university of groningen / faculty of science and engineering

Mechatronics

Week 8 Day 1

Previously

- We learned that delays may modify the stability properties of a system
- We studied the Nyquist's criterion as an alternative to check the stability of a system without computing its poles
- We learned to compute phase and gain margin of a system, and how to obtain them from their Nyquist plot
- We learned the phase margin of a system is helpful to determine how much the system is robust or not in the presence of delays

Today's lecture:

Absolute stability.

Stability of nonlinear system as per Circle criterion and Popov's criterion

Learning objectives

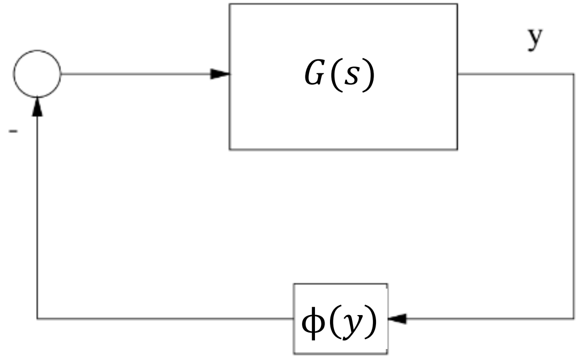
After today's lecture, you will be able to

- Check closed-loop stability of linear systems with sector nonlinearities

Linear Systems with Sector Nonlinearities

Linear systems with sector nonlinearities

A class of non-linear systems include linear time-invariant systems, with transfer function $G(s)$ and a feedback part with static nonlinearity.



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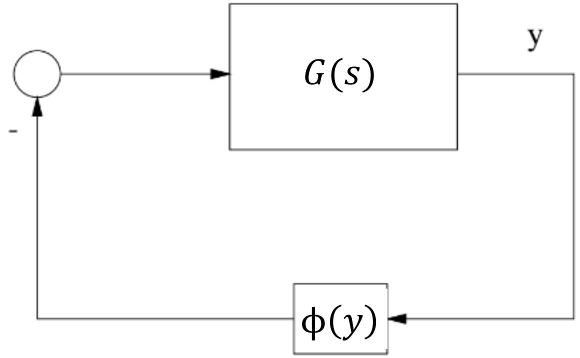

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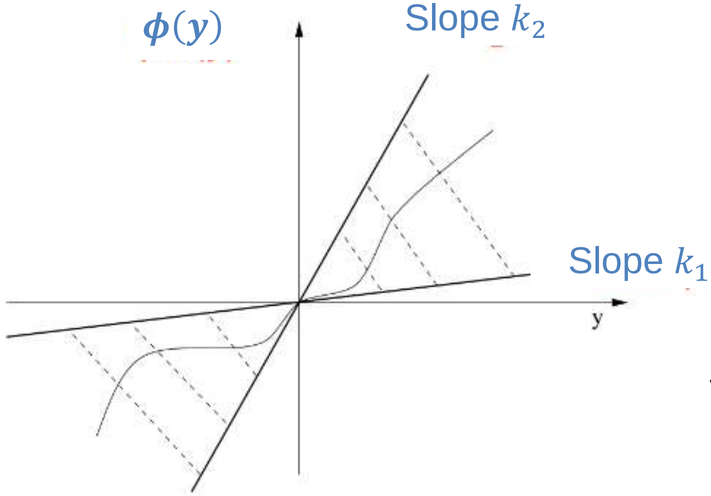
Special nonlinearities fall in the class of section nonlinearities. For example:

- Saturations
- Dead zones
- Hysteresis
- Relais

Only one equilibrium point is allowed!

Sector nonlinearities



Sector nonlinearities ϕ are continuous and belong to a sector $[k_1, k_2]$ if k_1 and k_2 exist such that

$$y \neq 0 \Rightarrow k_1 \leq \frac{\phi(y)}{y} \leq k_2$$

with $\phi(0) = 0, \phi(y)y \geq 0$

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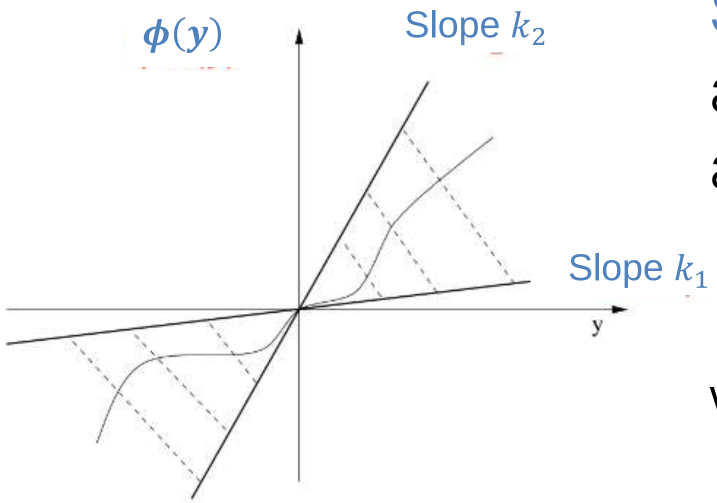

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*A special class is defined when $0 \leq \phi(y) \leq ky$

*If $k_1, k_2 \geq 0$, then ϕ lies only in the first and third quadrant

Popov's Criterion

Popov's criterion

Only useful for nonlinearities in sector $[\boldsymbol{0}, \boldsymbol{k}]$

Generalisation of Nyquist criterion with $(-1, 0)$ replaced with a line

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Generalisation of Nyquist criterion with $(-1, 0)$ replaced with a line

What line?

