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Answers

1. $\boxed{\text{2 points}}$ Ruben broke into a bank. He must deactivate the alarm system by entering a pin number. Before the robbery he has been able to gather the following information about the pin number: It consists of 4 digits, the digit 3 is used exactly 2 times and the other digits are distinct (for example 3319 or 2303 but not 3363 or 5353). He has made a list of all the possible combinations and constructed a little device that tries each of them in a random order, but never twice the same combination. When it inputs the wrong pin, nothing happens, and the device tries the next number exactly 0.1 second later. If it inputs the correct pin, the alarm is deactivated immediately and the device stops. As soon as it enters the first pin, a countdown starts. If the pin number is not found after 30.01 seconds, the alarm will go off.
What is the probability that the alarm will go off?

   We know that the digit 3 is used exactly 2 times and the other digits are never 3 and distinct in a four digit sequence. First, we will calculate the number of ways are there to pick a sequence of $k = 2$ distinct numbers chosen from $1, 2, 0, 4, \ldots, 9$, where 3 is replaced by 0.

$$(n)_k = \frac{n!}{(n-k)!} = \frac{9!}{7!} = 72 \tag{1}$$

   Secondly, let's calculate the number of positions these 3's can be at. We know there are $k!$ ways to rearrange a sequence of length k. For this sequence of $k = 4$ numbers, we have, therefore $4!$ ways. However, this is reduced to $\frac{4!}{4}$ because otherwise we'd be counting each number twice when we exchange the 3's positions and twice again when we permutate the non-3's into a combination of non-3's which we already accounted for in the first section.

   We can therefore compute that there are $6 \cdot 72 = 432$ possible PIN codes.

   If we had unlimited time, it would take $0.1 \cdot 432 - 0.1 = 43.1$ seconds to try them all, taking out the time of waiting after the last one. The probability that the alarm will go off is $1-$ the number of successful outcomes divided by the amount of total outcomes. Here let's compute the percentage of the time it would take to try all of the possible PIN numbers that is within the countdown, in other words, the probability to get a successful outcome out of all the available results.

$$\frac{30.01}{43.1} = 0.6962 \tag{2}$$

   Therefore the probability the alarm will go off is $1 - 0.6962 = 0.3037 \approx 0.3$

2. $\boxed{\text{3 points}}$ An investor has 19,000 euro to invest among 4 possible investments. Each investment must be in units of 1000 euro (0 euro, 1000 euro, 2000 euro, ...).

   1. If the total 19,000 euro is to be invested, how many different investment strategies are possible?

      We make a vector $\{0,1\}^m = \{0,1\}^{k+n+1}$ with exactly k ones that represents the 19 packages of a 1000 euros that we can distribute. The number of vectors that can have exactly k ones in n investment placements is given by

$$\binom{19+4-1=22}{19} = \frac{22!}{19!(3)!} = 1540 \tag{3}$$

      In other words, there are 1540 ways we can uniquely distribute the 19000 euro such that nothing goes without investing.

2. What if not all the money needs to be invested? One way to make this calculation is repeating the previous process summing all $0 \leq k \leq 19$ in which $k = i$ in the calculation below:

$$\sum_{i=0}^{n} \frac{(i+3)!}{3!i!} = 1540 + 1.330 + 1.140 + 969 + 816 + 680 + 560 + 455 + 364 + 286 + 220 + 165 \quad (4)$$

$$+ 120 + 84 + 56 + 35 + 20 + 10 + 4 + 1 = 8855 \quad (5)$$

In other words, there are 8855 ways we can uniquely distribute the 19000 euro such that some amount may not be invested.

3. $\boxed{\text{3 points}}$ A bin contains 3 types of disposable flashlights. The probability that a type 1 flashlight will give more than 100 hours of use is .7, with the corresponding probabilities for type 2 and type 3 flashlights being .4 and .3, respectively. Suppose that 20 percent of the flashlights in the bin are type 1, 30 percent are type 2, and 50 percent are type 3.

   1. What is the probability that a randomly chosen flashlight will give more than 100 hours of use?
   2. Given that a flashlight lasted more than 100 hours, what is the conditional probability that it was a type j flashlight, $j = 1, 2, 3$?

