Lecture notes for the course

Regeltechniek

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Contents

Preface

These course notes provide an introduction to physical modeling of dynamical systems. Dynamic modeling and physical understanding is of major importance in the Control Engineering practice. The first chapter deals with physical modeling based on classical methods and is based on lecture notes used for “Dynamische Regelsystemen deel I”, at a 2nd years B.Sc. course (2006) for Electrical Engineering students of Delft University of Technology, written by prof. dr. ir. Jacquelien M.A. Scherpen and dr. ing. Dimitri Jeltsema. The second chapter is partly covered by the course book of Dorf and Bishop, but explains nonlinear systems and nonlinear phenomena in more detail, and is provided for a more thorough understanding. In the Appendix, you will find a review on some Linear Algebra results that are used in this course, as well as a summary diagram of possible linear system descriptions in continuous time. These parts are based on the same lecture notes as mentioned above, and contains contributions of prof. dr. ir. Jacquelien M.A. Scherpen, dr. ing. Dimitri Jeltsema, dr. ir. Vincent Verdult and dr. ir. Ton van den Boom.

Chapter 1

Motion Equations and Models of Electromechanical Systems

After studying this chapter you are able to:

- derive the equations of motion of a mechanical, electrical and electromechanical system based on the energy in the dynamical elements.
- apply the Euler-Lagrange equations in modeling mechanical, electrical and electromechanical systems.
- go from the Euler-Lagrange equations to a first-order system description.
- set up the Hamiltonian system equations based on the internal energy of the system.

In classical mechanics there exists a rather complete description of electromechanical systems based on physical laws and principles. Based on the observation that nature acts in an 'economic' way, i.e., in nature a system usually follows the path of the least energy usage, models of such physical systems can be made. This means that the energy of the systems under consideration is a powerful tool for constructing a model of the system, and offers an opportunity to use physical insight for the analysis of the systems.

In this chapter we give an introduction in the way system theorists build models based on physical insight from classical mechanics. It is based on the idea that there are more general relationships that represent several of the different physical modeling domains (electrical, mechanical, hydraulic, etc.) in a similar manner. In this chapter, we present a technique that is based on a classical point of view with the final goal that the reader has a broader overview on the several physical modeling possibilities. The variables used in this approach are the generalized displacements and generalized momenta. The relation between these variables and the flow and effort variables are summarized in Table 1.1.

Table 1.1: Domains and variables.

	Effort	Flow	Generalized Displacement	Generalized Momentum
	e	f	q	p
Electric	voltage $u\ [V]$	current $i\ [A]$	charge $q\ [C]$	flux $\phi\ [Vs]$
Translation	force $F\ [N]$	velocity $v\ [m/s]$	displacement $x\ [m]$	momentum $p\ [Ns]$
Rotation	torque $M\ [Nm]$	angular velocity $\omega\ [rad/s]$	angular displacement $\theta\ [rad]$	rot. momentum b $[Nms]$
Hydraulic	pressure $p\ [N/m^2]$	vol. flow $Q\ [m^3/s]$	volume $V\ [m^3]$	momentum of flow tube $\Gamma\ [Ns/m^2]$
Thermodynamic	temp. $T\ [K]$	entropy flow $f_T\ [WK^{-1}]$	entropy $S\ [J/K]$	-

The chapter is organized as follows. First we treat Hamilton's principle very briefly in Section 1.1. Section 1.2 treats the Euler-Lagrange equation following from D'Alembert's principle, and reformulates it into a standard state-space form. Next, the classical approach to modeling of (possibly nonlinear) electrical networks is discussed. In Section 1.4, a Hamiltonian formulation is derived from the Euler-Lagrange equation, and Hamiltonian systems are introduced. Finally, in Section 1.5, the modeling procedure for electromechanical is outlined using a practical example. The chapter is closed with some concluding remarks in Section 1.6.

1.1 Hamilton's Principle

The principle of least action known as *Hamilton's principle*, is a general and universally applied method which permits us to construct mathematical models of heterogeneous physical structures, composed of electric, mechanical, pneumatic, and hydraulic elements. In order to state Hamilton's principle we introduce the *Lagrangian* function $\mathcal{L}$ of a system, which is given by the difference between the total *kinetic co-energy* (T^*) and the *potential energy* (V) of the system ($\mathcal{L} = T^* - V$).

Hamilton's principle can now be stated as *The motion of the system from time t_1 to time t_2 such that the line integral*

$$I = \int_{t_1}^{t_2} L\, dt, \tag{1.1}$$

has a stationary value for the correct path of the motion. That is, out of all possible paths by which the system state could travel from its position at time t_1 to its position at time t_2, it will actually travel along that path for which the value of the integral (1.1) is stationary (i.e., constant up to infinitesimal differences from the value and from the path), and thus minimal. The quantity I is referred to as the *action* or the *action integral*.

By techniques from the calculus of variation one can arrive at the Euler-Lagrange equations, but the calculus of variation is beyond the scope of this course. Another way of arriving at the Euler-Lagrange equations is discussed briefly in the next section.

1.2 The Euler-Lagrange Formalism

Let us consider a electromechanical system with n degrees of freedom. The dynamical equations of such a system can be represented in terms of n so called generalized displacement coordinates, $q = (q_1, \ldots, q_n)$. One can look at it as basic variables for a physical system. For example, in the translational mechanical domain, these coordinates represent the actual position of the system under consideration, in the rotational domain they represent the angle, and in the electrical domain the charge, (see Table 1.1). Similarly, we can speak about the generalized velocity coordinates, $\dot{q} = (\dot{q}_1, \ldots, \dot{q}_n)$, which are equal to the flow variables in bond graphs (and thus, equal to the flow variables in Table 1.1).

Now, after having introduced the generalized coordinates, we can derive a principle that is often called ***D'Alembert's principle*** from the *force-balance* of a system in equilibrium, i.e., the total force is zero, and some additional observations. The interested reader is referred to the book of Goldstein et al. (2002) for the derivations. From D'Alembert's principle we can derive the following equations of motion

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{q}_i}(q, \dot{q})\right) - \frac{\partial T^*}{\partial q_i}(q, \dot{q}) = F_i, \qquad i = 1, \ldots, n. \tag{1.2}$$

Here $T^*(q, \dot{q})$ denotes the total *kinetic co-energy* of the system, i.e., the left hand side of the equation are then the efforts related to "inductive storage elements", while F_i are the rest of the forces acting on the system.

Usually, the forces F_i are decomposed into three parts, $F_i = F_i^c + F_i^d + F_i^e$. The first part F_i^c are called the *conservative* forces, i.e., forces that are derivable from a *potential energy* which only depends on the generalized displacement coordinates, i.e., these are the effort variables related to "capacitive storage elements". The remaining part, consists of *dissipative* and generalized *external* forces, F_i^d and F_i^e respectively. Intuitively it is clear from the storage of energy that we consider the conservative forces as essential for the internal dynamics of the system, while the dissipative and generalized forces can be considered as external variables (remember that the dissipative forces represent the energy loss to the "external world", while the storage elements describe only the internal

dynamics). We denote the decomposition of the forces as follows

$$F_i = -\frac{\partial V}{\partial q_i}(q) + F_i^d + F_i^e, \quad i = 1, \ldots, n \tag{1.3}$$

with $V(q)$ being the potential energy function. Now defining the *Lagrangian function* $\mathcal{L}(q, \dot{q})$ as containing all information from the internal energy of the system as, $\mathcal{L}(q, \dot{q}) = T^*(q, \dot{q}) - V(q)$, we then obtain from (1.2) and (1.3) the *Euler-Lagrange equations*

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}(q, \dot{q})\right) - \frac{\partial \mathcal{L}}{\partial q_i}(q, \dot{q}) = F_i^d + F_i^e, \quad i = 1, \ldots, n. \tag{1.4}$$

From (1.4) a *conservative* control system is obtained by disregarding dissipative forces ($F_i^d = 0$) and interpreting the external forces F_i^e in (1.4) as input or control variables u_i. More generally, if m degrees of freedom can be directly controlled, then one obtains the control system

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}(q, \dot{q})\right) - \frac{\partial \mathcal{L}}{\partial q_i}(q, \dot{q}) = \begin{cases} u_i & i = 1, \ldots, m \\ 0 & i = m+1, \ldots, n \end{cases} \tag{1.5}$$

with $u_1, \ldots, u_m$ being the controls (inputs).

1.2.1 Lagrangian System Dynamics

The Euler-Lagrange equations can be derived following a few simple steps as follows:

1. Identify a minimum set of generalized displacement coordinates, q.
2. Express the kinetic co-energy and potential energy, i.e., $T^*(q, \dot{q})$ and $V(q)$, respectively, in terms of the generalized coordinates and velocities, and obtain the Lagrangian function by setting $\mathcal{L}(q, \dot{q}) = T^*(q, \dot{q}) - V(q)$.
3. Differentiate $\mathcal{L}(q, \dot{q})$ with respect to q_i and $\dot{q}_i$.
4. Write the Euler-Lagrange equation (1.5) for each $i = 1, \ldots, n$.

Let us illustrate these steps with a few simple examples.

Example 1.1 (Linear mass-spring system)

Consider the mass-spring system depicted in Figure 1.1. The mass is constrained to move only in one direction (the horizontal axis) and hence the position of the whole system can be defined by one generalized coordinate q (Step 1). In rest the position of the mass is set to $q = q^* = 0$.

The motion of the mass is governed by Newton's second law, which for a constant mass reads: mass $\times$ acceleration $=$ force, i.e.,

$$m\frac{d^2q}{dt^2} = -F^c, \tag{1.6}$$

where F^c represents the restoring force exerted by the spring which is considered negative because it acts in a direction opposite to q.

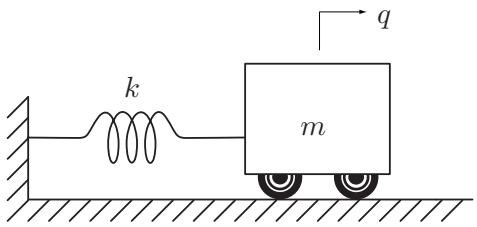

Figure 1.1: Linear mass-spring system.

Let us next derive the systems equation of motion using the Euler-Lagrange equation by following the remaining steps of the procedure above. When the spring is stretched a distance q the potential energy stored in the spring equals

$$V(q) = \frac{1}{2}k(q - q^*)^2 = \frac{1}{2}kq^2, \tag{1.7}$$

where k is the spring constant, and the kinetic co-energy is given by

$$T^*(\dot{q}) = \frac{1}{2}m\dot{q}^2. \tag{1.8}$$

Hence, the Lagrangian function reads (Step 2)

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2. \tag{1.9}$$

Now, differentiating the latter with respect to $\dot{q}$ and q, respectively, yields

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}) &= m\dot{q} \\ \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}) &= -kq = -F^c. \end{aligned}$$

Thus, the Euler-Lagrange equation (1.5) is given by

$$m\ddot{q} + F^c = 0, \tag{1.10}$$

which—as could be expected—is exactly the same as the equation of motion obtained using Newton's second law (1.6). Notice that if the mass is displaced by an external force F^e, the latter equation is extended to

$$m\ddot{q} + F^c = F^e, \tag{1.11}$$

In this simple example no particular advantage of using the Lagrangian methodology is visible. The advantage becomes more apparent in case of a nonlinear system.

Example 1.2 (One link rigid robot arm)

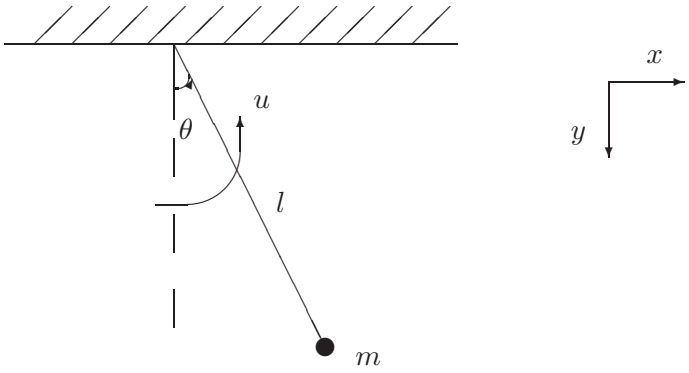

Figure 1.2: Schematic picture of a one-link rigid robot arm, where (x, y) are representing the translational coordinates for mass m.

Consider a frictionless, rigid one-link (one Degree of Freedom, 1 DOF) robot manipulator schematically presented in Figure 1.2 with external torque $F^e = u$ as input.

Let the generalized displacement coordinate be given by the angle $q = \theta$, and thus the generalized velocity coordinate given by the angular velocity, $\dot{q} = \dot{\theta}$. The Lagrangian function is given by

$$\mathcal{L}(\theta, \dot{\theta}) = T^*(\theta, \dot{\theta}) - V(\theta),$$

where T^* is the kinetic co-energy and V the potential energy. We know that the kinetic co-energy is given by $T^*(v) = \frac{1}{2}mv^2$ where $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ is the translational velocity of mass m. Since $x = l\sin(q)$, $y = l\cos(q)$, and thus $\dot{x} = l\dot{q}\cos(q)$, $\dot{y} = -l\dot{q}\sin(q)$, it then follows that the relation between v and $\dot{q}$ is given by $v = l\dot{q}$. Hence,

$$T^*(\dot{q}) = \frac{1}{2}ml^2\dot{q}^2.$$

The change in potential energy is given by the gravitational force, hence,

$$V(q) = -mg\,l\cos(q) + mgl,$$

where the additional term 'mgl' is given to have zero potential energy if $q = 0$. This implies that the Lagrangian is given by

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2}ml^2\dot{q}^2 + mg\,l\cos(q) - mgl.$$

Hence,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}) &= ml^2\dot{q} \\ \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}) &= -mgl\sin(q). \end{aligned}$$

Thus, the Euler-Lagrange equation (1.5) is given by

$$ml^2\ddot{q} + mgl\sin(q) = u.$$

1.2.2 First-Order Linear Systems Description

The above Euler-Lagrange equations are not yet in standard state space form, but are a set of implicit second-order differential equations. However, for linear systems with an even number of state variables, i.e., an even number of storage elements, we can easily give a more explicit form of the system equations. For linear systems, the kinetic co-energy is independent of the generalized position q, and can be given by the quadratic form

$$T^*(\dot{q}) = \frac{1}{2}\,\dot{q}^T M \dot{q},$$

where M is an $n \times n$ positive definite constant matrix (i.e., $M^T = M$, and $x^T M x > 0$ for all $x \neq 0$). The potential energy takes the form

$$V(q) = \frac{1}{2}\,q^T K q,$$

where K is an $n \times n$ positive semi-definite constant matrix, and the Lagrangian function is given by

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2}\,\dot{q}^T M \dot{q} - \frac{1}{2}\,q^T K q. \tag{1.12}$$

Then equation (1.5) becomes

$$M\,\ddot{q} + K\,q = B\,u, \quad \text{with } B = \begin{pmatrix} I_m \\ 0 \end{pmatrix}. \tag{1.13}$$

This is still a second-order differential equation, which can be brought into standard $2n$-dimensional state space form by noting that M is invertible since it is positive definite, and by taking $(q, \dot{q}) = (x_1, x_2)$ as the state coordinates:

$$\frac{d}{dt}\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}B \end{pmatrix} u \tag{1.14}$$

Example 1.3 (Example 1.1 Cont'd)

Consider again the mass-spring system of Example 1.1. The Euler-Lagrange equation was found to be

$$m\ddot{q} + kq = 0. \tag{1.15}$$

Thus, $M = m$ and $K = k$. Hence, by taking $x_1 = q$ and $x_2 = \dot{q}$, the state equation following from (1.13) reads

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{1.16}$$

or, in case an external force F^e is acting on the mass, $B = 1$, and thus

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} F^e. \tag{1.17}$$

1.2.3 First-Order Nonlinear System Description

As is observed in the case of a linear system, the system (1.5) is not yet in standard state space form, but is a set of implicit second-order differential equations. In this note we consider a large class of nonlinear physical systems with an explicit form for the kinetic co-energy $T^*(q,\dot{q})$, namely

$$T^*(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} \tag{1.18}$$

for some positive-definite (symmetric) matrix $M(q)$. This form for the kinetic co-energy implies that equation (1.5) takes in vector notation the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + k(q) = Bu, \tag{1.19}$$

with

$$k_i(q) = \frac{\partial V}{\partial q_i}(q), \quad i = 1,\ldots,n, \quad B = \begin{pmatrix} I_m \\ 0 \end{pmatrix}, \quad \text{and}$$

$$C_{li}(q,\dot{q}) = \frac{1}{2}\sum_{j=1}^{n}\left(\frac{\partial m_{il}}{\partial q_j}(q) + \frac{\partial m_{jl}}{\partial q_i}(q) - \frac{\partial m_{ij}}{\partial q_l}(q)\right)\dot{q}_j,$$

where $C_{li}(q,\dot{q})$ denotes the li-th element of $C(q,\dot{q})$. The term $C(q,\dot{q})\dot{q}$ is representing what is often called the *Coriolis* and *centrifugal forces*.