Consider $G(j\omega) = G_1(\omega) + jG_2(\omega)$ and correspondingly:

$$W(j\omega) = G_1(\omega) + j\omega G_2(\omega)$$

Then $\Re\{W(j\omega)\} = \Re\{G(j\omega)\}$ and $\Im\{W(j\omega)\} = \omega\Im\{G(j\omega)\}$.

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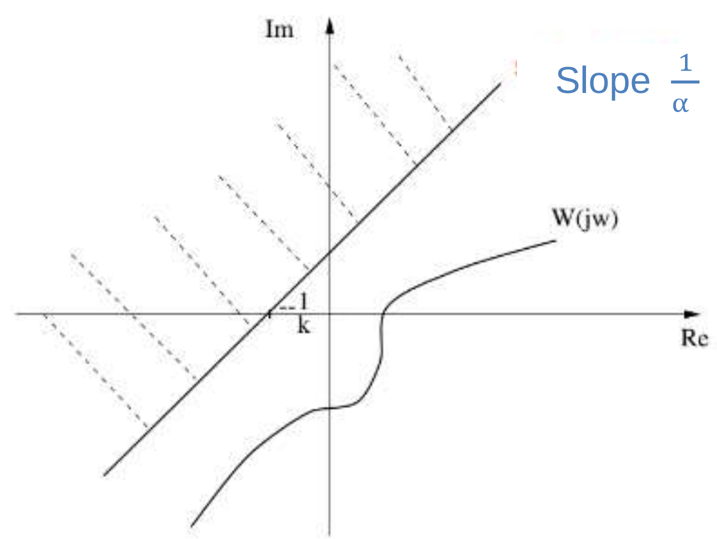
Consider the polar plot of W, called Popov plot, and take the line

$$\boxed{x - \alpha y + \frac{1}{k} = 0}$$



Popov's criterion

Proposition

If a linear system combines with a static nonlinearity in the feedback and fulfills

- A is asympt. stable (Hurwitz) i.e $\lambda_i(A) < 0$ for all i and (A, b) is controllable, i.e., $[b\ Ab\ \ \ldots A^{n-1}b]$ full rank
- Φ belongs to sector $[0, k]$
- There exists an $\alpha > 0$ such that for all $\omega \geq 0$

$$\Re\big((1 + j\alpha\omega)G(j\omega)\big) + \frac{1}{k} \geq \epsilon$$

for an arbitrarily small $\epsilon > 0$

then 0 is globally asymptotically stable

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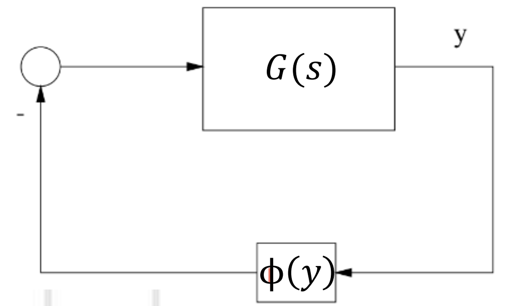
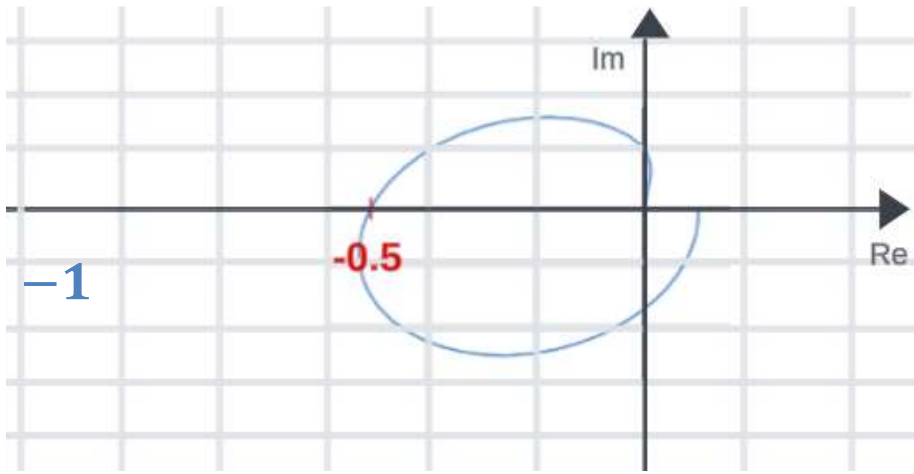
Only a sufficient condition!!

