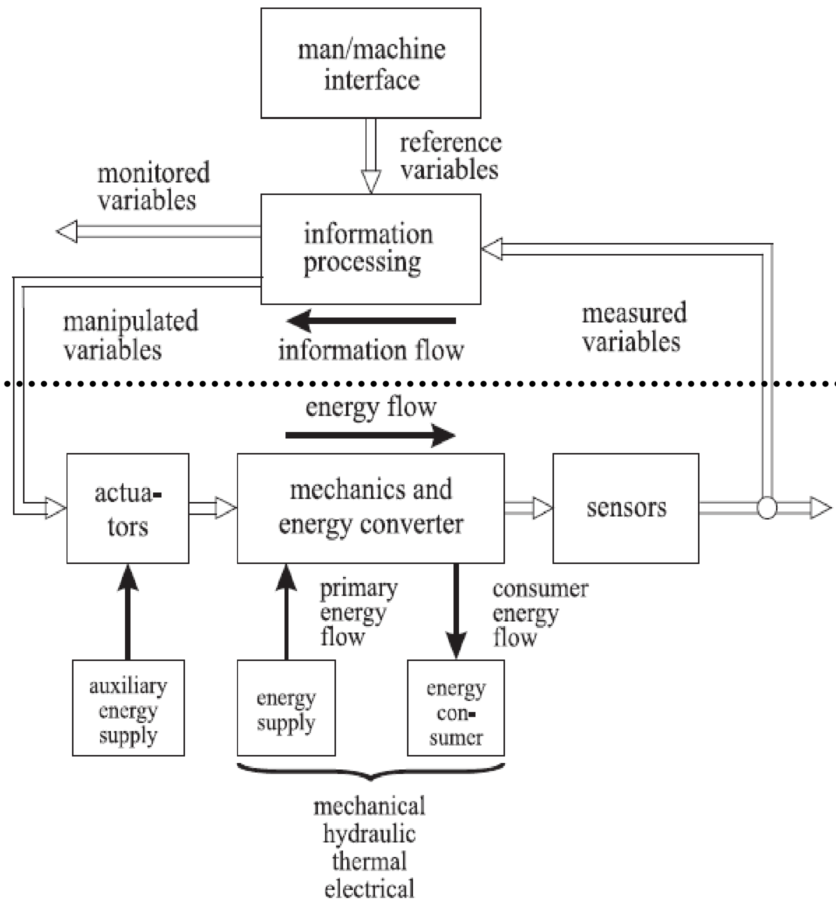
Mechatronics

Week 7 Day 1

Components of a Mechatronics System

Electronic domain

Mechanical, electrical, hydraulic, or thermal domain



The figure is taken from (Isermann, 2008).

Today's lecture:
Delayed systems

Learning objectives

After today's lecture, you will be able to

- analyse the stability properties of a system through its Nyquist plot
- determine the stability properties of delayed systems
- calculate the phase margin and critical time delay of a system

Additionally, we will refresh-or introduce-some concepts pertaining to Nyquist's criterion

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Time-delays can affect the stability of the system.

Delays effect on stability

The delay operator can be analysed in the frequency domain.

To this end, consider a delay operator of the form

$$y(t) = u(t - T),$$

where T is the time delay, the its transfer function is given by

$$\frac{Y(s)}{U(s)} = e^{-Ts}$$

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Our aim is to determine how time-delays affect the closed-loop system stability.

Delays Effect on Stability-Padé

To analyse the stability of the delayed system we can use the first-order Padé approximation

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Then, we can analyse the closed-loop system as usual:

- Calculate poles
- Check if they lie on the left half s-plane (negative real parts)

Example: Padé approximation

- Take system with transfer function

$$G(s) = \frac{1}{s+2}$$

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- We add a 2 second delay

$$G_d(s) = e^{-2s}\frac{1}{s+2}$$

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$$G_d(s) = e^{-2s}\frac{1}{s+2}$$

- Padé approximation of e^{-2s} can be computed

$$e^{-2s} = \frac{1-\frac{2}{2}s}{1+\frac{2}{2}s} = \frac{1-s}{1+s}$$

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$$G_d(s) = \frac{1-s}{1+s}\cdot\frac{1}{s+2} = \frac{1-s}{(1+s)(2+s)}$$

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$$G_d(s) = \frac{1-s}{1+s} \cdot \frac{1}{s+2} = \frac{1-s}{(1+s)(2+s)}$$

The system now has **two poles** at $s = -1$, $s = -2$, but it's **still stable**

Example: Padé approximation

Example: Stable System Becoming Unstable Due to Delay

Original System:

$$G(s) = \frac{1}{s+1}$$

- Stable: Pole at $s = -1$, which is in the left half of the s-plane.

With Delay:

Adding a time delay T, the system becomes:

$$G_{\text{delayed}}(s) = \frac{e^{-sT}}{s+1}$$

Using the **First-Order Pade Approximation**:

$$e^{-sT} \approx \frac{1 - \frac{sT}{2}}{1 + \frac{sT}{2}}$$

$$G_{\text{delayed}}(s) \approx \frac{1 - \frac{sT}{2}}{(s+1)\left(1 + \frac{sT}{2}\right)}$$

Denominator with Delay:

$$\text{Denominator: } \frac{s^2 T}{2} + \left(1 + \frac{T}{2}\right)s + 1$$

Stability Analysis:

- **Without Delay**: The pole is at $s = -1$, system is stable.
- **With Delay**: Poles are determined by the quadratic equation. For larger T, poles may cross to the right half-plane, causing instability.

Critical Delay:

For $T = 2$:

$$\text{Poles: } s = -1.0\,(\text{stable})$$

If T increases further, instability may occur.

Delays Effect on Stability-Padé

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Disadvantage: For high order systems this task can be challenging

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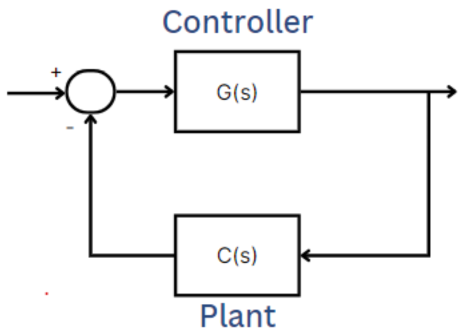
Alternative? Nyquist stability criterion

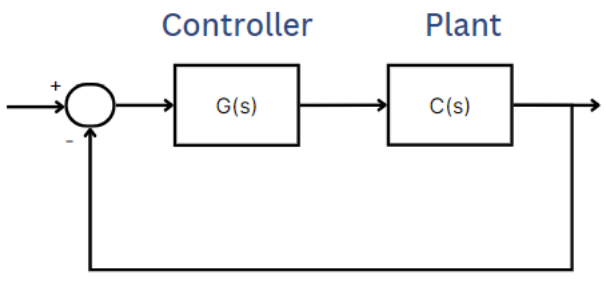
It tells us the number of closed-loop poles on the right half plane without calculating them

Nyquist’s stability criterion

Nyquist Stability Criteria-Preliminaries

For the systems





Nyquist Stability Criteria-Preliminaries

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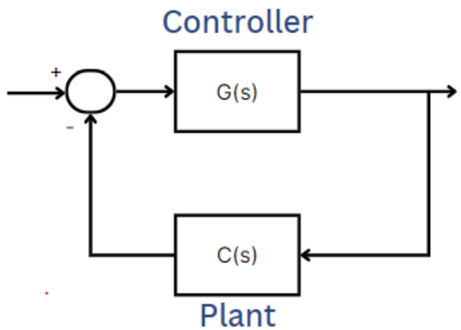

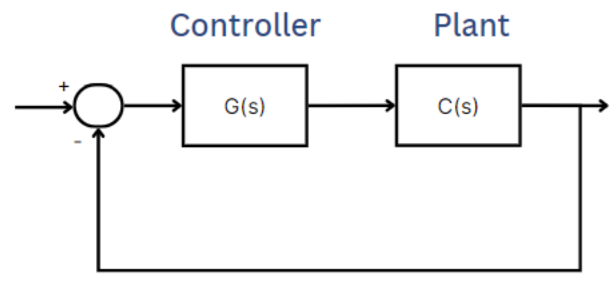

we have closed loop transfer functions

$$H_{CL}(s) = \frac{G(s)}{1+C(s)G(s)} \qquad H_{CL}(s) = \frac{C(s)G(s)}{1+C(s)G(s)}$$

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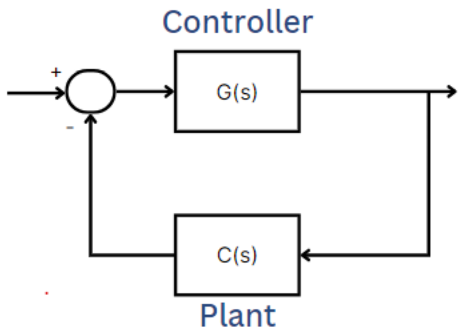

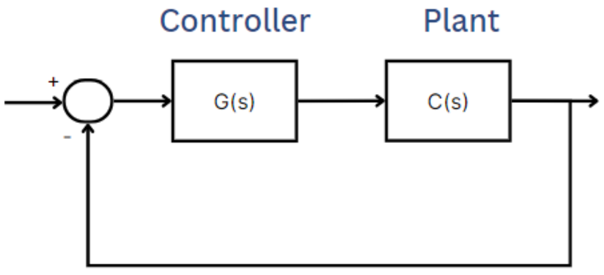

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In general, we have

$$\boxed{H_{CL}(s) = \frac{L(s)}{1 + C(s)G(s)}}$$

where characteristic polynomial $\chi(s) \coloneqq 1 + C(s)G(s)$ contains information about the closed loop system poles.

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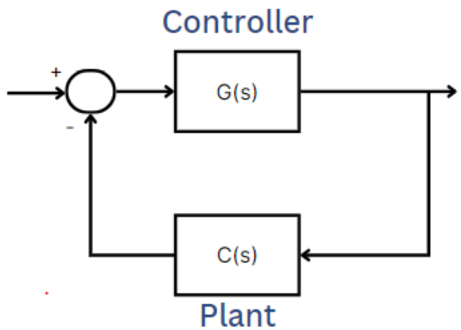
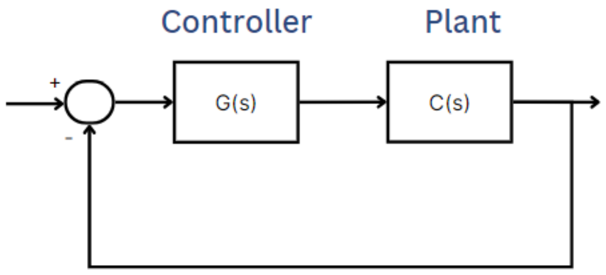
Nyquist Stability Criteria-Preliminaries

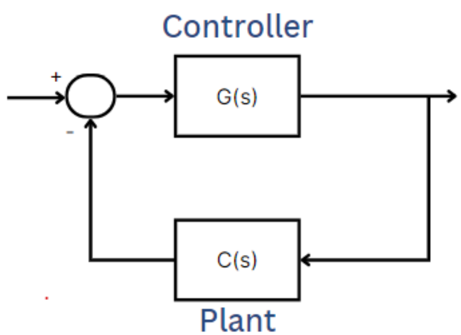


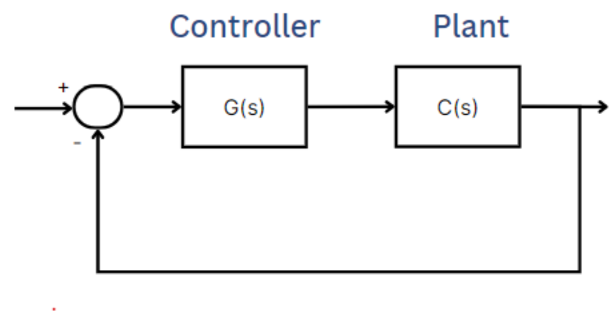


Let us establish three important concepts

- Relationship between the poles of $1 + C(s)G(s)$ and those of loop gain $C(s)G(s)$?

