Control Engineering 2019-2020
Exam – 18 June 2020
Prof. C. De Persis

- You have **3 hours** to complete the exam.
- You **can** use books and notes but **not** smart phones, computers, tablets and the like.
- Please write your answers using a pen, **not a pencil**.
- There are questions/exercises labeled as **Bonus**. These questions/exercises are optional and give you **extra** points if answered correctly.
- Hints are sometimes provided after a question to be used *only in the case* in which you did not answer a previous question.
- Please write down your Surname, Name, Student ID on each sheet.
- You will be given 2 sheets. If you need more, please ask. Please hand in **all the sheets** that you have used and the **text of the exam**.
- If you return the sheets, then your exam will be graded, unless you explicitly write "do not grade" on the first page.
- If your exam is graded, then the grade will be registered, even if the grade is lower than the one you got at the previous exam(s).

For the grader only

	Exercise 1	Exercise 2	Exercise 3	Exercise 4
Points				
Bonus	×	×	×	×

Exercise 1. State-feedback stabilization of a SIR model (10pt)

Epidemiological models are complex high-dimensional nonlinear systems used to approximate an optimal desired trajectory along which the original model is linearized such that the resulting linear model is used for control purposes. A much simplified version of one of such models takes the form

$$\dot{x} = \underbrace{\begin{bmatrix} -\epsilon & \mu_1 \\ \epsilon & -\gamma \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} \mu_2 \\ 0 \end{bmatrix}}_{B} u$$

where

- x_1 is (the deviation of) the number of exposed individuals (from the optimal trajectory) and x_2 is (the deviation of) the number of infected individuals (from the optimal trajectory);
- u represents the intensity of adopted public health measures;
- $\epsilon, \gamma, \mu_1, \mu_2$ are parameters obtained from fitting the model to the measured data. In the case of the outbreak in Hubei, the numerical values of the estimated parameters are approximated to $\epsilon = 0.1$ $\gamma = 0.2$. The values of μ_1, μ_2 are highly uncertain. We start with the nominal values $\mu_1 = 0.75$, $\mu_2 = 10$.

Provide an answer to the following equations:

a. (6pt) After checking whether or not the open-loop system is asymptotically stable, design a stabilizing state feedback control $u = -Kx$.

b. (4pt) Assume now that the two parameters μ_1, μ_2 are unknown and consider the closed-loop system $\dot{x} = (A - BK)x$, where K is a stabilizing control gain obtained for the nominal values of μ_1, μ_2. Take

$$K = \begin{bmatrix} 0 & 6 \end{bmatrix}$$

Note that the matrix $A - BK$ now depends on the unknown parameters μ_1, μ_2. Give a condition on μ_1, μ_2 under which the closed-loop system remains asymptotically stable.

Solution

a. The characteristic polynomial of A is given by

$$s^2 + (\epsilon + \gamma)s + \epsilon(\gamma - \mu_1) = s^2 + \underbrace{0.3}_{a_1}\, s \underbrace{-0.055}_{a_2}$$

which has one unstable root. Hence the open-loop system is unstable [1pt]. The reachability matrix is

$$W_r = \begin{bmatrix} \mu_2 & -\epsilon\mu_2 \\ 0 & \epsilon\mu_2 \end{bmatrix} = \begin{bmatrix} 10 & -1 \\ 0 & 1 \end{bmatrix}$$

which is nonsingular, hence the system is reachable [1pt]. We then compute

$$W_r^{-1} = \frac{1}{10}\begin{bmatrix} 1 & 1 \\ 0 & 10 \end{bmatrix} \quad [0.5\text{pt}]$$

The reachable canonical form is given by

$$\tilde{A} = \begin{bmatrix} -0.3 & 0.055 \\ 1 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad [1\text{pt}]$$

and its reachability matrix is

$$\tilde{W}_r = \begin{bmatrix} 1 & -0.3 \\ 0 & 1 \end{bmatrix} \quad [1\text{pt}]$$

Hence

$$\begin{aligned} K = & \begin{bmatrix} p_1 - 0.3 & p_2 + 0.055 \end{bmatrix} \begin{bmatrix} 1 & -0.3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 & 0.1 \\ 0 & 1 \end{bmatrix} \\ = & \begin{bmatrix} p_1 - 0.3 & p_2 + 0.055 \end{bmatrix} \begin{bmatrix} 0.1 & -0.2 \\ 0 & 1 \end{bmatrix} \qquad [1\text{pt}] \\ = & \begin{bmatrix} 0.1(p_1 - 0.3) & -0.2(p_1 - 0.3) + p_2 + 0.055 \end{bmatrix} \end{aligned}$$

A stabilizing gain K is obtained by replacing p_1, p_2 with the coefficients of any polynomial $s^2 + p_1 s + p_2$ whose roots have strictly negative real parts [0.5pt].

