Control Engineering
Lecture 6
ver. 1.5.2.2

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Previous Lecture

- Solutions to linear state space equations
- Diagonal forms and diagonalization
- Input/output response
  - Impulse, step response

Today

- Harmonic response
- Reachability
- Reachable canonical form
- An output regulation problem

Next lecture

- Solution to the output regulation problem
- Eigenvalue assignment
- Second order systems

Frequency or harmonic response

Harmonic response response to a sinusoidal input $u(t) = \cos\omega t$.
To compute the harmonic response, express $\cos\omega t$ via the Euler identity

$$\cos\omega t = \frac{1}{2}(\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\omega t}) = \frac{1}{2}(\cos(\omega t) + i\sin(\omega t) + \cos(\omega t) - i\sin(\omega t))$$

Let $s = i\omega$ and compute the output response for $u(t) = \mathrm{e}^{st}$

$$\begin{aligned} y(t) &= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Be^{s\tau}d\tau + De^{st} \\ &= Ce^{At}x(0) + Ce^{At}\int_0^t e^{(sI-A)\tau}Bd\tau + De^{st} \end{aligned}$$

<u>Hint</u> $\mathrm{e}^{A(t-\tau)} = \mathrm{e}^{At}\mathrm{e}^{-A\tau}$; $\mathrm{e}^{-A\tau}\mathrm{e}^{s\tau} = \mathrm{e}^{-A\tau}I\mathrm{e}^{s\tau} = \mathrm{e}^{-A\tau}\mathrm{e}^{Is\tau} = \mathrm{e}^{(sI-A)\tau}$
If s is not an eigenvalue of A, then $sI - A$ is nonsingular and

$$\begin{aligned} &\int_0^t \mathrm{e}^{(sI-A)\tau}d\tau = \int_0^t (sI-A)^{-1}(sI-A)\mathrm{e}^{(sI-A)\tau}d\tau = \int_0^t (sI-A)^{-1}\frac{d}{d\tau}\mathrm{e}^{(sI-A)\tau}d\tau \\ &= (sI-A)^{-1}\mathrm{e}^{(sI-A)\tau}\Big|_{\tau=0}^{\tau=t} = (sI-A)^{-1}\left(\mathrm{e}^{(sI-A)t} - \mathrm{e}^{(sI-A)0}\right) = (sI-A)^{-1}\left(\mathrm{e}^{(sI-A)t} - I\right) \end{aligned}$$

Frequency or harmonic response

The output response becomes

$$\begin{aligned}
y(t) &= Ce^{At}x(0) + Ce^{At}\left((sI-A)^{-1}e^{(sI-A)\tau}B\right)\Big|_0^t + De^{st} \\
&= Ce^{At}x(0) + Ce^{At}(sI-A)^{-1}\left(e^{(sI-A)t} - I\right)B + De^{st} \\
&= Ce^{At}x(0) + C(sI-A)^{-1}e^{st}B - Ce^{At}(sI-A)^{-1}B + De^{st}
\end{aligned}$$

Technical remark The third equality holds as a consequence of

$$\mathrm{e}^{At}(sI-A)^{-1} = (sI-A)^{-1}\mathrm{e}^{At}.$$

In turn, the latter identity holds because of two properties

1. $AB = BA \Rightarrow \mathrm{e}^{At}B = B\mathrm{e}^{At}$
2. $A(sI-A)^{-1} = (sI-A)^{-1}A$

Property 1 is proven leveraging the definition of matrix exponential.
Property 2 is shown as follows:

$$sA - A^2 = sA - A^2 \Leftrightarrow A(sI-A) = (sI-A)A \Leftrightarrow (sI-A)^{-1}A = A(sI-A)^{-1}$$

Frequency or harmonic response

The output response

$$\begin{aligned}
y(t) &= Ce^{At}x(0) + Ce^{At} \left((sI-A)^{-1} e^{(sI-A)\tau} B \right) \Big|_0^t + De^{st} \\
&= Ce^{At}x(0) + Ce^{At}(sI-A)^{-1}\left(e^{(sI-A)t} - I \right) B + De^{st} \\
&= Ce^{At}x(0) + C(sI-A)^{-1}e^{st}B - Ce^{At}(sI-A)^{-1}B + De^{st}
\end{aligned}$$

can be rearranged as

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI-A)^{-1}B \right)}_{\text{transient}} + \underbrace{\left(C(sI-A)^{-1}B + D \right) e^{st}}_{\text{steady-state}}.$$

