Control Engineering
Lecture 10
ver. 1.2.2.1

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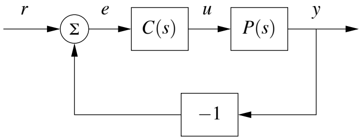
Today

- **Frequency domain analysis** (Chapter 9 of the textbook)
  - Nyquist plot
  - Nyquist criterion
  - Stability margins

The loop transfer function

Stability of the closed-loop system from the (open-)**loop transfer function**

$$L(s) = P(s)C(s) = \frac{n_p(s)n_c(s)}{d_p(s)d_c(s)} = \frac{n_L(s)}{d_L(s)}$$



Closed-loop $G_{yr}(s) = \frac{PC}{1+PC} = \frac{L}{1+L} = \frac{n_p(s)n_c(s)}{n_p(s)n_c(s)+d_p(s)d_c(s)} = \frac{n_L(s)}{n_L(s)+d_L(s)}$

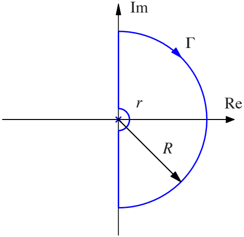
Stability $\Leftrightarrow$ $1 + L(s) = 0$ has all roots with strictly negative real part

Can we infer stability properties from $L(i\omega)$?

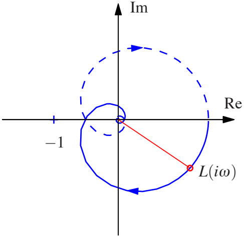
Nyquist contour

Nyquist plot

Graphical representation of the frequency response useful for stability analysis



(a) Nyquist D contour



(b) Nyquist plot

D contour $\Gamma_{0,\infty}$ - $L(s)$ no poles on the imaginary axis

Define for $R > 0$ the contour $\Gamma_{0,R}$

$$\Gamma_{0,R} = \{i\omega\colon\ -R \leq \omega \leq R\} \cup \{R\mathrm{e}^{i\theta}\colon\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$$

The Nyquist D contour $\Gamma_{0,\infty}$ is obtained from $\Gamma_{0,R}$ by letting $R \to +\infty$.

In words: the Nyquist D contour consists of the imaginary axis and an arc at infinity $s = R\mathrm{e}^{i\theta}$, $\theta : \frac{\pi}{2} \to -\frac{\pi}{2}$, $R \to +\infty$, enclosing the right-half complex plane

Nyquist plot

Plot of $L(s)|_{s\in\Gamma_{0,\infty}}$

Nyquist plot

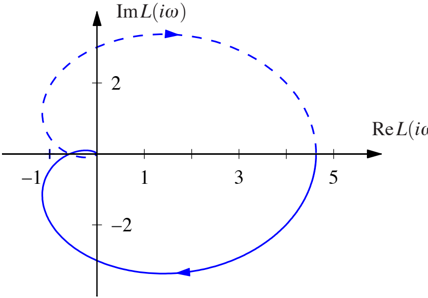
Example (Nyquist plot) $L(s) = \frac{1}{(s+a)^3}$

$$L(i\omega) = \frac{1}{(i\omega + a)^3}$$

$$\begin{aligned} |L(i\omega)| &= \frac{1}{(a^2+\omega^2)^{\frac{3}{2}}} \\ \arg L(i\omega) &= -3\arctan\frac{\omega}{a} \end{aligned}$$

Sample points

$$\begin{array}{lll} \omega = 0 & |L(i\omega)| = \frac{1}{a^3} & \arg L(i\omega) = 0 \\ \omega = \frac{\sqrt{3}}{3}a & |L(i\omega)| = \frac{3\sqrt{3}}{8a^3} & \arg L(i\omega) = -\frac{\pi}{2} \\ \omega = \sqrt{3}a & |L(i\omega)| = \frac{1}{8a^3} & \arg L(i\omega) = -\pi \\ \omega \to +\infty & |L(i\omega)| \to 0 & \arg L(i\omega) \to -\frac{3\pi}{2} \end{array}$$



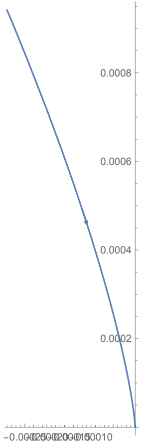
- Pointwise representation of $L(i\omega)$ as $\omega \in [0, +\infty)$
- $L(-i\omega) = M(\omega)\mathrm{e}^{-i\theta(\omega)}$ Nyquist plot symmetric wrt horizontal axis (recall the polar representation $L(i\omega) = M(\omega)\mathrm{e}^{i\theta(\omega)}$)

Nyquist plot

Example (Nyquist plot) $L(s) = \frac{1}{(s+a)^3}$

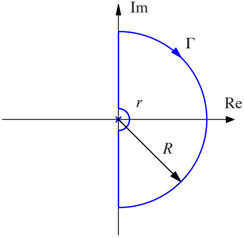
$$L(i\omega) = \frac{1}{(i\omega + a)^3}$$

$$\begin{aligned} |L(i\omega)| &= \frac{1}{(a^2+\omega^2)^{\frac{3}{2}}} \\ \arg L(i\omega) &= -3\arctan\frac{\omega}{a} \end{aligned}$$

