Mechatronics

Week 4 Day 1

Previously

- We distinguished between A, T, D-type elements, and A,T-variables
- We learned modelling multi-domain systems following a five-step procedure to obtain their state-space representation
- We modelled various multi-domain systems involving electrical, mechanical (Newtonian Approach, Vectorial Mechanics), and fluid domains
- We studied modelling mechanical systems via Euler-Largrange formalism (Analytical Mechanics)

Today's lecture:
Sampling and Discretization

Motivation

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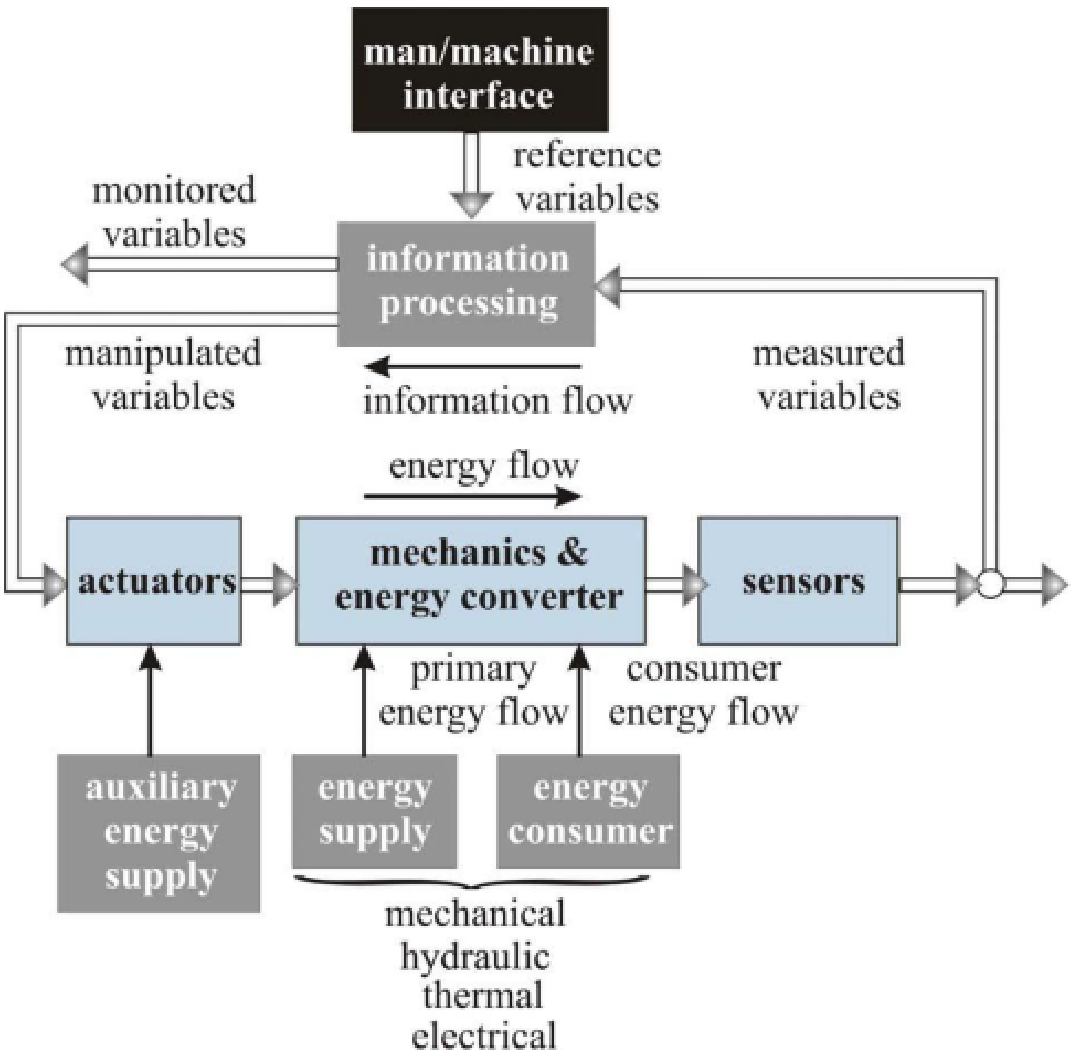
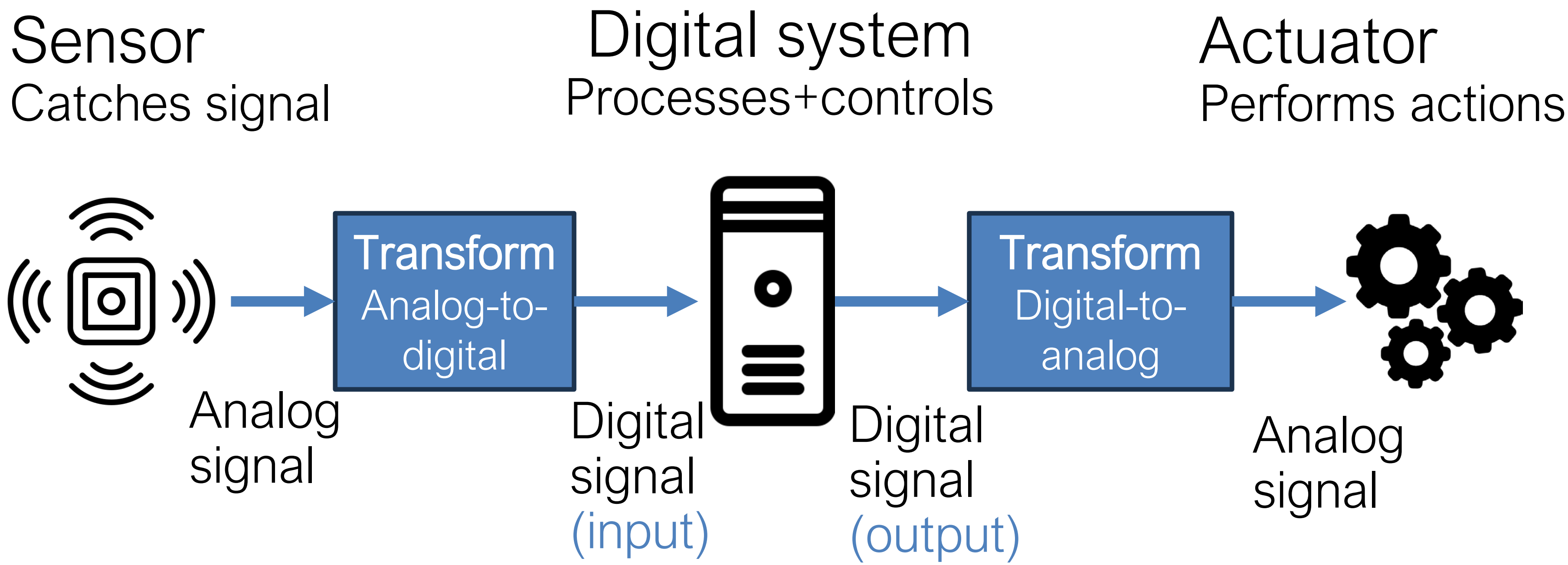


