1 a] $f(x) = \sin(\frac{x}{4}) - \cos(\frac{x}{4}) = 0$

$\boxed{7}$ (1) $f'(x) = \frac{1}{4}\cos(\frac{x}{4}) + \frac{1}{4}\sin(\frac{x}{4}) = 0$

$$x_{n+1} = x_n - \frac{\left(\sin(\frac{x_n}{4}) - \cos(\frac{x_n}{4})\right)}{\frac{1}{4}\left(\cos(\frac{x_n}{4}) + \sin(\frac{x_n}{4})\right)} \quad 1$$

$x_1 = 3.14165182$ 1

$\llcorner$ true error $5.91690\ 10^{-5}$

(2) $I_0 = [2.7\ \ 3.3]$ $m_0 = 3$

$$\left.\begin{array}{l} f(2.8) = -0.1206 < 0 \\ f(3.2) = 0.0206 > 0 \ \ 1 \\ f \text{ increasing on } I_0 \end{array}\right\} \Rightarrow \begin{array}{l} f(3) = -0.05 < 0 \\ I_1 = [3\ \ 3.3] \\ m_1 = 3.15 \ \ 1 \end{array}$$

$f(3.15) = +0.003 \Rightarrow I_2 = [3\ \ 3.15]$ $m_2 = 3.075$ 1

error for $m_2 = 3.075$: $0.066593\ldots$ 1

$(\frac{1}{2})^n\, 0.066593 < 5.9169\ 10^{-5}$ $n \geq 11$ 1

b] $g(x) = x + \cos(\frac{x}{4}) - \sin(\frac{x}{4})$

$\boxed{6}$ $g'(x) = 1 - \frac{1}{4}\left(\sin(\frac{x}{4}) + \cos(\frac{x}{4})\right)$

(1) $g'(\pi) = 0.64 < 1 \Rightarrow$ convergence 1

(2) $$\left.\begin{array}{l} K = g'(\pi) = 0.6464466 \ \ 1 \\ \hat{K} = \frac{x_4 - x_3}{x_3 - x_2} = 0.6464727 \ \ 1 \\ \text{reduction rate: } \frac{x_4 - \pi}{x_3 - \pi} = 0.6464521 \ \ 1 \end{array}\right\} \text{almost equal}$$

exact err

(3) $$\left.\begin{array}{l} \varepsilon_4 \leq \frac{K}{1-K}|x_4 - x_3| = 0.0247339 \ \ 1 \\ |x_4 - \pi| = 0.0247317 \ \ 1 \end{array}\right\} \text{almost equal}$$

2 a] 7

(1) $f(x) = e^{\sqrt{x}}$ $\quad f'(x) = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$ (1) $\quad f''(x) = e^{\sqrt{x}}\left(\frac{1}{4x} - \frac{1}{4x\sqrt{x}}\right)$ (1)

$f''(0) = -\infty$ not bounded $\Rightarrow$ convergence not optimal (1)

(2) 0 — $\frac{1}{2}$ — 1 $\quad \frac{1}{4}\left(f(0) + f(\tfrac{1}{2})\right) + \frac{1}{4}\left(f(\tfrac{1}{2}) + f(1)\right)$

$I_1(2) = \frac{1}{4}(1 + e^{\sqrt{1/2}}) + \frac{1}{4}(e^{\sqrt{1/2}} + e) = 1.9436279$ (1) (1)

global err. th. uses $M = \max|f''| = \infty \Longrightarrow$ err $\leq \infty$ (1)
useless

(3) $q = \left|\frac{I_{16} - I_{32}}{I_{32} - I_{64}}\right| = 2.74193\ldots$

not close to 4, not optimal convergence (1)

b] 6

(1) $q = 3.99956\ldots$ close to 4, now optimal convergence (1)

(2) $\varepsilon_{64} \simeq \frac{1}{3}(I_{64} - I_{32}) = 1.8051\ 10^{-4}$ (1)

(3) $T_2(16) = \frac{4}{3} I_{16} - \frac{1}{3} I_8 = 2.000001334$
$T_2(8) = \frac{4}{3} I_8 - \frac{1}{3} I_4 = 2.000021300$ (1)