Remark The gain K given in the part b. of the exercise is the one obtained by choosing $p_1 = 0.3$ and $p_2 = 5.945$ so that

$$K = \begin{bmatrix} 0 & 6 \end{bmatrix}$$

b.

$$A - BK = \begin{bmatrix} -0.1 & \mu_1 \\ 0.1 & -0.2 \end{bmatrix} - \begin{bmatrix} \mu_2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 6 \end{bmatrix} = \begin{bmatrix} -0.1 & \mu_1 - 6\mu_2 \\ 0.1 & -0.2 \end{bmatrix} \quad [0.5\text{pt}]$$

Its characteristic polynomial is given by

$$s^2 + 0.3s + (0.02 - 0.1(\mu_1 - 6\mu_2)) \quad [0.5\text{pt}]$$

The Routh-Hurwitz table is

$$\begin{array}{c|ll} 2 & 1 & 0.02 - 0.1(\mu_1 - 6\mu_2) \\ 1 & 0.3 & \\ 0 & 0.02 - 0.1(\mu_1 - 6\mu_2) & \end{array} \quad [1.5\text{pt}]$$

The closed-loop system remains stable as far as the condition

$$\mu_1 - 6\mu_2 < 0.2 \quad [1.5\text{pt}]$$

holds.

Exercise 2. Observer for a predator-prey system (10pt)

The linearized model of a predator-prey system that describes the evolution of the number of predators and prey in an ecosystem are given by

$$\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} 0.1 & -1 \\ 0.5 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 18 \\ 0 \end{bmatrix}}_{B} u \\ y &= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} x \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \dot{x}_1 &= 0.1x_1 - x_2 + 18u \\ \dot{x}_2 &= 0.5x_1 \\ y &= x_2 \end{aligned}$$

where x_1 is the number of preys, x_2 the number of predators and u a control input that modulates the food supply for the preys.

You want to design an observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

that estimates the number of preys using the number of predators. To this end:

a. (5pt) Determine *all* the observer gains L.

b. (5pt) Since one of the two states (the number of predators) is directly measured, i.e. $y = x_2$, you would like to design a reduced order observer that only estimates x_1. Design the scalar gain ℓ such that the estimate $\hat{x}_1$ produced by the reduced order observer

$$\begin{aligned} \dot{z} = &\ (0.1 - \ell)z + (\ell(0.1 - \ell) - 0.5)y + 9u \\ \hat{x}_1 = &\ 2(z + \ell y) \end{aligned}$$

satisfies $\lim_{t+\infty} e(t) = 0$, where $e = x_1 - \hat{x}_1 = x_1 - 2(z + \ell y)$.

Hint *Show that the estimation error dynamics is given by $\dot{e}(t) = (0.1 - \ell)e(t)$. To obtain it, bear in mind that $y = x_2$ and $\dot{y} = 0.5x_1$.*

Solutions.

a. The observability matrix is

$$W_0 = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} \quad [0.5\text{pt}]$$

which is nonsingular, hence observable, with

$$W_0^{-1} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \quad [0.5\text{pt}]$$

The characteristic polynomial of A is $s^2 \underbrace{-0.1}_{a_1} s + \underbrace{0.5}_{a_2}$ [0.5pt]. Hence, the observable canonical form is

$$\tilde{W}_o = \begin{bmatrix} 1 & 0 \\ -0.1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix} \quad [0.5\text{pt}]$$

We use

$$\begin{aligned} L = \; & W_o^{-1}\tilde{W}_o \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix} \begin{bmatrix} p_1 + 0.1 \\ p_2 - 0.5 \end{bmatrix} \text{[1pt]} \\ = & \begin{bmatrix} 0.2 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 + 0.1 \\ p_2 - 0.5 \end{bmatrix} = \begin{bmatrix} 0.2(p_1 + 0.1) + 2(p_2 - 0.5) \\ p_1 + 0.1 \end{bmatrix} \text{[1pt]} \end{aligned}$$

where p_1, p_2 is any pair of positive real numbers such that the polynomial $s^2 + p_1 s + p_2$ has all the roots with negative real parts [1pt].

