university of groningen

faculty of science and engineering

Mechatronics

Week 5 Day 1

Previous week

We studied

- sampling and discretization
- transformations from s-plane to z-plane
- obtaining transfer function of discrete-time system represented by a difference equation
- stability of discrete-time systems
- digital control systems

Today's lecture:

Optimal Controller Design

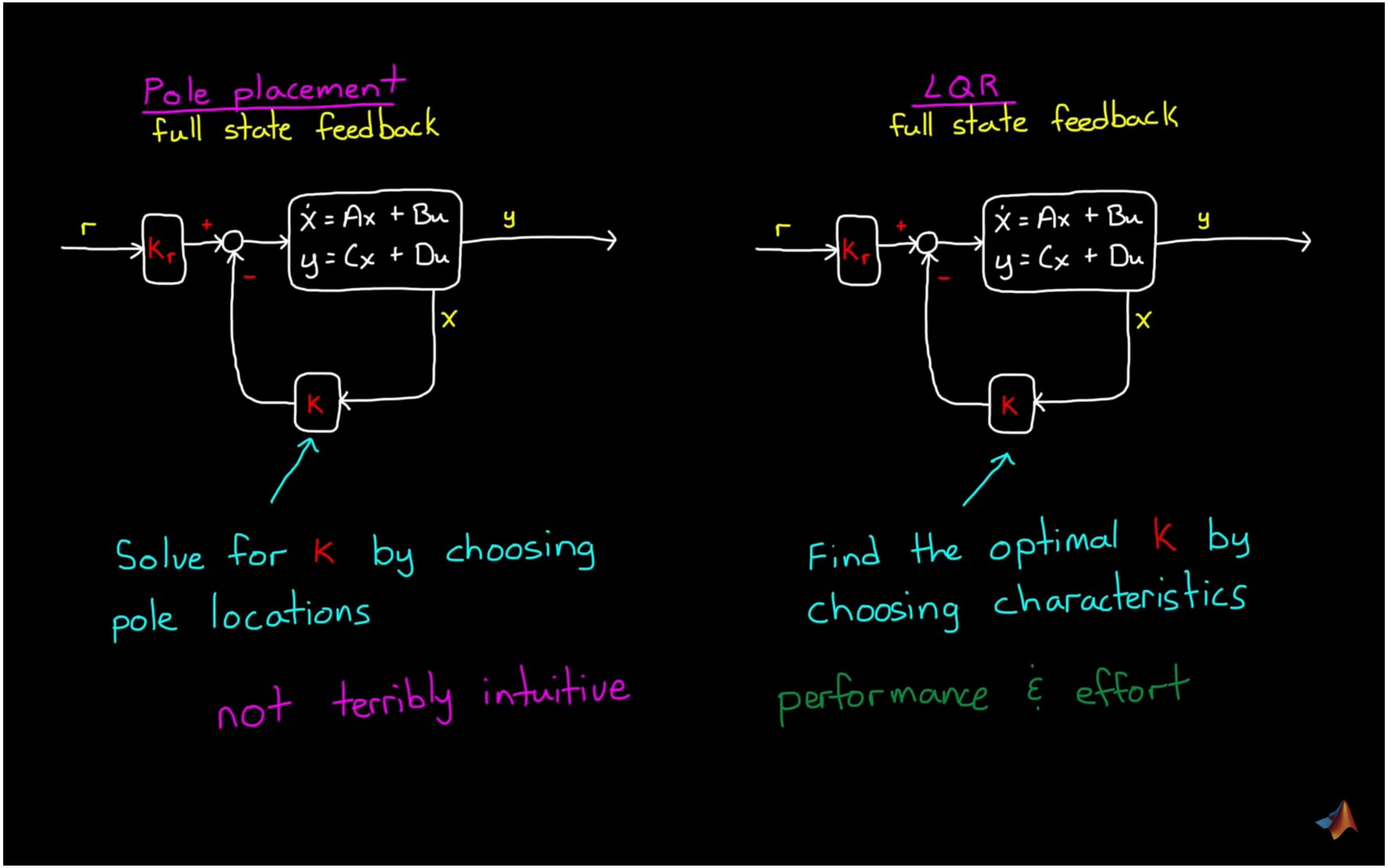
The Linear Quadratic Regulator (LQR)

Learning objectives

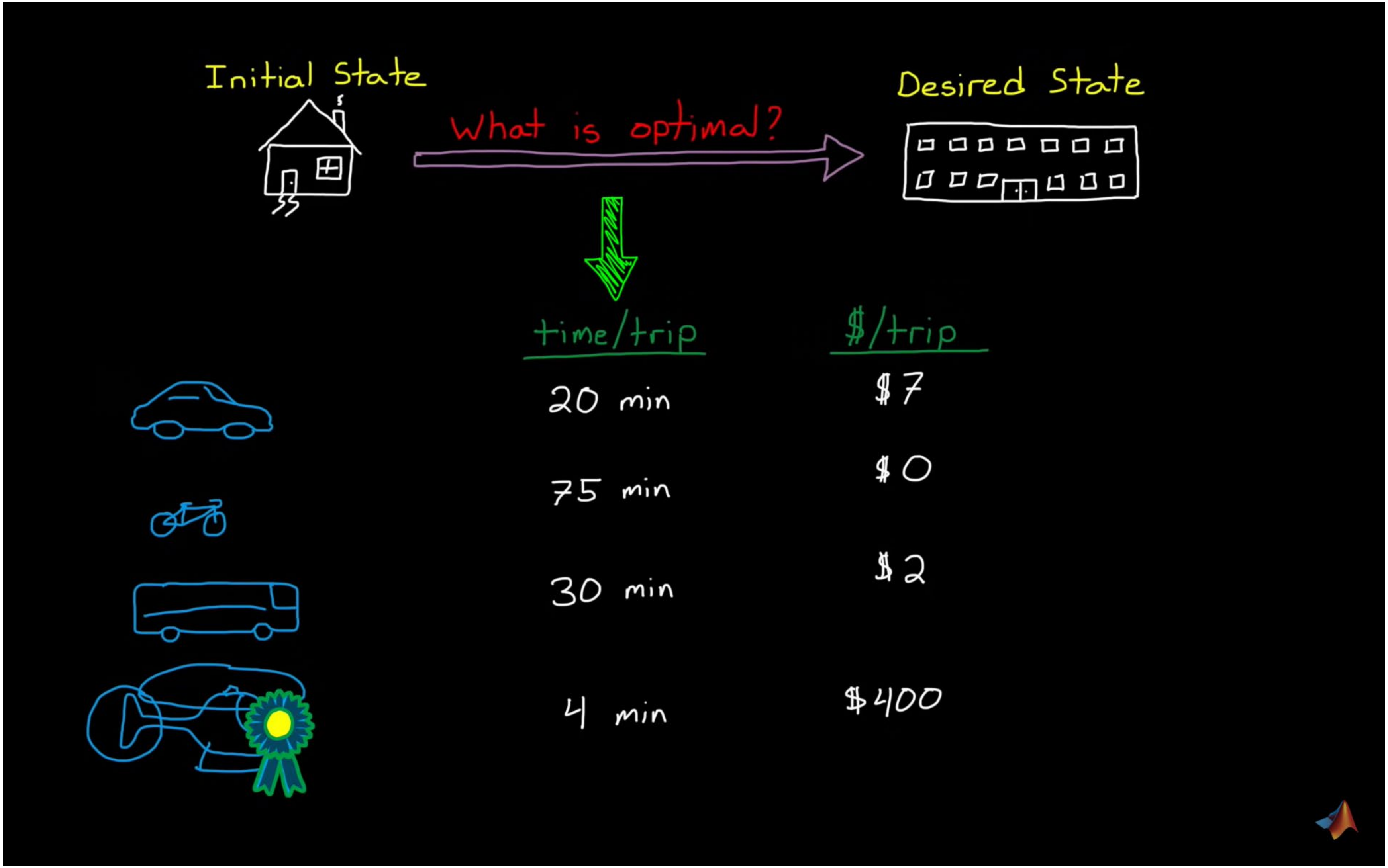
After today's lecture, you will be able to

