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Answers

1. $\boxed{\text{2 points}}$ Let X,Y, and Z be independent random variables, exponentially distributed with rate parameters λ, μ, and v, respectively

   - Find $\mathbb{P}(X < Y)$
     So we need to calculate $\mathbb{P}(X < Y)$. For this we first make use of the joint probability distribution which is the multiplication of the separate functions due to the independence of the variables. The functions follow the exponential distribution structure, so $f_{X,Y}(x, y) = f_X(x)f_Y(y) = \lambda e^{-\lambda x}\mu e^{-\mu y}$ for all $x, y \geq 0$. Because the exponential distributions are always continuous we can calculate the probability of $\mathbb{P}(X < Y)$ through an integral. Integrating the joint probability density function (pdf) of X and Y over the region where X<Y will give the accumulation of probability for X<Y.

$$\int_0^\infty \mu e^{-\mu y} \int_0^y \lambda e^{-\lambda x} dxdy = \int_0^\infty \frac{\lambda}{\lambda} e^{-\lambda x} \cdot \mu e^{-\mu y}|_0^y dy = \int_0^\infty (1 - e^{-\lambda y})\mu e^{-\mu y} dy$$
$$= \int_0^\infty -e^{-\lambda y}\mu e^{-\mu y} dy + \int_0^\infty \mu e^{-\mu y} dy$$
$$\int_0^\infty -e^{-\lambda y}\mu e^{-\mu y} dy = -\mu \int_0^\infty e^{-(\lambda+\mu)y} dy = -\mu \cdot \frac{1}{\lambda + \mu}$$
$$\int_0^\infty \mu e^{-\mu y} dy = \mu \int_0^\infty e^{-\mu y} dy = \mu \cdot \frac{1}{\mu} = 1$$
$$\int_0^\infty \int_0^y \lambda e^{-\lambda x}\mu e^{-\mu y} dxdy = \frac{-\mu}{\lambda + \mu} + 1 = \frac{\lambda}{\lambda + \mu}$$

     this is due to the fact that
$$\int_0^\infty e^{-kx} dx = 0 - (-\frac{1}{k}) = \frac{1}{k}$$
     So, to conclude $\mathbb{P}(X < Y) = \frac{\lambda}{\lambda+\mu}$

   - Find the distribution of min(Y,Z)
     Let W be min(Y,Z). We have, when $a \geq 0$

$$\mathbb{P}(W > a) = \mathbb{P}(\min(X, Y) > a) = \mathbb{P}(X > a, Y > a) = \mathbb{P}(X > a)\mathbb{P}(Y > a) = ve^{-va}\mu e^{-\mu a} = \mu v e^{-a(v+\mu)}$$

     In other words, the distribution of min(Y,Z) is defined by

$$min(Y, Z) = W = \begin{cases} \mu v e^{-a(v+\mu)} & a \geq 0 \\ 0 & otherwise \end{cases} \tag{1}$$

   - Find $\mathbb{P}(X < Y < Z)$
     Since X, Y, Z are independent we can say that $\mathbb{P}(X < Y < Z) = \mathbb{P}(X < Y) \cdot \mathbb{P}(Y < Z)$ and therefore

$$\mathbb{P}(X < Y < Z) = \mathbb{P}(X < Y) \cdot \mathbb{P}(Y < Z)$$
$$\int_0^\infty ve^{-vx} \int_0^y \mu e^{-\mu y} dydz = \frac{\mu}{\mu + v}$$
$$\mathbb{P}(X < Y < Z) = \frac{\lambda}{\lambda + \mu} \cdot \frac{\mu}{\mu + v} = \frac{\mu^2 + 2\mu\lambda + \lambda v}{\mu^2 + \lambda v + \lambda\mu + \mu v}$$

2. $\boxed{\text{3 points}}$ Let X and Y be independent random variables, each uniformly distributed on the interval $(0, 1)$. Set $W = XY$ and $Z = X/Y$. Find the joint pdf of W and Z. Remark: You will probably

consider a function g such that $(X, Y) = g(W, Z)$. You are allowed to give the range of g without justification. You might want to do a sketch of the the region R

Our goal is to make the joint pdf of W and Z, $f_{W,Z}(w, z)$.