Equation (1.19) is a well-known and frequently used representation for the dynamics of a robotic system and hence does occur often in the robotics literature. Recently, it also occurs more often in the electrical machine and network literature. Finally, from (1.19) we obtain the $2n$-dimensional standard nonlinear state space system with $(x_1, x_2) = (q, \dot{q})$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)(C(x_1,x_2)x_2 + k(x_1)) \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}(x_1)B \end{pmatrix} u \tag{1.20}$$

It can be readily checked that in the case of a linear system $M(x_1) = M$, $k(x_1) = Kx_1$, and $C(x_1,x_2)x_2 = 0$, due to the form of the kinetic energy that is independent of the generalized position coordinates.

Example 1.4 (Example 1.2 Cont'd)

For the frictionless rigid 1 DOF robot manipulator of Example 1.2 the inertia matrix is given by $M(q) = M = ml^2$, the Coriolis and centrifugal force is given by $C(q,\dot{q})\dot{q} = 0$, and the gravitational force is given by $k(q) = mg\,l\sin(q)$. Hence, the state equations follow from (1.20) as

$$\frac{d}{dt}\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \dot{q} \\ -\dfrac{mg\,l\sin(q)}{ml^2} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{ml^2} \end{pmatrix} u$$

Let us next study an example where the Coriolis and centrifugal force are nonzero, i.e., where $C(q,\dot{q})\dot{q} \neq 0$.

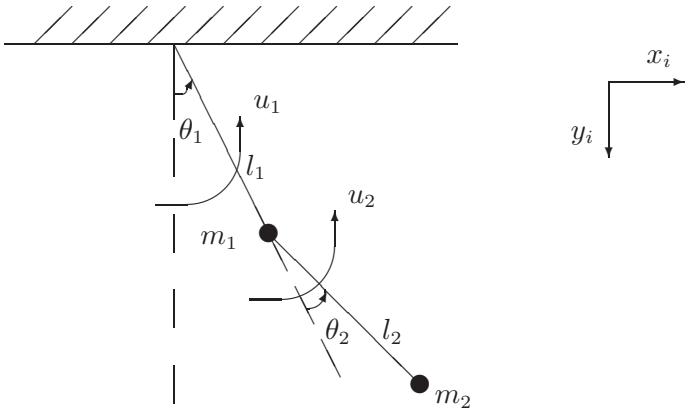

Figure 1.3: Two-link robot manipulator, where (x_i, y_i) represent the translational coordinates for mass m_i, $i = 1, 2$.

Example 1.5 (Two-link rigid robot arm)

Consider a frictionless, rigid two-link (2 DOF) robot manipulator schematically presented in Figure 1.3 with control torques u_1 and u_2 applied at the joints. Let the generalized position coordinates be given by the angles, i.e., $(q_1, q_2) = (\theta_1, \theta_2)$, see Table 1.1. Then the generalized velocity coordinates are given by the angular velocity, i.e., $(\dot{q}_1, \dot{q}_2) = (\dot{\theta}_1, \dot{\theta}_2)$. The Lagrangian function is given by

$$\mathcal{L}(\theta, \dot{\theta}) = T^*(\theta, \dot{\theta}) - V(\theta)$$

where T^* is the kinetic co-energy and V the potential energy. For the above configuration with rigid massless links the kinetic co-energy is computed as the sum of the kinetic co-energies T_1^* and T_2^* of the masses m_1 and m_2, respectively. We know that the kinetic co-energy is given by $T_i^* = \frac{1}{2} m_i v_i^2$, $i = 1, 2$, where v_i represents the translational velocity. Clearly the velocity of mass m_1 is given by $v_1 = l_1 \dot{\theta}_1$. The translational velocity of mass m_2 is given by $v_2 = (\dot{x}_2^2 + \dot{y}_2^2)^{1/2}$, where x_2 and y_2 represent the translational position of mass m_2. The position of mass m_2 can be related to the polar coordinates by the following:

$$\begin{aligned} x_2 &= l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \\ y_2 &= l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \end{aligned}$$

Differentiating the above equations, and combining it with the expression for v_2, results in the translational velocity of m_2. This yields

$$\begin{aligned} T_1^*(\dot{\theta}) &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 \\ T_2^*(\theta, \dot{\theta}) &= \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_2) \right), \end{aligned}$$

The potential energy of each mass is given by $V_i = -m_i g y_i$, $i = 1, 2$. The total potential energy V is the sum of the potential energies V_1 and V_2 of the two masses

$$\begin{aligned} V_1(\theta) &= -m_1 g l_1 \cos(\theta_1) \\ V_2(\theta) &= -m_2 g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_1 + \theta_2). \end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}(\theta,\dot{\theta}) &= \frac{1}{2}\,m_1\,l_1^2\,\dot{\theta}_1^2+\frac{1}{2}\,m_2\,\left(l_1^2\,\dot{\theta}_1^2+l_2^2\,(\dot{\theta}_1+\dot{\theta}_2)^2+2\,l_1\,l_2\,\dot{\theta}_1\,(\dot{\theta}_1+\dot{\theta}_2)\,\cos(\theta_2)\right)\\ &\quad +m_1\,g\,l_1\,\cos(\theta_1)+m_2\,g\,l_1\,\cos(\theta_1)+m_2\,g\,l_2\cos(\theta_1+\theta_2)\end{aligned}$$

Now the matrices in (1.19) are given by

$$M(\theta)=\begin{pmatrix} m_1\,l_1^2+m_2\,l_1^2+m_2\,l_2^2+2\,m_2\,l_1\,l_2\,\cos(\theta_2) & m_2\,l_2^2+m_2\,l_1\,l_2\,\cos(\theta_2)\\ m_2\,l_2^2+m_2\,l_1\,l_2\,\cos(\theta_2) & m_2\,l_2^2\end{pmatrix}$$

$$C(\theta,\dot{\theta})\dot{\theta}=\begin{pmatrix}-m_2\,l_1\,l_2\,\dot{\theta}_2\,(2\,\dot{\theta}_1+\dot{\theta}_2)\,\sin(\theta_2)\\ m_2\,l_1\,l_2\,\dot{\theta}_1^2\,\sin(\theta_2)\end{pmatrix}$$

$$k(\theta)=\begin{pmatrix} m_1\,g\,l_1\,\sin(\theta_1)+m_2\,g\,l_2\sin(\theta_1)+m_2\,g\,l_2\,\sin(\theta_1+\theta_2)\\ m_2\,g\,l_2\sin(\theta_1+\theta_2)\end{pmatrix}$$

$$B=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

for two control inputs ($m=2$), representing the torques applied on the joints.

Another method to obtain systems of first-order equations from the Euler-Lagrange equations is the Hamiltonian formulation. This will be discussed in Section 1.4. Let us next briefly discuss the inclusion of dissipation.

1.2.4 Adding Dissipation

So far, we only have discussed conservative systems. Dissipation however, is often desirable to have present in your model, because its effects often can not be neglected. Also, if we do not like oscillatory behavior, for example because it wears the material out (think about e.g. an oscillating bridge!), or because it is uncomfortable (think about e.g. a plane), we can design the system by adding dissipation. In the Euler-Lagrange equations it is often modeled with help of the Rayleigh dissipation function that is closely related to the total dissipated power. The Rayleigh dissipation function, denoted by D, is a function depending on the generalized velocity coordinates such that the associated dissipative forces F_i^d are given by

$$F_i^d=-\frac{\partial D}{\partial \dot{q}_i}(\dot{q}),\quad i=1,\ldots,n. \tag{1.21}$$

This means that the Euler-Lagrange equations (1.4) become:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}(q,\dot{q})\right)-\frac{\partial \mathcal{L}}{\partial q_i}(q,\dot{q})=-\frac{\partial D}{\partial \dot{q}_i}(\dot{q})+F_i^e,\quad i=1,\ldots,n. \tag{1.22}$$

Example 1.6 (Example 1.1 Cont'd)

Suppose the mass in the system of Example 1.1 exhibits linear friction when moving along the surface. In that case we may associate a Rayleigh dissipation of the form

$$D(\dot{q})=\frac{1}{2}b\dot{q}^2, \tag{1.23}$$

where b is the friction coefficient. Hence, the Euler-Lagrange equation including dissipation is given by

$$m\ddot{q} + kq = -b\dot{q}, \tag{1.24}$$

or in state-space description

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -b \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{1.25}$$

1.3 Lagrangian Formulation of Electrical Networks

So far we have introduced the Lagrangian formulism to elegantly describe a very broad class of mechanical systems. However, the same method can also be applied to obtain the differential equations of electrical networks. A particular advantage of using the same technique for both mechanical and electrical systems is that it offers a unified framework for electromechanical systems, i.e., systems with both electrical and mechanical components.

The first step in formulating the Lagrangian function for an electrical network is the selection of an appropriate set of generalized coordinates. We have seen that in a mechanical system the generalized coordinates are the displacement variables that are consistent with the mechanical constraints. In an electrical network such constraints are defined by the way in which the network elements are interconnected with each other, i.e., the network topology, and the electrical equivalent of displacement is electric charge (see Table 1.1). However, since the generalized coordinates have to be *independent*, we cannot choose the charges associated to each individual dynamic element (i.e., inductor charges, q_L, and capacitor charges, q_C) since these are constrained by Kirchhoff's laws. A more reasonable choice would be the selection of a set of loop (or mesh) charges as illustrated next.

1.3.1 RLC Networks with Voltage Sources

We first consider the case in which only voltage sources are present in the network; the general case in which both voltage and current sources are present is treated later on in Section 1.3.2. For this case, the basis network relationship, Kirchhoff's Voltage Law (KVL), applied to the set of independent network elements, i.e., the independent n_R resistors, n_L inductors, n_C capacitors, and n_E voltage sources, gives a set of k independent equations of the form

$$\underbrace{\left[\Psi_R \mid \Psi_L \mid \Psi_C\right]}_{=:\Psi}\begin{bmatrix} u_R \\ u_L \\ u_C \end{bmatrix} - \underbrace{\left[\Psi_R \mid \Psi_L \mid \Psi_C\right]u_E}_{=:E} = 0, \tag{1.26}$$

where

- $\Psi := \left[\Psi_R \mid \Psi_L \mid \Psi_C\right]$ is the network topology matrix following from KVL separated into three sets associated with the resistors, inductors, and capacitors, respectively.

- $u := [u_R, u_L, u_C]^T$, where u_R, u_L, and u_C are vectors whose components are the resistor, inductor, and capacitor voltages, respectively.
- u_E is the vector of voltage sources.

Now it is reasonable to try to express the Lagrangian function in terms of the magnetic co-energy and electric energy function associated with the sum of the magnetic co-energies of each individual inductor and the sum of electric energies of each individual capacitor, respectively. In particular, suppose that

$$T^*(i_L) = \int \phi_L^T di_L \tag{1.27}$$

represents the total magnetic co-energy associated with the inductors, where $\phi_L = \phi_L(i_L)$ denotes the vector of inductor flux-linkages (in the linear case, $\phi_L = Li_L$, and thus $T^*(i_L) = \frac{1}{2} i_L^T L i_L$). Similarly, suppose that

$$V(q_C) = \int u_C^T dq_C \tag{1.28}$$

represents the total electric energy associated with the capacitors (in the linear case, $u_C = C^{-1} q_C$, and thus $V(q_C) = \frac{1}{2} q_C^T C^{-1} q_C$). Furthermore, suppose that

$$D(i_R) = \int u_R^T di_R, \tag{1.29}$$

represents the Rayleigh dissipation function associated with the resistors (in the linear case, $u_R = Ri_R$, and thus $D(i_R) = \frac{1}{2} i_R^T R i_R$). In terms of the element variables we may then write

$$\begin{aligned} u_R &= \frac{\partial D}{\partial i_R}(i_R) \\ u_L &= \frac{d}{dt}\left(\frac{\partial T^*}{\partial i_L}(i_L)\right) \\ u_C &= \frac{\partial V}{\partial q_C}(q_C), \end{aligned} \tag{1.30}$$

so that (1.26) can be rewritten as

$$\Psi_R \frac{\partial D}{\partial i_R}(i_R) + \Psi_L \frac{d}{dt}\left(\frac{\partial T^*}{\partial i_L}(i_L)\right) + \Psi_C \frac{\partial V}{\partial q_C}(q_C) - E = 0. \tag{1.31}$$

Suppose now that we take as generalized coordinates the loop charges defined by

$$q := \int_0^t i_{\text{loop}}(\tau) d\tau = q_{\text{loop}}(t) - q_{\text{loop}}(0), \tag{1.32}$$

where i_{loop} is associated to the individual element currents i as

$$i = \begin{bmatrix} i_R \\ i_L \\ i_C \end{bmatrix} = \Psi^T i_{\text{loop}}, \tag{1.33}$$

and thus

$$i_R = \Psi_R^T i_{\text{loop}},\ i_L = \Psi_L^T i_{\text{loop}},\ \text{and}\ i_C = \Psi_C^T i_{\text{loop}}. \tag{1.34}$$

We therefore have that (1.31) can be rewritten as

$$\Psi_R \frac{\partial D}{\partial \Psi_R^T i_{\text{loop}}}(\Psi_R^T i_{\text{loop}}) + \Psi_L \frac{d}{dt}\left(\frac{\partial T^*}{\partial \Psi_L^T i_{\text{loop}}}(\Psi_L^T i_{\text{loop}})\right) + \Psi_C \frac{\partial V}{\partial \Psi_C^T q_{\text{loop}}}(\Psi_C^T q_{\text{loop}}) - E = 0, \tag{1.35}$$

or equivalently

$$\frac{\partial \bar{D}}{\partial i_{\text{loop}}}(i_{\text{loop}}) + \frac{d}{dt}\left(\frac{\partial \bar{T}^*}{\partial i_{\text{loop}}}(i_{\text{loop}})\right) + \frac{\partial \bar{V}}{\partial q_{\text{loop}}}(q_{\text{loop}}) - E = 0, \tag{1.36}$$

where $\bar{D}(i_{\text{loop}}) = D(\Psi_R^T i_{\text{loop}}) = D(i_R)$, etc.. Hence, by replacing $q := q_{\text{loop}}$ as the generalized coordinates and $\dot{q} := i_{\text{loop}}$ as the generalized velocities, and after rearranging the terms, we obtain the corresponding Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \bar{T}^*}{\partial \dot{q}}(\dot{q})\right) + \frac{\partial \bar{V}}{\partial q}(q) = -\frac{\partial \bar{D}}{\partial \dot{q}}(\dot{q}) + E, \tag{1.37}$$

or equivalently, with $\mathcal{L}(q, \dot{q}) = \bar{T}^*(\dot{q}) - \bar{V}(q)$,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q})\right) - \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}) = -\frac{\partial \bar{D}}{\partial \dot{q}}(\dot{q}) + E. \tag{1.38}$$

Example 1.7 (Linear RLC network driven by a voltage source (MacFarlane 1970))

For the network of Figure 1.4, take as generalized charge coordinates, the pair q_1 and q_2 such that the their time derivatives $\dot{q}_1$ and $\dot{q}_2$ are the pair of loop currents shown.

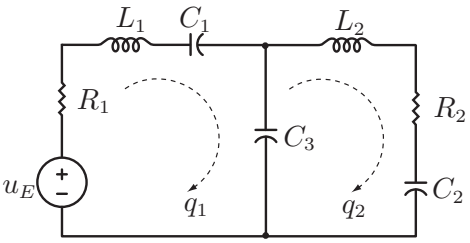

Figure 1.4: Linear RLC network driven by a voltage source.

We then have

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix},\ \dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix},\ E = \begin{bmatrix} u_E \\ 0 \end{bmatrix}$$

such that

$$T^*(\dot{q}) = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2\dot{q}_2^2,$$

$$V(q) = \frac{1}{2C_1}q_1^2 + \frac{1}{2C_2}q_2^2 + \frac{1}{2C_3}(q_1 - q_2)^2,$$

and

$$D(\dot{q}) = \frac{1}{2}R_1\dot{q}_1^2 + \frac{1}{2}R_2\dot{q}_2^2.$$

Thus,

$$\frac{\partial T^*}{\partial \dot{q}}(\dot{q}) = \begin{bmatrix} L_1\dot{q}_1 \\ L_2\dot{q}_2 \end{bmatrix}, \quad \frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{q}}(\dot{q})\right) = \begin{bmatrix} L_1\ddot{q}_1 \\ L_2\ddot{q}_2 \end{bmatrix}, \quad \frac{\partial D}{\partial \dot{q}}(\dot{q}) = \begin{bmatrix} R_1\dot{q}_1 \\ R_2\dot{q}_2 \end{bmatrix}$$

$$\frac{\partial V}{\partial q}(q) = \begin{bmatrix} \frac{1}{C_1}q_1 + \frac{1}{C_3}(q_1 - q_2) \\ \frac{1}{C_2}q_2 - \frac{1}{C_3}(q_1 - q_2) \end{bmatrix}.$$

This gives the network Euler-Lagrange equations

$$\begin{aligned} L_1\ddot{q}_1 + \tfrac{1}{C_1}q_1 + \tfrac{1}{C_3}(q_1 - q_2) &= -R_1\dot{q}_1 + u_E \\ L_2\ddot{q}_2 + \tfrac{1}{C_2}q_2 - \tfrac{1}{C_3}(q_1 - q_2) &= -R_2\dot{q}_2. \end{aligned}$$

1.3.2 RLC Networks with Current Sources

When only a number n_J of current sources are present, the basis network relationship, Kirchhoff's Current Law (KCL), applied to the set of independent network elements, gives a set of k independent equations of the form

$$\underbrace{\left[\Phi_R \mid \Phi_L \mid \Phi_C\right]}_{=:\Phi} \begin{bmatrix} i_R \\ i_L \\ i_C \end{bmatrix} - \underbrace{\left[\Phi_R \mid \Phi_L \mid \Phi_C\right] i_J}_{=:J} = 0, \tag{1.39}$$

where

- $\Phi := \left[\Phi_R \mid \Phi_L \mid \Phi_C\right]$ is the network topology matrix following from KCL separated into three sets associated with the resistors, inductors, and capacitors, respectively.
- $i := [i_R, i_L, i_C]^T$, where i_R, i_L, and i_C are vectors whose components are the resistor, inductor, and capacitor current, respectively.
- i_J is the vector of current sources.

