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Answers

1. $\boxed{\text{30 points}}$ **(Affine and projective classification of quadrics in $\mathbb{R}^3$)**

Let $M^2 = \{Q = 0\}$ be a non-singular and non-empty quadric surface in $\mathbb{R}^3$. Give a list of such quadric surfaces (each representing its unique equivalence class) up to the following transformations of $\mathbb{R}^3$:

The hint directs us to Lecture 5 for the list of such quadric surfaces up to Euclidean isometries. There we find that for any tangent plane T_pM^2 tangent to quadric surface M^2 at point p, there are three possible surface classes: hyperbolic, elliptic and parabolic.

a) Invertible affine transformations of $\mathbb{R}^3$.

We can interpret this surface $M^2 \in \mathbb{R}^3$ as a graph $z = z(x, y)$. Let $p \in M^2$ be a point on M^2 and T_pM^2 be the tangent space at p. By applying an affine mapping (affine transformation) $f(X) = AX + B$ where the A is invertible, we can give a list of quadric surfaces up to invertible affine transformations of $\mathbb{R}^3$. The general implicit form for a $\mathbb{R}^3$ quadric surface can be written in homogeneous x, y, z, w coordinates as:

$$ax^2 + 2bxy + 2cxz + 2dxw + ey^2 + 2fyz + 2gyw + hz^2 + 2izw + jw^2 = 0 \tag{1}$$

Setting $w = 1$, this provides the ability to position the quadrics in space. This equation can also be written in the matrix form:

$$x^tQx = 0 \tag{2}$$

where $x = \begin{bmatrix} x, y, z, w \end{bmatrix}$ and $Q = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}$. From this we calculate the matrix forms of all the basic quadrics

$$x^2 + y^2 + z^2 - 1 = 0, Q_{sphere} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3}$$

$$ax^2 + by^2 + cz^2 - 1 = 0, Q_{ellipsoid} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{4}$$

$$x^2 + z^2 - 1 = 0, Q_{cilinder} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{5}$$

$$x^2 + z^2 - y^2 = 0, Q_{cone} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

$$x^2 + z^2 - y = 0, Q_{paraboloid} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

$$x^2 - z^2 - y = 0, Q_{hyperbolic\ paraboloid} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{2} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

$$x^2 + z^2 - y^2 - 1 = 0, Q_{hyperboloid\ of\ one\ sheet} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{9}$$

$$x^2 + z^2 - y^2 + 1 = 0, Q_{hyperboloid\ of\ two\ sheets} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \tag{10}$$

the transformed quadric surface matrix Q' represents the same type of quadric surface as the original matrix Q, and verify that it remains unique under invertible affine transformations.

b) Invertible projective transformations of $\mathbb{R}^3$ (viewed as a subset of $P_3(\mathbb{R})$). Hint: Rewrite the polynomial equation $Q = 0$ defining the quadric in homogeneous coordinates (making Q a homogeneous polynomial) and apply diagonalization to the corresponding quadratic form.

c) Show the projective transformation bringing a quadric to its equivalence class representative can always be chosen to be a composition of an affine map and a perspectivity.

2. $\boxed{\text{30 points}}$ **(Quaternions and the orthogonal group O(3))** Consider $\mathbb{R}^4$ as a Euclidean vector space with the orthonormal basis $1, i, j, k$ and the algebra structure on it defined by the following rules:

- 1 is the unit element of the algebra
- $i^2 = j^2 = k^2 = -1$;
- $ij = k, jk = i$,and $ki = j$;
- $ij = -ji, jk = -kj$,and $ki = -ik$.

The numbers (elements of the algebra) $q = a_0 + ia_1 + ja_2 + ka_3, a_i \in \mathbb{R}$, are called quaternions. If $a_0 = 0$, then the number is called a pure quaternion. Note that pure quaternions thus span a Euclidean 3-space $\mathbb{R}^3$. It may be handy to look at the similar Exercise IV.45 in the book by M. Audin.

a) Prove that for each quaternion s of unit length, the mapping $q \to sqs^{-1}$, where q is a pure quaternion, is a rotation of $\mathbb{R}^3$;

For a quaternion of equation $q = a_0 + ia_1 + ja_2 + ka_3$, the norm is $|q| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$, so the quaternion s of unit length will be

$$s = \frac{a_0 + ia_1 + ja_2 + ka_3}{\sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}} \tag{11}$$

for which $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$, so we'll refer to it just as $a_0 + ia_1 + ja_2 + ka_3$. For quaternion q, however, we can deduce that $q = ia_1 + ja_2 + ka_3$, since it's a pure quaternion.
Now let's think about a mapping $q \to sqs^{-1}$. A quaternion inverse has the form

$$s^{-1} = \frac{a_0 - ia_1 - ja_2 - ka_3}{a_0^2 + a_1^2 + a_2^2 + a_3^2} = a_0 - ia_1 - ja_2 - ka_3 \tag{12}$$

