Mechatronics

Week 5 Day 2

Previously

- We studied how to build an optimal state controller using the LQR method

Today's lecture:
Observer for optimal control

Learning objectives

After today's lecture, you will be able to

- Design an observer to estimate states of a system from input and output
- Use an observer together with LQR method to control a system without measuring every state

Motivation

Optimal state controller $u = -Kx$ is ideal

Motivation

Optimal state controller $u = -Kx$ is ideal

PROBLEM: it requires measuring every state

Sensors are expensive so we try to use as few as possible

Motivation

Optimal state controller $u = -Kx$ is ideal

PROBLEM: it requires measuring every state

Sensors are expensive so we try to use as few as possible

SOLUTION: reconstruct state from input and output

Observer can estimate states

Controller is fed estimated states

Separation principle: separate designs of controller and observer

*Only applicable to linear systems

Observer or State Estimator

Observer

Consider the system $\begin{cases} \dot{x} = Ax + B\mathrm{u} \\ y = Cx + Du \end{cases}$

Let us denote the estimated state as $\hat{x}$.

Observer

Consider the system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

Let us denote the estimated state as $\hat{x}$.

Then, an observer, or state estimator is given by

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}),}$$

where $\hat{y} = C\hat{x} + Du$

and

L is a constant matrix to determine.

Observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ and } \hat{y} = C\hat{x} + Du$$

How to determine $\boldsymbol{L}$?

Observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ and } \hat{y} = C\hat{x} + Du$$

How to determine L?

Error between true state x and estimated state $\hat{x}$ can be defined as

$$e = x - \hat{x}$$

Observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ and } \hat{y} = C\hat{x} + Du$$

How to determine $\boldsymbol{L}$?

Error between true state x and estimated state $\hat{x}$ can be defined as

$$e = x - \hat{x}$$

We want $\lim_{t\to\infty} e(t) \to 0$ i.e $\lim_{t\to\infty} x = \hat{x}$

Observer

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ and } \hat{y} = C\hat{x} + Du}$$

How to determine L?

Error between true state x and estimated state $\hat{x}$ can be defined as

$$e = x - \hat{x}$$

We want $\lim_{t\to\infty} e(t) \to 0$ i.e $\lim_{t\to\infty} x = \hat{x}$.

It follows that $\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + \cancel{B}u - A\hat{x} - \cancel{B}u - L(y - \hat{y})$

$$= A(x - \hat{x}) - \mathrm{L}(\mathrm{C}x + \cancel{\mathrm{D}}\mathrm{u} - \mathrm{C}\hat{x} - \cancel{D}u)$$

$$= A(x - \hat{x}) - LC(x - \hat{x})$$

$$= (A - LC)(x - \hat{x}) = \boxed{(A - LC)e}$$

Observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ and } \hat{y} = C\hat{x} + Du$$

How to determine $\boldsymbol{L}$?

Error between true state x and estimated state $\hat{x}$ can be defined as

$$e = x - \hat{x}$$

We want $\lim_{t\to\infty} e(t) \to 0$ i.e $\lim_{t\to\infty} x = \hat{x}$.

It follows that $\dot{e} = (A - LC)e$

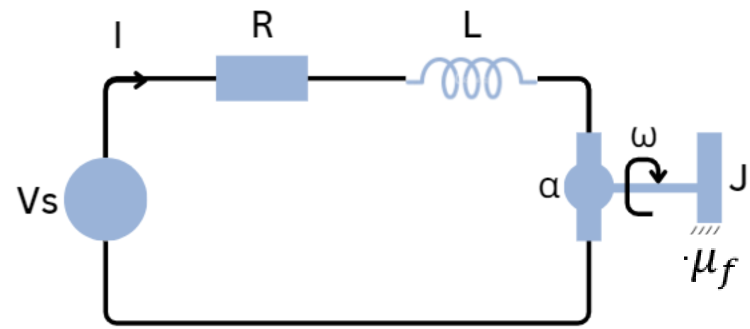
which implies we will choose $\boldsymbol{L}$ such that the eigevalues of $(\boldsymbol{A} - \boldsymbol{LC})$ have negative real parts

Example: Observer for a DC Motor

Given the DC motor for which the state-space is as follows:

$$\underbrace{\begin{bmatrix} \dot{I} \\ \dot{\omega} \\ \dot{\theta} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -R/L & -1/\alpha L & 0 \\ 1/\alpha J & -\mu_f/J & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I \\ \omega \\ \theta \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}_{B} \underbrace{V_S}_{u}$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} I \\ \omega \\ \theta \end{bmatrix}}_{x}$$