In this case, we need to express the Lagrangian function in terms of the magnetic energy and electric co-energy functions. Hence, suppose that

$$T(\phi_L) = \int i_L^T d\phi_L \tag{1.40}$$

represents the total magnetic energy associated with the inductors. Similarly, suppose that

$$V^*(u_C) = \int q_C^T du_C \tag{1.41}$$

represents the total electric energy associated with the capacitors. Furthermore, suppose that

$$D^*(u_R) = \int i_R^T du_R, \tag{1.42}$$

represents the Rayleigh *co-dissipation* function associated with the resistors. Hence,

$$\begin{aligned} i_R &= \frac{\partial D^*}{\partial u_R}(u_R) \\ i_L &= \frac{\partial T}{\partial \phi_L}(\phi_L) \\ i_C &= \frac{d}{dt}\left(\frac{\partial V^*}{\partial u_C}(u_C)\right), \end{aligned} \tag{1.43}$$

so that (1.39) can be rewritten as

$$\Phi_R \frac{\partial D^*}{\partial u_R}(u_R) + \Phi_L \frac{\partial T}{\partial \phi_L}(\phi_L) + \Phi_C \frac{d}{dt}\left(\frac{\partial V^*}{\partial u_C}(u_C)\right) - J = 0. \tag{1.44}$$

Suppose now that we take as generalized coordinates the node fluxes defined by

$$\phi := \int_0^t u_{\text{node}}(\tau) d\tau = \phi_{\text{node}}(t) - \phi_{\text{node}}(0), \tag{1.45}$$

where u_{node} is associated to the individual element voltages u as

$$u = \begin{bmatrix} u_R \\ u_L \\ u_C \end{bmatrix} = \Phi^T u_{\text{node}}, \tag{1.46}$$

and thus

$$u_R = \Phi_R^T u_{\text{node}},\ u_L = \Phi_L^T u_{\text{node}},\ \text{and}\ u_C = \Phi_C^T u_{\text{node}}. \tag{1.47}$$

We therefore have that (1.44) can be rewritten as

$$\Phi_R \frac{\partial D^*}{\partial \Phi_R^T u_{\text{node}}}(\Phi_R^T u_{\text{node}}) + \Phi_L \frac{\partial T}{\partial \Phi_L^T \phi_{\text{node}}}(\Phi_L^T \phi_{\text{node}}) + \Phi_C \frac{d}{dt}\left(\frac{\partial V^*}{\partial \Phi_C^T u_{\text{node}}}(\Phi_C^T u_{\text{node}})\right) - J = 0, \tag{1.48}$$

or equivalently

$$\frac{\partial \bar{D}^*}{\partial u_{\text{node}}}(u_{\text{node}}) + \frac{\partial \bar{T}}{\partial \phi_{\text{node}}}(\phi_{\text{node}}) + \frac{d}{dt}\left(\frac{\partial \bar{V}^*}{\partial u_{\text{node}}}(u_{\text{node}})\right) - J = 0, \tag{1.49}$$

where $\bar{D}^*(u_{\text{node}}) = D^*(\Phi_R^T u_{\text{node}}) = D^*(u_R)$, etc.. Hence, by replacing $\phi := \phi_{\text{node}}$ as the generalized displacements and $\dot{\phi} := u_{\text{node}}$ as the generalized velocities, and after rearranging the terms, we obtain the corresponding Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \bar{V}^*}{\partial \dot{\phi}}(\dot{\phi})\right) + \frac{\partial \bar{T}}{\partial \phi}(\phi) = -\frac{\partial \bar{D}^*}{\partial \dot{\phi}}(\dot{\phi}) + J, \tag{1.50}$$

or equivalently, with $\mathcal{L}^*(\phi,\dot{\phi}) = \bar{V}^*(\dot{\phi}) - \bar{T}(\phi)$,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}^*}{\partial \dot{\phi}}(\phi,\dot{\phi})\right) - \frac{\partial \mathcal{L}^*}{\partial \phi}(\phi,\dot{\phi}) = -\frac{\partial \bar{D}^*}{\partial \dot{\phi}}(\dot{\phi}) + J. \tag{1.51}$$

The latter equation is usually referred to as the *Co-Lagrangian* equation since it forms precisely the dual of (1.38).

Example 1.8 (Linear RLC network driven by current sources (MacFarlane 1970))

For the network of Figure 1.5, take node D as the reference node and the voltages of nodes A, B, and C as the generalized voltages u_{node_1}, u_{node_2} and u_{node_3}, respectively.

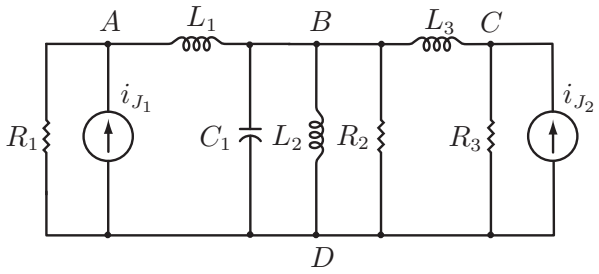

Figure 1.5: Linear RLC network driven by current sources.

Define the corresponding generalized flux-linkages as

$$\phi_i = \int_0^t u_{\text{node}_i}(\tau)d\tau, \quad i = 1,2,3, \tag{1.52}$$

and take these to be the network generalized coordinate set. Then

$$\begin{aligned} \text{flux-linkage of inductor } L_1 &= \phi_1 - \phi_2 \\ \text{flux-linkage of inductor } L_2 &= \phi_2 \\ \text{flux-linkage of inductor } L_3 &= \phi_2 - \phi_3. \end{aligned}$$

Thus we have

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \ \dot{\phi} = \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix}, \ J = \begin{bmatrix} i_{J_1} \\ 0 \\ i_{J_2} \end{bmatrix},$$

such that

$$T(\phi) = \frac{1}{2L_1}(\phi_1 - \phi_2)^2 + \frac{1}{2L_2}\phi_2^2 + \frac{1}{2L_3}(\phi_2 - \phi_3)^2,$$

$$V^*(\dot{\phi}) = \frac{1}{2}C_1\dot{\phi}_2^2,$$

and

$$D^*(\dot{\phi}) = \frac{1}{2}G_1\dot{\phi}_1^2 + \frac{1}{2}G_2\dot{\phi}_2^2 + \frac{1}{2}G_3\dot{\phi}_3^2,$$

where

$$G_k = \frac{1}{R_k}, \quad k = 1, 2, 3.$$

Thus,

$$\frac{\partial V^*}{\partial \dot{\phi}}(\dot{\phi}) = \begin{bmatrix} 0 \\ C_1\dot{\phi}_2 \\ 0 \end{bmatrix}, \quad \frac{d}{dt}\left(\frac{\partial V^*}{\partial \dot{\phi}}(\dot{\phi})\right) = \begin{bmatrix} 0 \\ C_1\ddot{\phi}_2 \\ 0 \end{bmatrix}, \quad \frac{\partial D^*}{\partial \dot{\phi}}(\dot{\phi}) = \begin{bmatrix} G_1\dot{\phi}_1 \\ G_2\dot{\phi}_2 \\ G_3\dot{\phi}_3 \end{bmatrix}$$

$$\frac{\partial T}{\partial \phi}(\phi) = \begin{bmatrix} \dfrac{\phi_1 - \phi_2}{L_1} \\ -\dfrac{\phi_1 - \phi_2}{L_1} + \dfrac{\phi_2}{L_2} + \dfrac{\phi_2 - \phi_3}{L_3} \\ -\dfrac{\phi_2 - \phi_3}{L_3} \end{bmatrix}.$$

This gives the network Euler-Lagrange equations

$$\begin{aligned} \frac{\phi_1 - \phi_2}{L_1} + G_1\dot{\phi}_1 &= i_{J_1} \\ C_1\ddot{\phi}_2 - \frac{\phi_1 - \phi_2}{L_1} + \frac{\phi_2}{L_2} + \frac{\phi_2 - \phi_3}{L_3} + G_2\dot{\phi}_2 &= 0 \\ -\frac{\phi_2 - \phi_3}{L_3} + G_3\dot{\phi}_3 &= i_{J_2}. \end{aligned} \tag{1.53}$$

Remark 1.1 In electrical network theory, the Rayleigh dissipation and co-dissipation function are known as the resistive *content* and *co-content*, respectively, see e.g., (MacFarlane 1970).

1.4 Hamiltonian Systems

The Lagrangian of mechanical systems is given in terms of the generalized kinetic *co*-energy and the generalized potential energy (see Section 1.2). In the slides we already emphasized that the energy (in contrast with the co-energy) results in the *most natural physical states* given by the generalized displacement and momenta coordinates. In this section, we present a very natural systematic system representation of physical systems given by the Hamiltonian framework (also stemming from classical mechanics), that can be obtained from the Euler-Lagrange equation.

Consider the conservative Lagrangian control system as given in equation (1.5), i.e., we disregard the dissipative forces. Now we continue with reformulating this into a so called Hamiltonian formulation. Therefore, we first need to introduce the *generalized momenta*

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}(q, \dot{q}), \qquad i = 1, \ldots, n. \tag{1.54}$$

In the previous section the physical meaning of the generalized momenta in the different domains can be found in Table 1.1. In general the $n \times n$ matrix N, with

$$N_{ij} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial \dot{q}_j}, \qquad i, j = 1, \ldots, n$$

will be non-singular for all q and $\dot{q}$, which implies that $p = (p_1, \ldots, p_n)$ are independent functions (in case of a kinetic co-energy of the form (1.18) we obtain $N(q, \dot{q}) = M(q)$). With the momenta as new coordinates, we can now introduce the *Hamiltonian function*

$$H(q,p) = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}(q,\dot{q}) = p^T \dot{q} - \mathcal{L}(q,\dot{q}) \tag{1.55}$$

where $\dot{q}$ and p are related by (1.54). We say that $H(p,q)$ is the so-called Legendre transform of $\mathcal{L}(q,\dot{q})$. An explanation on the Legendre transform falls beyond the scope of this course. Intuitively, it can be seen as some sort of "duality concept". The interested reader is referred to the book of Arnold et al. (1997).

1.4.1 Linear Systems

If we recall the Euler-Lagrange formulation for linear systems of the previous section, it follows from (1.12), (1.5), and (1.54) that the generalized momenta for a linear system are given by

$$p = M\dot{q}.$$

Notice (again) that this corresponds to the basic physics course knowledge that $p = mv$, with p the momentum, m the mass, and v the velocity. The Hamiltonian $H(q,p)$ is defined as the Legendre transform of $\mathcal{L}(q,\dot{q})$ as given in (1.12), i.e.,

$$H(q,p) = p^T\dot{q} - \frac{1}{2}\dot{q}^T M\dot{q} + \frac{1}{2}q^T Kq.$$

Substituting $\dot{q} = M^{-1}p$, we obtain

$$H(q,p) = \frac{1}{2}p^T M^{-1}p + \frac{1}{2}q^T Kq.$$

This Hamiltonian represents the total internal energy of the system in terms of the generalized position and momentum coordinates. The kinetic energy now takes the form $T(p) = \frac{1}{2}p^T M^{-1}p$, and the potential energy $V(q) = \frac{1}{2}q^T Kq$. It can be easily checked that the first order dynamical equations are given by

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & M^{-1} \\ -K & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} u$$

Check for yourself that this system is equivalent with (1.14) under the state transformation $q = q$, and $p = M\dot{q}$.

1.4.2 Nonlinear Systems

With (1.54), (1.55) and $B = I_n$ we obtain that the Euler-Lagrange equation (1.5) transforms into the *Hamiltonian* equations of motion

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}(q,p), \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}(q,p) + u_i, \end{aligned} \qquad i = 1, \ldots, n, \tag{1.56}$$

which we also call a *Hamiltonian control system*. A main advantage of (1.56) in comparison with (1.5) is that (1.56) immediately constitutes a control system in standard state space form, with the most natural physical state variables (q, p) (in physics usually called the phase variables). Moreover, the Hamiltonian $H(q, p)$ can be directly related to the energy of the system. Since $\mathcal{L}(q, \dot{q}) = T^*(q, \dot{q}) - V(q)$ it follows that $p = M(q)\dot{q}$, and

$$H(q,p) = \frac{1}{2} p^T M^{-1}(q) p + V(q), \tag{1.57}$$

which is the *internal energy* of the system, and results in the following Hamiltonian control system

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}(q,p), \qquad \left(\dot{q} = M^{-1}(q)p\right), \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}(q,p) + u_i = -\frac{1}{2}\frac{\partial}{\partial q_i}(p^T M^{-1}(q)p) - \frac{\partial V}{\partial q_i}(q) + u_i, \end{aligned} \qquad i = 1, \ldots, n. \tag{1.58}$$

A particulary nice feature of Hamiltonian systems is the following. Consider the time-derivative of $\dot{H}$ along the system (1.58):

$$\begin{aligned} \frac{dH}{dt}(q,p) &= \frac{\partial H}{\partial q}(q,p)\frac{dq}{dt} + \frac{\partial H}{\partial p}(q,p)\frac{dp}{dt} \\ &= \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}(q,p)\frac{\partial H}{\partial p_i}(q,p) - \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}(q,p)\frac{\partial H}{\partial q_i}(q,p) + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}(q,p) u_i \\ &= \sum_{i=1}^{n} \dot{q}_i u_i. \end{aligned}$$

This equation shows the power-balance of the system, stating that the rate of power preserved inside the system ($\dot{H}$) is equal to the power supplied to the system from the outside ($\dot{q}^T u$). This feature is a powerful property of Hamiltonian models of electromechanical systems and can be used for stability analysis.

Example 1.9 (One link rigid robot arm)

See Example 1.2. The generalized momenta coordinate is given by $p = ml^2\dot{q}$ and thus the Hamiltonian is $H(q,p) = \frac{1}{2ml^2}p^2 - mg\,l\cos q + mgl$. The Hamiltonian control system is given by

$$\begin{aligned} \dot{q} &= \frac{1}{ml^2}p \\ \dot{p} &= -mg\,l\sin q + u. \end{aligned}$$

Example 1.10 (Two links rigid robot arm)

Consider the rigid frictionless (no dissipation!) two-link robot manipulator of Example 1.5. We had $q_i = \theta_i$, $\dot{q}_i = \dot{\theta}_i$, $i = 1, 2$. By equation (1.54) we obtain the generalized momenta as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}) \qquad (p = M(q)\,\dot{q}).$$

In general $M(q)$ is positive definite, and the Hamiltonian function is given by (1.56) and (1.57) with $V(q) = V_1(q) + V_2(q)$. Take for simplicity $m_1 = m_2 = 1$ and $l_1 = l_2 = 1$, then the generalized momenta are given as

$$\begin{aligned} p_1 &= (3 + 2\cos q_2)\,\dot{q}_1 + (1 + \cos q_2)\,\dot{q}_2 \\ p_2 &= (1 + \cos q_2)\,\dot{q}_1 + \dot{q}_2 m \end{aligned}$$

while the Hamiltonian $H(q,p)$ is given as

$$\begin{aligned} H(q,p,u) = \; & \left(1 + \sin^2 q_2\right)^{-1} \left(\tfrac{1}{2}p_1^2 - (1 + \cos q_2)\,p_1 p_2 + \tfrac{1}{2}(3 + 2\cos q_2)\,p_2^2\right) \\ & -2g\cos q_1 - g\cos(q_1 + q_2). \end{aligned}$$

The Hamiltonian control system (1.58) is given by

$$\begin{aligned} \dot{q}_1 = \; & \left(1 + \sin^2 q_2\right)^{-1} p_1 - \left(1 + \sin^2 q_2\right)^{-1} (1 + \cos q_2)\,p_2 \\ \dot{q}_2 = \; & -\left(1 + \sin^2 q_2\right)^{-1} (1 + \cos q_2)\; p_1 + \left(1 + \sin^2 q_2\right)^{-1} (3 + 2\cos q_2)\; p_2 \\ \dot{p}_1 = \; & -2g\sin q_1 - g\sin(q_1 + q_2) + u_1 \\ \dot{p}_2 = \; & 2q_2\cos q_2 \left(1 + \sin^2 q_2\right)^{-2} \left(\tfrac{1}{2}p_1^2 - (1 + \cos q_2)\,p_1 p_2 + \tfrac{1}{2}(3 + 2\cos q_2)\,p_2^2\right) + \\ & \left(1 + \sin^2 q_2\right)^{-1} (p_2^2 \sin q_2 - p_1 p_2 \sin q_2) - g\sin(q_1 + q_2) + u_2. \end{aligned}$$

1.5 Electromechanical Systems

One of the main advantage of the Lagrangian or Hamiltonian formalism is that mechanical and electrical systems are treated analogously. This makes the methods particularly suitable for describing the dynamics of systems that have both electrical and mechanical components, i.e., electromechanical systems. A well-known example of an electromechanical systems is the electromechanical transducer. An electromechanical transducer is used to convert electrical energy into mechanical (or acoustical) energy, and vice versa. They are utilized for electrical actuation and sensing of mechanical displacements and forces in a wide variety of applications. An illustrative example of a sensing device is a microphone in which a sound pressure is converted into an electrical signal. In a microphone, the pressure acts upon a spring-supported mass, which usually consists of a stretched diaphragm. The generated mass (diaphragm) displacement is next converted into an electric output signal by means of an electromechanical transducer. In a loudspeaker, on the other hand, an electromechanical transducer is used to convert the electrical output signal of an audio amplifier into a force acting on the speaker diaphragm. This results in a displacement of the diaphragm, thereby generating sound waves. Other transducer examples are electromechanical filters, vibration sensors, devices employing feedback, and force and displacement sensors. In the remaining part of this chapter we will demonstrate the Lagrangian and Hamiltonian modeling procedure using a simplified model of a MEMS (micro electromechanical system) optical switch.