Advancements in digital technology have led to a rapid growth in the use of computers for

- data acquisition,
- information,
- processing, and
- control of continuous processes

Motivation

Continuous-time (analog) data cannot be directly handled by computers.
To use microcontrollers-based information processing data needs to be transformed.



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Modern Mechatronics Systems

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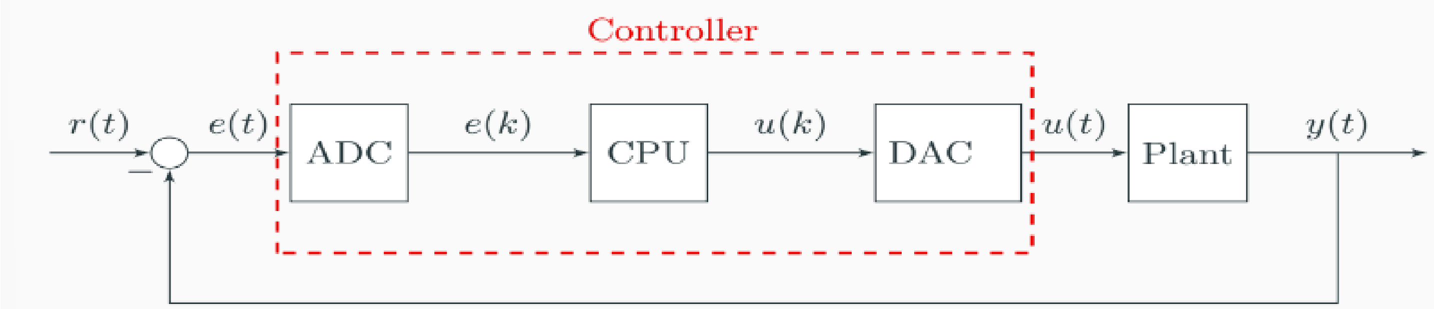
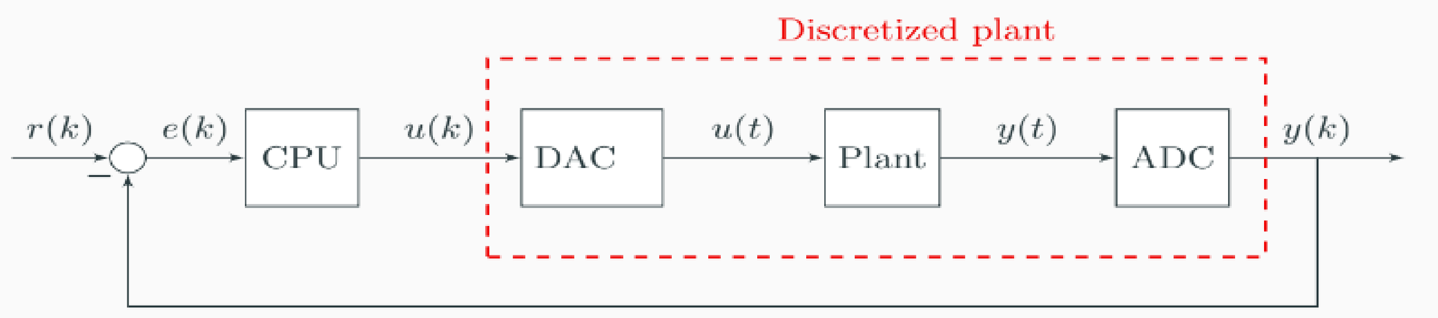
Modern Mechatronics Systems