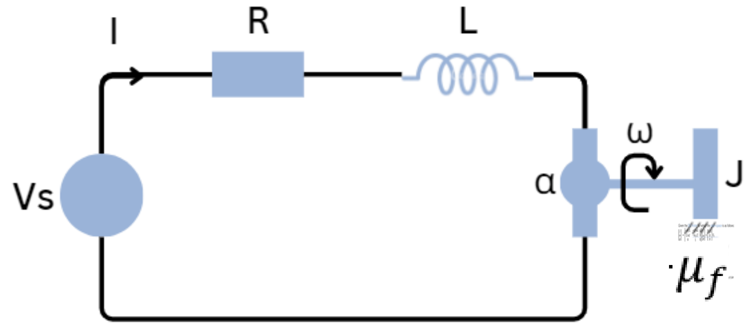


Example: Observer for a DC Motor

Given the DC motor for which the state-space is as follows:

$$\underbrace{\begin{bmatrix} \dot{I} \\ \dot{\omega} \\ \dot{\theta} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -R/L & -1/\alpha L & 0 \\ 1/\alpha J & -\mu_f/J & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I \\ \omega \\ \theta \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}_{B} \underbrace{V_S}_{u}$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} I \\ \omega \\ \theta \end{bmatrix}}_{x}$$



*Note that the **output** is the angular position $\boldsymbol{\theta}$

We can design a PID controller

$$V_s = K_p(r(t) - y(t)) + K_d(\dot{r}(t) - \dot{y}(t)) + K_i \int_0^t r(\tau) - y(\tau) d\tau$$

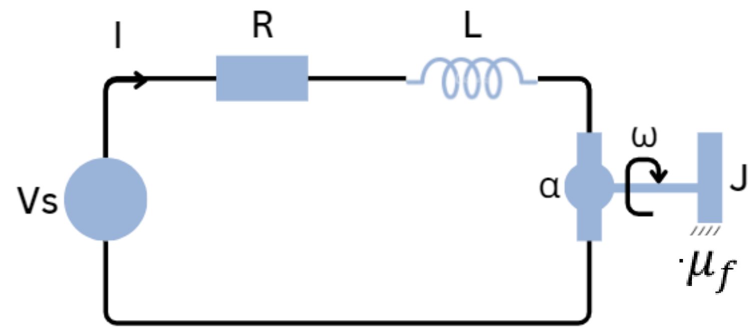
where r is a reference signal.

Example: Observer for a DC Motor

Given the DC motor for which the state-space is as follows:

$$\underbrace{\begin{bmatrix} \dot{I} \\ \dot{\omega} \\ \dot{\theta} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -R/L & -1/\alpha L & 0 \\ 1/\alpha J & -\mu_f/J & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I \\ \omega \\ \theta \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}}_{B} \underbrace{V_s}_{u}$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} I \\ \omega \\ \theta \end{bmatrix}}_{x}$$



*Note that the output is the angular position $\boldsymbol{\theta}$

We can design a PID controller

$$V_s = K_p(r(t) - y(t)) + K_d(\dot{r}(t) - \dot{y}(t)) + K_i \int_0^t r(\tau) - y(\tau) d\tau$$

where r is a reference signal.

❗We require $\dot{y} = \dot{\theta} = \omega$, but it is not available in measured output y

💡Solution: state-estimation

Example: Observer for a DC Motor

For state estimation we can build an observer with form:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ where } \hat{y} = C\hat{x} + Du$$

Example: Observer for a DC Motor

For state estimation we can build an observer with form:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ where } \hat{y} = C\hat{x} + Du$$

Since $\mathrm{D} = 0 \rightarrow y = Cx$ and $\hat{y} = C\hat{x} \rightarrow \boxed{\dot{\hat{x}} = A\hat{x} + Bu + LC(x - \hat{x})}$

Example: Observer for a DC Motor

For state estimation we can build an observer with form:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ where } \hat{y} = C\hat{x} + Du$$