, which, because it's a unit quaternion, is the same as the conjugate of s. Now $q \to sqs^{-1}$ results in a mapping $q \to d$, for which d can be calculated as the quaternion $d = d_0 + id_1 + jd_2 + kd_3$ for which

$$d_0 = d(de - cf + bg + ah) + c(ce + df + ag - bh) + b(be + af - dg + ch) + a(ae - bf - cg - dh) \tag{13}$$
$$d_1 = c(de - cf + bg + ah) - d(ce + df + ag - bh) + a(be + af - dg + ch) - b(ae - bf - cg - dh) \tag{14}$$
$$d_2 = -b(de - cf + bg + ah) + a(ce + df + ag - bh) + d(be + af - dg + ch) - c(ae - bf - cg - dh) \tag{15}$$
$$d_3 = a(de - cf + bg + ah) + b(ce + df + ag - bh) - c(be + af - dg + ch) - d(ae - bf - cg - dh) \tag{16}$$

Now we can rewrite that as a matrix multiplication

$$\begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} d & c & b & a \\ c & -d & a & -b \\ -b & a & d & -c \\ a & b & -c & -d \end{pmatrix} \left(\begin{pmatrix} e & -f & -g & -h \\ f & e & -h & g \\ g & h & e & -f \\ h & -g & f & e \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \right) \tag{17}$$

for which our rotation matrix would be

$$R = \begin{pmatrix} d & c & b & a \\ c & -d & a & -b \\ -b & a & d & -c \\ a & b & -c & -d \end{pmatrix} \begin{pmatrix} e & -f & -g & -h \\ f & e & -h & g \\ g & h & e & -f \\ h & -g & f & e \end{pmatrix} \tag{18}$$

b) We thus get a mapping from the sphere of unit quaternions $S^3 = \{s = s_0 + is_1 + js_2 + ks_3 \in \mathbb{R}^4 \mid s_0^2 + s_1^2 + s^2 + s_3^2 = 1\}$ to the group SO(3) of orthogonal matrices with unit determinant. Show that two unit quaternions s and τ represent the same rotation if and only if $\tau = -s$;

c) Deduce that SO(3) can naturally be identified with the real projective space $P_3(\mathbb{R})$;

The group SO(3), representing rotations in $\mathbb{R}^3$ with orthogonal matrices of unit determinant, can naturally be identified with the real projective space $P_3(\mathbb{R})$. This identification arises from the fact that rotations in $\mathbb{R}^3$ can be represented by unit quaternions, forming a group denoted as S^3, the unit sphere in $\mathbb{R}^4$. The mapping from S^3 to SO(3) associates each quaternion with a rotation matrix.

Additionally, defining an equivalence relation on S^3 such that $u \equiv v$ if and only if $u = -v$, captures the antipodal nature of rotations. Consequently, the group SO(3) can be naturally identified with the real projective space $P_3(\mathbb{R})$, highlighting their inherent structural similarities and establishing a homeomorphism between them.

d) Show that the full orthogonal group O(3) can also be represented using quaternions.

3. $\boxed{\text{20 points}}$ **(The 1-d projective groups over $\mathbb{R}$ and $\mathbb{C}$)**Prove that the group of projective transformations of $P_1(\mathbb{C})$ consists of the maps

$$f(z) = \frac{az + b}{cz + d'} \tag{19}$$

, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$.

Determine its subgroup that gives the group of orientation preserving isometries of the Poincaré unit disk $\mathbb{D} = \{z \in \mathbb{C} || z| < 1\}$ with the Riemannian metric

$$G(u, v) = \frac{4(du^2 + dv^2)}{(1 - u^2 - v^2)^2}, z = u + iv$$

.

A projective transformation of $P_1(\mathbb{C})$ can be defined as quotients of invertible linear transformations of $\mathbb{C}$. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a projective transformation.

The maps f(z) (equation 19) are defined everywhere in our projective space since this can be thought of as a completion of an affine space by points at infinity. These mappings are defined for points at infinity and when $cz + d = 0$. We also know that given $p_1(\mathbb{C}) = S^2$ we can rewrite an even number of reflections inversions as

$$g = \frac{az + b}{cz + d},\ ad - bc \neq 0$$

which is alike the group of maps given in equation 19.

As the determinant of the transformation $ad - bc$ is equal to one we know this projective transformation will preserve the lines. The subgroup will be orthogonal and made up of matrices of transformation such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since the reflection of the lines would preserve the angles, we can say that the group of orientation would preserve the isometries