If A is asymptotically stable, then the steady state response to $u(t) = \mathrm{e}^{st}$ is

$$y_{st}(t) = W(s)\mathrm{e}^{st}, \quad \text{with } W(s) = C(sI-A)^{-1}B + D, \quad s \in \mathbb{C}$$

Frequency or harmonic response

By linearity, the steady state response to $u(t) = \cos\omega t = \frac{1}{2}(\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\omega t})$ is

$$y_{st} = \frac{1}{2}W(i\omega)\mathrm{e}^{i\omega t} + \frac{1}{2}W(-i\omega)\mathrm{e}^{-i\omega t}$$

Consider for simplicity that the system has 1 input and 1 output. Then $W(i\omega)$ is a complex number with

$$\begin{aligned} W(i\omega) = &\quad \mathrm{Re}(W(i\omega)) + i\,\mathrm{Im}(W(i\omega)) \\ W(i\omega) = &\quad M(\omega)\mathrm{e}^{i\theta(\omega)} \quad \left(W(-i\omega) = M(\omega)\mathrm{e}^{-i\theta(\omega)}\right) \end{aligned}$$

where

$$\begin{aligned} \text{magnitude } M(\omega) = &\quad \sqrt{\mathrm{Re}(W(i\omega))^2 + \mathrm{Im}(W(i\omega))^2} = |W(i\omega)| \\ \text{phase } \theta(\omega) = &\quad \arctan\frac{\mathrm{Im}(W(i\omega))}{\mathrm{Re}(W(i\omega))} = \angle W(i\omega) \end{aligned}$$

The steady state value becomes

$$\begin{aligned} y_{st} &= \frac{1}{2}M(\omega)\mathrm{e}^{i(\omega t+\theta(\omega))} + \frac{1}{2}M(\omega)\mathrm{e}^{-i(\omega t+\theta(\omega))} \\ &= \frac{1}{2}M(\omega)(\cos(\omega t + \theta(\omega)) + i\sin(\omega t + \theta(\omega)) + \cos(\omega t + \theta(\omega)) - i(\omega t + \theta(\omega))) \\ &= M(\omega)\cos(\omega t + \theta(\omega)) \end{aligned}$$

Frequency or harmonic response

Harmonic response

The output response of the linear system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

to a sinusoidal input

$$u(t) = \cos \omega t$$

is a sinusoidal signal

$$y(t) = M(\omega)\cos(\omega t + \theta(\omega))$$

of the same frequency ω with amplitude

$$M(\omega) = \sqrt{\mathrm{Re}(W(i\omega))^2 + \mathrm{Im}(W(i\omega))^2}$$

and phase

$$\theta(\omega) = \arctan \frac{\mathrm{Im}(W(i\omega))}{\mathrm{Re}(W(i\omega))}$$

where

$$W(s) = C(sI - A)^{-1}B + D$$

Example

Example RLC circuit with $L = 1$, $C = 1$, $R = 1$

$$\begin{aligned} \frac{dx(t)}{dt} &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{aligned}$$

Recall

$$y_{st} = M(\omega)\cos(\omega t + \theta(\omega))$$

where

$$W(i\omega) = M(\omega)\mathrm{e}^{i\theta(\omega)} \quad \text{and} \quad W(s) = C(sI - A)^{-1}B + D$$

Example

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$$\begin{aligned} W(s) = C(sI - A)^{-1}B + D &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s^2 + s + 1} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{s}{s^2 + s + 1} \end{aligned}$$

Example

$$W(s) = C(sI - A)^{-1}B + D \quad = \quad \frac{s}{s^2 + s + 1}$$

$$W(i\omega) = \frac{i\omega}{i\omega + (1-\omega^2)} = \frac{i\omega[-i\omega + (1-\omega^2)]}{\omega^2 + (1-\omega^2)^2} = \frac{\omega^2 + i\omega(1-\omega^2)}{\omega^2 + (1-\omega^2)^2}$$

$$= \frac{\sqrt{\omega^4 + \omega^2(1-\omega^2)^2}}{\omega^2 + (1-\omega^2)^2} \mathrm{e}^{i \arctan \frac{\omega(1-\omega^2)}{\omega^2}} = \underbrace{\frac{\omega}{\sqrt{\omega^2 + (1-\omega^2)^2}}}_{M(\omega)} \mathrm{e}^{\underbrace{i \arctan \frac{\omega(1-\omega^2)}{\omega^2}}_{\theta(\omega)}}$$