b. The estimation error dynamics is

$$\begin{aligned} \dot{e} = \; & \dot{x}_1 - 2(\dot{z} + \ell\dot{y}) \\ = \; & 0.1x_1 - x_2 + 18u - 2((0.1-\ell)z + (\ell(0.1-\ell) - 0.5)y + 9u + 0.5\ell x_1) \text{ [0.5pt]} \\ = \; & 0.1x_1 - x_2 + 18u - 2(0.1-\ell)z - 2\ell(0.1-\ell)x_2 + x_2 - 18u - \ell x_1 \text{ [0.5pt]} \\ = \; & 0.1x_1 - 2(0.1-\ell)z - 2\ell(0.1-\ell)x_2 - \ell x_1 \text{ [0.5pt]} \\ = \; & (0.1-\ell)x_1 - 2(0.1-\ell)(z + \ell x_2) \text{ [0.5pt]} \\ = \; & (0.1-\ell)x_1 - (0.1-\ell)\hat{x}_1 \text{ [0.5pt]} \\ = \; & (0.1-\ell)e \text{ [0.5pt]} \end{aligned}$$

Hence, $g(\ell) = 0.1 - \ell$, and the error dynamics is asymptotically stable if $\ell > 0.1$ [1pt].

Exercise 3. PID control of a predator-prey system (10pt)

Consider the feedback loop represented in Fig. 1 where $P(s)$ is the transfer function (to be determined) of the predator-prey system considered in Exercise 3. Assume that $F(s) = 1$. The process is controlled via the ideal PID controller of the form

$$C(s) = \frac{k_d s^2 + k_p s + k_i}{s}$$

where k_p, k_i are parameters to be designed.

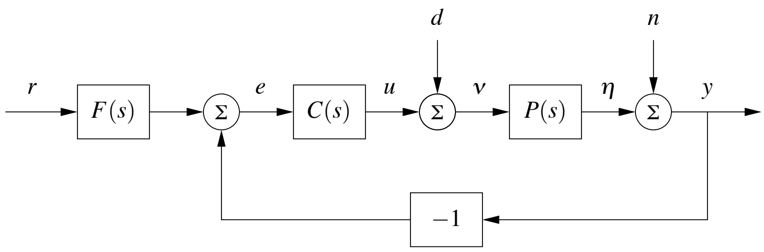

Figure 1: Feedback control system.

a. (8pt) **(Asymptotic stability)** Give conditions on the parameters k_d, k_p, k_i of the PID controller such that the closed loop system is asymptotically stable, i.e., the complementary sensitivity function $T(s) = G_{yr}(s)$ has all the poles with negative real parts.

b. (2pt) **(Model matching)** Discuss whether or not the parameters k_d, k_p, k_i of the PID controller can be designed in such a way that the complementary sensitivity function $T(s) = G_{yr}(s)$ coincides with a transfer function of the form

$$\frac{a_2 s^2 + a_1 s + a_0}{s^3 + b_2 s^2 + b_1 s + b_0}$$

for any given real numbers $a_1, a_0, b_3, b_2, b_1, b_0$.

Solutions.

a. The process transfer function is given by

$$
\begin{aligned}
P(s) = &\ C(sI-A)^{-1}B = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s-0.1 & 1 \\ -0.5 & s \end{bmatrix}^{-1} \begin{bmatrix} 18 \\ 0 \end{bmatrix} \\
= &\ \frac{1}{s^2-0.1s+0.5} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ 0.5 & s-0.1 \end{bmatrix} \begin{bmatrix} 18 \\ 0 \end{bmatrix} \\
= &\ \frac{1}{s^2-0.1s+0.5} \begin{bmatrix} 0.5 & s-0.1 \end{bmatrix} \begin{bmatrix} 18 \\ 0 \end{bmatrix} = \frac{9}{s^2-0.1s+0.5} \qquad [1\text{pt}]
\end{aligned}
$$

The complementary sensitivity function is given by

$$
\begin{aligned}
T(s) = &\ \frac{\frac{9(k_d s^2 + k_p s + k_i)}{(s^2-0.1s+0.5)s}}{1+\frac{9(k_d s^2 + k_p s + k_i)}{(s^2-0.1s+0.5)s}} \\
= &\ \frac{9(k_d s^2 + k_p s + k_i)}{(s^2-0.1s+0.5)s + 9(k_d s^2 + k_p s + k_i)} \\
= &\ \frac{9(k_d s^2 + k_p s + k_i)}{s^3 + (9k_d - 0.1)s^2 + (0.5+9k_p)s + 9k_i} \qquad (1\text{pt})
\end{aligned}
$$

Its poles have negative real part if and only if the first column of the Routh-Hurwitz table has no sign changes

$$
\begin{array}{c|ll}
3 & 1 & 0.5+9k_p \\
2 & 9k_d - 0.1 & 9k_i \\
1 & -\frac{1}{9k_d-0.1}(9k_i - (0.5+9k_p)(9k_d-0.1)) & \\
0 & 9k_i &
\end{array} \qquad [3\text{pt}]
$$