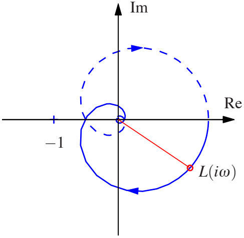


The graph on the right is the same Nyquist plot as before drawn for $\omega \in [10, 100]\ \mathrm{rad/sec}$. It shows that $|L(i\omega)| \to 0$ and $\arg L(i\omega) \to -\frac{3\pi}{2}$ as $\omega \to +\infty$

Nyquist plot



(a) Nyquist D contour



(b) Nyquist plot

D contour $\Gamma_{0,\infty}$ - $L(s)$ has one pole $s = 0$

Define for $0 < r < R$ the contour $\Gamma_{r,R}$ (see the figure above)

$$\Gamma_{r,R} = \{i\omega\colon -R \leq \omega \leq -r\} \cup \{r\mathrm{e}^{i\theta}\colon -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2}\} \\ \cup \{i\omega\colon r \leq \omega \leq R\} \cup \{R\mathrm{e}^{i\theta}\colon -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2}\}$$

The Nyquist D contour $\Gamma_{0,\infty}$ is the contour obtained from $\Gamma_{r,R}$ letting $r \to 0^+$ and $R \to +\infty$.
In words: if $L(s)$ has a pole at $s = 0$ the Nyquist contour is modified by a small semicircle around the pole $s = 0$

Nyquist plot

Plot of $L(s)|_{s \in \Gamma_{0,\infty}}$

Nyquist plot

Example $L(s) = \frac{1}{s}$; $L(i\omega) = \frac{1}{i\omega}$; $L(re^{i\theta}) = \frac{1}{i\theta}$

$$\begin{cases} |L(i\omega)| = & \frac{1}{\omega} \\ \arg L(i\omega) = & -\frac{\pi}{2} \quad r \le \omega \le R \end{cases}$$

$$\begin{cases} |L(re^{i\theta})| = & \frac{1}{r} \\ \arg L(re^{i\theta}) = & -\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \end{cases}$$

$$\begin{cases} |L(Re^{i\theta})| = & \frac{1}{R} \\ \arg L(Re^{i\theta}) = & -\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \end{cases}$$

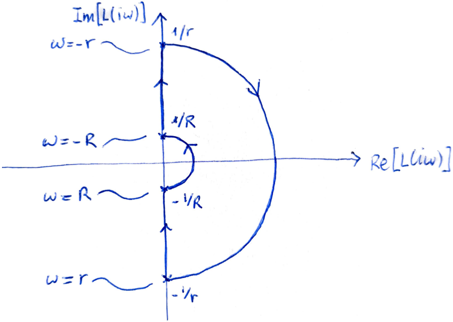


Figure: Plot of $L(s)|_{s\in\Gamma_{r,R}}$

Nyquist plot

Example $L(s) = \frac{1}{s}$; $L(i\omega) = \frac{1}{i\omega}$; $L(re^{i\theta}) = \frac{1}{re^{i\theta}}$

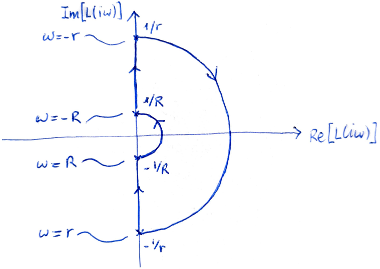


Figure: Plot of $L(s)|_{s\in\Gamma_{r,R}}$

$r \to 0^+$
$R \to +\infty$
$\Longrightarrow$

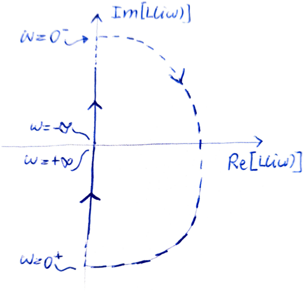
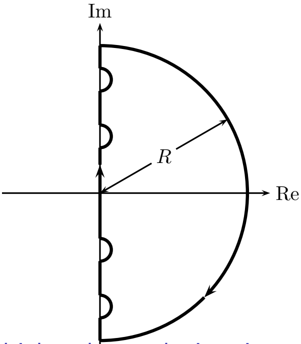


Figure: Plot of $L(s)|_{s\in\Gamma_{0,\infty}}$ – Nyquist plot

Nyquist plot



D contour $\Gamma_{0,\infty}$ - $L(s)$ has multiple poles on the imaginary axis

If $L(s)$ has multiple poles on the imaginary axis the Nyquist contour is modified by a small semicircle around any of such poles of $L(s)$. These semicircles have infinitesimal radius r and lie in the open right-half complex plane.

The mathematical description of the contour can be written similarly as before, although its expression will be more complicated.

