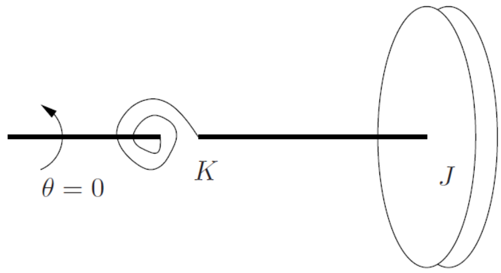
Control Engineering

Instruction Lecture 2: Solutions

Basic concepts modeling | Chapter 2,3 + reader chapter 1

Exercise 1. Euler-Lagrange equations

Consider a rotating mass connected to a spring. The mass has an intertia J and the rotational spring has a spring constant K. The left part of the spring is fixed (angle $\theta = 0$). An external torque $\tau(t)$ acts on the rotating mass.



1. Give the Euler-Lagrange equation of the system.
2. Determine the state space model from the Euler-Lagrange equation with the angular velocity $\dot{\theta}$ as an output.

SOLUTION

1. The Lagrangian (see "*Analogies for the different domains*" under Course Documents on Nestor) is given by
$$\mathcal{L} = \frac{1}{2}J\dot{\theta}^2 - \frac{1}{2}K\theta^2.$$
Using (1.5) from the reader you obtain
$$J\ddot{\theta} + K\theta = \tau.$$

2. Take as states $(x_1, x_2) = (\theta, \dot{\theta})$, and take as input $u = \tau$ such that
$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\frac{K}{J} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{J} \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

Exercise 2. State space models

Consider three masses evolving on a unitary circle (radius=1) which are connected by linear springs with coefficients a_{12}, a_{13}, a_{23} respectively, see Figure 1. A torque τ_1 is applied to mass 1, resulting in a circular motion of the mass. The variables $\theta_1, \theta_2, \theta_3$ are the angular displacements, $\omega_1, \omega_2, \omega_3$, the resulting angular velocities and J_1, J_2, J_3 the moment of inertia of the three rotating masses.

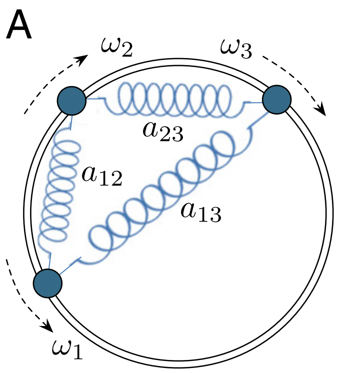


Figure 1: From Dörfler et al. Synchronization in complex oscillator networks and smart grids. PNAS 2013.

When neglecting the gravity and friction and assuming that the angle differences are small, the (linearized) equations of motion are given by

$$\begin{aligned} J_1\dot{\omega}_1 + a_{12}(\theta_1 - \theta_2) + a_{13}(\theta_1 - \theta_3) &= \tau_1 \\ J_2\dot{\omega}_2 - a_{12}(\theta_1 - \theta_2) + a_{23}(\theta_2 - \theta_3) &= 0 \\ J_3\dot{\omega}_3 - a_{13}(\theta_1 - \theta_3) - a_{23}(\theta_2 - \theta_3) &= 0. \end{aligned} \tag{1}$$

Derive a state space model for the system by introducing the state variables $x_1 = \theta_1, x_2 = \theta_2, x_3 = \theta_3$, $x_4 = \omega_1, x_5 = \omega_2$ and $x_6 = \omega_3$, with $u = \tau_1$ as an input, and the angular velocity $y = \omega_1$ as the output.
NOTE: $\dot{\theta}_i = \omega_i$, for $i = 1, 2, 3$.

SOLUTION

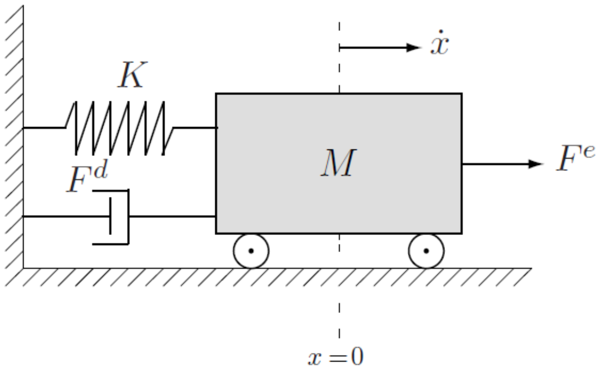
The state space model takes the form

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -(a_{12}+a_{13})/J_1 & a_{12}/J_1 & a_{13}/J_1 & 0 & 0 & 0 \\ a_{12}/J_2 & -(a_{12}+a_{23})/J_2 & a_{23}/J_2 & 0 & 0 & 0 \\ a_{13}/J_3 & a_{23}/J_3 & -(a_{13}+a_{23})/J_3 & 0 & 0 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/J_1 \\ 0 \\ 0 \end{pmatrix}}_{B} u$$

$$y = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}}_{C} x.$$

Exercise 3. Modeling of a mass-spring damper system

Consider a mechanical system with external force F^e. Mass $M > 0$ and spring constant $K > 0$ are constants. The damping force is given by F^d.



