1a) $f(x) = e^{-x} - 5x + 10 = 0$

(1) $x_2 = x_1 - f(x_1) \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} = 2.026387$

(2) $\left| \dfrac{x_2 - x_1}{x_2} \right| = 0.036327$

(3) order between linear and quadratic $\rightarrow$ probably near 6 iterations needed

b) $x_{n+1} = g(x_n) = x_n + a(e^{-x_n} - 5x_n + 10)$

$g'(x) = 1 + a(-e^{-x} - 5)$

$g'(2) = 1 + a(-e^{-2} - 5) = 0 \rightarrow a = \dfrac{-1}{-e^{-2} - 5} = 0.1947$

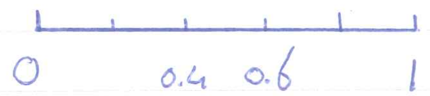
c) (1) $g(x) = \frac{1}{5} e^{-x} + 2 \qquad g'(x) = -\frac{1}{5} e^{-x}$

$g'(2) = -\frac{1}{5} e^{-2} = -0.027067$

fast linear convergence, factor error reduction is 0.027067

(2) $K = \dfrac{x_4 - x_3}{x_3 - x_2} = 0.026772$

$\varepsilon_4 \sim \left| \dfrac{K}{1-K} \right| |x_4 - x_3| = 1.329755\ E\text{-}8$

2a) 

$$\text{Trap}: \frac{0.2}{2}\left(\overset{3.4483}{f(0.4)}+\overset{2.9412}{f(0.6)}\right) = 0.63895$$

$$\text{Simp}: \frac{0.2}{6}\left(f(0.4)+4\underset{3.2}{f(0.5)}+f(0.6)\right) = 0.63965$$

b) $q = \dfrac{I_{64} - I_{128}}{I_{128} - I_{256}} = 3.9999 \approx 4 \rightarrow 2^{nd}$ order behaviour o.k.

$I_{128,ex} = \frac{4}{3} I_{128} - \frac{1}{3} I_{64} = 3.14159265333$

$I_{256,ex} = \frac{4}{3} I_{256} - \frac{1}{3} I_{128} =$ idem

$I_{256,ex,ex} =$ idem Does not make sense, because $I_{128,ex} = I_{256,ex}$; more decimals needed.

c) $\varepsilon_{256} = \frac{1}{3}(I_{256} - I_{128}) = 2.54333\ E-6$

$\left(\frac{1}{4}\right)^n \cdot \varepsilon_{256} < E-8 \rightarrow n = 4$

grid: $2^4 \cdot 256 = 4096$ segments

3 a) (1) $y(0.5) = 2 + 0.5\left(\frac{1}{10} \cdot 2^2 - \frac{2}{10}\right) = 2.1$

$y(1) = 2.1 + 0.5\left(\frac{1}{10} \cdot 2.1^2 - \frac{2}{10}\right) = 2.2205$

(2) $k_1 = 0.5\left(\frac{1}{10} \cdot 2^2 - \frac{2}{10}\right) = 0.1$

$k_2 = 0.5\left(\frac{1}{10}\, 2.1^2 - \frac{2}{10}\right) = 0.1205$

$y(0.5) = 2 + \frac{1}{2}(0.1 + 0.1205) = 2.11025$

b) (1) no stability limit visible
similar solutions on all grids

(2) $q = \frac{y_{10} - y_{20}}{y_{20} - y_{40}}$ at $x=4$: 3.776026

close to theoretical $q=4$
$\rightarrow$ 2^{nd} order convergence o.k.

