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Answers

1. $\boxed{\text{20 points}}$ **Conic sections**

   (a) $\boxed{\text{10 points}}$ Consider the cone

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\}$$

in $\mathbb{R}^3$ and let P be a 2-plane in $\mathbb{R}^3$ not passing through the origin. Show that every conic section $P \cap C$ is either an ellipse, parabola or hyperbola, i.e., $P \cap C$ is given by one of the following equations in suitable Cartesian coordinates on the 2-plane P:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (ellipse), \qquad y^2 = ax \quad (parabola) \qquad or \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (hyperbola)$$

   (b) $\boxed{\text{10 points}}$ Prove that conic sections arise as envelopes in the following way. Let C be a unit circle in $\mathbb{R}^2$ and let $A \in \mathbb{R}^2$ be a fixed point. Consider the family of perpendicular bisectors to the line segment AB where the point B traverses C. Prove that the envelope to this family of perpendicular bisectors is an ellipse or an hyperbola depending on whether A is inside or outside of C.

- Part 1

- Part 2

Given a unit circle $C : x^2 + y^2 = 1$ parametrised such that $x = r \cdot \cos(\theta), y = r \cdot \sin(\theta)$, we select two points A, B, for which B traverses C. In other words, $B = (r\cos(t), r\sin(t)$ and $A = (x_0, y_0)$.

Since we don't know if A is inside or outside of the circle, either $x_0^2 + y_0^2 > 0$ or $x_0^2 + y_0^2 < 0$.

To build the envelope of the family of perpendicular bisectors first we calculate the line segment $\vec{AB} = (x_0 - r\cos(\theta), x_0 - r\sin(\theta))$, for which the midpoint is $mdpt = (\frac{x_0 + r\cos(\theta)}{2}, \frac{y_0 + r\sin(\theta)}{2})$.

Let's make the set of lines $D_t = \{a(t)x + b(t)y + w(t) = 0\}$. Let us find the equation of the perpendicular bisector, knowing it's perpendicular to $\vec{AB}$ and so their dot product vanishes

$$\begin{aligned}
0 &= (x_0 - r\cos(t))x + (y_0 - r\sin(t))y + w(t) && (1)\\
a(t) &= x_0 - r\cos(t) && (2)\\
b(t) &= y_0 - r\sin(t) && (3)\\
w(t) &= -\frac{1}{2}((x_0 - r\cos(t))^2 + (y_0 - r\sin(t))^2) && (4)\\
&= -\frac{1}{2}(x_0^2 - 2rx_0\cos(t) + y_0^2 - 2ry_0\sin(t) + r^2(1)) && (5)\\
0 &= r\sin(t)x - r\cos(t)y - x_0 r\sin(t) + y_0 r\cos(t) && (6)
\end{aligned}$$

We calculate the envelope solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 - r\cos(t) & y_0 - r\sin(t) \\ r\sin(t) & -r\cos(t) \end{pmatrix}^{-1} \begin{pmatrix} -\frac{1}{2}(x_0^2 - 2rx_0\cos(t) + y_0^2 - 2ry_0\sin(t) + r^2) \\ ry_0\cos(t) - rx_0\sin(t) \end{pmatrix} \tag{7}$$

$$= \frac{1}{-rx_0\cos(t) - ry_0\sin(t) + r^2} \begin{pmatrix} -r\cos(t) & r\sin(t) - y_0 \\ -r\sin(t) & x_0 - r\cos(t) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}(x_0^2 - 2rx_0\cos(t) + y_0^2 - 2rx_0\sin(t) + r^2) \\ ry_0\cos(t) - rx_0\sin(t) \end{pmatrix} \tag{8}$$

(9)

