1 a) $\sin(\pi x) - x + 1.95 = 0$

(1) $\left.\begin{array}{l} f(1) = +0.95 > 0 \\ f(3/2) = -0.55 < 0 \end{array}\right\}$ f decreasing $\qquad I_0 = [1 \quad \frac{3}{2}] \quad m_0 = \frac{5}{4}$

$f(\frac{5}{4}) = -\frac{1}{2}\sqrt{2} - \frac{5}{4} + 1.95 < 0 \qquad I_1 = [1 \quad \frac{5}{4}] \quad m_1 = \frac{9}{8}$

$f(\frac{9}{8}) = \ldots > 0 \qquad I_2 = [\frac{9}{8} \quad \frac{5}{4}] \quad m_2 = \frac{19}{16}$

(2) $x_{n+1} = x_n - \dfrac{(\sin(\pi x_n) - x_n + 1.95)}{\pi\cos(\pi x_n) - 1} \qquad x_0 = \frac{5}{4}$

$$x_1 = \frac{5}{4} - \frac{(-\frac{1}{2}\sqrt{2} - \frac{5}{4} + 1.95)}{(\pi \cdot -\frac{1}{2}\sqrt{2} - 1)} = 1.247794$$

b)

(1) $g'(x) = 1 + 0.1(\pi\cos(\pi x) - 1)$

$g'(\frac{5}{4}) = 1 + 0.1(\pi \cdot -\frac{1}{2}\sqrt{2} - 1) \approx 0.67786$

$|g'(\frac{5}{4})| < 1 \longrightarrow$ convergence

(2) $K = \dfrac{x_4 - x_3}{x_3 - x_2} \approx 0.68098$

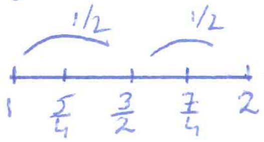
$\varepsilon_4 \approx \dfrac{K}{1-K} |x_4 - x_3| \approx 4{,}73085 \; 10^{-4}$

(3) $g'(\frac{5}{4}) = 1 + \alpha(\pi \cdot -\frac{1}{2}\sqrt{2} - 1) = 0 \implies \alpha = \dfrac{-1}{\pi \cdot -\frac{1}{2}\sqrt{2} - 1} = 0.31042$

2 a) (1) $f'(x) = \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$ $\qquad f''(x) = -\frac{\sin(\sqrt{x})}{4x} - \frac{\cos(\sqrt{x})}{4x\sqrt{x}}$

$f''(x)$ bounded on $[1,2]$, not singular $\longrightarrow$ optimal 2^{nd} order convergence

(2)



$I \approx \frac{1}{2} f\left(\frac{5}{4}\right) + \frac{1}{2} f\left(\frac{7}{4}\right) = \frac{1}{2}\sin\left(\sqrt{\frac{5}{4}}\right) + \frac{1}{2}\sin\left(\sqrt{\frac{7}{4}}\right)$

$= 0.934333$

$\varepsilon \approx \frac{(2-1)}{24}\left(\frac{1}{2}\right)^2 M = \frac{M}{96}$

$f''(1) = -\frac{\sin(1)}{4} - \frac{\cos(1)}{4} = -0.345443$

$f''(2) = -\frac{\sin(\sqrt{2})}{8} - \frac{\cos(\sqrt{2})}{8\sqrt{2}} = -0.137254$