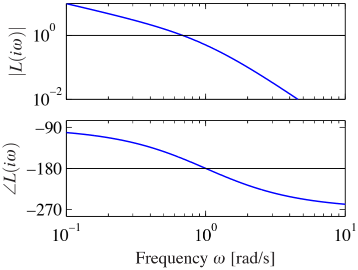
Nyquist plot

Plot of $L(s)|_{s \in \Gamma_{0,\infty}}$

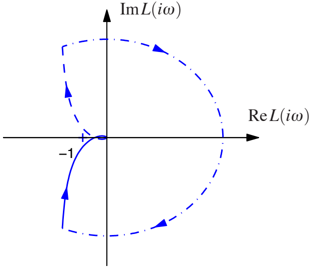
Nyquist plot

The Bode plot of the loop transfer function $L(s)$ provides us with enough information to draw the Nyquist plot, because at each frequency ω it returns the magnitude and the phase of the complex number $L(i\omega)$, therefore allowing us to draw the point $L(i\omega)$ on the complex plane $(\mathrm{Re} L(i\omega), \mathrm{Im} L(i\omega))$.

Example (Nyquist plot) $L(s) = \frac{k}{s(s+1)^2}$ $k = 1$ (pole at $s = 0$)



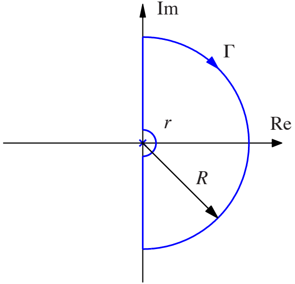
(a) Bode plot

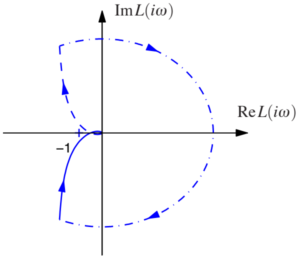


(b) Nyquist plot

Nyquist plot

Example (Nyquist plot) $L(s) = \frac{k}{s(s+1)^2}$ $k = 1$ (pole at $s = 0$)





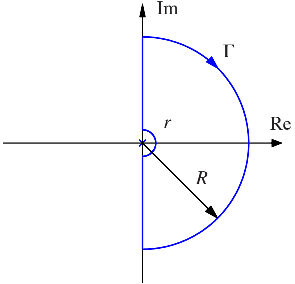
The small half-circle around the origin in the Γ contour (described by $s = r\mathrm{e}^{i\theta}$, with $r \to 0^+$ and $\theta : -\frac{\pi}{2} \to \frac{\pi}{2}$) is mapped in the large half-circle at infinity that encloses the right half-plane

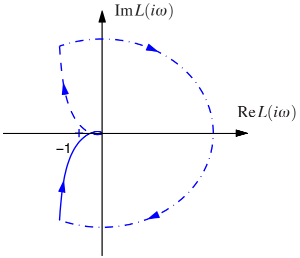
Let's see why once again

$$
\begin{aligned}
|L(r\mathrm{e}^{i\theta})| = &\quad \frac{k}{|r\mathrm{e}^{i\theta}||r\mathrm{e}^{i\theta}+1|^2} = \frac{k}{r[(1+r\cos\theta)^2 + (r\sin\theta)^2]} \\
\angle L(r\mathrm{e}^{i\theta}) = &\quad -\angle(r\mathrm{e}^{i\theta}) - 2\angle(r\mathrm{e}^{i\theta}+1) = -\theta - 2\arctan\frac{r\sin\theta}{1+r\cos\theta}
\end{aligned}
$$

Nyquist plot

Example (Nyquist plot) $L(s) = \frac{k}{s(s+1)^2}$ $k = 1$ (pole at $s = 0$)





The small half-circle around the origin in the Γ contour (described by $s = r\mathrm{e}^{i\theta}$, with $r \to 0^+$ and $\theta : -\frac{\pi}{2} \to \frac{\pi}{2}$) is mapped in the large half-circle that encloses the right half-plane

<u>Let's see why</u>

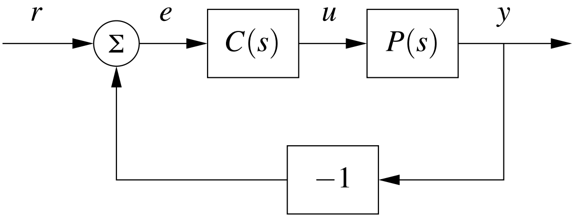
$$\begin{aligned}
|L(r\mathrm{e}^{i\theta})| = &\quad \frac{k}{|r\mathrm{e}^{i\theta}||r\mathrm{e}^{i\theta}+1|^2} = \frac{k}{r[(1+r\cos\theta)^2 + (r\sin\theta)^2]} \overset{r\to 0^+}{\to} +\infty \\
\angle L(r\mathrm{e}^{i\theta}) = &\quad -\angle(r\mathrm{e}^{i\theta}) - 2\angle(r\mathrm{e}^{i\theta}+1) = -\theta - 2\arctan\frac{r\sin\theta}{1+r\cos\theta} \overset{r\to 0^+}{\to} -\theta
\end{aligned}$$

