university of groningen

Exam Numerical Methods

November 9th 2017   18.30-21.30

It is allowed to use a book (paper version only) and lecture notes, as well as a (graphical) pocket calculator. The use of electronic devices (tablet, laptop, mobile phone, etc.) is not allowed.

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: $\boxed{10}$

Practica: $\boxed{18}$ For the 6 computer practica a maximum of 6*3=18 points can be earned.

1. Consider the equation $\sin(\frac{x}{4}) = \cos(\frac{x}{4})$, with exact solution $x = \pi$. To find the value of π, one could use an iterative method, with initial value $x_0 = 3$.

   (a) $\boxed{7}$ (1) Compute x_1 with Newton's method (one iteration), starting with $x_0 = 3$. Determine the exact error for x_1 (use exact value of π).
   (2) Compute two iterations with the Bisection method, with $I_0 = [2.7, 3.3]$ (and hence $m_0 = 3$) as initial search interval.
   How many additional iterations are needed to reach the same accuracy as in (1)?

   (b) $\boxed{6}$ Someone uses the iterative method $x_{n+1} = x_n + \cos(\frac{x_n}{4}) - \sin(\frac{x_n}{4})$, with $x_0 = 3$.
   The first 4 iterations are given by

x_0	x_1	x_2	x_3	x_4
3.00000000	3.05005011	3.08241246	3.10333506	3.11686095

   (1) Explain that this method will eventually converge.
   (2) Determine the theoretical error reduction and the convergence rate $\tilde{K}$.
   Compare these to the reduction rate of exact errors (use exact value of π).
   (3) Determine an error estimate for x_4 and compare it to the true error in x_4.

2. Consider the integral
$$I_1 = \int_0^1 e^{\sqrt{x}}\, dx = 2$$

   (a) $\boxed{7}$ (1) Will the Trapezoidal method give optimal 2nd order convergence? Explain.
   (2) Use the Trapezoidal method on a grid with two segments to approximate I_1.
   Is the global error theorem useful in this case? Explain.
   (3) The results for the Trapezoidal method on finer grids are given below (I_1 column)
   Compute the q-factor. What can you conclude?

n	$I_1(n)$	$I_2(n)$
4	1.97835496	2.04612896
8	1.99190779	2.01154821
16	1.99702730	2.00288805
32	1.99892076	2.00072208
64	1.99961132	2.00018052

   $I(n)$ is the approximation of the integral on a grid with n sub-intervals.

   (b) $\boxed{6}$ Through the substitution $u = \sqrt{x}$ the integral is converted: $I_2 = \int_0^1 2\, u\, e^{u}\, du = 2$
   (1) Compute the q-factor for the I_2 values (see table). What can you conclude?
   (2) Give an error estimate for $I_2(64)$ based on $I_2(n)$ values.
   (3) Compute improved solutions (T_2) for $I_2(8)$ and $I_2(16)$ by means of extrapolation.
   Combine these extrapolations into a highly accurate approximation $T_3(16)$.
   (4) Compare the result at (2) with the exact error in $T_3(16)$. How many intervals are required (powers of 2) for the Trapezoidal method to reach the accuracy of $T_3(16)$?

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3. Consider on [0 6] the o.d.e. $y'(x) = \dfrac{1}{y(x)} - x$, with boundary condition $y(0) = 1$.

(a) $\boxed{6}$ (1) Use explicit Euler to compute $y(x)$ at $x=1$ on a grid with $\Delta x = 0.5$ (2 steps).
(2) Use Heun's method (RK2) to compute $y(x)$ at $x=0.5$ on a grid with $\Delta x = 0.5$.
(3) Use the implicit(!) Euler method to compute $y(x)$ at $x=1$ on a grid with $\Delta x = 1$.