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + LC(x - \hat{x})}$$

For simpicity we'll take $R = L = J = \mu_f = \alpha = 1$

Then the observer looks as follows

$$\underbrace{\begin{bmatrix} \dot{\hat{I}} \\ \dot{\hat{\omega}} \\ \dot{\hat{\theta}} \end{bmatrix}}_{\dot{\hat{x}}} = \underbrace{\begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \hat{I} \\ \hat{\omega} \\ \hat{\theta} \end{bmatrix}}_{\hat{x}} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B} \underbrace{V_S}_{u} + \underbrace{\begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} I - \hat{I} \\ \omega - \hat{\omega} \\ \theta - \hat{\theta} \end{bmatrix}}_{x - \hat{x}}$$

Example: Observer for a DC Motor

For state estimation we can build an observer with form:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ where } \hat{y} = C\hat{x} + Du$$

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + LC(x - \hat{x})}$$

For simpicity we'll take $R = L = J = \mu_f = \alpha = 1$

Then the observer looks as follows

$$\underbrace{\begin{bmatrix} \dot{\hat{I}} \\ \dot{\hat{\omega}} \\ \dot{\hat{\theta}} \end{bmatrix}}_{\dot{\hat{x}}} = \underbrace{\begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \hat{I} \\ \hat{\omega} \\ \hat{\theta} \end{bmatrix}}_{\hat{x}} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B} \underbrace{V_S}_{u} + \underbrace{\begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} I - \hat{I} \\ \omega - \hat{\omega} \\ \theta - \hat{\theta} \end{bmatrix}}_{x - \hat{x}}$$

where $L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$ needs to be designed such that eigenvalues of $(\boldsymbol{A} - \boldsymbol{LC})$ are in the left hand plane. Convergence will be faster the farthest in the left hand plane they are positioned.

Example: Observer for a DC Motor

Let us compute $(\boldsymbol{A} - \boldsymbol{LC})$

$$(A - LC) = \begin{bmatrix} -1 & -1 & -l_1 \\ 1 & -1 & -l_2 \\ 0 & 1 & -l_3 \end{bmatrix}$$

Example: Observer for a DC Motor

Let us compute $(\boldsymbol{A}-\boldsymbol{LC})$

$$(A-LC)=\begin{bmatrix}-1 & -1 & -l_1\\ 1 & -1 & -l_2\\ 0 & 1 & -l_3\end{bmatrix}$$

Now we can get the characteristic polynomial

$$\det\big(\lambda I-(A-LC)\big)=\begin{vmatrix}\lambda+1 & 1 & l_1\\ -1 & \lambda+1 & l_2\\ 0 & -1 & \lambda+l_3\end{vmatrix}=0$$

$$\boxed{\lambda^3+(2+l_3)\lambda^2+(2+2l_3+l_2)\lambda+2l_3+l_2+l_1=0}$$

Example: Observer for a DC Motor

Say the designer requires the eigenvalues $\lambda_{1,2,3} = -100$, in the far left hand plane.

Then we'll have a target polynomial:

$$(\lambda + 100)(\lambda + 100)(\lambda + 100) = \lambda^3 + 300\lambda^2 + 30000\lambda + 100000 = 0$$

Example: Observer for a DC Motor

Say the designer requires the eigenvalues $\lambda_{1,2,3} = -100$, in the far left hand plane.

Then we'll have a target polynomial:

$$(\lambda + 100)(\lambda + 100)(\lambda + 100) = \lambda^3 + 300\lambda^2 + 30000\lambda + 100000 = 0$$

Which implies that

$$\|e(t)\| = \|x(t) - \hat{x}(t)\| \leq Ce^{-100t}\|(x(0) - \hat{x}(0)\|$$

i.e

$$\left\| \begin{matrix} I(t) - \hat{I}(t) \\ \omega(t) - \widehat{\omega}(t) \\ \theta(t) - \hat{\theta}(t) \end{matrix} \right\| \leq Ce^{-100t} \left\| \begin{matrix} I(0) - \hat{I}(0) \\ \omega(0) - \widehat{\omega}(0) \\ \theta(0) - \hat{\theta}(0) \end{matrix} \right\|$$

where $\|\cdot\|$ is euclidean norm and C is a positive constant

Definition of Euclidean norm:

$$\left\| \begin{matrix} a \\ b \\ c \end{matrix} \right\| = \sqrt{a^2 + b^2 + c^2}$$

Example: Observer for a DC Motor

We compare target polynomial

$$\lambda^3 + 300\lambda^2 + 30000\lambda + 100000 = 0$$

with characteristic polynomial

$$\lambda^3 + (2 + l_3)\lambda^2 + (2 + 2l_3 + l_2)\lambda + 2l_3 + l_2 + l_1 = 0$$