Simplified Nyquist criterion

Theorem

$L(s)$ has no poles in $\overline{\mathbb{C}^+}$ except for single poles on the imaginary axis

The closed-loop system is stable **if and only if** closed contour $L(s)|_{s\in\Gamma_{0,\infty}}$ (Nyquist plot) has no net encirclements of the point $(-1, i0)$



Simplified Nyquist criterion

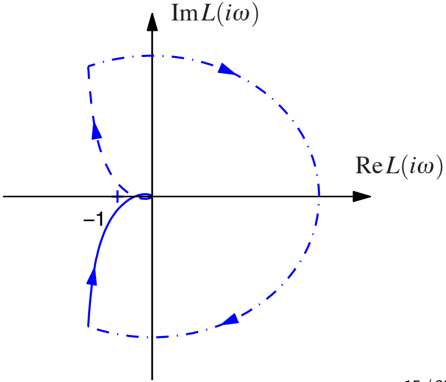
Theorem

$L(s)$ has no poles in $\overline{\mathbb{C}^+}$ except for single poles on the imaginary axis

The closed-loop system is stable **if and only if** closed contour $L(s)|_{s\in\Gamma_{0,\infty}}$ (Nyquist plot) has no net encirclements of the point $(-1, i0)$

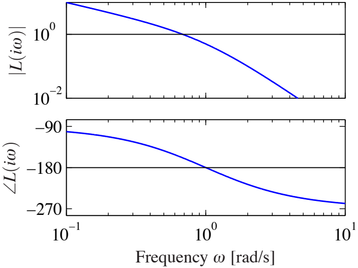
Net encirclement

- Place a pin at the point $(-1, i0)$
- Fix one end of a string at the pin at $(-1, i0)$ and let the other end be run across any point of the Nyquist plot
- No encirclements if the string does not wind up on the pin when the curve is encircled

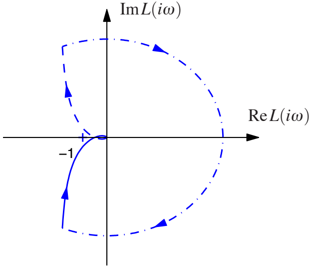


Nyquist criterion

Example (Nyquist plot) $L(s) = \frac{k}{s(s+1)^2}$, $k = 1$



(a) Bode plot



(b) Nyquist plot

- $L(s)$ has no pole in $\overline{\mathbb{C}^+}$ except a single pole at $s = 0$
- Net encirclements $= 0$

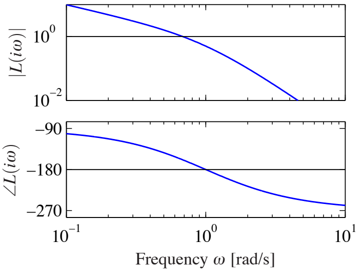
By Nyquist criterion, the closed-loop system is stable, i.e. the poles of

$$\frac{L}{1+L} = \frac{k}{s(s+1)^2 + k} = \frac{k}{s^3 + 2s^2 + s + k}$$

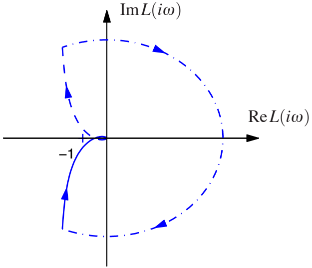
have all strictly negative real parts

Nyquist criterion

Example (Nyquist plot) $L(s) = \frac{k}{s(s+1)^2}$, $k = 1$



(a) Bode plot



(b) Nyquist plot

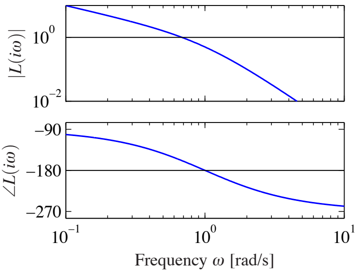
- $L(s)$ has no pole in $\overline{\mathbb{C}^+}$ except a single pole at $s = 0$
- Net encirclements $= 0$

<u>What happens if $k > 2$?</u>

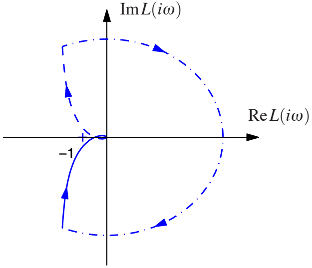
<u>What happens if $k < 0$?</u>

Nyquist criterion

Example (Nyquist plot) $L(s) = \frac{k}{s(s+1)^2}$, $k = 1$



(a) Bode plot



(b) Nyquist plot

What happens if $k > 2$?