Which results in

$$x(t) = \frac{\frac{r\cos(t)}{2}(x_0^2 - 2rx_0\cos(t) + y_0^2 - 2rx_0\sin(t) + r^2) + (y_0 - r\sin(t))(ry_0\cos(t) - rx_0\sin(t))}{-rx_0\cos(t) - ry_0\sin(t) + r^2} \tag{10}$$

$$= \frac{r\cos(t)(x_0^2 - 2rx_0\cos(t) + y_0^2 - 2rx_0\sin(t) + r^2) + (y_0 - r\sin(t))(ry_0\cos(t) - rx_0\sin(t))}{2r(-x_0\cos(t) - y_0\sin(t) + r)} \tag{11}$$

$$y(t) = \frac{(y_0 - r\sin(t))(x_0^2 - 2rx_0\cos(t) + y_0^2 - 2rx_0\sin(t) + r^2) + (x_0 - r\cos(t))(ry_0\cos(t) - rx_0\sin(t))}{2r(-x_0\cos(t) - y_0\sin(t) + r)} \tag{12}$$

As we can see if $x_0^2 + y_0^2 > r^2$ or $x_0^2 + y_0^2 < r^2$ makes a difference in the $w(t)$

2. $\boxed{\text{15 points}}$ **Envelopes** A family of lines parallel to the axis of a parabola is reflected by this parabola (so that the angle between the reflected ray and the normal is equal to the angle between this normal and the incident ray). What is the envelope of the reflected rays?

3. $\boxed{\text{35 points}}$ **Evolutes** Compute the evolutes and plot the wave-fronts for the following curves in $\mathbb{R}^2$:

   - Hippopede: $(x^2 + y^2)^2 = ax^2 + y^2$
   - Cubic curves: $y^2 + x^3 - ax = b$ where a and b are parameters

   Explain the relation between the wave-fronts and the evolutes (Huygens's principle) for these examples. You may use any software such as Mathematica or Geogebra.

   First let's compute the evolutes for the curves:

   For the hippopede, I first parametrise the curve in polar coordinate such that

$$r^4 = ar^2\cos^2(t) + r^2\sin^2(t) \tag{13}$$
$$r^2 = a\cos^2(t) + \sin^2(t) \tag{14}$$
$$r^2 = a(1 - \sin^2(t)) + \sin^2(t) \tag{15}$$
$$r^2 = a - a\sin^2(t) + \sin^2(t) \tag{16}$$
$$r^2 = a + (1 - a)\sin^2(t) \tag{17}$$
$$r = \sqrt{a + (1 - a)\sin^2(t)} \tag{18}$$

   and therefore I can make the hippopede $\gamma(t)$

$$\gamma(t) = \begin{cases} x(t) = \cos(t)\cdot\sqrt{a + (1 - a)\sin^2(t)} \\ y(t) = \sin(t)\cdot\sqrt{a + (1 - a)\sin^2(t)} \end{cases} \tag{19}$$

   The derivatives of this parametrised curve are

$$x'(t) = \frac{\cos(t)}{2\sqrt{2(a-1)\sin(t)\cos(t)}} - \sin(t)\sqrt{a + (1 - a)\sin^2(t)} \tag{20}$$
$$y'(t) = \frac{\sin(t)}{2\sqrt{2(a-1)\sin(t)\cos(t)}} + \cos(t)\sqrt{a + (1 - a)\sin^2(t)} \tag{21}$$
$$x''(t) = -\frac{\cos(t)\sin(t)}{\sqrt{a + (1 - a)\sin^2(t)}} - \sqrt{a + (1 - a)\sin^2(t)}\cdot\cos(t)(1 - a) \tag{22}$$
$$y''(t) = -\frac{\cos(t)\sin(t)}{\sqrt{a + (1 - a)\sin^2(t)}} + \sqrt{a + (1 - a)\sin^2(t)}\cdot\sin(t)(1 - a) \tag{23}$$

