**Examination Mechatronics**
Wednesday, January $20^{th}$ 2021, 8:30 – 11:30 hrs.

Last Name: ____________________ First Name: ____________________ StudentID: ____________

**INSTRUCTIONS:**

- This exam consists of a Nestor test and 7 exercises (open questions). Make sure you answer all the questions.
- Read the instructions of each question carefully and answer accordingly.
- The solution to each exercise (including all the arguments that led to the result) must be uploaded to Nestor. All the answers should be uploaded to the corresponding assignment in **PDF format. Answers in another format will not be accepted.**
- **Write down your Surname and Student ID on the name of each file you upload.**
- If your answer consists of more than one page, then enumerate the sheets to make clear the sequence of your solutions. If the procedure is not clear, or the file is not readable, the answer will be taken as wrong. **Providing only the final answers without proper argumentation is not acceptable and will not be graded**.
- Write neatly.
- Whenever we think appropriate, a follow-up **oral examination** to suspected cases will be arranged before the final grade is determined. In this case, the follow-up oral examination will be based on the questions of the exam, and the final grade will be based on the same weighting factor as before, where the adjusted grade from the oral examination will be used.
- **You have three hours to complete the test and thirty minutes to upload your answers. After the time is up, you will no longer have access to the exam, and, therefore, you will not be able to upload your answers. So, upload your solutions on time.**
- You can earn a maximum of 100 points in the exam. The amount of points spread over the exercises is 105 points, i.e., there are 5 bonus points to be earned.
- **You will only get a grade if you have finalized the practical, and you have signed the student pledge.**
- This is an **open** book exam.
- You have to work on your exam in full autonomy, **any form of collaboration with anyone will be severely punished**.
- In case of **urgent** situations that you want to communicate during the exam, you are allowed to use an electronic device to join the Nestor Collaborate session "Exam 20-01-2021" and privately communicate with the moderator in the chat, or you can send an email to `mechatronics.rug.20.21@gmail.com`. **Communication via other means is not possible.**

# Preliminaries

### Across and Through variable table

**Table 1.2.** Ideal system elements (linear)

| System type | Mechanical translational | Mechancial rotational | Electrical | Fluid | Thermal |
|---|---|---|---|---|---|
| A-type variable | Velocity, $v$ | Velocity, $\Omega$ | Voltage, $e$ | Pressure, $P$ | Temperature, $T$ |
| A-type element | Mass, $m$ | Mass moment of inertia, $J$ | Capacitor, $C$ | Fluid Capacitor, $C_f$ | Thermal capacitor, $C_h$ |
| Elemental equations | $F = m\frac{dv}{dt}$ | $T = J\frac{d\Omega}{dt}$ | $i = C\frac{de}{dt}$ | $Q_f = C_f\frac{dP}{dt}$ | $Q_h = C_h\frac{dT}{dt}$ |
| Energy stored | Kinetic | Kinetic | Electric field | Potential | Thermal |
| Energy equations | $\mathscr{E}_k = \frac{1}{2}mv^2$ | $\mathscr{E}_k = \frac{1}{2}J\Omega^2$ | $\mathscr{E}_e = \frac{1}{2}Ce^2$ | $\mathscr{E}_p = \frac{1}{2}C_f P^2$ | $\mathscr{E}_t = \frac{1}{2}C_h T^2$ |
| T-type variable | Force, $F$ | Torque, $T$ | Current, $i$ | Fluid flow rate, $Q_f$ | Heat flow rate, $Q_h$ |
| T-type element | Compliance, $1/k$ | Compliance, $1/K$ | Inductor, $L$ | Inertor, $I$ | None |
| Elemental equations | $v = \frac{1}{k}\frac{dF}{dt}$ | $\Omega = \frac{1}{K}\frac{dT}{dt}$ | $e = L\frac{di}{dt}$ | $P = I\frac{dQ_f}{dt}$ | |
| Energy stored | Potential | Potential | Magnetic field | Kinetic | |
| Energy equations | $\mathscr{E}_P = \frac{1}{2k}F^2$ | $\mathscr{E}_P = \frac{1}{2K}T^2$ | $\mathscr{E}_m = \frac{1}{2}Li^2$ | $\mathscr{E}_k = \frac{1}{2}IQ_f^2$ | |
| D-type element | Damper, $b$ | Rotational damper, $B$ | Resistor, $R$ | Fluid resistor, $R_f$ | Thermal resistor, $R_h$ |
| Elemental equations | $F = bv$ | $T = B\Omega$ | $i = \frac{1}{R}e$ | $Q_f = \frac{1}{R_f}P$ | $Q_h = \frac{1}{R_h}T$ |
| Rate of energy dissipated | $\frac{dE_D}{dt} = Fv = \frac{1}{b}F^2 = bv^2$ | $\frac{dE_D}{dt} = T\Omega = \frac{1}{B}T^2 = B\Omega^2$ | $\frac{dE_D}{dt} = ie = Ri^2 = \frac{1}{R}e^2$ | $\frac{dE_D}{dt} = Q_f P = R_f Q_f^2 = \frac{1}{R_f}P^2$ | $\frac{dE_D}{dt} = Q_h$ |

*Note:* A-type variable represents a spatial difference across the element.

