Control Engineering TBKRT05E

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Lecture 4
ver. 1.4.3

Last lecture

- Linear versus nonlinear systems
- Linearization

Today

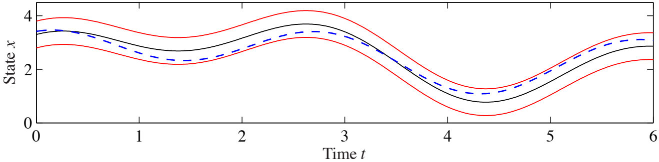
- Stability of linear systems
- Stability for nonlinear systems via linear approximation
- Routh Hurwitz theorem
- Matrix exponential e^{At}

Stability

Definition (Stable solution)

$x(t,a)$ is a stable solution IF $\forall \varepsilon$ there exists $\delta > 0$ such that

$$||b-a|| < \delta \quad \Rightarrow \quad ||x(t,b) - x(t,a)|| < \varepsilon, \quad \forall t \geq 0.$$



Definition (**Asymptotically** stable solution)

$x(t,a)$ is an asymptotically stable solution IF (i) it is stable and (ii) there exists $\delta > 0$ such that

$$||b-a|| < \delta \quad \Rightarrow \quad \lim_{t\to+\infty} ||x(t,b) - x(t,a)|| = 0$$

Stability

Definition (**Globally** asymptotically stable solution)

$x(t, a)$ is a globally asymptotically stable solution IF (i) it is stable and (ii) **for all** $b \in \mathbb{R}^n$

$$\lim_{t \to +\infty} ||x(t, b) - x(t, a)|| = 0$$

Remarks

- **Unstable** solutions are solutions that are not stable
- Stability notions also apply to equilibrium points (special case)
- Planar systems
  - Asymptotically stable (AS) equilibrium is a **sink**
  - Unstable equilibrium is a **source** (all trajectories move away from equilibrium)
  - Unstable equilibrium is a **saddle** (some trajectories move away, some converge to the equilibrium)
  - Equilibrium point that is stable but not AS is a **center**

Stability for linear systems

For linear systems (with input $u = 0$)

$$\dot{x} = Ax, \quad \in \mathbb{R}^n$$

the stability of $x = 0$ (always an equilibrium) depends on the eigenvalues of A

$$\lambda(A) = \{s \in \mathbb{C} : \det(sI - A) = 0\}.$$

Case I (A diagonal)

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

For all i, $\dot{x}_i = \lambda_i x_i \Rightarrow x_i(t) = \mathrm{e}^{\lambda_i t} x_i(0)$.
$x = 0$ is stable if $\lambda_i \leq 0$ for all i, asymptotically stable if $\lambda_i < 0$ for all i, unstable if $\lambda_i > 0$ for some i.

Stability for linear systems

Case II (A block diagonal)

$$\dot{x} = \begin{pmatrix} \sigma_1 & \omega_1 & \dots & 0 & 0 \\ -\omega_1 & \sigma_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma_n & \omega_n \\ 0 & 0 & \dots & -\omega_n & \sigma_n \end{pmatrix} x$$

That corresponds to A having all eigenvalues of the form $\lambda_j = \sigma_j \pm i\omega_j$.
The solution to the system are obtained solving the equations

$$\begin{pmatrix} \dot{x}_{2j-1}(t) \\ \dot{x}_{2j}(t) \end{pmatrix} = \begin{pmatrix} \sigma_j & \omega_j \\ -\omega_j & \sigma_j \end{pmatrix} \begin{pmatrix} x_{2j-1}(t) \\ x_{2j}(t) \end{pmatrix}, \; j = 1, 2, \dots, n$$

which return

$$\begin{pmatrix} x_{2j-1}(t) \\ x_{2j}(t) \end{pmatrix} = \mathrm{e}^{\sigma_j t} \begin{pmatrix} \cos(\omega_j t) & \sin(\omega_j t) \\ -\sin(\omega_j t) & \cos(\omega_j t) \end{pmatrix} \begin{pmatrix} x_{2j-1}(0) \\ x_{2j}(0) \end{pmatrix}, \; j = 1, 2, \dots, n.$$