Example: Observer for a DC Motor

We compare target polynomial

$$\lambda^3 + 300\lambda^2 + 30000\lambda + 100000 = 0$$

with characteristic polynomial

$$\lambda^3 + (2 + l_3)\lambda^2 + (2 + 2l_3 + l_2)\lambda + 2l_3 + l_2 + l_1 = 0$$

and get a system of equations for l_1, l_2, l_3

$$\begin{cases} 2 + l_3 = 300 \\ 2 + 2l_3 + l_2 = 300000 \\ 2l_3 + l_2 + l_1 = 1000000 \end{cases} \Longrightarrow \boxed{\begin{aligned} l_1 &= 970002 \\ l_2 &= 29402 \\ l_3 &= 298 \end{aligned}}$$

Example: Observer for a DC Motor

Taking $l_1 = 970002, l_2 = 29402$ and $l_3 = 298$, we can build a state observer for the DC motor system:

$$\underbrace{\begin{bmatrix} \dot{\hat{I}} \\ \dot{\hat{\omega}} \\ \dot{\hat{\theta}} \end{bmatrix}}_{\dot{\hat{x}}} = \underbrace{\begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \hat{I} \\ \hat{\omega} \\ \hat{\theta} \end{bmatrix}}_{\hat{x}} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B} \underbrace{V_S}_{u} + \underbrace{\begin{bmatrix} 970002 \\ 29402 \\ 298 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} I - \hat{I} \\ \omega - \hat{\omega} \\ \theta - \hat{\theta} \end{bmatrix}}_{x - \hat{x}}$$

Combination of optimal LQR with state observer

Combination LQR and observer

Consider the system

$$\begin{cases} \dot{x} = Ax + B\mathrm{u} \\ y = Cx \end{cases}$$

Combination LQR and observer

Consider the system

$$\begin{cases} \dot{x} = Ax + B\mathrm{u} \\ y = Cx \end{cases}$$

Then, to design a controller

$$u = -Kx$$

via LQR method, information of all states x is required!!

Combination LQR and observer

Consider the system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Then, to design a controller

$$u = -Kx$$

via LQR method, information of all states x is required!!

What if we don't have it? → Use **observer** to estimate it

Combination LQR and observer

For the system $\begin{cases} \dot{x} = Ax + B\mathrm{u} \\ y = Cx \end{cases}$

choose a controller $u = -K\hat{x}$

Combination LQR and observer

For the system $\begin{cases} \dot{x} = Ax + B\mathrm{u} \\ y = Cx \end{cases}$

choose a controller $u = -K\hat{x}$

which uses estimated state $\hat{x}$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ with } \hat{y} = C\hat{x}$$

Combination LQR and observer

For the system $\begin{cases} \dot{x} = Ax + B\mathrm{u} \\ y = Cx \end{cases}$

choose a controller $u = -K\hat{x}$

which uses estimated state $\hat{x}$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ with } \hat{y} = C\hat{x}$$

The error is defined as

$$e = x - \hat{x} \Rightarrow \dot{e} = (A - LC)e$$

Then, the closed loop system looks as follows

$$\dot{x} = Ax + \mathrm{B}u = Ax - \mathrm{BK}\hat{x} = Ax - \mathrm{B}K(x - e)$$
$$= (A - BK)x + BKe$$

Combination LQR and observer

For the system $\begin{cases} \dot{x} = Ax + B\mathrm{u} \\ y = Cx \end{cases}$

choose a controller $u = -K\hat{x}$

which uses estimated state $\hat{x}$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ with } \hat{y} = C\hat{x}$$

The error is defined as

$$e = x - \hat{x} \Rightarrow \dot{e} = (A - LC)e$$

Then, the closed loop system looks as follows

$$\dot{x} = A\mathrm{x} + \mathrm{B}u = (A - BK)x + BKe$$

In matrix form, we can write

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ \boxed{0} & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

Combination LQR and observer

For the system $\begin{cases} \dot{x} = Ax + B\mathrm{u} \\ y = Cx \end{cases}$

choose a controller $u = -K\hat{x}$

which uses estimated state $\hat{x}$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ with } \hat{y} = C\hat{x}$$

The error is defined as

$$e = x - \hat{x} \Rightarrow \dot{e} = (A - LC)e$$

Then, the closed loop system looks as follows

$$\dot{x} = A\mathrm{x} + \mathrm{B}u = (A - BK)x + BKe$$

In matrix form, we can write

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ \boxed{0} & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