**The other analogy for linear systems as was treated in Control Engineering, and is useful for Euler-Lagrange modeling.**

| | Kinetic coenergy | Potential energy | Rayleigh dissipation function |
|---|---|---|---|
| Translation | $T^*(\dot{x}) = \frac{1}{2}m\frac{dx}{dt}^2$ | $V(x) = \frac{1}{2}kx^2$ | $\mathcal{D}(\dot{x}) = \frac{1}{2}b\frac{dx}{dt}^2$ |
| Rotation | $T^*(\dot{\theta}) = \frac{1}{2}J\frac{d\theta}{dt}^2$ | $V(\theta) = \frac{1}{2}k\theta^2$ | $\mathcal{D}(\dot{\theta}) = \frac{1}{2}b\frac{d\theta}{dt}^2$ |
| Electric | $T^*(\dot{q}) = \frac{1}{2}L\frac{dq}{dt}^2$ | $V(q) = \frac{1}{2C}q^2$ | $\mathcal{D}(\dot{q}) = \frac{1}{2}R\frac{dq}{dt}^2$ |

## Euler-Lagrange equations

The Euler-Lagrange equation, considering external forces and dissipation, is given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q} = F - \frac{\partial \mathcal{D}(\dot{q})}{\partial \dot{q}} \tag{1}$$

where $\mathcal{L}(q,\dot{q})$ denotes the Lagrangian of the system, $q$ the position, and $\mathcal{D}(\dot{q})$ the Rayleigh dissipation function.

### Canonical forms

The state-space representation for a given transfer function is not unique, i.e., there are infinite-number of possibilities to express a given transfer function in state-space form. However, there are several forms that can be helpful in the design of controller or observer. Let us consider the following general transfer function for single-input single-output system:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}. \tag{2}$$

For this transfer function, the state-space representation in **canonical observable form** is given by

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix} &= \begin{bmatrix}0 & 0 & \cdots & 0 & -a_n\\ 1 & 0 & \cdots & 0 & -a_{n-1}\\ 0 & 1 & \cdots & 0 & -a_{n-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix} + \begin{bmatrix}b_n - a_n b_0\\ b_{n-1} - a_{n-1}b_0\\ \vdots\\ b_1 - a_1 b_0\end{bmatrix}u\\ y &= \begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix} + b_0 u.\end{aligned} \tag{3}$$

On the other hand, the state-space representation in the **canonical controllable form** is given by

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix} &= \begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ 1\end{bmatrix}u\\ y &= \begin{bmatrix}b_n - a_n b_0 & b_{n-1} - a_{n-1}b_0 & \cdots & b_2 - a_2 b_0 & b_1 - a_1 b_0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix} + b_0 u.\end{aligned} \tag{4}$$

**Z-transform.**

Denote by $Z\{u(n)\}$ the $Z$-transform of discrete-time signal $u(n)$ where $n = 0, 1, \ldots$.

- Unit step signal $u(n)$: $Z\{u(n)\} = \dfrac{1}{1 - z^{-1}}$
- Time-shifting property: $Z\{u(n-k)\} = z^{-k}U(z)$

**Transformation from $s$-domain to $z$-domain**

- The bilinear transformation: $s \mapsto \dfrac{2}{T_s}\dfrac{1 - z^{-1}}{1 + z^{-1}} = \dfrac{2}{T_s}\dfrac{z - 1}{z + 1}$
- The Euler backward approximation: $s \mapsto \dfrac{1}{T_s}(1 - z^{-1}) = \dfrac{z - 1}{T_s z}$
- The Euler forward approximation: $s \mapsto \dfrac{z - 1}{T_s}$

**Optimal state feedback control design(LQR)**

The Riccati equation, that is related to the optimal state feedback, reads as

$$A^T P + PA - PBR^{-1}B^T P + Q = 0, \tag{5}$$

where $P = P^T > 0$, and where $Q = Q^T > 0$ and $R = R^T > 0$ are related to the cost function

$$J = \int_0^\infty \left(x^T(\tau)Qx(\tau) + u^T Ru(\tau)\right) d\tau. \tag{6}$$

The optimal state feedback controller is given by $u(t) = -R^{-1}B^T Px(t)$.

**State observer design**

For a state-space system described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{7}$$

where $x$ is the actual state and $y$ is the measured signal, a state observer for such system has the following structure

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x} + Du, \end{aligned} \tag{8}$$

where $\hat{x}$ is the estimated state and $y$ is the corresponding estimated output.

**Padé approximation**

The first order Padé approximation of the delay transfer function $e^{-Ts}$ is given by

$$e^{-Ts} \approx \frac{1 - \frac{T}{2}s}{1 + \frac{T}{2}s}. \tag{9}$$

## Complex numbers

Consider a complex number

$$z = a + bj. \tag{10}$$

Then

$$\begin{aligned} |z| &= \sqrt{a^2 + b^2} \\ \angle z &= \tan^{-1}\left(\frac{b}{a}\right) \end{aligned} \tag{11}$$

## Phase margin

The phase margin of a system $G(s)$ can be computed as

$$P_m = -\pi - \angle G(j\omega_*), \tag{12}$$

where the frequency $\omega_*$ verifies

$$|G(j\omega_*)| = 1. \tag{13}$$

The phase margin of a system with delay is computed as follows

$$P_{m_d} = P_m + T\omega_*, \tag{14}$$

where $T$ is the time delay.

### Absolute stability

### Popov's criterion

**Proposition 1.** *If a linear system combines with a static nonlinearity in the feedback, and the following are fulfilled:*

- *$A$ (the system's matrix) is asymptotically stable*
- *The nonlinearity belongs to a sector $[0, k]$*
- *There exists a constant $\gamma \geq 0$ such that for all $\omega \geq 0$*

$$\Re\left((1 + j\gamma\omega)G(j\omega)\right) + \frac{1}{k} \geq \epsilon \tag{15}$$

*for arbitrarily small $\epsilon > 0$, then $0$ is globally asymptotically stable.*

### Circle's criterion

**Theorem 1.** *If a linear system combines with a static nonlinearity in the feedback, and the following are fulfilled:*