Stability for linear systems

Case II (A block diagonal)

$$\dot{x} = \begin{pmatrix} \sigma_1 & \omega_1 & \dots & 0 & 0 \\ -\omega_1 & \sigma_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma_n & \omega_n \\ 0 & 0 & \dots & -\omega_n & \sigma_n \end{pmatrix} x$$

Explicitly, the solutions to the system are

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{2n-1}(t) \\ x_{2n}(t) \end{pmatrix} = \begin{pmatrix} e^{\sigma_1 t}\cos(\omega_1 t) & e^{\sigma_1 t}\sin(\omega_1 t) & \dots & 0 & 0 \\ -e^{\sigma_1 t}\sin(\omega_1 t) & e^{\sigma_1 t}\cos(\omega_1 t) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e^{\sigma_n t}\cos(\omega_n t) & e^{\sigma_n t}\sin(\omega_n t) \\ 0 & 0 & \dots & -e^{\sigma_n t}\sin(\omega_n t) & e^{\sigma_n t}\cos(\omega_n t) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_{2n-1}(0) \\ x_{2n}(0) \end{pmatrix}$$

$x = 0$ is asymptotically stable if $\sigma_j = \mathrm{Re}(\lambda_j) < 0$ for all j.
$x = 0$ is stable if $\sigma_j = \mathrm{Re}(\lambda_j) \leq 0$ for all j and $\exists j$ such that $\sigma_j = 0$.
$x = 0$ is unstable if $\exists j$ such that $\sigma_j = \mathrm{Re}(\lambda_j) > 0$.

Stability for linear systems

Case III (A with distinct eigenvalues)
There exists a nonsingular matrix T such that [Textbook, Exercise 4.14][1]

$$\tilde{A} = T^{-1}AT$$

with

$$\tilde{A} = \begin{pmatrix} \Lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \Lambda_k \end{pmatrix}$$

where

$$\Lambda_i = \lambda_i \in \mathbb{R}, \quad \Lambda_i = \begin{pmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{pmatrix}$$

[1] Solution available in Brightspace "Handout on the diagonalization of a matrix".

Example

Consider the linear system

$$\dot{x} = Ax = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x$$

The eigenvalues of the matrix A are

$$\lambda(A) = \{s \in \mathbb{C} : s^2 + s + 1 = 0\} = \{-\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\}$$

with eigenvectors $v_1 = \begin{pmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix}$. Hence, the similarity transformation matrix is

$$T = \begin{pmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 & 1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} \frac{i}{\sqrt{3}} & \frac{i+\sqrt{3}}{2\sqrt{3}} \\ -\frac{i}{\sqrt{3}} & \frac{-i+\sqrt{3}}{2\sqrt{3}} \end{pmatrix}$$

which returns the diagonal matrix

$$\tilde{A} = T^{-1}AT = \begin{pmatrix} \frac{-1-i\sqrt{3}}{2} & 0 \\ 0 & \frac{-1+i\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \sigma + i\omega & 0 \\ 0 & \sigma - i\omega \end{pmatrix}$$

Example continued Following the "Handout on the diagonalization of a matrix", the similarity transformation

$$\hat{T} = TS = \begin{pmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}-1-i(\sqrt{3}-1)}{2\sqrt{2}} & -\frac{\sqrt{3}+1+i(\sqrt{3}+1)}{2\sqrt{2}} \\ \\ \frac{1-i}{\sqrt{2}} & \frac{1-i}{\sqrt{2}} \end{pmatrix}$$

returns the matrix

$$\tilde{A} = \hat{T}^{-1} A \hat{T} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$$