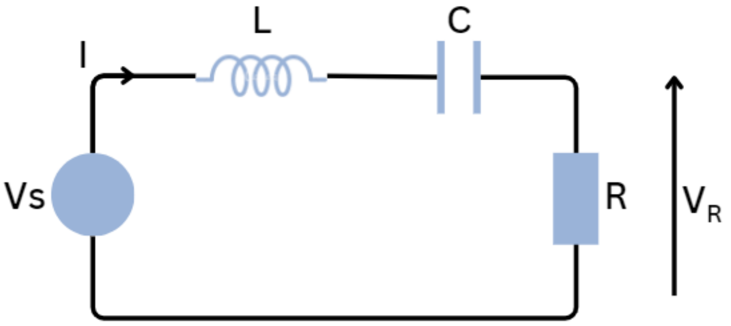
The matrix has upper block triangular form, which means we can separately design controller and observer (Separation principle)

Example: LQR with observer

Given the following electrical system with state space:

$$\begin{cases} \dfrac{dI}{dt} = -\dfrac{1}{2}V_c - \dfrac{1}{2}I + \dfrac{1}{2}V_s \\ \dfrac{dV_c}{dt} = \dfrac{1}{2}I \end{cases}$$

$$y = V_R = RI = I$$

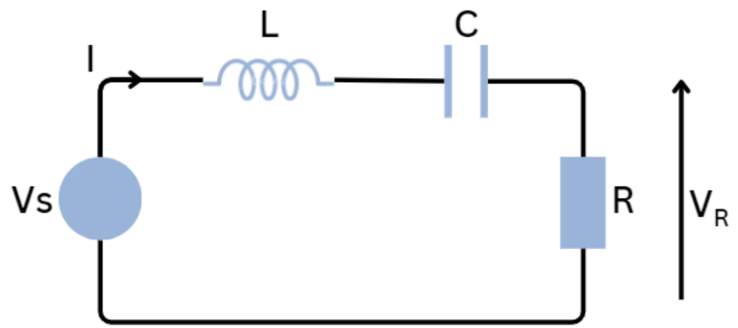


- The measured output is V_R → We measure I
- We don't measure state V_c

Example: LQR with observer

Given the following electrical system with state space:

$$\begin{cases} \dfrac{dI}{dt} = -\dfrac{1}{2}V_c - \dfrac{1}{2}I + \dfrac{1}{2}V_s \\ \dfrac{dV_c}{dt} = \dfrac{1}{2}I \end{cases}$$



$$y = V_R = RI = I$$

- The measured output is V_R → We measure I
- We don't measure state V_c

We can design an optimal controller and an observer **separately** to control the system solely using **measurement** of **output**

Example: LQR with observer

1. Optimal controller for the system

Remember that we need to optimise cost function:

$$J(x(0)) = \min_{u} \int_0^{\infty} x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau) d\tau$$

where Q and R are positive definite,

i.e $Q = Q^T > 0$ and $R = R^T > 0$

Example: LQR with observer

1. Optimal controller for the system

Remember that we need to optimise cost function:

$$J(x(0)) = \min_{u} \int_0^{\infty} x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau) d\tau$$

where Q and R are positive definite,

i.e $Q = Q^T > 0$ and $R = R^T > 0$

For our system

$$\begin{cases} \dfrac{dI}{dt} = -\dfrac{1}{2} V_c - \dfrac{1}{2} I + \dfrac{1}{2} V_s \\ \dfrac{dV_c}{dt} = \dfrac{1}{2} I \end{cases} \Rightarrow x = \begin{bmatrix} I(\tau) \\ V_c(\tau) \end{bmatrix}$$

And taking Q and R

$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R = 100$$

Example: LQR with observer

1. Optimal controller for the system

Remember that we need to optimise cost function:

$$J(x(0)) = \min_{u} \int_0^{\infty} x^T(\tau) Q x(\tau) + u^T(\tau)\, R u(\tau) d\tau$$

where Q and R are positive definite,

i.e $Q = Q^T > 0$ and $R = R^T > 0$

For our system

$$\begin{cases} \frac{dI}{dt} = -\frac{1}{2}V_c - \frac{1}{2}I + \frac{1}{2}V_s \\ \frac{dV_c}{dt} = \frac{1}{2}I \end{cases} \Rightarrow x = \begin{bmatrix} I(\tau) \\ V_c(\tau) \end{bmatrix}$$