Implementing mechatronics systems in practice

1. Implementation via discrete controller



2. Implementation by discretizing plant



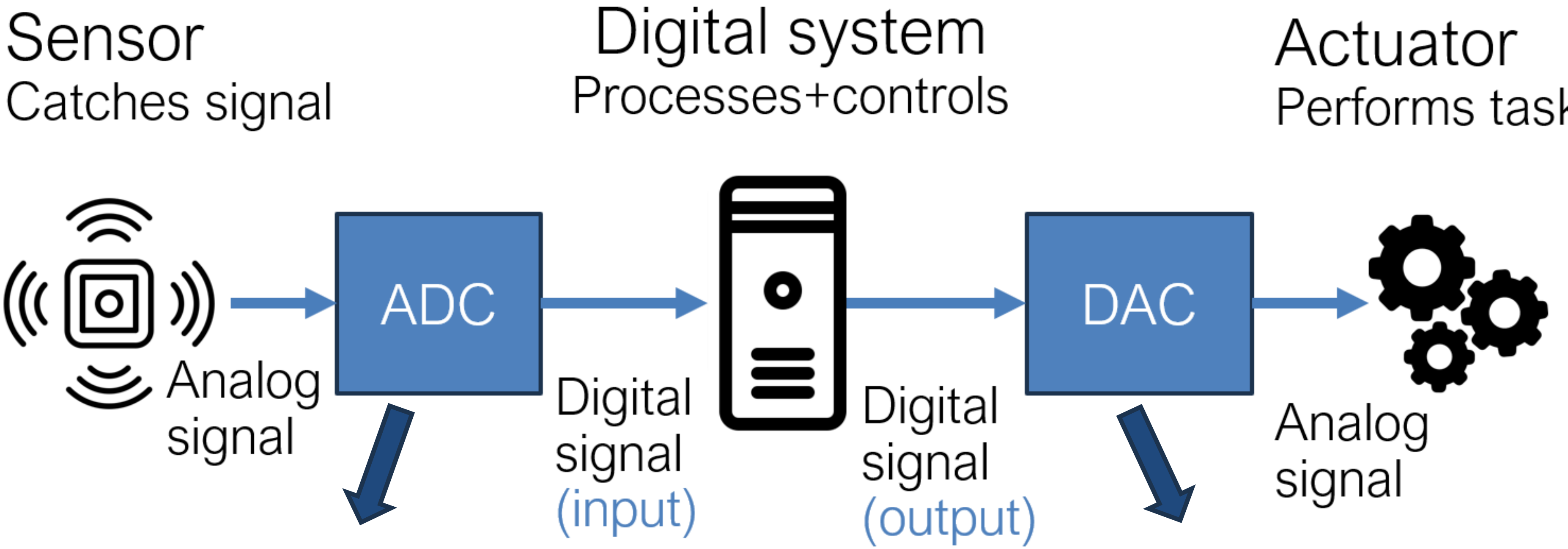
Learning objectives

After today's lecture, you will be able to

- Familiarize with the discretization of continuous-time signals and systems
- Appreciate the significance of choosing appropriate sample rate for the discretization
- Obtain the transfer function of a discrete-time system
- Learn various approximation methods and tools to discretize a continuous time system

Discretization and Reconstruction

Discretization and Reconstruction

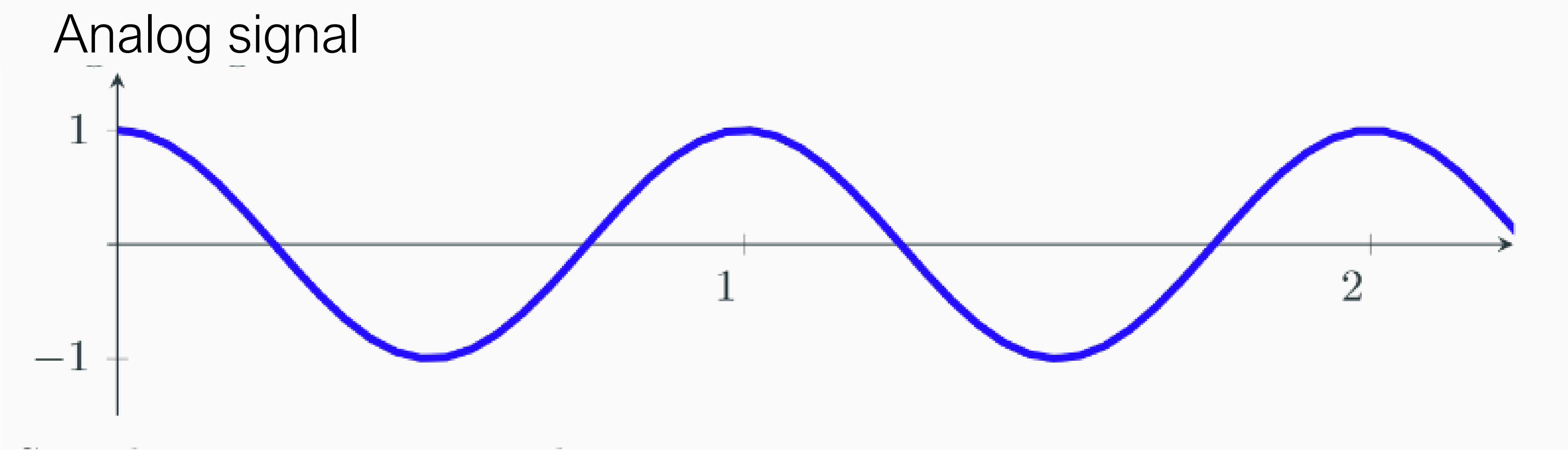


Analog-to-digital converter (**ADC**): transforms analog signal to discrete (discretization)

Digital-to-analog converter (**DAC**): transforms digital signal to analog (reconstruction)