For $\omega = \frac{1}{\sqrt{2}}$, $\theta(\omega) = \arctan \frac{1}{\sqrt{2}}$ and $M(\omega) = \sqrt{\frac{2}{3}}$. Hence

$$u(t) = \cos(\frac{1}{\sqrt{2}}t) \quad \Rightarrow \quad y(t) = \sqrt{\frac{2}{3}} \cos(\frac{1}{\sqrt{2}}t + \arctan \frac{1}{\sqrt{2}})$$

Frequency or harmonic response

By linearity, if

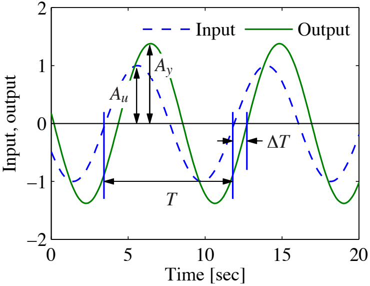
$$u(t) = A_u \cos(\omega t + \varphi_u)$$

then

$$\begin{aligned} y(t) &= M(\omega) \cdot A_u \cos(\omega t + \varphi_u + \theta(\omega)) \\ &=: A_y \cos(\omega t + \varphi_y) \end{aligned}$$

Gain $M(\omega) = \dfrac{A_y}{A_u}$

Phase $\theta(\omega) = \varphi_y - \varphi_u$



Frequency or harmonic response

Gain $M(\omega) = \dfrac{A_y}{A_u}$

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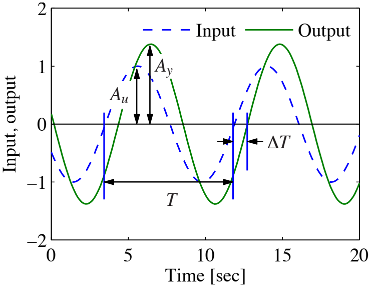


Phase $\theta(\omega)$ can be estimated graphically. Let t_u, t_y two adjacent times at which $u(t) = 0$, $y(t) = 0$.

$$\begin{aligned}
\cos(\omega t_u + \varphi_u) = 0 \quad &\Leftrightarrow \quad \omega t_u + \varphi_u = \tfrac{\pi}{2} \\
\cos(\omega t_y + \varphi_y) = 0 \quad &\Leftrightarrow \quad \omega t_y + \varphi_y = \tfrac{\pi}{2}
\end{aligned}$$

Then

$$\underbrace{\varphi_y - \varphi_u}_{\theta(\omega)} + \underbrace{\omega}_{\frac{2\pi}{T}} \underbrace{(t_y - t_u)}_{\Delta T} = 0 \quad \Leftrightarrow \quad \theta(\omega) = -\frac{2\pi}{T} \Delta T$$

Properties of the frequency response

Recall

$$y_{st} \quad = \quad M(\omega)\cos(\omega t + \theta(\omega))$$

where

$$W(i\omega) = M(\omega)\mathrm{e}^{i\theta(\omega)} \quad \text{and} \quad W(s) = C(sI - A)^{-1}B + D$$

Zero frequency or DC gain

$W(i\omega)|_{\omega=0} = C(-A)^{-1}B + D$

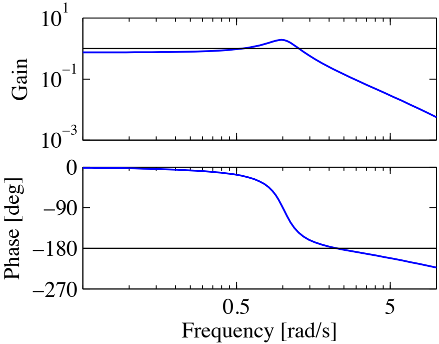
(defined only if A non-singular)

Bandwidth ω_b It is the frequency for which

$$\frac{M(\omega)}{M(0)} \geq \frac{1}{\sqrt{2}}, \quad \text{for all } \omega \in [0, \omega_b]$$

when $M(0) \neq 0$ (A non-singular).