This gives the conditions

$$
k_d > \frac{1}{90}, \quad k_i > 0, \quad 9k_p > \frac{9k_i}{9k_d - 0.1} - 0.5 \quad [3\text{pt}]
$$

<u>Comment</u> Most of the students will just try to find the parameters k_d, k_p, k_i that stabilize the closed-loop system. In that case, if their answer is correct, instead of giving them 6 points for the construction of the Routh table and the resulting discussion, you give them 2 points.

b. No, it is not possible to freely assign the closed-loop transfer function because assigning the denominator already fixes the parameters k_d, k_p, k_i and no degree of freedom is left to also assign the numerator.

Exercise 4. Loop shaping (10pt)

Consider a process whose dynamics is modeled via the transfer function $P(s)$, which has an unstable pole but it is otherwise unknown. The controller is a proportional controller

$$C(s) = 10k_p$$

with k_p initially set to $k_p = 1$. The Bode diagrams of the loop transfer function $L(s) = C(s)P(s)$ are obtained experimentally and given in Figure 2.

a. (3pt) Look at the Nyquist plots in Fig. 3, state which one corresponds to the Bode diagrams of Figure 2 and determine whether or not the closed-loop system is asymptotically stable or unstable.

b. (3pt) If the closed-loop system is asymptotically stable, determine the gain crossover frequency ω_{gc} and the phase $\angle L(i\omega_{gc})$. Use the latter value to determine the phase margin of the system.

c. (3pt) If the closed-loop system is asymptotically stable, find a positive value of the controller gain k_p such that the closed-loop system becomes unstable.

d. (1pt) If the closed-loop system is asymptotically stable, determine the steady output response of the closed-loop system to the reference step input $r(t) = 2 \cdot 1(t)$ (step response).

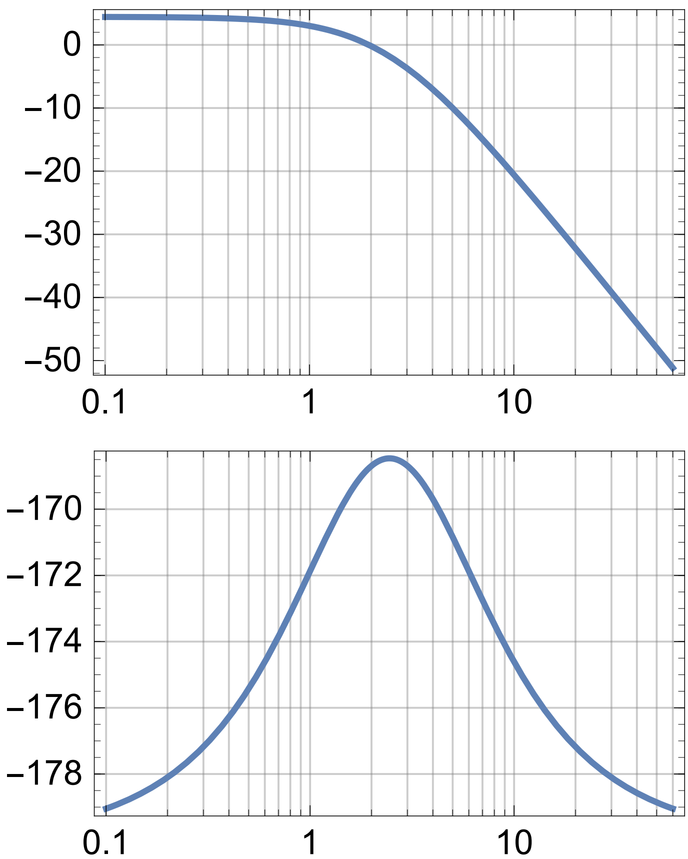

Figure 2: Bode diagram with log-linear scale for the gain plot (gain magnitude in dB).