Nyquist Stability Criteria-Preliminaries





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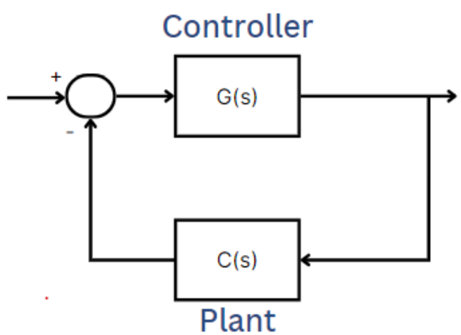
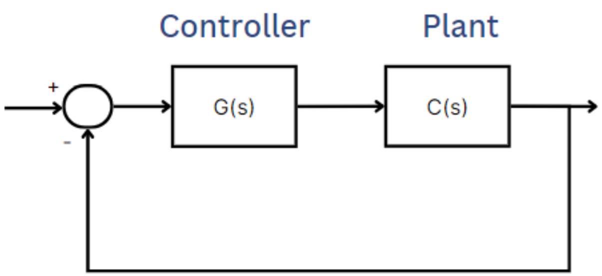
- Relationship between the poles of $1 + C(s)G(s)$ and those of loop gain $C(s)G(s)$?
- Relationship between the zeros of $1 + C(s)G(s)$ and the poles of the closed-loop transfer function H_{CL}(s)?

Nyquist Stability Criteria-Preliminaries





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- Cauchy's argument principle

Nyquist Stability Criteria-Preliminaries

Concept 1. What is the relationship between the poles of $1 + C(s)G(s)$ and those of loop gain $C(s)G(s)$?

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Let $C(s) := \frac{N_C(s)}{D_C(s)}$ and $\mathrm{G}(s) := \frac{N_G(s)}{D_G(s)}$, then

$$C(s)G(s) = \frac{N_C(s)N_G(s)}{D_C(s)D_G(s)}$$

and

$$1 + C(s)G(s) = \frac{N_C(s)N_G(s) + D_C(s)D_G(s)}{D_C(s)D_G(s)}$$

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Since the denominators are the same

Poles of $1 + C(s)G(s)$ =Poles of $C(s)G(s)$

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$$H_{CL}(s) = \frac{L(s)}{1 + C(s)G(s)}$$

It can be written as

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we can see that the denominator and numerator are the same

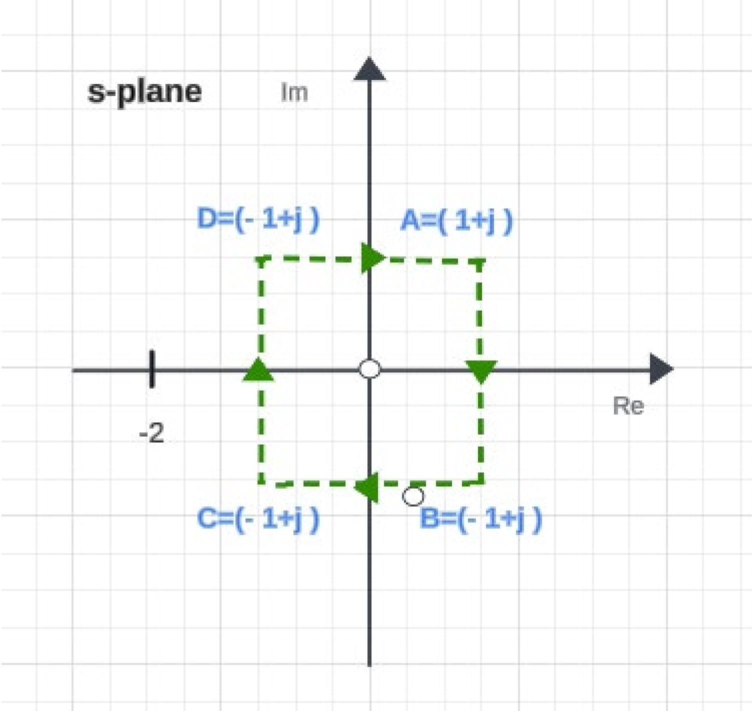
$$\boxed{\text{Zeros of } 1 + C(s)G(s) = \text{Poles of } H_{CL}(s)}$$

Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

Consider $G(s) = \frac{s}{s+2}$. Zero at $s = 0$

Take a contour in s-plane is encircling the zero of $G(s)$



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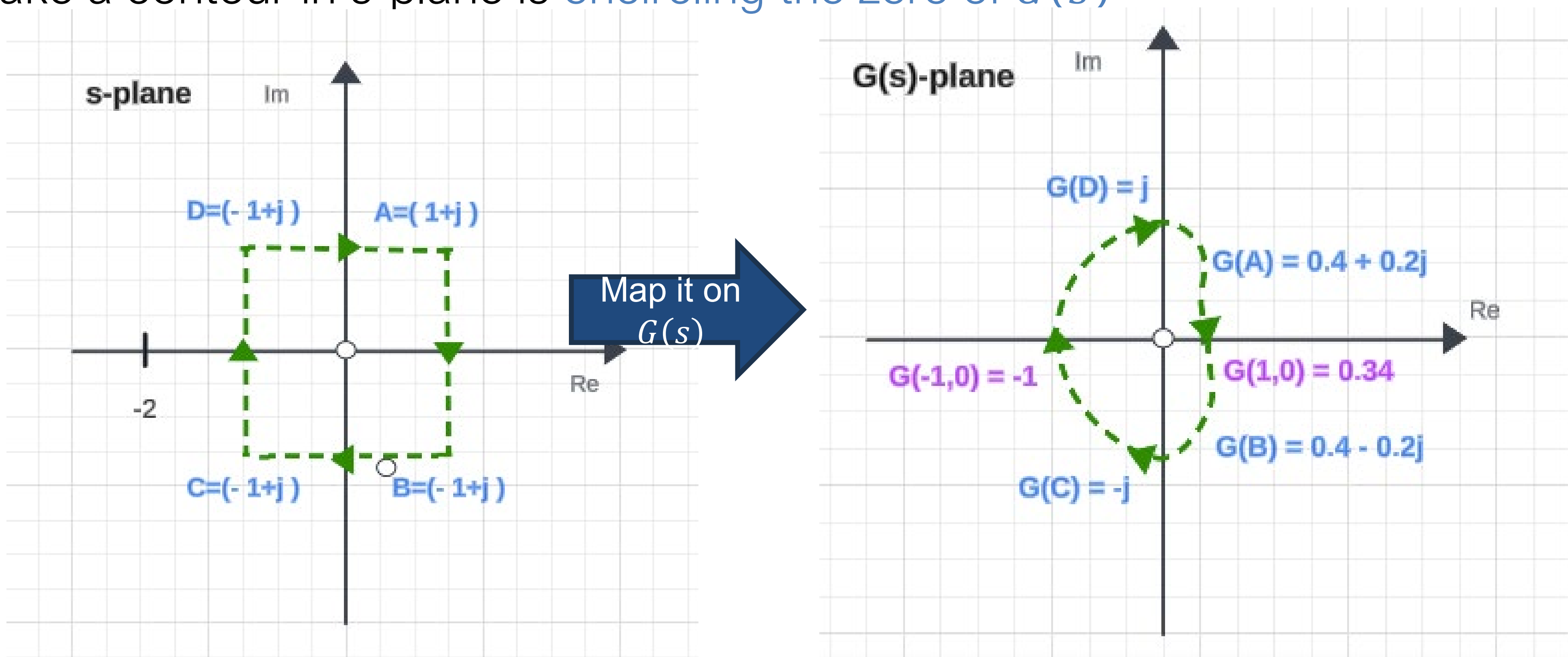

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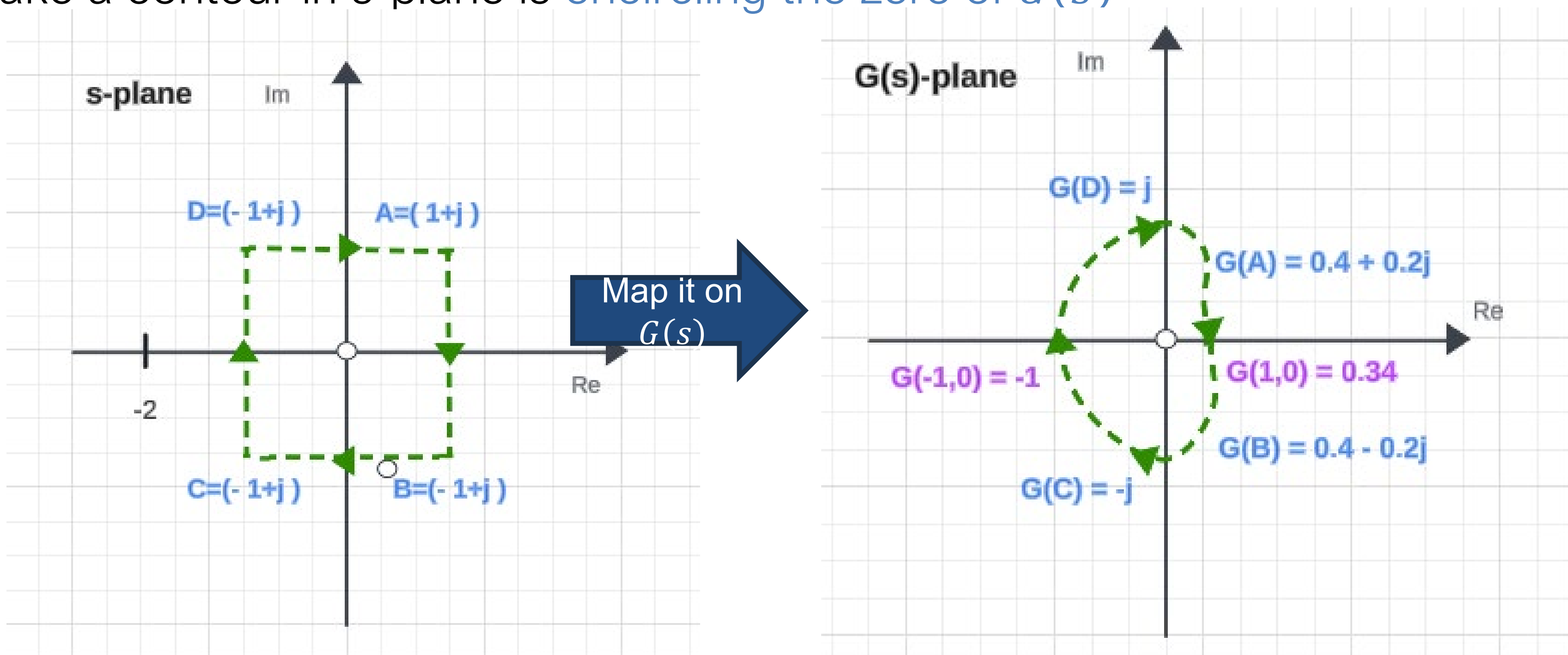

When you encircle a zero in the s-plane, you encircle the origin in G(s)-plane in clockwise direction

Nyquist Stability Criteria-Preliminaries

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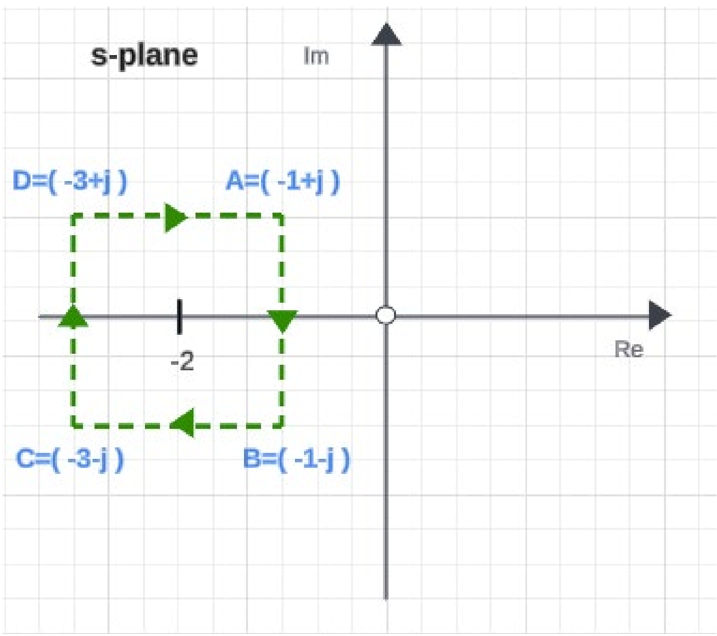