Example: Stability by Popov's criterion

Consider the nonlinear system on the right where nonlinearity $\phi(s)$ belongs to sector $[0,1]$



The Popov plot looks as follows:



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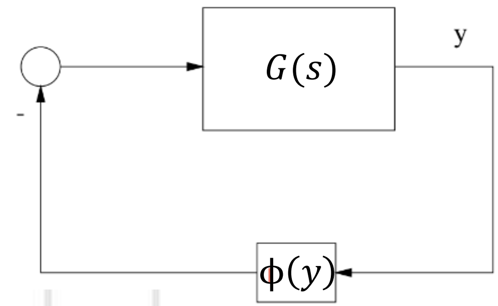

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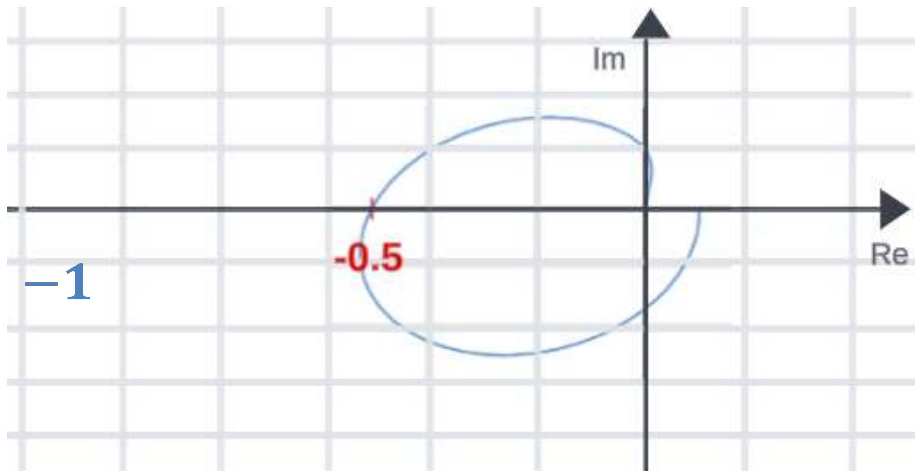

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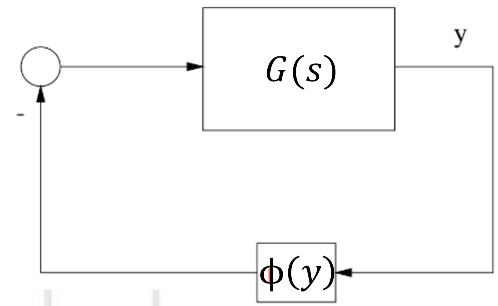

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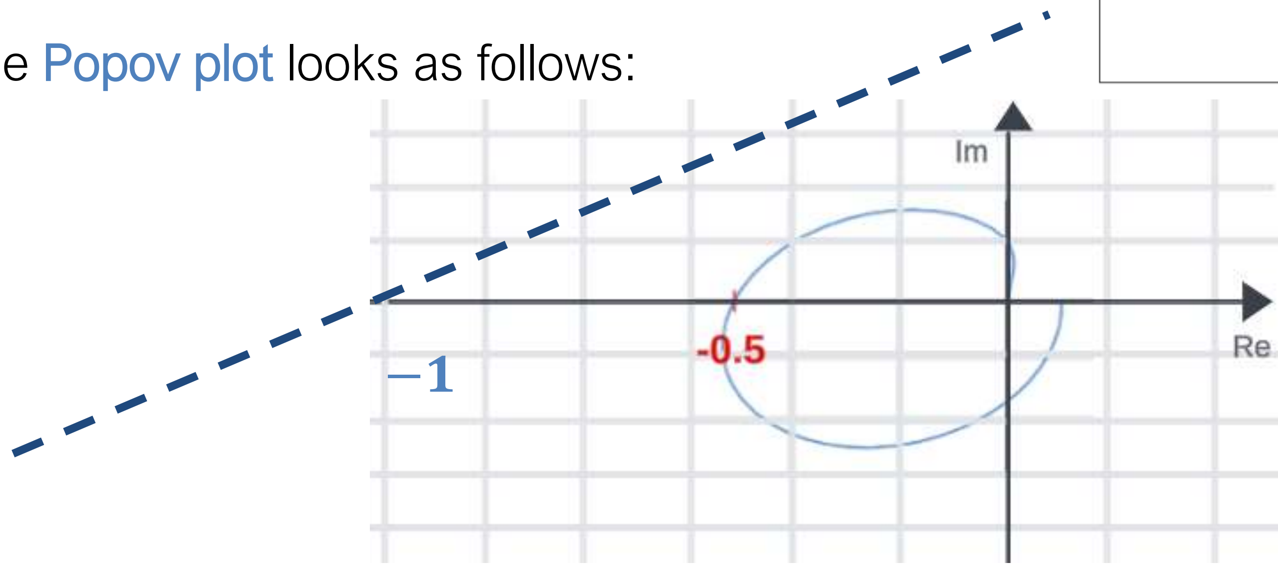