Example 1.11 (MEMS optical switch)

Figure 1.6 shows a picture of a MEMS optical switch. Optical switches are widely used in the optical fibre communication networks. It basically consists of three parts main parts: actuator (electrical), suspension beam (mechanical) and reflection mirror with optical fibre grooves (optical). Since

the optical model is simply a (static) function that connects the intensity of light to the position of the blade, we are in this example mainly interested in equating the dynamics of the actuator and the suspension beam.

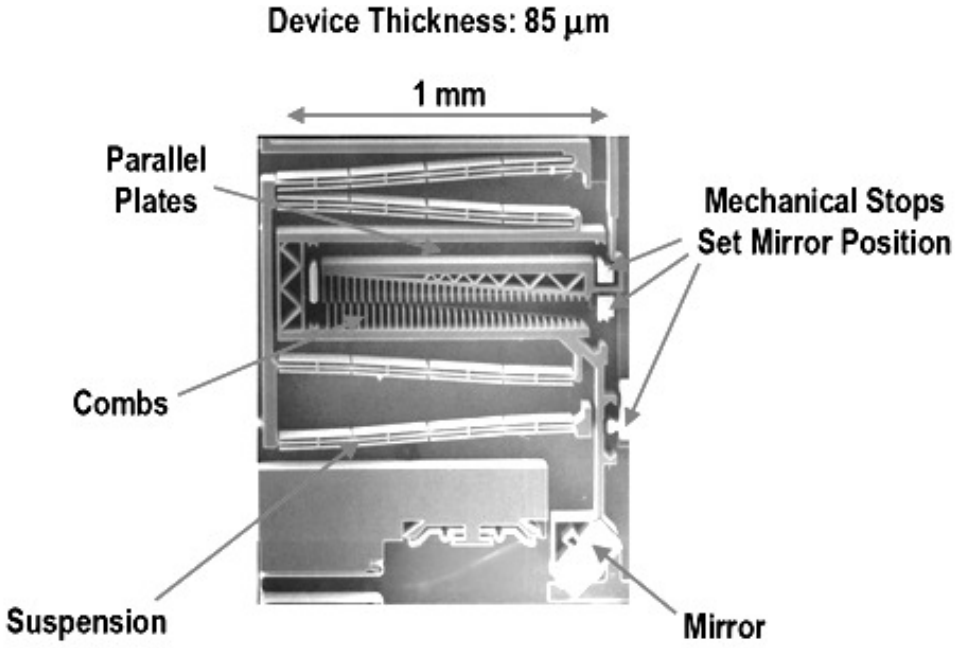

Figure 1.6: MEMS optical switch (courtesy of $io\lambda on$, Inc.).

Mechanical Model: The mechanical part of the system is determined by the following three parameters: (1) the effective mass m of the moving part; (2) the suspension stiffness k, which is generally a function of the mechanical displacement x

$$k(x) = k_1 x + k_2 x^3; \tag{1.59}$$

and (3) the damping coefficient b which will be assumed to be linear.

Taking as generalized coordinate and velocity pair the mechanical displacement $q_{\rm m} = x$ and its time-derivative $\dot{q}_{\rm m} = \dot{x}$, it easy to see that the stored mechanical kinetic co-energy is given by

$$T^*_{\rm m}(\dot{q}_{\rm m}) = \frac{1}{2} m \dot{q}^2_{\rm m} \tag{1.60}$$

and the stored mechanical potential energy

$$V_{\rm m}(q_{\rm m}) = \frac{1}{2} k_1 q^2_{\rm m} + \frac{1}{4} k_2 q^4_{\rm m}. \tag{1.61}$$

Furthermore, the dissipation can be described using a Rayleigh dissipation function of the form

$$D_{\rm m}(\dot{q}_{\rm m}) = \frac{1}{2} b \dot{q}^2_{\rm m}. \tag{1.62}$$

Electrical Model: The electrical part of the model considers generation of the electrostatic force by applying a voltage to terminals of the comb drive (see Figure 1.6). In principle this primarily consists of a capacitor with a position dependent capacity:

$$C(x) = \frac{\varepsilon_0 A}{\alpha} = \frac{2\eta\varepsilon_0\beta(x+x_0)}{\alpha}, \quad (x > -x_0), \tag{1.63}$$

where ε_0 is the dielectric constant of vacuum, η is the number of movable comb fingers, α is the gap between the fingers, and β is the thickness of the structural layer.

Let q_e denote the electrical generalized coordinate, then we have for the electrical energy storage

$$V_\mathrm{e}(q_\mathrm{m}, q_\mathrm{e}) = \frac{1}{2}C^{-1}(q_\mathrm{m})q_\mathrm{e}^2. \tag{1.64}$$

The actuator input voltage is represented by u_E, and the parasitic losses are assumed to be Ohmic such that

$$D_\mathrm{e}(\dot{q}_\mathrm{e}) = \frac{1}{2}r\dot{q}_\mathrm{e}^2. \tag{1.65}$$

The resulting Lagrangian for the overall system is now simply obtained by adding all the kinetic co-energy terms (which in this case is just the mechanical kinetic co-energy) and subtracting the sum of the potential energy contributions from both domains, i.e.,

$$\mathcal{L}(q_\mathrm{m}, q_\mathrm{e}, \dot{q}_\mathrm{m}) = \frac{1}{2}m\dot{q}_\mathrm{m}^2 - \left(\frac{1}{2}k_1 q_\mathrm{m}^2 + \frac{1}{4}k_2 q_\mathrm{m}^4 + \frac{1}{2}C^{-1}(q_\mathrm{m})q_\mathrm{e}^2\right). \tag{1.66}$$

Similarly, the total Rayleigh dissipation is given by

$$D(\dot{q}_\mathrm{m}, \dot{q}_\mathrm{e}) = \frac{1}{2}b\dot{q}_\mathrm{m}^2 + \frac{1}{2}r\dot{q}_\mathrm{e}^2. \tag{1.67}$$

Hence applying the Euler-Lagrange equations (1.22), with

$$q = \begin{bmatrix} q_\mathrm{m} \\ q_\mathrm{e} \end{bmatrix}, \ \dot{q} = \begin{bmatrix} \dot{q}_\mathrm{m} \\ \dot{q}_\mathrm{e} \end{bmatrix}, \text{ and } F^e = \begin{bmatrix} 0 \\ u_E \end{bmatrix}$$

yields

$$\frac{\partial\mathcal{L}}{\partial\dot{q}} = \begin{bmatrix} m\dot{q}_\mathrm{m} \\ 0 \end{bmatrix}, \quad \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) = \begin{bmatrix} m\ddot{q}_\mathrm{m} \\ 0 \end{bmatrix}, \quad \frac{\partial D}{\partial\dot{q}} = \begin{bmatrix} b\dot{q}_\mathrm{m} \\ r\dot{q}_\mathrm{e} \end{bmatrix}$$

$$\frac{\partial\mathcal{L}}{\partial q} = \begin{bmatrix} -k_1 q_\mathrm{m} - k_2 q_\mathrm{m}^3 - \frac{1}{2}\frac{\partial C^{-1}(q_\mathrm{m})}{\partial q_\mathrm{m}}q_\mathrm{e}^2 \\ -\frac{1}{C(q_\mathrm{m})}q_\mathrm{e} \end{bmatrix}.$$

This gives the systems Euler-Lagrange equations

$$\begin{aligned} m\ddot{q}_\mathrm{m} + k_1 q_\mathrm{m} + k_2 q_\mathrm{m}^3 + \frac{1}{2}\frac{\partial C^{-1}(q_\mathrm{m})}{\partial q_\mathrm{m}}q_\mathrm{e}^2 &= -b\dot{q}_\mathrm{m} \\ C^{-1}(q_\mathrm{m})q_\mathrm{e} &= -r\dot{q}_\mathrm{e} + u_E. \end{aligned}$$

Notice that the term

$$\frac{1}{2}\frac{\partial C^{-1}(q_\mathrm{m})}{\partial q_\mathrm{m}}q_\mathrm{e}^2 = -\frac{\alpha}{2\eta\varepsilon_0\beta(q_\mathrm{m}+x_0)^2}q_\mathrm{e}^2$$

represents the interaction force that links the mechanical and electrical subsystems.

Also notice that in this case the generalized mass matrix M is not invertible since

$$M = \begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix} \tag{1.68}$$

is singular (i.e., $|M| = 0$). Therefore we cannot extract a first-order model directly from the Euler-Lagrange equations using the technique shown in Subsection 1.2.3. However, we can still set $x_1 = q_\mathrm{m}$, $x_2 = \dot{q}_\mathrm{m}$, and $x_3 = q_\mathrm{e}$ to rewrite the Euler-Lagrange equations as three first-order differential equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}\left(-k_1 x_1 - k_2 x_1^3 + \frac{\alpha}{2\eta\varepsilon_0\beta(x_3+x_0)^2}x_3^2 - bx_2\right) \\ \dot{x}_3 &= \frac{1}{r}\left(-C^{-1}(x_1)x_3 + u_E\right). \end{aligned}$$

Another way to obtain a set of first-order differential equations is using the Hamiltonian formalism. The way the Hamiltonian equations are defined here do not admit the inclusion of dissipation. One possible solution to this problem is to add a Rayleigh dissipation term to the equations in a fashion similar to the Lagrangian equations. However, this is beyond the scope of this course. The interested reader is referred to the book of Nijmeijer and van der Schaft (1990).

Let us close this chapter by considering the Hamiltonian dynamics of the MEMS optical in case there is no loss.

Example 1.12 (MEMS optical switch (cont'd))

Consider again the MEMS optical switch of Figure 1.6. Suppose now that the system is lossless, i.e., $b = r = 0$, and assume that the system is actuated by a current source i_J.[1] Furthermore, define the Hamiltonian function

$$H(q_\mathrm{m}, q_\mathrm{e}, p_\mathrm{m}) = \dot{q}_\mathrm{m}p_\mathrm{m} - \frac{1}{2}m\dot{q}_\mathrm{m}^2 + \frac{1}{2}k_1 q_\mathrm{m}^2 + \frac{1}{4}k_2 q_\mathrm{m}^4 + \frac{1}{2}C^{-1}(q_\mathrm{m})q_\mathrm{e}^2,$$

which, since $\dot{q}_\mathrm{m} = \frac{\partial \mathcal{L}}{\partial \dot{q}_\mathrm{m}} = m^{-1}p_\mathrm{m}$, results in

$$H(q_\mathrm{m}, q_\mathrm{e}, p_\mathrm{m}) = \frac{1}{2m}p_\mathrm{m}^2 + \frac{1}{2}k_1 q_\mathrm{m}^2 + \frac{1}{4}k_2 q_\mathrm{m}^4 + \frac{1}{2}C^{-1}(q_\mathrm{m})q_\mathrm{e}^2.$$

Hence, we obtain

$$\begin{aligned} \dot{q}_\mathrm{m} &= \frac{\partial H}{\partial p_\mathrm{m}} = m^{-1}p_\mathrm{m} \\ \dot{q}_\mathrm{e} &= \frac{\partial H}{\partial p_\mathrm{e}} + i_J = i_J \\ \dot{p}_\mathrm{m} &= -\frac{\partial H}{\partial q_\mathrm{m}} = -k_1 q_\mathrm{m} - k_2 q_\mathrm{m}^3 - \frac{1}{2}\frac{\partial C^{-1}(q_\mathrm{m})}{\partial q_\mathrm{m}}q_\mathrm{e}^2. \end{aligned}$$

[1] Since there is no resistor in series with the capacitor and the voltage source, we need to replace the voltage source by a current source in order to avoid causality problems.

1.6 Concluding Remarks

In this chapter, an introduction in the classical way of modeling electrical, mechanical and electromechanical systems has been presented. The notation and approach stems from the classical mechanics literature. The Euler-Lagrange formalism, and the Hamiltonian formalism have been introduced and motivated for conservative systems without any algebraic constraints. A brief introduction to adding dissipation is also given. More extensions can be made for systems where dissipative forces play a role. Furthermore, extensions can be made for systems where algebraic constraints (often on the generalized velocity coordinates) play a role. It is also shown that the Lagrangian formalism needs extra treatment in case of electrical networks. Depending on the networks topology one must choose an appropriate set of generalized coordinates and velocities. We also briefly introduced the Brayton-Moser equations which can be obtained using a combination of co-energy functions and Rayleigh (co-)dissipation functions leading to a so-called mixed-potential function. The basic conclusion of this chapter is that we are now able to present a classical *energy storage based* framework in which first principle white box models can be put, so that the underlying physics is intuitively and clearly present in the system description.

Suggestions for further reading: An excellent recent treatment of Euler-Lagrange and Hamiltonian systems, including standard mechanical systems, switched-mode power converters, and electrical machines, is given in the book of Ortega et al. (1998) and the book of Kugi (2001). The latter also devotes a chapter to the Brayton-Moser equations. Some nice examples and problems can be found in the book of Banerjee (2005). A couple of good older books (if you can find them) include the book of Paynter (1960) or the book of MacFarlane (1970).

Chapter 2

From Nonlinear to LTI

After studying this chapter you are able to:

- distinct nonlinear systems from linear systems.
- apply linearization techniques to nonlinear systems.
- represent an LTI system as a state-space system, an input-output description or a transfer operator.
- convert between different LTI system descriptions.
- draw a block diagram of an LTI system.

Linear systems form an important class of systems, because on the one hand, many practical engineering systems may be accurately modeled as linear systems, while on the other hand, a large class of systems that cannot be modeled as a linear system can be approximated by a linear system using a *linearization procedure*. Furthermore, linear systems lend themselves well to mathematical analysis.

2.1 Linearization of Nonlinear Models

In the previous chapters we have presented some standard forms to describe the dynamical behavior of (possibly nonlinear) systems. It is often useful to determine analytically the parameter groups that are responsible for properties such as steady-state gain, time constants coupling coefficients, stability behavior, or sensitivity coefficients with respect to parameter variation. Unfortunately, for nonlinear systems, such analysis is in general rather complex as analytical solutions are usually not available. Often, we have to resort to numerical simulations and numerical computations for nonlinear systems. For example, a computer can provide a numerical solution, but in general does not provide background information about the relationship of this solution to the parameters of the model.

2.1.1 Nonlinear Phenomena

Because of the powerful tools we know for linear systems, the first step in analyzing a nonlinear system is usually to linearize it about some nominal operating point and analyze the resulting model. Most processes behave *globally* as *nonlinear* systems, whereas many processes behave *locally* as *linear* systems. Although it is appealing to use linear tools, since they are available and simpler than the existing nonlinear tools, we must keep the limitations of linearization in mind. There are two basic limitations of linearization. First, since linearization is an approximation in the neighborhood of an operating point, it can only predict "local" behavior of the nonlinear system in the vicinity of that point. It cannot predict the "nonlocal" behavior far from the the operating point, and certainly not the "global" behavior throughout the state-space. Second, the dynamics of a nonlinear system are much richer that the dynamics of a linear system. There are "essential nonlocal phenomena" that can take place only in the presence of nonlinearity. For that reason they cannot be described or predicted by linear models. Examples of essentially nonlinear phenomena are:

- *Finite escape time*: The state of an unstable linear system goes to infinity as time approaches infinity; a nonlinear system's state, however, can go to infinity in finite time.

- *Multiple isolated equilibria*: In a linear asymptotically stable system, all states approach zero (the origin of the state-space) as time increases, with no system input. For a nonlinear system, there may be a number of states that the system can approach as time increases, with no system input. These different states are called equilibrium states, of which some of them may unstable. The equilibrium state that the system approaches is determined by the system initial conditions. This condition is illustrated in a physical system that, when perturbed (disturbed), can settle to a number of different states, depending on the disturbance.

