university of groningen

Exam Numerical Methods

November 8th 2018 18.30-21.30

It is allowed to use a book (paper version only) and lecture notes, as well as a (graphical) pocket calculator. The use of electronic devices (tablet, laptop, mobile phone, etc.) is not allowed.

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: $\boxed{10}$

Practica: $\boxed{18}$ For the 6 computer practica a maximum of 6*3=18 points can be earned.

1. Consider the equation $\sin(\pi x) = x - 1.95$, with exact solution $x \approx \frac{5}{4}$. To find the solution, one could use an iterative method, with initial value $x_0 = \frac{5}{4}$.

   (a) $\boxed{6}$ (1) Compute two iterations with the Bisection method, with $I_0 = [1.0, 1.5]$ (and hence $m_0 = \frac{5}{4}$) as initial search interval.
   (2) Compute x_1 with Newton's method (one iteration), starting with $x_0 = \frac{5}{4}$.

   (b) $\boxed{7}$ By introducing a parameter $\alpha = 0.1$, someone derives the iterative method $x_{n+1} = x_n + 0.1(\sin(\pi x) - x + 1.95)$, with $x_0 = \frac{5}{4}$.
   The first 4 iterations are given by

x_0	x_1	x_2	x_3	x_4
1.25	1.249289	1.248808	1.248482	1.248260

   (1) Explain that this method will eventually converge.
   (2) Determine an error estimate for x_4.
   (3) Find the most optimal value for the parameter α.

2. Consider the integral $$I = \int_1^2 \sin(\sqrt{x})\,dx$$

   (a) $\boxed{7}$ (1) Will the Midpoint method give optimal 2nd order convergence? Explain.
   (2) Use the Midpoint method on a grid with two segments to approximate I.
   Apply the global error theorem to determine the accuracy of your approximation.
   Hint: you may use that $f''(x)$ has its extreme value at one of the ends.

   (b) $\boxed{6}$ The results for the Trapezoidal method on a number of grids are given below.

n	$I(n)$
4	0.931001
8	0.931839
16	0.932049
32	0.932102
64	0.932115

   $I(n)$ is the approximation of the integral on a grid with n sub-intervals.

   (1) Compute the q-factor. What can you conclude?
   (2) Compute improved solutions (T_2) for $I(8)$ and $I(16)$ by means of extrapolation.
   Does it make sense to further extrapolate into $T_3(16)$?
   (3) How many intervals (powers of 2) are needed with the (pure) Trapezium method (no extrapolation) to reach an accuracy 10^{-9}?

P.T.O.

3. Consider on $[-5\ 0]$ the o.d.e. $y'(x) = (x+1)y(x)$, with boundary condition $y(-5) = 2$.

(a) $\boxed{7}$ (1) Use Heun's method (RK2) to compute $y(x)$ at $x=-4.5$ on a grid with $\Delta x=0.5$.
(2) Use implicit(!) Euler to compute $y(x)$ at $x=-4.5$ on a grid with $\Delta x=0.5$.
(3) Is implicit Euler always stable in this case? If not, determine the stability limit.

(b) $\boxed{6}$ Heun's method RK2 is used on a number of grids ($N=40, 80, 160, 320$ segments). The table below shows solutions at a selection of x locations.

x_n	$N = 40$	$N = 80$	$N = 160$	$N = 320$
-4	7.071417 E-02	6.246023 E-02	6.086396 E-02	6.050695 E-02
-3	6.123759 E-03	5.188492 E-03	5.010061 E-03	4.970097 E-03
-2	1.381633 E-03	1.160728 E-03	1.118601 E-03	1.109151 E-03
-1	8.383685 E-04	7.041135 E-04	6.784912 E-04	6.727405 E-04
0	1.381090 E-03	1.160671 E-03	1.118594 E-03	1.109150 E-03

(1) Is there a stability limit visible? Explain.
(2) Compute the q-ratio for $x=0$. Conclusion?
(3) Give an error estimate for the solution at $x = 0$ on the finest grid.
(4) Which N is required (roughly) for an accuracy of 1.0E-8?
Is it advisable to use RK2 in that case? Explain.

4. The function $f(x) = \sin^2(\pi\, x)$ has several 'nice' f-values: $f(0) = 0$, $f(\frac{1}{6}) = \frac{1}{4}$, $f(\frac{1}{4}) = \frac{1}{2}$, $f(\frac{1}{3}) = \frac{3}{4}$, $f(\frac{1}{2}) = 1$. To approximate the location where $f(x) = 0.6$, we will use a linear fit through the employed x-values.

(a) $\boxed{7}$ (1) Apply a coordinate transformation to $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$, such that the new y-points are centered around $\hat{y}=0$.
(2) Use the transformations to set up a straight line through the original data.

(b) $\boxed{2}$ Find the approximate x location where $f(x) = 0.6$, using the straight line.

5. (a) $\boxed{7}$ Consider $A\vec{x} = \vec{b}$, and the initial vector $\vec{x}_0$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 4 & 1 & 0 \\ 0 & -2 & 15 & -1 \\ 0 & 0 & -2 & 5 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 3 \\ 7 \\ 3 \\ 7 \end{pmatrix} \quad \vec{x}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

(1) Compute $\|r^{(0)}\|_\infty$, i.e. the maximum-norm of the initial residual.
(2) How many Jacobi iterations are required to reduce $\|r^{(0)}\|_\infty$ with a factor 10^3?
(3) Compute $\vec{x}_1$, the result after 1 SOR iteration with $\omega = 1.5$.
(4) Will SOR converge for $\omega = 1.5$? Explain.

(b) $\boxed{6}$ Consider on $[0\ 1]$ the differential eqn. $y''(x) - \alpha\, y(x) = 3\, x^2$, with boundary conditions $y'(0) = 2$, $y'(1) = 0$ (notice the derivatives!).
(1) Take $\alpha = 2$. Give the matrix-vector system, when a grid is used with 3 segments of equal length, and the standard [1 -2 1]-formula is applied for $y''(x)$.
(2) Can you find the value for y at $x = \frac{1}{3}$ if $\alpha = 0$? Explain.

6. Consider the differential equation $\dfrac{\partial\phi}{\partial t} = 10^{-5}\dfrac{\partial^2\phi}{\partial x^2}$, with bound. cond. $\phi(0,t)=\phi(1,t)=20$ and initial condition $\phi(x,0) = A\sin(\pi x) + 20$. For $\partial^2/\partial x^2$ the usual [1 -2 1]-formula is used.

(a) $\boxed{4}$ (1) When a gridsize $\Delta x = 0.05$ is used for the explicit Euler method, which restriction follows for the timestep Δt?
(2) Does this change when the amplitude in the initial condition becomes $A = 0$?

(b) $\boxed{7}$ An extra source term is added to the equation: $\dfrac{\partial\phi}{\partial t} = 10^{-5}\dfrac{\partial^2\phi}{\partial x^2} + 2\,\phi$.
(1) Does this enhance stability in case of explicit Euler? Explain why.
(2) For the Crank-Nicolson method, each time step a system $Ax=b$ has to be solved. Give the i-th row of the system, i.e. $A(i,:)$ and $b(i)$, for general Δt and Δx.

Total: $\boxed{100}$