Stability for linear systems

Case III (A with distinct eigenvalues)
In the new coordinates

$$z = T^{-1}x \quad (x = Tz)$$

the system $\dot{x} = Ax$ becomes $\dot{z} = T^{-1}\dot{x} = T^{-1}ATz$, that is

$$\dot{z} = \tilde{A}z, \quad \text{with} \quad \tilde{A} = T^{-1}AT.$$

We have seen that if $\lambda_i < 0$ and $\sigma_i < 0$ then $z = \mathbf{0}$ is asymptotically stable. Since $\|T\| = \sup_{z\neq 0} \|Tz\|/\|z\|$

$$\|x(t)\| = \|Tz(t)\| \leq \|T\|\|z(t)\|$$

then one proves that $x = \mathbf{0}$ is also asymptotically stable.
In particular

$$\lim_{t\to+\infty} \|z(t)\| = 0$$

if and only if

$$\lim_{t\to+\infty} \|x(t)\| = 0.$$

Stability for linear systems

General result

Theorem - Stability of linear systems

The system

$$\dot{x} = Ax$$

is (globally) asymptotically stable if and only if the eigenvalues of A have all strictly negative real part, and is unstable if at least one eigenvalue of A has strictly positive real part.

We have shown the result for the case of distinct eigenvalues. The proof of the general case is based on the Jordan form of the matrix A

Remarks

- If the linear system is asymptotically stable, then it is globally asymptotically stable

Stability for linear systems

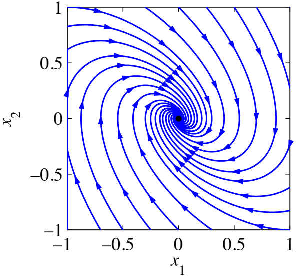
Example Consider the linear system

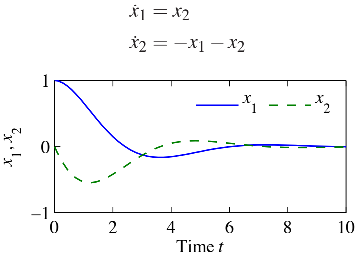
$$\dot{x} = Ax = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x$$

The eigenvalues of the matrix A are

$$\lambda(A) = \{s \in \mathbb{C} : s^2 + s + 1 = 0\} = \{-\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\}$$

Both eigenvalues have strictly negative real parts $\Rightarrow$ System asymptotically stable





Stability for linear systems

Example In fact consider the similarity transformation $z = T^{-1}x$ with

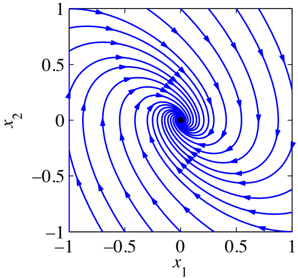
$$T = \begin{pmatrix} \frac{\sqrt{3}-1-i(\sqrt{3}-1)}{2\sqrt{2}} & -\frac{\sqrt{3}+1+i(\sqrt{3}+1)}{2\sqrt{2}} \\ \frac{1-i}{\sqrt{2}} & \frac{1-i}{\sqrt{2}} \end{pmatrix}$$

which returns the linear system and the solution in the z variables

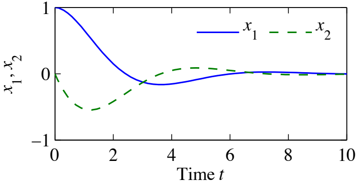
$$\dot{z} = \tilde{A}z = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} z, \quad z(t) = \mathrm{e}^{-\frac{1}{2}t} \begin{pmatrix} \cos(\frac{\sqrt{3}}{2}t) & -\sin(\frac{\sqrt{3}}{2}t) \\ \sin(\frac{\sqrt{3}}{2}t) & \cos(\frac{\sqrt{3}}{2}t) \end{pmatrix} z(0)$$

In the original x coordinates

$$x(t) = Tz(t) = \mathrm{e}^{-\frac{1}{2}t} T \begin{pmatrix} \cos(\frac{\sqrt{3}}{2}t) & -\sin(\frac{\sqrt{3}}{2}t) \\ \sin(\frac{\sqrt{3}}{2}t) & \cos(\frac{\sqrt{3}}{2}t) \end{pmatrix} T^{-1}x(0)$$