For this we will use Theorem 10.1.2 Let (X, Y) be a continuous random vector and let $g : D \to R$ one-to-one and differentiable, where $D := (x, y) \in \mathbb{R}_2 : f_{X,Y}(x, y) = 0, R = g(D) \subset \mathbb{R}_2$. If $h = g^{-1}$ and $(W, Z) = g(X, Y)$, then

$$f_{W,Z}(w,z) = \begin{cases} f_{X,Y}(g^{-1}(w,z)) \cdot |J(w,z)| & (w,z) \in \mathbb{R} \\ 0 & otherwise \end{cases}$$

Now if we make X,Y be in terms of W,Z we get $X = \sqrt{WZ}$ and $Y = \sqrt{\frac{W}{Z}}$ we can calculate the jacobian J and set $g^{-1}(w, z) = (x, y) = (\sqrt{wz}, \sqrt{\frac{w}{z}})$

$$J = |\frac{\partial(X,Y)}{\partial(W,Z)}| = \left| \begin{pmatrix} \frac{\partial X}{\partial W} & \frac{\partial X}{\partial Z} \\ \frac{\partial Y}{\partial W} & \frac{\partial Y}{\partial Z} \end{pmatrix} \right| = \left| \begin{pmatrix} \frac{\sqrt{Z}}{2\sqrt{W}} & \frac{\sqrt{W}}{2\sqrt{Z}} \\ \frac{1}{2\sqrt{WZ}} & \frac{-\sqrt{W}}{2Z\sqrt{Z}} \end{pmatrix} \right| = -\frac{\sqrt{W}\sqrt{Z}}{2Z\sqrt{Z}\sqrt{W}} - \frac{\sqrt{W}}{4Z^2\sqrt{W}} = -\frac{1}{2Z} - \frac{1}{4Z} = \frac{-3}{4Z}$$

and the joint probability distribution $f_{X,Y}(g^{-1}(w, z)) = f_{X,Y}(\sqrt{wz}, \sqrt{\frac{w}{z}}) = f_{X,Y}(x, y) = f_X(x)f_Y(y)$ due to the independence of X, Y. Each of those distributions are uniform and discrete so they follow the formula

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & otherwise \end{cases}$$

which means that if we assume that $0 < y < 1, 0 < x < 1$, then the joint probability distribution is

$$f_{W,Z}(w,z) = \begin{cases} \frac{-3}{4Z} & 0 < y < 1, 0 < x < 1 \\ 0 & otherwise \end{cases} = \begin{cases} \frac{-3}{4Z} & 0 < w < z < 1 \\ 0 & otherwise \end{cases}$$

since $0 < y < 1 = 0 < \sqrt{\frac{w}{z}} < 1 = 0 < w < z$ and $0 < x < 1 = 0 < \sqrt{wz} < 1 = 0 < wz < 1$

3. $\boxed{\text{3 points}}$ For $x > 0$ and $n \in \mathbb{N}$ set

$$A_{x,n} := \sum_{\substack{k\in\mathbb{N}: \\ \left|k-\frac{1}{2}n\right| \le \frac{1}{2}x\sqrt{n}}} \binom{n}{k}, \quad \text{and} \quad B_{x,n} := \sum_{\substack{k\in\mathbb{N}: \\ |k-n| \le x\sqrt{n}}} \frac{n^k}{k!}. \tag{2}$$

Show that for any fixed $x > 0$, one has $e^n A_{x,n}/2^n B_{x,n} \to 1$ as $n \to \infty$. Hint: You might want to use that the sum of n independent Poisson random variables with parameters $\lambda_1, \cdots, \lambda_n$ is Poisson distributed of parameter $\lambda_1 + \cdots + \lambda_n$. You might also want to use a well chosen limit theorem.