That is, $[0, \omega_b]$ is the range of frequencies ω over which the gain has decreased by not more than $\frac{1}{\sqrt{2}}$



Properties of the frequency response

Recall

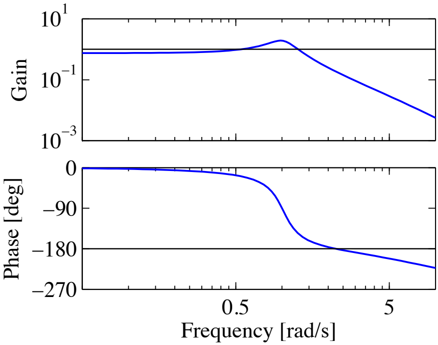
$$y_{st} \quad = \quad M(\omega)\cos(\omega t + \theta(\omega))$$

where

$$W(i\omega) = M(\omega)\mathrm{e}^{i\theta(\omega)} \quad \text{and} \quad W(s) = C(sI - A)^{-1}B + D$$

The **resonant peak** M_r is the largest value of the frequency/harmonic response

$$M_r = \max_{\omega \geq 0} M(\omega)$$



The **peak frequency** ω_{mr} is the frequency at which the reasonant peak is attained

$$\omega_{mr} = \arg\max_{\omega \geq 0} M(\omega)$$

Hence $M(\omega_{mr}) = M_r$.

Reachability

Reachability

Definition: A linear system is reachable if for any x_0, x_f, there exists a $T > 0$, and u defined on $[0, T]$ such that when $x(0) = x_0$, then $x(T) = x_f$.

Example Double integrator

$$\ddot{q} = u, \quad y = q \Leftrightarrow \dot{x} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u$$

Remember (Lecture 4) that

$$x(T) = \underbrace{\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}}_{\mathrm{e}^{AT}} x_0 + \int_0^T \underbrace{\begin{pmatrix} 1 & T-t \\ 0 & 1 \end{pmatrix}}_{\mathrm{e}^{A(T-t)}} \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u(t)dt$$

If $W(0,T) := \displaystyle\int_0^T \mathrm{e}^{A(T-t)} BB^T (\mathrm{e}^{A(T-t)})^T dt$ is invertible then

$$u(t) = B^T (\mathrm{e}^{A(T-t)})^T W(0,T)^{-1} \left(-\mathrm{e}^{AT} x_0 + x_f \right), \quad t \in [0, T]$$

guarantees $x(T) = x_f$. Why?

Reachability

Because, if in

$$x(T) = \underbrace{\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}}_{\mathrm{e}^{AT}} x_0 + \int_0^T \underbrace{\begin{pmatrix} 1 & T-t \\ 0 & 1 \end{pmatrix}}_{\mathrm{e}^{A(T-t)}} \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u(t)dt$$

we replace

$$u(t) = B^T(\mathrm{e}^{A(T-t)})^T W(0,T)^{-1}\left(-\mathrm{e}^{AT}x_0 + x_f\right), \quad t \in [0,T]$$

we obtain

$$\begin{aligned} x(T) = &\quad \mathrm{e}^{AT}x_0 + \int_0^T \mathrm{e}^{A(T-t)}BB^T(\mathrm{e}^{A(T-t)})^T W(0,T)^{-1}\left(-\mathrm{e}^{AT}x_0 + x_f\right) dt \\ = &\quad \mathrm{e}^{AT}x_0 + \overbrace{\int_0^T \mathrm{e}^{A(T-t)}BB^T(\mathrm{e}^{A(T-t)})^T dt}^{W(0,T)}\; W(0,T)^{-1}\left(-\mathrm{e}^{AT}x_0 + x_f\right) \\ = &\quad \mathrm{e}^{AT}x_0 + \left(-\mathrm{e}^{AT}x_0 + x_f\right) = x_f \end{aligned}$$

Reachability

Definition: A linear system is reachable if for any x_0, x_f, there exists a $T > 0$, and u such that when $x(0) = x_0$, then $x(T) = x_f$.

Example Double integrator

$$W(0,T) := \int_0^T \mathrm{e}^{A(T-t)} B B^T (\mathrm{e}^{A(T-t)})^T dt \stackrel{\theta = T-t}{=} \int_0^T \mathrm{e}^{A\theta} B B^T (\mathrm{e}^{A\theta})^T d\theta$$

$$\int_0^T \begin{pmatrix} \theta \\ 1 \end{pmatrix} \begin{pmatrix} \theta & 1 \end{pmatrix} d\theta = \int_0^T \begin{pmatrix} \theta^2 & \theta \\ \theta & 1 \end{pmatrix} d\theta = \begin{pmatrix} \frac{\theta^3}{3} & \frac{\theta^2}{2} \\ \frac{\theta^2}{2} & \theta \end{pmatrix}_{\theta=0}^{\theta=T}$$

$$\Rightarrow \det W(0,T) \neq 0$$

The double integrator is a reachable system.

$W(0,T)$ nonsingular is independent of T, hence the reachability property is independent of T (if the system is reachable for some T, then it is reachable for any T).