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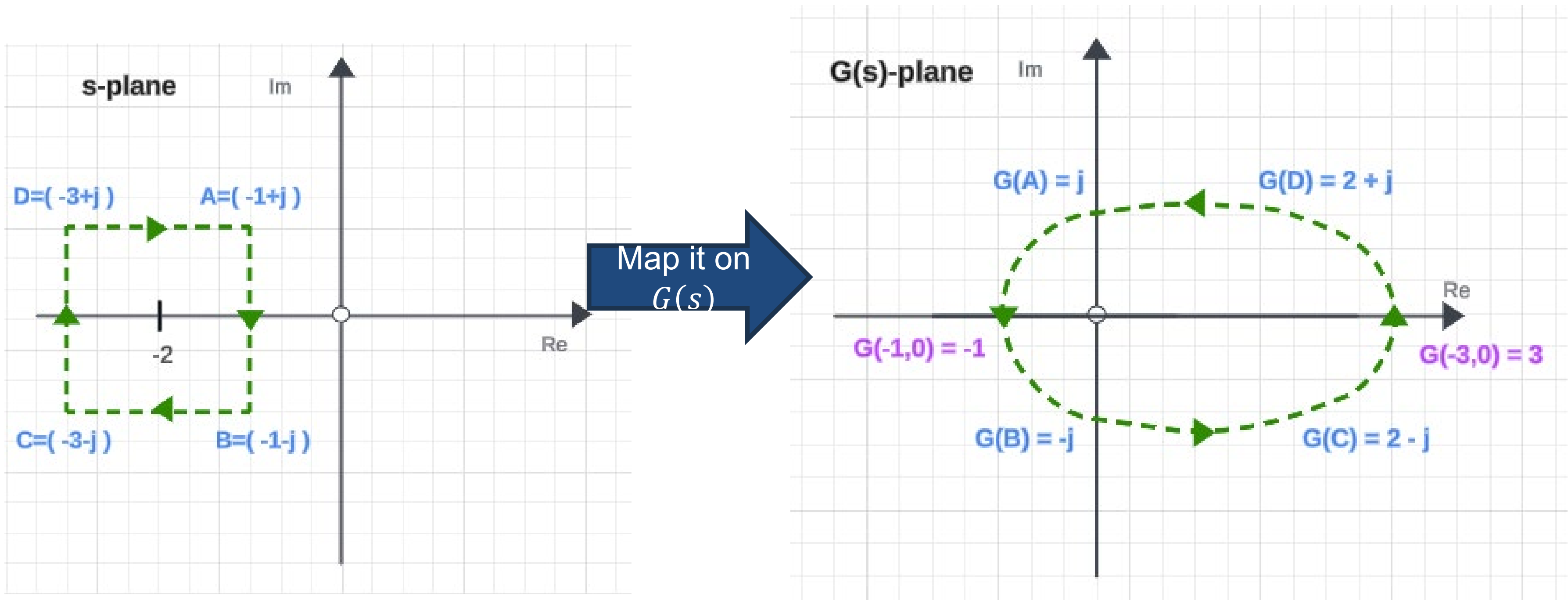

Nyquist Stability Criteria-Preliminaries

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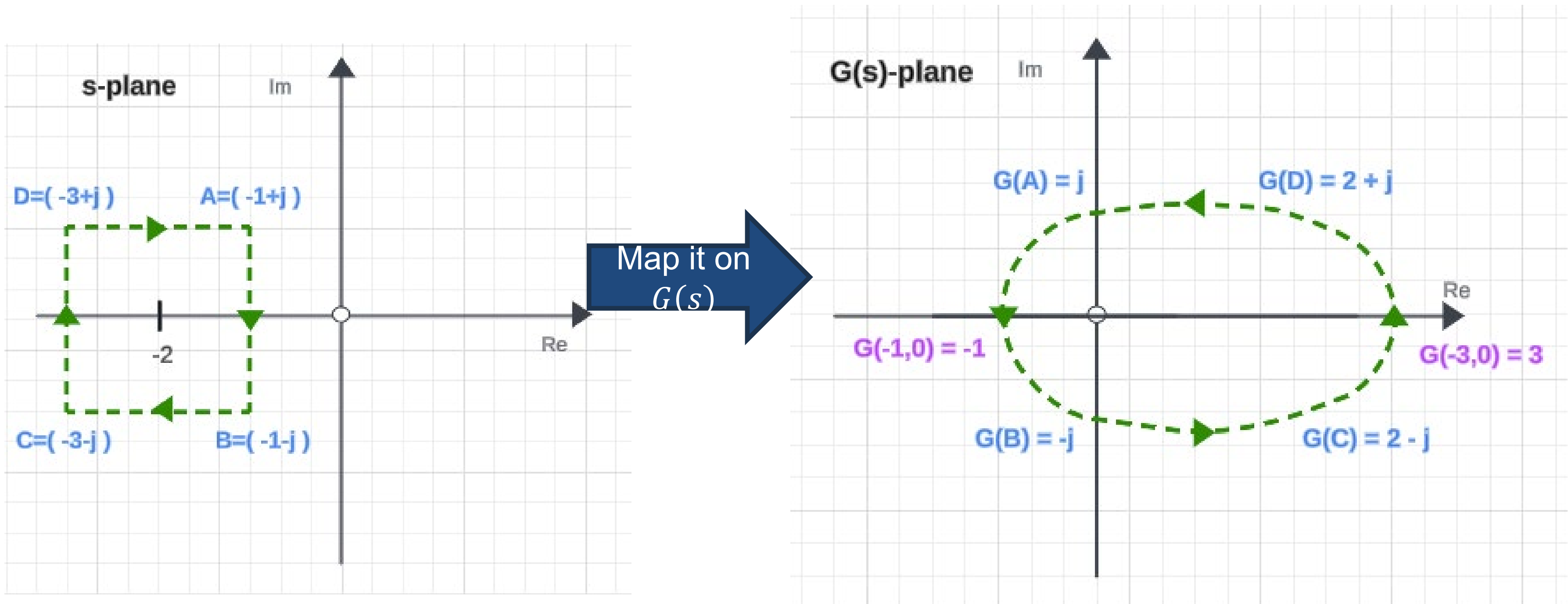


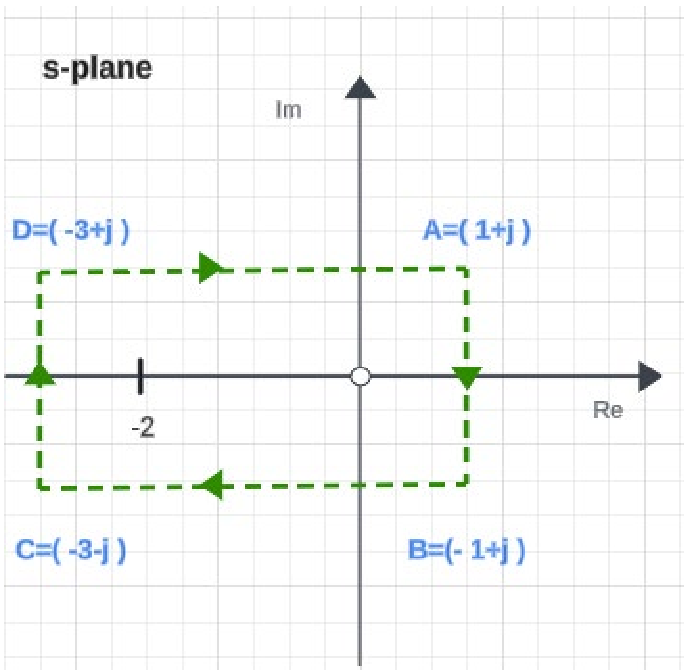
When you encircle a pole in the s-plane, you encircle the origin in G(s)-plane in counter clockwise direction

Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

Consider $G(s) = \frac{s}{s+2}$. Zero $s = 0$, pole $s = -2$

Take a contour in s-plane is encircling both zero and pole of $G(s)$



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Nyquist Stability Criteria-Preliminaries

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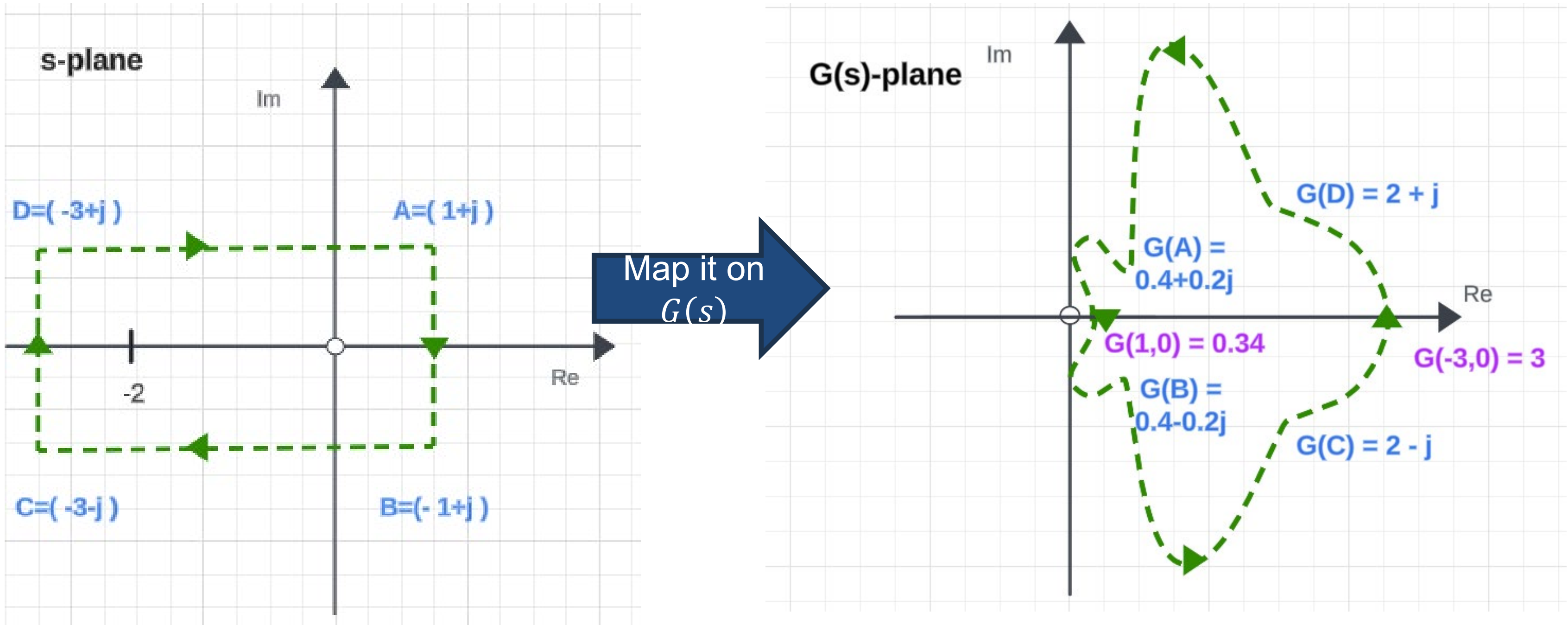

