Geometry 2023
Resit, Thursday 29 June, 15:00-17:00

- Below you can find the resit exam questions. There are 4 questions summing up to 85 points; you get extra 15 points for a clear writing of solutions.
- You may consult two A4 pages handwritten or typeset by you for formulas, results, etc. treated in this course. Other materials are not allowed.
- When handing in your solutions, please do not forget to write your name and student number on the envelope. Good luck!

Solutions

1. $\boxed{10+15 = 25 \text{ pts}}$ a) Compute the evolute of the following curve in $\mathbb{R}^2$:

$$\gamma(t) = (t^3, t^4) \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}^2.$$

b) Show that for every $k \in \mathbb{N} \cup \{0\}$, the curvature of a smooth regular curve $\gamma \colon \mathbb{R} \to \mathbb{R}^n,\ n \geq 2$, can have exactly k singular points.

Hint for part b): Euclidean Frenet-Serret formulas.

a) Recall that the the evolute of a parametrised curve $\gamma = (x(t), y(t))$ is given by the general formula

$$C(t) = \gamma(t) + \rho(t) n(t),$$

where $\rho(t)$ is the curvature radius and $n(t)$ the unit normal to $\gamma(t)$. In Cartesian coordinates,

$$C(t) = (x(t), y(t)) + \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}(-\dot{y}, \dot{x}),$$

In our case, $x(t) = t^3$ and $y(t) = t^4$. A direct computation yields

$$\rho(t) = \frac{t^4}{12}(9 + 16t^2)^{3/2}$$

and

$$C(t) = (-2t^3 - \frac{16}{3}t^5, 5t^4 + \frac{9}{4}t^2).$$

b) First observe that it suffices to let $n = 2$, as any plane curve naturally sits in $\mathbb{R}^n$ for $n \geq 2$.

Next recall that the Frenet-Serret formulas in dimension 2 have the form

$$\begin{pmatrix} \frac{d\tau(s)}{ds} \\ \frac{dn(s)}{ds} \end{pmatrix} = \begin{pmatrix} 0 & K(s) \\ -K(s) & 0 \end{pmatrix} \begin{pmatrix} \tau(s) \\ n(s) \end{pmatrix}, \tag{1}$$

where $\tau(s)$ is the unit tangent and $n(s)$ is the unit normal of a regular curve γ parametrised by arc-length.

Let $a = a(s)$ be a bounded curve with exactly k zeros and define $K = \int_0^s a(s)ds$. Then K has exactly k singular points.

Solving the Frenet-Serret equations (1), with the initial conditions $\tau(0) = (1,0)$ and $n(0) = (0,1)$, we get a curve γ with the required property that its curvature $K(s)$ has exactly k singular points.

2. $\boxed{5+15+10 = 30 \text{ pts}}$ Consider a 2-surface $M^2 \subset \mathbb{R}^3$ given by the implicit equations

$$M^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^4 + y^4 + z^4 = 1\}.$$

i) Prove that M^2 is a regular surface in $\mathbb{R}^3$;

ii) Prove that the intersections of M^2 with the planes $x = 0$, $y = 0$ and $z = 0$ are geodesics of M^2;

iii) Using part ii), subdivide M^2 into 8 geodesic triangles. Deduce that

$$\frac{1}{2\pi} \int_{M^2} K \mathrm{d}S = 2,$$

where $K \colon M^2 \to \mathbb{R}$ is the Gaussian curvature of M^2 and $\mathrm{d}S$ is the area element.

Hint for part iii): local Gauss-Bonnet theorem.

i) Let $f = x^4 + y^4 + z^4 \colon \mathbb{R}^3 \to \mathbb{R}$ and observe that the gradient $\mathrm{grad} f \neq 0$ on M^2. It follows that M^2 is a regular surface.

ii) By the symmetry of the variables, it suffices to consider the intersection of M^2 with the plane $x = 0$.

Consider the reflection s_x of $\mathbb{R}^3$ in this plane, $s_x(x,y,z) = (-x,y,z)$.

Then s_x is an isometry of M^2 and hence maps geodesics to geodesics.