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2$$



Stability analysis via linear approximation I

Consider the nonlinear system

$$\dot{x} = f(x) \quad \text{(NL)}$$

with equilibrium point $\bar{x}$ (i.e., $f(\bar{x}) = 0$).
Consider the associated linearized system

$$\Delta\dot{x} = \left.\frac{\partial f(x)}{\partial x}\right|_{x=\bar{x}} \Delta x = A\Delta x.$$

Can we infer stability properties of $\bar{x}$ studying $\Delta\dot{x} = A\,\Delta x$?

Yes!

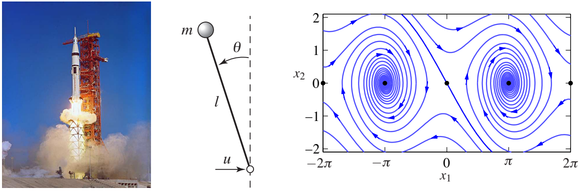
Theorem

If $\Delta\dot{x} = A\,\Delta x$ is (globally) asymptotically stable, then $\bar{x}$ is a (locally) asymptotically stable equilibrium of (NL).
If $\Delta\dot{x} = A\,\Delta x$ is unstable, then $\bar{x}$ is an unstable equilibrium of (NL).

Stability analysis via linear approximation II

Example Consider the inverted pendulum

$$\dot{x} = f(x, u) = \begin{pmatrix} x_2 \\ \sin x_1 - c x_2 + u \cos x_1 \end{pmatrix}, \quad c > 0$$



(a) (b) (c)

Equilibrium points for $\bar{u} = 0$ are

$$\bar{x} = \begin{pmatrix} \pm k\pi \\ 0 \end{pmatrix}, \quad k \in \mathbb{N}$$

Stability analysis via linear approximation III

Example We want to study stability of the equilibrium (upright position) with no torque applied

$$\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \bar{u} = 0$$

What do you expect?

Linearized system at $\bar{x}, \bar{u}$

$$\Delta\dot{x} = A\,\Delta x = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix} \Delta x, \quad c > 0$$

$\lambda(A) = \{-\frac{c}{2} \pm \frac{\sqrt{c^2+4}}{2}\} \Rightarrow$ one eigenvalue with positive real part: $\bar{x}$ is an unstable equilibrium point (in fact $\bar{x}$ is a saddle).

What do you expect if you analyze the other equilibrium (downward position)?

Routh Hurwitz Theorem

The Routh Hurwitz theorem permits to determine the sign of the real part of the roots of a polynomial (hence, of the eigenvalues of a matrix) without explicitly determining the roots.

Polynomial

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

Routh's table

n even

$$\begin{array}{r||cccccc} \text{n} & 1 & a_2 & a_4 & \ldots & a_{n-2} & a_n \\ \text{n-1} & a_1 & a_3 & a_5 & \ldots & a_{n-1} & \\ \text{n-2} & & & & & & \\ \vdots & & & & & & \\ 2 & & & & & & \\ 1 & & & & & & \\ 0 & & & & & & \end{array}$$

n odd

$$\begin{array}{r||ccccc} \text{n} & 1 & a_2 & a_4 & \ldots & a_{n-1} \\ \text{n-1} & a_1 & a_3 & a_5 & \ldots & a_n \\ \text{n-2} & & & & & \\ \vdots & & & & & \\ 2 & & & & & \\ 1 & & & & & \\ 0 & & & & & \end{array}$$