- Design an optimal controller via the linear quadratic regulator method (LQR)

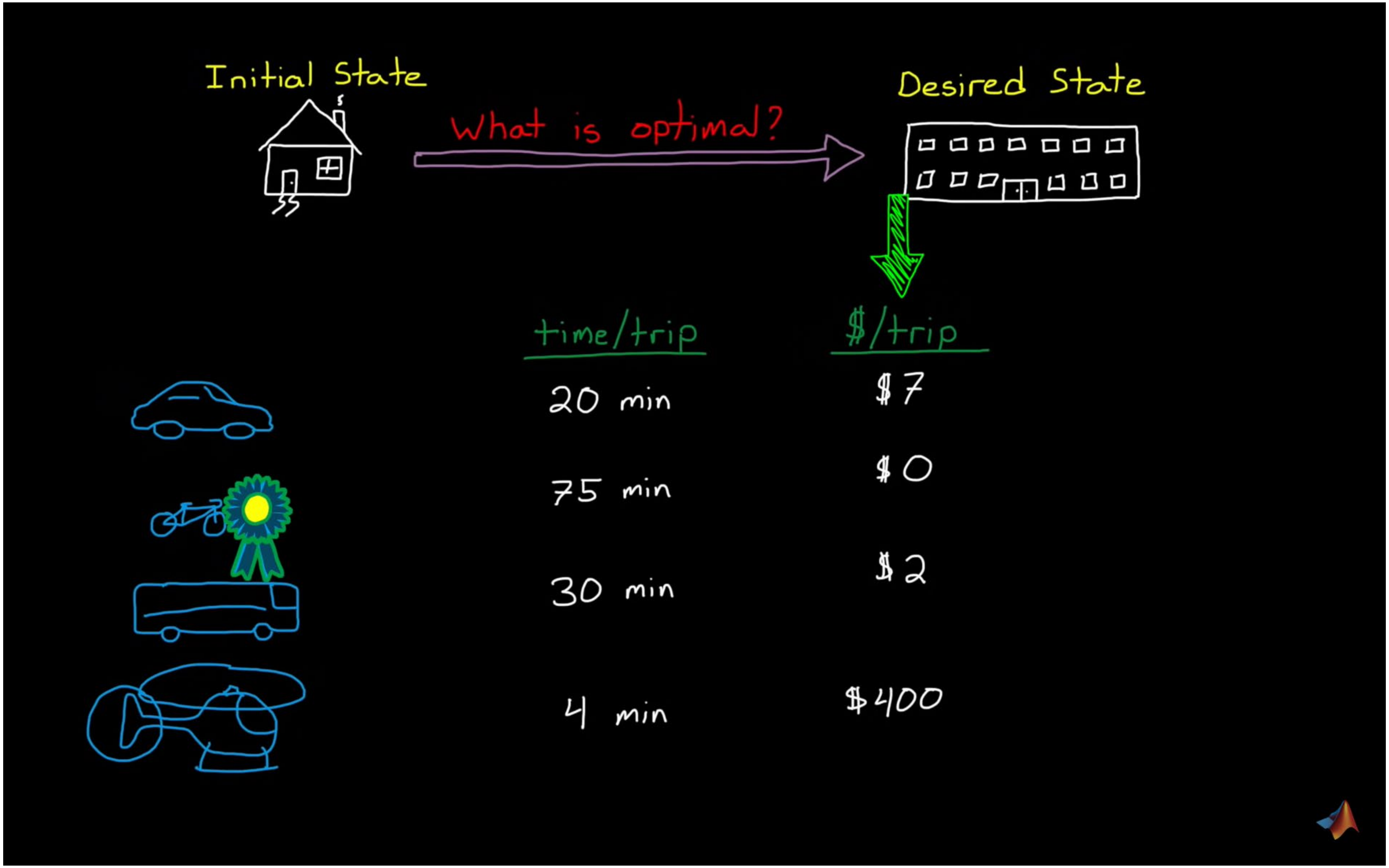
Pole Placement vs LQR Method



LQR Method


Initial State
What is optimal?
Desired State
time/trip
$/trip
20 min
$7
75 min
$0
30 min
$2
4 min
$400