In order to work with more clean formulas, more similar to already known distributions, we will slightly change the statement to $2^{-n}A_{x,n}/e^{-n}B_{x,n} \to 1$.

$$2^{-n}A_{x,n} := \sum_{\substack{k\in\mathbb{N}: \\ \left|k-\frac{1}{2}n\right| \le \frac{1}{2}x\sqrt{n}}} \binom{n}{k}\frac{1}{2^n} = \sum_{\substack{k\in\mathbb{N}: \\ \left|k-\frac{1}{2}n\right| \le \frac{1}{2}x\sqrt{n}}} \binom{n}{k} p^k(1-p)^{n-k}$$

$$\frac{1}{2^n} = \frac{1}{2^{n-k+k}} = \frac{1}{2^k} \cdot \frac{1}{2^{n-k}} = \frac{1}{2}^k \cdot \frac{1}{2}^{n-k} = p^k(1-p)^{n-k} \qquad (\text{for } p = \tfrac{1}{2})$$

This is the exact formula of a Bernoulli distribution with $\mathbb{E}[X] = \frac{n}{2}, \sigma = \frac{\sqrt{n}}{2}$.

$$e^{-n}B_{x,n} := \sum_{\substack{k\in\mathbb{N}:\\ |k-n|\le x\sqrt{n}}} e^{-n}\frac{n^k}{k!}$$

This is the exact formula of a Poisson distribution parametrised by $\lambda = n$ with $\mathbb{E}[X] = \lambda = n$ and $\sigma = \sqrt{\lambda} = \sqrt{n}$.

Both distributions are sums over some bounds $|k - \mathbb{E}[X]| \le x\sigma_x$ that can be interpreted as

$$\mathbb{P}(\mathbb{E}[X] + x\sigma_x < k < \mathbb{E}[X] - x\sigma_x) = \sum_{|k-\mathbb{E}[X]|\le x\sigma_x} f_X(X = k)$$

As $n \to \infty$, by the central limit theorem, both of these distributions converge to a normal distribution. Due to the $k - \mathbb{E}[X]$ they are centered around 0 and due to $x\sigma_x$, the variance will be normalised to 1. Therefore, as $n \to \infty$ they become the same distribution, we can say that $e^n A_{x,n}/2^n B_{x,n} \to 1$.

4. $\boxed{\text{points}}$ Let X be a Poisson distributed random variable with parameter 1. Show that $\mathbb{P}(X \ge t) \le e^{t-1}/t^t$ for $t \ge 1$ Hint: You might want to first show that $\mathbb{P}(X \ge t) \le e^{-\theta t}e^{e^\theta - 1}$ for $\theta \ge 0$.
Remark: This bound can be used to show the following threshold phenomena: Let M_n be the maximum of n i.i.d. Poisson distributed random variables of parameter 1. For any fixed $a \in R$, as $n \to \infty$, we have that

$$\mathbb{P}(M_n \ge (1+a)\log n/\log\log n) \to \begin{cases} 1 & if a < 0 \\ 0 & if a > 0 \end{cases} \tag{3}$$

If you want an extra challenge, you can try to prove this as well (but it is not required, and it will not be graded).

Let's use Markov's inequality: Let $a > 0$ and Y be any non-negative random variable. Then, $\mathbb{P}(Y \ge a) \le \frac{1}{a}\mathbb{E}[Y]$. With this we can show a relation between $Y = e^{\theta X}$ and $a = e^{\theta t}$. We'll also use the moment generating function to calculate the expectation:

$$\mathbb{P}(e^{\theta X} \ge e^{\theta t}) \le \frac{\mathbb{E}[e^{\theta X}]}{e^{\theta t}}$$
$$M_X(\theta) = \mathbb{E}[e^{\theta X}] = e^{\lambda(e^\theta - 1)} = e^{e^\theta - 1}$$
$$\mathbb{P}(X \ge t) \le \frac{e^{e^\theta - 1}}{e^{\theta t}} = e^{e^\theta - 1 - \theta t} = e^{-\theta t}e^{e^\theta - 1}$$

We want to find the optimal value of θ to minimize the bound $e^{-\theta t}e^{e^\theta - 1}$.

Let's choose $\theta = \log t, e^\theta = t$ to substitute into the bound:

$$e^{-\theta t}e^{e^\theta - 1} = e^{-(\log t)t}e^{t-1}$$
$$e^{-(\log t)t} = e^{-t\log t} = \frac{1}{t^t}$$
$$e^{-(\log t)t}e^{t-1} = \frac{e^{t-1}}{t^t}$$

So to conclude:

$$\mathbb{P}(X \ge t) \le e^{-(\log t)t}e^{t-1} = \frac{e^{t-1}}{t^t}$$