Routh Hurwitz Theorem

n even (similarly for n odd)

$$
\begin{array}{r||llllll}
\text{n} & 1 & a_2 & a_4 & \dots & a_{n-2} & a_n \\
\text{n-1} & a_1 & a_3 & a_5 & \dots & a_{n-1} & \\
\text{n-2} & b_1 & b_2 & b_3 & \dots & & \\
\vdots & & & & & & \\
 & h_1 & h_2 & h_3 & \dots & & \\
 & k_1 & k_2 & k_3 & \dots & & \\
 & \ell_1 & \ell_2 & \ell_3 & \dots & & \\
\vdots & & & & & & \\
2 & & & & & & \\
1 & & & & & & \\
0 & & & & & &
\end{array}
$$

where

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} 1 & a_2 \\ a_1 & a_3 \end{vmatrix}, \quad b_2 = -\frac{1}{a_1}\begin{vmatrix} 1 & a_4 \\ a_1 & a_5 \end{vmatrix}, \quad \dots$$

and the rest of the table is filled via the formula

$$\ell_i = -\frac{1}{k_1}\begin{vmatrix} h_1 & h_{i+1} \\ k_1 & k_{i+1} \end{vmatrix}, \quad i = 1, 2, \dots$$

Routh Hurwitz Theorem

n even (similarly for n odd)

$$\begin{array}{r||cccccc}
\text{n} & 1 & a_2 & a_4 & \dots & a_{n-2} & a_n \\
\text{n-1} & a_1 & a_3 & a_5 & \dots & a_{n-1} & \\
\text{n-2} & b_1 & b_2 & \dots & & & \\
\vdots & \vdots & \vdots & \vdots & & & \\
2 & x_1 & x_2 & & & & \\
1 & y_1 & & & & & \\
0 & z_1 & & & & &
\end{array}$$

where all the elements of the table are filled using the formula above.

Theorem

Consider a polynomial

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

and its table constructed as before. Then the roots have all real part strictly less than zero if and only if the signs of the coefficients of the first column are all the same.

Routh Hurwitz Theorem

Example $p(s) = s^4 + s^3 + 3s^2 + 2s + 1$

Special cases (not treated in detail)

1. The first element of a row is 0 but there is at least another term that is non zero.
   Replace the 0 element with $\varepsilon > 0$, complete the table and study the number of sign changes letting $\varepsilon \to 0^+$.
   Example $p(s) = s^5 + 3s^4 + 2s^3 + 6s^2 + 3s + 3$
2. All the elements of a row are zero. Suppose this happens for row $n-k$. Then the original polynomial can be factorized by the polynomial $p_a(s) = \alpha_{k+1}s^{n-k+1} + \alpha_{k+3}s^{n-k-1} + \ldots$, where the coefficients α are obtained from the row $n-k+1$. Being a polynomial of even or odd powers only, it is not Hurwitz.
   Example $p(s) = s^6 + 5s^5 + 2s^4 + 5s^3 + 4s^2 + 15s + 3$

Routh-Hurwitz Theorem is not in your textbook. For a reference, please check e.g. Goodwin-Graebe-Salgado, Control System Design, Section 5.5.3. See also a document on Routh-Hurwitz analysis in Nestor – Additional Material.

Routh Hurwitz Theorem

Example $p(s) = s^4 + s^3 + 3s^2 + 2s + 1$

$$\begin{array}{c||ccc} 4 & 1 & 3 & 1 \\ 3 & 1 & 2 & \\ 2 & b_1 & b_2 & \\ 1 & y_1 & & \\ 0 & z_1 & & \end{array}$$

$$b_1 = -\frac{1}{1}\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = 1 \quad b_2 = -\frac{1}{1}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$$y_1 = -\frac{1}{b_1}\begin{vmatrix} 1 & 2 \\ b_1 & b_2 \end{vmatrix} = -\frac{1}{b_1}(b_2 - 2b_1) = 1$$

$$z_1 = -\frac{1}{y_1}\begin{vmatrix} b_1 & b_2 \\ y_1 & 0 \end{vmatrix} = b_2 = 1$$

Hence the roots have all strictly negative real parts.