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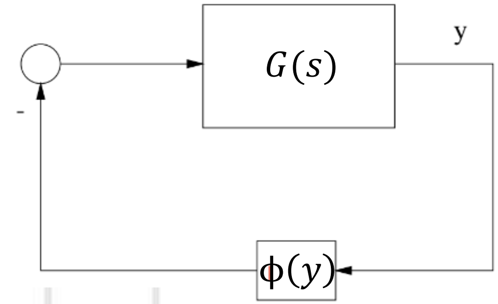

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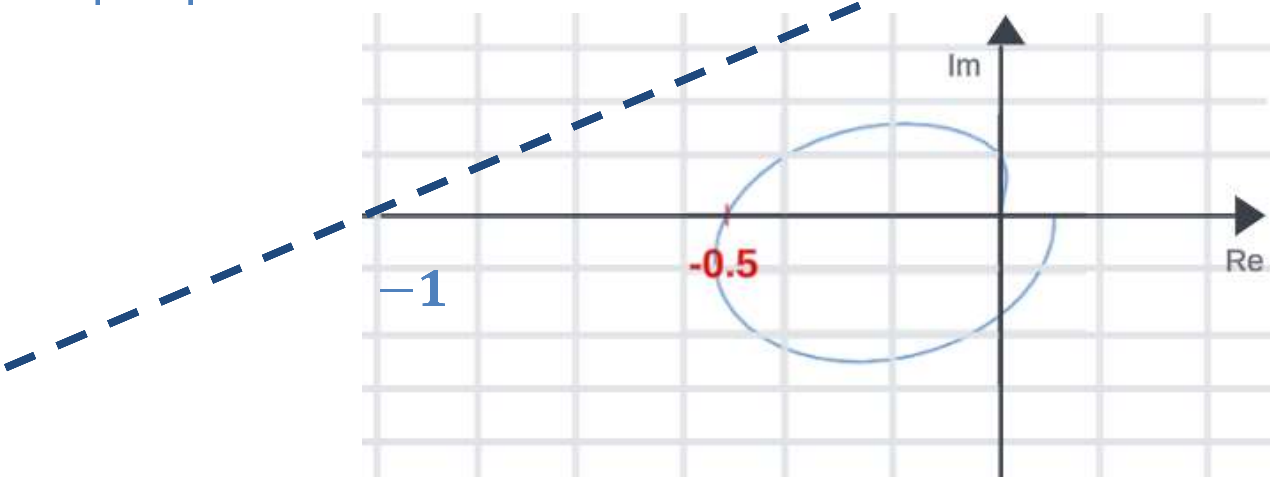

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We can draw a line cutting the real line at -1 with slope $\frac{1}{\alpha}$ with $\alpha>0$

There exists a line for which the whole Popov's plot stays underneath, therefore the closed loop system is stable

Circle criterion

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Useful for sector nonlinearities $\boldsymbol{\Phi}$ in sector $[\boldsymbol{k_1}, \boldsymbol{k_2}]$

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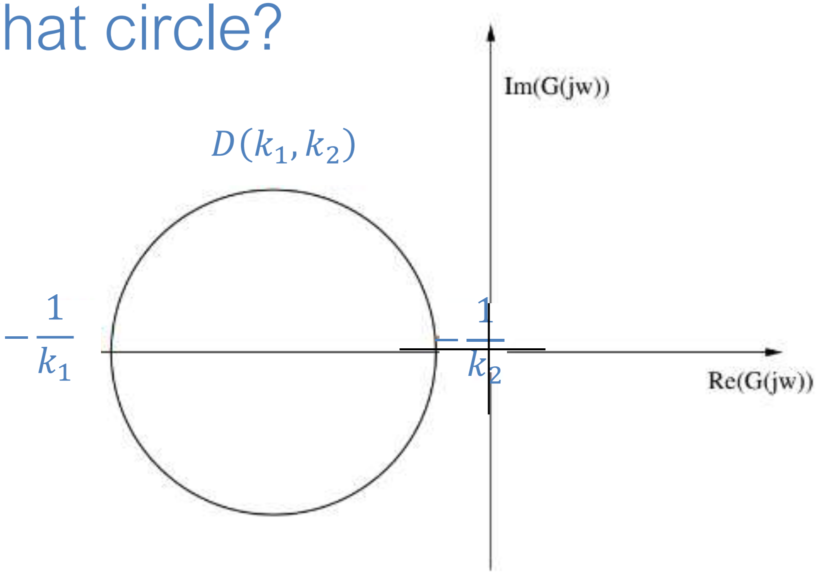