- *$A$ (the system's matrix) has no eigenvalues on the $j\omega$-axis and $\rho$ eigenvalues in the RHP*
- *The nonlinearity belongs to the sector $[k_1, k_2]$*
- *One of the following holds*
  - *$0 < k_1 \leq k_2$, the Nyquist plot of $G(j\omega)$ does not enter the disk $D(k_1, k_2)$ and encircles it $\rho$ times anti-clockwise*
  - *$0 = k_1 < k_2$, the Nyquist plot stays to the right of $\Re(s) > -\frac{1}{k_2}$*
  - *$k_1 < 0 < k_2$, the Nyquist plot of $G(j\omega)$ stays in $D(k_1, k_2)$*
  - *$k_1 < k_2 < 0$, the Nyquist plot of $-G(j\omega)$ does not enter $D(-k_1, -k_2)$ and encircles it $\rho$ times anti-clockwise*

*Then $0$ is globally asymptotically stable.*

1. (20 TOTAL points) Complete the test "Exercise 1" on Nestor

   Check the solution on Nestor

2. (20 TOTAL points) **Modeling**. Let the first digit (from right to left) of your student number be denoted as $\alpha$. Then, solve the corresponding case (e.g., if your student number is `sXXXXXX6`, then solve the problem given in **C4**).

**C1** ($\alpha = 0$ or $\alpha = 1$) Obtain the state-space model of the system depicted in Fig. 1. To this end, consider

$$f_r(\dot{z}_2) = b_1\dot{z}_2 + b_2\tanh(\dot{z}_2), \tag{16}$$

and the relationships

$$\begin{aligned} F &= DP_{3r} \\ \dot{z}_1 &= \frac{1}{D}Q_m, \end{aligned} \tag{17}$$

where $D$ is a constant parameter.

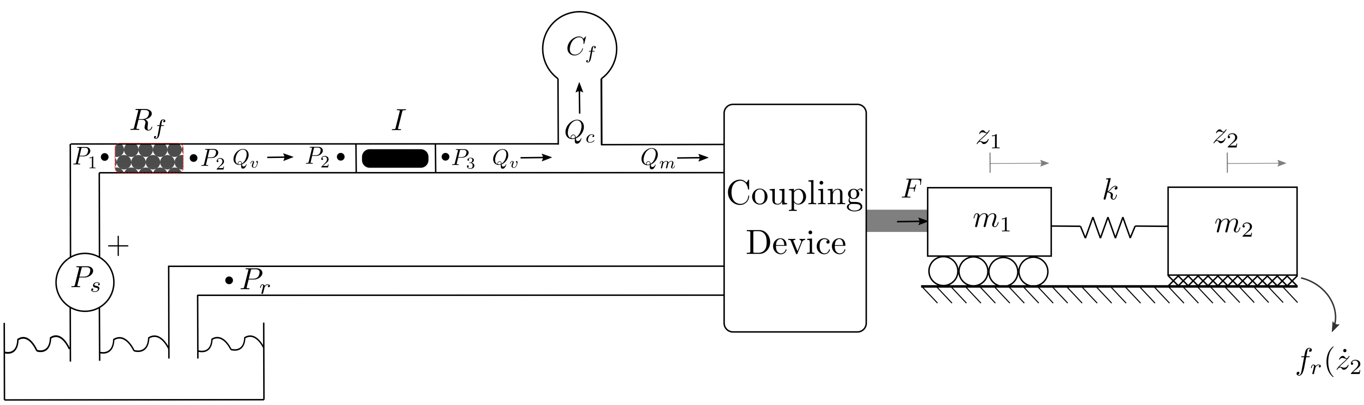


Figure 1: Multidomain system for the case **C1**.

**C2** ($\alpha = 2$ or $\alpha = 3$) Obtain the state-space model of the system depicted in Fig. 2. To solve this problem consider the following relationships

$$\begin{aligned} T_T &= DP_{3r}, \quad & T_G &= \frac{1}{\beta}i_G, \\ \omega_T &= \frac{1}{D}Q, \quad & \omega_G &= \beta V_G, \end{aligned} \tag{18}$$

where $D$ and $\beta$ are constant parameters.

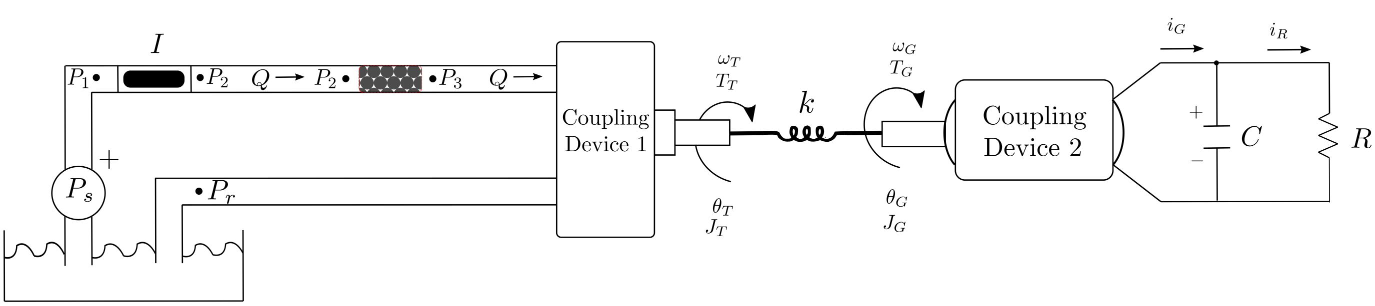


Figure 2: Multidomain system for the case **C2**.

**C3** ($\alpha = 4$ or $\alpha = 5$)
Obtain the state-space model of the system depicted in Fig. 3. To this end, consider

$$Q_H = \beta V_{R_H} i_{R_H}, \tag{19}$$

where $\beta$ is a constant parameter.