$\Big\}\ M = 0.345443$

$\Rightarrow \varepsilon \approx 3.5984\ 10^{-3}$

b) (1) $q = \frac{I_{16} - I_{32}}{I_{32} - I_{64}} = 4.076923$

$q \approx 4 \longrightarrow 2^{nd}$ order convergence, according to theory

(2) I_4

$I_8 \rightarrow T_2(8) = \frac{4}{3} I(8) - \frac{1}{3} I(4) = 0.932118$

$I_{16} \rightarrow T_2(16) = \frac{4}{3} I(16) - \frac{1}{3} I(8) = 0.932119$

} not enough digits available

T_3 makes no sense

(3) $\varepsilon_{64} = \frac{1}{3}(I_{64} - I_{32}) = 4.333333\ 10^{-6}$

$\left(\frac{1}{4}\right)^n * 4.33\ 10^{-6} < 10^{-9} \implies n \geq 7$

segments: $64 * 2^7 = 8192$

3 a) (1) $k_1 = \frac{1}{2}(-5+1) \cdot 2 = -4$

$k_2 = \frac{1}{2}(-4\frac{1}{2}+1)(2+k_1) = -\frac{7}{4}(-2) = +3\frac{1}{2}$

$y(-4\frac{1}{2}) = 2 + \frac{1}{2}(-4+3\frac{1}{2}) = +7/4$

(2) $y = 2 + \frac{1}{2} \cdot (-4\frac{1}{2}+1)\, y \Rightarrow y = 8/11$

(3) $y_{n+1} = y_n + h(x_{n+1}+1)\, y_{n+1}$

$(1 - h(x_{n+1}+1))\, y_{n+1} = y_n \Rightarrow A = \frac{1}{1-h(x_{n+1}+1)}$

$|A| < 1 \Leftrightarrow 1 - h(x_{n+1}+1) > 1$ OR < -1

$\Leftrightarrow h(x_{n+1}+1) < 0$ OR > -2

for x in $[-5,-1]$ first case satisfied $(h>0)$
for x in $[-1,0]$ second case satisfied
$\Big\}$ always stable

b) (1) no point to point oscillations $\rightarrow$ stable
(increase from $x=-1$ to $x=0$ is part of solution)

(2) $\frac{y_{80} - y_{160}}{y_{160} - y_{320}}$ for $x=0$: $q = 4.455421$ convergence follows 2nd order (according to theory)

(3) $\frac{1}{3}(y_{320} - y_{160}) = 0.003148$

(4) $\left(\frac{1}{4}\right)^n * 0.003148 < 10^{-8}$ $n \geq 10$ grid $N = 2^{10} * 320 = 327\,680$

alternative: RK4 method or extrapolation much faster

$$4 \quad \begin{array}{ccccc} 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ \hline 0 & 1/6 & 1/4 & 1/3 & 1/2 \end{array}$$

0.6

$f(x) \to \tilde{x}$

$x \to \tilde{y}$

$f(x) = \frac{6}{10} : \tilde{x} = \frac{6}{10}$

$x?$ $\quad \tilde{y}?$

a) $\hat{x} = \dfrac{\tilde{x} - \frac{1}{2}}{1/2} = 2\tilde{x} - 1$

$$\begin{array}{c|ccccc} \hat{x} & -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 \\ \hline \tilde{y} & 0 & 1/6 & 1/4 & 1/3 & 1/2 \end{array}$$

$M_0 = 5 \quad M_1 = 0 \quad M_2 = 2\tfrac{1}{2} \quad F_0 = 1\tfrac{1}{4} \quad F_1 = 7/12$

$\begin{pmatrix} 5 & 0 \\ 0 & 5/2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5/4 \\ 7/12 \end{pmatrix} \quad a = 1/4, \; b = 7/30 \quad \tilde{y} = \frac{1}{4} + \frac{7}{30}\hat{x} = \frac{1}{4} + \frac{7}{30}(2\tilde{x} - 1)$

$= \frac{7}{15}\tilde{x} + \frac{1}{60}$

b) $\tilde{y}\left(\frac{6}{10}\right) = \frac{7}{15} \cdot \frac{6}{10} + \frac{1}{60} = \frac{89}{300} \leftarrow$ x coordinate

5 a) (1) $r^{(0)} = Ax_0 - b = \begin{pmatrix} 0 \\ 4 \\ -3 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ -6 \\ -2 \end{pmatrix} \leftarrow$ max norm = 6

(2) worst diag. dom. is row 2: $\mu = 3/4$

$\left(\frac{3}{4}\right)^n < 10^{-3}$ $\quad n \geq 25$ iterations

(3) $\hat{x}_1 = \frac{1}{1}(3 - 0 \cdots 0) = 3$ $\qquad x_1 = (1.5)*3 - (0.5)*0 = 4\frac{1}{2}$