Nyquist Stability Criteria-Preliminaries

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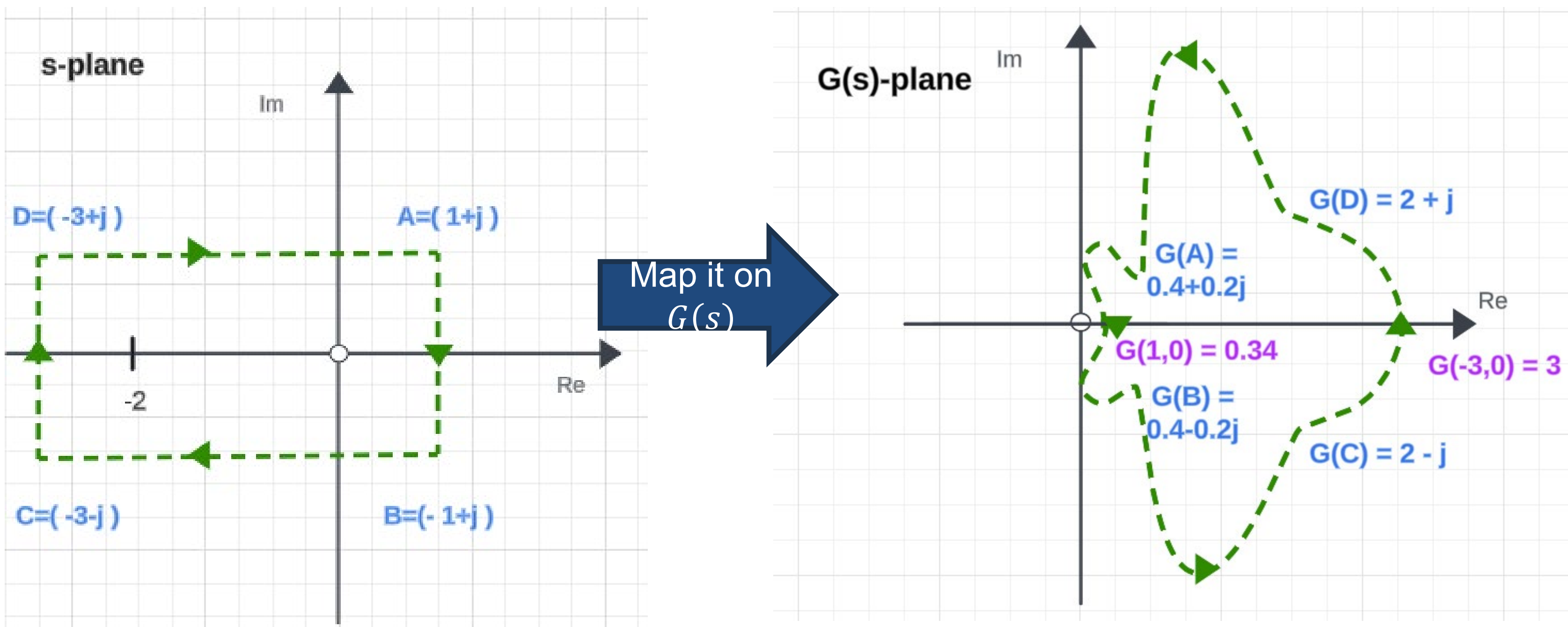


When you encircle both a pole and zero in the s-plane, you do not encircle the origin in G(s)-plane

Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy’s argument principle

- The encirclement (in clockwise direction) of the poles and zeros in a contour in the s-plane is related to the encirclement of the origin in the $G(s)$-plane
- The contour must not touch any pole or zero

Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

- The encirclement (in clockwise direction) of the poles and zeros in a contour in the s-plane is related to the encirclement of the origin in the $G(s)$-plane
- The contour must not touch any pole or zero

Generally we have

$$N = Z - P,$$

where

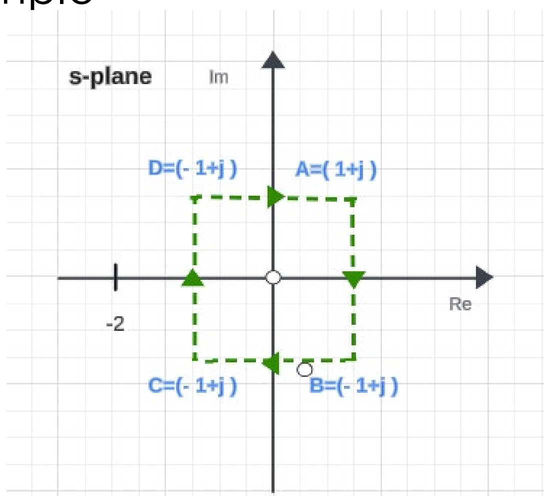
- N is the number of encirclements of origin in the $\mathrm{G}(s)$-plane
- Z is the number of encircled zeros of $\mathrm{G}(s)$ in the s-plane
- P is the number of encircled poles of $\mathrm{G}(s)$ in the s-plane

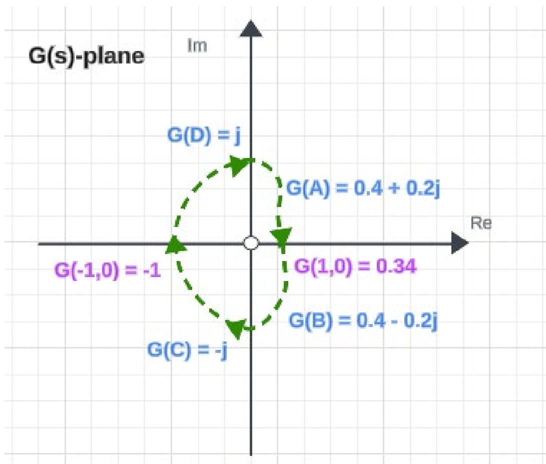
Nyquist Stability Criteria-Preliminaries

Concept 3. Cauchy's argument principle

Consider $G(s) = \frac{s}{s+2}$

In the example





we have $Z = 1$, $P = 0$, then

$$N = Z - P = 1$$

which implies one encirclement of origin in the G(s)-plane in clock-wise direction

Nyquist Stability Criteria-Preliminaries

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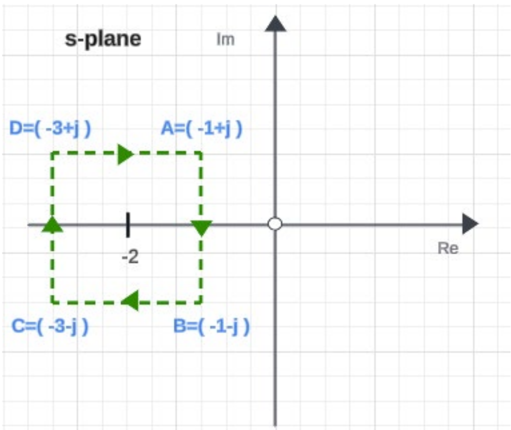

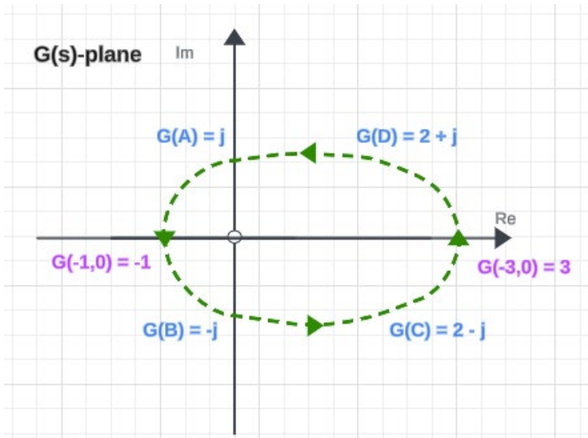

we have $Z = 0$, $P = 1$, then

$$N = Z - P = -1$$

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Nyquist Stability Criteria-Preliminaries

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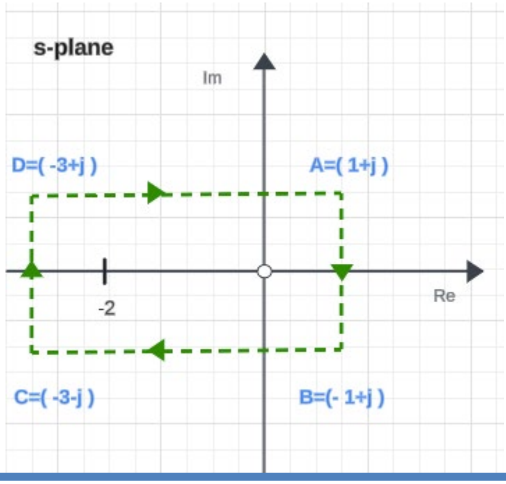

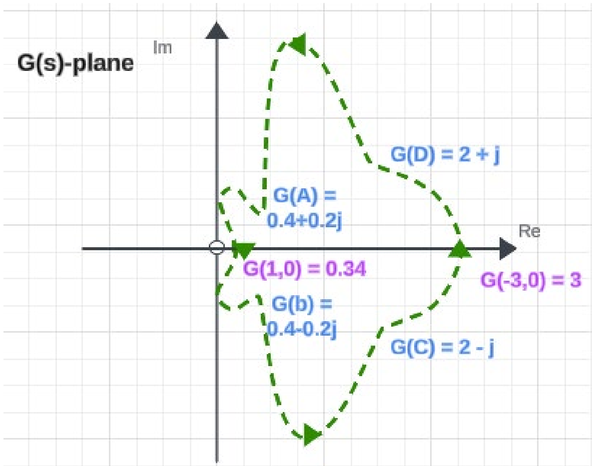

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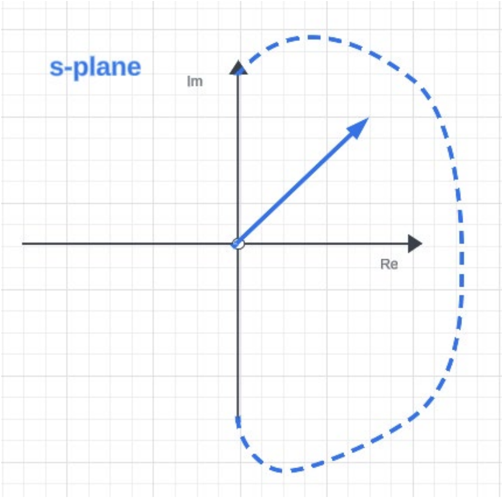
$$N = Z - P = 0$$

which implies no encirclement of origin in the G(s)-plane

Nyquist Stability Criterion

How do we use Nyquist Stability Criterion?

- We can construct a contour that covers the entire right-half plane

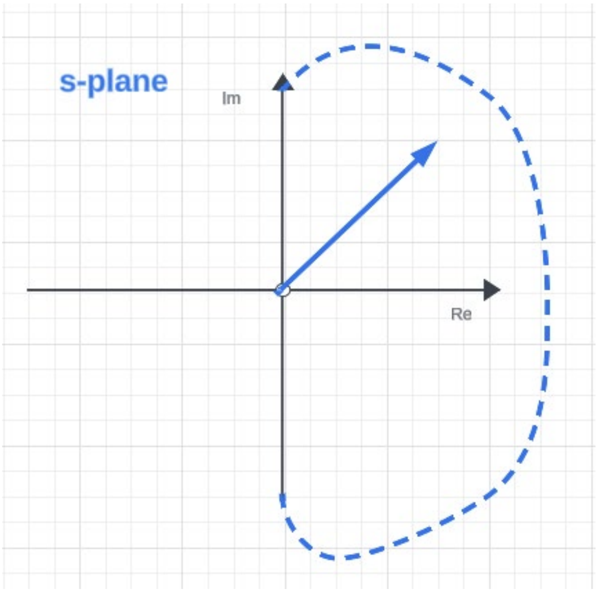


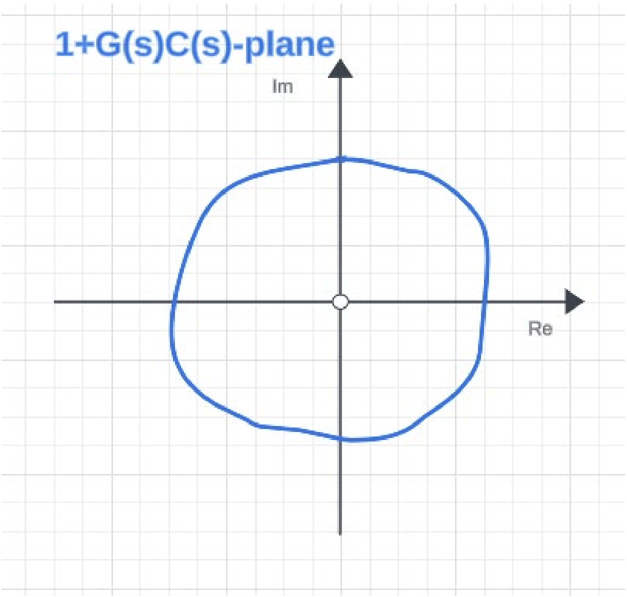
Nyquist Stability Criterion

How do we use Nyquist Stability Criterion?