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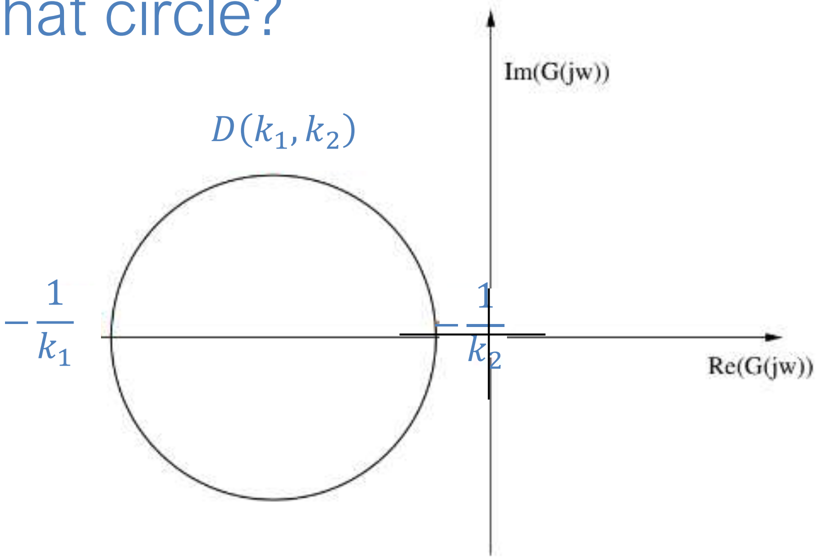

*Note that if $k_2 \to k_1$:

-sector becomes thinner
-nonlinearity → linear

-circle → point

Then circle criterion becomes Nyquist criterion

Circle criterion

Theorem

If a linear system combines with a static nonlinearity in the feedback and fulfills

- A no eigenvalues on $\mathrm{j}\omega$ −axis and ρ eigenvalues in RHP
- Φ belongs to sector $[k_1, k_2]$
- One of the following holds

  $0 < k_1 \leq k_2$, Nyquist plot of $G(j\omega)$ does not enter $\mathcal{D}(k_1, k_2)$ and encircles it ρ times anti-clockwise

  $0 = k_1 < k_2$, Nyquist plot of $G(j\omega)$ stays to the right of $\Re(s) > -\frac{1}{k_2}$

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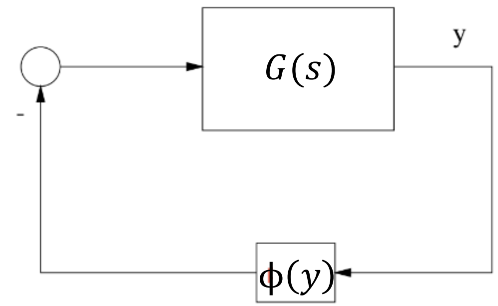

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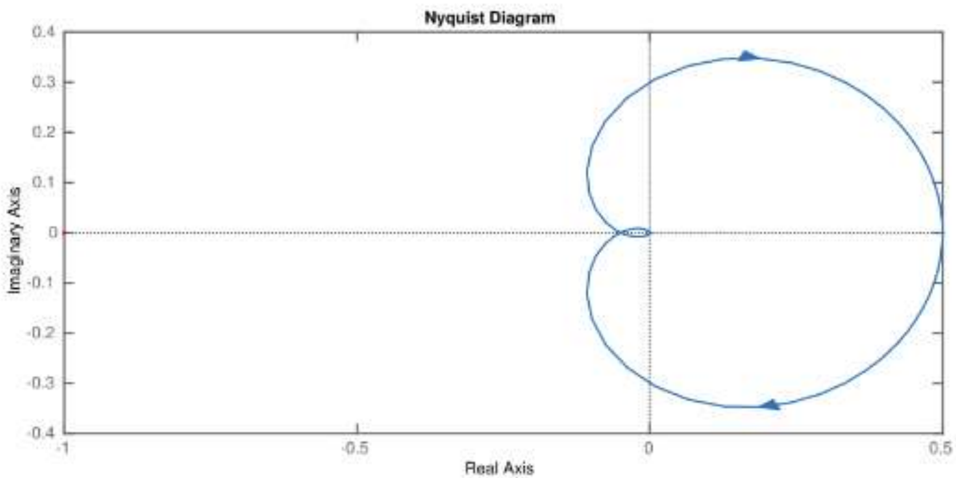

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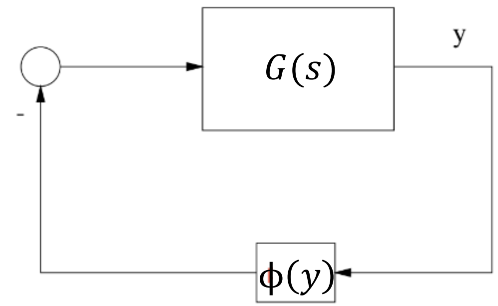

