Lecture 2: Dynamic Mode Decomposition

Dr. Saeed Ahmed

Learning-based Control

Department of Mechanical and Process Engineering,
University of Kaiserslautern

Dynamic Mode Decomposition (DMD)

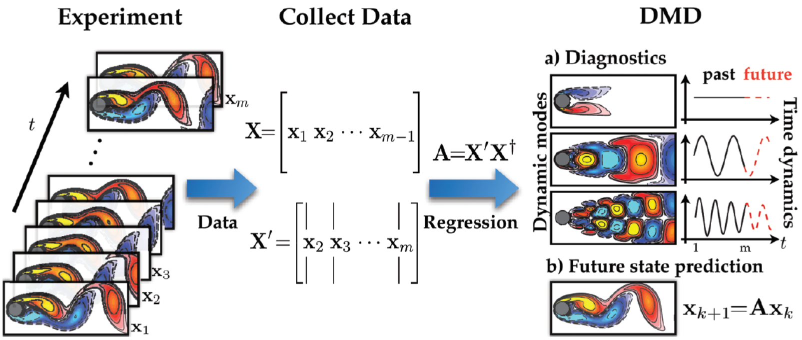


Figure 1: DMD illustrated on the fluid flow past a circular cylinder

Objective of DMD

The DMD algorithm seeks the leading spectral decomposition (i.e., eigenvalues and eigenvectors) of the best-fit linear operator A that relates the two snapshot matrices X and X' in time:

$$X' \approx AX$$

The best fit operator A then establishes a linear dynamical system that best advances snapshot measurements forward in time.

Overview of Singular Value decomposition (SVD)

Generally, we are interested in analyzing large data set $X \in \mathbb{R}^{n\times m}$:

$$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_m \\ | & | & & | \end{bmatrix}$$

The singular value decomposition is a unique matrix decomposition that exists for every matrix $X \in \mathbb{R}^{n\times m}$:

$$X = U\Sigma V^T = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{bmatrix} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_m \\ | & | & & | \end{bmatrix}^T$$

where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ are unitary matrices with orthonormal columns i.e. $UU^T = U^TU = I_{n\times n}$ and $VV^T = V^TV = I_{m\times m}$. $\Sigma \in \mathbb{R}^{n\times m}$ is a diagonal matrix with real, nonnegative entries on the diagonal called signular values ($\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_m \geq 0$) and zeros off the diagonal.

The DMD Algorithm

Step 1: Find the truncated SVD of X

Compute the singular value decomposition of X:

$$X = U\Sigma V^* = \begin{bmatrix} \tilde{U} & \tilde{U}_{rem} \end{bmatrix} \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & \Sigma_{rem} \end{bmatrix} \begin{bmatrix} \tilde{V}^* \\ \tilde{V}^*_{rem} \end{bmatrix} \approx \tilde{U}\tilde{\Sigma}\tilde{V}^*,$$

where $U \in \mathbb{R}^{n\times n}$, $\Sigma \in \mathbb{R}^{n\times m-1}$, $\tilde{V}^* \in \mathbb{R}^{m-1\times m-1}$, $\tilde{U} \in \mathbb{R}^{n\times r}$, $\tilde{\Sigma} \in \mathbb{R}^{r\times r}$, $\tilde{V}^* \in \mathbb{R}^{r\times m-1}$, $_{rem}$ indicates the remaining $m-1-r$ singular values, and * denotes the complex conjugate transpose.

Remark 1a. The columns of the matrix $\tilde{U}$ are also known as Proper Orthogonal Decomposition (POD) modes, and they satisfy $\tilde{U}^*\tilde{U} = I$. Similarly, columns of $\tilde{V}$ are orthonormal and satisfy $\tilde{V}^*\tilde{V} = I$. In practice, choosing the approximate rank r is one of the most important and subjective steps in DMD.

Remark 1b. The truncated SVD with a truncation value r and eliminating the reminder (rem) allows for the pseudoinverse to be accomplished since $\tilde{\Sigma}$ is square.

The DMD Algorithm

Step 2. Compute reduced-order approximation $\tilde{A}$

The full matrix A may be obtained by computing the pseudo-inverse of X:

$$A = X' X^{\dagger} = X' \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}^{*}.$$

However, we are only interested in the leading r eigenvalues and eigenvectors of A, and we may thus project A onto the POD modes in U:

$$\tilde{A} = \tilde{U}^{*} A \tilde{U} = \tilde{U}^{*} X' \tilde{V} \tilde{\Sigma}^{-1}$$

Remark 2a. The key observation here is that the reduced matrix $\tilde{A}$ has the same nonzero eigenvalues as the full matrix A. Thus, we need only compute the reduced $\tilde{A}$ directly, without ever working with the high-dimensional A matrix.