And taking Q and R

$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$$
$$R = 100$$

$$J = \int_0^{\infty} \underbrace{[I(\tau) \quad V_c(\tau)]}_{x^T(\tau)} \underbrace{\begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} I(\tau) \\ V_c(\tau) \end{bmatrix}}_{x(\tau)} + \underbrace{100}_{R} \underbrace{V_S(\tau)}_{u(\tau)} d\tau$$

Example: LQR with observer

1. Optimal controller for the system

We have cost function:

$$J = \int_0^\infty [I(\tau) \quad V_c(\tau)] \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I(\tau) \\ V_c(\tau) \end{bmatrix} + 100 V_S(\tau) d\tau$$

Example: LQR with observer

1. Optimal controller for the system

We have cost function

$$J = \int_0^\infty [I(\tau) \quad V_c(\tau)] \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I(\tau) \\ V_c(\tau) \end{bmatrix} + 100 V_S(\tau) d\tau$$

We can solve LQR problem for P

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

Example: LQR with observer

1. Optimal controller for the system

We have cost function

$$J = \int_0^\infty [I(\tau) \quad V_c(\tau)] \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I(\tau) \\ V_c(\tau) \end{bmatrix} + 100 V_S(\tau) d\tau$$

We can solve LQR problem for P

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

with A, B from state-space

$$\begin{cases} \dfrac{dI}{dt} = -\dfrac{1}{2}V_c - \dfrac{1}{2}I + \dfrac{1}{2}V_s \\ \dfrac{dV_c}{dt} = \dfrac{1}{2}I \end{cases} \Rightarrow \boxed{A = \begin{bmatrix} -1/2 & -1/2 \\ 1/2 & 0 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}}$$

and $\boxed{Q = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}, R = 100}$

Example: LQR with observer

1. Optimal controller for the system

We have cost function

$$J = \int_0^{\infty} [I(\tau) \quad V_c(\tau)] \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I(\tau) \\ V_c(\tau) \end{bmatrix} + 100 V_S(\tau) d\tau$$

We can solve LQR problem for P

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

with A, B from state-space

$$\begin{cases} \dfrac{dI}{dt} = -\dfrac{1}{2}V_c - \dfrac{1}{2}I + \dfrac{1}{2}V_s \\ \dfrac{dV_c}{dt} = \dfrac{1}{2}I \end{cases} \Rightarrow A = \begin{bmatrix} -1/2 & -1/2 \\ 1/2 & 0 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

and $Q = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 100$

$$\boxed{P \approx \begin{bmatrix} 83.547 & 0.9975 \\ 0.9975 & 84.961 \end{bmatrix}}$$

Example: LQR with observer

1. Optimal controller for the system

We have cost function

$$J = \int_0^\infty [I(\tau) \;\; V_c(\tau)] \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I(\tau) \\ V_c(\tau) \end{bmatrix} + 100 V_S(\tau) d\tau$$

With solution to LQR problem

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \Rightarrow \boxed{P \approx \begin{bmatrix} 83.547 & 0.9975 \\ 0.9975 & 84.961 \end{bmatrix}}$$

Example: LQR with observer

1. Optimal controller for the system

We have cost function

$$J = \int_0^\infty [I(\tau) \;\; V_c(\tau)] \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I(\tau) \\ V_c(\tau) \end{bmatrix} + 100 V_S(\tau) d\tau$$

With solution to LQR problem

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \;\Rightarrow P \approx \begin{bmatrix} 83.547 & 0.9975 \\ 0.9975 & 84.961 \end{bmatrix}$$

Then optimal feedback controller is

$$V_s = -R^{-1}B^T P \begin{bmatrix} I \\ V_c \end{bmatrix} = -\frac{1}{100} [1/2 \;\; 0] \begin{bmatrix} 83.547 & 0.9975 \\ 0.9975 & 84.961 \end{bmatrix}$$

$$= -\frac{1}{200} [83.547 \;\; 0.9975] \begin{bmatrix} I \\ V_c \end{bmatrix}$$

$$= \boxed{-\frac{83.547}{200} \mathrm{I} - \frac{0.9975}{200} V_c}$$

Example: LQR with observer

1. Optimal controller for the system

We have cost function:

$$J = \int_0^{\infty} [I(\tau) \quad V_c(\tau)] \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I(\tau) \\ V_c(\tau) \end{bmatrix} + 100 V_S(\tau) d\tau$$