For a randomly chosen flashlight, the probability that it will be type 1 is 0.2, type 2 0.3 and type 3 0.5. For each type we have been given the probability $\mathbb{P}$ that it works, and the probability that it doesn't will be $1 - \mathbb{P}$. Taken this into account,

1. We make this calculation derived from the `Law of total probabilities`, visible in equation 6 and Theorem 3.1.1 of the lecture notes

$$\mathbb{P}(B) = \sum \mathbb{P}(B \mid A_i) \cdot \mathbb{P}(A_i) \quad (6)$$

here the event B is giving more than 100 hours worth of light and each A_i is each of the flashlight types. The probability that a randomly chosen flashlight will give more than a 100 hours of use is $0.2 \cdot 0.7 + 0.3 \cdot 0.4 + 0.5 \cdot 0.3 = 0.14 + 0.12 + 0.15 = 0.41$.

2. Given this setup we assume that we have to calculate some $\mathbb{P}(B \mid A_i), \forall i = j = 1, 2, 3$ in which B is the event of lasting more than 100 hours, A is the flashlight type and i entails the exact type number.

$$\mathbb{P}(A_i \mid B) = \frac{\mathbb{P}(A_i \cap B)}{P(B)} \quad (7)$$

From the previous section we know $P(B) = \sum \mathbb{P}(B \mid A_i)\mathbb{P}(A_i) = 0.41$, so we have left to calculate the intersection of the two events. For this calculation we use `Bayes formula` visible below in equation 8 and Theorem 3.1.1 of the lecture notes

$$\mathbb{P}(A_i \mid B) = \frac{\mathbb{P}(B \mid A_i)\mathbb{P}(A_i)}{\sum \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)} \quad (8)$$

we calculate for each of the 3 cases of flashlight types

$$\mathbb{P}(1 \mid 100h) = \frac{\mathbb{P}(100h \mid 1)\mathbb{P}(1)}{\sum \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)} = \frac{0.7 \cdot 0.2}{0.41} = \frac{0.14}{0.41} \approx 0,34 \quad (9)$$

$$\mathbb{P}(2 \mid 100h) = \frac{\mathbb{P}(100h \mid 2)\mathbb{P}(2)}{\sum \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)} = \frac{0.3 \cdot 0.4}{0.41} = \frac{0.12}{0.41} \approx 0,29 \quad (10)$$

$$\mathbb{P}(3 \mid 100h) = \frac{\mathbb{P}(100h \mid 3)\mathbb{P}(3)}{\sum \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)} = \frac{0.5 \cdot 0.3}{41} = \frac{0.15}{0.41} \approx 0,36 \quad (11)$$

4. $\boxed{\text{2 points}}$ Let λ be a positive constant. The probability mass function of a random variable X taking values in N_0 is given by $p(i) = \frac{c\lambda^{2i}}{i!}, i = 0, 1, 2, \ldots$, where c is some constant.

1. Find the value of c.
   To do this we use the fact that the sum of all probabilities add up to one, and the properties fo the exponential function

$$\sum_{i=0}^{\infty} \frac{c\lambda^{2i}}{i!} = 1 \tag{12}$$

$$c\sum_{i=0}^{\infty} \frac{(\lambda^2)^i}{i!} = c \cdot e^{\lambda^2} = 1 \tag{13}$$

$$c = e^{-\lambda^2} \tag{14}$$

   where e^{λ^2} is the Taylor series expansion of e^x evaluated at $x = \lambda^2$

2. Find $\mathbb{P}(X = 0)$
   For a probability mass function $f_X(x) = \mathbb{P}(X = x), \forall x$, we know that

$$p(0) = \frac{c\lambda^{2*0}}{0!} = \frac{c}{1} = c = e^{-\lambda^2} \tag{15}$$

3. Assume $\lambda = 1$. Find $\mathbb{P}(X > 2)$

$$\mathbb{P}(X > 2) = 1 - \mathbb{P}(X \leq 2) = 1 - F_X(2) = 1 - \frac{e^{-\lambda^2}\lambda^{2*2}}{2!} - \frac{e^{-\lambda^2}\lambda^{2*1}}{1!} - e^{-\lambda^2} \tag{16}$$

$$= 1 - \frac{e^{-1^2}1^4}{2!} - \frac{e^{-1^2}1^2}{1!} - e^{-1^2} = 1 - \frac{1}{2e} - \frac{2}{e} = 1 - \frac{5}{2e} \approx 0.919 \tag{17}$$