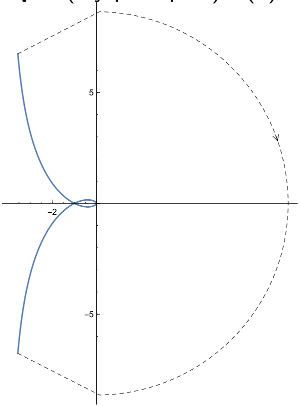
What happens if $k < 0$?

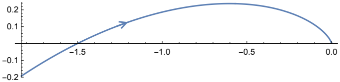
Routh-Hurwitz theorem
Study of the roots of the equation
$s^3 + 2s^2 + s + k = 0$

3	1	1
2	2	k
1	$-(\frac{k}{2} - 1)$	
0	k	

Nyquist criterion

Example (Nyquist plot) $L(s) = \frac{k}{s(s+1)^2}$, $k = 3$





Number of net encirclements of the point $(-1, i0)$ is equal to 2

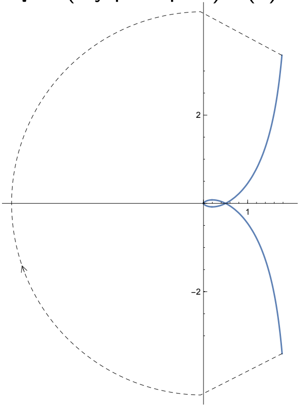
What happens if $k > 2$? There are 2 poles with positive real part

Routh-Hurwitz theorem
Study of the roots of the polynomial
$s^3 + 2s^2 + s + k = 0$

3	1	1
2	2	k
1	$-(\frac{k}{2} - 1)$	
0	k	

Nyquist criterion

Example (Nyquist plot) $L(s) = \frac{k}{s(s+1)^2}$, $k = -1$



Number of net encirclements of the point $(-1, i0)$ is equal to 1

What happens if $k < 0$? There is 1 pole with positive real part

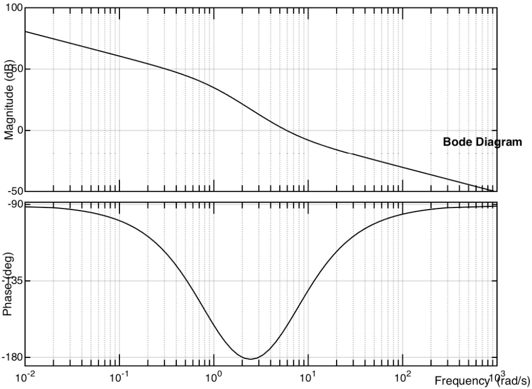
Routh-Hurwitz theorem
Study of the roots of the polynomial
$s^3 + 2s^2 + s + k = 0$

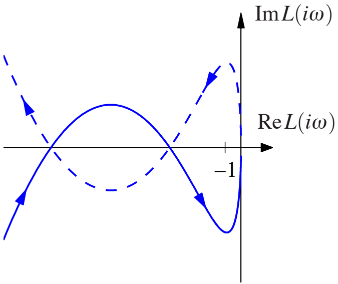
$$
\begin{array}{c||ll}
3 & 1 & 1 \\
2 & 2 & k \\
1 & -(\frac{k}{2} - 1) & \\
0 & k &
\end{array}
$$

Conditional stability

Stability of the closed-loop system changes as gain k changes

$$L(s) = \frac{3(s+6)^2}{s(s+1)^2}$$

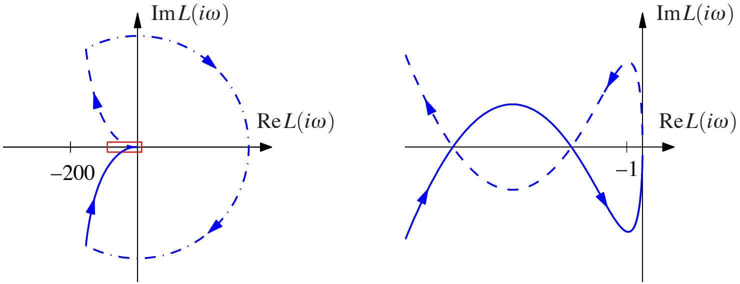




Conditional stability

Stability of the closed-loop system changes as gain k changes

$$L(s) = \frac{3(s+6)^2}{s(s+1)^2}$$



Net encirclements $= 0$

$$L(s) = \frac{k(s+6)^2}{s(s+1)^2}$$

stable $\quad 0 < k < \frac{1}{4},\ k > \frac{2}{3}$

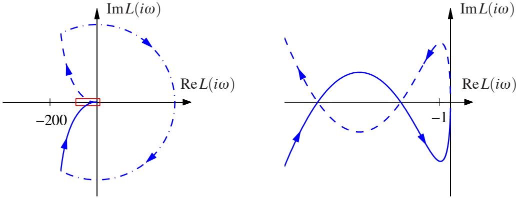
unstable $\quad k < 0,\ \frac{1}{4} < k < \frac{2}{3}$