Consider the geodesic γ passing through the point $p = (0,0,1) \in M^2$ in the direction $v = (0,1,0)$.

Since $s_x(p) = p$ and $ds_x(v) = v$, it follows that $s_x(\gamma) = \gamma$.

But there is only one such curve, namely the intersection of M^2 with the plane $x = 0$ (parametrised by constant speed).

iii) The 3 geodesics $M^2 \cap \{x = 0\}$, $M^2 \cap \{y = 0\}$, and $M^2 \cap \{z = 0\}$, subdivide M^2 into 8 isometric triangles with interior angles equal to $\frac{\pi}{2}$.

According to the local Gauss-Bonnet theorem, for one such triangle ABC, we have

$$\int_{M^2} K \mathrm{d}S = \frac{3\pi}{2} - \pi = \frac{\pi}{2}.$$

Hence

$$\frac{1}{2\pi} \int_{M^2} K \mathrm{d}S = \frac{4\pi}{2\pi} = 2.$$

3. $\boxed{5+13 = 18\text{pts}}$ Consider the affine 3-space $(\mathbb{Z}_2)^3$, where $\mathbb{Z}_2$ is the two-element field.

   i) Are two non-intersecting lines in $(\mathbb{Z}_2)^3$ always parallel?

   ii) State and prove an affine version of Desargues's theorem in $(\mathbb{Z}_2)^3$.

i) Two non-intersecting lines in an affine space are parallel iff they have the same direction. Consider the lines $\ell_1 = \{(0,0,0),(0,0,1)\}$ and $\ell_2 = \{(1,0,0),(1,1,1)\}$. These lines do not intersect, but they are still not parallel since they are directed by the vectors $(0,0,1)$ and $(0,1,1)$, respectively, which are linearly independent.

ii) Consider two proper triangles $A_1B_1C_1$ and $A_2B_2C_2$ in $(\mathbb{Z}_2)^3$ with non-coinciding vertices and whose sides are respectively parallel. (Here proper means that the points A_1, B_1, and C_1 are not collinear; In this case it is in fact implied by the the 6 points $A_1B_1C_1$ and $A_2B_2C_2$ being distinct.) Then the lines A_1A_2, B_1B_2 and C_1C_2 are parallel.

For the proof, consider the translation $f(P) = P + \overrightarrow{A_1A_2}$.

Since translations map parallel lines to parallel lines, B_1 must be sent to B_2; Indeed, A_1B_1 is parallel to A_2B_2.

Similarly, C_1 must be sent to C_2. It follows that the three lines A_1A_2, B_1B_2 and C_1C_2 are parallel.

4. $\boxed{12\text{pts}}$ Consider two distinct lines ℓ_1 and ℓ_2 in $\mathbb{R}^2$ and a perspectivity $f\colon \ell_1 \to \ell_2$ with center $O \in \mathbb{R}^2 \setminus (\ell_1 \cup \ell_2)$. Show that f is a Euclidean isometry if and only if ℓ_1 and ℓ_2 are parallel and the distance from O to ℓ_1 and ℓ_2 is the same.

Hint: recall the definition of a perspectivity f: for a point $x \in \ell_1$, $f(x)$ is defined as the intersection point of the line Ox with ℓ_2 if these lines intersect (if Ox is parallel to ℓ_2, then $f(x)$ is defined to be the point at infinity of ℓ_2).

First observe that ℓ_1 and ℓ_2 must be parallel, for otherwise f sends a finite point of ℓ_1 to the point at infinity of ℓ_2.

Let now ℓ_1 and ℓ_2 be parallel. Without loss of generality, O is the origin in $\mathbb{R}^2$ and ℓ_i is given by $\{(x, y) \in \mathbb{R}^2 \mid y = c_i\},\ c_i \neq 0$.

Then the perspectivity f from ℓ_1 to ℓ_2 with center O is given by the formula

$$(x, c_1) \mapsto (\frac{c_2}{c_1}x, c_2).$$

But the map $f(x) = \frac{c_2}{c_1}x$ is a Euclidean isometry if and only if $c_1 = \pm c_2$.

Since ℓ_1 and ℓ_2 are distinct, $c_1 = -c_2$.

End of exam