- *Limit cycles*: A periodic oscillation in a nonlinear system is called a limit cycle. In the phase plane, a limit cycle is defined as an isolated closed curve on which the system trajectory will stay for ever. The closeness and isolation of the trajectory indicates the periodic and limiting nature of the motion (with nearby trajectories converging or diverging from it). In general, limit cycles are non-sinusoidal. A periodic oscillation in a linear time-invariant system is always sinusoidal with the amplitude of oscillation a function of both the amplitude of the system excitation (applied inputs) and the initial conditions. In certain nonlinear systems, the amplitude of oscillation is independent of the system excitation or initial conditions.

- *Subharmonic, harmonic or almost-periodic oscillations*: A nonlinear system with a periodic input may exhibit a periodic output whose frequency is either a subharmonic or a harmonic of the input frequency. For example, an input of frequency 10 Hz may result in an output of 5 Hz for the subharmonic case or 30 Hz for the harmonic case.

- *Bifurcation*: As the parameters of nonlinear dynamic systems are changed, the stability of the equilibrium point can change (as it does also in linear systems) and so can the number of equilibrium points. Values of these parameters at which the qualitative nature of the system's motion changes are known as critical or bifurcation values.

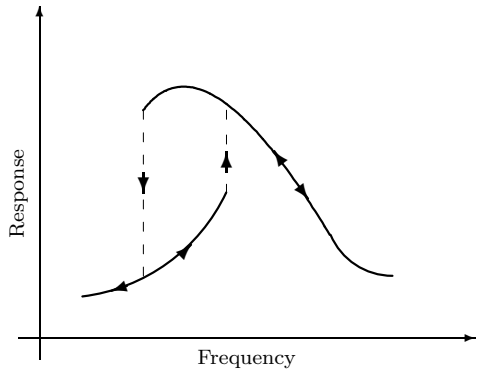

Figure 2.1: Jump resonance in a frequency response.

- *Chaos*: For stable linear systems, small differences in initial conditions can only cause small differences in the output. Nonlinear systems, however, can display a phenomenon called chaos, by which we mean that the system output is extremely sensitive to initial conditions. The essential feature of chaos is unpredictability of the system output. Chaos must be distinguished from random motion. In random motion, the system model or input contain uncertainty and, as a result, the time variation of the output cannot be predicted exactly (only statistical measures are available). On the other hand, in chaotic motion, the involved problem is deterministic, and there is little uncertainty in the system model, input, or initial conditions.

- *Jump phenomenon*: A jump phenomenon is illustrated in Figure 2.1. Here a nonlinear system frequency response that exhibits the jump phenomenon is shown, which is called jump resonance. Suppose that the nonlinear system input is a sinusoid of constant amplitude. Then, as the frequency of the input sinusoid is increased, a discontinuity (jump) occurs in the amplitude of the response. As the frequency of the input is deceased, again a jump occurs but at a different frequency.

2.1.2 Linearization

The linearization process of a nonlinear time-invariant system is basically as follows. Let $\overline{x}, \overline{u}$ be the states and inputs corresponding to an operating point for the nonlinear system model

$$\dot{x} = f(x, u). \tag{2.1}$$

Hence, suppose that a (small) perturbation occurs such that

$$x = \overline{x} + \Delta x, \quad u = \overline{u} + \Delta u, \tag{2.2}$$

and expand the function $f(x, u)$ in a Taylor series around the nominal values $\overline{x}, \overline{u}$, i.e.,

$$f(x, u) = f(\overline{x}, \overline{u}) + \frac{\partial f}{\partial x}\Big|_{\overline{x},\overline{u}} \Delta x + \frac{\partial f}{\partial u}\Big|_{\overline{x},\overline{u}} \Delta u + \text{higher-order terms}. \tag{2.3}$$

Since $\dot{x} = \dot{\overline{x}} + \Delta\dot{x}$, we have that

$$\dot{\overline{x}} + \Delta\dot{x} = f(\overline{x}, \overline{u}) + \frac{\partial f}{\partial x}\Big|_{\overline{x},\overline{u}} \Delta x + \frac{\partial f}{\partial u}\Big|_{\overline{x},\overline{u}} \Delta u + \text{higher-order terms.} \tag{2.4}$$

For sufficiently smooth f and small Δx and Δu, the higher-order terms are small compared to the linear terms and can be neglected. The same procedure can be applied to the output equations h.

Often the operating point is taken to be an equilibrium of the nonlinear system. By doing this, we have that $f(\overline{x}, \overline{u}) = 0$, and hence we approximate the nonlinear equations by linear ones and this leads to the approximating linear state-space model

$$\begin{aligned} \Delta\dot{x} &= A\Delta x + B\Delta u, \quad \Delta x(t_0) = \Delta x_0 \\ \Delta y &= C\Delta x + D\Delta u, \end{aligned} \tag{2.5}$$

where

$$A = \frac{\partial f}{\partial x}\Big|_{\overline{x},\overline{u}},\ B = \frac{\partial f}{\partial u}\Big|_{\overline{x},\overline{u}},\ C = \frac{\partial h}{\partial x}\Big|_{\overline{x},\overline{u}},\ D = \frac{\partial h}{\partial u}\Big|_{\overline{x},\overline{u}}. \tag{2.6}$$

Notice that if the nominal conditions $\overline{x}, \overline{u}$ constitute a *nominal trajectory* for the system, we have to linearize along a (given) nominal trajectory. Although it is possible to consider linearizations along time varying trajectories (Rugh 1996), we will restrict ourselves to the approximation around an equilibrium or operating point.

Example 2.1 (Scalar system)

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) \\ y &= x, \end{aligned}$$

where x is a scalar and $f(x) = x^2$. Suppose we want to approximate the nonlinear system in the operating point $\overline{x} = 1$. In that case, $\overline{x} = f(\overline{x}) = 1$, and it is easy to see that

$$f(x) \approx f(\overline{x}) + \frac{\partial f}{\partial x}\Big|_{\overline{x}=1} \Delta x = 1 + 2\Delta x.$$

This is illustrated in Figure 2.2a. Taking $\Delta x = x - \overline{x}$ yields the system $\Delta\dot{x} \approx 2\Delta x$. This system describes the behavior of x relative to the point $\overline{x}$. This is illustrated in Figure 2.2b.

Example 2.2 (A nonlinear differential equation)

Consider the nonlinear differential equation

$$\ddot{\gamma} + (1 + \gamma)\dot{\gamma} = u.$$

The state-space equations are determined by defining $x_1 = \gamma$ and $x_2 = \dot{\gamma}$, i.e.,

$$\begin{aligned} \dot{x}_1 &= x_2 = f_1(x_2) \\ \dot{x}_2 &= -x_1 x_2 - x_2 + u = f_2(x_1, x_2, u). \end{aligned}$$

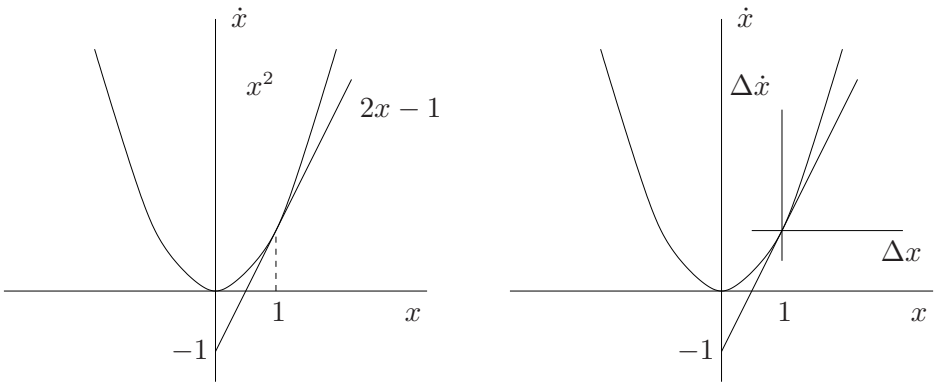

Figure 2.2: (a) Linearization of $f(x) = x^2$ in the operating point $(1, 1)$. (b) Local coordinates Δx with respect to the operating point $(1, 1)$.

Then, from (2.6) (verify!)

$$A = \left[\begin{array}{cc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{array}\right]_{\overline{x},\overline{u}} = \left[\begin{array}{cc} 0 & 1 \\ -x_2 & -x_1 - 1 \end{array}\right]_{\overline{x},\overline{u}} \text{ and } B = \left[\begin{array}{c} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{array}\right]_{\overline{x},\overline{u}} = \left[\begin{array}{c} 0 \\ 1 \end{array}\right].$$

With the system at the operating point $\overline{x} = \mathrm{col}(\overline{x}_1, \overline{x}_2), \overline{u}$, the linearized state-space equations are

$$\begin{aligned} \Delta\dot{x}_1 &= \Delta x_2 \\ \Delta\dot{x}_2 &= (-\overline{x}_2)\Delta x_1 - (\overline{x}_1 + 1)\Delta x_2 + \Delta u. \end{aligned}$$

Example 2.3 (Satellite) (Rugh 1996)

An Earth satellite of unit mass can be modeled as a point mass moving in a plane while attracted to the origin of the plane by an inverse square law force. It is convenient to choose polar coordinates, with r the radius from the origin of the mass and θ the angle from an appropriate axis. Assuming the satellite can apply force u_1 in the radial direction and u_2 in the tangential direction, as shown in Figure 2.3, the equations of motion have the form

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2 - \frac{\beta}{r^2} + u_1 \\ \ddot{\theta} &= \frac{1}{r}\left(-2\dot{r}\,\dot{\theta} + u_2\right), \end{aligned}$$

where β is a constant.

To construct a state-space system, let

$$x_1 = r, \quad x_2 = \dot{r}, \quad x_3 = \theta, \quad x_4 = \dot{\theta}.$$

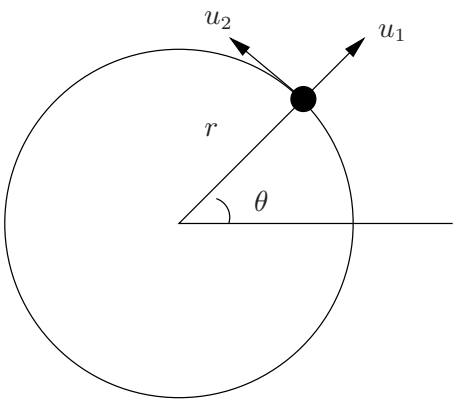

Figure 2.3: An Earth satellite in gravitational orbit.

If we take r and θ as outputs, the state-space system becomes

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_2 \\ x_1x_4^2 - \dfrac{\beta}{x_1^2} + u_1 \\ x_4 \\ \dfrac{-2x_2x_4}{x_1} + \dfrac{u_2}{x_1}\end{bmatrix}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}.$$

Assume that the input u_1 is constant: $u_1 = \overline{u}_1 \neq 0$ and that the input u_2 is identically zero: $u_2 = 0$. It is easy to verify that for these nominal inputs an equilibrium point of the system is given by

$$\overline{x}_1 = r_0, \quad \overline{x}_2 = 0, \quad \overline{x}_3 = \theta_0, \quad \overline{x}_4 = 0,$$

with $r_0 = (\beta/\overline{u}_1)^{1/2}$. Denoting the state equation as

$$\dot{x} = f(x,u),$$

we can derive

$$\frac{\partial f}{\partial x}(x,u) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ x_4^2 + 2\dfrac{\beta}{x_1^3} & 0 & 0 & 2x_1x_4 \\ 0 & 0 & 0 & 1 \\ \dfrac{2x_2x_4}{x_1^2} - \dfrac{u_2}{x_1^2} & -2\dfrac{x_4}{x_1} & 0 & -2\dfrac{x_2}{x_1}\end{bmatrix},$$

and

$$\frac{\partial f}{\partial u}(x,u) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{x_1} \end{bmatrix}.$$

Introducing the signals $\Delta u = u - \overline{u}$, $\Delta x = x - \overline{x}$, and $\Delta y = y - \overline{y}$, with

$$\overline{y} = \begin{bmatrix} r_0 \\ \theta_0 \end{bmatrix},$$

the linearized state-space system is given by

$$\begin{aligned} \Delta\dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2\beta/r_0^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Delta x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/r_0 \end{bmatrix} \Delta u \\ \Delta y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Delta x. \end{aligned}$$

2.1.3 Equilibrium Point Shifting

In linear systems analysis and design, we often transform the linear equations in such a way that the equilibrium point is the origin of the state-space. In most cases this will lead to a reduction of notational and analytical complexity, especially when we want to analyze the system's stability behavior. We can do the same thing for nonlinear systems, about a specific equilibrium point. Suppose that the equilibrium point of interest is x^*. Then, by introducing new variables

$$x' = x - x^*, \quad u' = u - u^*,$$

and substituting $x = x' + x^*$ and $u = u' + u^*$ into equation (2.1), a new set of equations with respect to the variables x' and u' are obtained

$$\dot{x} = f(x' + x^*, u' + u^*). \tag{2.7}$$

One can easily verify that there is a one-to-one correspondence between the solutions of (2.1) and those of (2.7), and that in addition, $x' = 0$, the solution corresponding to $x = x^*$, is an equilibrium point of (2.7). Therefore, instead of studying the behavior of equation (2.1) in the neighborhood of x^*, one can equivalently study the behavior of the equation (2.7) in the vicinity of the origin.

2.2 System Transfer Operator

A linear time-invariant (LTI) system may be described in terms of a *functional operator* relating the output y to the input u through a well-defined mathematical relationship. A system may be described operationally as

$$y(t) = H\{u(t)\}$$

with the interpretation that the output function $y(t)$ is the result of a mathematical operation on the input function $u(t)$. This mathematical operation is described by the *operator* H. The operator H is the *dynamic transfer operator* and forms an alternative description of the dynamics of the system.

An operator, such as H, represents a mathematical transformation on a system function. It is similar to an algebraic function in the sense that both transform an input within their domain to an output. Unlike a function, which maps instantaneous values of the input to the output, an operator maps an *input function* to an *output function*. At any time instant the output of an operator may depend not only on the current value of the input, but also on its complete time history.

Example 2.4 (Linear operators) (Rowell and Wormley 1997)

In Figure 2.4 two simple linear systems are shown. The output of each system can be related to its input by a dynamical transfer function operator.

The first example (a) is an electric voltage divider. The input to the system is the voltage $v_1(t)$ and the output is $v_2(t)$. The system can be described with the following operator:

$$v_2(t) = H\{v_1(t)\} = \frac{R_2}{R_1 + R_2} v_1(t)$$

The output of this system only depends on the current value of the input; the operator H is a *static operator*.

The second example (b) is a fluid tank with vertical walls. The input to the system is the flow $Q(t)$ and the output the pressure $P(t)$. The system can be described as follows:

$$P(t) = H\{Q(t)\} = \frac{1}{C} \int_0^t Q(\tau) d\tau + P(0)$$

The operator H is a *dynamic operator*, since it computes the integral of the input from time $t = 0$ to the present time. In other words, the output depends on the time history of the input.

We want to derive the transfer operator for an LTI state-space system given by

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \tag{2.8}$$

$$y(t) = Cx(t) + Du(t) \tag{2.9}$$

We start by introducing the *differential operator*. If a function $x(t)$ is differentiable and has the property that $x(t) = 0$ at time instant $t = 0$, then the differential operator, denoted by S, generates the time derivative of the $x(t)$ for time $t > 0$, that is,

$$S\{x(t)\} = \frac{dx(t)}{dt}$$

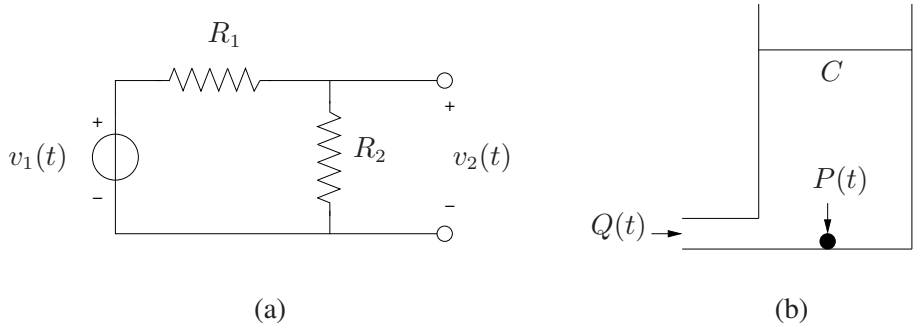

Figure 2.4: Two simple linear systems: a) an electric voltage divider, b) a fluid tank.

