Introduction to Image Processing

Lecturer:
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Week 4 – Morphological Filters

November 20, 2024

Basics of morphological filters

Morphological Image Processing

- emphasis on *shapes* in spatial image domain

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- *set-theoretical* approach to analysis of *binary images*

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- *set-theoretical* approach to analysis of *binary images*
- our constraining focus: discrete space and operations
- object-shape intuition → algebra of image operations → algorithms
- treatment of both binary- and *scalar* images

Sets

Set-theory symbolics

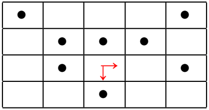
union	$\cup$
intersection	$\cap$
set inclusion	$\subseteq$
element of	$\in$
universal set	E
empty set	$\varnothing$
set difference	$A \backslash B = \{a \in A : a \notin B\}$
complement	$A^c = E \backslash A$

Set Algebra

Recall the basic algebra rules for sets:

$$
\begin{aligned}
A \cup B &= B \cup A \\
A \cap B &= B \cap A \\[6pt]
(A \cup B) \cup C &= A \cup (B \cup C) \\
(A \cap B) \cap C &= A \cap (B \cap C) \\[6pt]
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\[6pt]
(A \cup B)^c &= A^c \cap B^c \\
(A \cap B)^c &= A^c \cup B^c \\[6pt]
(A \subseteq B \ \wedge \ B \subseteq A) &\Leftrightarrow A = B
\end{aligned}
$$

Binary images as sets

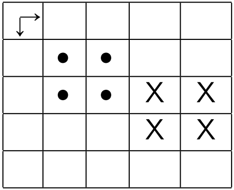


Set of *kernel* pixels:

$$\{(-2,-2),\ (2,-2),\ (-1,-1),\ (0,-1),\ (1,-1),\ (-1,0),\ (2,0),\ (0,1)\}$$

Note: the origin $(0,0)$ is denoted as $\ulcorner\!\!\rightarrow$

Set translation



Set X (black dots) and its shift X_h (crosses) by $h = (2, 1)$.

$$X_h = \{x + h : x \in X\},$$

is the *translate* of X over the vector $h \in \mathbb{E}$. $\mathbb{E}$ is the Euclidean space.

Notation

Let X, A be two sets, combined with a morphological operation op.

$$X \text{ op } A = \{x \text{ op } a : x \in X, a \in A\}$$

Then, the notation follows:

$$\begin{aligned} X &: \quad \text{(binary) image} \\ A &: \quad \textit{structuring element (SE)}\text{; kernel} \\ \text{op} &: \quad \text{morphological operation} \end{aligned}$$

Note: the size of the structuring element A can differ from the size of image X.

Binary erosion- and dilation filters

Minkowski operations: addition

Let X, A be sets.

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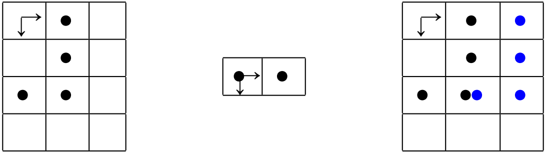

(left) binary image $X = \{(0,2),\ (1,0),\ (1,1),\ (1,2)\}$
(middle) structuring element $A = \{(0,0),\ (1,0)\}$.
(right) $X \oplus A : x \in X, a \in A$;
black dots are $X + a_0$, blue dots are $X + a_1$.

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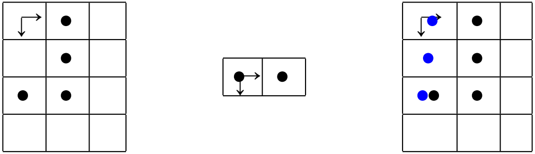

(left) binary image $X = \{(0,2),\ (1,0),\ (1,1),\ (1,2)\}$
(middle) structuring element $A = \{(0,0),\ (1,0)\}$.
(right) $X \ominus A : x \in X, a \in A$;
black dots are $X - a_0$, blue dots are $X - a_1$. The result includes only cells with *both* dots $\rightarrow X \ominus A = \{(0,2)\}$.