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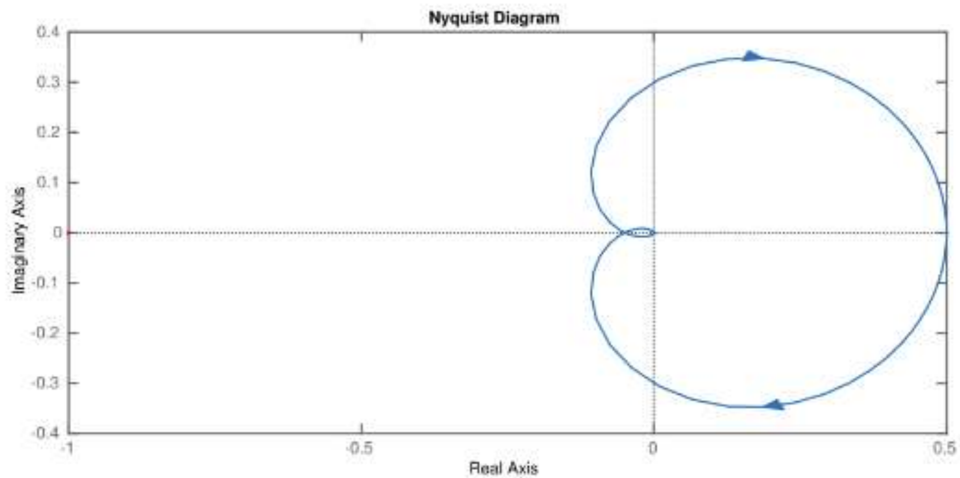

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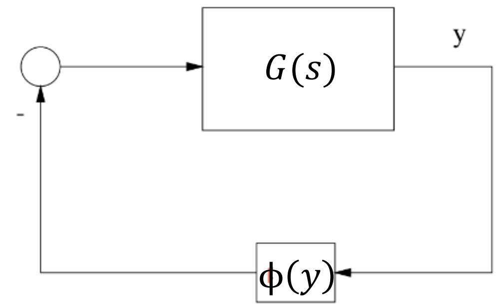

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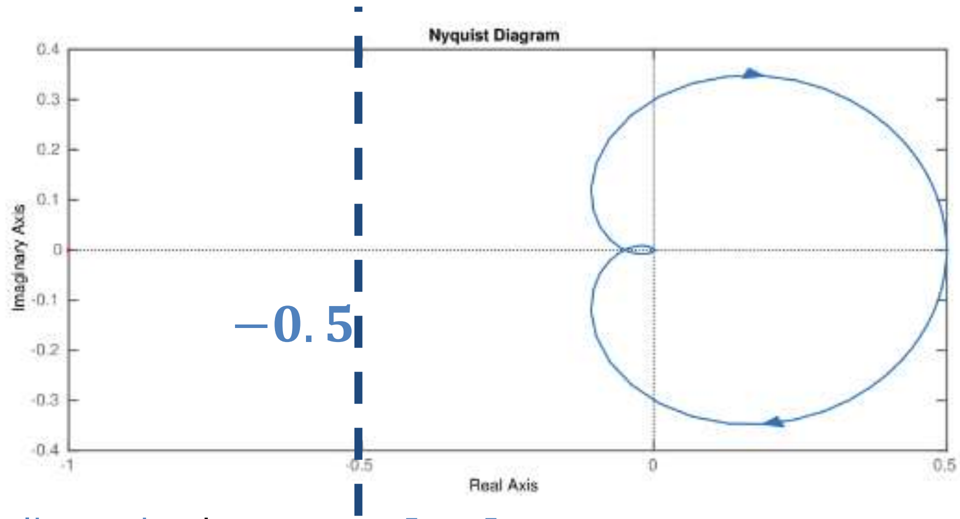

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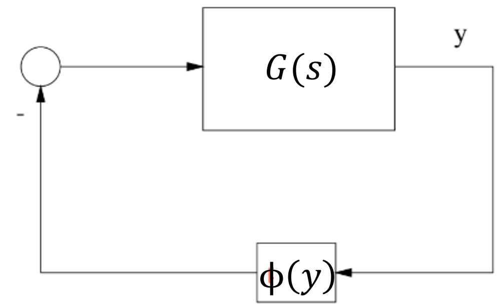

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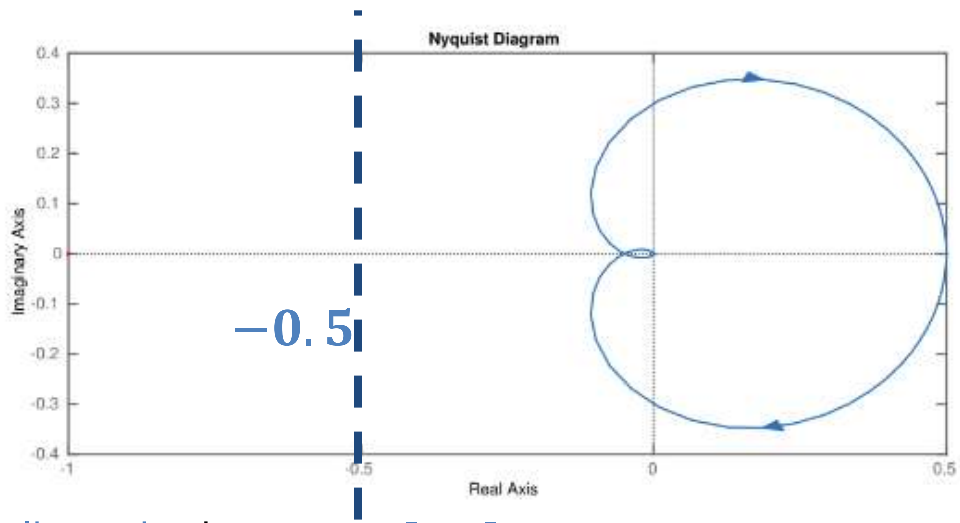