$T_3(16) = \frac{16}{15} T_2(16) - \frac{1}{15} T_2(8) = 2.00000000284$ (1)

$\rightarrow$ error $2.84\ 10^{-9}$

(4) $\left(\frac{1}{4}\right)^m \varepsilon_{64} < 2.84\ 10^{-9}$ (1)

$m \geq 8$ refinements $\Rightarrow 64 \times 2^8$ segments
16384
(1)

3 a] $y_{n+1} = y_n + h\left(\frac{1}{y_n} - x_n\right)$

6 (1) $y(\tfrac{1}{2}) = 1 + \frac{1}{2}\left(\frac{1}{1} - 0\right) = 1\frac{1}{2}$ 1

$y(1) = 1\frac{1}{2} + \frac{1}{2}\left(\frac{1}{1\frac{1}{2}} - \frac{1}{2}\right) = 1\ 7/12 = 1.5833\ldots$ 1

(2) $\left.\begin{aligned} k_1 &= \tfrac{1}{2}\left(\tfrac{1}{1} - 0\right) = \tfrac{1}{2} \\ k_2 &= \tfrac{1}{2}\left(\frac{1}{1+\frac{1}{2}} - \tfrac{1}{2}\right) = \tfrac{1}{12} \end{aligned}\right\}$ 1

$y(\tfrac{1}{2}) = 1 + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{12}\right) = 1\frac{7}{24} = 1.2916$ 1

(3) $y_{n+1} = y_n + h\left(\frac{1}{y_{n+1}} - x_{n+1}\right)$ 1

$y_{n+1} = 1 + 1\left(\frac{1}{y_{n+1}} - 1\right) \Rightarrow y_{n+1} = \frac{1}{y_{n+1}}$ $\quad$ $y_{n+1} = 1$ (OR $y_{n+1} = -1$) 1

7 b] (1) $q = \left|\frac{y_{960} - y_{1920}}{y_{1920} - y_{3840}}\right| = 8.3332\ldots$ 1

close to 2^3, according to theory (3rd order method) 1

(2) $\varepsilon = \frac{1}{7}\left(y_{3840} - y_{1920}\right) = 7.3057\ 10^{-11}$ 1

1

(3) $\frac{8}{7} y_{3840} - \frac{1}{7} y_{1920} = 0.20166901]647457$ 1

(4) wiggles for $N=30$, solution o.k. for $N=60$ 1

stability limit between $N=30$ and $N=60$ (probably) 1

4 a) $\boxed{2}$

0	1.2	2.4
1	e	e^3

$\longrightarrow$ (1)

-1	0	1
0	1	3

(1) $\hat{x} = \dfrac{x-1.2}{1.2}$

b) $\boxed{7}$

(1) $\left.\begin{matrix} M_0 = 3 \\ M_1 = 0 \\ M_2 = 2 \end{matrix}\right\}$ (1) $\left.\begin{matrix} F_0 = 4 \\ F_1 = 3 \end{matrix}\right\}$ (1)

$$\left(\begin{array}{cc|c} 3 & 0 & 4 \\ 0 & 2 & 3 \end{array}\right) \qquad \hat{a} = 4/3 \ (1) \qquad \hat{b} = 3/2 \ (1)$$

$$y = 4/3 + 3/2\,\hat{x} \implies y = e^{4/3} e^{3/2\,\hat{x}} \quad (1)$$

$$y = e^{\frac{4}{3}} e^{\frac{3}{2}\frac{x-1.2}{1.2}} = e^{\frac{4}{3}} e^{\frac{5}{4}x - \frac{3}{2}} = e^{-\frac{1}{6}} e^{5/4x} \quad (1)$$

(2) $y(3.6) = e^{-1/6} e^{\frac{5}{4}\times 3.6} = e^{4\frac{1}{3}}$ (1)

5a) row 1: $\frac{0}{2} = 0$ row 2: $\frac{1.5}{3} = \frac{1}{2}$ row 3: $\frac{3}{\alpha}$ row 4: $\frac{1}{4}$