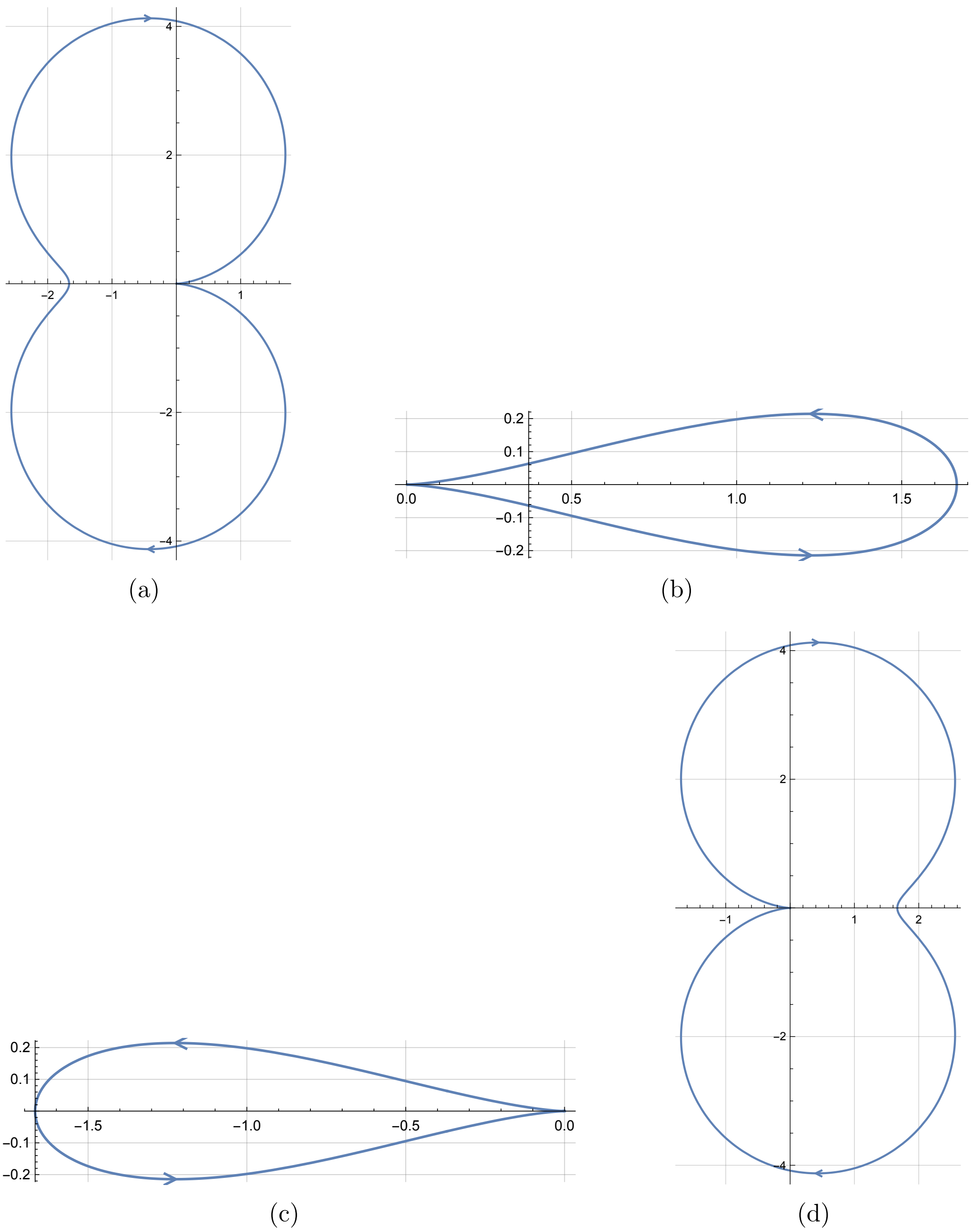

Figure 3: Nyquist plots

Solutions.

a. The correct Nyquist plot is the one in (c). The number of net clockwise encirclements of -1 is $N = -1$ (one counterclockwise encirclement). Since $P = 1$, by Nyquist general theorem, the number of poles of the closed-loop system in the right-half plane is $Z = N + P = 0$. Hence, the closed-loop system is asymptotically stable.

b. The gain crossover frequency is the frequency at which $|L(i\omega)| = 1$. From the Bode diagrams, $\omega_{gc} = 2$ rad/sec (1pt) and $\angle L(i2) = -169^\circ$ (1pt). from which $m_\varphi = 11^\circ$ (1pt).

c. From the Bode diagram we see that the DC gain is given by $L(0)|_{\text{dB}} = 4\text{dB}$ and $\angle L(0) = -180°$, that is $L(0) = -1.58$. Using the Nyquist diagram we see that for any $0 < k_p < (1.58)^{-1} \approx 0.63$, the intercept of the Nyquist plot with the negative real axis is on the right of -1 and thus the number of net clockwise encirclements of -1 is $N = 0$. By the general Nyquist theorem, we conclude that the closed-loop system becomes unstable for any $0 < k_p < 0.63$.

d. Since $L(0) = -1.58$ (see previous answer), then $L(0)/(1+L(0)) = -2.58/(-0.58) = 4.45$ and the step response is given by $y_{\text{steady state}} = 9.8 \cdot 1(t)$.