Sampling theorem

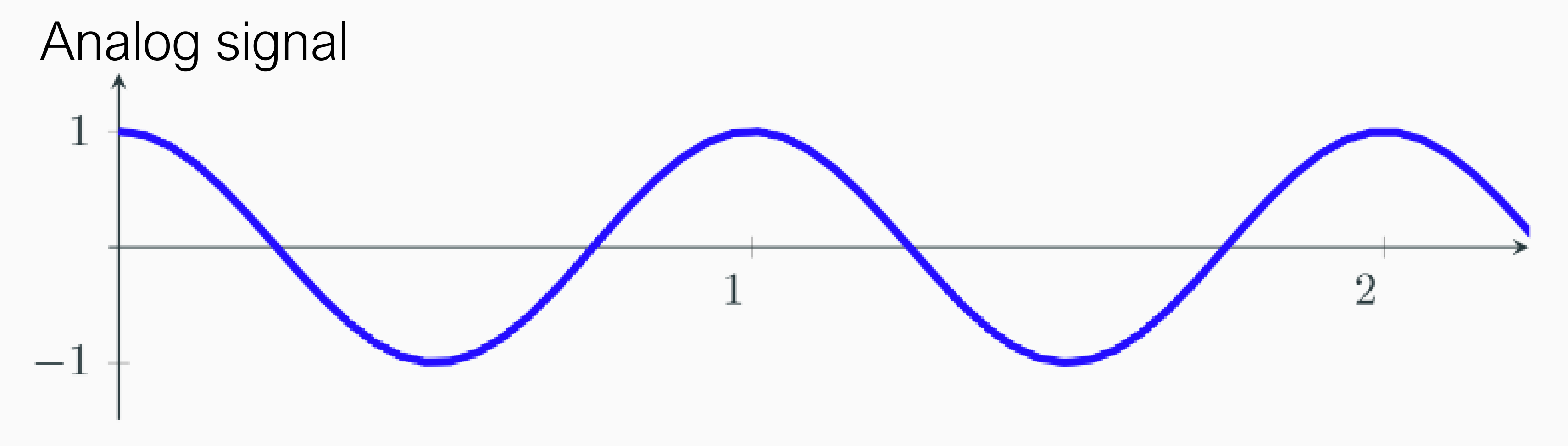
Nyquist-Shannon sampling theorem

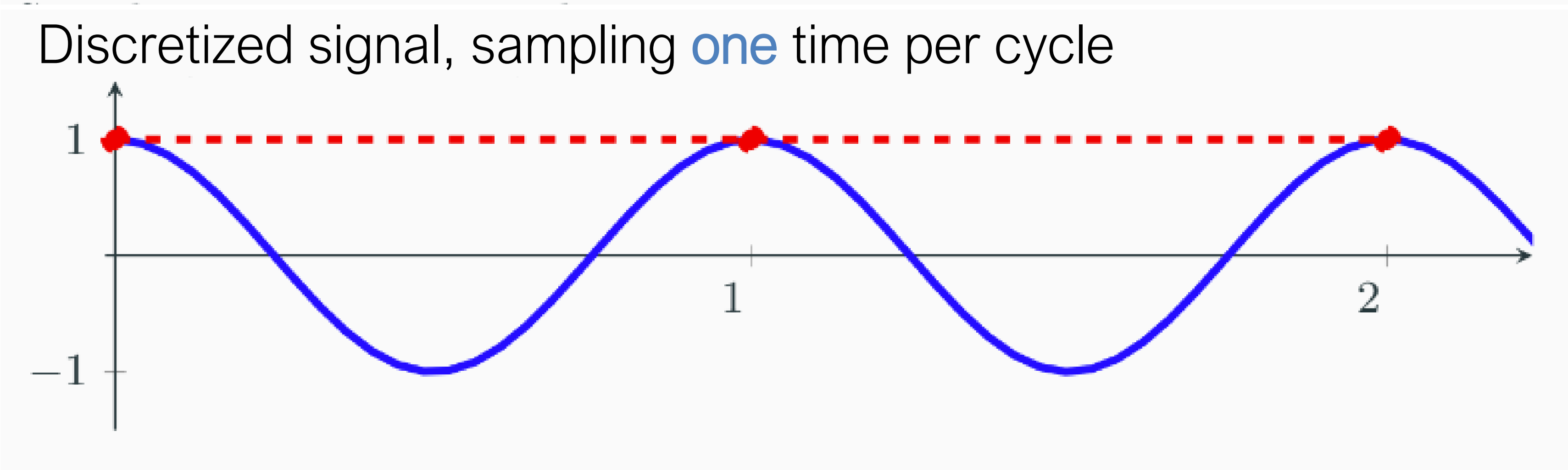


Frequency (ω): number of cycles completed in a second

For the given signal, $\omega = 1Hz$

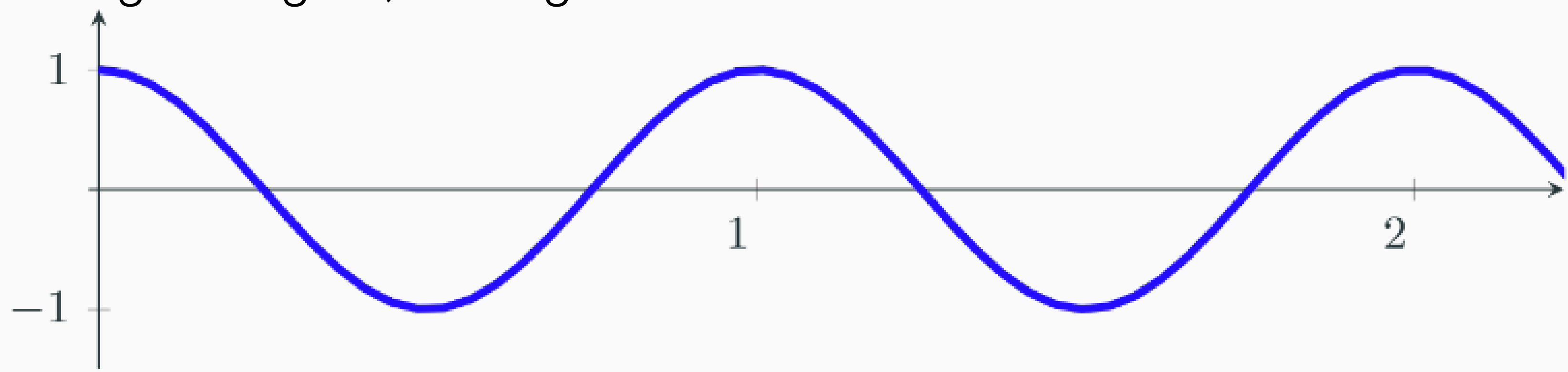
Nyquist-Shannon sampling theorem



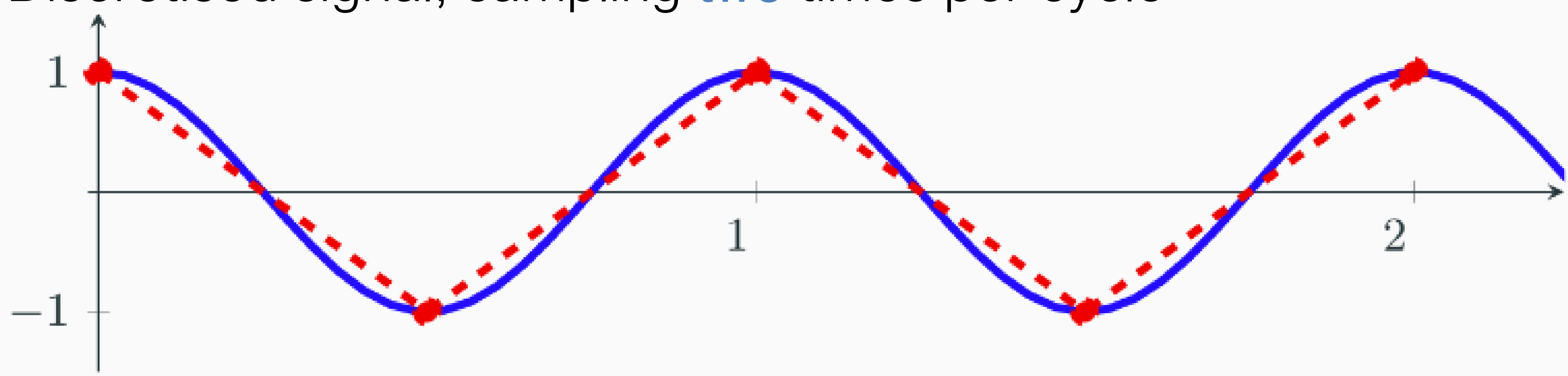


Nyquist-Shannon sampling theorem

Original signal, analog $\omega = 1Hz$

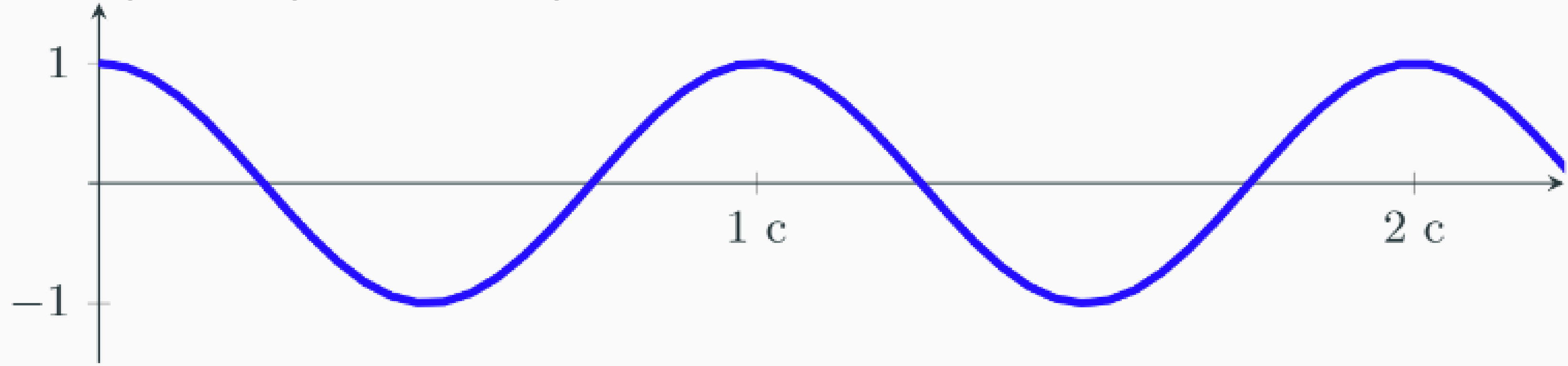


Discretised signal, sampling two times per cycle