Routh Hurwitz Theorem

Example $p(s) = s^5 + 3s^4 + 2s^3 + 6s^2 + 3s + 3$

$$
\begin{array}{c||ccc}
5 & 1 & 2 & 3 \\
4 & 3 & 6 & 3 \\
3 & \color{red}{0} & 2 & \\
2 & & & \\
1 & & & \\
0 & & &
\end{array}
\qquad
\begin{array}{c||ccc}
5 & 1 & 2 & 3 \\
4 & 3 & 6 & 3 \\
3 & \color{red}{\varepsilon} & 2 & \\
2 & \frac{6\varepsilon-6}{\varepsilon} & 3 & \\
1 & -\frac{3\varepsilon^2-12\varepsilon+12}{6\varepsilon-6} & & \\
0 & 3 & &
\end{array}
$$

As $\varepsilon \to 0^+$, the first element of row "2" converges to a negative value, hence there is a sign change in the coefficients of the first column: not all the roots have all strictly negative real parts.

(in fact there are two complex conjugate roots $\approx 0.3 \pm i1.22$ with positive real parts)

Routh Hurwitz Theorem

Example $p(s) = s^6 + 5s^5 + 2s^4 + 5s^3 + 4s^2 + 15s + 3$

$$
\begin{array}{c||cccc}
6 & 1 & 2 & 4 & 3 \\
5 & 5 & 5 & 15 & \\
4 & 1 & 1 & 3 & \\
3 & 0 & 0 & & \\
2 & & & & \\
1 & & & & \\
0 & & & &
\end{array}
$$

All the elements of row labeled "3" are zero (hence, $n - k = 3$), which implies that the polynomial can be factorized by

$$p_a(s) = s^4 + s^2 + 3$$

where the coefficients are taken from the row labelled "3+1=4". In fact, carrying out the polynomial division, we obtain

$$s^6 + 5s^5 + 2s^4 + 5s^3 + 4s^2 + 15s + 3 = (s^4 + s^2 + 3)(s^2 + 5s + 1)$$

Routh Hurwitz Theorem

The Routh-Hurwitz criterion is particularly useful when the polynomial coefficients depend on some parameters and we want to study the nature of the roots as a function of these parameters.

Example $p(s) = s^3 + (2+\beta)s^2 + (1+2\beta)s + \alpha + \beta$

$$
\begin{array}{c||cc}
3 & 1 & 1+2\beta \\
2 & 2+\beta & \alpha+\beta \\
1 & \frac{2(\beta+1)^2-\alpha}{\beta+2} & \\
0 & \alpha+\beta &
\end{array}
$$

The roots have all strictly negative real parts for all values of (β, α) that belong to the region

$$\{(\beta,\alpha) \in \mathbb{R}^2 : \beta > -2, -\beta < \alpha < 2(\beta+1)^2\}$$

State-space description

Solutions to linear state space equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Autonomous state equation:

$$\frac{dx(t)}{dt} = Ax(t)$$

Initial condition: $x(0) = x_0$.

To compute an explicit expression of this response, we need to introduce the

Matrix Exponential

Matrix Exponential

Consider

$$\frac{dx(t)}{dt} = Ax(t), \quad x(0) = x_0$$

By the fundamental theorem of calculus

$$\int_0^t \frac{dx(\tau_1)}{d\tau_1} d\tau_1 = \int_0^t Ax(\tau_1)d\tau_1 \Rightarrow x(t) = x_0 + \int_0^t Ax(\tau_1)d\tau_1$$

By the same reason, we have $x(\tau_1) = x_0 + \int_0^{\tau_1} Ax(\tau_2)d\tau_2$, which gives

$$\begin{aligned} x(t) = & \; x_0 + \int_0^t Ax(\tau_1)d\tau_1 = x_0 + \int_0^t A[x_0 + \int_0^{\tau_1} Ax(\tau_2)d\tau_2]d\tau_1 \\ = & \; x_0 + \int_0^t Ax_0 d\tau_1 + \int_0^t \int_0^{\tau_1} A^2 x(\tau_2)d\tau_2 d\tau_1 \\ = & \; x_0 + Atx_0 + \int_0^t \int_0^{\tau_1} A^2 x(\tau_2)d\tau_2 d\tau_1 \end{aligned}$$