With solution to LQR problem:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \ \Rightarrow P \approx \begin{bmatrix} 83.547 & 0.9975 \\ 0.9975 & 84.961 \end{bmatrix}$$

Then optimal feedback controller is

$$\boxed{V_s = -\frac{83.547}{200}\mathrm{I} - \frac{0.9975}{200} V_c}$$

Our output does not contain this

Example: LQR with observer

2. State estimator (observer) for the system

Remember that for state estimation we can build an observer with form:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ where } \hat{y} = C\hat{x} + Du$$
$$\dot{\hat{x}} = A\hat{x} + Bu + LC(x - \hat{x})$$

Example: LQR with observer

2. State estimator (observer) for the system

Remember that for state estimation we can build an observer with form:

$$\boxed{\begin{gathered}\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ where } \hat{y} = C\hat{x} + Du \\ \dot{\hat{x}} = A\hat{x} + Bu + LC(x - \hat{x})\end{gathered}}$$

For our system

$$\begin{cases} \dfrac{dI}{dt} = -\dfrac{1}{2}V_c - \dfrac{1}{2}I + \dfrac{1}{2}V_s \\ \dfrac{dV_c}{dt} = \dfrac{1}{2}I \end{cases}$$

$$y = V_R = RI = I$$

$$x = \begin{bmatrix} I(\tau) \\ V_c(\tau) \end{bmatrix}, A = \begin{bmatrix} -1/2 & -1/2 \\ 1/2 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, C = [1 \quad 0]$$

Example: LQR with observer

2. State estimator (observer) for the system

Remember that for state estimation we can build an observer with form:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \text{ where } \hat{y} = C\hat{x} + Du$$
$$\dot{\hat{x}} = A\hat{x} + Bu + LC(x - \hat{x})$$

For our system

$$\begin{cases} \dfrac{dI}{dt} = -\dfrac{1}{2}V_c - \dfrac{1}{2}I + \dfrac{1}{2}V_s \\ \dfrac{dV_c}{dt} = \dfrac{1}{2}I \end{cases}$$
$$y = V_R = RI = I$$

$$\Rightarrow \quad x = \begin{bmatrix} I(\tau) \\ V_c(\tau) \end{bmatrix}, A = \begin{bmatrix} -1/2 & -1/2 \\ 1/2 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, C = [1 \quad 0]$$

The observer takes form

$$\boxed{\underbrace{\begin{bmatrix} \dot{\hat{I}} \\ \dot{\hat{V_c}} \end{bmatrix}}_{\dot{\hat{x}}} = \underbrace{\begin{bmatrix} -1/2 & -1/2 \\ 1/2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \hat{I} \\ \hat{V_c} \end{bmatrix}}_{\hat{x}} + \underbrace{\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}}_{L} \underbrace{[1 \quad 0]}_{C} \underbrace{\begin{bmatrix} I - \hat{I} \\ V_c - \hat{V_c} \end{bmatrix}}_{x - \hat{x}} + \underbrace{\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}}_{B} \underbrace{V_s}_{u}}$$

Example: LQR with observer

2. State estimator (observer) for the system

The observer has form

$$\begin{bmatrix} \dot{\hat{I}} \\ \dot{\hat{V}}_c \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} \hat{I} \\ \hat{V}_c \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} I - \hat{I} \\ V_c - \hat{V}_c \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} V_s$$

with $\mathrm{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$ designed such that eigenvalues of $(A - LC)$ are in the left hand plane.

Example: LQR with observer

2. State estimator (observer) for the system

The observer has form

$$\begin{bmatrix} \dot{\hat{I}} \\ \dot{\hat{V}}_c \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} \hat{I} \\ \hat{V}_c \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} I - \hat{I} \\ V_c - \hat{V}_c \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} V_s$$

with $\mathrm{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$ designed such that eigenvalues of $(A - LC)$ are in the left hand plane.

We can compute the characteristic polynomial

$$\det\big(\lambda I - (A - LC)\big) = \begin{vmatrix} \lambda + \frac{1}{2} + l_1 & \frac{1}{2} \\ -\frac{1}{2} + l_2 & \lambda \end{vmatrix} = \boxed{\lambda^2 + \left(\frac{1}{2} + l_1\right)\lambda + \left(\frac{1}{4} + \frac{1}{2} l_2\right) = 0}$$

Example: LQR with observer

2. State estimator (observer) for the system

The observer has form

$$\begin{bmatrix} \dot{\hat{I}} \\ \dot{\hat{V}}_c \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} \hat{I} \\ \hat{V}_c \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} I - \hat{I} \\ V_c - \hat{V}_c \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} V_s$$

with $\mathrm{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$ designed such that eigenvalues of $(A - LC)$ are in the left hand plane.