$\hat{x}_2 = \frac{1}{4}(7 - (-2)(4\frac{1}{2}) - (1)(0)) = 4$ $\quad x_2 = (1.5)*4 - (0.5)*1 = 5\frac{1}{2}$

$\hat{x}_3 = \frac{1}{15}(3 - (-2)(5\frac{1}{2}) - (-1)(1)) = 1$ $\quad x_3 = (1.5)*1 - (0.5)*0 = 1\frac{1}{2}$

$\hat{x}_4 = \frac{1}{5}(7 - (-2)(1\frac{1}{2})) = 2$ $\qquad x_4 = (1.5)*2 - (0.5)*1 = 2\frac{1}{2}$

SOR will diverge because of x_1 values in combi with $w = 1.5$

b) $y_1 \quad y_2 \quad y_3 \quad y_4$

$0 \quad \frac{1}{3} \quad \frac{2}{3} \quad 1$

$y_2 - y_1 = 2 * \Delta x = 2/3$

$y_4 - y_3 = 0$

(1) $\dfrac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2} - 2y_i = 3x_i^2$

$$\left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 2/3 \\ 9 & -20 & 9 & 0 & 1/3 \\ 0 & 9 & -20 & 9 & 4/3 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right)$$

(2) When $\alpha = 0$ the matrix is singular $\Rightarrow$ no solution

$$\begin{matrix} -1 & 1 & 0 & 0 \\ 9 & -18 & 9 & 0 \\ 0 & 9 & -18 & 9 \\ 0 & 0 & -1 & 1 \end{matrix} \rightarrow \begin{matrix} -1 & 1 & 0 & 0 \\ 0 & -9 & 9 & 0 \\ 0 & 9 & -9 & 0 \\ 0 & 0 & -1 & 1 \end{matrix} \rightarrow y_1 = y_2 = y_3 = y_4$$

no info to determine value

because of two bound. cond. in terms of $y'(x)$

6 a) (1) $R = \frac{\Delta t D}{\Delta x^2} < \frac{1}{2} \Rightarrow \Delta t < \frac{1}{2}\frac{\Delta x^2}{D} = \frac{1}{2}\frac{(0{,}05)^2}{10^{-5}} = 125$

(2) if $A = 0$: $\left.\begin{array}{l}\phi(x,0) = 20 \\ \phi(0,t) = \phi(1,t) = 20\end{array}\right\}$ solution : $\phi(x,t) = 20$

solution will stay 20 during time steps, Δt limit not important

b)

(1) $\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = D\,\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2} + 2\phi_i^n$

$$\Rightarrow \phi_i^{n+1} = \phi_i^n + R(\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n) + 2\Delta t\,\phi_i^n$$

$$= R\phi_{i-1}^n + \underbrace{(1-2R+2\Delta t)}_{>0}\phi_i^n + R\phi_{i+1}^n$$

$$\Leftrightarrow R < \frac{1+2\Delta t}{2} = \frac{1}{2} + \Delta t$$

still stable for larger R values, stability enhanced

(2) $\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \frac{1}{2}D\,\frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2} + \frac{1}{2}D\,\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2} + \phi_i^{n+1} + \phi_i^n$

$$\phi_i^{n+1} - \phi_i^n = \frac{1}{2}R(\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}) + \frac{1}{2}R(\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n) + (\phi_i^{n+1} + \phi_i^n)\Delta t$$

$$\underbrace{-\frac{1}{2}R}_{La}\phi_{i-1}^{n+1} + \underbrace{(1+R-\Delta t)}_{Da}\phi_i^{n+1} \underbrace{-\frac{1}{2}R}_{Ra}\phi_{i+1}^{n+1} = \frac{1}{2}R\phi_{i-1}^n + (1-R+\Delta t)\phi_i^n + \frac{1}{2}R\phi_{i+1}^n$$