1 a) $f(x) = 2\cos(x) - x - 1 = 0$ $\qquad x_{n+1} = x_n - \dfrac{(2\cos(x_n) - x_n - 1)}{-2\sin(x_n) - 1}$ (1)

4 $\quad f'(x) = -2\sin(x) - 1$ (1)

(1) $x_0 = 0.6 \rightarrow x_1 = 0.62380$ (1)

(2) $\varepsilon = \dfrac{x_1 - x_0}{x_1} = 0.038153$ (1)

b) (1) $K = \dfrac{x_5 - x_4}{x_4 - x_3} = -0.083766$ $\qquad \varepsilon \approx \left| \dfrac{K}{1-K}(x_5 - x_4) \right| = 8.7339E-8$ (1) (1)

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(2) error reduction factor 0.083766 (2)

(3) $g'(x) = -2a\sin(x) + 1 - a$ (1)

$g'(x_5) \approx 0 \quad \rightarrow a = \dfrac{1}{2\sin(x_5) + 1} = 0.46128$ (1) (1)

2 a) $-1 \quad -\frac{1}{2} \quad 0 \quad \frac{1}{2} \quad 1$ $\qquad h = 1$ $\qquad I = 1 \cdot f(-\tfrac{1}{2}) + 1 f(\tfrac{1}{2}) = 1(0) + 1(2e^{1/2}) = 2\sqrt{e}$ (1) (1)

7 $\quad$ (1) $\qquad (= 3.2974)$

(2) $f'(x) = (2x+3)e^x$

$f''(x) = (2x+5)e^x$ (1) $\qquad \varepsilon \leq \frac{2}{24} \cdot 1^2 \cdot 7e = \frac{7e}{12}$ (1)

$M = f''(1) = 7e$ (1) $\qquad (= 1.5857)$

(3) grid too coarse $\longrightarrow$ higher order terms in Taylor can't be neglected (1)

M belongs to one segment (not general) $\longrightarrow$ less suitable other segments (1)

b) (1) $E \approx \frac{1}{3} | I_{256} - I_{128} | = 6.7257E-5$ (1)

6

$(\frac{1}{4})^n \varepsilon < 1.0E-8 \quad n \geq 7 \longrightarrow$ segments $2^7 \cdot 256 = 32768$ (1) (1)

(2) $T_2(128) = \frac{4}{3} I_{128} - \frac{1}{3} I_{64} = 3.8219\,2015$ (1)

error $= T_2(128) - (\frac{3}{e} + e) \approx 1.36E-9$ (1)

(3) Trapezoidal err level for $n = 8$: $1.02E-9 \longrightarrow \dfrac{65536}{128+64} \approx 340$ gain factor (1)

8

3a) $y_{n+1} = y_n + h(4x_n y_n^2)$ 1

(1) $y(\frac{1}{2}) = 1 + \frac{1}{2}(4 \cdot 0 \cdot 1^2) = 1$ 1

$y(\frac{1}{4}) = 1 + \frac{1}{4}(4 \cdot 0 \cdot 1^2) = 1$

$y(\frac{1}{2}) = 1 + \frac{1}{4}(4 \cdot \frac{1}{4} \cdot 1^2) = 5/4$ 1

$2 \cdot \frac{5}{4} - 1 \cdot 1 = 3/2$ 1

(2) no, only holds for linear test eqn, this ODE is non-linear 1

(3) $y_{n+1} = y_n + h(4x_{n+1} y_{n+1}^2)$ 1 non-linear eqn to solve 1

$y(\frac{1}{2}) = 1 + \frac{1}{2}(4 \cdot \frac{1}{2} \cdot y^2(\frac{1}{2})) \rightarrow y^2(\frac{1}{2}) - y(\frac{1}{2}) + 1 = 0$

$D = 1 - 4 = -3 \rightarrow$ complex solutions 1

b) (1) $q(0.4) = 4.3450 \approx 2^2$ 1 according to 2^{nd} order convergence

5

(2) $\mathcal{E}(0.3) = 2.83199\text{E}-5$ 1 (here $\frac{1}{3}$ coeff used)

$\mathcal{E}(0.4) = 1.29896\text{E}-4$ 1 $\leftarrow$ larger, because of err. accumulatio 1

(3) $\frac{4}{3} y(0.0125) - \frac{1}{3} y(0.025)$ at $x = 0.4 \rightarrow 1.470596\ldots$ 1