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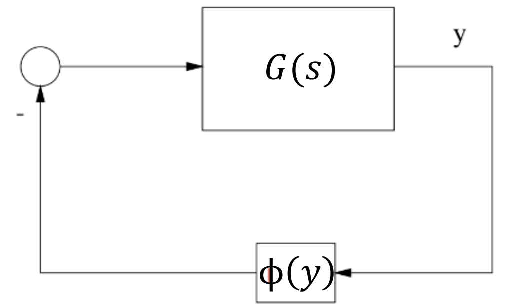
Since Nyquist plot stays to the right of real line crossing at $-\frac{1}{2}$, the closed loop system is stable

Example: Stability by Circle criterion

Consider the nonlinear system on the right where nonlinearity $\phi(s)$ belongs to sector $[-1,2]$



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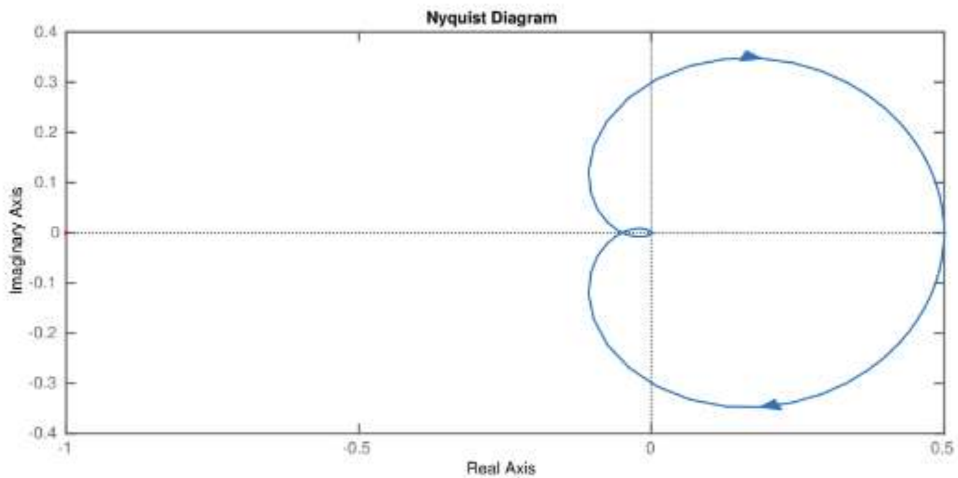

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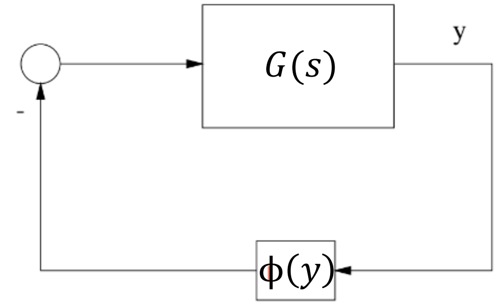

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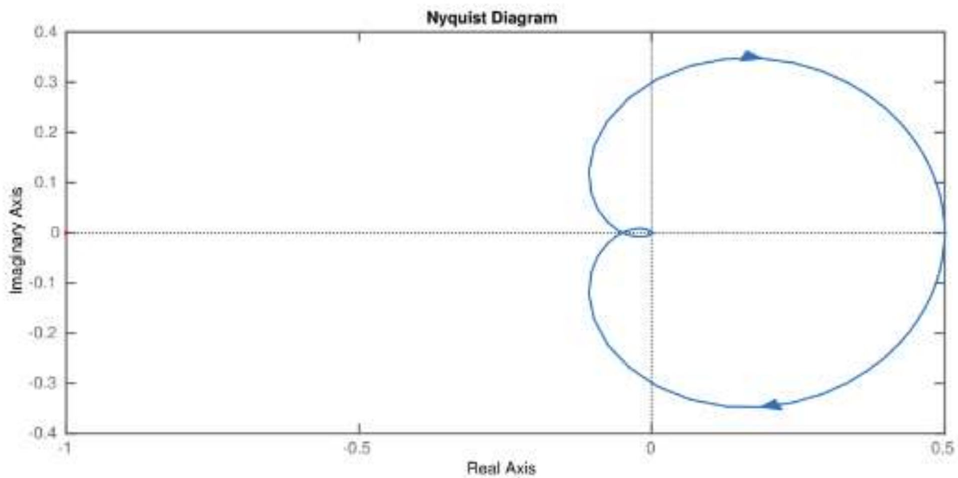