for which the normal is

$$n(t) = \frac{\dot{\gamma}}{||\dot{\gamma}||} = \frac{1}{||\dot{\gamma}||}\begin{pmatrix} \frac{\cos(t)}{2\sqrt{2(a-1)\sin(t)\cos(t)}} - \sin(t)\sqrt{a+(1-a)\sin^2(t)} \\ \frac{\sin(t)}{2\sqrt{2(a-1)\sin(t)\cos(t)}} + \cos(t)\sqrt{a+(1-a)\sin^2(t)} \end{pmatrix} \tag{24}$$

$$||\dot{\gamma}|| = \frac{1}{8a} - \frac{1}{8\sin(t)\cos(t)} + a - (1-a)\sin^2(t)(1+\cos^2(t)) \tag{25}$$

and the curvature is

$$k(t) = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(x^2+y^2)^{3/2}} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(a+\sin^2(t))^{3/2}} \tag{26}$$

We can say that evolute is

$$X(t) = x(t) + \frac{1}{k(s)}n(t) \tag{27}$$

$$Y(t) = y(t) + \frac{1}{k(s)}n(t) \tag{28}$$

where $k(t)$ is the curvature and $n(t)$ is the normal vector.
For the cubic curve first we parametrise the curve such that $x = t, y = \sqrt{b+t^3-at}$. We calculate the derivatives

$$x'(t) = 1, y'(t) = \frac{3t^2-a}{2\sqrt{b+t^3-at}} \tag{29}$$

$$x''(t) = 0, y''(t) = -\frac{3t(3t^2-a)}{4(b+t^3-at)^{3/2}} \tag{30}$$

$$\tag{31}$$

and use them to get the curvature

$$k(t) = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(x^2+y^2)^{3/2}} = \frac{\frac{3t^2-a}{2\sqrt{b+t^3-at}} - 0}{(t^2+b+t^3-at)^{3/2}} = \frac{3t^2-a}{(t^2+b+t^3-at)^{3/2}2\sqrt{b+t^3-at}} \tag{32}$$

We can also get $n(t)$ by doing

$$n(t) = \frac{(-y'(t), x'(t))}{||\dot{\gamma}||} = \frac{1}{\sqrt{1+(\frac{3t^2-a}{2\sqrt{b+t^3-at}})^2}}(-\frac{3t^2-a}{2\sqrt{b+t^3-at}}, 1) \tag{33}$$

$$= \frac{1}{\sqrt{1+\frac{(3t^2-a)^2}{4(b+t^3-at)}}}(-\frac{3t^2-a}{2\sqrt{b+t^3-at}}, 1) = (-\frac{2\sqrt{b+t^3-at}}{(3t^2-a)\sqrt{1+\frac{(3t^2-a)^2}{4(b+t^3-at)}}}, \frac{1}{\sqrt{1+\frac{(3t^2-a)^2}{4(b+t^3-at)}}}) \tag{34}$$

And by that we get evolute

$$X(t) = t + \frac{(t^2+b+t^3-at)^{3/2}2\sqrt{b+t^3-at}}{3t^2-a} \cdot -\frac{2\sqrt{b+t^3-at}}{(3t^2-a)\sqrt{1+\frac{(3t^2-a)^2}{4(b+t^3-at)}}} \tag{35}$$

$$Y(t) = \sqrt{b+t^3-at} + \frac{(t^2+b+t^3-at)^{3/2}2\sqrt{b+t^3-at}}{3t^2-a} \cdot \frac{1}{\sqrt{1+\frac{(3t^2-a)^2}{4(b+t^3-at)}}} \tag{36}$$

Huygens's principle states that the curve f_a must have a singular point for any value of s such that $\rho(s) = a$. The set of these singular points (when a varies) is the evolute of the curve γ.

4. $\boxed{\text{15 points}}$ **Elliptic cubic curves** Consider again the equation of a cubic curve

$$y^2 + x^3 - ax = b$$

where now the variables x, y and the parameters a, b are in the complex affine line $\mathbb{C}$. Prove that such a curve determines a real 2-surface in $\mathbb{C}^2 \simeq \mathbb{R}^4$. Show that such a surface is always connected. When is it non-singular?