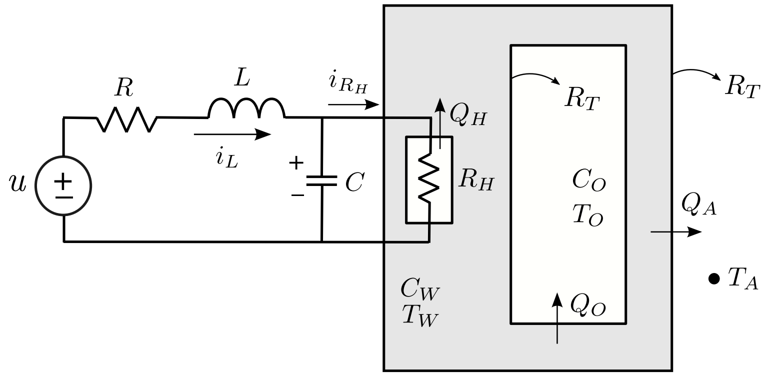


Figure 3: Multidomain system for the case **C3**.

**C4** ($\alpha = 6$ or $\alpha = 7$) Obtain the equations of motion, via the Euler-Lagrange formalism, of the system depicted in Fig. 4. The force that the electro-magnet is exerting on the metallic surface is denoted by $f_m(i)$. This force is coupling the electrical part of the system with the mechanical one, and it is given by

$$f_m(i) = \frac{(\beta i)^2 \gamma}{2}, \tag{20}$$

where $\beta$ and $\gamma$ are constant parameters. Once you get the equations of motion, obtain the state-space model that describes this system. **Hint:** remember that $i = \frac{dq}{dt}$, where $q$ is the electric charge.

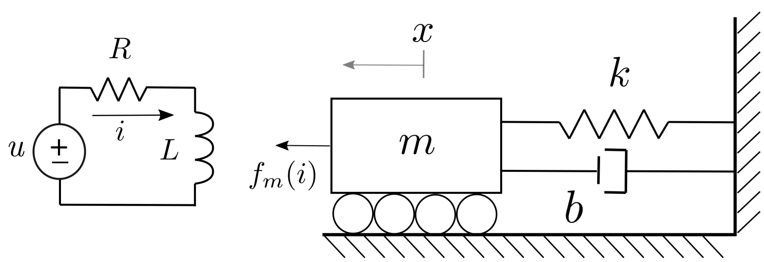


Figure 4: Simplified magnetic lock.

**C5** ($\alpha = 8$ or $\alpha = 9$) Obtain the equations of motion, vie the Euler-Lagrange formalism, of the system depicted in Fig. 5. To this end, denote with $m_1$ the mass of the rod, and with $m_2$ the mass of the wheel. The distance from the reference to the center of masses is indicated in red. **Hint:** the angular displacement of the mass is $\theta_1 + \theta_2$. Don't forget the gravity. The kinetic energy of each element has two components, one related to its inertia and the other related to the velocities in $X$ and $Y$.

Check the solution on Nestor

3. (10 TOTAL points) **Discretization**. Consider the system

$$\dot{x} = Ax(t), \quad A := \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & -a \end{bmatrix} \tag{21}$$

where $a$ is given by the first digit (from right to left) of your student number, i.e., if your student number is `sXXX213`, then $a = 3$. **Note**: if $a$ equals zero in your case, then consider $a = 10$.

(a) (4 points) Consider the forward Euler approximation of the time derivative, namely,

$$\dot{x} \approx \frac{x(k+1) - x({\color{red}k})}{T}, \quad T = b_1 + \frac{b_2}{10}, \tag{22}$$

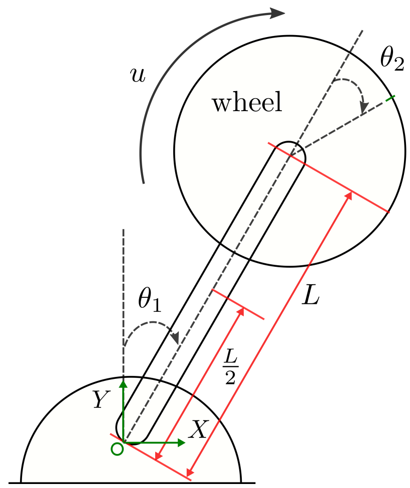


Figure 5: Mechanical system composed of a wheel and a rod.

where $b_1$ and $b_2$ are the third and second digit (from right to left) of your student number, i.e., if your student number is `sXXXX213`, then $b_1 = 2$, $b_2 = 1$, and $T = 2.1$. Obtain the dynamics of $x(k+1)$, and determine whether the discretized system is stable or not (justify your answer).

(b) (6 points) Compute the largest sampling time before the discretized system becomes unstable (justify your answer).

**(a)** Given (22), the discretized system is

$$x(k+1) = TAx(k) + Ix(k) = A_d x(k), \tag{23}$$

where

$$A_d := \begin{bmatrix} 1 & T \\ -\frac{T}{4} & 1 - aT \end{bmatrix}. \tag{24}$$

To determine if the system is stable or not, we compute the eigenvalues of $A_d$. If they are inside the unit circle, then the system is stable. Otherwise, the system is unstable.