- We can construct a contour that covers the entire right-half plane

- We map the contour on right half s-plane to the $1 + C(s)G(s)$ –plane. We call this Nyquist plot





Nyquist Stability Criterion

- Then, we use Cauchy's argument principle (Concept 3)

$$N = Z - P,$$

where

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    Concept 2. Zeros of $1 + C(s)G(s)$ =Poles of $\boldsymbol{H}_{\boldsymbol{CL}}(\boldsymbol{s})$ which we can use to analyse closed loop stability

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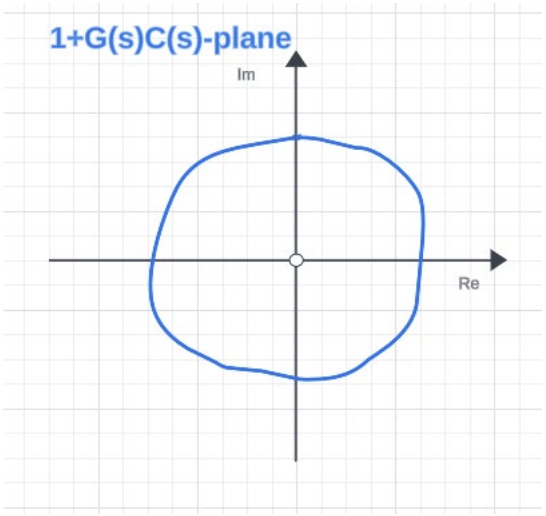
- Note that for stability of the closed loop system we want no closed-loop poles on the right half s-plane. So we require no zeros of $1 + \mathrm{C}(s)\mathrm{G}(s)$ encircled, which means $Z = 0$. Then the stability criteria becomes

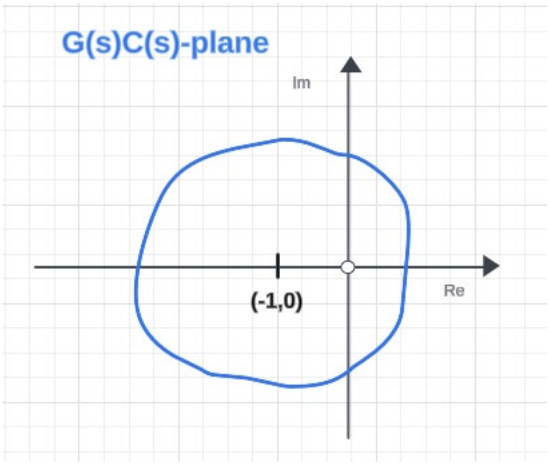
$$\boxed{\mathbf{0} = \boldsymbol{N + P}}$$

Nyquist Stability Criterion

- Instead of mapping $1 + \mathrm{C}(s)\mathrm{G}(s)$ we can directly map $\mathrm{C}(s)\mathrm{G}(s)$, and look for encirclements of the point (-1,0)

$$\boxed{1 + \mathrm{C}(s)\mathrm{G}(s) \Leftrightarrow C(s)G(s) = -1}$$





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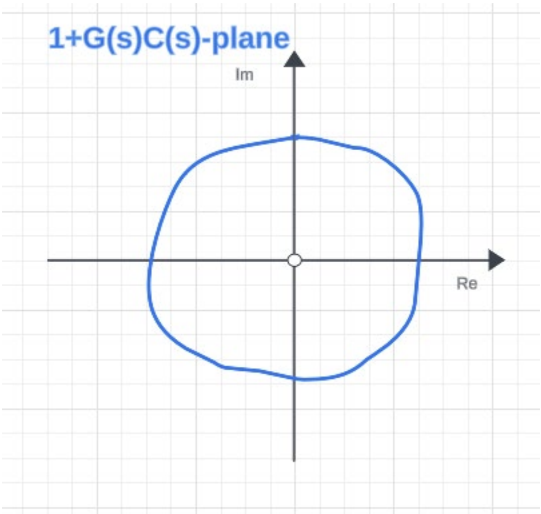

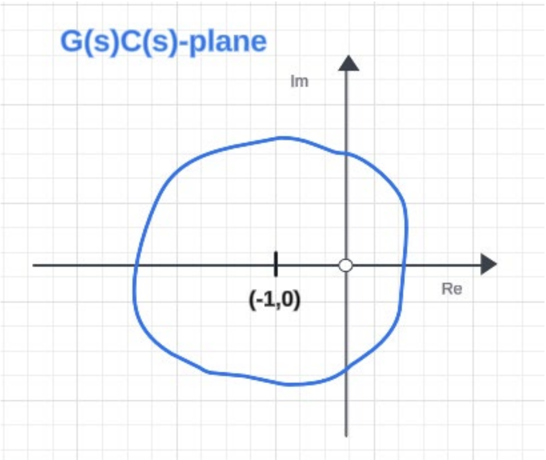

Then we can get $\boldsymbol{N}$ by simply looking at encirclements of map $\boldsymbol{C(s)G(s)}$ around point $(\mathbf{-1, 0})$

Nyquist Stability Criterion

In conclusion,

- We can get $\boldsymbol{N}$ by simply looking at encirclements of map $\boldsymbol{C(s)G(s)}$ around point $(\boldsymbol{-1}, \boldsymbol{0})$
- We can get $\boldsymbol{P}$ by calculating poles of $\boldsymbol{C(s)G(s)}$
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- We have stability if $\boldsymbol{Z = 0}$

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Nyquist criterion:
A feedback control system is stable if and only if the number of counterclockwise encirclements of the point (-1,0) is equal to the number of poles of $\boldsymbol{G(s)C(s)}$ on the right-half of the complex plane

Nyquist Stability Criterion

- Given the contour on the entire right half s-plane, its maping to the $C(s)G(s)$-plane (of $C(s)G(s)$) is called a **Nyquist plot**
- The Nyquist plot is the plot of $C(s)G(s)$ with $s = j\omega$ where ω goes from $-\infty$ to ∞
- From the Nyquist plot, we can check the number of counterclockwise encirclements of the point $(-1,0)$. Thus, we can determine the **stability** of the closed-loop system.

Example: Construction of Nyquist plot

Consider

$$C(s)G(s) = \frac{s+3}{s^2+2s+1} \text{ with } G(s) = \frac{1}{s^2+2s+1}, \qquad C(s) = s+3$$

Replacing $s = j\omega$, we obtain

$$C(j\omega)G(j\omega) = \frac{3+j\omega}{1-\omega^2+j2\omega}$$

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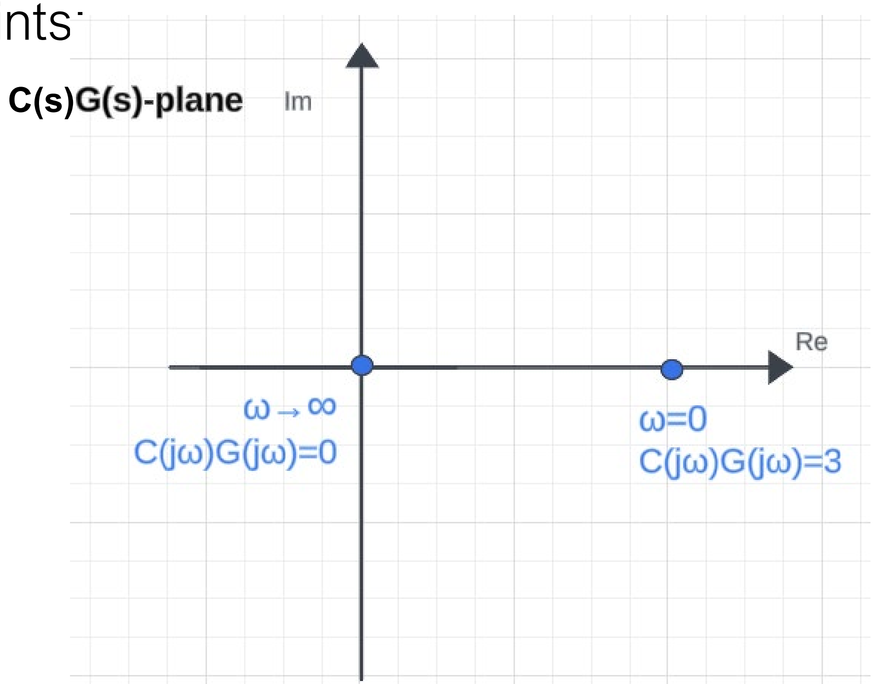
We can plot this points:



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which gives

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We can calculate value of $\Im\{C(j\omega)G(j\omega)\}$ at which $\Re\{C(j\omega)G(j\omega)\} = 0$

$$\Im\{C(j\sqrt{3})G(j\sqrt{3})\} = \Im\{C(-j\sqrt{3})G(-j\sqrt{3})\} = -\frac{\sqrt{3}}{2}$$

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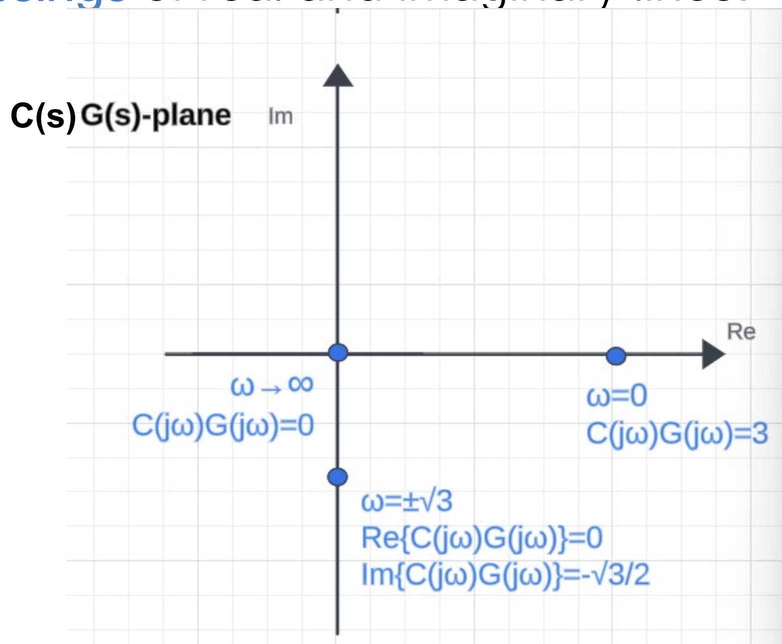
We can plot the crossings of real and imaginary lines:



Example: Construction of Nyquist plot

We have plotted $C(j\omega)G(j\omega)$ for $\omega = 0$ and $\omega \to \pm\infty$, and real and imaginary line crossings.
*Also note that $\Re\{C(j\omega)G(j\omega)\} < 0$ for large values of ω, so negative values of $\boldsymbol{\Re\{C(j\omega)G(j\omega)\}}$ are expected before approaching $\omega \to \pm\infty$

