Homework 2
(due on Tuesday 21 May)

Version: April 24, 2024

Chapters covered: from 1 to 9.
Upload your solutions on Brightspace until **Tuesday 21 May, 23:59**. If you hand in late then that will count as not handed in.
For each problem you need to justify your result, in a clear way. If you provide a numerical result without explanations then your answer will be ignored.
There is a total of 5+4+2+6=17 points. If it counts in your final grade (depending on the result of your exam, as explained on Brightspace), it will be worth 10% of the final grade.

Problem 1; 5 points In a small town, there is a yearly tradition where three contestants are chosen to participate in a series of challenges. Each contestant is given a unique challenge, and either succeeds or fails. The challenges vary in difficulty, and are not necessarily identical for each contestant. The success of one contestant might influence the success of the others.

The town's historian has kept records of the event for many years, and has found that on average, 1.8 of the three contestants succeed in their challenges each year. This year, three new contestants - Alice, Bob, and Charlie - are stepping up to the plate. Let X_1, X_2, X_3 be random variables that represent the success of the trials of Alice, Bob and Charlie (taking value 1 if success, 0 otherwise). Let $X = X_1 + X_2 + X_3$ denote the number of successes among Alice, Bob, and Charlie.

Given the historical average, we know that $\mathbb{E}[X] = 1.8$, but we do not know the joint distribution of (X_1, X_2, X_3).

(a) What is the largest possible value of $\mathbb{P}(X = 3)$?

(b) What is the smallest possible value of $\mathbb{P}(X = 3)$?

(c) What is the answer to these questions if the chances of success of the contestants are independent and identically distributed?

Remark: In both cases, you need to justify your answer. That means that you need to give an argument why the value can not be larger or cannot be smaller, and construct an example of a (joint) distribution of the three trials for which the claimed value is obtained (and the expectation of X is 1.8).
Hint: Let $p_i = \mathbb{P}(X_i = 1)$. You might want to find a relation between the expectation of X and the numbers p_1, p_2 and p_3. You might want to find a relation between $\mathbb{P}(X = 3)$ and the numbers p_1, p_2 and p_3 (this might be in term of their minimum or maximum).

Problem 2; 4 points Two players take turns shooting at a target, with each shot by player i hitting the target with probability p_i, and miss with probability $q_i = 1 - p_i$, $i = 1, 2$. We assume that the players' shots are (mutually) independent. Shooting ends when two consecutive shots hit the target (i.e. either when player 1 hits the target right after player hits it, or the other way round).

If player 1 starts, we denote N_1 the number of shots taken. Similarly, if player 2 starts, we denote N_2 the number of shots taken.

Let $\mu_i = \mathbb{E}[N_i]$ denote the mean number of shots taken when player i shoots first, $i = 1, 2$. Let H_i denote the event $\{$the i-th shot is a hit$\}$.

(a) Express $\mathbb{E}[N_1 | H_1 \cap H_2]$, $\mathbb{E}[N_1 | H_1 \cap H_2^c]$ and $\mathbb{E}[N_1 | H_1^c]$ in terms of $\mu_1, \mu_2, p_1, p_2, q_1, q_2$.
Remark: You don't need to necessarily need to use all the parameters. Don't forget to give a concise explanation of your reasoning.

(b) Show that $\mu_1(1 - p_1 q_2) = 1 + p_1 + \mu_2 q_1$.
Hint: You might want to prove first that $\mu_1 = 2p_1 p_2 + (2 + \mu_1) p_1 q_2 + (1 + \mu_2) q_1$ (using the results of the previous question).

(c) Assume that $p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{4}$. Find μ_1 and μ_2.
Hint: First, derive formula for μ_2 similar as the one of the previous question (no justification needed for this step); then solve a linear system of equations.

Problem 3; 2 points. Let X_1, X_2, X_3 and X_4 be independent continuous random variables with a common distribution function F. Find the probability

$$p = \mathbb{P}(X_1 < X_2 > X_3 < X_4).$$

Hint: You might want to invoke some combinatorial reasoning. OR you might want to consider the random variables $Y_i = F(X_i)$, explicit their distribution and compute the probability via integration.
Remark 1: In case this needs clarification, the event $\{X_1 < X_2 > X_3 < X_4\}$ is the intersection of the events $\{X_1 < X_2\}$, $\{X_2 > X_3\}$, and $\{X_3 < X_4\}$.
Remark 2: Don't forget to give a clear explanation of your reasoning.
Remark 3: You are allowed to assume that F is strictly increasing on some interval $[a,b]$ and constant outside of this interval. (this is not needed in principle, but it can make justifications easier)

Problem 4; 6 point A drone is programmed to randomly drop a package within a predefined unit square area (considered the drop zone). The drone has been programmed in such a way that it's more likely to drop the package towards the upper right corner of the drop zone.

Let's denote the drop location by the random variables X and Y, representing the horizontal and vertical coordinates of the drop location, respectively. The joint probability density function of X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{5}(2xy + y^2), & 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Check that f is indeed a joint pdf.

(b) Find the marginal density function of X.

(c) Find the marginal density function of Y.

(d) Prove or disprove that X and Y are independent.

(e) Compute the probability $\mathbb{P}(X < Y)$.

(f) Compute the expectation $\mathbb{E}[X^2 Y]$.