(b) $\boxed{7}$ The explicit 3rd(!) order RK3 method is used on 2 coarse grids ($N=30, 60$ segments), and 3 fine grids. The table below shows solutions at a selection of x locations.

x_n	$N = 30$	$N = 60$	$N = 960$	$N = 1920$	$N = 3840$
4.8	0.20100403	0.21028199	0.2103099530989	0.2103099575739	0.2103099581125
5.0	0.90758040	0.20164462	$\underline{0.2016690}068014$	$\underline{0.2016690}110630$	$\underline{0.2016690}115744$
5.2	0.26123492	0.19370630	0.1937273518644	0.1937273559404	0.1937273564279
5.4	0.42414186	0.18638281	0.1864008393445	0.1864008432572	0.1864008437237
5.6	0.53003059	0.17960342	0.1796187654317	0.1796187691996	0.1796187696472
5.8	0.31511378	0.17330818	0.1733211909195	0.1733211945577	0.1733211949884
6.0	1.78800005	0.16744589	0.1674569010218	0.1674569045435	0.1674569049589

(1) Compute the q-ratio for $x=5.0$ (equal digits are underlined). Conclusion?
(2) Give an error estimate for the solution at $x = 5.0$ on the finest grid.
(3) Compute an improvement for the solution at $x = 5.0$ (extrapolation).
(4) Is there a stability limit visible? Explain.

4. We will fit an exponential curve through 3 data points: $y(0) = 1$, $y(1.2) = e$, $y(2.4) = e^3$.

(a) $\boxed{2}$ Apply a coordinate transformation, such that the $\hat{x}_i$ points are centered around $x=0$ as follows: $\hat{x}_1=-1$, $\hat{x}_2=0$, $\hat{x}_3=1$. Then apply the ln-function to the measured data.

(b) $\boxed{7}$ (1) Use the transformations to set up the exponential curve (through original data).
(2) Predict the y value at $x = 3.6$.

5. (a) $\boxed{6}$ Consider $A\vec{x} = \vec{b}$, and the initial vector $\vec{x}_0$

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0.5 & 0 \\ 1 & -1 & \alpha & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 6 \\ \beta \\ 6 \\ 7 \end{pmatrix} \quad \vec{x}_0 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

(1) For which values of α and β will the Jacobi method converge?
(2) Let $\alpha = \beta = 10$. How many iterations are required with Jacobi to reduce the initial error in $\vec{x}_0$ with a factor 100?
(3) Let $\alpha = \beta = 10$. Compute $\vec{x}_1$, the result after 1 Gauß-Seidel iteration.

(b) $\boxed{6}$ Consider on [0 1] the differential eqn. $y''(x) + 2\,y(x) = 4\,x$, with boundary conditions $y(0) = 2$, $y'(1) = 0$ (notice the derivative!).
(1) Give the matrix-vector system, when a grid is used with 2 segments of equal length, and the standard [1 -2 1]-formula is applied for $y''(x)$.
(2) Solve the system to find the values in the interior point and at $x = 1$.

6. Initially, an iron bar (length $1m$, diffusion const. $\kappa=2.26.10^{-5}$) is at room-temperature 20^oC. During 30 seconds the bar is heated with a flame in the middle (width $5cm$, $Q=70^oC/s$):

$$\frac{\partial T(x,t)}{\partial t} = \kappa\,\frac{\partial^2 T(x,t)}{\partial x^2} + Q(x,t), \quad T(x,0) = T(0,t) = T(1,t) = 20^oC.$$

For $\partial^2/\partial x^2$ the standard [1 -2 1]-formula is applied.

(a) $\boxed{6}$ Suppose the Explicit Euler method is used on an equidistant grid with 400 segments.
(1) What is the maximum time step? Is it advisable to use that time step?
(2) Explain in detail how/where the sourceterm $Q(x,t)$ goes in your numerical model.

(b) $\boxed{6}$ (1) Give 2 pro's and 2 con's of the Crank Nicolson method, compared to the Explicit Euler method on the same spatial grid (400 segments).
(2) Explain in detail how/where the sourceterm goes in your numerical model in case of the Crank Nicolson method, with -1 at the left/right diagonal positions.

Total: $\boxed{100}$