Minkowski operations: dilation & erosion

- *Dilation* by structuring element A:

$$\delta_A(X) = X \oplus A$$

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- *Dilation* by structuring element A:

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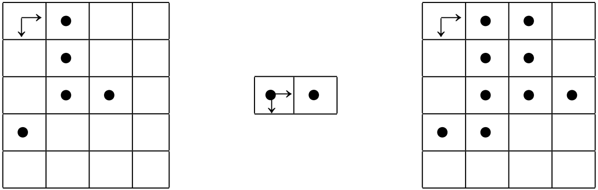
- *Erosion* by structuring element A:

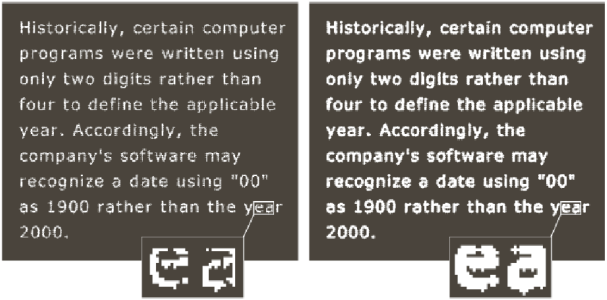
$$\varepsilon_A(X) = X \ominus A$$

Dilation: discrete case example



(left) binary image X.
(middle) structuring element A.
(right) dilation of X by A.

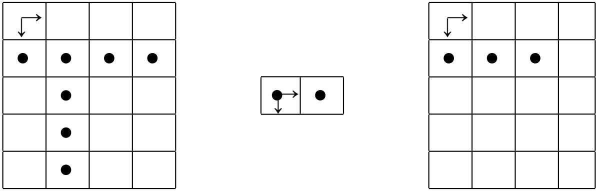
Image example: dilation



0	1	0
1	1	1
0	1	0

(left) input. (right) dilation of input by 'cross' structuring element.

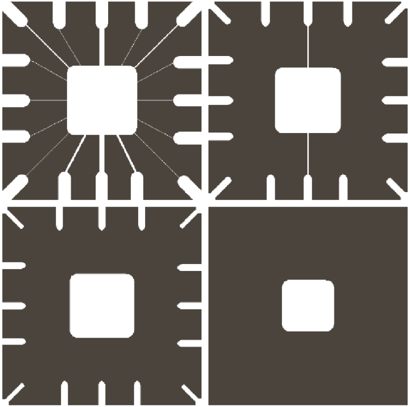
Erosion: discrete case example



(left) binary image X.
(middle) structuring element A.
(right) erosion of X by A.

Image example: erosion

Erosion with *square* structuring element ($a \times b$ kernel, $x = 1 \; \forall x \in X$, center origin)



(top-left) Input. (top-right) erosion, kernel 11×11.
(bottom-left) erosion, kernel 15×15. (bottom-right) erosion, kernel 45×45.

Erosion of a set by itself

$$X \ominus A = \{h \in E : A_h \subseteq X\}$$

As a consequence, the erosion of some set X using itself as structuring element equals the origin $(0, 0)$:

