university of groningen

Exam Numerical Methods

November 7th 2023   18:15-20:15   (**2 hours**)

It is allowed to use a book (paper version only) and lecture notes, as well as a (graphical) pocket calculator. The use of electronic devices (tablet, laptop, mobile phone, etc.) is not allowed.

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: $\boxed{10}$

Practica: $\boxed{18}$ For the 6 computer practica a maximum of 6*3=18 points can be earned.

1. Consider the equation $2\cos(x) = x + 1$, with solution $x \approx 0.62$.

   (a) $\boxed{4}$ (1) Perform 1 iteration with Newton's method, starting with $x_0 = 0.6$.
   (2) Give an error estimate for the result.

   (b) $\boxed{8}$ Someone considers a method to solve the equation:

   $$x_{n+1} = a(2\cos(x_n) - 1) + (1-a)x_n, \text{ with } x_0 = 0.6$$

   Here a serves as a relaxation parameter. For $a = 0.5$ the first iterations are given by

n	1	2	3	4	5
x_n	0.6253 3561	0.6234 3451	0.6235 9534	0.6235 8185	0.6235 8298

   (1) Determine an error estimate.
   (2) How fast will errors decrease (approximately) if $a = 0.5$?
   (3) Determine the optimal value of a, for fast (linear) convergence.

2. Consider the integral $I = \int_{-1}^{1} (2x+1)\, e^{x}\, dx = \frac{3}{e} + e \approx 3.82.$

   (a) $\boxed{7}$ (1) Approximate I using the Midpoint method with two subintervals.
   (2) Apply the "global error theorem" to this case with two subintervals.
   Hint: the relevant extrema are located at $x = -1$ and $x = 1$.
   (3) Give two reasons why the error estimate is not accurate in this case.

   With the Trapezoidal method the following results are obtained, with $I(n)$ the approximation of the integral on a grid with n segments. The method converges according to theory ($q \approx 4$).

n	32	64	128	256
$I(n)$	3.8262 2421	3.8229 9626	3.8221 8918	3.8219 8741

   (b) $\boxed{6}$ (1) How many segments (use powers of 2) are required for an accuracy of 1.0E-8?
   (2) Compute the improved solution $T_2(128)$ and its exact error.
   (3) Describe the gain in efficiency by comparing both error levels and segments.

3. Consider the diff. eqn. $y'(x) = 4\,xy^2$, with boundary condition $y(0) = 1$.

   (a) $\boxed{8}$ (1) Apply explicit Euler to determine $y(\frac{1}{2})$ on two grids, with $h = \frac{1}{2}$ and $h = \frac{1}{4}$.
   Then use extrapolation to improve your result in $x = \frac{1}{2}$.
   (2) Is the stability requirement $|1 + ah| < 1$ useful for this problem? Explain.
   (3) Explain the problem(s) encountered when implicit(!) Euler is used with $h = 1/2$.

P.T.O.

(b) $\boxed{5}$ With a 2nd order RK method, the result at a selection of x locations is as follows:

x_n	$\Delta x = 0.1$	$\Delta x = 0.05$	$\Delta x = 0.025$	$\Delta x = 0.0125$
0.1	1.0200 0000	1.0203 5441	1.0204 0108	1.0204 0721
0.2	1.0858 8914	1.0867 9464	1.0869 3055	1.0869 5193
0.3	1.2166 3073	1.2189 7121	1.2194 0379	1.2194 8875
0.4	1.4609 5759	1.4683 8356	1.4700 7679	1.4704 6648

(1) Show that the employed method converges according to theory.
(2) Give error estimates for the solutions at $x = 0.3$ and $x = 0.4$ on the fine grid. Which of these error estimates is bigger? Explain why.
(3) Give an extrapolation for the value at $x = 0.4$.

4. $\boxed{9}$ Fit the function $y(x) = a\,e^{bx}$ through the set of measured points given below. Hint: it helps to apply a transformation first.

x_n	-1	1	3
y_n	e^{-1}	1	e^2

5. Consider the linear equation $A\vec{x} = \vec{b}$, with $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & -20 & 4 & 0 \\ 0 & 4 & -20 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$

The initial vector for the iterative methods is: $\vec{x}_0 = (1, 1, 1, 1)^T$.

(a) $\boxed{4}$ (1) Compute $\|r_{(0)}\|_\infty$, i.e. the max.-norm of the initial residual for $\vec{x}_0$.
(2) How many Jacobi iterations are needed to reduce this error with a factor 100?

(b) $\boxed{5}$ (1) Perform one SOR iteration with $\omega = \frac{1}{2}$, for the given $\vec{x}_0$.
(2) Does it make sense to use SOR with $\omega = \frac{1}{2}$ for the given $A\vec{x} = \vec{b}$? Explain why.

(c) $\boxed{4}$ The given $A\vec{x} = \vec{b}$ can be related to the 2nd order o.d.e. $y'' + ay = 0$, with boundary conditions $y(0) = y(1) = 2$. Determine the value of a in the o.d.e.

6. Consider for $0 \le x \le 9$ the equation $\dfrac{\partial u}{\partial t} = D\,\dfrac{\partial^2 u}{\partial x^2} + \beta\,(u + x)$, with $D = 0.09$.

The boundary and initial conditions are $u(0,t) = 20$, $u(9,t) = -20$, $u(x,0) = 20\cos(\frac{\pi}{9}x)$.
For $\partial^2/\partial x^2$ the usual [1 -2 1]-formula is used.

A spatial grid is used with 2 internal mesh points (and of course 2 boundary points).

(a) $\boxed{6}$ Take $\beta = 0$.
(1) What is the max. possible time step in case of the explicit Euler method?
(2) Determine the solution after one time step of $\delta t = 10$ with explicit Euler.

(b) $\boxed{6}$ Take $\beta = 0.1$.
Give the 2nd row of the linear system (matrix and RHS vector) when the implicit Euler method is used for this problem with $\delta t = 10$.

Total: $\boxed{100}$