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Answers

1. 30 points **(Riemannian metric on models of the hyperbolic plane)** Let

$$\mathbb{D} = \{z \in \mathbb{C} \mid\mid z \mid< 1\} \quad and \quad \mathbb{H} = \{z = (x + iy) \in \mathbb{C} \mid y > 0\} \tag{1}$$

be the open unit disk and the upper-half plane in the complex plane C, respectively. Consider the so-called Cayley transform

$$f = \frac{z - i}{z + i} : \mathbb{C} \to \mathbb{C} \tag{2}$$

a) Show that f gives a complex-analytic diffeomorphism between $\mathbb{H}$ and $\mathbb{D}$, i.e. the restriction of f to $\mathbb{H}$ is a complex-analytic bijection between $\mathbb{H}$ and $\mathbb{D}$ with a complex analytic inverse.

According to the definition of homeomorphism, in order to determine if f is a complex-analytic diffeomorphism, we should check both differentiable map bijection f and its inverse f^{-1} are differentiable.

$$f = \frac{z - i}{z + i} \tag{3}$$

$$\dot{f} = -\frac{z - i}{(z + i)^2} + \frac{1}{z + i} \tag{4}$$

$$f^{-1} = g = -\frac{i(z + 1)}{z - 1} \tag{5}$$

$$\dot{g} = -\frac{i}{z - 1} + \frac{i(1 + z)}{(z - 1)^2} \tag{6}$$

When analysing f we realise that for $z = -i$, there is a division by 0. This result if possible for values $r\cos(\theta) + ir\sin(\theta) = -i$ which means that $\sin(\theta) = -1$ and $r = 1$. However, even when $\theta = --90 + 2k\pi$, according to the definition of $\mathbb{D}$ is always $\mid z \mid< 1$, so it will only approximate 1. Similarly, if $z = 1$, then g would have a division by 0, but this wouldn't be possible for the same reasons given.

To further prove these are differentiable we make use of the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{4xy + 4x}{(x^2 + y^2 + 2y + 1)^2} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{2y^2 + 4y - 2x^2 + 1}{(x^2 + y^2 + 2y + 1)^2} = -\frac{\partial v}{\partial x}$$

b) Prove that the pull-back of the Riemannian metric

$$G(u, v) = \frac{4(du^2 + dv^2)}{(1 - (u^2 + v^2))^2}, z = u + iv \tag{7}$$

and on $\mathbb{D}$ under f has the form

$$(f * G)(x, y) = \frac{dx^2 + dy^2}{y^2} \tag{8}$$

and get the $f(z)$ in terms of

$$f(z) = u(x, y) + iv(x, y) = \frac{x + i(y - 1)}{x + i(y + 1)} = \frac{x + i(y - 1)}{x + i(y + 1)} \frac{x - i(y + 1)}{x - i(y + 1)} = \frac{x^2 + y^2 - 1 - 2ix}{x^2 + (y + 1)^2} \tag{9}$$

$$\Im f(x, y) = v(x, y) = \frac{-2x}{x^2 + (y + 1)^2} \tag{10}$$

$$\Re f(x, y) = u(x, y) = \frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2} \tag{11}$$

$$f(z) = \frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2} + i\frac{-2x}{x^2 + (y + 1)^2} \tag{12}$$

and calculate in terms of x and y using mathematica's `FULLSIMPLIFY`

$$1-u^2-v^2=\frac{4y}{x^2+(1+y)^2} \tag{13}$$

$$du=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=\frac{4x(1+y)}{(x^2+(1+y)^2)^2}dx+\frac{-2x^2+2(1+y)^2}{x^2+(1+y)^2)^2} \tag{14}$$

$$dv=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=\frac{2(x^2-(1+y)^2)}{(x^2+(1+y)^2)^2}dx+\frac{4x(1+y)}{(x^2+(1+y)^2)^2}dy \tag{15}$$

$$du^2=\frac{4x^2(1+y)^2}{x^2+(1+y)^2)^4}dx^2+\frac{16x^2(1+y)^2}{x^2+(1+y)^2)^4}dy^2+\frac{16x(1+y)(x^2-(1+y)^2)}{x^2+(1+y)^2)^4}dxdy \tag{16}$$

$$dv^2=\frac{16x^2(1+y)^2}{x^2+(1+y)^2)^4}dx^2+\frac{-2x^2+2(1+y)^2)^2}{(x^2+(1+y)^2)^4}dy^2+\frac{8x(1+y)(-2x^2+2(1+y)^2)}{(x^2+(1+y)^2)^4}dxdy \tag{17}$$

$$du^2+dv^2=(\frac{4x^2(1+y)^2}{x^2+(1+y)^2)^4}+\frac{16x^2(1+y)^2}{x^2+(1+y)^2)^4})dx^2 \tag{18}$$