Conditional stability

Routh-Hurwitz table
Study of the roots of the polynomial
$s(s+1)^2 + k(s+6)^2$

3	1	12k+1
2	k+2	36 k
1	$\frac{12k^2-11k+2}{k+2}$	
0	36k	

Note that the 3rd row is negative for $k < -2$ and $1/4 < k < 2/3$, and positive otherwise.



$$L(s) = \frac{k(s+6)^2}{s(s+1)^2}$$

stable	$0 < k < \frac{1}{4},\ k > \frac{2}{3}$
unstable	$k < 0,\ \frac{1}{4} < k < \frac{2}{3}$

Generalized Nyquist criterion

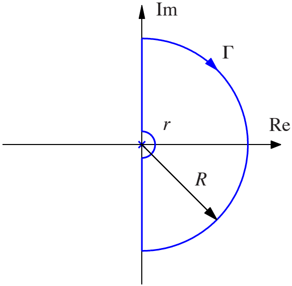
In the simplified Nyquist criterion, $L(s)$ must have no poles in $\overline{\mathbb{C}^+}$ except for single poles on the imaginary axis

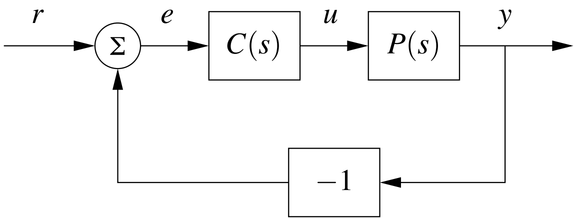
Nyquist's Stability Theorem

- $L(s)$ has P poles enclosed in the Nyquist contour Γ
- N number of net **clockwise** encirclements of $(-1, i0)$ by $L(s)|_{s \in \Gamma_{0,\infty}}$

$$\Downarrow$$

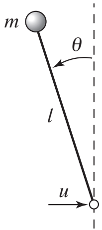
Closed-loop has $Z = N + P$ poles in the right half-plane



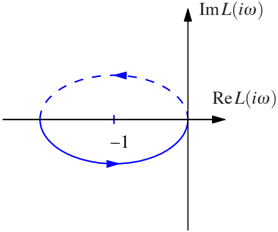


Generalized Nyquist criterion

Example Stabilized inverted pendulum



(a) Inverted pendulum



(b) Nyquist plot

Pendulum

$Y(s) = P(s)U(s) = \dfrac{1}{s^2 - 1} U(s)$

input = pivot acceleration

output = pendulum angle

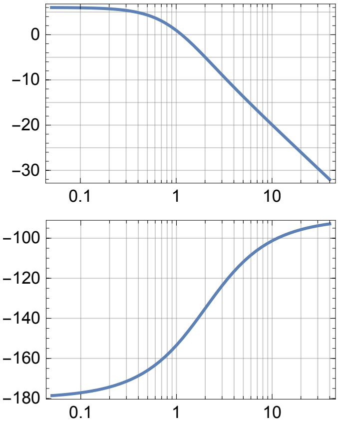
PD Controller

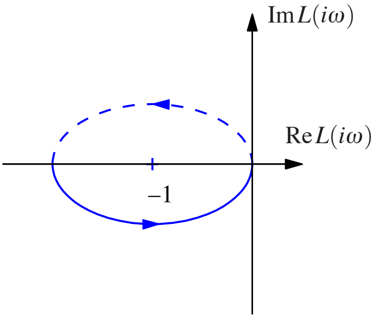
$U(s) = C(s)E(s) = k(s + 2)E(s)$

$k = 1$

Generalized Nyquist criterion

Example Stabilized inverted pendulum





Pendulum

$$Y(s) = P(s)U(s) = \frac{1}{s^2-1}U(s)$$

input = pivot acceleration

output = pendulum angle

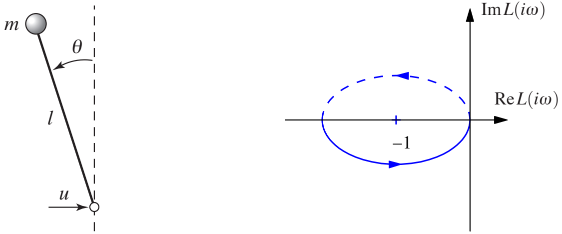
PD Controller

$$U(s) = C(s)E(s) = k(s+2)E(s)$$

$k = 1$

Generalized Nyquist criterion

Example Stabilized inverted pendulum



(a) Inverted pendulum (b) Nyquist plot

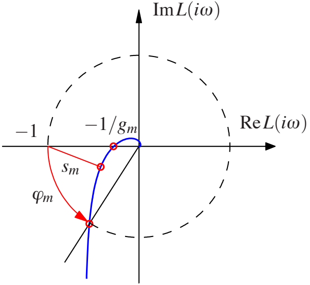
$P = 1$ number of poles enclosed in the Nyquist contour Γ
$N = -1$ number of net **clockwise** encirclements of -1 by $L(s)$ when s moves along Γ in **clockwise** direction
$Z = N + P = 0$ number of poles in the right-half plane (stable)
If $k < 0.5$ no encirclements and $Z = 1$ (unstable)