$$X \ominus X = \{h \in E : X_h \subseteq X\} = \{(0, 0)\}$$

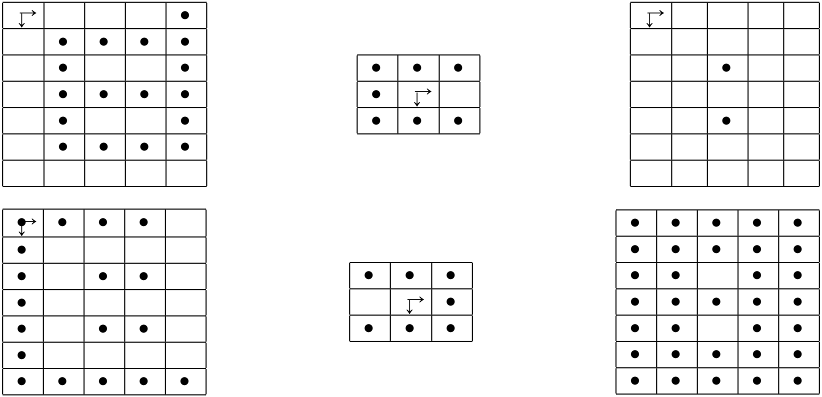
Duality w.r.t. set-complementation

Let X^c denote the complement of X, and $\check{A}$ the reflect (i.e. x-y axis mirror) of A. Then:

$$X \oplus A = (X^c \ominus \check{A})^c$$

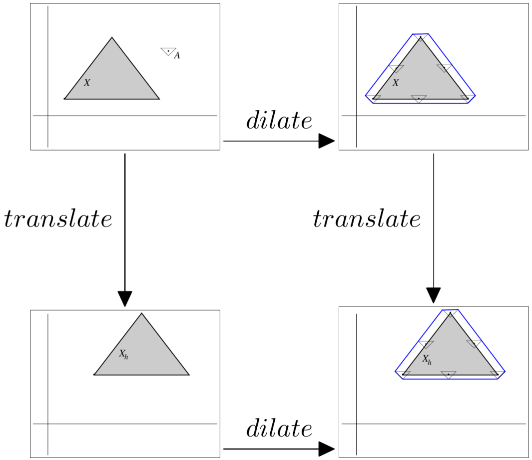
In words: *dilating* an image by A gives the same result as *eroding* the *background* by $\check{A}$ and taking the complement.

Duality: example for erosion



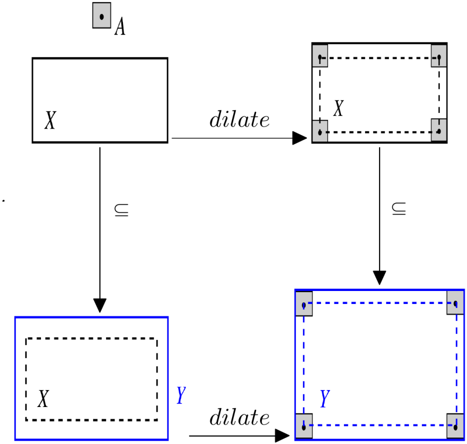
Top: (left) binary image X; (middle) structuring element A; (right) erosion of X by A.
Bottom: (left) binary image X^c; (middle) reflected structuring element $\check{A}$; (right) dilation of X^c by $\check{A}$.

Translation invariance



$$(X \oplus A)_h = X_h \oplus A$$

Scaling subsets



$$X \subseteq Y \implies X \oplus A \subseteq Y \oplus A$$

Boundary extraction via morphology

Let B be a structuring element containing the origin. The *morphological boundary* of A is defined by:

$$\beta(A) = A \backslash (A \ominus B)$$

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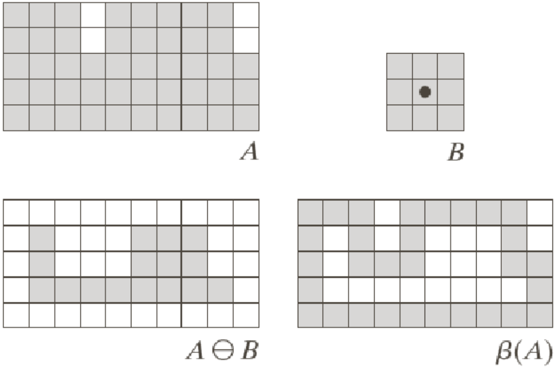

Boundary extraction

Algebraic properties

$X \oplus A = A \oplus X$ *commutativity*

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$$X \oplus A = A \oplus X \qquad \textit{commutativity}$$