We can compute the characteristic polynomial

$$\det\big(\lambda I - (A - LC)\big) = \begin{vmatrix} \lambda + \frac{1}{2} + l_1 & \frac{1}{2} \\ -\frac{1}{2} + l_2 & \lambda \end{vmatrix} = \lambda^2 + \left(\frac{1}{2} + l_1\right)\lambda + \left(\frac{1}{4} + \frac{1}{2} l_2\right) = 0$$

Let's choose desired eigenvalues at $\lambda_{1,2} = -100$, then target polynomial

$$(\lambda + 100)(\lambda + 100) = \boxed{\lambda^2 + 200\lambda + 1000 = 0}$$

Example: LQR with observer

2. State estimator (observer) for the system

Comparing characteristic polynomial

$$\lambda^2 + \left(\frac{1}{2} + l_1\right)\lambda + \left(\frac{1}{4} + \frac{1}{2}l_2\right) = 0$$

and target polynomial

$$\lambda^2 + 200\lambda + 1000 = 0$$

Example: LQR with observer

2. State estimator (observer) for the system

Comparing characteristic polynomial

$$\lambda^2 + \left(\frac{1}{2} + l_1\right)\lambda + \left(\frac{1}{4} + \frac{1}{2}l_2\right) = 0$$

and target polynomial

$$\lambda^2 + 200\lambda + 1000 = 0$$

we get a system of equations for l_1, l_2

$$\begin{cases} \frac{1}{2} + l_1 = 200 \\ \frac{1}{4} + \frac{1}{2}l_2 = 1000 \end{cases} \Longrightarrow \boxed{\begin{matrix} l_1 = 199.5 \\ l_2 = -19999.5 \end{matrix}}$$

Example: LQR with observer

2. State estimator (observer) for the system

Comparing characteristic polynomial

$$\lambda^2 + \left(\frac{1}{2} + l_1\right)\lambda + \left(\frac{1}{4} + \frac{1}{2} l_2\right) = 0$$

and target polynomial

$$\lambda^2 + 200\lambda + 1000 = 0$$

we get a system of equations for l_1, l_2

$$\begin{cases} \frac{1}{2} + l_1 = 200 \\ \frac{1}{4} + \frac{1}{2} l_2 = 1000 \end{cases} \Longrightarrow \begin{array}{c} l_1 = 199.5 \\ l_2 = -19999.5 \end{array}$$

Then our observer:

$$\boxed{\begin{bmatrix} \dot{\hat{I}} \\ \dot{\hat{V}}_c \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} \hat{I} \\ \hat{V}_c \end{bmatrix} + \underbrace{\begin{bmatrix} 199.5 \\ -19999.5 \end{bmatrix}}_{L} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} I - \hat{I} \\ V_c - \hat{V}_c \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} V_s}$$

Example: LQR with observer

3. Put observer and controller together

Thanks to the separation principle, we can use the designed optimal controller, together with the estimated states from the observer.

Example: LQR with observer

3. Put observer and controller together

Thanks to the separation principle, we can use the designed optimal controller, together with the estimated states from the observer.

The controller is

$$V_s = -\frac{1}{200}[83.547 \quad 0.9975]\begin{bmatrix}\hat{I}\\ \hat{V}_c\end{bmatrix},$$

with observed/estimated states given by

$$\begin{bmatrix}\dot{\hat{I}}\\ \dot{\hat{V}}_c\end{bmatrix} = \begin{bmatrix}-1/2 & -1/2\\ 1/2 & 0\end{bmatrix}\begin{bmatrix}\hat{I}\\ \hat{V}_c\end{bmatrix} + \begin{bmatrix}199.5\\ -19999.5\end{bmatrix}[1 \quad 0]\begin{bmatrix}I-\hat{I}\\ V_c-\hat{V}_c\end{bmatrix} + \begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}V_s$$

Summary

- An observer can be designed as $\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$, where

  $\dot{\hat{x}}$ estimated state,

  $\hat{y} = C\hat{x} + Du$

  L is a constant matrix to determine

- L can be chosen such that the eigenvalues of $(A - LC)$ have negative real parts
- The estimated states from the observer can be used in an optimal controller
- The controller and observer can be designed separately

Next week:

No lecture

After break:

Delayed systems