LQR Method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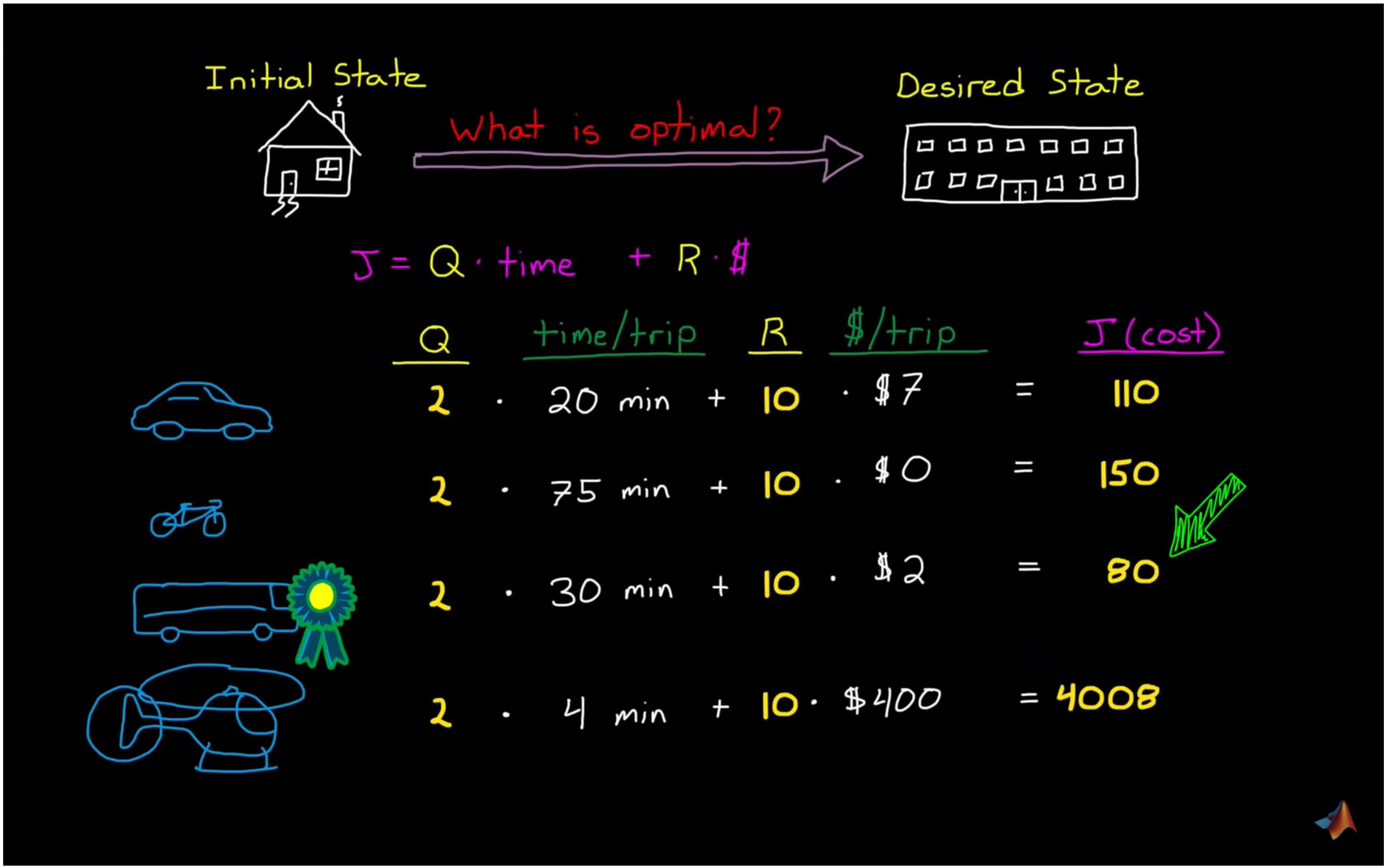
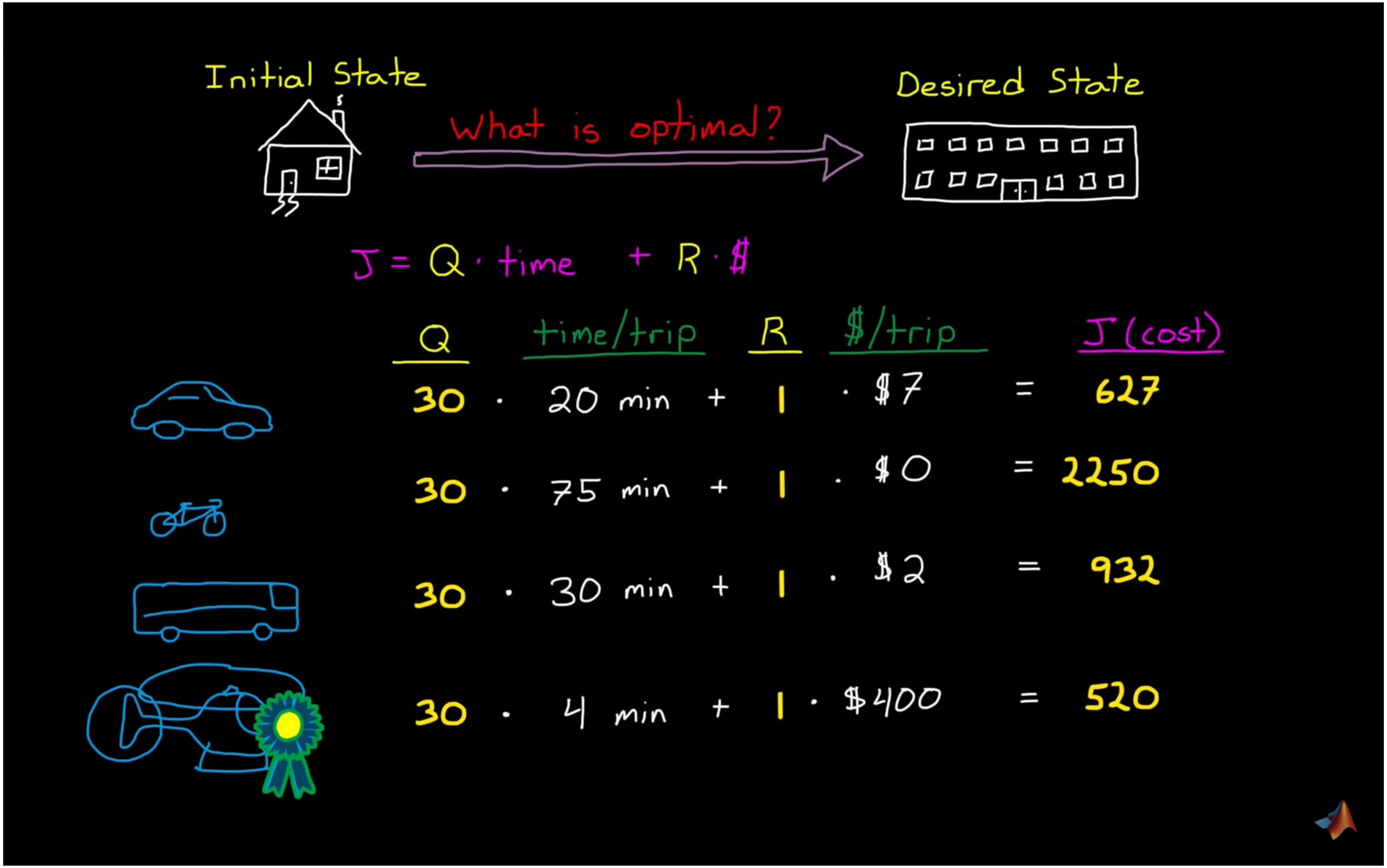
LQR Method