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X \oplus A = A \oplus X & \textit{commutativity} \\
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(X \cup Y) \oplus A = (X \oplus A) \cup (Y \oplus A) & \textit{distributivity} \\
(X \cap Y) \ominus A = (X \ominus A) \cap (Y \ominus A) & \textit{distributivity}
\end{array}
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$X \subseteq Y \implies X \oplus A \subseteq Y \oplus A$	*increasing*
$X \subseteq Y \implies X \ominus A \subseteq Y \ominus A$	*increasing*
$(X \ominus A) \ominus B = X \ominus (A \oplus B)$	*iteration*

Decomposition of a structuring element

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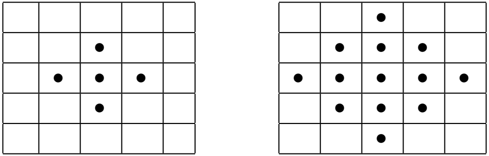

Left: the set $B_0 = B$. Right: the set B_1.

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where L_r is a row of k 1-pixels,

$$L_r = \{(0,0),\ (1,0),\ (2,0), \ldots, (k-1,0)\}$$

and L_c is a column of k 1-pixels,

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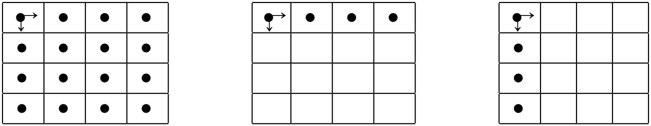

Left: square B_1. Middle: the set L_r. Right: the set L_c.

Combined morphological filters - opening and closing

Opening

The *opening* γ_A is an erosion followed by a dilation.

$$\gamma_A(X) = X \circ A := (X \ominus A) \oplus A$$

That is, $\gamma_A = \delta_A \varepsilon_A$.

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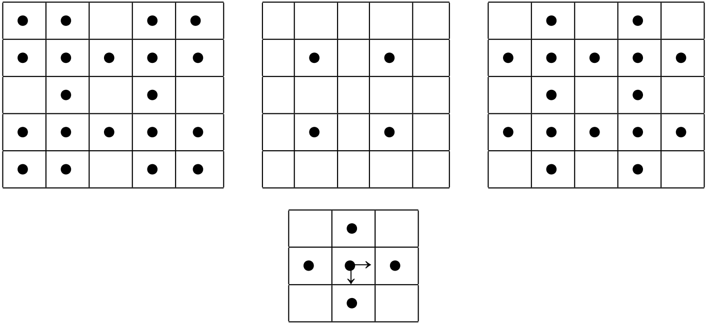

(a) image X. (b) $\delta_A = X \ominus A$. (c) $\gamma_A = \delta_A \oplus A$. (d) structuring element A.

Closing

The *closing* ϕ_A is a dilation followed by an erosion.

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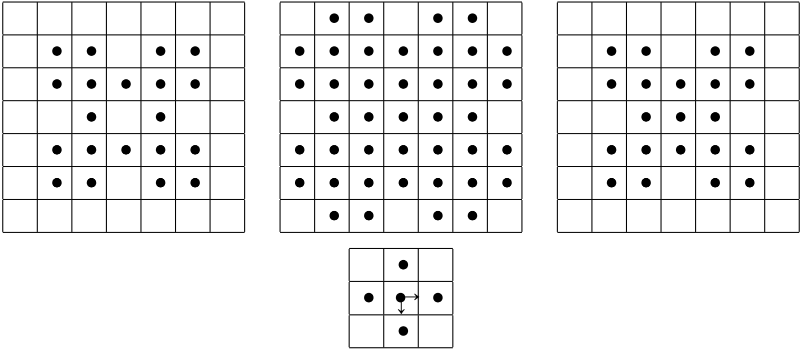
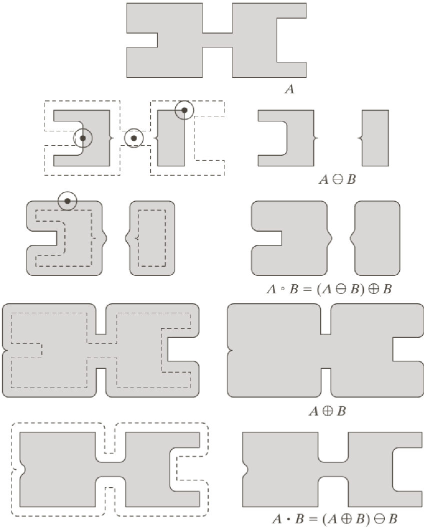