Remark 2b. Computing the eigendecomposition of A versus $\tilde{A}$ can be a computationally crucial step for efficiency. For instance, in fluid dynamics or epidemiology problems, we can have an arbitrarily large dimension n. The direct solution of the $n \times n$ eigenvalue problem might not be feasible; thus solving the $r \times r$ is substantially more attractive.

The DMD Algorithm

Step 3. Investigate the dynamic properties of $\tilde{A}$

The spectral decomposition of $\tilde{A}$ is computed:

$$\tilde{A}W = W\Lambda.$$

Remark 3. The entries of the diagonal matrix Λ are the DMD eigenvalues, which also correspond to eigenvalues of the full A matrix.

Step 4. Solve for the dynamic modes of A

The DMD Modes Φ are eigenvectors of the high-dimensional A matrix corresponding to the eigenvalues in Λ:

$$\Phi = X'\tilde{V}\tilde{\Sigma}^{-1}W$$

Remark 4. Let us prove this fact.

$$\begin{aligned} A\Phi &= (X'\tilde{V}\tilde{\Sigma}^{-1}\underbrace{\tilde{U}^*)(X'\tilde{V}\tilde{\Sigma}^{-1}}_{\tilde{A}}W) = X'\tilde{V}\tilde{\Sigma}^{-1}\tilde{A}W \\ &= X'\tilde{V}\tilde{\Sigma}^{-1}W\Lambda \\ &= \Phi\Lambda. \end{aligned}$$

DMD MATLAB Code

```
function [Phi, Lambda, b] = DMD(X,Xprime,r)

[U,Sigma,V] = svd(X,'econ');        % Step 1
Ur = U(:,1:r);
Sigmar = Sigma(1:r,1:r);
Vr = V(:,1:r);

Atilde = Ur'*Xprime*Vr/Sigmar;      % Step 2
[W,Lambda] = eig(Atilde);           % Step 3

Phi = Xprime*(Vr/Sigmar)*W;         % Step 4
alpha1 = Sigmar*Vr(1,:)';
b = (W*Lambda)\alpha1;
```

Dynamic Mode Decomposition with Control (DMDc)

The goal of DMDc is to analyze the relationship between a future system measurement x_{k+1} with the current measurement x_k and the current control u_k. For each trio of measurement data, a pair of linear operators provides the following relationship:

$$x_{k+1} \approx Ax_k + Bu_k$$

The problem is to find the best-fit approximations to the mappings A and B utilizing the trio of measurement data.

The DMDc Algorithm

Step1a. Collect and construct the measurement and control input snapshot matrices

Collect the measurement and control input snapshot matrices:

$$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_{m-1} \\ | & | & & | \end{bmatrix}, \; X' = \begin{bmatrix} | & | & & | \\ x_2 & x_3 & \dots & x_m \\ | & | & & | \end{bmatrix},$$

$$\Upsilon = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_{m-1} \\ | & | & & | \end{bmatrix},$$

then the equation $x_{k+1} \approx Ax_k + Bu_k$ can be written as:

$$X' \approx AX + B\Upsilon.$$

Utilizing the three data matrices X', X, and Υ, DMDc is focused on finding best-fit approximations to the mappings A and B.

The DMDc Algorithm

Step1b. Stack the date matrices X and Υ to construct the matrix Ω

The approximate relationship $X' \approx AX + B\Upsilon$ between the data matrices X, Υ, and X' can be written in the following form:

$$X' \approx G\Omega,$$

where

$$G = \begin{bmatrix} A & B \end{bmatrix} \text{ and } \Omega = \begin{bmatrix} X \\ \Upsilon \end{bmatrix}.$$

The DMDc Algorithm

Step 2. Compute the SVD of the input space Ω

Using the pseudo-inverse:

$$\begin{aligned} G &= X'\Omega^{\dagger}, \\ \begin{bmatrix} A & B \end{bmatrix} &= X' \begin{bmatrix} X \\ \Upsilon \end{bmatrix} \end{aligned}$$

where Ω contains both the measurement and control snapshot information. Computing the SVD of $\Omega = U\Sigma V^* \approx \tilde{U}\tilde{\Sigma}\tilde{V}^*$ with truncation value p provides an approximation of G:

$$G \approx \bar{G} = X'\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^*$$

We can find the approximations of the matrices A and B by breaking the linear operator $\tilde{U}$ into two separate components:

$$\begin{bmatrix} A, & B \end{bmatrix} \approx \begin{bmatrix} \bar{A}, & \bar{B} \end{bmatrix} \approx \begin{bmatrix} X'\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}_1{}^*, & X'\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}_2{}^* \end{bmatrix}$$

where

$$\tilde{U}^* = \begin{bmatrix} \tilde{U}_1{}^*, & \tilde{U}_2{}^* \end{bmatrix}.$$

The DMDc Algorithm

Step 3. Compute the SVD of the output space X'

Compute the SVD of X', thereby obtaining the decomposition $X' \approx \hat{U}\hat{\Sigma}\hat{V}^*$ with truncation value r.

