Geometry 2023
Exam, Tuesday 11 April, 08:30-10:30

- Below you can find the exam questions. There are 4 questions summing up to 85 points; you get extra 15 points for a clear writing of solutions.
- You may consult two A4 pages handwritten or typeset by you for formulas, results, etc. treated in this course. Other materials are not allowed.
- When handing in your solutions, please do not forget to write your name and student number on the envelope. Good luck!

Solutions

1. $\boxed{5+15 = 20 \text{ pts}}$ Consider the following parametric equation of a tractrix:

$$\gamma = \big(t - \tanh t, \frac{1}{\cosh t}\big) : \mathbb{R} \to \mathbb{R}^2.$$

   i) Determine whether $\gamma = \gamma(t)$ is an everywhere regular curve;

   ii) Compute the evolute of γ and state its relation to Huygens's principle.

i) First observe that γ is a smooth (even real-analytic) curve. The derivative of γ is given by $\gamma'(t) = \big(\frac{\sinh^2 t}{\cosh^2 t}, -\frac{\sinh t}{\cosh^2 t}\big)$, which is zero iff $t = 0$.

Hence γ is singular at $t = 0$ and regular for $t \neq 0$.

ii) Recall that the the evolute of a parametrised curve $x = x(t), y = y(t)$ is given by the general formula

$$C(t) = \gamma(t) + \rho(t)n(t),$$

where $\rho(t)$ stands for the curvature radius and $n(t)$ for the unit normal to $\gamma(t)$ (directed so that $(\gamma'(t), n(t))$ gives a positive basis of $\mathbb{R}^2$). Using Cartesian coordinates,

$$C(t) = (x(t), y(t)) + \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}(-\dot{y}, \dot{x}),$$

In our case, $x(t) = t - \tanh t$ and $y(t) = \frac{1}{\cosh t}$. A direct computation yields $\rho(t) = \sinh t$ and the following equation for the evolute:

$$C(t) = (t, \cosh t).$$

According to Huygens's principle, the evolute $C = C(t)$ is given by the set of the singular points

$$\{\gamma(t) + an(t) \mid \gamma'(t) + an'(t) = 0, \quad a, t \in \mathbb{R}\}$$

of the wave-fronts $\gamma(t) + an(t)$.

2. $\boxed{10+10+15 = 35 \text{ pts}}$ Consider the torus of revolution T^2 in $\mathbb{R}^3$ given by parametric equations

$$\begin{aligned} x &= (2 + \cos\psi)\cos\varphi, \\ y &= (2 + \cos\psi)\sin\varphi, \\ z &= \sin\psi, \quad (\varphi, \psi) \in [0, 2\pi]^2. \end{aligned}$$

i) Prove that the meridians $\varphi = \text{const}$ and the two parallels $\psi = 0, \pi$ are geodesics of T^2.

ii) Prove that there exists a geodesic triangle on the torus T^2 for which the angle sum is a) strictly greater than π and b) strictly less than π.

iii) Using part i), subdivide T^2 into 4 geodesic triangles. Deduce that $\frac{1}{2\pi}\int_{T^2} K \mathrm{d}S = 0$, where $K\colon T^2 \to \mathbb{R}$ is the Gaussian curvature of T^2 and $\mathrm{d}S$ is the area element.

Hint: local Gauss-Bonnet theorem. You may use without proof that every pair of points on T^2 is connected by a geodesic.

i) Recall that the geodesic equations for 2-surfaces in $\mathbb{R}^3$ have the form

$$\gamma''(s) \mid\mid n(\gamma(s)),$$

where $n(\gamma(s))$ is orthogonal to the surface at the point $\gamma(s)$ and s stands for (a constant multiple of) the arc-length parameter. But the meridians $\varphi = \text{const}$ and parallels $\psi = 0, \pi$ are plane curves in $\mathbb{R}^3$ and the normal vector $n(\gamma(s))$ lies in their plane. It follows that $\gamma''(s) \mid\mid n(\gamma(s))$.

An alternative solution is based on the following symmetry argument. The mapping $(\varphi, \psi) \mapsto (2\varphi_0 - \varphi, \psi)$, induced by the reflection in the 2-plane in $\mathbb{R}^3$ spanned by the vector $(\cos\varphi_0, \sin\varphi_0)$ and the z-axis, is a smooth isometry of T^2. Observe that the meridian $\varphi = \varphi_0$ is fixed under this isometry. Since $\varphi = \varphi_0$ are parametrised by the arc-length parameter ψ, the existence and uniqueness theorem applied to the geodesic equations implies $\varphi = \varphi_0$ are geodesics of T^2.