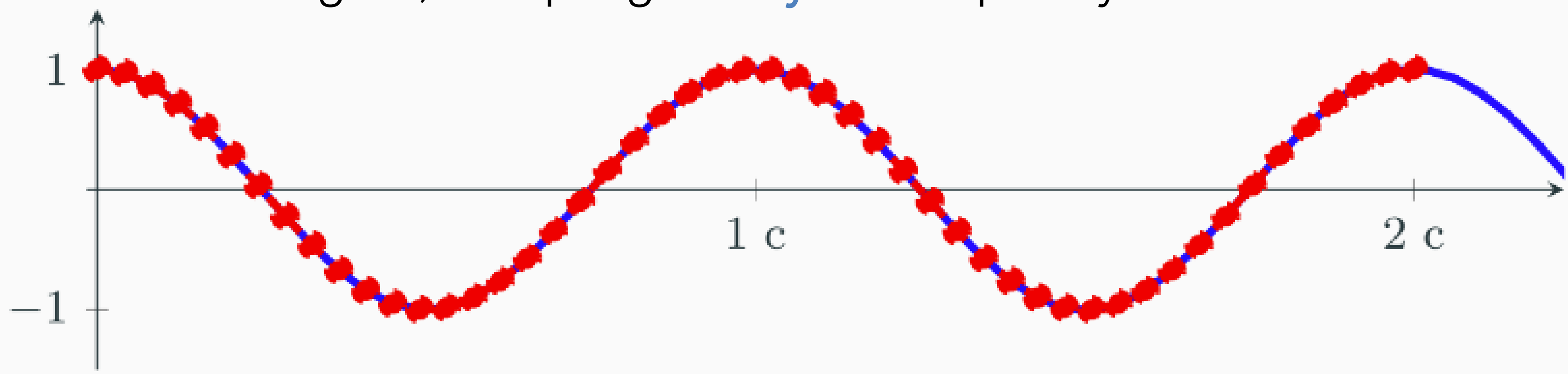


Nyquist-Shannon sampling theorem

Original signal, analog $\omega = 1Hz$



Discretised signal, sampling ‘many’ times per cycle



Nyquist-Shannon sampling theorem

A digital signal has a fixed sampling time T_s per cycle.

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Caveat: there is an approximation error as a result of discretization (loss of information)

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Then…
How do we know what sample rate (no of samples per cycle) to use?

