Control Engineering

Instruction Lecture 3

Dynamic systems, linearization | Chapter 4,5

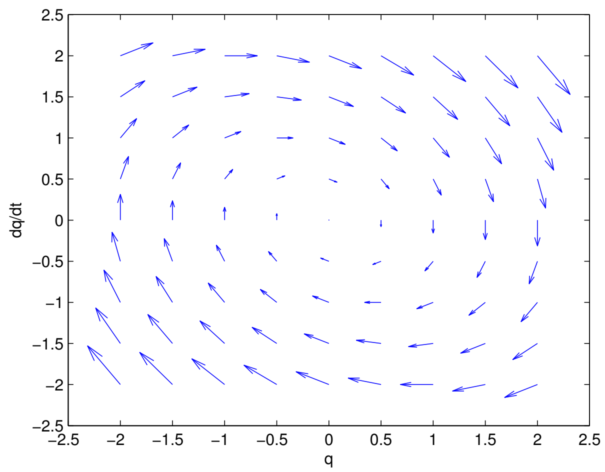
Exercise 1. Phase portraits

Consider a mass-spring-damper system modeled as

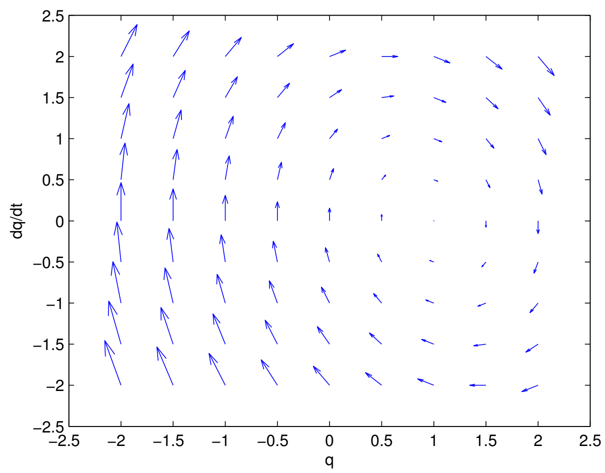
$$m\frac{d^2q}{dt^2} + b\frac{dq}{dt} + kq = u, \tag{1}$$

with q the position of mass m, b the damping coeffient, and k the spring constant.

1. Derive a state space model from (1) with state $x_1 = q, x_2 = \dot{q}$ and the position q as the output.
2. From the state space model determine the equilibrium point(s) for the system for $u = 0$. Check their stability using the phase portrait below ($m = 2kg$, $b = 1kg/s$, and $k = 2kg/s^2$).



3. Assume that the control objective is to stabilize the system at the position $q = 2$. Without any control ($u = 0$), is the control objective achieved?
4. Now consider the case where we apply a proportional controller $u = k_p(r^* - q)$, with proportional gain $k_p = 2$ and reference signal $r^* = 2$. Compare the phase portrait below with the one in step 2. Does the proportional control achieve the control goal? If not, what kind of control would you recommend to be used?



Exercise 2. Root locus plot

Consider the linear system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} u, \quad y = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} x \tag{2}$$

with the feedback $u = -ky$. In Figure 1 below the location of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of are plotted as a function of the parameter k. For $k = 0$ the eigenvalues are located at $\lambda_1 = 0.38, \lambda_2 = 2.61, \lambda_3 = 0$ (denoted by $\times$ in Figure 1). For an increasing k the locations of $\lambda_1, \lambda_2, \lambda_3$ follow respectively the green, blue and red lines.

