Geometry 2024, homework set 3

- Below you can find your third homework assignment. Please upload it to BrightSpace by **Friday April 5**. The deadline is strict, so late homework will not be graded.
- The number of points per question is given in a box. 20 extra points are given for a clear writing of solutions. Note that high marks for the homeworks contribute to the final grade.

QUESTIONS

1. $\boxed{10+10+10 = 30\text{pts}}$ (**Affine and projective classification of quadrics** in $\mathbb{R}^3$)

   Let $M^2 = \{Q = 0\}$ be a non-singular and non-empty quadric surface[1] in $\mathbb{R}^3$. Give a list of such quadric surfaces (each representing its unique equivalence class) up to the following transformations of $\mathbb{R}^3$:

   a) Invertible affine transformations of $\mathbb{R}^3$;

   b) Invertible projective transformations of $\mathbb{R}^3$ (viewed as a subset of $P_3(\mathbb{R})$).

   c) Show the projective transformation bringing a quadric to its equivalence class representative can always be chosen to be a composition of an affine map and a perspectivity.

   Hint for part b): Rewrite the polynomial equation $Q = 0$ defining the quadric in homogeneous coordinates (making Q a homogeneous polynomial) and apply diagonalization to the corresponding quadratic form.

2. $\boxed{10+10+5+5 = 30\text{pts}}$ (**Quaternions and the orthogonal group** $O(3)$) Consider $\mathbb{R}^4$ as a Euclidean vector space with the orthonormal basis $1, i, j, k$ and the algebra structure on it defined by the following rules:

   - 1 is the unit element of the algebra;
   - $i^2 = j^2 = k^2 = -1$;
   - $ij = k,\ jk = i$, and $ki = j$;
   - $ij = -ji,\ jk = -kj$, and $ki = -ik$.

   The numbers (elements of the algebra) $q = a_0 + ia_1 + ja_2 + ka_3,\ a_i \in \mathbb{R}$, are called *quaternions*. If $a_0 = 0$, then the number is called a *pure quaternion*. Note that pure quaternions thus span a Euclidean 3-space $\mathbb{R}^3$.

[1] See Lecture 5 for the list of such quadric surfaces up to Euclidean isometries.

a) Prove that for each quaternion s of unit length, the mapping $q \mapsto sqs^{-1}$, where q is a pure quaternion, is a rotation of $\mathbb{R}^3$;

b) We thus get a mapping from the sphere of unit quaternions

$$S^3 = \{s = s_0 + is_1 + js_2 + ks_3 \in \mathbb{R}^4 \mid s_0^2 + s_1^2 + s_2^2 + s_3^2 = 1\}$$

to the group $SO(3)$ of orthogonal matrices with unit determinant. Show that two unit quaternions s and τ represent the same rotation if and only if $\tau = -s$;

c) Deduce that $SO(3)$ can naturally be identified with the real projective space $P_3(\mathbb{R})$;

d) Show that the full orthogonal group $O(3)$ can also be represented using quaternions.

Hint: It may be handy to look at the similar **Exercise IV.45** in the book by M. Audin.

3. $\boxed{10+10 = 20\text{pts}}$ (**The 1-d projective groups over $\mathbb{R}$ and $\mathbb{C}$**) Prove that the group of projective transformations of $P_1(\mathbb{C})$ consists of the maps

$$f = \frac{az+b}{cz+d}, \qquad \text{where } a, b, c, d \in \mathbb{C} \quad \text{and} \quad ad - bc = 1.$$

Determine its subgroup that gives the group of orientation preserving isometries of the Poincaré unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ with the Riemannian metric $G(u, v) = \frac{4(\mathrm{d}u^2+\mathrm{d}v^2)}{(1-u^2-v^2)^2}$, $z = u + iv$.

End of homework