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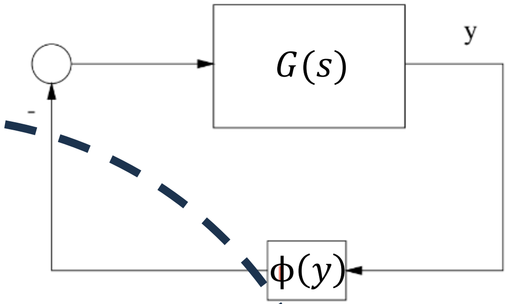

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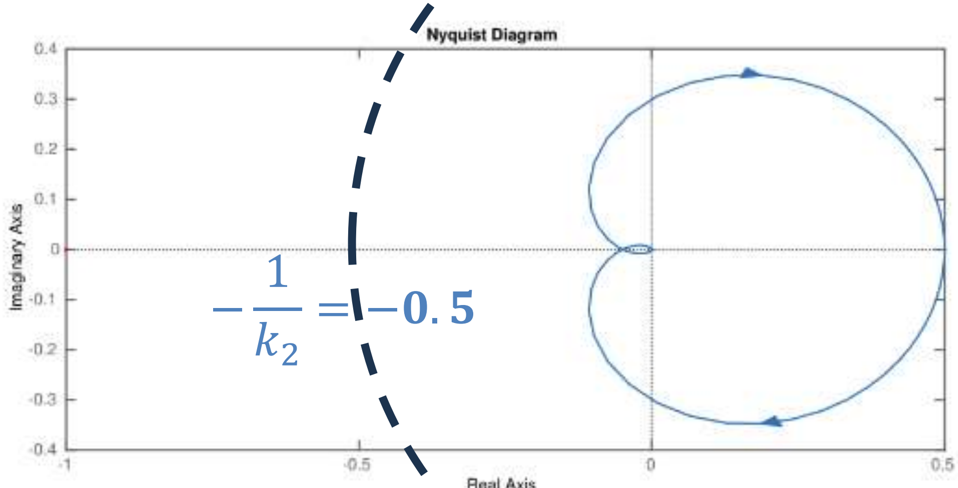

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We have a nonlinearity in sector $[-1,2]$
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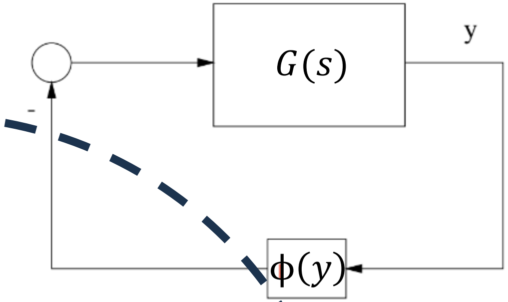

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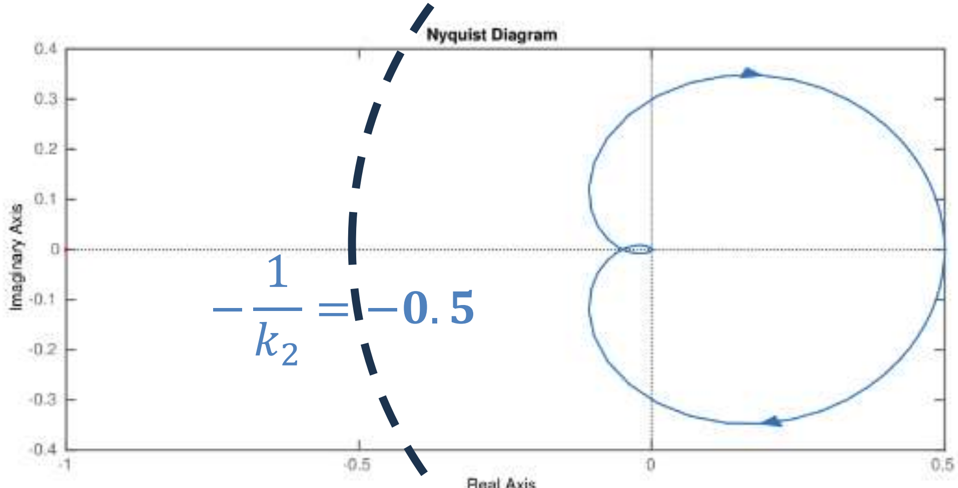

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We need to take circle $D(-1,2)$ and check that Nyquist plot stays inside

Since the Nyquist plot does not exist the circle, we conclude the closed loop system is stable

Summary

- Circle criterion is a generalisation of Nyquist criterion with point $(-1,0)$ replaced with a circle
- Popov's criterion is a generalisation of Nyquist criterion with point $(-1,0)$ replaced with a line
- Circle and Popov's criterion can be used to study closed loop stability of linear systems with sector nonlinearities
- They only provide sufficient stability conditions

The End