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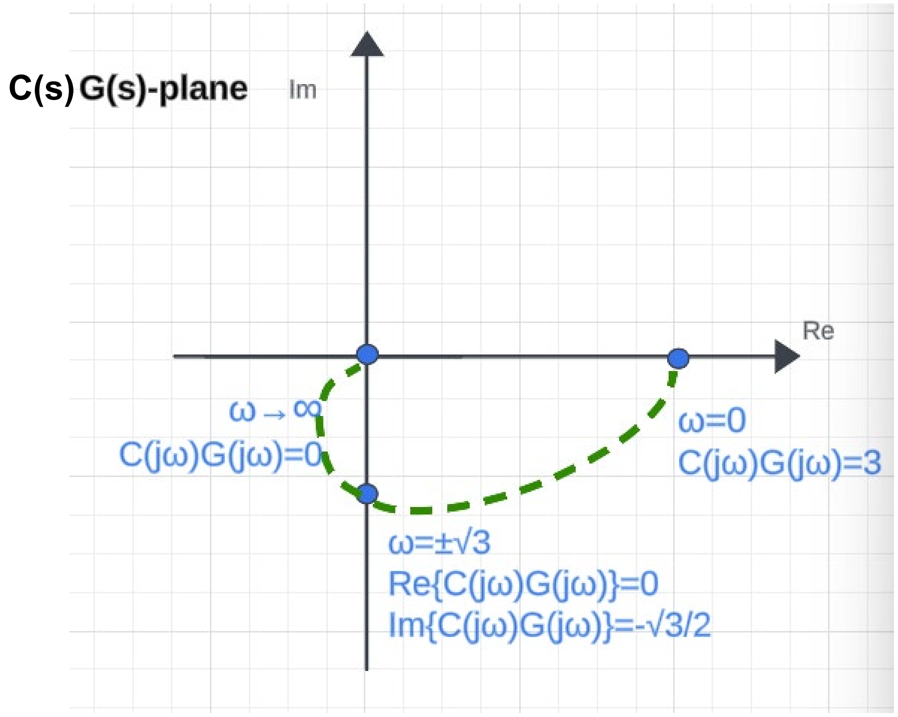
- Knowing this we can approximately connect the points



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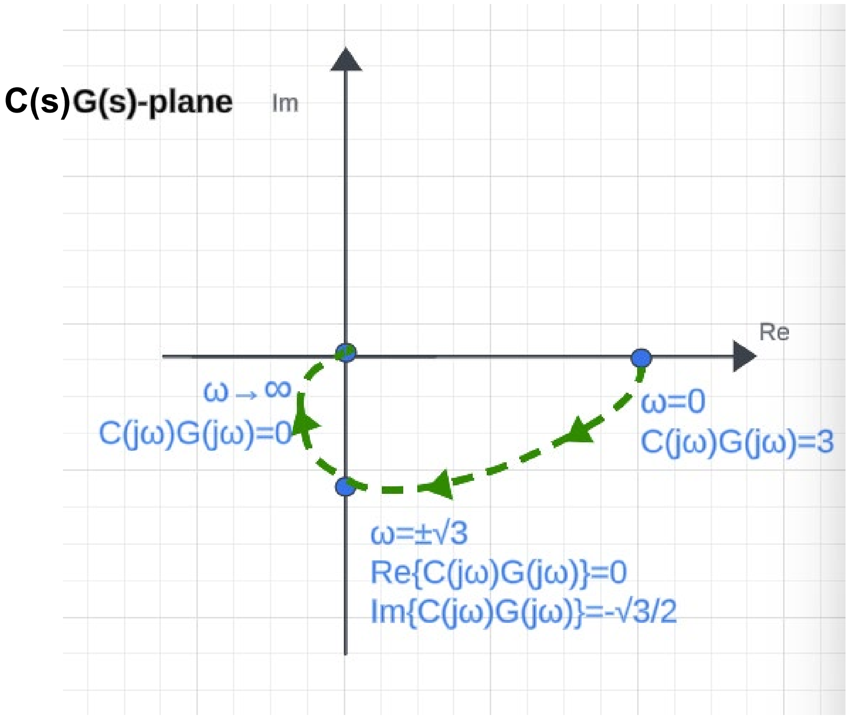
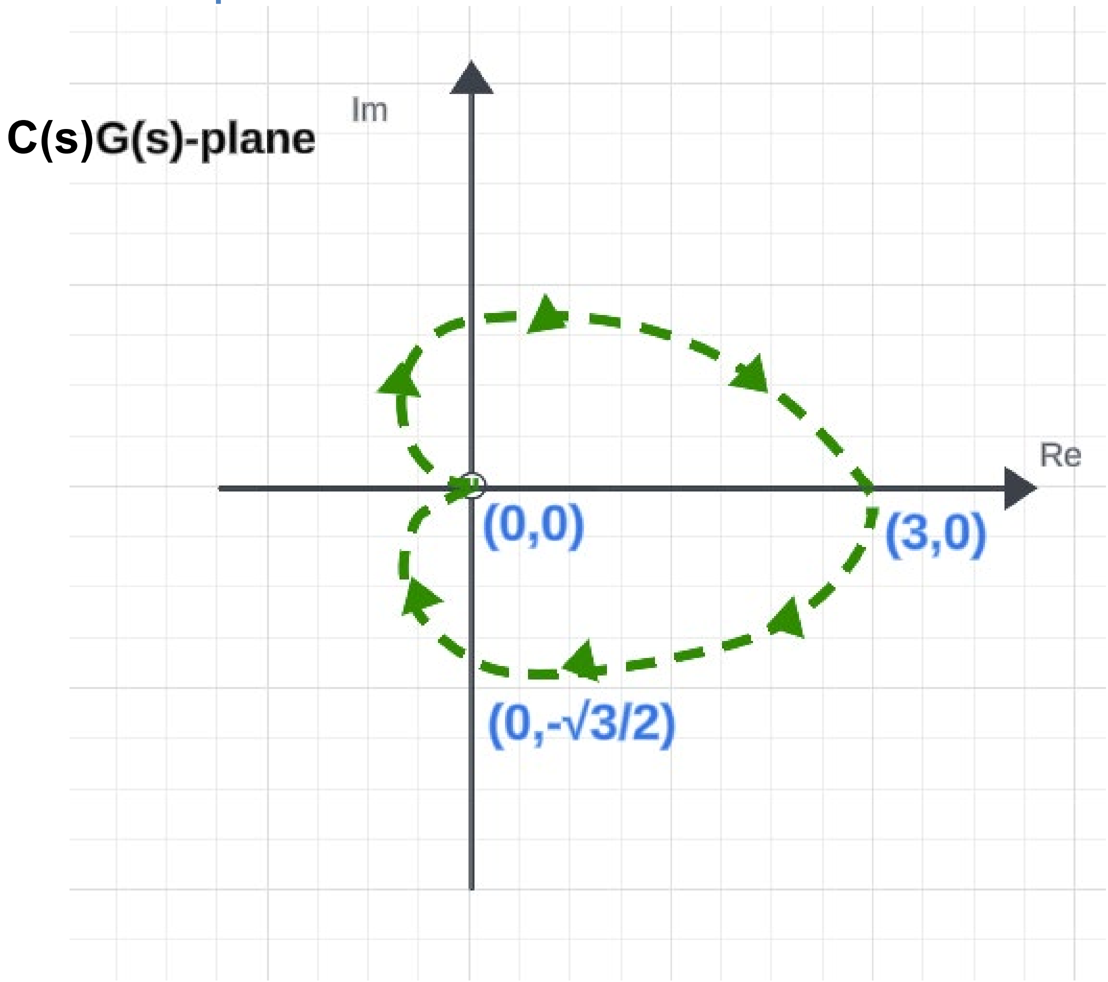
- Knowing this we can approximately connect the points
- The direction is plotted starting from $\omega = 0$ towards $\omega \to \infty$



Example: Construction of Nyquist plot

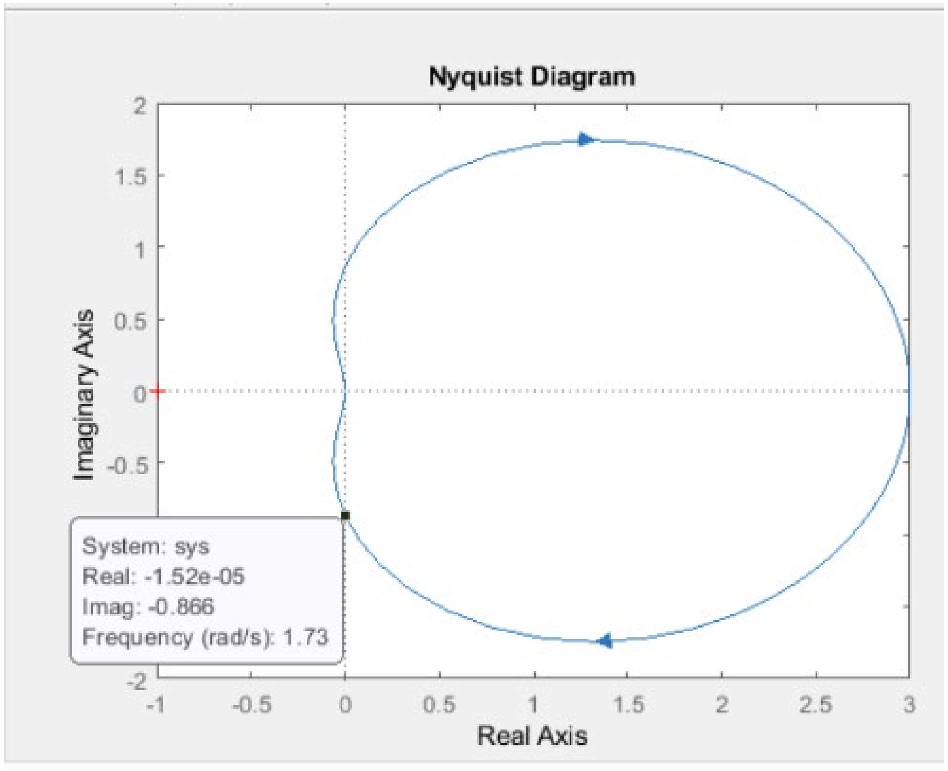
The Nyquist plot is symmetric around the Real line, so we can get an estimation of the full plot:



Example: Construction of Nyquist plot

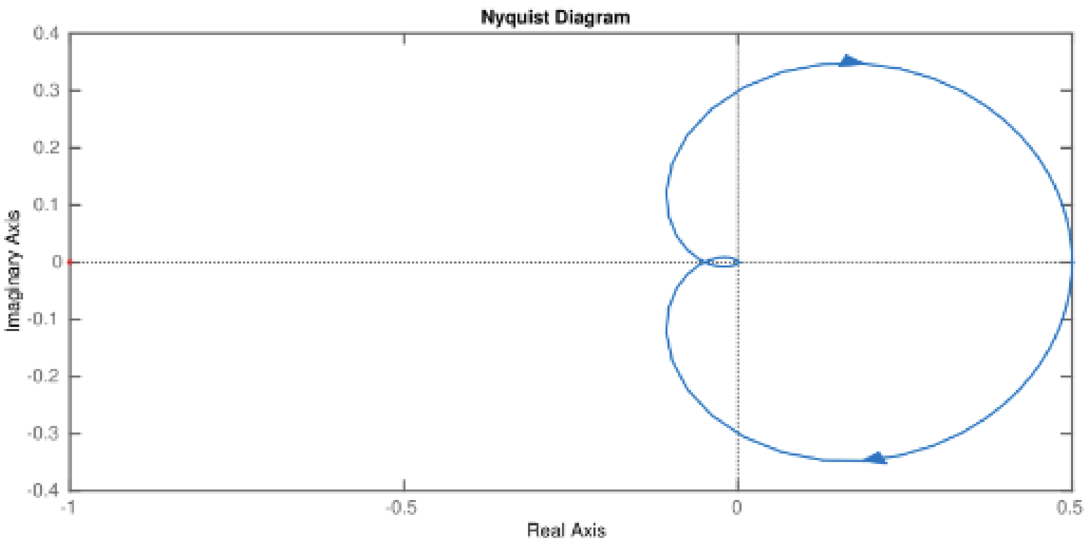
In MATLAB

Create the system: sys=tf([0 1 3], [1 2 1])
Use function to obtain plot: nyquist(sys)



Example: Stability using Nyquist

Take a system with transfer function $G(s)$ and controller with transfer function $C(s)$.
The loop gain is defined as $L(s) := G(s)C(s)$ and is stable
The Nyquist plot looks as follows:



