbb{P}(X = 3) = p_1 \cdot p_2 \cdot p_3 \tag{3}$$

     .

2. $\boxed{\text{4 points}}$ Two players take turns shooting at a target, with each shot by player i hitting the target with probability p_i, and miss with probability $q_i = 1 - p_i, i = 1, 2$. We assume that the players' shots are (mutually) independent. Shooting ends when two consecutive shots hit the target (i.e. either when player 1 hits the target right after player hits it, or the other way round). If player 1 starts, we denote N_1 the number of shots taken. Similarly, if player 2 starts, we denote N_2 the number of shots taken. Let $\mu_i = \mathbb{E}[N_i]$ denote the mean number of shots taken when player i shoots first, $i = 1, 2$. Let H_i denote the event the i-th shot is a hit

- Express $\mathbb{E}[N_1|H_1 \cap H_2]$, $\mathbb{E}[N_1|H_1 \cap H_2^c]$ and $\mathbb{E}[N_1|H_1^c]$ in terms of $\mu_1, \mu_2, p_1, p_2, q_1, q_2$.
  There are three possible scenarios
  1. $\mathbb{E}[N_1|H_1 \cap H_2]$ is the expected number of shots taken before the end of the episode when both the first and second shot hit. These two successful hits fulfill the finishing condition, and therefore $\mathbb{E}[N_1|H_1 \cap H_2] = 2$.
  2. $\mathbb{E}[N_1|H_1 \cap H_2^c]$ is the expected number of shots taken taken before the end of the episode when the first shot hit but the second doesn't. Because we don't know how long it will take until the finishing condition is fulfilled, we can say that $\mathbb{E}[N_1|H_1 \cap H_2^c] = 2 + \mu_1$ where $\mu_1 = \mathbb{E}[N_1]$
  3. $\mathbb{E}[N_1|H_1^c]$ is the expected number of shots taken taken before the end of the episode when the first shot doesn't hit. Using the same reasoning as in the last case we can say $\mathbb{E}[N_1|H_1^c] = 1 + \mu_2$ where $\mu_2 = \mathbb{E}[N_2]$
- Show that $\mu_1(1 - p_1q_2) = 1 + p_1 + \mu_2q_1$.
  We can argue that $\mu_i = \mathbb{E}[N_i] = \mathbb{E}[N_1|H_1 \cap H_2]\mathbb{P}(H_1 = 1 \cap H_2 = 1) + \mathbb{E}[N_1|H_1 \cap H_2^c]\mathbb{P}(H_1 = 1 \cap H_2 = 0) + \mathbb{E}[N_1|H_1^c]\mathbb{P}(H_1 = 0)$ given that it encompasses every possible scenario leading from an episode in which the first shooter starts, weighted by the probability of it happening.
  Because the players shots are mutually independent, the probabilities of two events happening will be the multiplication of their individual probabilities, so $\mathbb{P}(H_1 = 1 \cap H_2 = 1) = \mathbb{P}(H_1 = 1)\mathbb{P}(H_2 = 1) = p_1p_2$. This can be translated into the following equations, which lead to the final conclusion

$$\mu_1 = 2p_1p_2 + (2 + \mu_1)p_1q_2 + (1 + \mu_2)q_1 \tag{4}$$
$$\mu_1 = 2p_1p_2 + 2p_1q_2 + \mu_1p_1q_2 + q_1 + \mu_2q_1 \tag{5}$$
$$\mu_1 - \mu_1p_1q_2 = 2p_1(p_2 + q_2) + q_1 + \mu_2q_1 \tag{6}$$
$$\mu_1(1 - p_1q_2) = p_1 + p_1 + q_1 + \mu_2q_1 \tag{7}$$
$$\mu_1(1 - p_1q_2) = 1 + p_1 + \mu_2q_1 \tag{8}$$

- Assume that $p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{4}$. Find μ_1 and μ_2. Hint: First, derive formula for μ_2 similar as the one of the previous question (no justification needed for this step); then solve a linear system of equations.

$$\mu_2 = \frac{1 + p_2 + \mu_1q_2}{1 - p_2q_1}, \tag{9}$$
$$\mu_1 = \frac{1 + p_1 + \mu_2q_1}{1 - p_1q_2} \tag{10}$$
$$\mu_2 = \frac{10}{7} + \frac{6}{7}\mu_1, \tag{11}$$
$$\mu_1 = \frac{4}{5}\mu_2 + \frac{12}{5} \tag{12}$$
$$\mu_1 = \frac{4}{5}(\frac{10}{7} + \frac{6}{7}\mu_1) + \frac{12}{5} = \frac{8}{7} + \frac{24\mu_1}{35} + \frac{12}{5} \tag{13}$$
$$\mu_1(1 - \frac{24}{35}) = \frac{124}{25}, \tag{14}$$
$$\mu_1 = \frac{\frac{124}{35}}{\frac{11}{35}} = \frac{124}{11} \approx 11.27 \tag{15}$$
$$\mu_2 = \frac{2}{7}(5 + 3\frac{124}{11}) = \frac{122}{11} \approx 11.09 \tag{16}$$