**(b) Note: the answer to this exercise is greatly simplified if you know the value of $a$. Here we present the general version of the solution.**

We need to obtain the eigenvalues of $A_d$ in terms of $T$. To this end, we compute

$$\begin{aligned} \mathtt{det}(\lambda I - A_d) &= \mathtt{det}\left(\begin{bmatrix} \lambda - 1 & -T \\ \frac{T}{4} & \lambda - (1 - aT) \end{bmatrix}\right) \\ &= (\lambda - 1)(\lambda - 1 + aT) + \frac{T^2}{4} \\ &= \lambda^2 + (aT - 2)\lambda + \frac{T^2}{4} + 1 - aT. \end{aligned} \tag{25}$$

Hence, the eigenvalues of $A_d$ are given by

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}\left\{2 - aT \pm \sqrt{(aT-2)^2 - T^2 - 4 + 4aT}\right\} \\ &= \frac{1}{2}\left\{2 - aT \pm \sqrt{a^2T^2 - 4aT + 4 - T^2 - 4 + 4aT}\right\} \\ &= \frac{1}{2}\left\{2 - aT \pm T\sqrt{a^2 - 1}\right\} \\ &= 1 + \frac{T}{2}\left\{-a \pm \sqrt{a^2 - 1}\right\}. \end{aligned} \tag{26}$$

Since $a \geq 1$, we get that $-a \pm \sqrt{a^2-1} < 0$. Therefore, $|\lambda_{1,2}| = 1$ when

$$\frac{T}{2}\left\{-a \pm \sqrt{a^2-1}\right\} = -2. \tag{27}$$

Hence,

$$T = -\frac{4}{\left\{-a \pm \sqrt{a^2-1}\right\}}. \tag{28}$$

Note that the eigenvalue corresponding to the case $-a-\sqrt{a^2-1}$ "grows" faster in magnitude. Accordingly,

$$T = -\frac{4}{\left\{-a - \sqrt{a^2-1}\right\}} = \frac{4}{\left\{a + \sqrt{a^2-1}\right\}}. \tag{29}$$

4. (12 TOTAL points) **Discrete-time PD controller**. Consider the system

$$G(s) = \frac{b+1}{s-(a+1)} \tag{30}$$

and the feedback scheme provided in Fig. 6.

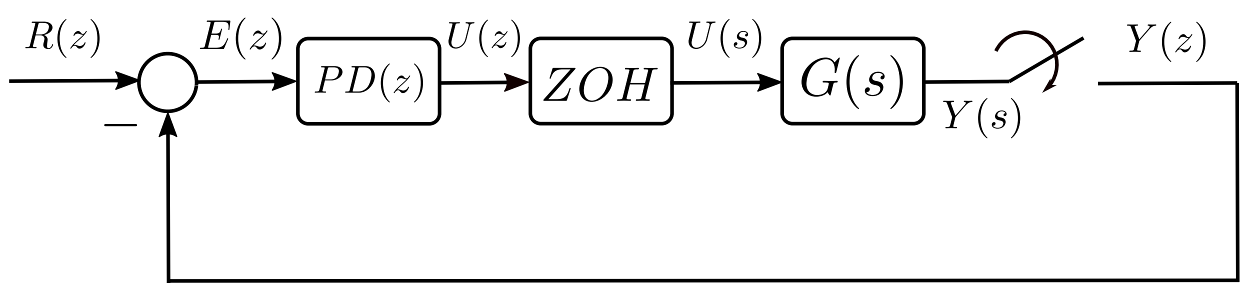


Figure 6: Diagram of a closed-loop digital controller. The switch converts the analog signal $Y(s)$ into a digital one.

Use the Backward Euler approximation of $s$ to discretize the plant and a sampling time

$$T = c_1 + \frac{c_2}{10}, \tag{31}$$

where $a, b, c_1, c_2$, and $k_p$ are determined by the first five digits (from right to left) of your student number ($\mathtt{sXXabc_1c_2k_p}$). Propose a gain $k_d$ such that the closed-loop system is stable. **Hint:** remember that the transfer function of the zero-order hold can be represented by

$$ZOH = \frac{(1-z^{-1})}{s}. \tag{32}$$

Note that the transfer function of the ZOH interconnected (in cascade) with the plant is given by

$$H_{\mathtt{ZOH}} = (1-z^{-1})\left(\frac{1}{s}\cdot\frac{b+1}{s-(a+1)}\right). \tag{33}$$

Hence, using the backward Euler approximation, we get

$$
\begin{aligned}
H_{\texttt{ZOH}}(z) &= \frac{z-1}{z}\left(\frac{Tz}{z-1}\cdot\frac{b+1}{(\frac{z-1}{Tz})-(a+1)}\right) \\
&= \frac{T(b+1)}{\left(\frac{z-1-Tz(a+1)}{Tz}\right)} \\
&= \frac{T^2z(b+1)}{z-1-Tz(a+1)}.
\end{aligned} \tag{34}
$$

On the other hand, the transfer function of the discrete PD controller is given by

$$
C(z) = \frac{(TK_p + K_d)\,z - K_d}{Tz}. \tag{35}
$$

Thus, the open-loop transfer function is given by

$$
\begin{aligned}
H(z) &= C(z)H_{\texttt{ZOH}}(z) \\
&= \frac{(TK_p+K_d)\,z-K_d}{Tz}\cdot\frac{T^2z(b+1)}{z-1-Tz(a+1)} \\
&= \frac{T\,(TK_p+K_d)\,(b+1)z-TK_d(b+1)}{z-1-Tz(a+1)}.
\end{aligned} \tag{36}
$$

Remember that the closed-loop transfer function has the following structure

$$
H_{\texttt{CL}}(z) = \frac{(\cdot)}{1+H(z)} \tag{37}
$$

where the pole is determined by the numerator of $1+H(z)$. Then, we compute

$$
1+H(z) = \frac{T\,(TK_p+K_d)\,(b+1)z-TK_d(b+1)+z-1-Tz(a+1)}{z-1-Tz(a+1)}. \tag{38}
$$

Hence, the characteristic polynomial of the closed-loop system is given by

$$
\{T\,(TK_p+K_d)\,(b+1)+1-T(a+1)\}\,z-TK_d(b+1)-1. \tag{39}
$$

The remaining of the solution depends on your student number and the value of $K_d$ you have chosen.