Initial State
Desired State
What is optimal?
J = Q · time + R · $
Q
time/trip
R
$/trip
J (cost)
2 · 20 min + 10 · $7 = 110
2 · 75 min + 10 · $0 = 150
2 · 30 min + 10 · $2 = 80
2 · 4 min + 10 · $400 = 4008

LQR Method


Initial State
What is optimal?
Desired State
J = Q · time + R · $
Q
time/trip
R
$/trip
J (cost)
30 · 20 min + 1 · $7 = 627
30 · 75 min + 1 · $0 = 2250
30 · 30 min + 1 · $2 = 932
30 · 4 min + 1 · $400 = 520

LQR Method

For a given state space equation $\begin{cases} \dot{x} = Ax + B\mathrm{u} \\ y = Cx + Du \end{cases}$

LQR Method

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$$u = -Kx$$

LQR Method

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$$u = -Kx,$$

where the gain K is to be designed such that

- the closed loop system $\dot{x} = (A - BK)\mathrm{x}$ is asymptotically stable

LQR Method

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The LQR problem aims to assign a state feedback

$$u = -Kx,$$

where the gain K is to be designed such that

- the closed loop system $\dot{x} = (A - BK)\mathrm{x}$ is asymptotically stable
- it optimizes the cost function

$$\mathrm{J}\big(\mathrm{x}(0)\big) = \begin{matrix} min \\ u \end{matrix} \int_0^\infty x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau)d\tau,$$

where Q and R are symmetric positive definite, i.e., $Q = Q^T > 0$ and $R = R^T > 0.$

LQR Method

Remember the system $\begin{cases} \dot{x} = Ax + B\mathrm{u} \\ y = Cx + Du \end{cases}$ with controller

$$u = -Kx$$

and cost function

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LQR Method

Remember the system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ with controller

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where $Q = Q^T > 0$ and $R = R^T > 0$.

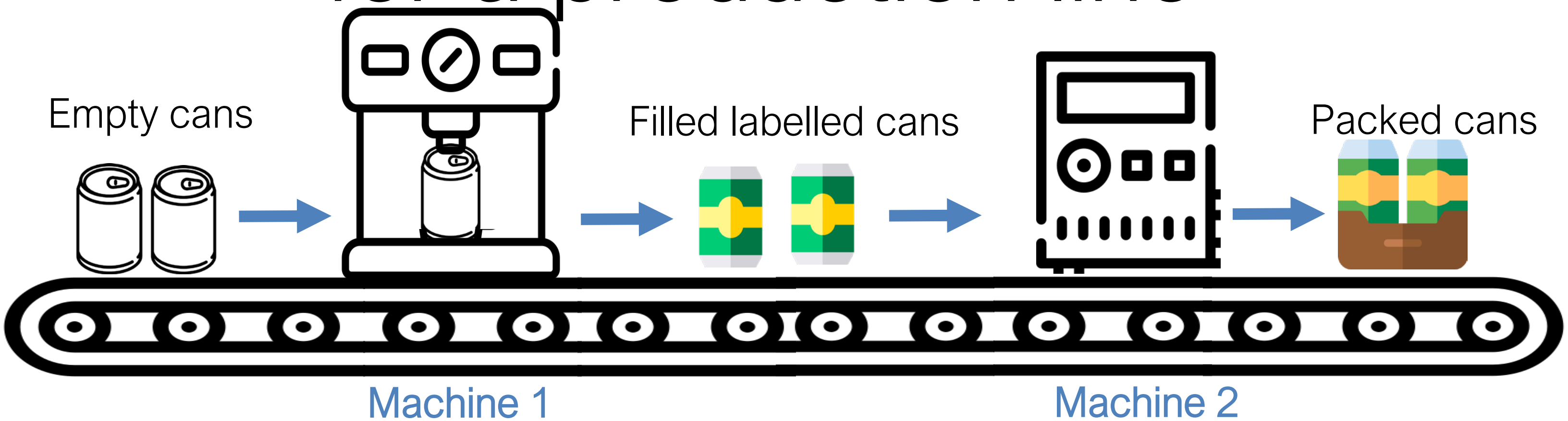
The solution to the LQR problem is given by

$$u = \underbrace{-R^{-1}B^T P}_{K} x,$$

where $\boldsymbol{P}$ is a symmetric positive definite matrix $(P = P^T > 0)$ that is solution to the algebraic Riccati equation (ARE)

$$A^T P + PA - PBR^{-1}B^T P + Q = 0.$$

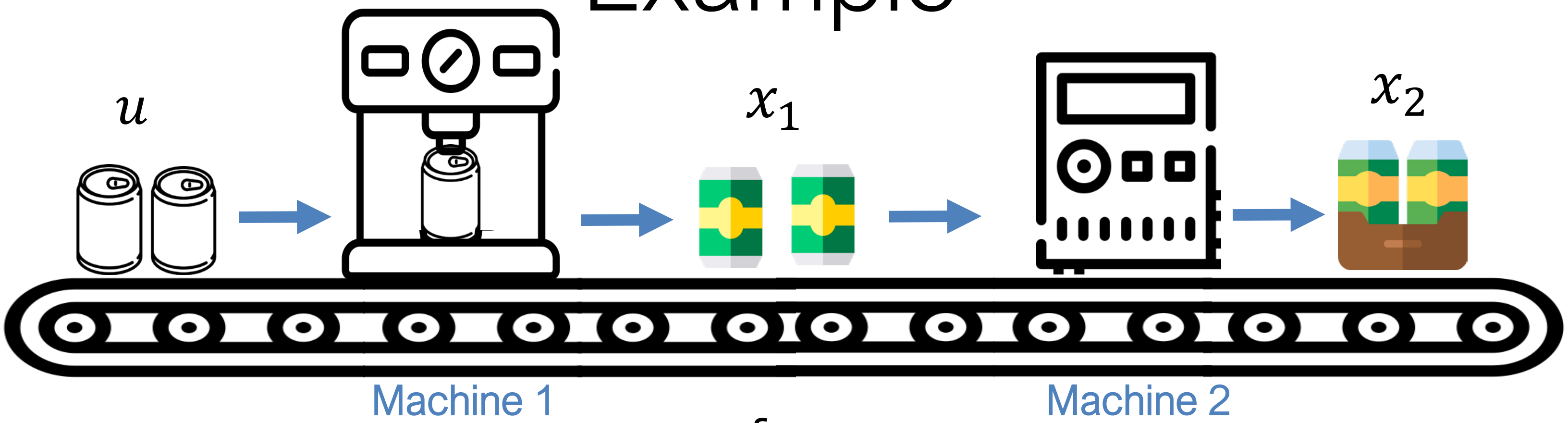
Example: Optimal controller design for a production line