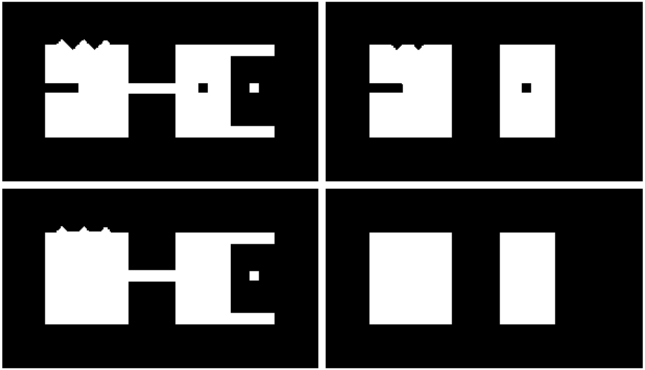
(a) image X. (b) $\varepsilon_A = X \oplus A$. (c) $\phi_A = \varepsilon_A \ominus A$. (d) structuring element A.

Opening and closing - implications

- An opening smooths contours, cuts narrow bridges, removes small islands and sharp corners.
- A closing fills narrow channels and small holes.

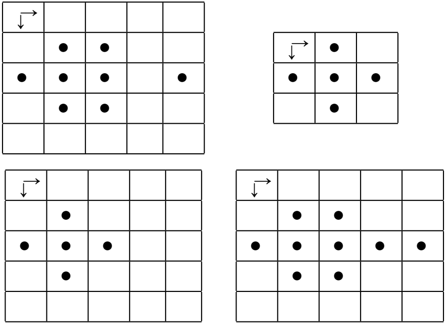


Opening and closing: example



Upper left: Original. Upper right: Opening.
Lower left: Closing. Lower right: Closing of the opening.
The structuring element is a square.

Opening & Closing: discrete-case examples



Upper left: binary image X. Upper right: structuring element A.
Lower left: *opening* of X by A. Lower right: *closing* of X by A.

Example


A
$A \ominus B$
B
1 1 1
1 1 1
1 1 1
$(A \ominus B) \oplus B = A \circ B$
$(A \circ B) \oplus B$
$[(A \circ B) \oplus B] \ominus B = (A \circ B) \bullet B$

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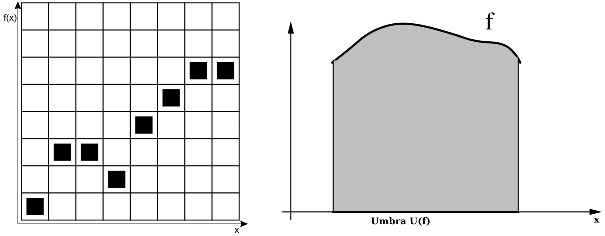
  2. *idempotent*: $\phi_A(\phi_A(X)) = \phi_A(X)$
  3. extensive: for every X, $\phi_A(X) \supseteq X$

Morphological operations on scalar functions (1D) and images (2D)

Grayscale morphology - 1D

We start the explanation by looking at morphological filtering of *digitized* 1D functions $f(x)$.

$$f = f(x) : x \in X$$



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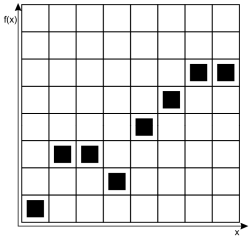
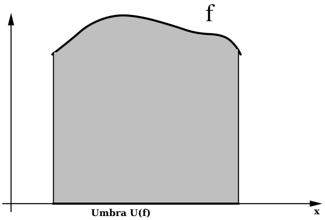
We start the explanation by looking at morphological filtering of *digitized* 1D functions $f(x)$.

$$f = f(x) : x \in X$$





In order to dilate or erode the *curve*, we require the *integral* under the curve. This is what we call *the umbra*.

$$U(f) = \int f(x)$$

The Umbra function

Umbra means shadow.