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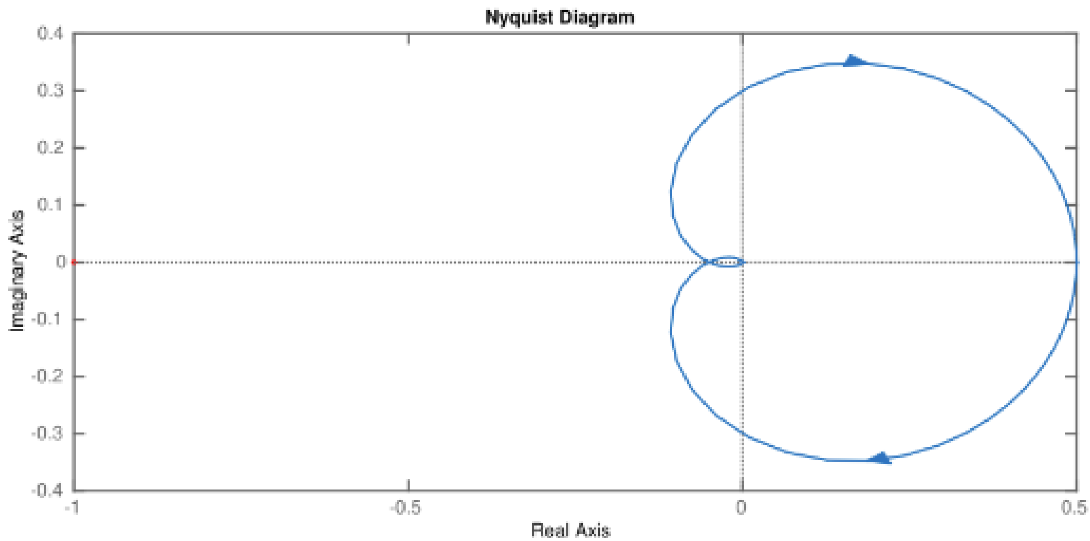

Since $L(s)$ is stable we have no poles in the right-half plane.

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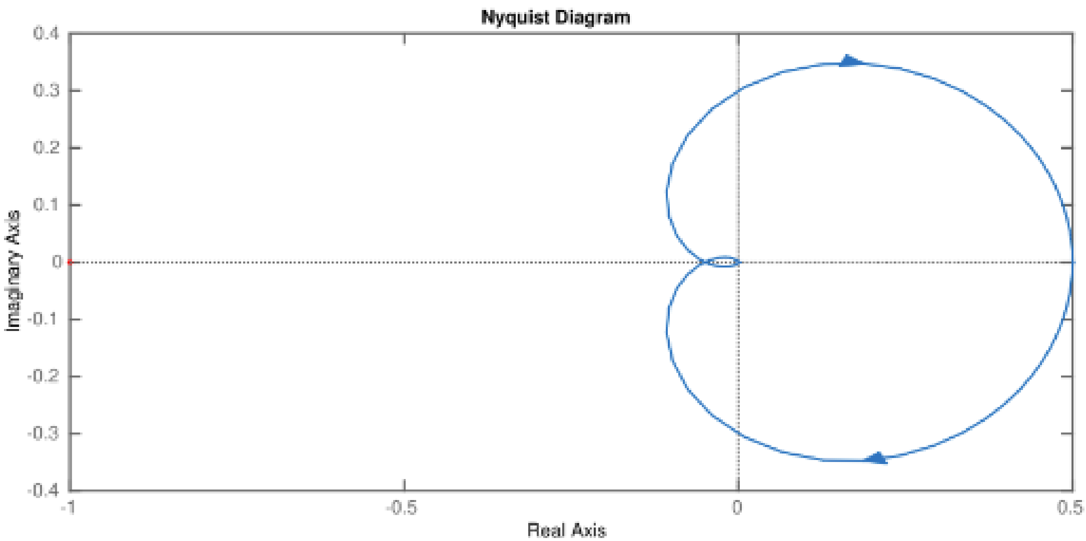

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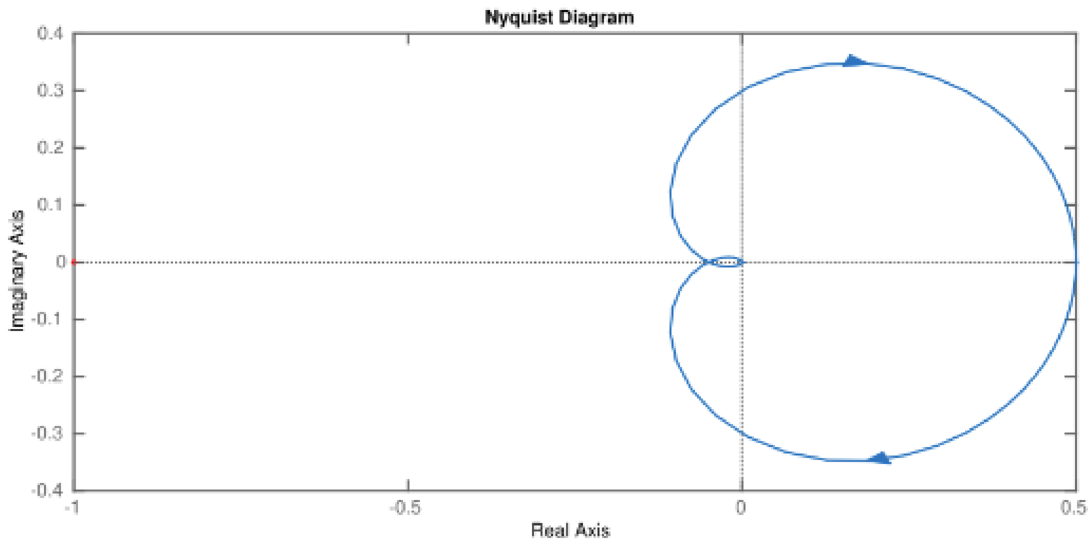

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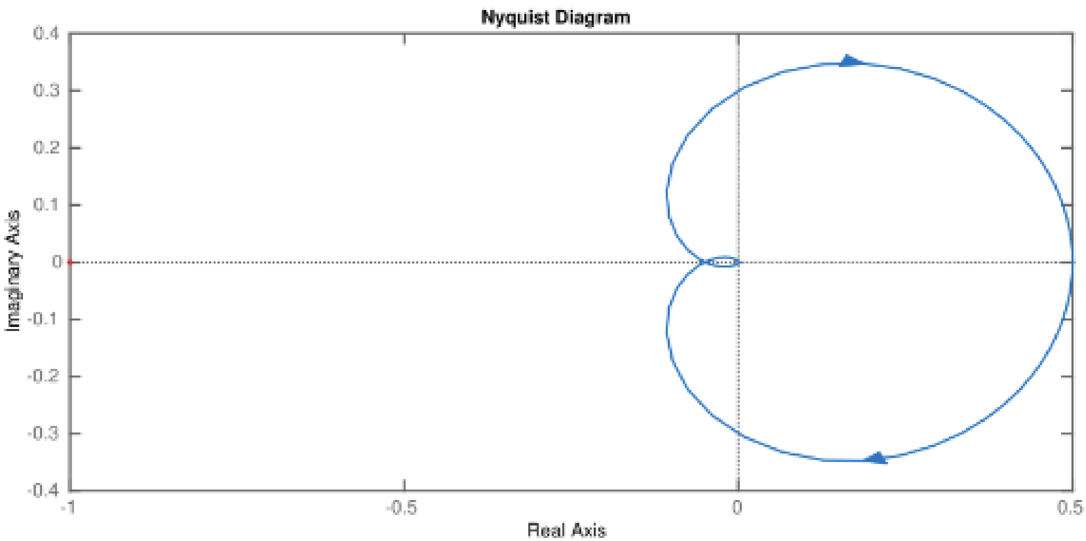

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How do we take into account delays? Phase and gain margin

Gain margin and phase margin

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- The gain margin and phase margin are two quantitative measures of the stability of the system

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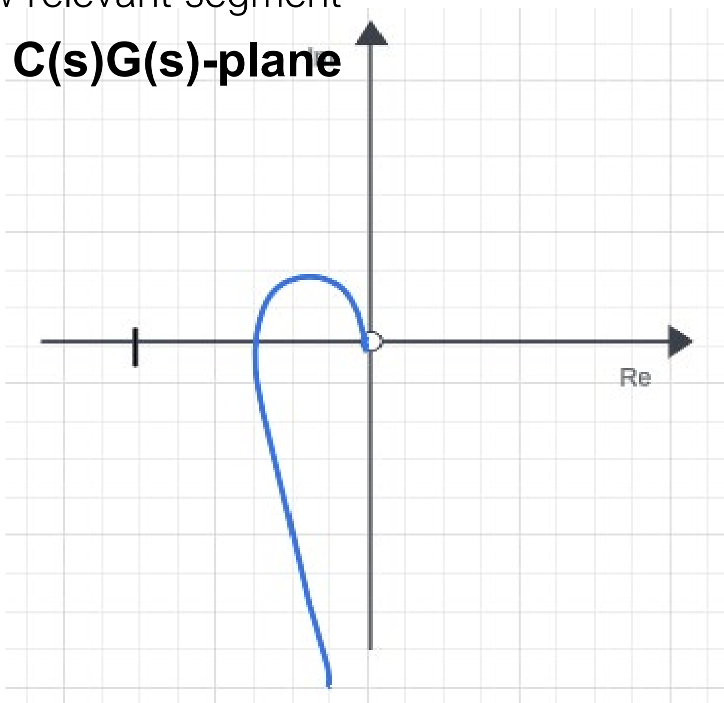
- They determine how much the gain and phase of the system can shift before they reach instability
- They can be computed for a system
- They can be obtained from Nyquist plot
- The critical time delay the system can undergo until reaching instability can be computed from phase margin. This shows us how robust the system is to delays.

Gain and phase margin from Nyquist

Take a Nyquist plot

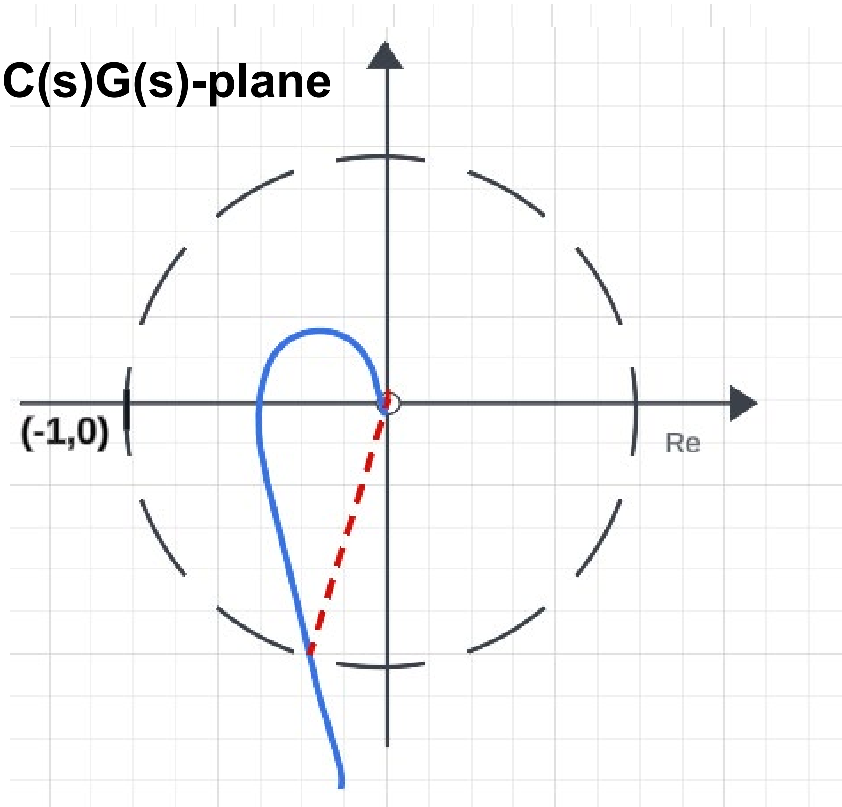
*Due to symmetry only one half is enough