Consider the production line for cans, where

- $\boldsymbol{u}$ is the number of raw material (empty unlabelled cans) ,
- $\boldsymbol{x_1}$ is the number of intermediate product (filled labelled cans) ,
- $\boldsymbol{x_2}$ is the number of final product (can packs) ,
- $\boldsymbol{r}$ is the demand (reference signal)

Example

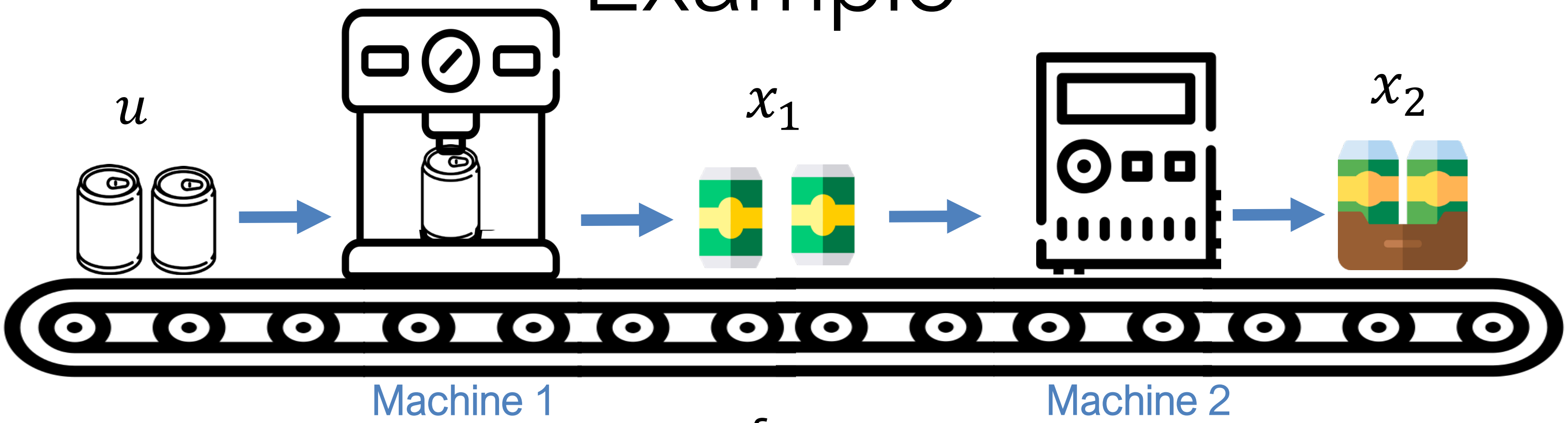


r reference

For maintaining occupancy of machine 2, the rate of production of machine 1 $\dot{x}_1$ has to be regulated by u

$$\dot{x}_1 = -10x_1 - 11x_2 + u$$

Example



r reference

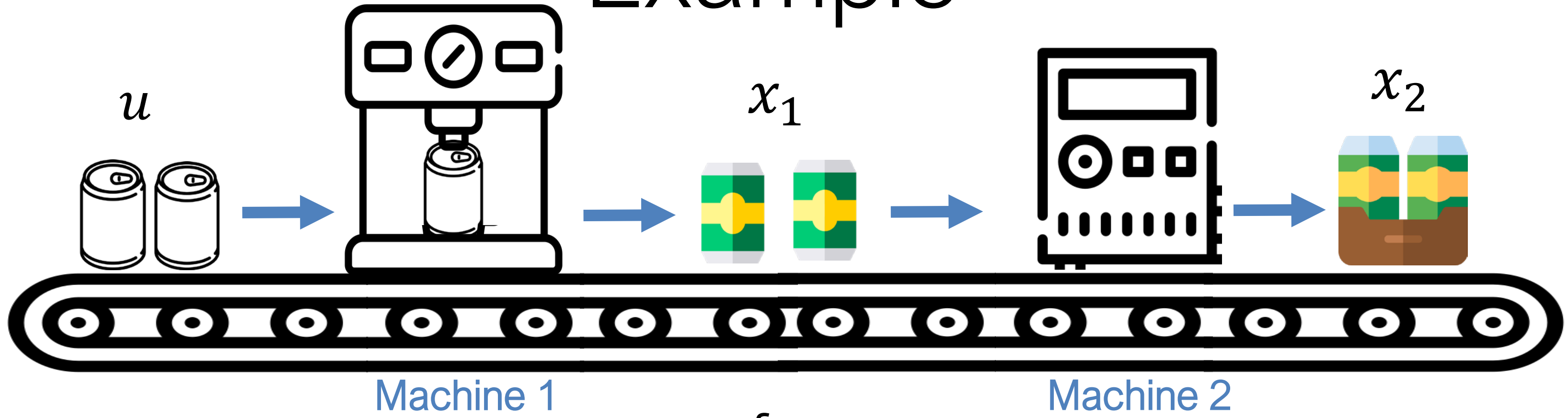
For maintaining occupancy of machine 2, the rate of production of machine 1 $\dot{x}_1$ has to be regulated by u

$$\dot{x}_1 = -10x_1 - 11x_2 + u$$

The rate of production of machine 2 $\dot{x}_2$ is proportional to the intermediate product

$$\dot{x}_2 = x_1$$

Example



r reference

Then, with equations $\dot{x}_1 = -10x_1 - 11x_2 + u$ and $\dot{x}_2 = x_1$ we can build a state-space representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -10 & -11 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u$$

$$y(t) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad D = 0,$$

considering the number of final products x_2 as output.

