Geometry 2024, Homework set 1

- Below you can find your first homework assignment. Please upload it to BrightSpace by **Monday March 4**. The deadline is strict, so late homework will not be graded.
- The number of points per question is given in a box. 15 extra points are given for a clear writing of solutions. Note that high marks for the homeworks contribute to the final grade.

QUESTIONS

1. $\boxed{10+10 = 20\text{pts}}$ (**Conic sections**)

   a) Consider the cone
   $$C = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 - z^2 = 0\}$$
   in $\mathbb{R}^3$ and let P be a 2-plane in $\mathbb{R}^3$ **not passing through the origin**. Show that every conic section $P \cap C$ is either an ellipse, parabola or hyperbola, i.e., $P \cap C$ is given by one of the following equations in suitable Cartesian coordinates on the 2-plane P:
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(ellipse)}, \qquad y^2 = ax \quad \text{(parabola)} \qquad \text{or} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{(hyperbola)}.$$

   b) Prove that conic sections arise as envelopes in the following way. Let C be a unit circle in $\mathbb{R}^2$ and let $A \in \mathbb{R}^2$ be a fixed point. Consider the family of perpendicular bisectors to the line segment AB, where the point B traverses C. Prove that the envelope to this family of perpendicular bisectors is an ellipse or hyperbola depending on whether A is inside or outside of C; see Figure 1 below.

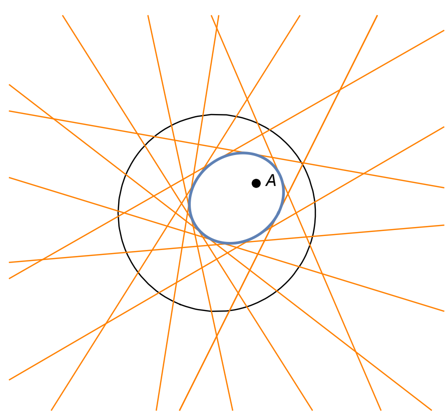

Figure 1: Ellipse as an envelope of a family of rays.

2. $\boxed{\text{15pts}}$ (**Envelopes**) Exercise 13 from Ch 7 of the book by M. Audin.

3. $\boxed{17{+}18 = 35\text{pts}}$ (**Evolutes**) Compute the evolutes and plot the wave-fronts for the following curves in $\mathbb{R}^2$:

   - Hippopede: $(x^2 + y^2)^2 = ax^2 + y^2$, where $a \neq 0$ is a parameter;
   - Cubic curves: $y^2 + x^3 - ax = b$, where a and b are parameters (see Figure 2).

   Explain the relation between the wave-fronts and the evolutes (Huygens's principle) for these examples. You may use any software such as Mathematica or GeoGebra.

4. $\boxed{\text{15pts}}$ (**Elliptic (cubic) curves**) Consider again the equation of a cubic curve

$$y^2 + x^3 - ax = b,$$

   where now the variables x, y and the parameters a, b are in the complex affine line $\mathbb{C}$. Prove that such a curve determines a real 2-surface in $\mathbb{C}^2 \simeq \mathbb{R}^4$. Show that such a surface is always connected (compare with Figure 2). When is it non-singular?

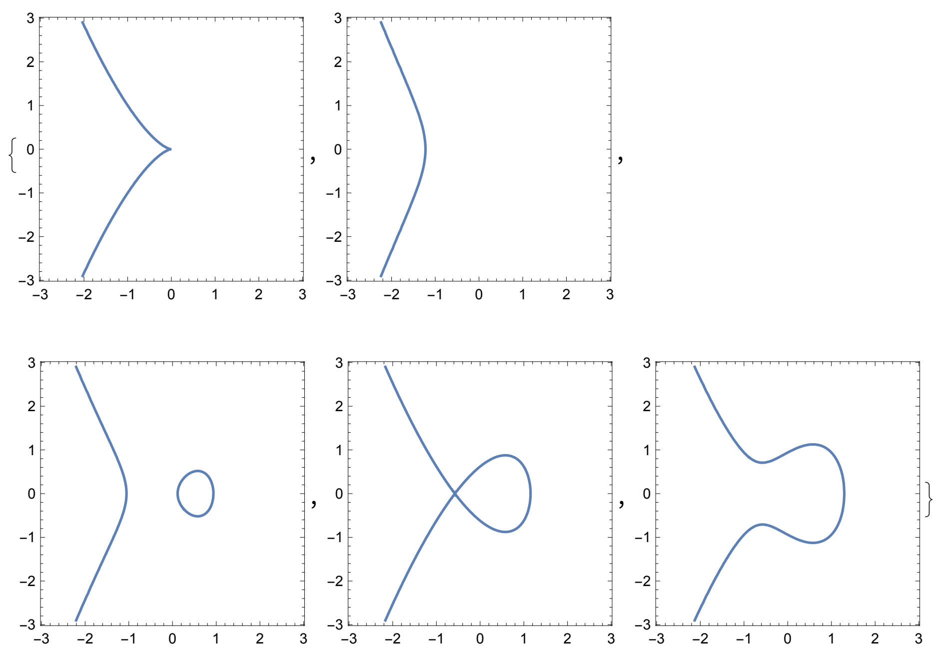

Figure 2: Cubics in $\mathbb{R}^2$ for $a = b = 0$ and $a = 1$ and several choices of b.

End of homework