Nyquist-Shannon sampling theorem

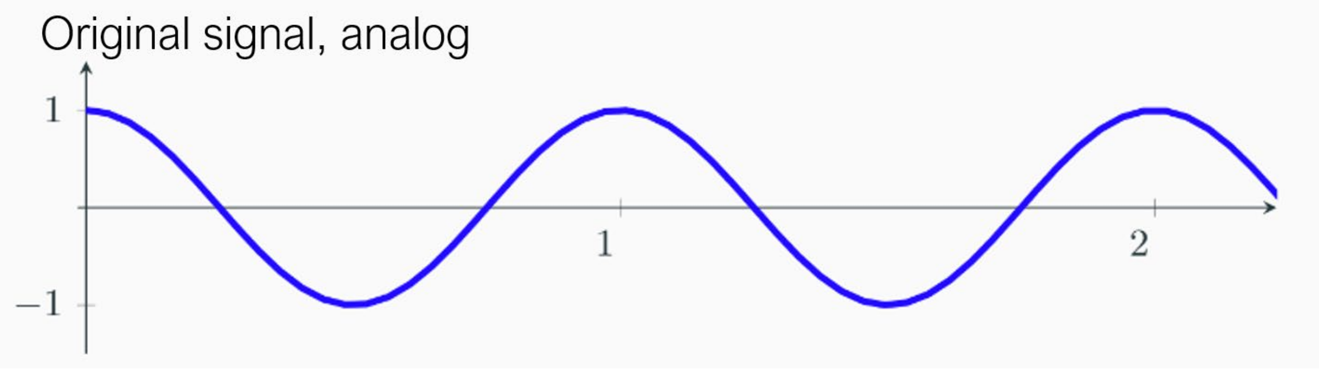
Nyquist-Shannon Theorem

If a function $x(t)$ contains no frequencies higher than ω Hertz, it is completely determined by giving its ordinates as a series of points spaced $\frac{1}{2\omega}$ seconds apart.

A sufficient sample rate is 2ω samples/second, where ω frequency of the signal

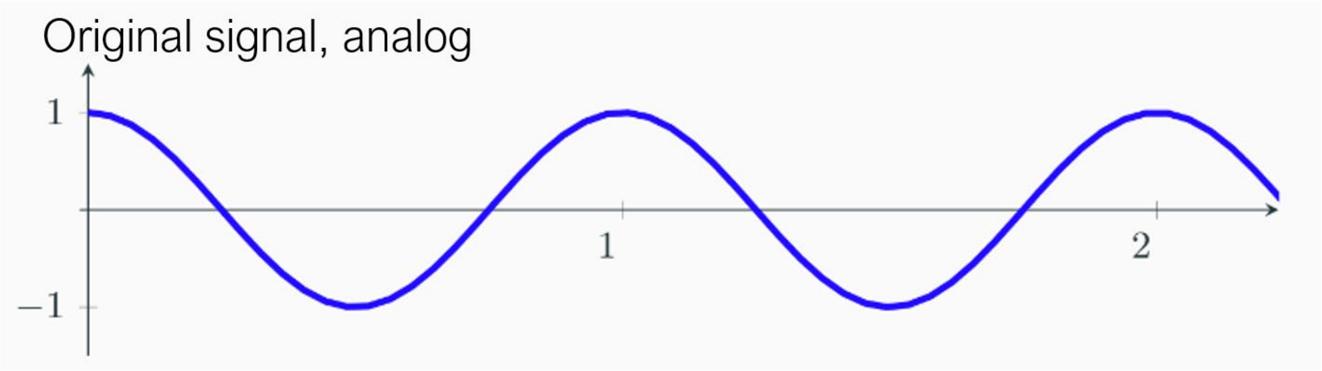
Nyquist-Shannon sampling theorem

For the example signal with $\omega = 1Hz$

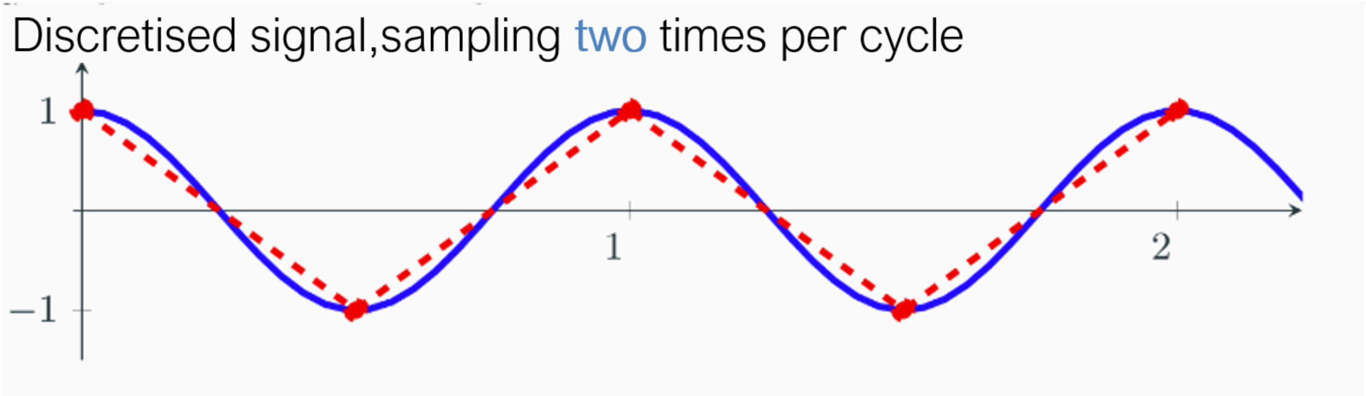


Nyquist-Shannon sampling theorem

For the example signal with $\omega = 1Hz$



A sampling of $2\omega = 2$ *samples/second is deemed sufficient*



z-transform

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- z-transform is an important tool to study discrete time signals and systems

Given $x(k) \in \mathbb{R},\ k = 0,1,\ldots\infty$, assuming $x(k) = 0$ for all $k < 0$

Definition (z-transform of $x(k)$)

$$\boxed{\mathrm{X}(\mathrm{z}) = \mathcal{Z}\{x(k)\} = \sum_{k=0}^{\infty} x(k) z^{-k}}$$

where $z \in \mathbb{C}$ is the independent variable.