5. (10 TOTAL points) **LQR**. Consider the system

$$
\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & a_1+1\\ a_2+1 & a_3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ b+1\end{bmatrix}u, \tag{40}
$$

and the cost function

$$
J = \int_0^\infty \left\{(c_1+1)x_1^2(t) + 2x_1(t)x_2(t) + c_2x_2^2(t) + u^2(t)\right\}dt, \tag{41}
$$

where $a_1, a_2, a_3, b, c_1$, and $c_2$ are determined by the first six digits (from right to left) of your student number ($\mathtt{sXa_1a_2a_3bc_1c_2}$), i.e., if your student number is `s2834651`, then

$$
A = \begin{bmatrix}0 & 9\\ 4 & 4\end{bmatrix}, \quad B = \begin{bmatrix}0\\ 7\end{bmatrix}, \quad J = \int_0^\infty \left\{6x_1^2(t) + 2x_1(t)x_2(t) + x_2^2(t) + u^2(t)\right\}dt. \tag{42}
$$

Design an LQR that stabilizes the system and minimizes (41).

Note that

$$
Q = \begin{bmatrix}c_1+1 & 1\\ 1 & c_2\end{bmatrix}, \quad R = 1. \tag{43}
$$

**Note: a particular case takes place when $c_2 = 0$. However, you should be able to continue with the exercise and obtain a conclusion.**
Consider

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}. \tag{44}$$

To solve this exercise, we need to compute

$$PA = \begin{bmatrix} p_2(a_2+1) & p_1(a_2+1) + a_3p_2 \\ p_3(a_2+1) & p_2(a_2+1) + a_3p_3 \end{bmatrix}, \quad PB = \begin{bmatrix} p_2(b+1) \\ p_3(b+1) \end{bmatrix}. \tag{45}$$

Hence,

$$A^TP + PA - PBR^{-1}B^TP + Q =$$

$$\begin{bmatrix} 2p_2(a_2+1) - p_2^2(b+1)^2 + c_1 + 1 & p_1(a_2+1) + a_3p_2 + p_3(a_2+1) - p_2p_3(b+1)^2 + 1 \\ p_1(a_2+1) + a_3p_2 + p_3(a_2+1) - p_2p_3(b+1)^2 + 1 & 2p_2(a_2+1) + 2a_3p_3 - p_3^2(b+1)^2 + c_2 \end{bmatrix}. \tag{46}$$

Therefore, to obtain $P$, you need to solve three equations, namely,

$$\begin{aligned} 2p_2(a_2+1) - p_2^2(b+1)^2 + c_1 + 1 &= 0 \\ 2p_2(a_2+1) + 2a_3p_3 - p_3^2(b+1)^2 + c_2 &= 0 \\ p_1(a_2+1) + a_3p_2 + p_3(a_2+1) - p_2p_3(b+1)^2 + 1 &= 0. \end{aligned} \tag{47}$$

The control law is given by

$$u = -\begin{bmatrix} p_2(b+1) & p_3(b+1) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{48}$$

6. (10 TOTAL points) **Observer**. Consider the system

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t); \quad A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \\ e(t) &= Cx(t); \qquad\qquad C = \begin{bmatrix} c_1 & c_2 \end{bmatrix}, \end{aligned} \tag{49}$$

where the parameters $a_1, a_2, a_3, a_4, c_1$, and $c_2$ are determined by the first six digits (from right to left) of your student number $\mathtt{sXa_1a_2a_3a_4c_1c_2}$. Design an observer that guarantees that the error $\tilde{x}(t) := x(t) - \hat{x}(t)$ has a rate of convergence of 10. Justify your answer.

These kinds of problems were extensively studied during the course. Consequently, we only present a short answer.
Define

$$A_o := A - LC = \begin{bmatrix} a_1 - c_1L_1 & a_2 - c_2L1 \\ a_3 - c_1L_2 & a_4 - c_2L2 \end{bmatrix}. \tag{50}$$

To solve this problem, we need to obtain the characteristic polynomial of $A_o$ and equate it with the desired one. Thus, we compute

$$\begin{aligned} p(A_o) &= \mathtt{det}(I\lambda - A_o) \\ &= \mathtt{det}\left(\begin{bmatrix} \lambda + c_1L_1 - a_1 & c_2L_1 - a_2 \\ c_2L_2 - a_3 & \lambda + c_2L_2 - a_4 \end{bmatrix}\right) \\ &= \lambda^2 + \{c_2L_2 + c_1L_1 - (a_1 + a_4)\}\lambda + a_1a_4 - a_2a_3 + c_1L_2a_2 - c_2L_2a_1 - c_1L_1a_4 + c_2L_1a_3. \end{aligned} \tag{51}$$

On the other hand, the desired characteristic polynomial is given by

$$\lambda^2 + (p_1 + p_2)\lambda + p_1p_2 \tag{52}$$

where you need to choose $p_1 \geq 10$ and $p_2 \geq 10$. Hence, the values of $L$ are obtained by solving the set of equations

$$\begin{aligned} c_2L_2 + c_1L_1 - (a_1 + a_4) &= p_1 + p_2 \\ a_1a_4 - a_2a_3 + c_1L_2a_2 - c_2L_2a_1 - c_1L_1a_4 + c_2L_1a_3 &= p_1p_2. \end{aligned} \tag{53}$$

7. (13 TOTAL points) **Delayed systems**. Consider a linear system, whose transfer function is given by

$$G(s) = \frac{1}{a_1s^2 + (a_2+1)s + a_3}, \tag{54}$$

in closed-loop with a PD controller and a time delay $T = b_1 + \frac{b_2}{10}$. The parameters $a_1, a_2, a_3, K_p, b_1$, and $b_2$ are determined by the first six digits (from right to left) of your student number ($\mathtt{sXa_1a_2a_3k_pb_1b_2}$).