*We care about at intersection with unit circle and real axis so for simplicity we only draw relevant segment



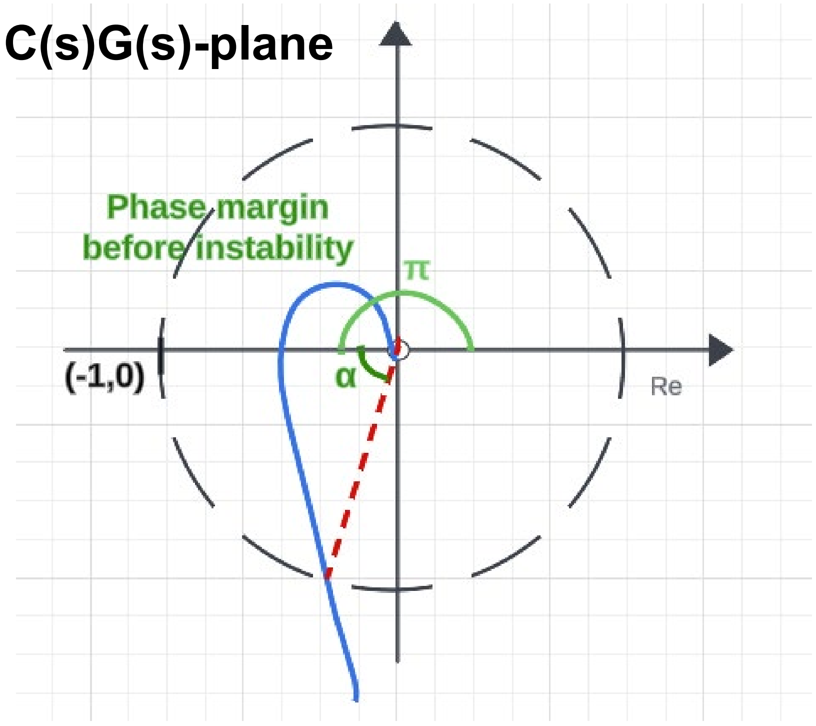
Gain and phase margin from Nyquist

We find crossing of Nyquist plot and unitary circle



Gain and phase margin from Nyquist

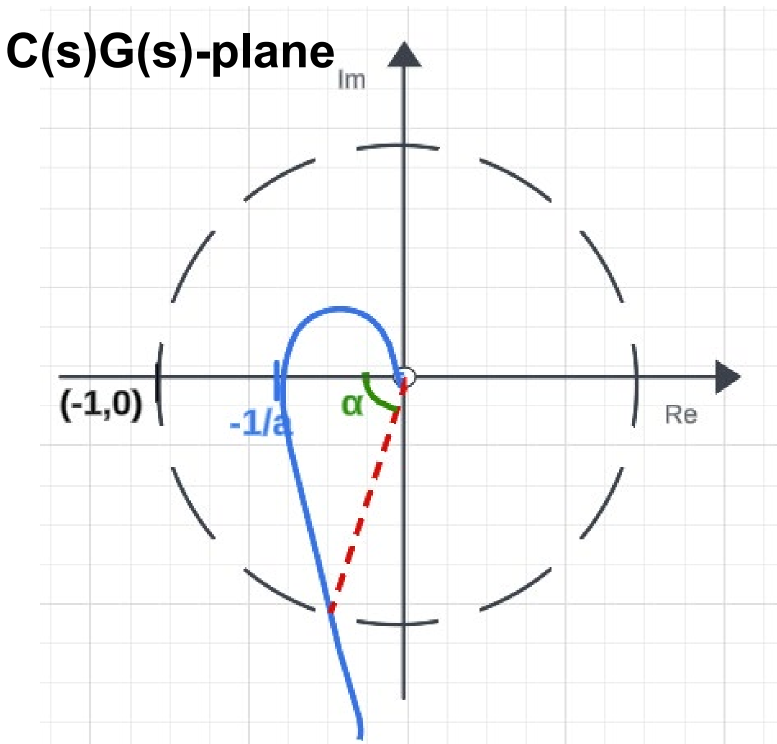
We find crossing of Nyquist plot and unitary circle
The angle between positive real line and said crossing is phase margin



$$P_m = -\pi - \alpha$$

Gain and phase margin from Nyquist

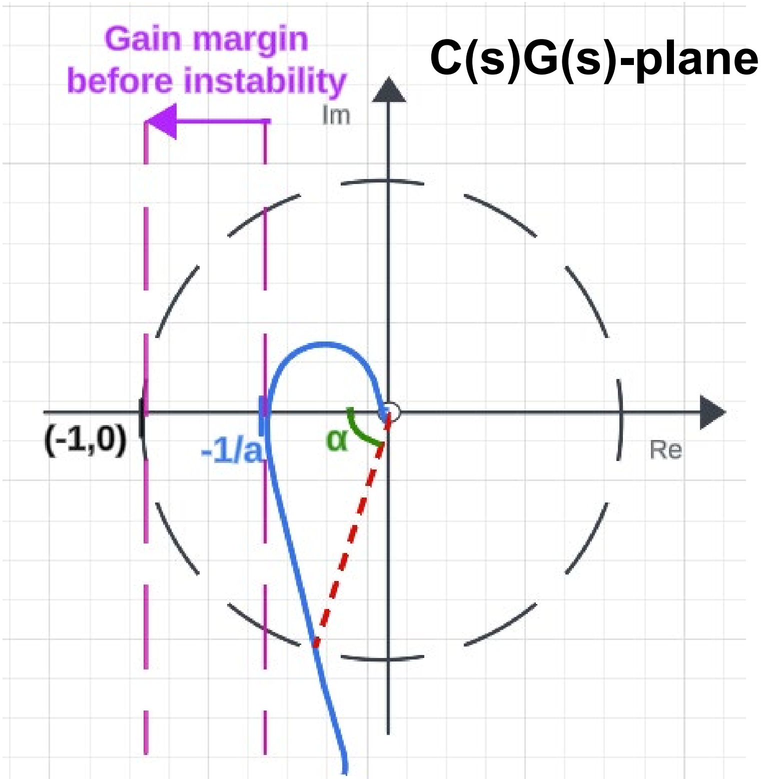
Crossing of Nyquist plot and real line gives point (-1/a, 0)



$$G_m = 20 log a$$

Gain and phase margin from Nyquist

Crossing of Nyquist plot and real line gives point (-1/a, 0)
Gain margin is difference until point (-1/a,0) reaches (-1,0)



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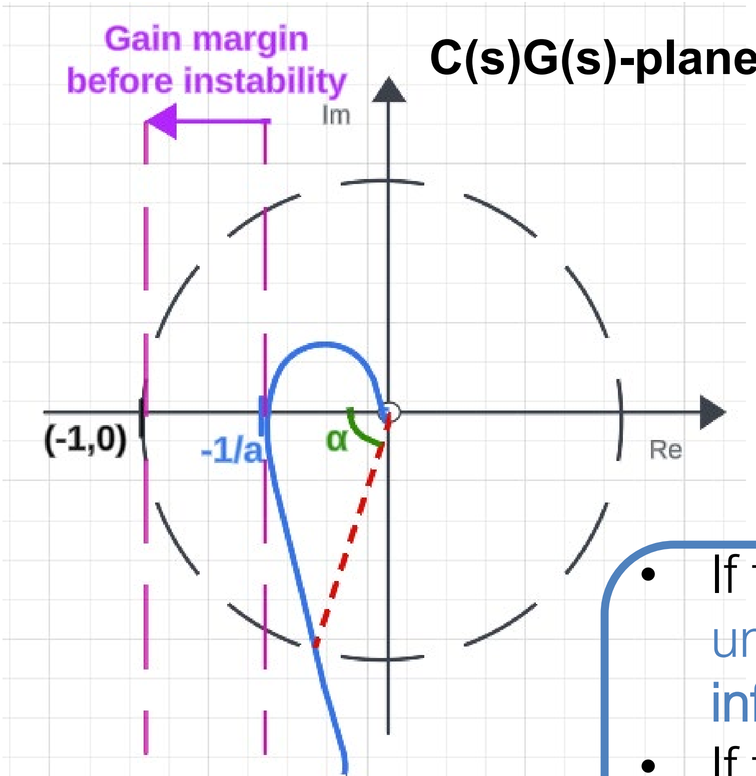
Gain and phase margin from Nyquist



$$P_m = -\pi - \alpha$$

$$G_m = 20 log a$$

IMPORTANT Remark

- If the Nyquist plot does not cross the unitary circle, then then **phase margin** is **infinite**
- If the Nyquist plot does not cross the real line in the left half plane, then the **gain margin** is **infinite**

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- It is expressed in dBs of phase shift required at $\pi(180°)$ to make the closed-loop system unstable.

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- For this purpose, we find ω^* such that $\angle G(j\omega^*)C(j\omega^*) = -\pi$. Then G_m is given by

$$\boxed{G_m = \frac{1}{|G(j\omega^*)C(j\omega^*)|}}$$

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Stability of delayed systems

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If we compute the magnitude and phase of the transfer function for $s = j\omega$, with ω starting from $\omega = -\infty$ to $\omega = \infty$, we notice that

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-magnitude is always one.

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Consider ω^* such that $|G(j\omega^*)C(j\omega^*)| = 1$. If there exists a time delay, the closed-loop system is stable if the shift phase ω^* does not exceed P_m, that is,

$$P_m + (T\omega^*) < 0$$

Therefore, the critical time delay is given by

$$T = -\frac{P_m}{\omega^*}$$

Example: computation of critical time delay

Consider

$C(s)G(s) = \frac{s+3}{s^2+2s+1}$ with $G(j\omega) = \frac{1}{1-\omega^2+j2\omega}$ and $C(j\omega) = 3 + j\omega$

We previously calculated real and imaginary parts:

$$\Re\{C(j\omega)G(j\omega)\} = \frac{3-\omega^2}{1+2\omega^2+\omega^4}, \Im\{C(j\omega)G(j\omega)\} = \frac{-\omega(5+\omega^2)}{1+2\omega^2+\omega^4}$$

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Step 1. Find ω^* such that $|C\{j\omega^*\}G\{j\omega^*\}| = 1$

This implies that

$$\left|\frac{3+j\omega^*}{(1-\omega^{*2})+j2\omega^*}\right| = 1 \Rightarrow \frac{\sqrt{9+(\omega^*)^2}}{\sqrt{(1-(\omega^*)^2)^2+4(\omega^*)^2}} = 1 \Rightarrow (\omega^*)^4 + (\omega^*)^2 - 8 = 0$$

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Letting $\alpha := (\omega^*)^2$,we have

$$\alpha^2 + \alpha - 8 = 0$$

which yields $\alpha = 2.3723$ or $\alpha = -3.3723$. Therefore

$$\boxed{\omega^* = \sqrt{\alpha} = \sqrt{2.3723} = 1.5402\text{rad/sec}}$$

*We choose only positive value of α because fequency ω^*should be real number

Example: computation of critical time delay

Step 2. Compute $\angle C(j\omega^*)G(j\omega^*)$

This implies that

$$\angle C(j\omega^*)G(j\omega^*) = \frac{\Im\{C(j\omega^*)G(j\omega^*)\}}{\Re\{C(j\omega^*)G(j\omega^*)\}} = \frac{-\omega^*(5+(\omega^*)^2)}{3-{\omega^*}^2} = \frac{-1.5402(5+2.3723)}{3-2.3723}$$

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Step 4: Compute $P_{m_d} = P_m + (T\omega^*)$ and $P_{m_d} < 0$ for stability

Step 5: The critical time delay is given by

$$\boxed{T = -\frac{P_m}{\omega^*} = \frac{1.16260}{1.5402} = 1.0557\ sec}$$

This is the time delay that the system can handle before instability

Summary

- Delays may modify the stability properties of a system
- The Nyquist's criterion offers an alternative to check the stability of a system without computing its poles
- The phase margin of a system is helpful to determine whether the system is robust or not in the presence of delays

Next week:

Absolute stability.

Stability of nonlinear system as per Circle criterion and Popov's criterion