Matrix Exponential

$$
\begin{aligned}
x(t) = & \quad x_0 + \int_0^t Ax(\tau_1)d\tau_1 = x_0 + \int_0^t A[x_0 + \int_0^{\tau_1} Ax(\tau_2)d\tau_2]d\tau_1 \\
= & \quad x_0 + \int_0^t Ax_0 d\tau_1 + \int_0^t \int_0^{\tau_1} A^2 x(\tau_2)d\tau_2 d\tau_1 \\
= & \quad x_0 + Atx_0 + \int_0^t \int_0^{\tau_1} A^2 x(\tau_2)d\tau_2 d\tau_1
\end{aligned}
$$

This computation can be repeated infinite times (replace $x(\tau_2)$ etc) to obtain

$$
x(t) = x_0 + Atx_0 + A^2\frac{t^2}{2!}x_0 + A^3\frac{t^3}{3!}x_0 + \ldots = [I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \ldots]x_0
$$

The series $I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \ldots$ converges for every finite t and the resulting matrix is denoted by e^{At}, the exponential of the matrix At
Hence

$$
x(t) = \mathrm{e}^{At}x_0
$$

Matrix Exponential

Example Double integrator (approximation of nonholonomic mobile robots)

$$\ddot{q} = u, \quad y = q$$

State space form ($x = (q\ \dot{q})^T$)

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Notice that

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

therefore

$$\mathrm{e}^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = I + At = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Hence the homogenous solution ($u = 0$) to the double integrator is

$$x_h(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x_0 = \begin{pmatrix} x_{01} + x_{02}t \\ x_{02} \end{pmatrix}, \quad y_h(t) = q(t) = x_1(t) = x_{01} + x_{02}t$$

Matrix Exponential

Example Harmonic oscillator (spring-mass system)

$$\ddot{q} + \omega_0^2 q = u, \quad y = q$$

State space form ($x = (\omega_0 q \; \dot{q})^T$)

$$\dot{x} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

The homogenous solution is

$$x_h(t) = \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix} x_0, \quad y(t) = \frac{1}{\omega_0} x_{h1}(t)$$

For instance, if $x(0) = (0 \; \omega_0)^T$, then

$$y(t) = \sin(\omega_0 t)$$

The displacement of a mass-spring system that starts from the rest position with an initial velocity ω_0 has a sinusoidal evolution.

Matrix Exponential

Example Harmonic oscillator (spring-mass system)
If

$$A = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}$$

then

$$\begin{aligned} I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots = & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} t + \begin{pmatrix} -\omega_0^2 & 0 \\ 0 & -\omega_0^2 \end{pmatrix} \frac{t^2}{2!} \\ & + \begin{pmatrix} 0 & -\omega_0^3 \\ \omega_0^3 & 0 \end{pmatrix} \frac{t^3}{3!} + \dots \\ = & \begin{pmatrix} 1 - \omega_0^2 \frac{t^2}{2!} + \dots & \omega_0 t - \omega_0^3 \frac{t^3}{3!} + \dots \\ -\omega_0 t + \omega_0^3 \frac{t^3}{3!} - \dots & 1 - \omega_0^2 \frac{t^2}{2!} + \dots \end{pmatrix} \\ = & \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix} \end{aligned}$$

Hence

$$\mathrm{e}^{At} = \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix}$$

Today

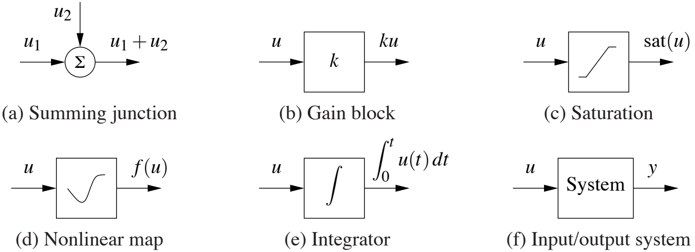
- Stability of linear systems
- Stability for nonlinear systems via linear approximation
- Routh Hurwitz theorem
- Matrix exponential e^{At}
- **LTI system block diagram**

