1 a) $g(x) = \ln(\frac{2}{x})$ $g'(x) = -\frac{1}{x}$ (1) $g'(0.8) = -\frac{5}{4}$ (1) $|g'(x_0)| > 1$ divergence (1)

3

b) $x_{n+1} = x_n - \frac{x_n e^{x_n} - 2}{e^{x_n} + x_n e^{x_n}}$ (1) $x_1 = 0.85481\ldots$ (1)

3

$err \approx \left|\frac{x_1 - x_0}{x_0}\right| = 0.06851\ldots$ (1)

c) $K = \frac{x_5 - x_4}{x_4 - x_3} = -0.85045\ldots$ (1) (2) $\mathcal{E} \approx \frac{K}{1-K}|x_5 - x_4| = 0.023968\ldots$ (1)

7

(1) $\Rightarrow$ slow convergence: $|g'(p)| \approx |K| \approx 0.85$ not close to zero (1)

(3) $0.85045^n \mathcal{E} < E{-4}$ (1) $\longrightarrow n \geq 34$ (1)

(4) $2x_5 - x_4 = 0.92894$ (1) $\longrightarrow$ does not yet enhance accuracy more iterations required (1)

2 a) $\Delta x = 0.4$; -1, -0.2, 0.2, 1 $\quad M = 0.4\frac{1}{0+2} = 0.2$ (1)

7

$T = \frac{0.4}{2}\left(\frac{1}{-0.2+2} + \frac{1}{0.2+2}\right) = 0.20202\ldots$ (1)

$S = \frac{0.4}{6}\left(\frac{1}{-0.2+2} + \frac{4}{0+2} + \frac{1}{0.2+2}\right) = 0.200673\ldots$ (1)

$\mathcal{E} \approx \frac{0.4}{12}(0.4)^2 M$ $\quad f(x) = \frac{1}{x+2}$ $\quad f'(x) = \frac{-1}{(x+2)^2}$ $\quad f''(x) = \frac{2}{(x+2)^3}$ (1)

$M = \frac{2}{(-0.2+2)^3}$ (1) $\Longrightarrow \mathcal{E} = 1.828989\,E{-3}$ (1)

6

b) (1) $\frac{I_{64} - I_{32}}{I_{32} - I_{16}} = 3.9889\ldots \approx 4 \rightarrow 2^{nd}$ order according to theory (1)

(2) $\frac{4}{3}I_{32} - \frac{1}{3}I_{16} = 1.09861185\ldots$ (1) $\quad$ error $= 4.3533\,E{-7}$ (1)

(3) $\mathcal{E} \approx \frac{1}{3}(I_{256} - I_{128}) = 2.2599\ldots E{-6}$ (1)

(4) $(\frac{1}{4})^n * 2.2599\,E{-6} < 4.3533\,E{-7}$ (1) $\rightarrow n \geq 2 \longrightarrow 1024$ segments (1)

4

3 a) $y_{n+1} = y_n + \Delta x (4 x_n y_n)$

(1) $\Delta x = 0.5$ $y(0.5) = 1 + 0.5(4*0*1) = 1$

$y(1.0) = 1 + 0.5(4*0.5*1) = 2$ (1)

(2) $\Delta x = 0.25$ $y(0.25) = 1 + 0.25(4*0*1) = 1$

$y(0.5) = 1 + 0.25(4*0.25*1) = 5/4$ (1)

$y(0.75) = 5/4 + 0.25(4*0.5*5/4) = 15/8$

$y(1.0) = 15/8 + 0.25(4*0.75*15/8) = 105/32$ (1)

(3) errors: $\sqrt{e} - 1 \hat{=} 0.649$

$\sqrt{e} - 5/4 \hat{=} 0.399$ → factor $0.61 \simeq \frac{1}{2}$ approx. linear (1)

4 b) $k_1 = 0.5(2*0*1) = 0$ (1) RK2:

(1) $k_2 = 0.5(2*0.5*(1+0)) = 1/2$ (1) $y(0.5) = 1 + \frac{1}{2}(0 + \frac{1}{2}) = 5/4$ (1)

(2) $y(\frac{1}{2}) = y(0) + 0.5(2*0.5*y(\frac{1}{2}))$ Impl. Euler

$= 1 + 0.5\, y(\frac{1}{2})$ (1) $y(0.5) = 2$ (1)

5

c) (1) for $\Delta x = 0.4$ oscillation visible, different from other solutions → unstable (1)

(2) apparently it takes more small errors to become visible will be visible for larger x (1)