Reachability

Definition: A linear system is reachable if for any x_0, x_f, there exists a $T > 0$, and u such that when $x(0) = x_0$, then $x(T) = x_f$.

The result about the non singularity of the reachability Gramian $W(0, T)$ and reachability holds for every linear system.

Theorem – Reachability Gramian condition

A linear system $\dot{x} = Ax + Bu$ is reachable **if and only if**

$$W(0, T) = \int_0^T \mathrm{e}^{A\theta} B B^T \mathrm{e}^{A^T \theta} d\theta$$

is **invertible**

The proof of the sufficiency goes as in the case of the double integrator and is constructive (a control input is constructed).

Reachability

Definition: A linear system is reachable if for any x_0, x_f, there exists a $T > 0$, and u such that when $x(0) = x_0$, then $x(T) = x_f$.

To check the reachability of a linear system we checked invertibility of $W(0, T)$

Is there a more handy method to test reachability of a system?

Theorem – Reachability rank condition

A linear system $\dot{x} = Ax + Bu$ is reachable **if and only if**

$$W_r = [B \ AB \ A^2B \cdots A^{n-1}B] \quad \text{reachability matrix}$$

is **invertible**

<u>Remark</u> If the number of inputs $m > 1$, then reachability matrix W_r is an $n \times mn$ matrix. In this case, $\dot{x} = Ax + Bu$ is reachable **if and only if** the rank of W_r is n (W_r is a full row rank matrix)

Reachability

Example Double integrator

$$\ddot{q} = u, \quad y = q \Leftrightarrow \dot{x} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u$$

Reachability matrix

$$\begin{aligned} W_r &= [B \; AB \; A^2B \cdots A^{n-1}B] = [B \; AB] \\ &= \left[\begin{array}{c|c} 0 & 1 \\ 1 & 0 \end{array} \right] \end{aligned}$$

W_r is nonsingular, hence the double integrator is reachable.

Reachability Rank Theorem - Proof (Sketch, sufficiency)

Sufficiency W_r invertible $\Rightarrow W(0,T)$ invertible.

Suppose by contradiction that $W(0,T)$ is singular, i.e. there exists $v \neq 0$ such that $W(0,T)v = 0$. Hence, $v^T W(0,T)v = 0$ or $\int_0^T v^T \mathrm{e}^{A\theta} B B^T \mathrm{e}^{A^T\theta} v d\theta = 0$ and therefore $B^T \mathrm{e}^{A^T\theta} v = 0$ for all $\theta \in [0,T]$.

At $\theta = 0$, $B^T v = 0$. Take the derivative with respect to time of $B^T \mathrm{e}^{A^T\theta} v = 0$, i.e. $B^T A^T \mathrm{e}^{A^T\theta} v = 0$, and compute it at $\theta = 0$. Then $B^T A^T v = 0$. Differentiate other $n-2$ times and compute the resulting matrix at $\theta = 0$, to obtain that $B^T (A^T)^k v = 0$, $k = 2, 3, \ldots n-1$.

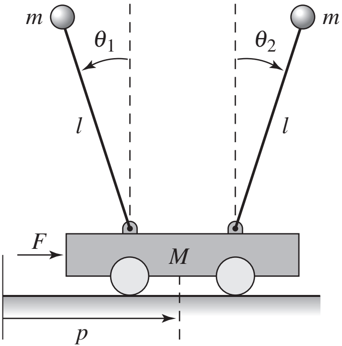
Vectorize the previous identities as

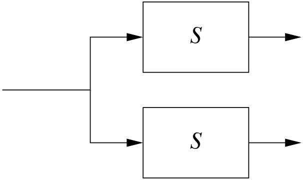
$$\begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} v = 0 \Leftrightarrow v^T \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} = 0$$

This implies that $W_r = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}$ is singular as well, that is a contradiction.

Reachability

Can you think of a simple physical system that is **not** reachable?

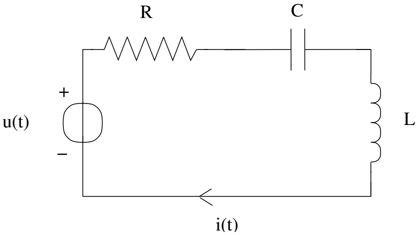




Example: RLC circuit (mass-spring-damper system)

Input:
$u(t)$

Output:
$y(t) = i(t)$.



$$\begin{aligned}\frac{dx(t)}{dt} &= \begin{bmatrix} 0 & 1/L \\ -1/C & -R/L \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1/L \end{bmatrix} x(t)\end{aligned}$$

Is this system reachable?