$$+(\frac{16x^2(1+y)^2}{x^2+(1+y)^2)^4}+\frac{-2x^2+2(1+y)^2)^2}{(x^2+(1+y)^2)^4})dy^2 \tag{19}$$

$$+(\frac{16x(1+y)(x^2-(1+y)^2)}{x^2+(1+y)^2)^4}+\frac{8x(1+y)(-2x^2+2(1+y)^2)}{(x^2+(1+y)^2)^4})dxdy \tag{20}$$

$$=\frac{4}{(x^2+(1+y)^2)^2}dx^2+\frac{4}{(x^2+(1+y)^2)^2}dy^2=\frac{4dx^2+4dy^2}{(x^2+(1+y)^2)^2} \tag{21}$$

and with that we can start rewriting the pullback

$$(f*G)(x,y)=\frac{4(du^2+dv^2)}{(1-(u^2+v^2))^2}=\frac{\frac{16(dx^2+dy^2)}{(x^2+(1+y)^2)^2}}{\left(\frac{4y}{x^2+(1+y)^2}\right)^2}=\frac{\frac{16(dx^2+dy^2)}{(x^2+(1+y)^2)^2}}{(\frac{(4y)^2}{(x^2+(1+y)^2)^2}}=\frac{16(dx^2+dy^2)}{(4y)^2} \tag{22}$$

$$=\frac{dx^2+dy^2}{y^2} \tag{23}$$

c) Conclude that H and D with these metrics are isometric.

They are isometric because when we make the pullback $g^*\circ f^*=(f\circ g)^*\equiv$ chain rule for change of variables

2. $\boxed{\text{20 points}}$ **(Gaussian curvature of Plücker's conoid)** Consider the following surface in affine Euclidean 3-space

$$M^2=(x,y,z)\in\mathbb{R}^3|z(x^2+y^2)=xy \tag{24}$$

a)Prove that M^2 is a ruled surface. Determine at which points is this surface regular. Is M^2 orientable?

By using cylindrical coordinates in space, we can write the above function into parametric equations $f(u,v)=(x=v\cos u, y=v\sin u, z=\sin 2u)$.

First, a 2-surface in $\mathbb{R}^3$ is ruled if exists a curve $\gamma=\gamma(t), t\in(a,b)$ or $[a,b]$ and a vector-valued function $h(t)$ with some range as parameter t, and $M^2=\gamma(t)+s\vec{h}(t)\mid t\in(a,b), s\in R$.

Based on this definition, let us consider the curve $\gamma(v)$ parametrised by $u\mapsto(0,0,\sin 2u)$, i.e. the z-axis. At any point $p=(0,0,sin2u_0)$ on this curve, we associate a vector $w(p)=(\cos u_0,\sin u_0, sin2u_0)$. Then $w(p)-p$ defines a straight line given by $y\sin u_0+x\cos u_0=0$. We can extend this line in the x- and y-directions by considering $v(p)=(v\cos u_0, v\sin u_0, sin2u_0)$ for $v\in R$. Thus, the lines extending from p draws the surface, it is a ruled surface.

Now, let's determine the points in which this surface is regular. For that we use definition 2.2 from the textbook ([1]), which states that a geometric surface Σ defined by a parametrisation $f:U\to E$ is said to be regular if the differential of f at any point of U has rank 2. So let's calculate the differential

$$df_{(u,v)} = \begin{pmatrix} -vsin(u) & cos(u) \\ vcos(u) & sin(u) \\ 2cos(2u) & 0 \end{pmatrix}$$

we can observe that it has rank 2 because for the column vectors n, m there is no constants a, b such that $a\vec{n_u} + b\vec{m_v} = 0$.

Let $n = (u, v)^T$, define $d_f \cdot n$ as

$$\begin{pmatrix} -vsin(u) & cos(u) \\ vcos(u) & sin(u) \\ 2cos(2u) & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -vusin(u) + vcos(u) \\ uvcos(u) + vsin(u) \\ 2ucos(2u) \end{pmatrix} \tag{25}$$

Since $df \cdot n \neq \mathbf{0}$, we can assume that the surface M^2 is orientable.

b) Compute the Gaussian curvature of Plücker's conoid at its regular points. The Gaussian curvature can be calculated as the division between the determinant of the second and first fundamental form.