$\boxed{6}$ (1) convergence if $|\alpha| > 3$ 1

β value not important 1

(2) $\alpha = 10$, worst factor $= \frac{1}{2}$ (row 2) 1

$\left(\frac{1}{2}\right)^n < \frac{1}{100} \Rightarrow n \geq 7$ iterations 1

(3)

$x_0(3)$ $x_0(4)$

$$\left.\begin{aligned}
x_1 &= \tfrac{1}{2}\left(6 - 0 - 0\right) = 3 \\
x_2 &= \tfrac{1}{3}\left(10 - (-1*3) - (\tfrac{1}{2}*2)\right) = 4 \\
x_3 &= \tfrac{1}{10}\left(6 - (1*3 + (-1)*4) - (-1*3)\right) = 1 \\
x_4 &= \tfrac{1}{4}\left(7 - (-1*1)\right) = 2
\end{aligned}\right\} 2$$

b) $\boxed{6}$

y_1 y_2 y_3

0 $\frac{1}{2}$ 1

$y(0) = 2 \rightarrow y_1 = 2$

$y'(1) = 0 \rightarrow -y_2 + y_3 = 0$ 1

$$\frac{y_1 - 2y_2 + y_3}{(1/2)^2} + 2y_2 = 4 * \frac{1}{2} \quad 1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & -6 & 4 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \overset{1}{=} \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -6 & 4 & -6 \\ 0 & -1 & 1 & 0 \end{array}\right) \overset{1}{\rightarrow} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

interior point: $y_2 = 3$ 1 $y(1) = y_3 = 3$ 1

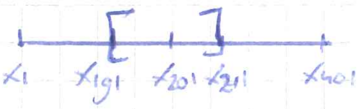
6 a] 400 segments : $\Delta x = 1/400$, $k = 2.26\,10^{-5}$

$\boxed{6}$ (1) $R = \dfrac{\Delta t\, k}{\Delta x^2} = \dfrac{1}{2} \Rightarrow \Delta t = \dfrac{1}{2} \times \left(\dfrac{1}{400}\right)^2 \times \dfrac{1}{2.26\,10^{-5}} = 0.13827$ 1 1

large timestep for $O(\Delta t)$ method $\rightarrow$ low accuracy $\rightarrow$ not advisable 1

(2) $$\frac{T_i^{n+1} - T_i^n}{\Delta t} = k\,\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + Q(x_i, t_n)$$

$$\Rightarrow T_i^{n+1} \overset{1}{=} T_i^n + \underbrace{\boxed{\frac{k\Delta t}{\Delta x^2}}}_{R=}\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right) + Q(x_i, t_n)\,\Delta t$$ 1



- in RHS, factor Δt
- for x_i locations between 0.475 and 0.525 (x_{191}, x_{211})
- if $t_n \leq 30$ seconds

1

b] $\boxed{6}$ pro's : $O(\Delta t^2, \Delta x^2)$ instead of $O(\Delta t, \Delta x^2)$, better accuracy in Δt 1

implicit method, no stability limit (always stable) 1

con's : more complicated
set up linear system each timestep
solve linear system each timestep

1

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{1}{2}k\,\frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{\Delta x^2} + \frac{1}{2}k\,\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + Q(x_i, t_n)$$

$$\Rightarrow T_i^{n+1} - \frac{R}{2}\left(T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}\right) = T_i^n + \frac{R}{2}\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right) + Q(x_i, t_n)\,\Delta t$$

$$\Rightarrow -T_{i+1}^{n+1} + \left(2 + \frac{2}{R}\right)T_i^{n+1} - T_{i-1}^{n+1} \overset{1}{=} T_{i+1}^n + \left(-2 + \frac{2}{R}\right)T_i^n + T_{i-1}^n + Q(x_i, t_n)\,\Delta t \cdot \frac{2}{R}$$ 1

- in RHS, factor $\Delta t \cdot \frac{2}{R}$
- for x_i locations between 0.475 and 0.525 (x_{191}, x_{211})
- if $t_n \leq 30$ seconds

1