Example

For the given state-space

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -10 & -11 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u$$

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LQR Problem: Find a regulator u in order to control the production such that it meets the demand r in an optimal manner, i.e,

1. Optimise cost function $\mathrm{J}(\mathrm{x}(0)) = \int_0^\infty x^T(\tau)\, Qx(\tau) + u^T(\tau)\, Ru(\tau) d\tau$,

where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\mathrm{R} = 5$ (u is scalar)

2. Make sure that the system is asymptotically stable

Example

Step 1. Optimise cost function

Cost function given by $J(x(0)) = \int_0^\infty x^T(\tau)\, Qx(\tau) + u^T(\tau)\, Ru(\tau) d\tau$,

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where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $R = 5$ (u is scalar)

Then the cost function is

$$\boxed{J = \int_0^\infty {x_1}^2 + 2{x_2}^2 + 5u^2 d\tau}$$

Example

Step 1. Optimise cost function

Cost function given by $J(x(0)) = \int_0^{\infty} x^T(\tau)\,Qx(\tau) + u^T(\tau)\,Ru(\tau)d\tau$,

where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $R = 5$ (u is scalar)

Then the cost function is

$$\boxed{J = \int_0^{\infty} \underbrace{x_1{}^2}_{\text{Discrepancy in production in machine 1}} + \underbrace{2x_2{}^2}_{\text{Discrepancy in production in machine 2}} + \underbrace{5u^2}_{\text{Discrepancy in supply}} d\tau}$$

Intuitive meaning: discrepancy in machine 2 is costlier than machine 1. Discrepancy in supply is even costlier than the machines

Example

Step 1. Optimise cost function

$$J = \int_0^\infty {x_1}^2 + 2{x_2}^2 + 5u^2 d\tau$$

How to optimise?

There exists a $u = -Kx$ that will optimise the cost funtion.

Example

Step 1. Optimise cost function

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How to optimise?

There exists a $u = -Kx$ that will optimise the cost funtion.

According to LQR theory

- We first need to solve the following Ricatti equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

for a matrix P which is symmetric positive definite ($P = P^T > 0$).

Note that A, B are known from state space; and Q and R are known from the cost function.

Example

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  for a matrix P which is symmetric positive definite ($P = P^T > 0$).

  Note that A, B are known from state space; and Q and R are known from the cost function.

- After solving the Ricatti equation for P we can find $K = R^{-1}B^T P$, so that $u = -Kx$ will optimise the cost function

Example

Step 1. Optimise cost function. **Solving Ricatti equation**

Note: In Matlab, you can use command **lqr** to directly solve the Ricatti equation for P.

Example

Step 1. Optimise cost function. **Solving Ricatti equation Manually:**

Choose $\mathrm{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$, symmetric $P_{12} = P_{21}$

Example

Step 1. Optimise cost function. Solving Ricatti equation Manually:

Choose P $= \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$, symmetric $P_{12} = P_{21}$

Then $A^T P + PA - PBR^{-1}B^T P + Q = 0$ becomes

$$\underbrace{\begin{bmatrix} -10 & 1 \\ -11 & 0 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} -10 & -11 \\ 1 & 0 \end{bmatrix}}_{A}$$

$$- \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}}_{P} \underbrace{\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} \underbrace{\frac{1}{5}}_{R^{-1}} \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{B^T}}_{\frac{1}{5}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{Q} = 0$$

Example

Step 1. Optimise cost function. Solving Ricatti equation

Performing matrix multiplications:

$$\begin{bmatrix} -10P_{11}+P_{12} & -10P_{12}+P_{22} \\ -11P_{11} & -11P_{12} \end{bmatrix} + \begin{bmatrix} -10P_{11}+P_{12} & -11P_{11} \\ -10P_{12}+P_{22} & -11P_{12} \end{bmatrix}$$

$$-\frac{1}{5}\begin{bmatrix} {P_{11}}^2 & P_{11}P_{12} \\ P_{11}P_{12} & {P_{12}}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example

Step 1. Optimise cost function. Solving Ricatti equation

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which gives the system of equations

$$\begin{cases} -20P_{11} + 2P_{12} - \frac{1}{5}{P_{11}}^2 + 1 = 0 \\ -10P_{12} + P_{22} - 11P_{11} - \frac{1}{5}P_{11}P_{12} = 0 \\ -22P_{12} - \frac{1}{5}{P_{12}}^2 + 2 = 0 \end{cases}$$

Example

Step 1. Optimise cost function. Solving Ricatti equation

The system of equations is solved getting expressions for P_{11}, P_{12}, P_{22}

$$\begin{cases} -\frac{1}{5}{P_{11}}^2 - 20P_{11} + 2P_{12} + 1 = 0 & (1) \\ -\frac{1}{5}P_{11}P_{12} - 10P_{12} + P_{22} - 11P_{11} = 0 & (2) \\ -\frac{1}{5}{P_{12}}^2 - 22P_{12} + 2 = 0 & (3) \end{cases}$$

Example

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From (3), $P_{12} = -\frac{22}{2/5} \pm \frac{\sqrt{22^2+8/5}}{2/5}$

Example

Step 1. Optimise cost function. Solving Ricatti equation

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From (3), $P_{12} = -\frac{22}{2/5} \pm \frac{\sqrt{22^2+8/5}}{2/5}$

From (1), $P_{11} = -\frac{20}{2/5} \pm \frac{\sqrt{20^2+4/5(2P_{12}+1)}}{2/5}$

Example

Step 1. Optimise cost function. **Solving Ricatti equation**

The system of equations is solved getting expressions for P_{11}, P_{12}, P_{22}

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From (1), $P_{11} = -\frac{20}{2/5} \pm \frac{\sqrt{20^2+4/5(2P_{12}+1)}}{2/5}$

From (2), $P_{22} = \frac{1}{5}P_{11}P_{12} + 10P_{12} + 11P_{11}$

Example

Step 1. Optimise cost function. **Solving Ricatti equation**

There are multiple solution combinations for the system of equations

We need to look for the solution (**trial and error**) that:

- Makes all entries $\boldsymbol{P}_{11}, \boldsymbol{P}_{12}, \boldsymbol{P}_{22}$ real
- Makes $\boldsymbol{P}$ positive definite
  - $P_{11} > 0$
  - $P_{22} > 0$