z-transform

It is posible to relate **Laplace** and ***z*** **transforms** with the following methods:

- $z = e^{sT}$ (Impulse Invariant Method)

- $s = \frac{2(z-1)}{T(z+1)}$ (Bilinear Transform Method)

Example 1: Infinite sequence

Let $x(k+1) = ax(k)$ with $a \in \mathbb{R}$. This has solution: $x(k) = a^k x(0)$

Therefore, we have the sequence

$$\{x(k)\} = \{x(0), ax(0), a^2x(0) \ldots\},$$

which is infinite and converges if and only if $|a| < 1$

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Then, the z-transform of $x(k)$

$$X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} = x(0)\sum_{k=0}^{\infty} a^k z^{-k} = x(0)\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k,$$

which follows the geometric series formula

Recall geometric series formula:

For a sequence $c_k = c^k$ with $|c| < 1$. The sum of the elements in the sequence is

$$\sum_{k=0}^{\infty} c^k = \frac{1}{1-c} \; for \; |c| < 1$$

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$|z| > |a|$ is called the Region of Convergence (ROC) of the z-transform

Example 2: Finite sequence

Consider {x(k)}= $\{a, b, c, 0,0,0\ \ldots\}$, with $a, \mathrm{b}, \mathrm{c} \in \mathbb{R}$

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Consider $\{x(k)\} = \{a, b, c, 0,0,0\ \ldots\}$, with $a, b, c \in \mathbb{R}$

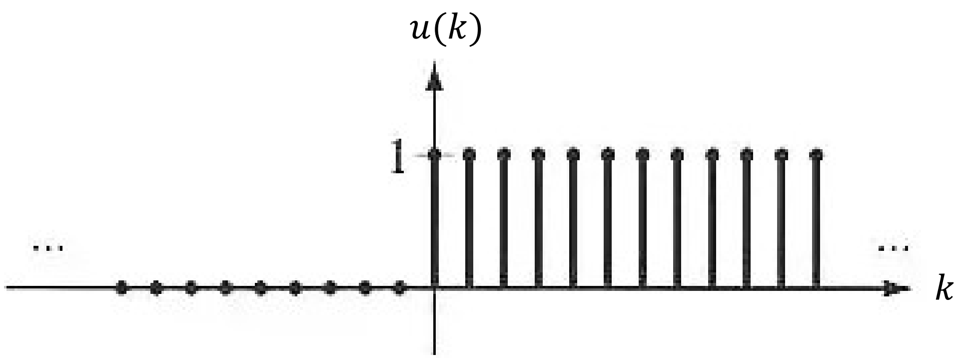
Then, the z-transform of $x(k)$

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} = x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} = a + bz^{-1} + cz^{-2}$$

Example 3: Unit Step

Consider unit step function

$$u(k) = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases}$$



Example 3: Unit Step

Consider unit step function

$$u(k) = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases}$$

Then, the z-transform of $\mathrm{u}(k)$

Geometric series

$$U(z) = \sum_{k=0}^{\infty} u(k) z^{-k} = \sum_{k=0}^{\infty} 1 z^{-k} = \frac{1}{1 - z^{-1}} = \frac{1}{z - 1}$$

when $|z| > 1$

z-transform pairs

$x(k), \mathrm{k} \geq 0$	z-Transform $\mathrm{X}(z)$	Region of Convergence
$x(\mathrm{k})$	$\sum_{k=0}^{\infty} x(k) z^{-k}$	
$\delta(\mathrm{k})$	1	$\lvert z\rvert > 0$
$au(\mathrm{k})$	$\frac{az}{z-1}$	$\lvert z\rvert > 1$
$\mathrm{k}u(\mathrm{k})$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$k^2 u(\mathrm{k})$	$\frac{\mathrm{z}(\mathrm{z}+1)}{(\mathrm{z}-1)^3}$	$\lvert z\rvert > 1$
$a^k u(\mathrm{k})$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$e^{-ak} u(\mathrm{k})$	$\frac{z}{(z-e^{-a})}$	$\lvert z\rvert > e^{-a}$
$ka^k u(\mathrm{k})$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
$sin(ak)u(\mathrm{k})$	$\frac{z\, sin(a)}{z^2 - 2z\, cos(a) + 1}$	$\lvert z\rvert > 1$
$cos(ak)u(\mathrm{k})$	$\frac{z[z - cos(a)]}{z^2 - 2z\, cos(a) + 1}$	$\lvert z\rvert > 1$
$a^k sin(bk)u(\mathrm{k})$	$\frac{[a\, sin(b)]z}{z^2 - [2a\, cos(b)]z + a^2}$	$\lvert z\rvert > \lvert a\rvert$
$a^k cos(bk)u(\mathrm{k})$	$\frac{[z - a\, cos(b)]z}{z^2 - [2a\, cos(b)]z + a^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$e^{-ak} sin(bk)u(\mathrm{k})$	$\frac{[e^{-a} sin(b)]z}{z^2 - [2e^{-a} cos(b)]z + e^{-2a}}$	$\lvert z\rvert > e^{-a}$
$e^{-ak} cos(bk)u(\mathrm{k})$	$\frac{[z - e^{-a} cos(b)]z}{z^2 - [2e^{-a} cos(b)]z + e^{-2a}}$	$\lvert z\rvert > e^{-a}$