The differential operator S is a *linear operator*. A linear operator is defined as one that satisfies the principle of superposition. When the input $f(t)$ to a linear operator H is the sum of two component variables, that is, $f(t) = f_1(t) + f_2(t)$, the principle of superposition requires that

$$H\{f_1(t) + f_2(t)\} = H\{f_1(t)\} + H\{f_2(t)\}$$

The result of any linear operator H acting on the combined variable $f_1(t) + f_2(t)$ is the same as the sum of the effects of that operator acting on the two variables independently. Linear operators can be used in cascade, for example

$$\left.\begin{array}{l} y(t) = H_1\{z(t)\} \\ z(t) = H_2\{x(t)\} \end{array}\right\} \quad y(t) = H_1\{H_2\{x(t)\}\}$$

Often for ease of notation, only the braces around the original input quantity are retained:

$$H_1\{H_2\{x(t)\}\} = H_1H_2\{x(t)\}$$

Some other notation simplifications are:

$$\begin{aligned} H\{H\{x(t)\}\} &= H^2\{x(t)\} \\ H^n\{x(t)\} &= H\{H\cdots\{H\{H\{x(t)\}\}\}\cdots\} \end{aligned}$$

and

$$H_1\{x(t)\} + H_2\{x(t)\} = [H_1 + H_2]\{x(t)\}$$

Cascaded linear operators may be combined into an equivalent single operator using the standard rules of algebra. For example,

$$y(t) = [S+2][S+1]\{x(t)\} = [S^2 + 3S + 2]\{x(t)\}$$

implies that

$$y(t) = \frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t).$$

An inverse operator H^{-1} has the property

$$H^{-1}\{H\{x(t)\}\} = x(t)$$

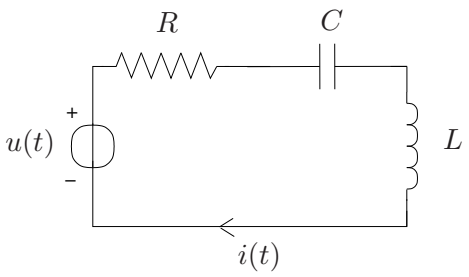

Figure 2.5: An RLC circuit driven by a voltage source $u(t)$.

Hence, H^{-1} restores the operation of H. The inverse of the differential operator S is the integral operator, denoted by S^{-1}, (or sometimes as $1/S$) given by

$$S^{-1}\{x(t)\} = \int_{-\infty}^{t} x(\tau)\, d\tau$$

The concept of an inverse operator is important in the description of dynamic models. Given the relation

$$H_2\{y(t)\} = H_1\{u(t)\}$$

with the assumption that H_2^{-1} exists, an explicit operational expression for the output $y(t)$ may be found by operating on both sides of the equation with the inverse operator H_2^{-1}; this result in

$$H_2^{-1}H_2\{y(t)\} = H_2^{-1}H_1\{u(t)\}$$

or equivalently

$$y(t) = H_2^{-1}H_1\{u(t)\}.$$

Example 2.5 (RLC circuit)

Consider the RLC circuit shown in Figure 2.5 driven by a voltage source $u(t)$. This system can be described by the differential equation

$$\frac{du(t)}{dt} = R\frac{di(t)}{dt} + \frac{i(t)}{C} + L\frac{d^2i(t)}{dt^2}. \tag{2.10}$$

This equation can be written in operator form as follows:

$$\left[S^2 + \frac{R}{L}S + \frac{1}{LC}\right]\{y(t)\} = \left[\frac{1}{L}S\right]\{u(t)\}.$$

Therefore, the transfer operator describing the system is

$$H(S) = \frac{\dfrac{1}{L}S}{S^2 + \dfrac{R}{L}S + \dfrac{1}{LC}}. \tag{2.11}$$

Now we will derive the transfer operator for the LTI system (2.8)–(2.9). In operational form our LTI system becomes:

$$\begin{aligned} S\{x(t)\} &= A\{x(t)\} + B\{u(t)\} \\ y(t) &= C\{x(t)\} + D\{u(t)\} \end{aligned}$$

The state equation can be rewritten as

$$[SI - A]\{x(t)\} = B\{u(t)\}$$

Assuming that the inverse operator $[SI - A]^{-1}$ exists, we get

$$x(t) = [SI - A]^{-1}B\{u(t)\}$$

and

$$\begin{aligned} y(t) &= C[SI - A]^{-1}B\{u(t)\} + D\{u(t)\} \\ &= \Big[C[SI - A]^{-1}B + D\Big]\{u(t)\} \end{aligned} \tag{2.12}$$

Using Cramer's rule for matrix inversion (see the Appendix A), the inverse operator can be written as

$$[SI - A]^{-1} = \frac{\text{adj}(SI - A)}{\det(SI - A)},$$

where det is the determinant, and adj is the adjugate; therefore

$$y(t) = \frac{C\text{adj}(SI - A)B + \det(SI - A)D}{\det(SI - A)}\{u(t)\}$$

This equation is often denote simply as

$$y(t) = H(S)u(t)$$

where the system transfer operator is given by

$$H(S) = \frac{C\text{adj}(SI - A)B + \det(SI - A)D}{\det(SI - A)} \tag{2.13}$$

For a system with m inputs, $u_1(t), u_2(t), \ldots, u_m(t)$ and p outputs $y_1(t), y_2(t), \ldots, y_p(t)$ is a matrix operator with p rows and m columns.

Example 2.6 (RLC circuit)

We will show that using the state-space description of the RLC circuit in Figure 2.5, we arrive at the same system transfer function as in Example 2.5 on page 34. The state-space description is

$$\begin{aligned} \frac{dx(t)}{dt} &= \begin{bmatrix} 0 & 1/L \\ -1/C & -R/L \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 0 & 1/L \end{bmatrix} x(t). \end{aligned}$$

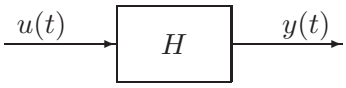

Figure 2.6: Block diagram of a system represented by the functional operator H.

We use equation (2.12) to derive the transfer function $H(S)$. We have

$$\begin{aligned}
[SI - A]^{-1} &= \begin{bmatrix} S & -1/L \\ 1/C & S + R/L \end{bmatrix}^{-1} \\
&= \frac{1}{S(S + R/L) + 1/(LC)} \begin{bmatrix} S + R/L & 1/L \\ -1/C & S \end{bmatrix}
\end{aligned}$$

where we used a well-known formula for the inverse of a 2×2 matrix (see Appendix A). The expression for $H(S)$ based on equation (2.12) becomes

$$\begin{aligned}
H(S) &= \begin{bmatrix} 0 & 1/L \end{bmatrix} \left(\frac{1}{S(S + R/L) + 1/(LC)} \begin{bmatrix} S + R/L & 1/L \\ -1/C & S \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{S(S + R/L) + 1/(LC)} \begin{bmatrix} 0 & 1/L \end{bmatrix} \begin{bmatrix} 1/L \\ S \end{bmatrix} \\
&= \frac{\frac{1}{L}S}{S^2 + \frac{R}{L}S + \frac{1}{LC}}.
\end{aligned}$$

which is equal to equation (2.11).

2.3 Block Diagrams

A system description can be depicted graphically in an *operational block diagram*. The system is represented as a processing block with inputs and outputs. An arrow pointing into the block represents an input, an arrow pointing out of the block represents an output. Block diagrams can be drawn at many levels of detail, from a complete description of the system in a single block to a detailed interconnection of several blocks. Figure 2.6 shows a block diagram in which the system is represented as one single block; it provides the lowest level of detail.

A more detailed block diagram can be obtained by representing the system by interconnections of primitive mathematical operators. The block diagram for any given system is not unique; a system may be represented by many equivalent block diagrams. All linear time-invariant systems may be represented by an interconnection of three primitive operators:

The constant scaling operator: The scaling operator multiplies the input function by a constant factor. It is denoted by the value of the constant, either numerically or symbolically. A scaling by

a yield the following input-output description:

$$y(t) = au(t).$$

The differential operator: If a function $u(t)$ is differentiable and has the property that $u(t) = 0$ at time instant $t = 0$, then the differential operator, denoted by S, generates the time derivative of the input $u(t)$ for time $t > 0$. The following input-output description corresponds to the differential operator:

$$y(t) = S\{u(t)\} = \frac{du(t)}{dt}.$$

Note that the output is only defined for $t > 0$.

The integral operator: The integral operator, denoted by S^{-1} is given by

$$y(t) = S^{-1}\{u(t)\} = \int_0^t u(\tau)\, d\tau + y(0),$$

where $y(0)$ takes into account the influence of the initial condition.

Figure 2.7 shows the block diagrams of these primitive operators.

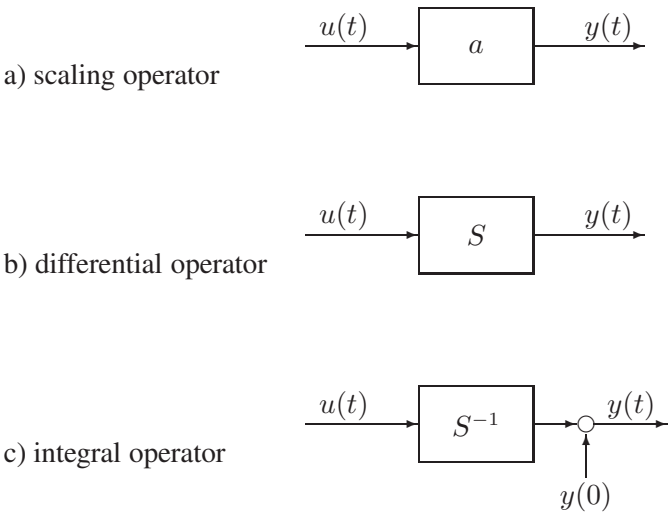

Figure 2.7: Block diagrams of the primitive linear operators for LTI systems.

While transformations on single variables are described by operators, two or more system variables are combined through arithmetic operations. The four operations for combining these variables are: addition, subtraction, multiplication and division. Figure 2.8 shows these operations as they appear in a block diagram. Note that to describe linear systems, only the addition and subtraction operations are needed. Subtraction can of course be regarded as a combination of a negative scaling operator and the addition operator.

In constructing block diagrams for an LTI system, the goal is to avoid differentiation blocks, if possible such blocks should be converted into integrator blocks.

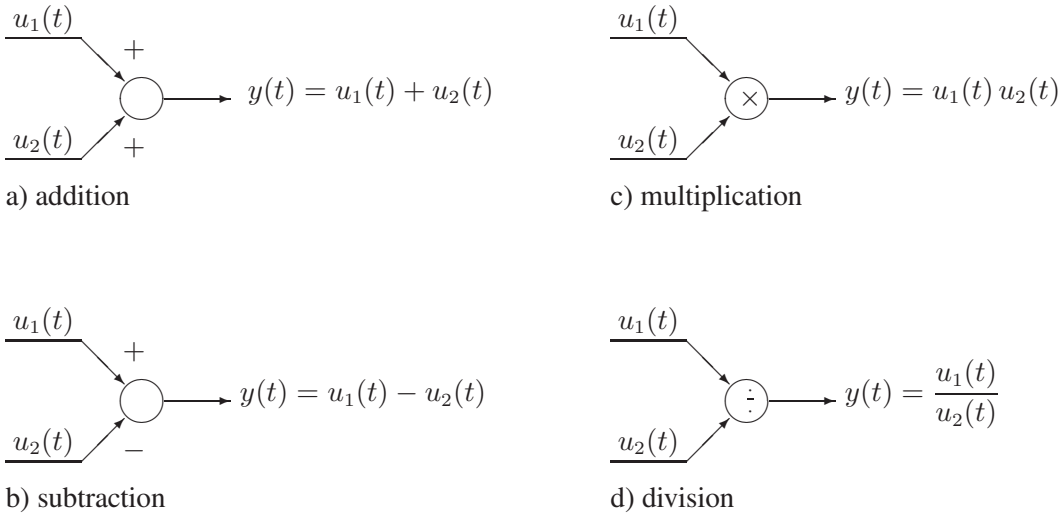

Figure 2.8: Arithmetic operations to combine system variables.

Another graphical representation of a system is a bondgraph, we are not going into the details of that here.

The construction of a block diagram will now be illustrated with an example.

Example 2.7 (Block diagram)

Consider the RLC circuit shown in Figure 2.9 together with its bondgraph. The equations describing the system are:

$$\begin{aligned} u(t) &= V_R(t) + V_C(t) + V_L(t) \\ i(t) &= \frac{1}{L}\int_0^t V_L(\tau)d\tau + i(0) \\ V_C(t) &= \frac{1}{C}\int_0^t i(\tau)d\tau + V_C(0) \\ V_R(t) &= iR \end{aligned}$$

Block diagrams of the elements are shown in Figure 2.10. The summation of the voltages can be represented as in Figure 2.11. The resulting block diagram is shown in Figure 2.12

The state-space description can also be represented as a block diagram. In such a diagram the state vector is represented as the direct result of vector integration. Figure 2.13 shows the block diagram of an LTI state-space system.

In drawing block diagrams for LTI state-space systems, the states are always taken as the outputs of integrator blocks. Hence, a system of order n has n integrators in its block diagram. The derivative of the state variables are the inputs to the integrator blocks, and each state equation expresses a derivative as a sum of weighted state variables and inputs.

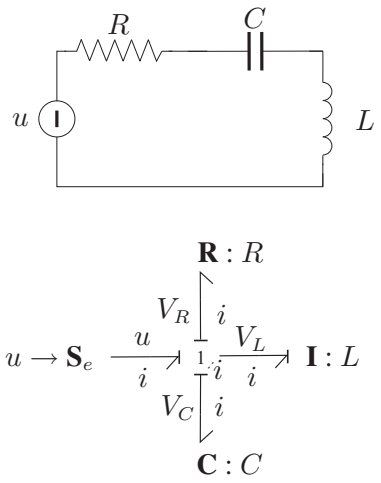

Figure 2.9: A series RLC circuit: circuit diagram (*top*) and bond graph (*bottom*).

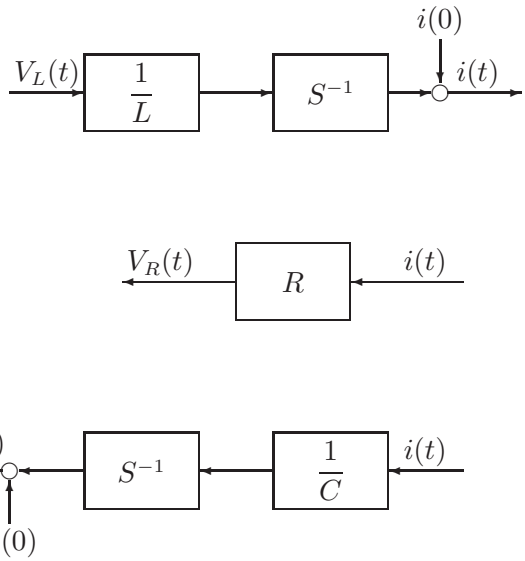

Figure 2.10: Block diagrams of the equations for the R, L and C element.

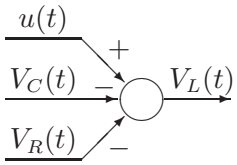

Figure 2.11: Summation of voltages.

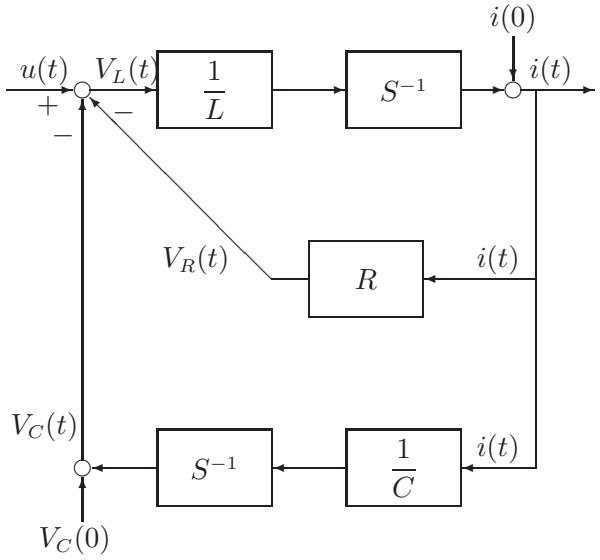

Figure 2.12: Block diagram of the RLC circuit.

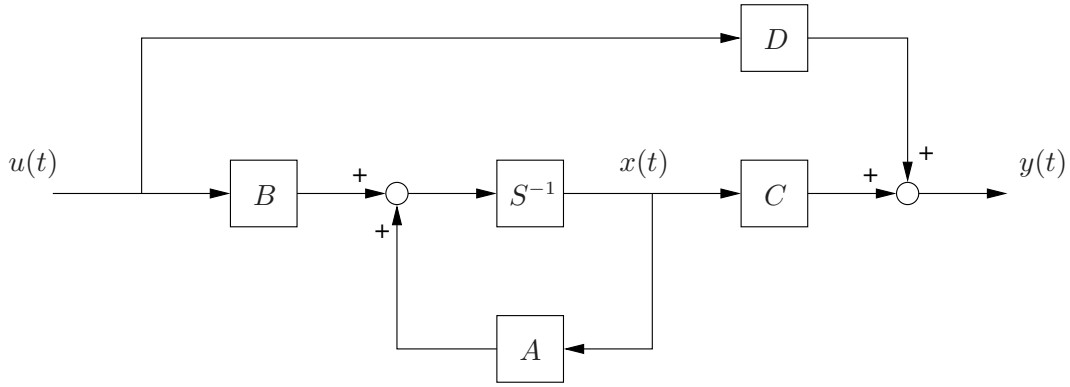

Figure 2.13: Vector block diagram for an LTI state-space system.

