university of groningen

Exam Numerical Methods

November 8th 2022 18:15-20:15 (**2 hours**)

It is allowed to use a book (paper version only) and lecture notes, as well as a (graphical) pocket calculator. The use of electronic devices (tablet, laptop, mobile phone, etc.) is not allowed.

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: $\boxed{10}$

Practica: $\boxed{18}$ For the 6 computer practica a maximum of 6*3=18 points can be earned.

1. Consider the equation $\sin(\pi x) + x = 5,$ with solution x in $[4\ ,\ 4.5]$.

   (a) $\boxed{5}$ (1) Give the iteration formula when Newton's method is used for this problem.
   (2) Use this method to compute x_1; start with $x_0 = 4$, and give an error estimate.

   (b) $\boxed{8}$ Consider the iterative method $x_{n+1} = x_n + \alpha(\sin(\pi x_n) + x_n - 5),$ with $x_0 = 4.25$.
   For $\alpha = -0.5$ the first iterations are given by

n	0	1	2	3	4
x_n	4.25	4.27144661	4.25916893	4.26599503	4.26213187

   (1) Will this method give fast convergence? Explain.
   (2) Suppose the convergence is linear. Determine an improved value (for x_4) via extrapolation, using a technique similar as for Euler's method (o.d.e.'s). Does this approach enhance the accuracy in general? Explain.
   (3) Determine the optimal value for α, based on the initial value $x_0 = 4.25$.

2. Consider the integral $I_p = \displaystyle\int_0^2 \frac{1}{x^p + 4}\,dx,$ with values $I_1 = \ln(\frac{3}{2})$ and $I_2 = \frac{\pi}{8}$.

   (a) $\boxed{6}$ (1) Take $p = 1$. Approximate I_1 on a grid with 2 segments, using both the Midpoint method and the Trapezoidal method.
   (2) Give an error bound for the Trapezoidal value, using the "global error" theorem.

   For $p = 2$, using the Trapezoidal method the following results are obtained on a number of subsequently refined grids (indicated with segments n)

n	8	16	32	64
$I_2(n)$	0.3923 7356	0.3926 1770	0.3926 7874	0.3926 9400

   (b) $\boxed{8}$ (1) Explain that an extrapolation for $I(64)$ is allowed.
   (2) How many segments (use powers of 2) are required for an accuracy of 1.0E-8?
   (3) Compute the improved solution for $I(64)$ by means of extrapolation.
   (4) How many function evaluations are needed for the extrapolation of $I(64)$, in order to compute it in the most efficient way?

P.T.O.

3. Consider the differential equation $y'(x) = 4(x - y)$, with boundary condition $y(0) = 2$. The exact solution is $y(x) = \frac{9}{4}e^{-4x} + x - \frac{1}{4}$.

(a) $\boxed{6}$ (1) Use explicit Euler, with two steps of $\Delta x = 0.25$, to approximate $y(0.5)$.
(2) Use Heun's RK2 method, with one step of $\Delta x = 0.5$, to approximate $y(0.5)$.
(3) Which result for $y(0.5)$ is more accurate? How come?

(b) $\boxed{8}$ For x in $[0\ ,\ 2]$, the solution is determined using a 3rd(!) order RK method. The result at a selection of x locations, for a number of grids, is as follows

x_n	$\delta x = 1/4$	$\delta x = 1/8$	$\delta x = 1/16$	$\delta x = 1/32$
0.0	2.0000 0000	2.0000 0000	2.0000 0000	2.0000 0000
0.5	0.5000 0000	0.5497 8477	0.5540 2005	0.5544 4961
1.0	0.7777 7778	0.7899 4262	0.7910 7920	0.7911 9536
1.5	1.2530 8642	1.2553 2186	1.2555 5062	1.2555 7418
2.0	1.7503 4294	1.7507 0907	1.7507 5000	1.7507 5425

(1) Is there a stability limit visible? Explain.
(2) The stability region for RK3 is between that of RK2 and RK4.
Determine the stability limit (on the safe side) for this equation.
(3) Compute the q-factor at $x = 1.0$. What is your conclusion?
(4) Which δx (powers of 2) is needed for an accuracy of 1.0E-8 everywhere?
Hint: take into account the result obtained at (3).

4. Consider the coordinate points $f(0) = 2$, $f(\frac{1}{2}) = 10$, $f(1) = 24$. To approximate $f(\frac{3}{2})$, we will use a straight-line fit $y = a + bx$ through the given points.

(a) $\boxed{7}$ (1) Determine a coordinate transformation such that the new x-points are -1,0,1.
(2) Set up a least-squares fit through the original data.

(b) $\boxed{3}$ (1) Use the curve-fit to approximate $f(\frac{3}{2})$.
(2) Determine the error for $x = 1$.

5. Consider the equation $A\vec{x} = \vec{b}$, with

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -2.5 & 1 & 0 \\ 0 & 1 & -2.5 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

(a) $\boxed{5}$ The equation is related to the differential eqn. $y''(x) = p\,y(x) + q\,x$ on $[0\ ,\ 6]$, with boundary conditions $y(0) = a$ and $y(6) = b$. Determine p, q, a, b.

(b) $\boxed{5}$ (1) Compute $\|r^{(0)}\|_\infty$, i.e. the max.-norm of the initial residual for $\vec{x}_0 = (0\ 1\ 2\ 3)^T$.
(2) How many Jacobi iterations are needed to reduce this error with a factor 100?

(c) $\boxed{4}$ Similar to the SOR method (i.e. Gauss-Seidel with relaxation ω), relaxation can also be applied during the Jacobi method. This approach is called JOR.
(1) Perform one JOR iteration with $\omega = \frac{1}{2}$, and initial vector $\vec{x}_0 = (0\ 1\ 2\ 3)^T$.
(2) Does it make sense to use JOR with $\omega = \frac{1}{2}$ for the given $A\vec{x} = \vec{b}$? Explain why.

6. Consider for x in $[0\ ,\ 1]$ the partial diff. eqn. $\phi_t = 0.5\,\phi_{xx} + a\,\phi$, with $a = 0$ or $a = 1$.
The boundary and initial conditions are: $\phi(0, t) = 0$, $\phi(1, t) = 1$, $\phi(x, 0) = x$.
For $\partial^2/\partial x^2$ the usual [1 -2 1]-formula is used.
Use a grid size $\Delta x = 0.5$ (i.e. 2 segments, 3 grid points in total).

(a) $\boxed{5}$ (1) For which time step Δt will the explicit Euler method be stable if $a = 0$?
(2) Determine how the stability limit depends on a.
(3) For which time step Δt will the explicit Euler method be stable if $a = 1$?

(b) $\boxed{4}$ Let $a = 1$. Give the solution at $x = 0.5$ after 1 time step of $\Delta t = 1$ with explicit Euler.

Total: $\boxed{100}$