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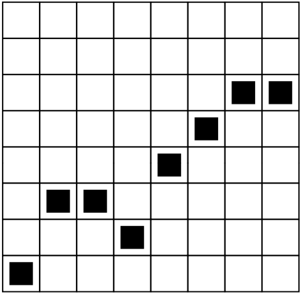
Umbra of a **set** X includes both X and the volume of units in its "shadow".

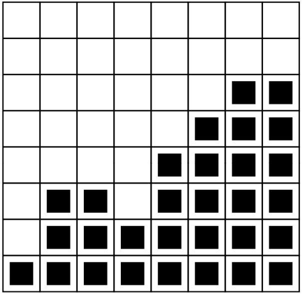
Umbra function: 1D

Formally, the umbra is a mapping $\mathbb{Z}^1 \rightarrow \mathbb{Z}^2$ of a 1D function $f(x)$ where

$$y_{max} = f(x)$$
$$U(f(x)) = \{(x, y) : y \leqslant y_{max}\}$$



f

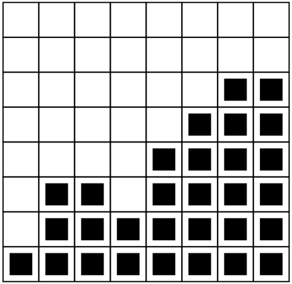


$U(f)$

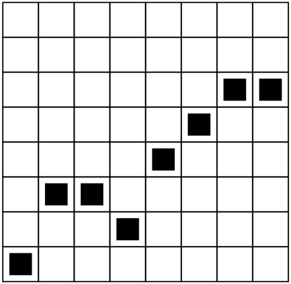
Top function: 1D

Let R be set of samples in $\mathbb{Z}^2$. Then the top funcion is a mapping $\mathbb{Z}^2 \rightarrow \mathbb{Z}^1$, where

$$R = f(x, y)$$
$$T(R) = \{y : f(x, y) > f(x, y + 1)\}$$



R



$T(R)$

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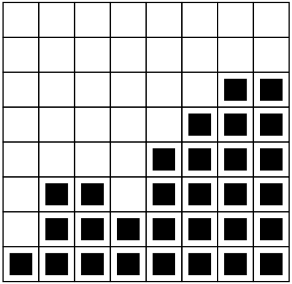

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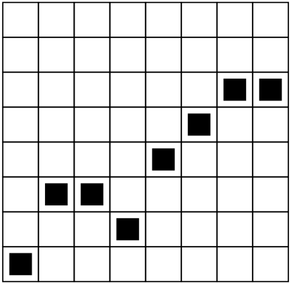
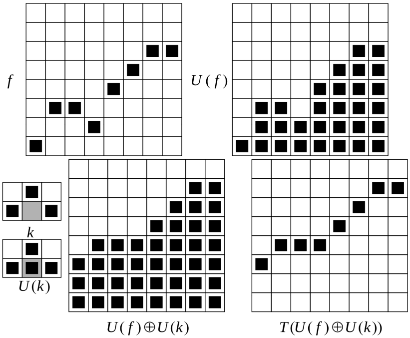


$T(R)$

So, we have $T(U(f)) = f$.

Gray level dilation: 1D

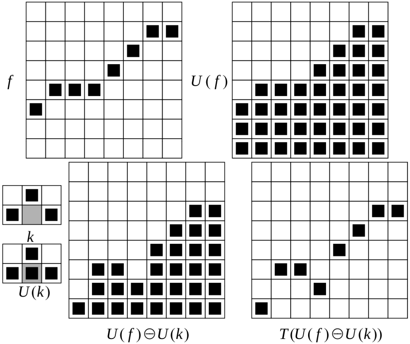
- grayscale dilation: $f \oplus_{gray} k = T(U(f) \oplus U(k))$



Note: kernel k above is meant as an illustration, not as a value equivalent.