Block diagrams

Block diagrams

Graphical representation of the information flow in control systems

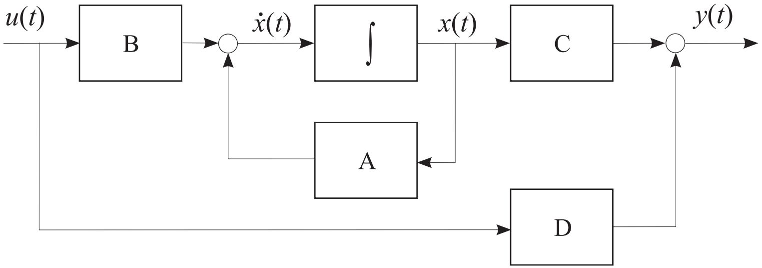
- **Boxes** represent different system elements
- **Arrows pointing toward** the boxes are the inputs
- Arrows **going out** of the boxes are the outputs



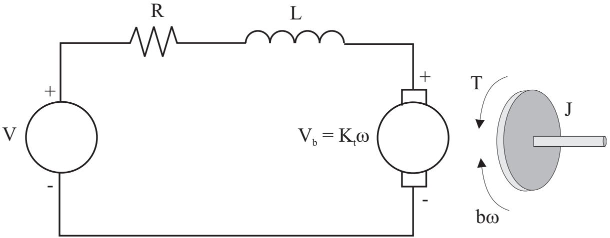
LTI system block diagram

LTI, **L**inear **T**ime **I**nvariant, state space system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$



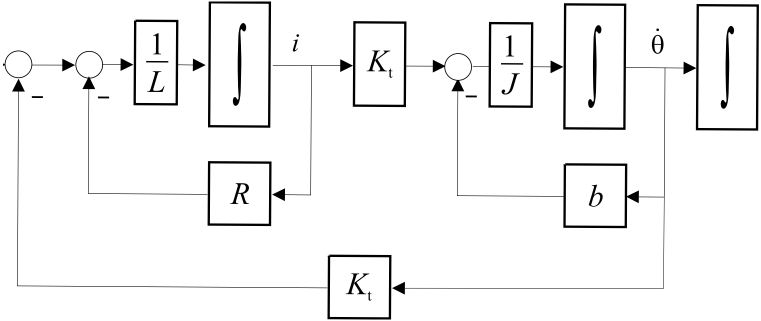
Example: DC Motor



Differential equation:

$$\begin{aligned}
L\frac{di(t)}{dt} + \overbrace{Ri(t)}^{dissipation} &= V(t) - \overbrace{K_t\frac{d\theta(t)}{dt}}^{back\ emf} && \text{electrical part} \\
J\frac{d^2\theta(t)}{dt^2} + \underbrace{b\frac{d\theta(t)}{dt}}_{viscous\ friction} &= \underbrace{K_t i(t)}_{torque} && \text{mechanical part}
\end{aligned}$$

Example: DC Motor, block diagram



$$L\frac{di(t)}{dt} + \overbrace{Ri(t)}^{dissipation} = V(t) \overbrace{-K_t\frac{d\theta(t)}{dt}}^{back\ emf} \quad \text{electrical part}$$

$$J\frac{d^2\theta(t)}{dt^2} + \underbrace{b\frac{d\theta(t)}{dt}}_{viscous\ friction} = \underbrace{K_t i(t)}_{torque} \quad \text{mechanical part}$$

Summary

Today

- Stability of linear systems
- Stability for nonlinear systems via linear approximation
- Routh Hurwitz theorem
- Matrix exponential e^{At}
- Block diagrams

Next lecture

- Basic properties of e^{At}
- Solutions to linear state space equations
- Reachability and reachable canonical form