Stability margins

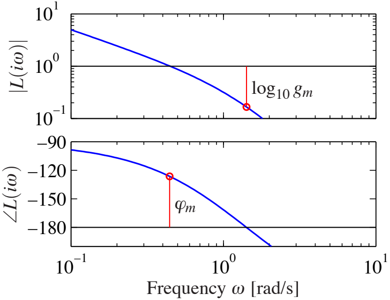
Measures of how a system is robustly stable wrt perturbations

Gain margin g_m

Smallest amount that the open-loop gain k in $kL(s)$ can be increased before the closed-loop system goes unstable



(a) Nyquist plot



(b) Bode plot

$L(s)$ with monotonically decreasing phase

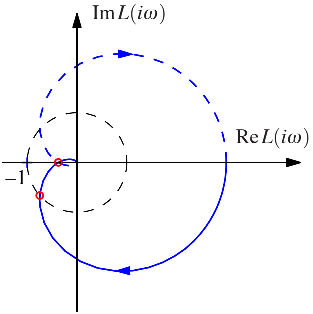
Phase cross-over frequency $\boldsymbol{\omega_{pc}}$ frequency at which $\angle L(i\omega_{pc}) = -180^\circ$

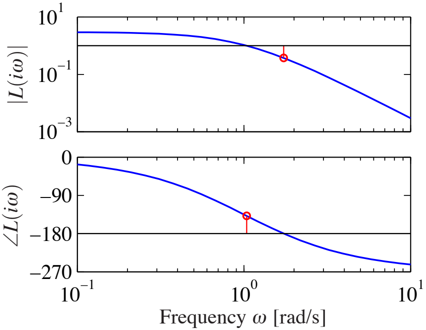
$\rightarrow$ compute $|L(i\omega_{pc})|$

Gain margin g_m such that $g_m \cdot |L(i\omega_{pc})| = 1 \Rightarrow g_m = |L(i\omega_{pc})|^{-1}$

Stability margins

Example (Gain margin) $L(s) = \dfrac{3}{(s+1)^3}$





Phase cross-over frequency $\angle L(i\omega_{pc}) = -180^\circ$

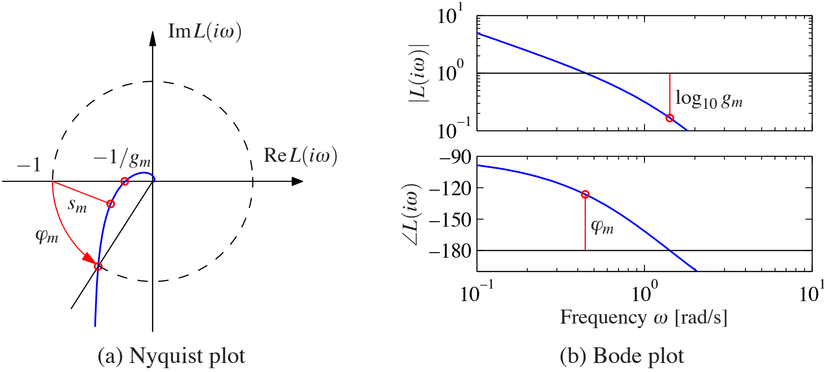
$$\angle L(i\omega) = -3\frac{180^\circ}{\pi} \arctan \omega \Rightarrow \omega_{pc} = \sqrt{3}$$

$$|L(i\omega)| = \frac{3}{(1+\omega^2)^{3/2}} \Rightarrow |L(i\omega_{pc})| = \frac{3}{(1+\sqrt{3}^2)^{3/2}} = \frac{3}{8} \Rightarrow g_m = \frac{8}{3} \approx 2.67$$

Stability margins

Phase margin φ_m

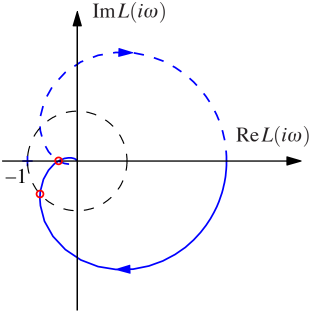
Smallest amount of phase lag needed for the system to reach the stability limit

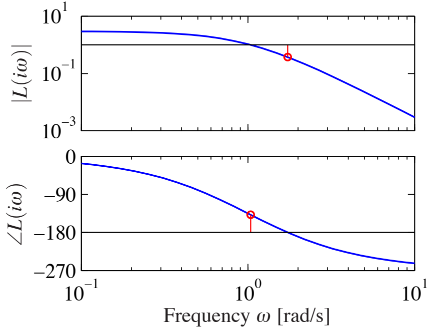


Gain cross-over frequency ω_{gc} frequency at which $|L(i\omega_{gc})| = 1$
Phase margin $\varphi_m = 180° + \angle L(i\omega_{gc})$

