1 5

a) $x_{n+1} = x_n - \dfrac{(\sin(\pi x_n) + x_n - 5)}{\pi\cos(\pi x_n) + 1}$ (1) (1)

(1)

(2) $x_1 = 4 - \dfrac{(0+4-5)}{\pi+1} = 4 + \dfrac{1}{\pi+1} = 4.24145$ (1) (1)

$\varepsilon = \dfrac{x_1 - 4}{x_1} = 5.6926\,E{-2}$ (1)

8 b) $g(x) = x - \frac{1}{2}(\sin(\pi x) + x - 5)$

(1) $g'(x) = 1 - \frac{1}{2}(\pi\cos(\pi x) + 1)$ (1)

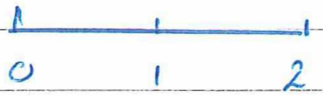
$g'(4.25) = 1 - \frac{1}{2}\left(\pi\frac{\sqrt{2}}{2} + 1\right) = -0.61072$ (1) not that fast, value not close to zero (1)

(2) $2x_4 - x_3 = 4.2583$ (1) this only helps if you are closer to convergence (1)

(3) $g'(x) = 1 + \alpha(\pi\cos(\pi x) + 1)$ (1)

$g'(4.25) = 1 + \alpha\left(\pi\frac{\sqrt{2}}{2} + 1\right) = 0$ (1) $\longrightarrow \alpha = \dfrac{-1}{\pi\frac{\sqrt{2}}{2}+1} = -0.31042$ (1)

2 a) 6



$M = 1\dfrac{1}{\frac{1}{2}+4} + 1\dfrac{1}{\frac{3}{2}+4} = \dfrac{40}{99} = 0.40404$ (1)

6 (1)

(1) $T = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} + \frac{1}{6}\right) = \frac{49}{120} = 0.40833$ (1)

(2) $f(x) = \dfrac{1}{x+4}$ $\quad f''(x) = \dfrac{2}{(x+4)^3}$ (1) $\longrightarrow M = \frac{1}{32}$ (1) $\quad \varepsilon < \frac{1}{12}(2-0)(1)^2\frac{1}{32} = \frac{1}{192} = 5.2083E$ (1)

8 b)

(1) $q = \dfrac{I_{16} - I_{32}}{I_{32} - I_{64}} \approx 4$, according to 2nd order theory (1)

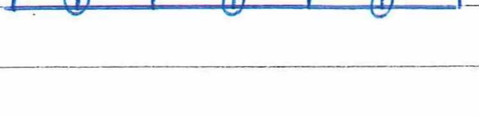
(3) $\frac{4}{3} I_{64} - \frac{1}{3} I_{32} = 0.39269909$ (1)

(2) $\text{err} = \frac{1}{3}(I_{64} - I_{32}) = 5.08667\,E{-6}$ (1) (1)

$(5.08667E{-6})\left(\frac{1}{4}\right)^n < E{-8} \longrightarrow n \geq 5$ (1)

segments: $2^5 \times 64 = 2048$ (1)

(4)



$I_{32} \rightarrow 33$ function calls

$I_{64} \rightarrow \underline{32}$ new function calls

$\phantom{I_{64} \rightarrow}\ 65^{+}$ total calls (2)

3 a) $\underset{0}{+}\!\!-\!\!\underset{1/4}{+}\!\!-\!\!\underset{1/2}{+}$ (1) $y(\tfrac14) = y(0) + \tfrac14 * 4(0 - y(0)) = 2 + (0-2) = 0$ (1)

6] $y(\tfrac12) = y(\tfrac14) + \tfrac14 * 4(\tfrac14 - y(\tfrac14)) = 0 + (\tfrac14 - 0) = 1/4$ (1)

(2) $\left.\begin{aligned} k_1 &= \tfrac12 * 4(0 - y(0)) = 2(0-2) = -4 \\ k_2 &= \tfrac12 * 4(\tfrac12 - (y(0) - 4)) = 2(\tfrac12 + 2) = 5 \end{aligned}\right\}$ (1)

$y(\tfrac12) = 2 + \tfrac12(-4+5) = 2\tfrac12$ (1)

(3) exact val: $y(\tfrac12) \approx 0.5545$ Euler more accurate because of finer grid (2)

b) (1) no limit visible, all solutions same pattern (1)

8] (2) $\left.\begin{aligned} RK2&: -2 < -4h < 0 \\ RK4&: -2.7 < -4h < 0 \end{aligned}\right\}$ for safety: $h < 1/2$ (1)

(3) $q = \left.\dfrac{y_{1/8} - y_{1/16}}{y_{1/16} - y_{1/32}}\right|_{at\ x=1} \approx 9.7846$ (1), not close enough to $2^3 = 8$ (1) (theoretically expected)

(4) $\varepsilon = \tfrac17 (y_{1/32} - y_{1/16})|_{at\ x=2} \approx 6.0714\ E{-7}$ (1)

$(\tfrac18)^n * \varepsilon < 10^{-8}$ (1) $\longrightarrow n \geq 2$ take $n=3$ for safety (1)

