Diagonalization of a matrix with all distinct eigenvalues (Textbook, Exercise 4.14)

Let $A \in \mathbb{R}^n \times \mathbb{R}^n$ be a square matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $v_1, \ldots v_n$.

(a) *Show that if the eigenvalues are distinct ($\lambda_i \neq \lambda_j$ for $i \neq j$), then $v_i \neq v_j$ for $i \neq j$.*

Let $v_i = v_j = \overline{v}$ for some pair of indices i, j and some vector $\overline{v}$. Note that by definition of eigenvector $\overline{v} \neq 0$, and $Av_i = \lambda_i v_i$, $Av_j = \lambda_j v_j$. Hence $A\overline{v} = \lambda_i \overline{v}$, and $A\overline{v} = \lambda_j \overline{v}$. By subtracting the two identities above, we obtain $0 = (\lambda_i - \lambda_j)\overline{v}$. Since $\overline{v} \neq 0$, it must be true that $0 = \lambda_i - \lambda_j$, which is a contradiction. Hence, $v_i \neq v_j$.

(b) *Show that if the eigenvalues are distinct ($\lambda_i \neq \lambda_j$ for $i \neq j$), then the eigenvectors are all linearly independent.*

Suppose that the maximum number of linearly independent eigenvectors is m and that $m < n$. Notice that $m \geq 1$ because the eigenvectors are nonzero vectors. Assume without loss of generality that the linearly independent eigenvectors are the first m, namely $v_1, v_2, \ldots, v_m$. Since v_{m+1} must be linearly dependent on $v_1, v_2, \ldots, v_m$, we must have

$$v_{m+1} = \sum_{i=1}^{m} \beta_i v_i, \tag{1}$$

with $\beta = [\beta_1 \ldots \beta_m]^T \neq 0$. Multiply both sides by A to obtain

$$\lambda_{m+1} v_{m+1} = \sum_{i=1}^{m} \beta_i \lambda_i v_i.$$

Also multiply both sides of (1) by λ_{m+1} to obtain

$$\lambda_{m+1} v_{m+1} = \sum_{i=1}^{m} \beta_i \lambda_{m+1} v_i.$$

Subtract the two identities to obtain

$$0 = \sum_{i=1}^{m} \beta_i (\lambda_i - \lambda_{m+1}) v_i$$

Since $\lambda_i - \lambda_{m+1} \neq 0$, this is equivalent to say that the m vectors $v_1, \dots, v_m$ are linearly dependent, which contradicts the assumption that $v_1, \dots, v_m$ are linearly independent. Hence the maximum number of linearly independent vectors must be $m = n$.

(c) *Let $T = [v_1 \dots v_n]$ and show that $T^{-1}AT$ is a diagonal matrix.*

By definition $Av_i = \lambda_i v_i$ for all i. In matrix form, these identities write as

$$[Av_1 \dots Av_n] = [\lambda_1 v_1 \dots \lambda_n v_n],$$

or, equivalently,

$$A[v_1 \dots v_n] = [v_1 \dots v_n] \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}.$$

Hence, it has been shown that

$$AT = T\Lambda.$$

Being a matrix whose columns are all eigenvectors, T is nonsingular (see (b)) and therefore, multiplying both sides by T^{-1}

$$T^{-1}AT = \Lambda.$$

(d) *Show that it can always be found a nonsingular matrix $\hat{T}$ such that*

$$\hat{T}^{-1}A\hat{T} = \begin{bmatrix} \Lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Lambda_k \end{bmatrix},$$

with $\Lambda_i = \lambda_i \in \mathbb{R}$ or

$$\Lambda_i = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}.$$

It is enough to show that for a (2×2) matrix A with complex eigenvalues $\sigma \pm i\omega$, there exists a nonsingular matrix $\hat{T}$ such that

$$\hat{T}^{-1} A \hat{T} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}.$$

It is known from question (c), that there exists a matrix T such that

$$T^{-1} A T = \begin{bmatrix} \sigma + i\omega & 0 \\ 0 & \sigma - i\omega \end{bmatrix}$$

Consider the nonsingular matrix

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Straightforward but tedious calculations show that

$$S^{-1} T^{-1} A T S = S^{-1} \begin{bmatrix} \sigma + i\omega & 0 \\ 0 & \sigma - i\omega \end{bmatrix} S = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}.$$

We conclude that $\hat{T} = TS$ leads to the desired result. In fact any nonsingular matrix

$$S = \begin{bmatrix} a & -ia \\ b & ib \end{bmatrix},$$

with $a, b \in \mathbb{R}$ such that $ab \neq 0$, achieves the same result.