Stability margins

Example (Phase margin) $L(s) = \dfrac{3}{(s+1)^3}$





Gain cross-over frequency $|L(i\omega_{gc})| = 1$ with $|L(i\omega)| = \frac{3}{(1+\omega^2)^{3/2}}$

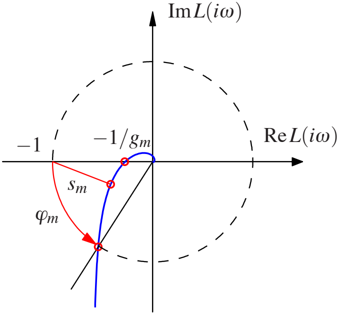
$$|L(i\omega_{gc})| = \frac{3}{(1+\omega_{gc}^2)^{3/2}} = 1 \Leftrightarrow 3 = (1+\omega_{gc}^2)^{3/2} \Leftrightarrow \omega_{gc} = \sqrt{3^{2/3}-1} \approx 1.04$$

$$\angle L(i\omega) = -3\arctan\omega \Rightarrow \varphi_m = 180^\circ + \angle L(i\omega_{gc}) = 180^\circ - 138.4^\circ \approx 41.6^\circ$$

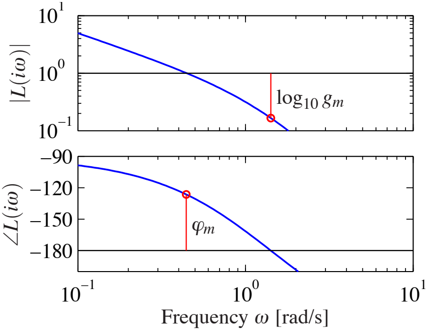
Stability margins

Stability margin s_m

Shortest distance from the Nyquist curve to the critical point



(a) Nyquist plot



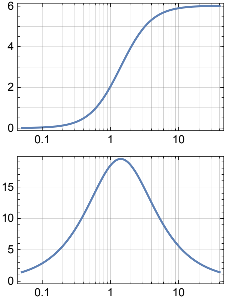
(b) Bode plot

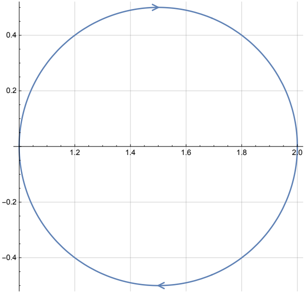
$$\begin{aligned} s_m &= \min_{\omega\geq 0} |(-1+i0) - (\mathrm{Re}(L(i\omega)) + i\,\mathrm{Im}(L(i\omega))| \\ &= \min_{\omega\geq 0} |-1 - \mathrm{Re}(L(i\omega)) - i\,\mathrm{Im}(L(i\omega))| \\ &= \min_{\omega\geq 0} \sqrt{(1+\mathrm{Re}(L(i\omega)))^2 + (\mathrm{Im}(L(i\omega))^2} \end{aligned}$$

Stability margins

Stability margin s_m

Shortest distance from the Nyquist curve to the critical point





Example $L(s) = 2\frac{s+1}{s+2}$. Hence, $L(i\omega) = 2\frac{\omega^2+2}{\omega^2+4} + i2\frac{\omega}{\omega^2+4}$ In fact

$$s_m \quad = \quad \min_{\omega\geq 0}\sqrt{\left(1+2\frac{\omega^2+2}{\omega^2+4}\right)^2 + \frac{4\omega^2}{(\omega^2+4)^2}}$$

Since $\sqrt{\cdot}$ is monotonically increasing, this is equivalent to $\min_{\omega\geq 0}\left(1+2\frac{\omega^2+2}{\omega^2+4}\right)^2 + \frac{4\omega^2}{(\omega^2+4)^2}$

Stability margins

Stability margin s_m

Shortest distance from the Nyquist curve to the critical point

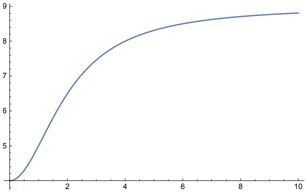
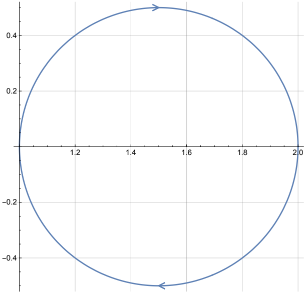


Figure: Plot of $\left(1+2\frac{\omega^2+2}{\omega^2+4}\right)^2+\frac{4\omega^2}{(\omega^2+4)^2}$



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Since $\sqrt{\cdot}$ is monotonically increasing, this is equivalent to $\min_{\omega\geq 0}\left(1+2\frac{\omega^2+2}{\omega^2+4}\right)^2+\frac{4\omega^2}{(\omega^2+4)^2}$. This has solution $\omega=0$ (see Figure on the left). Then $s_m=2$.

Next Lecture

- **Assignment for this lecture**: study Sections 9.1-9.3 and read Section 9.4
- **Next lecture**: PID control (Chapter 10)