Now, when the equation is treated as defining a curve in $\mathbb{C}^2$, it represents a complex curve. However, when considering the real part $\Re(F_{ab}) = 0$ and the imaginary part $\Im(F_{ab}) = 0$, you obtain two real equations, which define a real 2-surface in $\mathbb{C}^2$ (or equivalently, a real 4-dimensional space $\mathbb{R}^4$).

To show that the curve determines a real 2-surface, we can express the equation in terms of real and imaginary parts, such that defining the surface involve only real variables and can be parametrised by real numbers. Let u, v, p, q be real numbers and define x, y such that

$$x = u + iv \tag{37}$$
$$y = p + iq \tag{38}$$
$$b = (p + iq)^2 + (u + iv)^3 - a(u + iv) \tag{39}$$
$$b = p^2 - q^2 + 2ipq + (u^2 - v^2 + 2iuv)(u + iv) - au - iav \tag{40}$$
$$b = p^2 - q^2 + 2ipq + (u^2 - v^2 + 2iuv)(u + iv) - au - iav \tag{41}$$
$$b = p^2 - q^2 + 2ipq + u^3 + iu^2v - iv^3 - uv^2 + 2iu^2v - 2uv^2 - au - iav \tag{42}$$
$$b = p^2 - q^2 + 2ipq + u^3 + i(3u^2v - v^3 - av) - 3uv^2 - au \tag{43}$$

Since the real and imaginary parts can be separated in a way that allows to express the equation as a system of real equations involving only real variables, then we can say that the cubic curve in the complex plane defines a real 2-surface in $\mathbb{R}^4$.

$$\Re : p^2 - q^2 + u^3 - 3uv^2 - au - b = 0 \tag{44}$$
$$\Im : 2ipq + 3iu^2v - iv^3 - iav = 0 \tag{45}$$

To check if the surface is connected we go to the definition of path-connected. For $x, y \in X$, a path in X from x to y:

$$f : [0, 1] \subsetneq \mathbb{E} \to^{cont} X \begin{cases} f(t_0 = 0) = x \\ f(t = 1) = y \end{cases} \tag{46}$$

then that path f joins x and y. In other words, a topological space X is called path-connected if for any two points $x, y \in X$ there exists a continuous map $f : [0, 1] \to X$ such that $f(0) = x, f(1) = y$ (a path from x to y). If X is path-connected, then it is also connected. To prove this for the cubic curve we'll say that $f(t) = (x(t), y(t))$ for $x(t) = t$ and $y(t)^2 = b - x(t)^3 + ax(t) = b - t^3 + at$ which is equivalent to $y(t) = \sqrt{b - t^3 + at}$. Now we can state that $f(0) = (0, \sqrt{b})$ and $f(1) = (1, \sqrt{b - 1 + a})$. Because when we give $x = 0$, we get $y^2 = b$ therefore $y = \sqrt{b}$ and $x = 1, y = \sqrt{b - 1 + a}$ we can say that the mapping is continuous and the surface, path-connected, and therefore, connected.

As for non-singularity, a surface is considered non-singular at a point if the gradient (or Jacobian) of the defining equations is nonzero at that point. So let's analyze the partial derivatives with respect to x and y to check for non-singularity.

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 3u^2 - 3v^2 - a & -6uv \\ 6uv & 3u^2 - a - 3v^2 \end{bmatrix} \tag{47}$$

where F_1 and F_2 are the real and imaginary parts of the equation.

We can assume that the following points will lead to singularities:

- $u = 0, a = -3v^2$ since $3u^2 - a - 3v^2 = 0 + 3v^2 - 3v^2 = 0$ and $6uv = 0$
- $v = 0, a = 3u^2$, similarly as the previous point
- $a = v = u = 0$, trivially will turn all derivatives to 0