Similarly, the reflection $(x, y, z) \mapsto (x, y, -z)$ fixes the parallels $\psi = 0$ and $\psi = \pi$. Since φ is a multiple of the arc-length parameter for these parallels, we get that $\psi = 0, \pi$ are indeed geodesics of T^2.

ii) First recall that the Gaussian curvature of the torus is positive for $\psi \in (-\pi/2, \pi/2)$ mod 2π and negative for $\psi \in (\pi/2, 3\pi/2)$. This follows since the points of T^2 are elliptic in the first case an hyperbolic in the second. It then suffices to take a small geodesic triangle on T^2 entirely contained in the region $\psi \in (-\pi/2, \pi/2)$ for part a), and $\psi \in (\pi/2, 3\pi/2)$ for part b). The result then follows from the local Gauss-Bonnet theorem.

iii) From part i) we get that the meridian $\varphi = 0$ and the parallels $\psi = 0, \pi$ are geodesics. They subdivide the torus into 2 geodesic quadrangles. These quadrangles can in turn be divided into 4 geodesic triangles by connecting the opposite vertices with geodesics. By the local Gauss-Bonnet theorem, for each of these individual triangles ABC,

$$\int_{ABC} K \mathrm{d}S = \alpha + \beta + \gamma - \pi.$$

But the total angle-sum of the 4 triangles is 4π, since there are 2 vertices in total. Hence

$$\int_{T^2} K \mathrm{d}S = 4\pi - 4\pi = 0.$$

3. $\boxed{8+7 = 15\text{pts}}$ Consider the affine 3-space $\mathbb{Q}^3$, where $\mathbb{Q}$ is the field of rational numbers. Let A, B, C, and A', B', C' be 6 distinct points in $\mathbb{Q}^3$. Assume that the line AB is parallel to $A'B'$, BC is parallel to $B'C'$, and AC is parallel to $A'C'$. Prove that[1]

   i) If two of the lines AA', BB', and CC' intersect, then these three lines are concurrent (i.e., intersect at a single point);

   ii) If two of the lines AA', BB', and CC' are parallel, then all three lines are parallel (i.e., AA', BB', and CC' intersect at infinity).

This is a 3-dimensional Desargues's theorem, the proof of which is essentially the same as in the 2-dimensional case.

i) First assume that AA' and BB' intersect at a single point O. Then

$$\overrightarrow{OB'} = \lambda \overrightarrow{OB}, \quad \text{where} \quad \lambda = \frac{\overrightarrow{OA'}}{\overrightarrow{OA}},$$

by a theorem of Thales. Consider the scaling $P \mapsto O + \lambda \overrightarrow{OP}$ and let the point $C'' = O + \lambda \overrightarrow{OC}$. Since scalings map parallel lines to parallel lines, we have on the one hand that C'' belongs

[1] **Here we need to assume that A, B, C are not collinear, so that they form a proper triangle, and that no two of the lines AA', BB', CC' coincide**. You get full points for attempting both parts of the exercise.

to the affine line $A'C'$, and on the other hand that it belongs to $B'C'$. Since the latter lines intersect in the single point C', we must have $C'' = C'$. (Indeed, A, B, C being non-collinear implies that so are A', B', C', hence the lines $A'C'$ and $B'C'$ are distinct.) It follows that O is the intersection point of AA', BB', and CC'.

ii) Now assume AA' and BB' are parallel. Since AA' and BB' are also disjoint, for a vector $u \in \mathbb{Q}^3 \setminus \{0\}$, $A' = A + u$ and $B' = B + u$. Let the point $C'' = C + u$. Then, on the one hand, we have that C'' belongs to the affine line $A'C'$, and on the other hand it belongs to the affine line $B'C'$. Since the latter lines intersect in the single point C', we must have $C'' = C'$. It follows that AA', BB', and CC' are parallel.

4. $\boxed{\text{15pts}}$ Let (e_1, e_2) be a basis of $\mathbb{C}^2$ and $P(\mathbb{C}^2)$ be realised as the projective completion of the affine line $\{ze_1 + e_2 \mid z \in \mathbb{C}\}$ in $\mathbb{C}^2$. Similarly, let (e^1, e^2) be the dual basis of $(\mathbb{C}^2)^*$ and $P((\mathbb{C}^2)^*)$ be the projective completion of the affine line $\{ve^1 + e^2 \mid v \in \mathbb{C}\}$. Let $A \colon \mathbb{C}^2 \to \mathbb{C}^2$ be a linear isomorphism and $A^* \colon (\mathbb{C}^2)^* \to (\mathbb{C}^2)^*$ be its dual. Let f and f^* be the corresponding projective transformations of $P(\mathbb{C}^2)$ and $P((\mathbb{C}^2)^*)$. Compute f^* in the coordinate v when $f = z + 1$ is a translation.

First note that the translation $f = z + 1$ can be represented in matrix form as

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{i.e.} \quad f = \frac{z+1}{0 \cdot z + 1}.$$

This means that the linear isomorphism of $\mathbb{C}^2$ given by A induces $f = z + 1$. Hence the dual map f^* is induced by the linear isomorphism $A^* \colon (\mathbb{C}^2)^* \to (\mathbb{C}^2)^*$ given by

$$A^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We conclude that f^* can be written as

$$f^* = \frac{v}{v+1}.$$

End of exam