4) 9

$\hat{x} = x - 1$ $-2 \quad 0 \quad 2$ 1

$\hat{y} = \ln(y)$ $-1 \quad 0 \quad 2$ 1

$M_0 = 3, M_1 = 0, M_2 = 8$

$F_0 = 1, F_1 = 6$ 2

$\left(\begin{array}{cc|c} 3 & 0 & 1 \\ 0 & 8 & 6 \end{array}\right)$ 1 $a = 1/3$, $b = 3/4$ 1

$y(x) = e^{1/3} \cdot e^{3/4(x-1)} = e^{-5/12} e^{3/4 x}$ 1

2

5a) $r_0 = Ax_0 - b = \begin{pmatrix} -1 \\ -12 \\ -12 \\ -1 \end{pmatrix}$ $\|r_0\|_\infty = 12$ 1

4 (1)

(2) diag dominance: $8/20 = 2/5$ 1 $\left(\frac{2}{5}\right)^n < 0.01$ 1 $n \geq 6$ iterates 1

b) $\hat{x}_1 = \frac{1}{1}(2 - 0(1) - 0(1) - 0(1)) = 2$ $x_1 = \frac{1}{2}.2 + \frac{1}{2}.1 = \frac{3}{2}$

5 $\hat{x}_2 = \frac{1}{-20}(0 - 4(\frac{3}{2}) - 4(1) - 0(1)) = 1/2$ $x_2 = \frac{1}{2}.\frac{1}{2} + \frac{1}{2}.1 = \frac{3}{4}$ 3

(1) $\hat{x}_3 = \frac{1}{-20}(0 - 0(\frac{3}{2}) - 4(\frac{3}{4}) - 4(1)) = 7/20$ $x_3 = \frac{1}{2}.\frac{7}{20} + \frac{1}{2}.1 = \frac{27}{40}$

$\hat{x}_4 = \frac{1}{1}(2 - 0(\frac{3}{2}) - 0(\frac{3}{4}) - 0(\frac{27}{40})) = 2$ $x_4 = \frac{1}{2}.2 + \frac{1}{2}.1 = \frac{3}{2}$

(2) Jacobi converges $\Longrightarrow$ GS converges 1 $\Longrightarrow \omega = \frac{1}{2}$ makes it slower 1

c) $h = 1/3$ 1 $\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + a y_i = 0$ 1

4 0 $\frac{1}{3}$ $\frac{2}{3}$ 1

$\Rightarrow y_{i+1} - 2y_i + y_{i-1} + h^2 a y_i = 0$

$\Rightarrow 4y_{i+1} - 8y_i + 4y_{i-1} + 4h^2 a y_i = 0$ 1 $-8 + 4(\frac{1}{3})^2 a = -20$

$a = -27$ 1

6a) $\Delta x = 3$ $R = \frac{0.09}{3^2}\delta t < \frac{1}{2} \longrightarrow \delta t < 50$ 1

6 (1) 0 3 6 9

(2) $R = \frac{0.09}{3^2}.10 = 0.1$ 1 initial values 20 10 -10 -20 at 0 3 6 9 1

at $t = 1.\delta t = 10$: $u_1 = 20$, 1 $u_4 = -20$ (bound. cond)

$u_2 = 0.1(20) + 0.8(10) + 0.1(-10) = 9$ 1

$u_3 = 0.1(10) + 0.8(-10) + 0.1(-20) = -9$ 1

b) $R = 0.1$ $\frac{u_i^{n+1} - u_i^n}{\delta t}$ 1 $= \frac{D}{\Delta x^2}(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) + 0.1(u_i^{n+1} + x_i)$ 1

6

$-Ru_{i-1}^{n+1} + (2R + 1 - \overset{0.1\delta t}{.1})u_i^{n+1} - Ru_{i+1}^{n+1} = u_i^n + \overset{0.1\delta t}{.1}x_i$

$-u_{i-1}^{n+1} + (2 + \frac{1}{R} - \frac{1}{R})u_i^{n+1} - u_{i+1}^{n+1} = \frac{1}{R}u_i^n + \frac{1}{R}x_i$ 2

$\rightarrow [-1 \quad 2 \quad -1 \quad 0]$ 2nd row A 1

$10(10) + 10(3) = [130]$ 1 2nd element b