$$I = \begin{pmatrix} \frac{\delta \vec{f}}{\delta u}, \frac{\delta \vec{f}}{\delta u} & \frac{\delta \vec{f}}{\delta u}, \frac{\delta \vec{f}}{\delta v} \\ \frac{\delta \vec{f}}{\delta v}, \frac{\delta \vec{f}}{\delta u} & \frac{\delta \vec{f}}{\delta v}, \frac{\delta \vec{f}}{\delta v} \end{pmatrix} = \begin{pmatrix} 2 + v^2 + 2\cos(4u) & 0 \\ 0 & 1 \end{pmatrix} \tag{26}$$

$$det(I) = \frac{1}{2 + v^2 + 2\cos(4u)} \tag{27}$$

$$n(u, v) = \frac{\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}}{||\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}||} = \frac{(-2\cos(2u)\sin(u), 2\cos(u)\cos(2u), -v)}{\sqrt{v^2 + 2 + 2\cos(4u)}} \tag{28}$$

$$II = \begin{pmatrix} \langle \vec{f_{xx}}, \vec{n} \rangle & \langle \vec{f_{xy}}, \vec{n} \rangle \\ \langle \vec{f_{yx}}, \vec{n} \rangle & \langle \vec{f_{yy}}, \vec{n} \rangle \end{pmatrix} = \begin{pmatrix} \frac{4v\sin(2u)}{\sqrt{2+v^2+2\cos(4u)}} & \frac{2\cos(2u)}{\sqrt{2+v^2+2\cos(4u)}} \\ \frac{2\cos(2u)}{\sqrt{2+v^2+2\cos(4u)}} & 0 \end{pmatrix} \tag{29}$$

$$= \frac{1}{\sqrt{2 + v^2 + 2\cos(4u)}} \begin{pmatrix} 4v\sin(2u) & 2\cos(2u) \\ 2\cos(2u) & 0 \end{pmatrix} \tag{30}$$

$$K = \frac{det(II)}{det(I)} = -\frac{4\cos(2u)^2}{(2 + v^2 + 2\cos(4u))^2} \tag{31}$$

3. $\boxed{\text{35 points}}$ **(Geodesics)** Consider the two models of hyperbolic geometry given in exercise 1 (the upper-half plane $\mathbb{H}$ and the Poincaré disk $\mathbb{D}$ models).

a) Write down the geodesic equations for these models

For the Poincaré disk we get the metric from equation 7 and get its matrix form for the calculation of Christoffel symbols.

$$g_{ij} = \begin{pmatrix} g_{ii} & g_{ij} \\ g_{ji} & g_{jj} \end{pmatrix} = \begin{pmatrix} \frac{4}{(1-(u^2+v^2))^2} & 0 \\ 0 & \frac{4}{(1-(u^2+v^2))^2} \end{pmatrix} \tag{32}$$

$$(g_{ij})^{-1} = \begin{pmatrix} \frac{(1-(u^2+v^2))^2}{4} & 0 \\ 0 & \frac{(1-(u^2+v^2))^2}{4} \end{pmatrix} \tag{33}$$

we can tell that $u_1 = u(s)$ and $u_2 = v(s)$ for which we assume s is proportional to arc-length, and we notice $g_{11} = g_{22}, (g_{11})^{-1} = (g_{22})^{-1}$.

We calculate the Christoffel Symbols Γ^i_{kj}, resulting in

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial u^1} + \frac{\partial g_{11}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^1}\right) = \frac{1}{2}g^{11}\frac{\partial g_{11}}{\partial u^1} = \frac{1}{2}\frac{(1-(u^2+v^2))^2}{4}\frac{16u}{(1-(u^2+v^2))^3} \tag{34}$$

$$= \frac{2u}{1-(u^2+v^2)} \tag{35}$$

$$\Gamma^1_{21} = \Gamma^1_{12} = \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial u^2} + \frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^1}\right) = \frac{1}{2}g^{11}\frac{\partial g_{11}}{\partial u^2} = \frac{1}{2}\frac{(1-(u^2+v^2))^2}{4}\frac{16v}{(1-(u^2+v^2))^3} \tag{36}$$

$$= \frac{2v}{1-(u^2+v^2)} \tag{37}$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left(\frac{\partial g_{21}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^1}\right) = \frac{-1}{2}g^{11}\frac{\partial g_{22}}{\partial u^1} = \frac{-2u}{1-(u^2+v^2)} \tag{38}$$

$$\Gamma^2_{11} = \frac{1}{2}g^{22}\left(\frac{\partial g_{12}}{\partial u^1} + \frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2}\right) = \frac{-1}{2}g^{22}\frac{\partial g_{11}}{\partial u^2} = \frac{-2v}{1-(u^2+v^2)} \tag{39}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\left(\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^2}\right) = \frac{1}{2}g^{22}\frac{\partial g_{22}}{\partial u^1} = \frac{2u}{1-(u^2+v^2)} \tag{40}$$