Step 4. Compute the approximation of the operators $G = [A\ B]$

Using the transformation $x = \hat{U}\tilde{x}$, the following reduced-order approximations of A and B can be computed:

$$\tilde{A} = \hat{U}^* \bar{A} \hat{U} = \hat{U}^* X' \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}_1{}^* \hat{U}$$

$$\tilde{B} = \hat{U}^* \bar{B} = \hat{U}^* X' \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}_2{}^*$$

The DMDc Algorithm

Step 5. Perform the eigenvalue decomposition of $\tilde{A}$

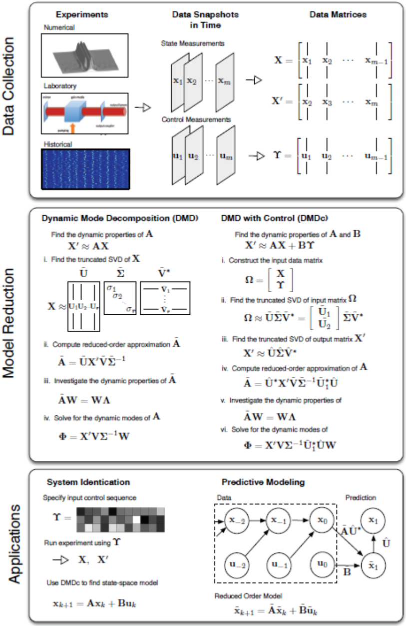
The spectral decomposition of $\tilde{A}$ is computed:

$$\tilde{A}W = W\Lambda.$$

Step 6. Compute the dynamic modes of the operator A

The dynamic modes Φ of the operator A are computed as

$$\Phi = X'\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}_1{}^*\hat{U}W.$$


Data Collection
Experiments
Numerical
Laboratory
Historical
Data Snapshots in Time
State Measurements
Control Measurements
Data Matrices
Model Reduction
Dynamic Mode Decomposition (DMD)
Find the dynamic properties of $\mathbf{A}$
$\mathbf{X}' \approx \mathbf{A}\mathbf{X}$
i. Find the truncated SVD of $\mathbf{X}$
ii. Compute reduced-order approximation $\tilde{\mathbf{A}}$
$\tilde{\mathbf{A}} = \tilde{\mathbf{U}}\mathbf{X}'\tilde{\mathbf{V}}\tilde{\mathbf{\Sigma}}^{-1}$
iii. Investigate the dynamic properties of $\tilde{\mathbf{A}}$
$\tilde{\mathbf{A}}\mathbf{W} = \mathbf{W}\mathbf{\Lambda}$
iv. Solve for the dynamic modes of $\mathbf{A}$
$\mathbf{\Phi} = \mathbf{X}'\mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{W}$
DMD with Control (DMDc)
Find the dynamic properties of $\mathbf{A}$ and $\mathbf{B}$
$\mathbf{X}' \approx \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{\Upsilon}$
i. Construct the input data matrix
ii. Find the truncated SVD of input matrix $\mathbf{\Omega}$
iii. Find the truncated SVD of output matrix $\mathbf{X}'$
$\mathbf{X}' \approx \hat{\mathbf{U}}\hat{\mathbf{\Sigma}}\hat{\mathbf{V}}^*$
iv. Compute reduced-order approximation of $\mathbf{A}$
v. Investigate the dynamic properties of
$\tilde{\mathbf{A}}\mathbf{W} = \mathbf{W}\mathbf{\Lambda}$
vi. Solve for the dynamic modes of
Applications
System Identication
Specify input control sequence
Run experiment using $\mathbf{\Upsilon}$
Use DMDc to find state-space model
$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k$
Predictive Modeling
Data
Prediction
Reduced Order Model
$\tilde{\mathbf{x}}_{k+1} = \tilde{\mathbf{A}}\tilde{\mathbf{x}}_k + \tilde{\mathbf{B}}\tilde{\mathbf{u}}_k$