A block diagram of a state-space system of order n can be constructed with the following procedure:

Step 1: Draw n integrator blocks and assign a state variable to the output of each block.

Step 2: Draw a summing element at the input of each block.

Step 3: Use the state equations to connect state variables and inputs to the summing elements through scaling operators.

Step 4: Expand the output equations and sum the state variables and inputs through scaling operators to form the components of the output.

2.4 Input-Output Descriptions

Input-output descriptions are system descriptions that do not involve the state variables. In principle, state variables are not needed to describe the input-output behavior of the system, one can put everything into differential equations with only the input and output variables. Below we will first indicate how a state-space system can be converted into an input-output description. Subsequently we discuss one way to obtain a state-space description from an input-output description. A more elaborate discussion on transforming input-output descriptions into state-space descriptions is provided in the course book. We limit ourselves to single input, single output (SISO) systems. For systems with multiple inputs and outputs (MIMO), the procedure is more involved and will not be considered here (see for example Rugh 1996; Kailath 1996).

From State-Space to Input-Output Description

A state-space description of a single-input, single output LTI system can be converted into an input-output description using the transfer function operator. The first step is to derive the transfer function operator for the state-space system, as described in the Subsection 2.2. The second step is to use this transfer operator to derive a single differential equation between the output variable and the system input. The following example illustrates the outlined procedure.

Example 2.8 (From state-space to input-output description)

Consider the state-space system

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}-0.5 & 1.5\\ -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}2\\ 0\end{bmatrix}u\\ y &= \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + 2u.\end{aligned}$$

The determinant of $SI - A$ is

$$\det(SI - A) = S^2 - 1.5S + 0.5$$

and the adjugate of $SI - A$ is

$$\text{adj}\begin{bmatrix} S+0.5 & -1.5 \\ 1 & S-2 \end{bmatrix} = \begin{bmatrix} S-2 & 1.5 \\ -1 & S+0.5 \end{bmatrix}.$$

Further we compute

$$\begin{aligned} C\text{adj}(SI-A)B + \det(SI-A)D & \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} S-2 & 1.5 \\ -1 & S+0.5 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + (S^2 - 1.5S + 0.5)2 \\ &= (2S-6) + (2S^2 - 3S + 1) \\ &= 2S^2 - S - 5. \end{aligned}$$

So the related differential equation is given by:

$$(S^2 - 1.5S + 0.5)\{y(t)\} = (2S^2 - S - 5)\{u(t)\}$$

or

$$\frac{d^2y(t)}{dt^2} - 1.5\frac{dy(t)}{dt} + 0.5y(t) = 2\frac{d^2u(t)}{dt^2} - \frac{du(t)}{dt} - 5u(t).$$

From Input-Output to State-Space Description

An input-output description of an LTI system can be converted into a state-space system in many different ways. Reason being that the state-space representation is not unique. Below we describe one convenient state equation formulation that is widely used in control systems theory. Some other forms will be discussed in the course book.

The general form of the input-output description of an LTI system

$$a_N\frac{d^Ny(t)}{dt^N} + a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + a_0y(t) = b_M\frac{d^Mu(t)}{dt^M} + b_{M-1}\frac{d^{M-1}u(t)}{dt^{M-1}} + \cdots + b_0u(t),$$

can be converted into the following operational form:

$$[a_NS^N + a_{N-1}S^{N-1} + \cdots + a_1S + a_0]\{y\} = [b_MS^M + b_{M-1}S^{M-1} + \cdots + b_1S + b_0]\{u\}.$$

Without loss of generality it is assumed that the order of the polynomials operators that act on $u(t)$ and $y(t)$ are the same. Let this order be n, then the system can be described by

$$[a_nS^n + a_{n-1}S^{n-1} + \cdots + a_0]\{y(t)\} = [b_nS^n + b_{n-1}S^{n-1} + \cdots + b_0]\{u(t)\}.$$

The differential operators can be eliminated by operating with S^{-n} on both sides to obtain

$$[a_n + a_{n-1}S^{-1} + \cdots + a_0S^{-n}]\{y(t)\} = [b_n + b_{n-1}S^{-1} + \cdots + b_0S^{-n}]\{u(t)\}. \tag{2.14}$$

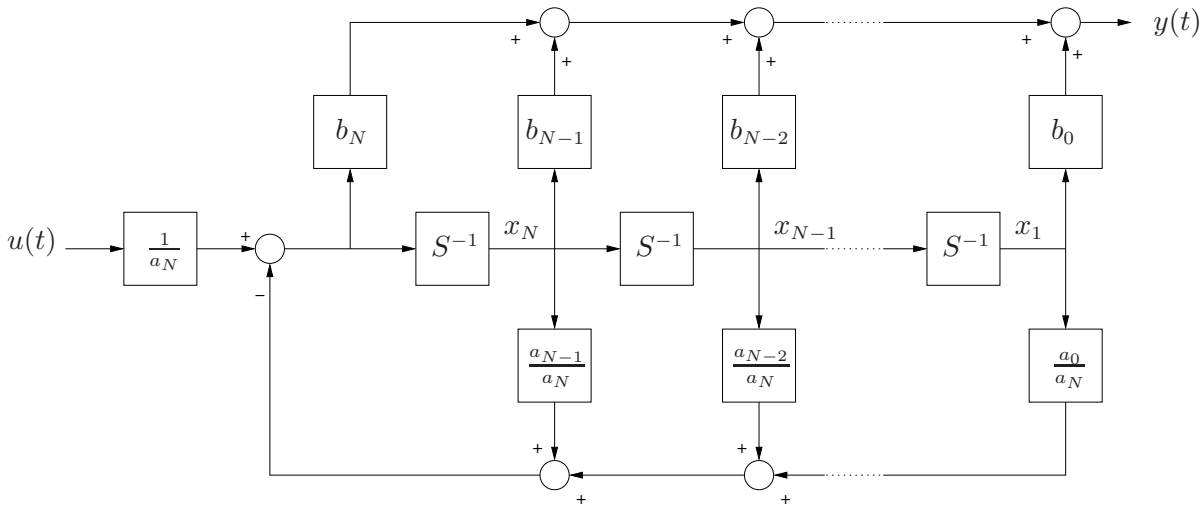

Figure 2.14: Block diagram of a system represented by a differential equation.

We define a dummy variable $z(t)$

$$z(t) = [a_n + a_{n-1}S^{-1} + \cdots + a_1 S^{-n+1} + a_0 S^{-n}]^{-1}\{u(t)\}, \tag{2.15}$$

then equation (2.14) becomes

$$y(t) = [b_n + b_{n-1}S^{-1} + \cdots + b_1 S^{-n+1} + b_0 S^{-n}]\{z(t)\}. \tag{2.16}$$

Equation (2.15) can be solved for $u(t)$, yielding

$$u(t) = [a_n + a_{n-1}S^{-1} + \cdots + a_1 S^{-n+1} + a_0 S^{-n}]\{z(t)\}.$$

By rearranging terms we obtain a *feedback structure*:

$$z(t) = \frac{1}{a_n}u(t) - [\frac{a_{n-1}}{a_n}S^{-1} + \cdots + \frac{a_0}{a_n}S^{-n}]\{z(t)\}. \tag{2.17}$$

The dummy variable $z(t)$ is specified in terms of the system input $u(t)$ and a weighted sum of successive integrations of itself. Figure 2.14 shows a block diagram representation of equations (2.16) and (2.17). A string of n cascaded integrator blocks with $z(t)$ defined as the input to the first block is used to generate the feedback terms $S^{-i}\{z(t)\}$, $i = 1, 2, \ldots, n$.

A set of state equations may be found from the block diagram by assigning the state variable $x_i(t)$ to the output of ith integrator. Because of the direct cascade connection of the integrators, the state equations take a very simple form. By inspection we get

$$x_i(t) = S^{i-n-1}\{z(t)\} \quad , \quad \text{for } i = 1, \ldots, n$$

then

$$z(t) = \frac{1}{a_n}u(t) - \frac{a_0}{a_n}x_1(t) - \cdots - \frac{a_{n-1}}{a_n}x_n(t)$$

Thus,

$$\begin{aligned} x_1(t) &= S^{-n}\{z(t)\} \\ x_2(t) &= S^{-n+1}\{z(t)\} \\ &\vdots \\ x_{n-1}(t) &= S^{-2}\{z(t)\} \\ x_n(t) &= S^{-1}\{z(t)\}, \end{aligned}$$

and therefore

$$\begin{aligned} \frac{dx_1(t)}{dt} &= S\{S^{-n}\{z(t)\}\} &&= x_2(t) \\ \frac{dx_2(t)}{dt} &= S\{S^{-n+1}\{z(t)\}\} &&= x_3(t) \\ &\vdots \\ \frac{dx_{n-1}(t)}{dt} &= S\{S^{-2}\{z(t)\}\} &&= x_n(t) \\ \frac{dx_n(t)}{dt} &= S\{S^{-1}\{z(t)\}\} &&= \frac{1}{a_n}u(t) - \frac{a_0}{a_n}x_1(t) - \cdots - \frac{a_{n-1}}{a_n}x_n(t). \end{aligned}$$

In matrix form these equations are

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0/a_n & -a_1/a_n & \cdots & -a_{n-2}/a_n & -a_{n-1}/a_n \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1/a_n \end{bmatrix} u(t).$$

or

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t).$$

This is a common form of state equations used in control systems theory and known as the *controller canonical* form. This form lead to a set of state variables that may not be physical variables within the system.

The corresponding output relationship is specified by equation (2.16). Noting that

$$x_i(t) = S^{-(n+1-i)}\{z(t)\}, \quad i = 1, 2, \ldots, n,$$

then

$$\begin{aligned} y(t) &= b_0 x_1(t) + \cdots + b_{n-1} x_n(t) + b_n z(t) \\ &= \left(b_0 - \frac{b_n a_0}{a_n}\right) x_1(t) + \cdots + \left(b_{n-1} - \frac{b_n a_{n-1}}{a_n}\right) x_n(t) + \frac{b_n}{a_n} u(t). \end{aligned}$$

Which in matrix-vector notation becomes

$$y(t) = \left[\begin{array}{ccc} \left(b_0 - \frac{b_n a_0}{a_n}\right) & \cdots & \left(b_{n-1} - \frac{b_n a_{n-1}}{a_n}\right) \end{array} \right] \left[\begin{array}{c} x_1(t) \\ \vdots \\ x_n(t) \end{array} \right] + \frac{b_n}{a_n} u(t),$$

or

$$y(t) = Cx(t) + Du(t).$$

Example 2.9 (From input-output to state-space description)

Consider a system described by the differential equation

$$16\frac{d^3y(t)}{dt^3} + 20\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + y(t) = 5\frac{d^2u(t)}{dt^2} - 7\frac{du(t)}{dt} + 2u(t).$$

The state-space description is given by

$$\begin{aligned} \left[\begin{array}{c} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{array} \right] &= \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/16 & -8/16 & -20/16 \end{array} \right] \left[\begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \end{array} \right] + \left[\begin{array}{c} 0 \\ 0 \\ 1/16 \end{array} \right] u(t) \\ y(t) &= \left[\begin{array}{ccc} 2 & -7 & 5 \end{array} \right] \left[\begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \end{array} \right]. \end{aligned}$$

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Appendix A

Review of Linear Algebra

This appendix summarizes some important concepts of linear algebra that are used in this course. It is based on a similar chapter in the book by Verhaegen and Verdult (2003). For a more elaborate introduction to the topics, the reader is referred to the book by Strang (1988).

A.1 Vectors

A *vector* is an array of real or complex numbers. Throughout this book we use $\mathbb{R}$ to denote the set of real numbers and $\mathbb{C}$ to denote the set of complex numbers. Vectors come in two flavors, *column vectors* and *row vectors*. The column vector that consists of the elements $x_1, x_2, \ldots, x_n$, $x_i \in \mathbb{C}$ will be denoted as x, i.e.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

In this book a vector denoted by a lower case character will always be a column vector. Row vectors are denoted as x^T, i.e.

$$x^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

Note that the row vector x^T is also called the *transpose* of the column vector x. The number of elements in a vector is called the *dimension* of the vector. A vector having n elements is referred to as an n-dimensional vector. We use the notation $x \in \mathbb{C}^n$ to denote an n-dimensional vector that has complex-valued elements. Obviously, an n-dimensional vector with real-valued elements is denoted as $x \in \mathbb{R}^n$. In this book, most vectors that we encounter will be real-valued, therefore in the remaining part of this chapter we will restrict ourselves to real-valued vectors. Note however that most results can be readily extended to complex-valued vectors

The multiplication of a vector $x \in \mathbb{R}^n$ with a scalar $\alpha \in \mathbb{R}$ is defined as

$$\alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

The sum of two vectors $x, y \in \mathbb{R}^n$ is defined as

$$x + y = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

The *inner product* of two vectors $x, y \in \mathbb{R}^n$ is equal to

$$x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Two vectors are called *orthogonal* if their inner product equals zero. The *norm* of a vector x, denoted as $||x||$, is the square root of the inner product of this vector with itself, i.e

$$||x|| = \sqrt{x^T x}$$

Any two vectors $x, y \in \mathbb{R}^n$ satisfy the *Cauchy-Schwartz inequality*

$$|x^T y| \leq ||x||||y||$$

where equality only holds if $x = \alpha y$ for some scalar $\alpha \in \mathbb{R}$.

A set of m vectors $x_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$ is *linearly independent*, if the linear combination of these vectors

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m \tag{A.1}$$

with $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$ can only become zero if $\alpha_i = 0$, for all $i = 1, 2, \ldots, m$. If there exist coefficients α_i, $i = 1, 2, \ldots, m$ which are not all equal to zero, but still make (A.1) zero, the vectors are *linearly dependent*. In this case at least one of them, say x_m, maybe expressed as a linear combination of the others, i.e.

$$x_m = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{m-1} x_{m-1}$$

for some scalars $\beta_i \in \mathbb{R}$, $i = 1, 2, \ldots, m-1$ with at least one of these scalars unequal to zero. Note that a set of m vectors in $\mathbb{R}^n$ must be linearly dependent if $m > n$. Let us illustrate linear independence with an example.

Example A.1 (Linear independence)

Consider the vectors

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad x_3 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

The vectors x_1 and x_2 are linearly independent, because

$$\alpha_1 x_1 + \alpha_2 x_2 = 0$$

implies $\alpha_1 = 0$ and $\alpha_1 + \alpha_2 = 0$ and thus $\alpha_1 = \alpha_2 = 0$. The vectors x_1, x_2 and x_3 are linearly dependent, because $x_3 = 2x_1 + x_2$.

The collection of all n-dimensional vectors forms a *vector space*. If every vector in a vector space $\mathcal{V}$ can be expressed as a linear combination of the vectors $x_i \in \mathcal{V}$, $i = 1, 2, \ldots, \ell$, these vectors x_i *span* the space. In other words, every vector $y \in \mathcal{V}$ can be written as

$$y = \sum_{i=1}^{\ell} \alpha_i x_i$$

with $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, \ell$. A *basis* for a vector space is a set of vectors that span the space and are linearly independent. An *orthogonal basis* is a basis in which every vector is orthogonal to all the other vectors contained in the basis. The number of vectors in a basis is referred to as the dimension of the space. Therefore, to span an n-dimensional vector space, we need at least n vectors.

A *subspace* $\mathcal{U}$ of a vector space $\mathcal{V}$, denoted as $\mathcal{U} \subset \mathcal{V}$ is a nonempty subset that satisfies two requirements:

1. $x + y \in \mathcal{U}$ for all $x, y \in \mathcal{U}$.
2. $\alpha x \in \mathcal{U}$ for all $\alpha \in \mathbb{R}$ and $x \in \mathcal{U}$.

By taking $\alpha = 0$ in the second rule, it follows that the zero vector belongs to every subspace. Note that every subspace is again a vector space. Two subspaces $\mathcal{U}$ and $\mathcal{W}$ of the same space $\mathcal{V}$ are orthogonal if every vector in $\mathcal{U}$ is orthogonal to every vector in $\mathcal{W}$. Given a subspace $\mathcal{U}$ of $\mathcal{V}$, the space of all vectors orthogonal to $\mathcal{U}$ is called the *orthogonal complement* of $\mathcal{U}$, denoted by $\mathcal{U}^{\perp}$.

Example A.2 (Vector spaces)

The vectors

$$x_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \qquad x_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

are linearly independent, and form a basis for the vector space $\mathbb{R}^3$. They do not form an orthogonal basis, since x_2 is not orthogonal to x_3 The vector x_1 spans a one-dimensional subspace of $\mathbb{R}^3$. This subspace is the orthogonal complement of the two-dimensional subspace of $\mathbb{R}^3$ spanned by x_2 and x_3, since x_1 is orthogonal to both x_2 and x_3.