$$\Gamma^2_{22} = \frac{1}{2}g^{22}\left(\frac{\partial g_{22}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^2}\right) = \frac{1}{2}g^{22}\frac{\partial g_{22}}{\partial u^2} = \frac{2v}{1-(u^2+v^2)} \tag{41}$$

and we substitute for the geodesic equation

$$\ddot{u}^i + \Gamma^i_{kj}\dot{u}^k\dot{u}^j = 0 \tag{42}$$

obtaining the two differential equations with each of the variables $u_1 = u$ and $u_2 = v$

$$\ddot{u} + \Gamma^1_{11}\dot{u}^2 + 2\Gamma^1_{12}\dot{u}\dot{v} + \Gamma^1_{22}\dot{v}^2 = 0 \tag{43}$$

$$\ddot{v} + \Gamma^2_{11}\dot{u}^2 + 2\Gamma^2_{12}\dot{u}\dot{v} + \Gamma^2_{22}\dot{v}^2 = 0 \tag{44}$$

$$\tag{45}$$

therefore we can say that the geodesics are

$$\ddot{u} + \frac{2u}{1-(u^2+v^2)}\dot{u}^2 + \frac{4v}{1-(u^2+v^2)}\dot{u}\dot{v} - \frac{2u}{1-(u^2+v^2)}\dot{v}^2 = 0 \tag{46}$$

$$(1-(u^2+v^2))\ddot{u} + 2u\dot{u}^2 + 4v\dot{u}\dot{v} - 2u\dot{v}^2 = 0 \tag{47}$$

$$\ddot{v} - \frac{2v}{1-(u^2+v^2)}\dot{u}^2 + \frac{4u}{1-(u^2+v^2)}\dot{u}\dot{v} + \frac{2v}{1-(u^2+v^2)}\dot{v}^2 = 0 \tag{48}$$

$$(1-(u^2+v^2))\ddot{v} + 2u\dot{u}^2 + 4v\dot{u}\dot{v} + 2u\dot{v}^2 = 0 \tag{49}$$

$$\ddot{v} - \ddot{u} = 0 \tag{50}$$

This equations are difficult to solve, however, we know that the upper-half plane is isometric to the Poincaré disk with the metrics given. We consider for the half-plane in hyperbolic geometry

$$\mathbb{H} = \{z = (x+iy), y > 0\}, g = \frac{1}{y^2}(dx^2 + dy^2) \tag{51}$$

we have the metric given by $g_{\mathbb{H}}(x,y) = \frac{1}{x^2}(dx^2 + dy^2)$.

$$g_{ij} = \begin{pmatrix} y^{-2} & 0 \\ 0 & y^{-2} \end{pmatrix} \tag{52}$$

$$g_{ij}^{-1} = \begin{pmatrix} y^{2} & 0 \\ 0 & y^{2} \end{pmatrix} \tag{53}$$

$$\Gamma_{11}^1 = 0 \tag{54}$$

$$\Gamma_{12}^1 = \frac{-1}{y} \tag{55}$$

$$\Gamma_{22}^1 = 0 \tag{56}$$

$$\Gamma_{11}^2 = \frac{1}{y} \tag{57}$$

$$\Gamma_{21}^2 = 0 \tag{58}$$

$$\Gamma_{22}^2 = \frac{-1}{y} \tag{59}$$

geodesics are given by

$$\ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0 \tag{60}$$

$$\ddot{y} - \frac{1}{y}(\dot{x}^2 + \dot{y}^2) = 0 \tag{61}$$

b) Verify that circular arcs in $\mathbb{H}$, respectively, $\mathbb{D}$, meeting the boundary of $\mathbb{H}$, resp., $\mathbb{D}$, orthogonally are geodesics (when parametrised by constant speed).

$$\frac{d}{ds}\left(\frac{\dot{x}}{y^2}\right) = \frac{\ddot{x}y^2 - \dot{x}2x\dot{y}}{4} = 0 \tag{62}$$

$$\frac{\dot{x}}{y^2} = c \Leftrightarrow \dot{x} = cy^2 \tag{63}$$

$$\tag{64}$$

so if $c = 0$, $x = a$ and is a straight line in the hyperbolic plane

Consider now an ellipsoid in $\mathbb{R}^3$:

$$E^2 = \{(x, y, z) \in \mathbb{R}^3 | \frac{x2}{a2} + \frac{y2}{b2} + \frac{z2}{c2} = 1\} \tag{65}$$

c) Prove that the intersections of E^2 with coordinate xy, xz, yz 2-planes are geodesics on $E2$,that is, $\gamma_x = E^2 \cap x = 0, \gamma_y = E^2 \cap y = 0$, and $\gamma_z = E^2 \cap z = 0$, are geodesics (when parametrised by constant speed).

d) Does it follow from part c) that γ_x, γ_y, and γ_z are the only closed geodesics of E^2?