Gray level erosion: 1D

- grayscale dilation: $f \oplus_{gray} k = T(U(f) \oplus U(k))$
- grayscale erosion: $f \ominus_{gray} k = T(U(f) \ominus U(k))$



Note: kernel k above is meant as an illustration, not as a value equivalent.

Gray level opening & closing: 1D

- grayscale dilation: $f \oplus_{gray} k = T(U(f) \oplus U(k))$
- grayscale erosion: $f \ominus_{gray} k = T(U(f) \ominus U(k))$
- grayscale opening/closing: as before using $\oplus_{gray}$ and $\ominus_{gray}$.

From now on we simply write $\oplus$ instead of $\oplus_{grey}$. The meaning of $\oplus$ and $\ominus$ depend on the context.

Gray level dilation/erosion: 1D

Let K be the domain of the function k.

- grayscale dilation:
  $(f \oplus_{gray} k)(x) = (T(U(f) \oplus U(k)))(x)$

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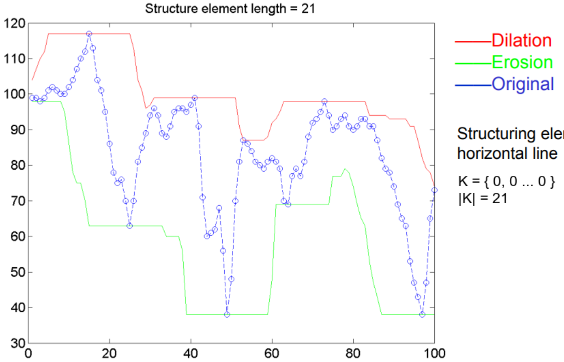
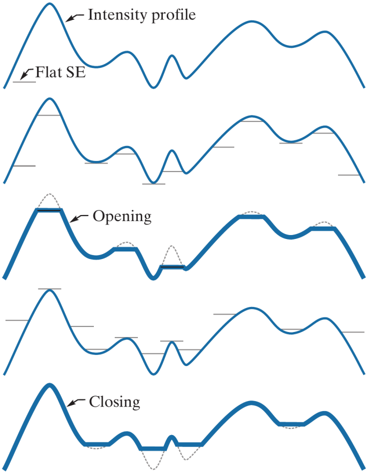


Structuring element:
horizontal line

K = { 0, 0 ... 0 }
|K| = 21

Flat 1-D opening and closing



Scalar image dilation & erosion: 2D

Let $f : F \rightarrow G$ be a gray value image with domain F and $b : B \rightarrow G$ a structuring element with domain B. The gray value range is G.

Scalar image dilation & erosion: 2D

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$$(f \oplus b)(x, y) = \max_{(s,t)\in B} [f(x - s, y - t) + b(s, t)]$$

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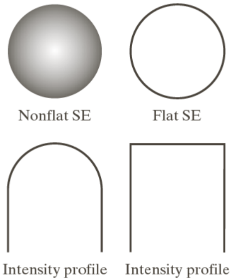
$$(f \ominus b)(x, y) = \min_{(s,t) \in B} \left[f(x + s, y + t) - b(s, t) \right]$$

Structuring functions: 2D



- For processing a gray value image $f(x, y)$ one can use another gray value image $b(x, y)$, called a *nonflat* structuring function (or element).
- If $b(x, y) = constant$ we call it a *flat* structuring function (or element).