(a) (5 points) Propose a derivative gain $k_d$ such that the phase margin of $H(s) := C(s)G(s)$ ensures the stability of the closed-loop system (without delay).

(b) (5 points) Will be the delayed closed-loop system stable? Justify your answer.

(c) (3 points) Compute the critical time delay for this system.

**Note: we are following the convention that we discuss during the lectures.**
**(a)** The PD transfer function is given by

$$C(s) = K_ds + K_p. \tag{55}$$

Therefore, the open-loop transfer function is

$$H(s) = C(s)G(s) = \frac{K_ds + K_p}{a_1s^2 + (a_2+1)s + a_3}. \tag{56}$$

Replacing $s$ by $j\omega$, we get

$$H(j\omega) = \frac{K_dj\omega + K_p}{(a_3 - a_1\omega^2) + (a_2+1)j\omega} \tag{57}$$

The next step is to obtain $\omega^*$ such that the modulus of $H(j\omega)$ equals one. Thus, we compute

$$\frac{K_d^2\omega^2 + K_p^2}{(a_3 - a_1\omega^2)^2 + (a_2+1)^2\omega^2} = 1, \tag{58}$$

which leads to

$$a_1^2\omega^4 + \left[(a_2+1)^2 - 2a_1a_3 - K_d^2\right]\omega^2 + a_3^2 - K_p^2 = 0. \tag{59}$$

Hence, you need to propose a $K_d$ such that one of the solutions of the polynomial above is purely real and positive. That solution will be $\omega^*$. Then, you need to compute the phase of $H(j\omega^*)$ and the phase margin of the system. The rest of the solution to this part of the problem relies on your selection of $K_d$. An easy way to check if the phase margin of the system corresponds to a stable system is using the MATLAB function `margin` (remember that MATLAB considers a different convenction for the phase margin).

**(b)** Based on the answer to the previous item, you only need to check if

$$PM + T\omega^* < 0, \tag{60}$$

where PM denotes the phase margin of $H(j\omega)$. If yes, then the system is stable. If the inequality becomes equality, then $T$ is the critical time delay. Otherwise, the system is unstable.

**(c)** You need to compute

$$T = -\frac{PM}{\omega^*}, \tag{61}$$

where PM denotes the phase margin of $H(j\omega)$.

8. (10 TOTAL points) **Plot analysis**. Let $G(s)$ be the transfer function of a linear continuous-time stable system.

(a) (5 points) Consider the following

- If the first digit (from right to left) of your student number is $0, 1, 2$, or $3$, the Nyquist plot of $G(s)$ is depicted in Fig. 7.
- If the first digit (from right to left) of your student number is $4, 5$, or $6$, the Nyquist plot of $G(s)$ is depicted in Fig. 8.
- If the first digit (from right to left) of your student number is $7, 8$, or $9$, the Nyquist plot of $G(s)$ is depicted in Fig. 9.

Suppose that there is a delay that shifts the phase (towards $-1$) of the system in $\frac{\pi}{a}$ radians, where $a$ is the last digit (from right to left) of your student number. Is the closed-loop (delayed) system stable? Justify your answer.

(b) (5 points) Consider that the linear system $G(s)$ is interconnected with a nonlinearity $\varphi(y)$.

- If the first digit (from right to left) of your student number is 0 or 1, then the Nyquist plot of $G(s)$ is depicted in Fig. 7, and $\varphi(y) \in [-b_1/10, b_2 + 1]$, where $b_1$ is the second digit (from right to left) of your student number, and $b_2$ is the third digit, e.g., for `s5792256` $\varphi(y) \in [-0.5, 3]$.
- If the first digit (from right to left) of your student number is 2 or 3, then the Nyquist plot of $G(s)$ is depicted in Fig. 8, and $\varphi(y) \in [k_1, k_2]$, where $k_1$ is the smallest digit of your student number and $k_2$ is the biggest number, e.g., for `s5792256` $k_1 = 2$ and $k_2 = 9$.
- If the first digit (from right to left) of your student number is 4 or 5, then the Nyquist plot of $G(s)$ is depicted in Fig. 9, and $\varphi(y) \in [0, k_2]$, where $k_2$ is the biggest digit of your student number, e.g., for `s5792256` $k_2 = 9$.
- If the first digit (from right to left) of your student number is 6 or 7, then the Popov plot of $G(s)$ is depicted in Fig. 10, and $\varphi(y) \in [0, k]$, where $k$ is the second digit (from right to left) of your student number. If such a digit is zero, then consider the next one (from right to left).
- If the first digit (from right to left) of your student number is 8 or 9, then the Popov plot of $G(s)$ is depicted in Fig. 11, and $\varphi(y) \in [0, k]$, where $k$ is the second digit (from right to left) of your student number. If such a digit is zero, then consider the next one (from right to left).

What can be said about the stability of the closed-loop system? Justify your answer

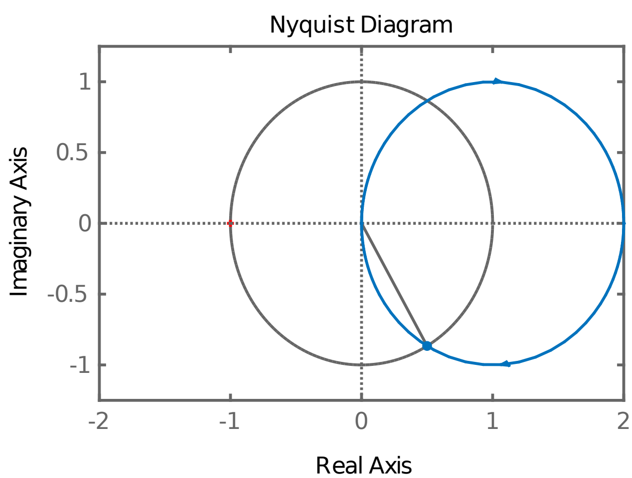


Figure 7: Nyquist plot of $G(s)$.