z-transform properties

Linearity

$$\mathcal{Z}\big(a_1 x_1(k) + a_2 x_2(k)\big) = a_1 X_1(z) + a_2\, X_2(z)$$

Time shift

$$\mathcal{Z}(x(k - k_0)) = z^{-k_0}\, X(z)$$

Time reversal

$$\mathcal{Z}\big(x(-k)\big) = X(z^{-1})$$

First difference

$$\mathcal{Z}(x(k) - x(k-1)) = (1 - z^{-1})X(z)$$

z-transform properties

$x(n), n \geq 0$	Sequence	Transform	Region of Convergence
	$x[k]$ $x_1[k]$ $x_2[k]$	$X(z)$ $X_1(z)$ $X_2(z)$	R R_1 R_2
Linearity	$\left(a_1 x_1(k) + a_2 x_2(k)\right)$	$a_1 X_1(z) + a_2\, X_2(z)$	At least intersection of R_1 and R_2
Time shifting	$x[k - k_0]$	$z^{-n_0}\, X(z)$	R except possible addition or deletion of origin
Scaling in z-domain	$e^{j\omega_0 k} x[k]$ $\mathrm{z}_0^{\mathrm{k}}\mathrm{x}[\mathrm{k}]$ $\mathrm{a}^{\mathrm{k}}\mathrm{x}[\mathrm{k}]$	$X(e^{-j\omega_0} z)$ $X\left(\frac{z}{z_0}\right)$ $X(\mathrm{a}^{-1} z)$	R $z_0 R$ Scaled version of R
Time reversal	$\mathrm{x}[-\mathrm{k}]$	$X(z^{-1})$	Inverted R
Conjugation	$\mathrm{x}^*[\mathrm{k}]$	$\mathrm{X}^*(\mathrm{z}^*)$	R
Convolution	$\mathrm{x}_1[\mathrm{k}] * \mathrm{x}_2[\mathrm{k}]$	$X_1(z) X_2(z)$	At least intersection of R_1 and R_2
First difference	$x[k] - x[k-1]$	$(1 - z^{-1}) X(z)$	At least intersection of R and $\lvert z \rvert > 0$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}} X(z)$	At least intersection of R and $\lvert z \rvert > 1$
Differenciation in z-domain	$kx[\mathrm{k}]$	$-z \frac{dX(z)}{dz}$	R

Transfer functions

Consider the n-th order difference equation

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n)$$

$$= b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n)$$

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Applying z-transformation

$$(1 + a_1 z^{-1} + \cdots + a_n z^{-n}) Y(z) = (b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}) U(z)$$

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Applying z-transformation

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Then, the transfer function

$$\boxed{\mathrm{G(z)} = \frac{\mathrm{Y(z)}}{\mathrm{U(z)}} = \frac{\mathrm{b_0} + b_1 z^{-1} + \cdots + \mathrm{b_n z^{-n}}}{1 + \mathrm{a_1 z^{-1}} + \cdots + \mathrm{a_n z^{-n}}} = \frac{\mathrm{b_0 z^n} + \mathrm{b_1 z^{n-1}} + \cdots + \mathrm{b_n}}{\mathrm{z^n} + \mathrm{a_1 z^{n-1}} + \cdots + \mathrm{a_n}}}$$

State space representation and transfer functions

Just as in continuous time

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k) + Du(k)$$

where $\mathrm{x}(\mathrm{k}) \in \mathbb{R}^{n}$, $\mathrm{u}(\mathrm{k}) \in \mathbb{R}^{p}$, $\mathrm{y}(\mathrm{k}) \in \mathbb{R}^{m}$ is a state space representation of an n-th order difference equation

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Applying z-transformation

$$\boxed{\boldsymbol{G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D}}$$

From continuous to discrete transfer function

Depending on the approximation method, there are several ways to relate s and z. The most important ones are

Euler forward

Substitute

$$G(s) \longrightarrow \boxed{s \simeq \frac{z-1}{T_s}} \longrightarrow G(z)$$

Euler backward

Substitute

$$G(s) \longrightarrow \boxed{s \simeq \frac{z-1}{zT_s}} \longrightarrow G(z)$$

Bilinear (Tustin)

Substitute

$$G(s) \longrightarrow \boxed{s \simeq \frac{2}{T_s}\frac{z-1}{z+1}} \longrightarrow G(z)$$

From discrete to continuous transfer function

The reverse transformations to go from z to s are shown below for the most important methods:

Euler forward

Substitute

$$G(s) \longleftarrow \boxed{z \simeq \frac{1}{1 - T_s s}} \longleftarrow G(z)$$

Euler backward

Substitute

$$G(s) \longleftarrow \boxed{z \simeq \frac{1 + \frac{T_s}{2} s}{1 - \frac{T_s}{2} s}} \longleftarrow G(z)$$

Bilinear (Tustin)

Substitute

$$G(s) \longleftarrow \boxed{z \simeq T_s s + 1} \longleftarrow G(z)$$

Example

- Consider the system

$$\frac{dx(t)}{dt} = \underbrace{\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, D = 0$$

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- Can we use $C(z) = C(zI - A)^{-1}B + D$ to obtain discrete time TF?

Example

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- Can we use $C(z) = C(zI - A)^{-1}B + D$ to obtain a discrete time TF?
- NO! It's a continous-time system represented by a differential equation, not a discrete-time system represented by a difference

Example

- Consider the system

$$\frac{dx(t)}{dt} = \underbrace{\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, D = 0$$

- Can we use $C(z) = C(zI - A)^{-1}B + D$ to obtain a discrete time TF?
- NO! It's a continous-time system represented by a differential equation, not a discrete-time system represented by a difference
- We instead need to use $\boldsymbol{C(s) = C(sI - A)^{-1}B + D}$

Example

- It follows that

$$C(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & -3 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+1 & 3 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3}{(s+1)(s+2)}$$

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$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+1 & 3 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3}{(s+1)(s+2)}$$

- Using Euler forward approximation (for example), $s \simeq \frac{z-1}{T_s}$, we get the transfer function in z:

$$\boxed{\mathbf{G(z)} = \frac{\mathbf{3T_s{}^2}}{\mathbf{(z + T_s - 1)(z + 2T_s - 1)}}}$$

Summary

- Discretization plays important role in information processing part of a Mechatronics system

- Nyquist-Shannon sampling theorem establishes necessary and sufficient conditions to select adequate sample rate

- z-transform, analogous to Laplace, is useful to obtain transfer function of a discrete-time system

Next lecture:

Discrete-time systems and control