Flat scalar image dilation & erosion: 2D

In practice, the function b is replaced by zero-constant $b : B \to \{0\}$, i.e. a flat structuring element with value 0. This results in:

- The *flat gray value dilation* of f by b is defined as

$$(f \oplus b)(x, y) = \max_{(s,t)\in B} [f(x - s, y - t)]$$

(*local maximum filter*)

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- associated with *gray value smoothing*, defined as

$$(f \tilde{,} b)(x, y) = \operatorname*{median}_{(s,t)\in B} [f(x + s, y + t)]$$

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- those three operations compose the *rank-order filters*

Flat Gray value dilation and erosion

Original

Dilation

Erosion

Gray value opening and closing

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- The *gray value closing* of f by b is the function $f \bullet b$ defined by $f \bullet b = (f \oplus b) \ominus b$

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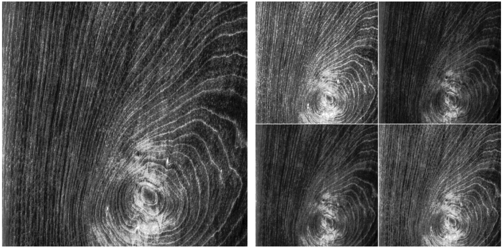

Left: a gray value image. Right: upper left: gray value dilation; upper right: gray value erosion; lower left: gray value opening; lower right: gray value closing. (*flat* structuring element)

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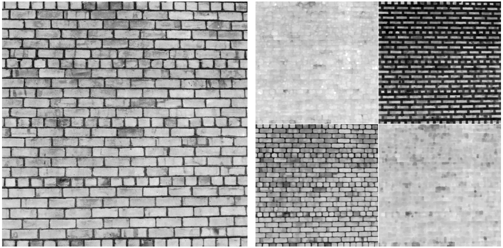
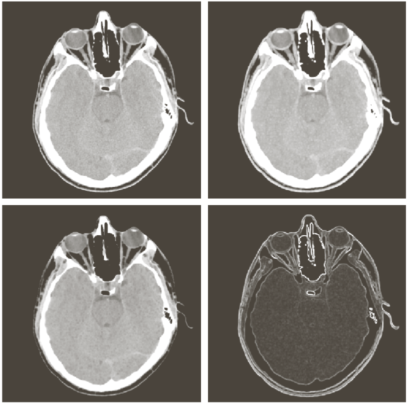


Left: a gray value image. Right: upper left: gray value dilation; upper right: gray value erosion; lower left: gray value opening; lower right: gray value closing. (*flat* structuring element)

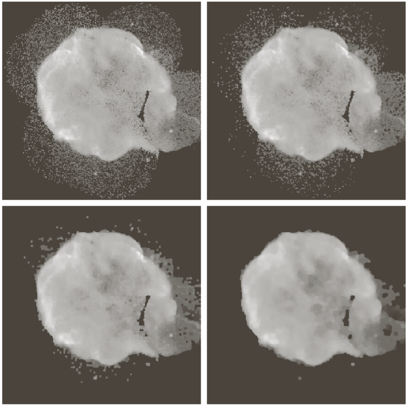
Morphological gradient

$$g = (f \oplus b) - (f \ominus b)$$



(a) CT image. (b): Dilation. (c) Erosion. (d) Morphological gradient.

Morphological smoothing



(a) Image of supernova.
(b)-(d): opening-closing of (a) with flat disk of radius 2, 3, 5.

Top/bottom-hat filter

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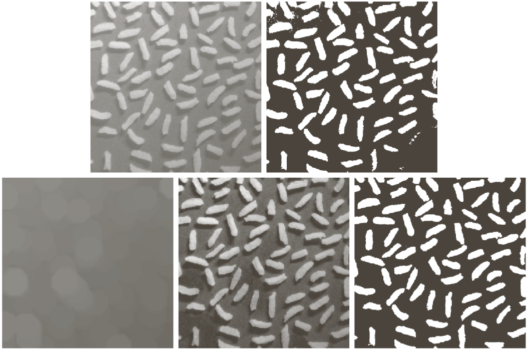
- Useful for correcting uneven illumination.



(a) Input. (b) Thresholded image. (c) Opening of (a) by disk of radius 40.
(d) Top-hat of (a). (e) Thresholded top-hat.

That's it for this week!