**(a)** Depending on your student number, we have the following conclusions

- if $a = 2$, the delayed system will be stable only for the case depicted in Fig. 7.
- for the system depicted in Fig. 8, the delayed system will be stable only if $a > 2$.
- for the system depicted in Fig. 9, the delayed system will be unstable as $a$ is 2, 3, or 4.

**(b)** The solution to this question depends on your student number.

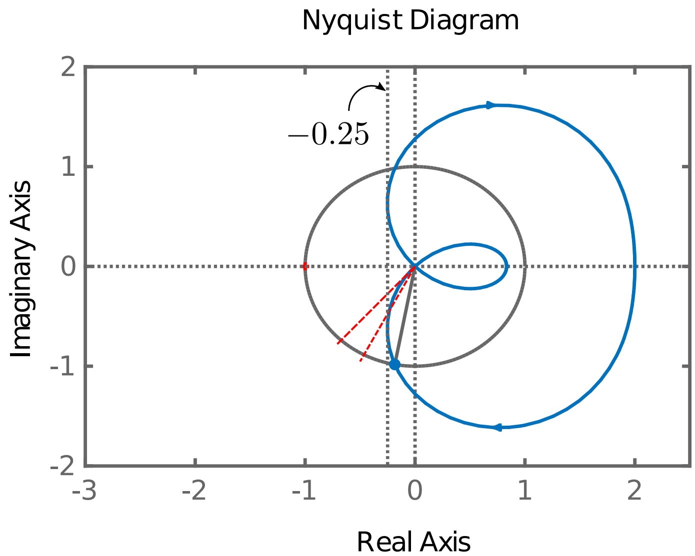


Figure 8: Nyquist plot of $G(s)$. The red lines indicate the angles $\pi/4$ and $\pi/3$, respectively.

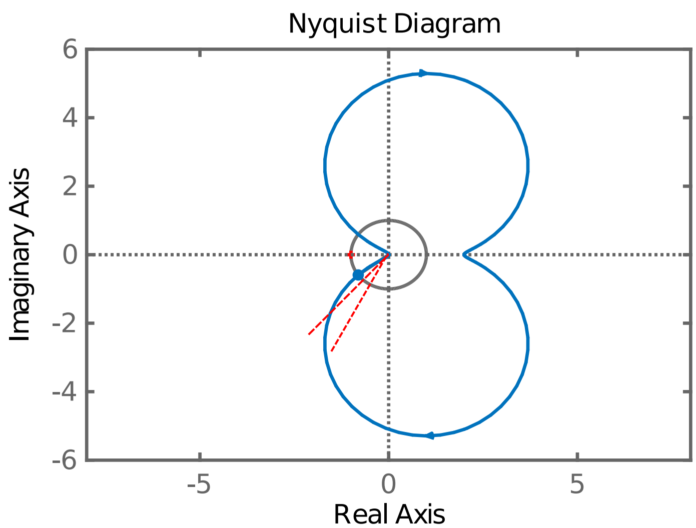


Figure 9: Nyquist plot of $G(s)$. The red lines indicate the angles $\pi/4$ and $\pi/3$, respectively.

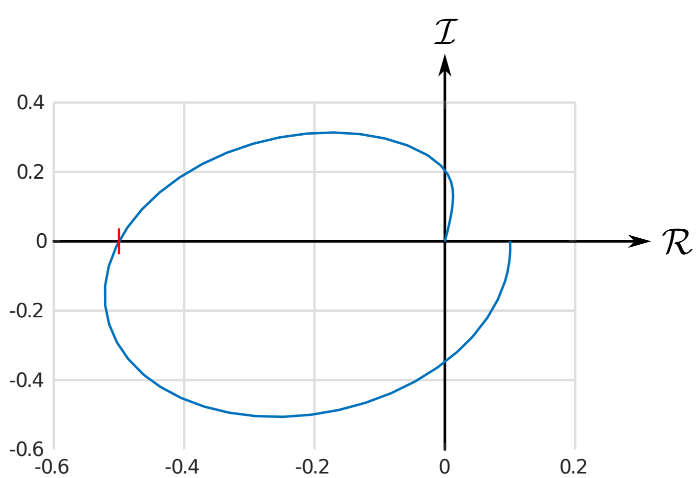


Figure 10: Popov's plot of $G(s)$. The red mark is located at $-0.5$.

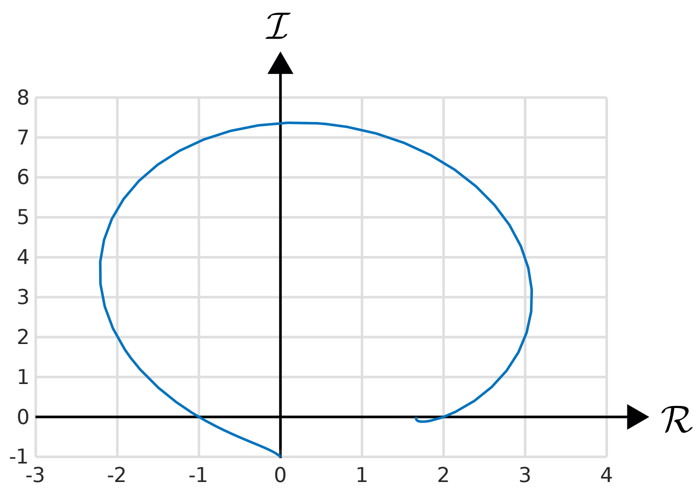


Figure 11: Popov's plot of $G(s)$.