Example

Step 1. Optimise cost function. **Solving Ricatti equation**

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- Makes **$\boldsymbol{P}$ positive definite**
  - $P_{11} > 0$
  - $P_{22} > 0$

That solution for the Ricatti equation is

$$\boxed{P_{12} \approx 0.0908;\ P_{11} \approx 0.0590; P_{22} \approx 1.5581}$$

Example

Step 1. Optimise cost function. Building optimal controller

With solution for the Ricatti equation:

$$P = \begin{bmatrix} 0.0590 & 0.0908 \\ 0.0908 & 1.5581 \end{bmatrix}$$

Example

Step 1. Optimise cost function. **Building optimal controller**

With solution for the Ricatti equation

$$P = \begin{bmatrix} 0.0590 & 0.0908 \\ 0.0908 & 1.5581 \end{bmatrix},$$

we can build $u = -Kx$ that optimises J, using gain $K = R^{-1}B^TP$

$$u = -\underbrace{\underbrace{\tfrac{1}{5}}_{R^{-1}} \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{B^T} \underbrace{\begin{bmatrix} 0.0590 & 0.0908 \\ 0.0908 & 1.5581 \end{bmatrix}}_{P}}_{K} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

Example

Step 1. Optimise cost function. **Building optimal controller**
With solution for the Ricatti equation:

$$P = \begin{bmatrix} 0.0590 & 0.0908 \\ 0.0908 & 1.5581 \end{bmatrix},$$

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$$\boxed{u = -0.012x_1 - 0.02x_2 \text{ optimises } J}$$

Example

Step 2. Make sure that the system is asymptotically stable

$$\dot{x} = Ax + Bu \text{ with } u = -K\tilde{x} \text{ yields}$$
$$\dot{x} = (A - BK)x$$

So $(A - BK)$ should be Hurwitz or asymptotically stable

Example

Step 2. Make sure that the system is asymptotically stable

$\dot{x} = Ax + Bu$ with $u = -K\tilde{x}$ yields

$$\dot{x} = (A - BK)x$$

So $(\mathrm{A} - \mathrm{B}K)$ should be Hurwitz or asymptotically stable

Lets compute $(\mathrm{A} - \mathrm{BK})$

$$\boxed{(\mathrm{A} - \mathrm{B}K) = \begin{bmatrix} -10 & -11 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [0.012 \quad 0.02] = \begin{bmatrix} -10.012 & -11.020 \\ 1 & 0 \end{bmatrix}}$$

Example

Step 2. Make sure that the system is asymptotically stable

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Now to obtain eigenvalues we get the characteristic equation

$$\boxed{|\lambda I - (\mathrm{A} - \mathrm{B}K)| = \begin{vmatrix} \lambda + 10.012 & 11.02 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 10.012\lambda + 11.02}$$

Example

Step 2. Make sure that the system is asymptotically stable

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Now to obtain eigenvalues we get the characteristic equation

$$|\lambda I - (\mathrm{A} - \mathrm{B}K)| = \begin{vmatrix} \lambda + 10.012 & 11.02 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 10.012\lambda + 11.02$$

Solving for λ

$$\boxed{\lambda_{1,2} = -\frac{10.012}{2} \pm \frac{1}{2}\sqrt{10.012^2 - 4(11.020)}}$$

Since $\lambda_{1,2} < 0$ the system is asymptotically stable

Proof

How do we know solution to Riccatti equation optimises cost function?

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How do we know solution to Riccati equation optimises cost function?

Step 1. Consider the cost function

$$J(x(t)) = \frac{1}{2}\int_t^{\infty} x^T(\tau)Qx(\tau)+ u^T(\tau)\,Ru(\tau)d\tau$$

It will be optimised by the solution where the derivative is minimised ($\frac{d}{dt}J(x(t)) = 0$)

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How do we know solution to Riccatti equation optimises cost function?

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$$\mathrm{J}(\mathrm{x}(\mathrm{t})) = \frac{1}{2}\int_t^{\infty} x^T(\tau)Qx(\tau) + u^T(\tau)\, Ru(\tau)d\tau$$

It will be optimised at the solution where the derivative is minimised ($\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{J}(\mathrm{x}(\mathrm{t})) = 0$)

To find that solution:

Step 2. Perform the derivative

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{J}(\mathrm{x}(\mathrm{t})) = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}\int_t^{\infty} x^T(\tau)Qx(\tau) + u^T(\tau)\, Ru(\tau)d\tau\right)$$

Proof

How do we know solution to Riccati equation optimises cost function?

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Step 2. Perform the derivative

$$\frac{d}{dt}J(x(t)) = \frac{d}{dt}\left(\frac{1}{2}\int_t^{\infty} x^T(\tau)Qx(\tau) + u^T(\tau)\, Ru(\tau)d\tau\right)$$

Step 3. By using the fundamental theorem $\int_a^b f(\tau)d\tau = -\int_b^a f(\tau)d\tau$

$$\frac{d}{dt}J\big(x(t)\big) = \frac{d}{dt}\left(-\frac{1}{2}\int_{\infty}^{t} x^T(\tau)Qx(\tau) + u^T(\tau)\, Ru(\tau)d\tau\right)$$

Proof

How do we know solution to Riccatti equation optimises cost function?

Step 1. Consider the cost function

$$J(x(t)) = \frac{1}{2}\int_t^{\infty} x^T(\tau)Qx(\tau) + u^T(\tau)\, Ru(\tau)d\tau$$

It will be optimised at the solution where the derivative is minimised ($\frac{d}{dt}J(x(t)) = 0$)

To find that solution:

Step 2. Perform the derivative

$$\frac{d}{dt}J(x(t)) = \frac{d}{dt}(\frac{1}{2}\int_t^{\infty} x^T(\tau)Qx(\tau) + u^T(\tau)\, Ru(\tau)d\tau)$$

Step 3. By using the fundamental theorem $\int_a^b f(\tau)d\tau = -\int_b^a f(\tau)d\tau$

$$\frac{d}{dt}J\big(x(t)\big) = \frac{d}{dt}(-\frac{1}{2}\int_{\infty}^{t} x^T(\tau)Qx(\tau) + u^T(\tau)\, Ru(\tau)d\tau)$$

Step 4. By using the fundamental theorem $\frac{d}{dt}\int_a^t f(\tau)d\tau = f(t)$

$$\frac{d}{dt}J\big(x(t)\big) = -\frac{1}{2}x^T(t)Qx(t) - \frac{1}{2}\, u^T(t)\, Ru(t)$$