(3) $q = \dfrac{y_{0.1} - y_{0.05}}{y_{0.05} - y_{0.025}}$ at $x = 1.2$ → $q \simeq 9.27$ close enough to 8 → according to theory (1)

(4) $\frac{8}{7} y_{0.025} - \frac{1}{7} y_{0.05}$ at $x = 1.2$ → 5.613512 E-2 (1) (1)

4 a)

2	3	4	
1	5	12	

$\hat{x} = x - 3$ (1)

-1	0	1
1	5	12

7

$M_0 = 3$ $M_1 = 0$ $M_2 = 2$

$F_0 = 18$ $F_1 = 11$ (2)

$\left(\begin{array}{cc|c} 3 & 0 & 18 \\ 0 & 2 & 11 \end{array}\right)$ (1) $a = 6$, $b = 5\frac{1}{2}$ (1)

$y \overset{(1)}{=} 6 + 5\frac{1}{2}(x-3) = -10\frac{1}{2} + 5\frac{1}{2}x$ (1)

2 b) (1) $y(1) = -10\frac{1}{2} + 5\frac{1}{2} = -5$ (1)

(2) $y(2) = -10\frac{1}{2} + 11 = +\frac{1}{2}$, should be 1 → error $\frac{1}{2}$ (1)

5 a) $A\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}9\\9\\9\end{pmatrix}=\begin{pmatrix}1\\-3\\-2\end{pmatrix}-\begin{pmatrix}9\\9\\9\end{pmatrix}=\begin{pmatrix}-8\\-12\\-11\end{pmatrix}$ (1) $\leftarrow \|r^0\|_\infty = 12$ (1)

4

(1)

(2) row 1: 1 row 2: $3/4$ row 3: $3/5$ (1)

$(3/4)^n < 0.01 \quad n \geq 17$ (1)

6

b) (1) $x_1 = \frac{1}{1}(9-(0)(1)-(0)(1)) = 9$

$x_2 = \frac{1}{-4}(9-(-1)(1)-(2)(1)) = -2$ (1)

$x_3 = \frac{1}{-5}(9-(2)(1)-(1)(1)) = -6/5$ (1)

(2) $\tilde{x}_1 = \frac{1}{1}(9-(0)(1)-(0)(1)) = 9 \rightarrow x_1 = 2*9-1 = 17$ (1)

$\hat{x}_2 = \frac{1}{-4}(9-(-1)(17)-(2)(1)) = -6$ (1) $\rightarrow x_2 = 2*-6-1 = -13$ (1)

$\hat{x}_3 = \frac{1}{-5}(9-(2)(17)-(1)(-13)) = \frac{12}{5} \rightarrow x_3 = 2*\frac{12}{5}-1 = 19/5$ (1)

2

c) use row 1 to eliminate $A(3,1) \Longrightarrow$
subtract row 1 twice from row 3 (1)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ -1 & -4 & 2 & 9 \\ 0 & 1 & -5 & -9 \end{array}\right)$$ (1)

6 a) (1) $R = \frac{\Delta t\, k}{\Delta x^2} = \frac{\Delta t\ 0.01}{(0.5)^2} < \frac{1}{2} \Rightarrow \Delta t < 12\frac{1}{2}$ (1) (1)

7

(2) 1 — 2 — 3 at 0, 0.5, 1; $\phi_1 = 20,\ \phi_3 = 20$ (bound. cond) (1)

$$\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} = k\,\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2} + 400(x_i - x_i^2)$$

$$\Longrightarrow \phi_i^{n+1} = R\phi_{i-1}^n + (1-2R)\phi_i^n + R\phi_{i+1}^n + 400\Delta t(x_i - x_i^2)$$ (1)

$i=2$: $x_2 = 0.5 \quad \phi_2 = 0.01(20) + (1-0.02)(20) + 0.01(20) + 100\ (1/4)$ (1)

$R = \frac{0.25\ 0.01}{(0.5)^2} = 0.01$ (1) $\quad = 20 + 25 = 45$ (1)

b) $$\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} = k\,\frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2} + 400(x_i - x_i^2)$$ (1)

5

$$\Longrightarrow -R\phi_{i-1}^{n+1} + (2R+1)\phi_i^{n+1} - R\phi_{i+1}^{n+1} = \phi_i^n + 400\Delta t(x_i - x_i^2)$$ (1)

$\Longrightarrow$ i^{th} row A: $0 \cdots 0\ \underset{L}{-R}\ \underset{D}{2R+1}\ \underset{R}{-R}\ 0 \cdots 0$ (2) $\quad \| \quad b(i) = \phi_i^n + 400\Delta t(x_i - x_i^2)$ (1)