3. $\boxed{\text{2 points}}$ Let X_1, X_2, X_3 and X_4 be independent continuous random variables with a common distribution function F. Find the probability

$$p = \mathbb{P}(X_1 < X_2 > X_3 < X_4) = \mathbb{P}(X1 < X2 \cap X2 > X3 \cap X3 < X4)$$

Given any four distinct numbers, there are 4!=24 possible permutations. Out of these, we need to identify the permutations that satisfy the given inequalities

$$\begin{aligned} X_1 < X_2 > X_3 < X_4 = \{&(X_1 < X_3 < X_2 < X_4), && (17)\\ &(X_1 < X_3 < X_4 < X_2), && (18)\\ &(X_3 < X_1 < X_2 < X_4), && (19)\\ &(X_3 < X_1 < X_4 < X_2), && (20)\\ &(X_3 < X_4 < X_1 < X_2))\} && (21)\end{aligned}$$

Each valid permutation has an equal probability of occurring since the $Y_i = F(X_i)$'s are uniformly distributed. Therefore, the probability p is given by:

$$p = \frac{\text{Number of Valid Permutations}}{\text{Total Number of Permutations}} = \frac{5}{24} \tag{22}$$

4. $\boxed{\text{6 points}}$ A drone is programmed to randomly drop a package within a predefined unit square area (considered the drop zone). The drone has been programmed in such a way that it's more likely to drop the package towards the upper right corner of the drop zone. Let's denote the drop location by the random variables X and Y, representing the horizontal and vertical coordinates of the drop location, respectively. The joint probability density function of X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{5}(2xy + y^2), & 0 \le x \le 1 \ \ and \ \ 0 \le y \le 1 \\ 0, & otherwise \end{cases} \tag{23}$$

- Check that f is indeed a joint pdf
  To show that is a joint pdf we will check that it ranges a probability equal to 1.

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)dxdy = \int_0^1\int_0^1 \frac{6}{5}(2xy+y^2)dxdy = \frac{6}{5}\int_0^1 (x^2y + y^2x)|_0^1 dy = \frac{6}{5}\int_0^1 (y+y^2)dy \tag{24}$$

$$= \frac{6}{5}(\frac{y^2}{2} + \frac{y^3}{3})|_0^1 = \frac{6}{5}(\frac{1}{2} + \frac{1}{3}) = 1 \tag{25}$$

- Find the marginal density function of X

$$f_X(x) = \int_0^1 \frac{6}{5}(2xy+y^2)dy = \frac{6}{5}(xy^2 + \frac{y^3}{3})|_0^1 = \frac{6}{5}(x + \frac{1}{3}) \tag{26}$$

- Find the marginal density function of Y

$$f_Y(y) = \int_0^1 \frac{6}{5}(2xy+y^2)dx = \frac{6}{5}\int_0^1 (x^2y + y^2x)|_0^1 dy = \frac{6}{5}(y+y^2) \tag{27}$$

- Prove or disprove that X and Y are independent.
  We'll use proposition 9.2.1, which states that if $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$ then X, Y are independent. To test this we multiply the marginal density function that we calculated in the previous two subsections:

$$f_X(x) \cdot f_Y(y) = \frac{6}{5}(x + \frac{1}{3}) \cdot \frac{6}{5}(y + y^2) = \frac{36}{25}(xy + xy^2 + \frac{y}{3} + \frac{y^2}{3}) \tag{28}$$

  Since this result doesn't coincide with the given joint pdf function, we conclude that the variables X and Y are not independent.

- Compute the probability $\mathbb{P}(X < Y)$
  To calculate this probability we will calculate the function integration such that it's limited to $X = x < y$, therefore effectively calculating the probability $\mathbb{P}(X < Y)$

$$\int_0^1 \int_0^y \frac{6}{5}(2xy + y^2)dxdy = \frac{6}{5}\int_0^1 (x^2y + y^2x)|_0^y dy \tag{29}$$

$$= \frac{6}{5}\int_0^1 (y^3 + y^3)dy = \frac{6}{5}\frac{1}{4}(y^4 + y^4)|_0^1 = \frac{12}{20} = \frac{3}{5} \tag{30}$$

$$\tag{31}$$

- Compute the expectation $\mathbb{E}[X^2Y]$
  We compute the expectation by adding the x^2y into the double integral as follows

$$\int_0^1 \int_0^1 \frac{6}{5}x^2y(2xy + y^2)dxdy = \frac{6}{5}\int_0^1 \int_0^1 (2x^3y^2 + y^3x^2)dxdy = \frac{6}{5}\int_0^1 \frac{x^4}{2}y^2 + y^3\frac{x^3}{3}|_0^1 dy \tag{32}$$

$$= \frac{6}{5}\int_0^1 \frac{1}{2}y^2 + y^3\frac{1}{3}dy = \frac{6}{5}\left(\frac{1}{2}\frac{y^3}{3} + \frac{y^4}{4}\frac{1}{3}\right)|_0^1 = \frac{6}{5}(\frac{1}{6} + \frac{1}{12}) = \frac{3}{10} \tag{33}$$