δx-grid $(\tfrac12)^3 * \tfrac{1}{32} = \tfrac{1}{256}$ (1)

4a) $\dfrac{0 \quad \tfrac12 \quad 1}{2 \quad 10 \quad 24} \xrightarrow{\hat{x} = 2x - 1} \dfrac{-1 \quad 0 \quad 1}{2 \quad 10 \quad 24}$ (1) $\left.\begin{aligned} &M_0 = 3 \quad M_1 = 0 \quad M_2 = 2 \\ &F_0 = 36 \quad F_1 = 22 \end{aligned}\right\}$ (2)

7]

$y = 12 + 11\hat{x}$ (1) $= 22x + 1$ (1) $\quad \left.\begin{matrix} 3 & 0 \\ 0 & 2 \end{matrix}\right| \begin{matrix} 36 \\ 22 \end{matrix} \longrightarrow a = 12$ (1), $b = 11$ (1)

3] b) $f(3/2) = 22 * \tfrac32 + 1 = 34$ (1), err at $x=1$: $|22*1 + 1 - 24| = 1$ (1) (1)

5a) $y_1 \quad y_2 \quad y_3 \quad y_4$

5 $\quad 0 \quad 2 \quad 4 \quad 6 \qquad \left.\begin{array}{l} y_1 = a = 0 \\ y_4 = b = 3 \end{array}\right\}$ (1)

$\frac{y_{i+1} - 2y_i + y_{i-1}}{(2)^2} = p y_i + q x_i$ (1) $\Rightarrow y_{i+1} + (-2-4p) y_i + y_{i-1} = 4 q x_i$ (1)

$-2 - 4p = -2.5 \rightarrow p = \frac{1}{8}$ (1)

$4q2 = 1$ and $4q4 = 2 \rightarrow q = \frac{1}{8}$ (1)

5

b) (1) $r_0 = Ax_0 - b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -2.5 & 1 & 0 \\ 0 & 1 & -2.5 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ (1) $= \begin{pmatrix} 0 \\ -1.5 \\ -3 \\ 0 \end{pmatrix} \leftarrow \|r_0\|_\infty = 3$ (1)

(2) diag dom: row 1 $\rightarrow \frac{0}{1}$ row 2 $\rightarrow \frac{2}{2.5}$ row 3 $\rightarrow \frac{2}{2.5}$ row 4 $\rightarrow \frac{0}{1}$

$\frac{4}{5}$ (1)

$\left(\frac{4}{5}\right)^n < 0.01$ (1) $\rightarrow n \geq 21$ (1) iterations

4

c) (1) $\left.\begin{array}{l} \hat{x}_1 = \frac{1}{1}(0 - 0(1) - 0(2) - 0(3)) = 0 \\ \hat{x}_2 = \frac{1}{-2.5}(1 - 1(0) - 1(2) - 0(3)) = 2/5 \\ \hat{x}_3 = \frac{1}{-2.5}(2 - 0(0) - 1(1) - 1(3)) = 4/5 \\ \hat{x}_4 = \frac{1}{1}(3 - 0(0) - 0(1) - 0(2)) = 3 \end{array}\right\}$ (2) $\left.\begin{array}{l} x_1 = \frac{1}{2}(0) + \frac{1}{2}(0) = 0 \\ x_2 = \frac{1}{2}\left(\frac{2}{5}\right) + \frac{1}{2}(1) = 7/10 \\ x_3 = \frac{1}{2}\left(\frac{4}{5}\right) + \frac{1}{2}(2) = 7/5 \\ x_4 = \frac{1}{2}(3) + \frac{1}{2}(3) = 3 \end{array}\right\}$ (1)

(2) Jacobi already converges $\rightarrow \omega = \frac{1}{2}$ makes it slower (1)

6 a) 5

(1) $0 \quad \frac{1}{2} \quad 1 \qquad \Delta x = \frac{1}{2} \qquad R = \frac{k\Delta t}{(\Delta x)^2} < \frac{1}{2} \qquad \Delta t < \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2}{1/2} = 1/4$ (1)

(2) $\phi_i^{n+1} - \phi_i^n \stackrel{(1)}{=} R(\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n) + a\phi_i^n \Delta t$

$\Rightarrow \phi_i^{n+1} \stackrel{(1)}{=} R\phi_{i+1}^n + \underbrace{(1 - 2R + a\Delta t)}_{>0}\phi_i^n + R\phi_{i-1}^n \quad (*)$

$> 0 \longrightarrow R < \frac{1 + a\Delta t}{2}$ (1)

(3) $R = \frac{k}{(\Delta x)^2}\Delta t = 2\Delta t \Rightarrow 2\Delta t < \frac{1}{2} + \frac{1}{2}\Delta t \Rightarrow \Delta t < \frac{1}{3}$ (1)

b) $R = \frac{k}{(\Delta x)^2}\Delta t = \frac{0.5}{(0.5)^2}2(1) = 2$ (1)

4

$\phi^{n+1}_{(x=0.5)} \stackrel{(*)}{=} 2(1) + (1 - 2(2) + 1(1))\left(\frac{1}{2}\right) + 2(0) = 1$

(2) (1)