(3) $\frac{1}{3}(y_{40} - y_{20})$ at $x=3$: 3.606 E-3
$x=4$: 4.182 E-3
$\rightarrow$ larger, because of error accumulation

(4) $\frac{4}{3}y_{40} - \frac{1}{3}y_{20}$ at $x=4$: -1.509063

(5) $|1+ah|<1$ restriction only for test equation $y'=ay$,
in case of $y' = \frac{1}{10}y^2 - \frac{1}{10}$ not valid

4a) $\hat{x} = \frac{x-5}{2} \rightarrow$

-3	-1	1	3
4.8	5.9	7.6	8.0

b) $M_0 = 4$ $M_1 = 0$ $M_2 = 20$

$F_1 = 26.3$ $F_2 = 11.3$

$$\left.\begin{matrix} 4 & 0 \\ 0 & 20 \end{matrix}\right| \begin{matrix} 26.3 \\ 11.3 \end{matrix} \qquad \begin{matrix} a = 6.575 \\ b = 0.565 \end{matrix}$$

$$\ell(x) = 6.575 + 0.565\hat{x} = 6.575 + 0.565\left(\frac{x-5}{2}\right)$$
$$= 5.1625 + 0.2825 \cdot x$$

$$\ell(10) = 7.9875$$

5a)

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ -1 & -3 & 1 & 0 & 4 \\ 0 & 1 & -3 & 2 & 2 \\ 1 & 0 & 0 & 2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ -1 & -3 & 1 & 0 & 4 \\ 0 & 1 & -3 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right)$$

(1) use row 1 to eliminate A_{41}

(2)

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 1 & -3 & 1 & 0 & 2 \\ 0 & 1 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

y_1	y_2	y_3	y_4
0	1/3	2/3	1

bound. cond. = $y_1 = -1$, $y_4 = 1$

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{(1/3)^2} + \alpha y_i = \frac{\beta}{x_i}$$

$$\Rightarrow y_{i-1} + \left(-2 + \frac{\alpha}{9}\right) y_i + y_{i+1} = \frac{\beta}{9 x_i}$$

Row 2, Row 3: $-2 + \frac{\alpha}{9} = -3$

Row 2: $\frac{\beta}{9(1/3)} = 2$

Row 3: $\frac{\beta}{9(2/3)} = 1$

$$\Rightarrow \alpha = -9, \ \beta = 6$$

b) (1) err $= (0, -3, 0, 0)$ max err $= -3$ $\|r^0\|_\infty = 3$

(2) diag. dominance: $\left(\frac{0}{1}, \frac{2}{3}, \frac{3}{3}, \frac{1}{2}\right)$

→ conv. problems may occur (but not for sure divergence)

(3) $\hat{x}_1 = \frac{1}{1}(-1-0) = -1$ $x_1 = 2*(-1) - 1*(-1) = -1$

$\hat{x}_2 = \frac{1}{-3}(4 - (-1)(-1) - 1*0) = -1$ $x_2 = 2*(-1) - 1*(0) = -2$

$\hat{x}_3 = \frac{1}{-3}(2 - (1)(-2) - 2*1) = -2/3$ $x_3 = 2*(-\frac{2}{3}) - 1*(0) = -4/3$

$\hat{x}_4 = \frac{1}{2}(1 - (1)(-1)) = 1$ $x_4 = 2*(1) - 1*(1) = 1$

6) a) (1) $R = \frac{\Delta t k}{\Delta x^2} = \Delta t \frac{10^{-3}}{(1/200)^2} < \frac{1}{2} \rightarrow \Delta t < 0.0125$

(2) this relatively large Δt causes weak accuracy
$\rightarrow$ not very wise

b) advantage: no instabilities, larger Δt possible
disadvantage: each time step $Ax = b$ has to be solved, which may take a long time

c) $$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + \phi_i^{n+1} + \phi_i^n = k \frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}$$

$$\Rightarrow \phi_i^{n+1} + \Delta t \phi_i^{n+1} - R(\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}) = \phi_i^n - \Delta t \phi_i^n$$

$$\Rightarrow -R\phi_{i-1}^{n+1} + (1 + \underset{\substack{\uparrow \\ \text{extra}}}{\Delta t} + 2R)\phi_i^{n+1} - R\phi_{i+1}^{n+1} = (1 - \underset{\substack{\uparrow \\ \text{extra}}}{\Delta t})\phi_i^n$$