Proof

Step 5. Since, by chain rule

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{J}\big(\mathrm{x(t)}\big) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{J}\big(\mathrm{x(t)}\big)\right)^T \dot{x} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{J}\big(\mathrm{x(t)}\big)\right)^T (Ax + Bu)$$

we can write:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{J}\big(\mathrm{x(t)}\big)\right)^T (Ax + Bu) = -\frac{1}{2}x^T(t)Qx(t) - \frac{1}{2}\,u^T(t)\,Ru(t)$$

Proof

Step 5. Since, by chain rule

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{J}(\mathrm{x}(\mathrm{t})) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{J}(\mathrm{x}(\mathrm{t}))\right)^T \dot{x} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{J}(\mathrm{x}(\mathrm{t}))\right)^T (Ax + Bu)$$

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Step 6. Let us take a guess solution for a linear system

$$J(x) = \frac{1}{2}x^T Px, \text{ where } P = P^T > 0 \Rightarrow \frac{dJ}{dx}(x) = Px$$

Then we can write the equation as

$$x^T P(Ax + Bu) + \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru = 0$$

or

$$x^T PAx + \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru + x^T PBu = 0$$

Proof

Step 5. Since, by chain rule

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{J}(\mathrm{x}(\mathrm{t})) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{J}(\mathrm{x}(\mathrm{t}))\right)^T \dot{x} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{J}(\mathrm{x}(\mathrm{t}))\right)^T (Ax + Bu)$$

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$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{J}(\mathrm{x}(\mathrm{t}))\right)^T (Ax + Bu) = -\frac{1}{2}x^T(t)Qx(t) - \frac{1}{2}\, u^T(t)\, Ru(t)$$

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or

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Step 7. Using the completion of squares

$$x^T PAx + \frac{1}{2}x^T Qx + \frac{1}{2}\left(u^T R^{\frac{1}{2}} + x^T PBR^{-\frac{1}{2}}\right)\left(R^{\frac{1}{2}}u + R^{-\frac{1}{2}}B^T Px\right) - \frac{1}{2}x^T PBR^{-1}B^T Px = 0$$

Proof

Step 8. For the equation

$$x^T PAx + \frac{1}{2}x^T Qx + \frac{1}{2}\left(u^T R^{\frac{1}{2}} + x^T PBR^{-\frac{1}{2}}\right)\left(R^{\frac{1}{2}}u + R^{-\frac{1}{2}}B^T Px\right) - \frac{1}{2}x^T PBR^{-1}B^T Px = 0$$

we need to find an input $\boldsymbol{u}$ that will allow us to minimise it

Proof

Step 8. For the equation

$$x^T PAx + \frac{1}{2}x^T Qx + \frac{1}{2}\left(u^T R^{\frac{1}{2}} + x^T PBR^{-\frac{1}{2}}\right)\left(R^{\frac{1}{2}}u + R^{-\frac{1}{2}}B^T Px\right) - \frac{1}{2}x^T PBR^{-1}B^T Px = 0$$

we need to find an input $\boldsymbol{u}$ that will allow us to minimise it

We can make the parts depending on u equal to $\mathbf{0}$ by taking

$$R^{\frac{1}{2}}u + R^{-\frac{1}{2}}B^T Px = 0\,,$$

which results in the input

$$\boxed{u = -R^{-1}B^T Px}$$

Proof

Step 8. For the equation

$$x^T P A x + \frac{1}{2} x^T Q x + \frac{1}{2}\left(u^T R^{\frac{1}{2}} + x^T P B R^{-\frac{1}{2}}\right)\left(R^{\frac{1}{2}} u + R^{-\frac{1}{2}} B^T P x\right) - \frac{1}{2} x^T P B R^{-1} B^T P x = 0$$

we need to find an input $\boldsymbol{u}$ that will allow us to minimise it

We can make the parts depending on u equal to $\mathbf{0}$ by taking

$$R^{\frac{1}{2}} u + R^{-\frac{1}{2}} B^T P x = 0\,,$$

which results in the input

$$\boxed{u = -R^{-1} B^T P x}$$

Step 9. Using the input leaves us with an equation only depending on x

$$x^T P A x + \frac{1}{2} x^T Q x - \frac{1}{2} x^T P B R^{-1} B^T P x = 0$$

for which the value of solution P that minimises the equation will optimise the cost function

Proof

Step 9. Differentiating twice w.r.t x, we have

$$PA + A^T P + \underbrace{\frac{1}{2} Q + \frac{1}{2} Q^T}_{Q} \underbrace{- \frac{1}{2} PBR^{-1}B^T P - \frac{1}{2} PBR^{-1}B^T P}_{-PBR^{-1}B^T P} = 0$$

Proof

Step 9. Differentiating twice w.r.t x, we have

$$PA + A^T P + \underbrace{\frac{1}{2}Q + \frac{1}{2}Q^T}_{Q} \underbrace{- \frac{1}{2}PBR^{-1}B^T P - \frac{1}{2}PBR^{-1}B^T P}_{-PBR^{-1}B^T P} = 0$$

Which results in the Ricatti equation:

$$\boxed{\begin{array}{l} A^T P + PA - PBR^{-1}B^T P + Q = 0 \\ \text{where } P = P^T > 0 \\ \text{and } A - BK = A - BR^{-1}B^T P \text{ is asymptotically stable} \end{array}}$$

Demonstrating we can solve Ricatti equation to get P so that $u = -R^{-1}B^T Px$ is optimal

Summary

- An optimal controller can be designed with form

$$u = -Kx$$

- The controller has to be defined such that

$\dot{x} = (A - BK)\mathrm{x}$ is asymptotically stable

- The cost function below is optimized

$$\mathrm{J}\big(\mathrm{x}(0)\big) = \min_{u} \int_0^\infty x^T(\tau)Qx(\tau) + u^T(\tau)\,Ru(\tau)d\tau,$$

where Q and R are positive definite, i.e., $Q = Q^T > 0$ and $R = R^T > 0$

- The solution to the problem is $K = R^{-1}B^TP$ with P solution to the Ricatti equation.

Next lecture (Preview)

Optimal state controller $u = -Kx$ is ideal

PROBLEM: it requires measuring every state

What if we cannot measure every state?

SOLUTION: reconstruct state from input and output

HOW?

Designing an **Observer**