Reachable canonical form

Coordinate invariance (Lecture 5): state space transformation $z = Tx$ does not change the input-output behavior of the dynamical system

Hence, is there a state space form for which it is "easier" to study **reachability**? YES

Reachable canonical form

Reachable canonical form

Special form of the **reachable** linear system

$$\dot{x} = Ax + Bu$$

that is related to the concept of reachability and is very useful for feedback design (immediate recognition of characteristic polynomial, which is important for stability)

Based on characteristic polynomial given by

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

Canonical form:

$$\dot{z} = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

Reachable canonical form

Canonical form:

$$\dot{z} = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

To obtain this representation consider the change of coordinates $z = Tx$, with T nonsingular to be determined. In the new coordinates the system is

$$\dot{z} = T\dot{x} = T(Ax + Bu) = T(AT^{-1}z + Bu) = \underbrace{TAT^{-1}}_{\tilde{A}} z + \underbrace{TB}_{\tilde{B}} u$$

How to compute *T* Compute reachability matrix

$$\begin{aligned} \tilde{W}_r &= [\tilde{B}\ \tilde{A}\tilde{B}\ \tilde{A}^2\tilde{B} \cdots \tilde{A}^{n-1}\tilde{B}] \\ &= [TB\ \ TAT^{-1}TB\ \ TA^2T^{-1}TB \cdots TA^{n-1}T^{-1}TB] \\ &= T[B\ AB\ A^2B \cdots A^{n-1}B] \\ &= TW_r \end{aligned}$$

Hence

$$T = \tilde{W}_r W_r^{-1}$$

Reachable canonical form

Example Consider the two-dimensional system

$$\frac{dx}{dt} = \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

The associated reachable canonical form is

$$\tilde{A} = \begin{pmatrix} -a_1 & -a_2 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where the coefficients a_1, a_2 are obtained from the characteristic equation

$$\lambda(s) = \det(sI - A) = s^2 - 2\alpha s + (\alpha^2 + \omega^2) \quad \Longrightarrow \quad \begin{aligned} a_1 &= -2\alpha, \\ a_2 &= \alpha^2 + \omega^2. \end{aligned}$$

Reachable canonical form

Example Compute now the reachability matrices for both systems

$$W_r = \begin{pmatrix} 0 & \omega \\ 1 & \alpha \end{pmatrix}, \qquad \tilde{W}_r = \begin{pmatrix} 1 & -a_1 \\ 0 & 1 \end{pmatrix}$$

Then, the matrix T is given by

$$T = \tilde{W}_r W_r^{-1} = \begin{pmatrix} -(a_1+\alpha)/\omega & 1 \\ 1/\omega & 0 \end{pmatrix} = \begin{pmatrix} \alpha/\omega & 1 \\ 1/\omega & 0 \end{pmatrix}$$

and the corresponding change of coordinates $z = Tx$ becomes

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Tx = \begin{pmatrix} \alpha x_1/\omega + x_2 \\ x_1/\omega \end{pmatrix}$$

<u>Exercise</u> Show explicitly that the change of coordinates above transforms the system (A, B) into the system $(\tilde{A}, \tilde{B})$

Today

- Harmonic response
- Reachability
- Reachable canonical form
- **An output regulation problem**

A first control problem

Problem Given the system

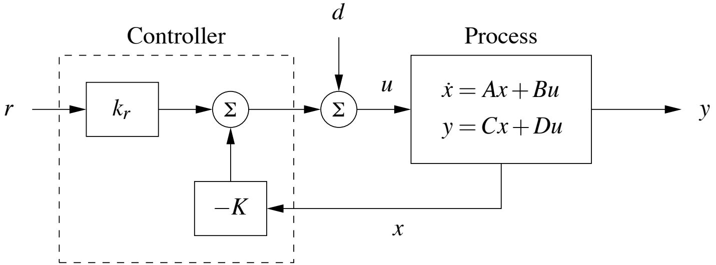
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

find a state feedback control of the form

$$u = -Kx + k_r r$$

such that the output response of the closed-loop system converges to r, i.e.

$$y(t) \to r \quad \text{as} \quad t \to +\infty.$$



Today

- Harmonic response
- Reachability
- Reachable canonical form
- An output regulation problem

Reading assignment Reachability is treated in [Textbook, Section 6.1]

Next lecture

- Solution to the output regulation problem
- Eigenvalue assignment
- Second order systems