A.2 Matrices

A vector is an array that has either one column or one row. An array that has m rows and n columns is called an m by n dimensional *matrix*, or shortly an $m \times n$ matrix. An example of an $m \times n$ matrix

is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The scalar a_{ij} is referred to as the (i,j)th entry or element of the matrix. In this book matrices will be denoted by upper case letters. We use the notation $A \in \mathbb{C}^{m\times n}$ to denote an $m \times n$ matrix with complex-valued entries and $A \in \mathbb{R}^{m\times n}$ to denote an $m \times n$ matrix with real-valued entries. Most matrices that we encounter will be real-valued so in this chapter we will mainly discuss real-valued matrices.

An n-dimensional column vector can of course be viewed as an $n\times 1$ matrix and an n-dimensional row vector can be viewed as a $1 \times n$ matrix. A matrix that has the same number of columns and rows is called a *square matrix*. Square matrices have some special properties which are described in section A.3.

The matrix $A \in \mathbb{R}^{m\times n}$ can be viewed as a collection of n column vectors of dimension m

$$A = \begin{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} & \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} & \cdots & \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \end{bmatrix}$$

or as a collection of m row vectors of dimension n

$$A = \begin{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \end{bmatrix} \\ \begin{bmatrix} a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \end{bmatrix}$$

This shows that we can partition a matrix into column vectors or row vectors. A more general partitioning is a partitioning into *submatrices*, for example

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

with $A_{11} \in \mathbb{R}^{p\times q}$, $A_{12} \in \mathbb{R}^{p\times(n-q)}$, $A_{21} \in \mathbb{R}^{(m-p)\times q}$, and $A_{22} \in \mathbb{R}^{(m-p)\times(n-q)}$ for certain $p < m$ and $q < n$.

The *transpose* of a matrix A, denoted by A^T is the $n \times m$ matrix that is obtained by interchanging the rows and columns of A, i.e.

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nm} \end{bmatrix}$$

It immediately follows that $(A^T)^T = A$.

A special matrix is the *identity matrix*, denoted by I. This is a square matrix that has only nonzero entries along its diagonal, and these entries are all equal to one, i.e.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{A.2}$$

If we multiply the matrix A with a scalar $\alpha \in \mathbb{R}$ we get a matrix which (i,j)th entry is given by αa_{ij} The sum of two matrices A and B of equal dimensions yields a matrix of the same dimensions which (i,j)th entry is given by $a_{ij}+b_{ij}$. Obviously, $(A+B)^T = A^T+B^T$. The product of the matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ yields an $m \times p$ matrix which (i,j)th entry is given by $\sum_{k=1}^{n} a_{ik}b_{kj}$. It is important to note that in general we have $AB \neq BA$. We also have $(AB)^T = B^T A^T$.

A matrix that has the property that $A^T A = I$ is called an *orthogonal matrix*. Hence, the column vectors a_i of an orthogonal matrix are orthogonal vectors that in addition satisfy $a_i^T a_i = 1$

The *rank* of the matrix A, denoted as $\mathrm{rank}(A)$, is defined as the number of linearly independent columns. It is easy to see that the number of linearly independent columns must equal the number of linearly independent rows. Hence $\mathrm{rank}(A) = \mathrm{rank}(A^T)$ and $\mathrm{rank}(A) \leq \min(m,n)$ if A is $m \times n$. The matrix A has *full rank* if $\mathrm{rank}(A) = \min(m,n)$.

With the definition of the rank of a matrix we can now define the *four fundamental subspaces* of a matrix. Consider the matrix $A \in \mathbb{R}^{m\times n}$ with $\mathrm{rank}(A) = r$. The columns of A span a vector space called the *column space* of the matrix A, denoted as $\mathrm{range}(A)$. The column space of A is a subspace of $\mathbb{R}^m$. The r linearly independent columns of A form a basis for its column space. Similarly, the rows of A span a vector space called the *row space* of the matrix A, which is a subspace of $\mathbb{R}^n$. The row space is denoted as $\mathrm{range}(A^T)$. The r linearly independent rows of A form a basis for its row space. Two other important subspaces associated with the matrix A are the *null space* and the *left null space*. The null space, denoted as $\ker(A)$ consists of all vectors $x \in \mathbb{R}^n$ that satisfy $Ax = 0$. The left null space, denoted as $\ker(A^T)$ consists of all vectors $y \in \mathbb{R}^m$ that satisfy $A^T y = 0$. To summarize, the four fundamental subspace of the matrix $A \in \mathbb{R}^{m\times n}$ are

$$\begin{aligned} \mathrm{range}(A) &= \{y \in \mathbb{R}^m : y = Ax \text{ for some } x \in \mathbb{R}^n\} \\ \mathrm{range}(A^T) &= \{x \in \mathbb{R}^n : x = A^T y \text{ for some } y \in \mathbb{R}^m\} \\ \ker(A) &= \{x \in \mathbb{R}^n : Ax = 0\} \\ \ker(A^T) &= \{y \in \mathbb{R}^m : A^T y = 0\} \end{aligned}$$

We have the following important relation between these subspaces

Theorem A.1 *(Strang 1988) Given a matrix $A \in \mathbb{R}^{m\times n}$. The null space of A is the orthogonal complement of the row space of A in $\mathbb{R}^n$, and the left null space of A is the orthogonal complement of the column space of A in $\mathbb{R}^m$.*

From this we see that the null space has dimension $n-r$, and the left null space has dimension $m-r$.

Example A.3 (Fundamental subspaces)

Given the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

It is easy to see that the first and second column of this matrix are linearly independent. The third and fourth column are scaled versions of the second column, and thus $\mathrm{rank}(A) = 2$. The first two columns are a basis for the two-dimensional column space of A. This column space is a subspace of $\mathbb{R}^3$. The left null space of A is the orthogonal complement of the column space in $\mathbb{R}^3$. Hence, it is a one-dimensional space, a basis for this space is the vector

$$y = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

because it is orthogonal to the columns of the matrix A. It is easy to see that indeed $A^T y = 0$. The row space of A is spanned by the first two rows. This two-dimensional space is a subspace of $\mathbb{R}^4$. The two-dimensional null space is the orthogonal complement of this space. The null space can be spanned by the vectors

$$x_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

because they are linearly independent and are orthogonal to the rows of the matrix A. It is easy to see that indeed $Ax_1 = 0$ and $Ax_2 = 0$.

A matrix $A \in \mathbb{R}^{m \times n}$ can be thought of as a mapping that transforms the space $\mathbb{R}^n$ into its column space. To see this, note that every vector $x \in \mathbb{R}^n$ can be decomposed into two orthogonal components: x_r which lies in the row space of A and x_n which lies in the null space of A. From $Ax = Ax_n + Ax_r = Ax_r$ we see that x_n is mapped to the zero vector in $\mathbb{R}^m$ and that x_r is mapped to a vector in the column space of A, since Ax_r is a linear combination of the columns of A.

A.3 Square Matrices

Square matrices have some special properties of which some important ones are described in this section. If a square matrix A has full rank, there exists a unique matrix A^{-1} called the inverse of A, such that

$$AA^{-1} = A^{-1}A = I$$

A square matrix that has full rank is often called an *invertible* or *nonsingular* matrix. We have $(A^T)^{-1} = (A^{-1})^T$. If A is a square orthogonal matrix, $A^T A = I$, and thus $A^{-1} = A^T$ and also $AA^T = I$. If A and B are invertible matrices then $(AB)^{-1} = B^{-1}A^{-1}$.

The *determinant* of a square $n \times n$ matrix, denoted as $\det(A)$ is a scalar that is defined recursively as

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

where A_{ij} is the $(n-1) \times (n-1)$ matrix that is formed by deleting the ith row and jth column of A. The determinant of a 1×1 matrix A (a scalar) is equal to the matrix itself, i.e. $\det(A) = A$. Some important properties of the determinant are the following:

1. $\det(A^T) = \det(A)$ for all $A \in \mathbb{R}^{n \times n}$.
2. $\det(\alpha A) = \alpha^n \det(A)$ for all $A \in \mathbb{R}^{n \times n}$ and $\alpha \in \mathbb{R}$.
3. $\det(AB) = \det(A)\det(B)$ for all $A, B \in \mathbb{R}^{n \times n}$.
4. $\det(A) \neq 0$ if and only if $A \in \mathbb{R}^{n \times n}$ is invertible.
5. $\det(A^{-1}) = \dfrac{1}{\det(A)}$ for all invertible matrices $A \in \mathbb{R}^{n \times n}$.

The determinant and the inverse of a matrix are related as follows

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)} \tag{A.3}$$

where the *adjugate* of A, denoted as $\text{adj}(A)$ is an $n \times n$ matrix with its (i,j)th entry given by $(-1)^{i+j}\det(A_{ji})$ and A_{ji} is the $(n-1) \times (n-1)$ matrix that is formed by deleting the jth row and ith column of A.

Example A.4 (Determinant and inverse of a 2×2 matrix)

Consider the matrix $A \in \mathbb{R}^{2 \times 2}$ given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The determinant of this matrix is given by

$$\det(A) = (-1)^{1+1} a_{11} \det(a_{22}) + (-1)^{2+1} a_{21} \det(a_{12}) = a_{11}a_{22} - a_{21}a_{12}$$

The *adjugate* of A is given by

$$\text{adj}(A) = \begin{bmatrix} (-1)^2 a_{22} & (-1)^3 a_{12} \\ (-1)^3 a_{21} & (-1)^4 a_{11} \end{bmatrix}$$

Hence, the inverse of A equals

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

It is easily verified that indeed $AA^{-1} = I$.

An *eigenvalue* of the square matrix $A \in \mathbb{R}^{n\times n}$ is a scalar $\lambda \in \mathbb{C}$ that satisfies

$$Ax = \lambda x \tag{A.4}$$

for some nonzero vector $x \in \mathbb{C}^n$. The vectors x that satisfy (A.4) on page 56 are called the *eigenvectors* of A. Note that even if the matrix A is real-valued, the eigenvalues and eigenvectors are often complex-valued. Since (A.4) on page 56 is equivalent to $(\lambda I - A)x = 0$, the eigenvectors of A are in the null space of the polynomial matrix $(\lambda I - A)$. To find the (nonzero) eigenvectors, λ should be such that the polynomial matrix $(\lambda I - A)$ is singular. This is equivalent to the condition

$$\det(\lambda I - A) = 0$$

The left hand side of this equation is a polynomial in λ of degree n and is often referred to as the *characteristic polynomial*. Hence, the eigenvalues of a matrix A are equal to the roots of the characteristic polynomial. The eigenvalues of a full rank matrix are all nonzero. This follows from the fact that a zero eigenvalue turns equation (A.4) into $Ax = 0$ which has no nonzero solution x if A has full rank. From this it also follows that a singular matrix A has $\mathrm{rank}(A)$ nonzero eigenvalues and $n - \mathrm{rank}(A)$ eigenvalues that are equal to zero. The eigenvalues of the matrix A can be related to its determinant. Let the eigenvalues of A be given by λ_i, $i = 1, 2, \ldots, n$ then

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n = \prod_{i=1}^{n} \lambda_i$$

The eigenvalues can also be related to the *trace*. The trace of a square matrix $A \in \mathbb{R}^{n\times n}$, denoted as $\mathrm{tr}(A)$ is the sum of its diagonal entries, i.e.

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$$

We have

$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \cdots \lambda_n = \sum_{i=1}^{n} \lambda_i$$

Example A.5 (Eigenvalues and eigenvectors)

Consider the matrix

$$\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

We have $\det(A) = -2$ and $\mathrm{tr}(A) = 1$. The eigenvalues of this matrix are computed as the roots of

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - 4 & 5 \\ -2 & \lambda + 3 \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2)$$

Hence, the eigenvalues are -1 and 2. We see that indeed $\det(A) = -1 \cdot 2$ and $\mathrm{tr}(A) = -1 + 2$. The eigenvector x corresponding to the eigenvalue of -1 is found by solving $Ax = -x$. Let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then it follows that

$$\begin{aligned} 4x_1 - 5x_2 &= -x_1 \\ 2x_1 - 3x_2 &= -x_2 \end{aligned}$$

or equivalently $x_1 = x_2$, thus an eigenvector corresponding to the eigenvalue -1 is given by

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly, it can be found that

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue 2.

An important property of the eigenvalues of a matrix is summarized in the next lemma.

Lemma A.1 *(Strang 1988) Suppose that the matrix $A \in \mathbb{R}^{n \times n}$ has a complex valued eigenvalue λ, with corresponding eigenvector x. Then the complex conjugate of λ is also an eigenvalue of A and the complex conjugate of x is the corresponding eigenvector.*

An important question is: when are the eigenvectors of a matrix linearly independent? The following lemma provides a partial answer to this question.

Lemma A.2 *(Strang 1988) The eigenvectors $x_1, x_2, \ldots, x_k$, corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, $k \leq n$ of the matrix $A \in \mathbb{R}^{n \times n}$ are linearly independent.*

Eigenvectors corresponding to repetitive eigenvalues can be linearly independent, but this is not necessary. This is illustrated in the next example.

Example A.6 (Linear dependent eigenvectors)

Consider the matrix

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

The two eigenvalues of this matrix are both equal to 3. Solving $Ax = 3x$ to determine the eigenvectors shows that all eigenvectors are a multiple of

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This example shows that a matrix that has repetitive eigenvalues can have a set of eigenvectors that are not linearly independent.

Now we will look at some special square matrices. A *diagonal matrix* is a matrix with only nonzero entries along its diagonal. An example of diagonal matrix is the identity matrix, see equation (A.2) on page 53. A diagonal matrix $A \in \mathbb{R}^{n\times n}$ is often denoted as $A = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$. Hence, $I = \text{diag}(1, 1, \ldots, 1)$.

An *upper triangular matrix* is a square matrix in which all the entries below the diagonal are equal to zero. A *lower triangular matrix* is a square matrix in which all the entries above the diagonal are equal to zero. Examples of such matrices are:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Triangular matrices have some interesting properties. The transpose of an upper triangular matrix is a lower triangular matrix and vice versa. The product of two upper (lower) triangular matrices is again upper (lower) triangular. The inverse of an upper (lower) triangular matrix is also upper (lower) triangular. Finally, the determinant of a triangular matrix $A \in \mathbb{R}^{n\times n}$ equals the product of its diagonal entries, i.e.

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

A square matrix that satisfies $A^T = A$ is called a *symmetric matrix*. The inverse matrix of a symmetric matrix is again symmetric.

Lemma A.3 *(Strang 1988) The eigenvalues of a symmetric matrix $A \in \mathbb{R}^{n\times n}$ are real and its eigenvectors corresponding to distinct eigenvalues are orthogonal.*

A square symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called *positive definite* if for all nonzero vectors $x \in \mathbb{R}^n$ it satisfies $x^T Ax > 0$. We often use $A > 0$ to denote that the matrix A is positive definite. A square symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called *positive semidefinite* if for all nonzero vectors $x \in \mathbb{R}^n$ it satisfies $x^T Ax \geq 0$. This is denoted as $A \geq 0$. By reversing the inequality signs we can define *negative definite* and *negative semidefinite* matrices in a similar way. A positive semidefinite matrix has eigenvalues λ_i that satisfy $\lambda_i \geq 0$. A positive definite matrix has only positive eigenvalues.

Example A.7 (Positive definite matrix)

Consider the symmetric matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

The eigenvalues of this matrix are 2 and 4. Therefore this matrix is positive definite.

A.4 Eigenvalue Decomposition

In this section we look at a useful matrix decompositions called the *eigenvalue decomposition*. Suppose that the matrix $A \in \mathbb{R}^{n\times n}$ has n linear independent eigenvectors. Let these eigenvectors be

the columns of the matrix V, then it follows that $AV = V\Lambda$ where Λ is a diagonal matrix with the eigenvalues of the matrix A along its diagonal. Since the eigenvectors are assumed to be linearly independent, the matrix V is invertible and we have the following important result.

Theorem A.2 (Eigenvalue decomposition) *(Strang 1988) Any matrix $A \in \mathbb{R}^{n\times n}$ that has n linear independent eigenvectors can be decomposed as*

$$A = V\Lambda V^{-1}$$

where $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix containing the eigenvalues of the matrix A, and the columns of the matrix $V \in \mathbb{R}^{n\times n}$ are the corresponding eigenvectors.

Note that not all matrices have n linear independent eigenvectors. The eigenvalue decomposition can only be performed for matrices with n independent eigenvectors, these matrices are called *diagonalizable*, since $V^{-1}AV = \Lambda$. With lemma A.2 on page 57 it follows that any matrix with distinct eigenvalues is diagonalizable. The converse is not true: there exist matrices with repeated eigenvalues that are diagonalizable. An example is the identity matrix I, it has all eigenvalues equal to 1, but does have n linearly independent eigenvectors if we take $V = I$.

For symmetric matrices we have a theorem that is somewhat stronger than theorem A.2 and is related to lemma A.3 on page 58.

Theorem A.3 (Spectral theorem) *(Strang 1988) Any symmetric matrix $A \in \mathbb{R}^{n\times n}$ can be diagonalized by an orthogonal matrix*

$$Q^T AQ = \Lambda$$

where $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix containing the eigenvalues of the matrix A and the columns of the matrix $Q \in \mathbb{R}^{n\times n}$ form a complete set of orthogonal eigenvectors, such that $Q^TQ = I$.

Appendix B

Summary of Continuous-Time System Descriptions