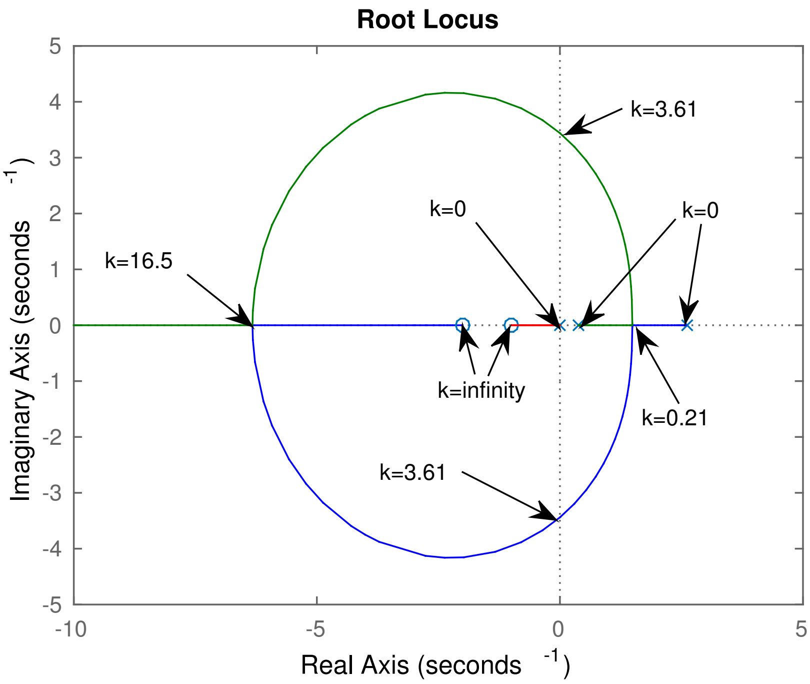


Figure 1: Rootlocus plot of the system (2).

Explain the system behavior for the following values of k. The corresponding locations of the eigenvalues are depicted in the figure above.

1. $0 \leq k \leq 0.21$
2. $0.21 < k < 3.61$
3. $k = 3.61$
4. $3.61 < k < 16.5$
5. $k \geq 16.5$

Exercise 3. Linearization

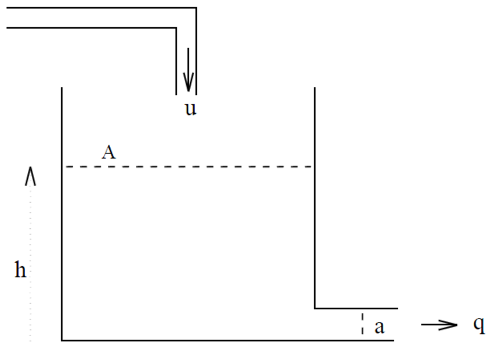
Consider a scalar system

$$\frac{dx}{dt} = -x^2 - 4x - 3 + 2u, \tag{3}$$

with output $y = 5x$. Compute the equilibrium points for the unforced system ($u = 0$) and use (5.33)-(5.34) of the book to compute the linearization around each equilibrium point. What can be said about the stability of the linearized system(s) if the input is chosen equal to zero?

Exercise 4. Linearization

Consider the water tank given below.



$u\,[\mathrm{m}^3/\mathrm{s}]$ is the in-flow, $h\,[\mathrm{m}]$ is the height of the water column, $q\,[\mathrm{m}^3/\mathrm{s}]$ is the out-flow, $\mathcal{A}\,[\mathrm{m}^2]$ us the surface of the water, and $a\,[\mathrm{m}^2]$ is the surface of the outflow area. We assume that the local pressure caused by the in-flow is neglectable.
According to the law of Bernoulli we have $q(t) = a\sqrt{2gh(t)}$, with $g = 9.81\,\mathrm{m}/\mathrm{s}^2$ the gravitation constant. Then, the velocity at which the water height in the tank changes is equal to in-flow minus out-flow, divided by the surface area. Then we obtain the following state space model with input u and output q:

$$\begin{aligned} \frac{d}{dt}h(t) &= -\frac{a\sqrt{2g}}{\mathcal{A}}\sqrt{h(t)} + \frac{1}{\mathcal{A}}u(t) \\ q(t) &= a\sqrt{2g}\sqrt{h(t)} \end{aligned} \tag{4}$$

1. Determine the equilibrium point for $u(t) = 0$. What is the physical meaning?
2. Determine the equilibrium point (h_e, u_e) of (4) for a constant in-flow $u(t) = u_e = 10$.
3. Linearize the system around the equilibrium point for $u(t) = u_e = 10$.