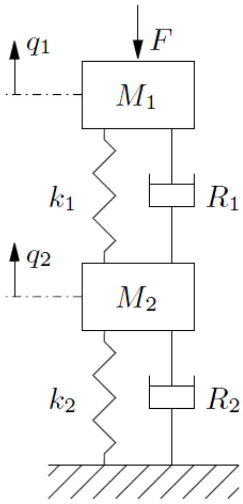
1. Give the Lagrangian for this system.
2. Assume $F^d = 0$ and give the Euler-Lagrange equations for this system.
3. Assume $F^d = -\dot{x} - 2\dot{x}^3$, and determine the Rayleigh dissipation function.
4. Derive the equations of motion from the Euler-Lagrage equations.
5. Derive a state space model from the equations of motion found above and take $\dot{x}$ as an output.

SOLUTION

1. The Lagrangian is given by
$$\mathcal{L} = \frac{1}{2}M\dot{x}^2 - \frac{1}{2}Kx^2$$
2. For $F^d = 0$ the Euler-Lagrange equation is given by (1.5) from the reader
$$M\ddot{x} + Kx = F^e$$
3. For $F^d = -\dot{x} - 2\dot{x}^3$ the Rayleigh dissipation function follows from (1.21) from the reader
$$D = -\int F^d \, d\dot{x} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{x}^4$$
4. The Euler-Lagrange equation including damping is given by (1.22) from the reader
$$M\ddot{x} + Kx = F^e - \dot{x} - 2\dot{x}^3$$
5. Take as the states $(x_1, x_2) = (x, \dot{x})$ and $u = F^e$ as the input we obtain the state equations
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{K}{M}x_1 - \frac{1}{M}x_2 - \frac{2}{M}x_2^3 + \frac{1}{M}u. \end{aligned}$$
Note that this equation can not be put into the linear matrix form $\dot{x} = Ax + Bu$ due to the nonlinear damping term $\frac{2}{M}x_2^3$. The output equation is given by
$$y = x_2 = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Exercise 4.

Consider the linear mass-spring-damper system with input force F depicted below. Throughout this exercise gravity may be neglected.



1. Determine the Lagrangian of the system.
2. Determine the Rayleigh dissipation function.
3. Determine the equations of motion, using the Lagragian.
4. Give a state space model with the positions of the two masses as the output.
5. Give the equivalent electrical circuit.

SOLUTION

1. The Lagrangian is given by

$$\mathcal{L} = \frac{M_1}{2}\dot{q}_1^2 + \frac{M_2}{2}\dot{q}_2^2 - \frac{k_1}{2}\left(q_1 - q_2\right)^2 - \frac{k_2}{2}q_2^2$$

2. The Rayleigh dissipation function follow from (1.23) directly as

$$D(\dot{q}) = \frac{R_1}{2}\left(\dot{q}_1 - \dot{q}_2\right)^2 + \frac{R_2}{2}\dot{q}_2^2.$$

3. The equations of motion are given by (1.22) in the reader. First note that $\frac{\partial \mathcal{L}}{\partial q_1} = -k_1\left(q_1 - q_2\right), \frac{\partial \mathcal{L}}{\partial q_2} = k_1\left(q_1 - q_2\right) - k_2 q_2, \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = M_1\dot{q}_1, \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = M_2\dot{q}_2$, such that the equations of motion are given by

$$\begin{aligned} M_1\ddot{q}_1 + k_1 q_1 - k_1 q_2 &= -F - R_1\left(\dot{q}_1 - \dot{q}_2\right) \\ M_2\ddot{q}_2 - k_1\left(q_1 - q_2\right) + k_2 q_2 &= R_1\left(\dot{q}_1 - \dot{q}_2\right) - R_2\dot{q}_2 \end{aligned}$$

4. Take as the states $(x_1, x_2, x_3, x_4) = (q_1, q_2, \dot{q}_1, \dot{q}_2)$ and input $u = F$ we obtain the state equation

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -\frac{k_1}{M_1}x_1 + \frac{k_1}{M_1}x_2 - \frac{R_1}{M_1}\left(x_3 - x_4\right) - \frac{1}{M_1}u \\ \dot{x}_4 &= \frac{k_1}{M_2}x_1 - \frac{(k_1 + k_2)}{M_2}x_2 + \frac{R_1}{M_2}x_3 - \frac{(R_1 + R_2)}{M_2}x_4. \end{aligned}$$

Put into matrix form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{M_1} & \frac{k_1}{M_1} & -\frac{R_1}{M_1} & \frac{R_1}{M_1} \\ \frac{k_1}{M_2} & -\frac{(k_1+k_2)}{M_2} & \frac{R_1}{M_2} & -\frac{(R_1+R_2)}{M_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{M_1} \\ 0 \end{pmatrix} u$$

with output equation

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

5. To obtain the equivalent electrical circuit, first consider the following analogies:

Mechanical		**Electrical**		**Mechanical**	**Electrical**
Spring k	$F_s = kx$	Capacitor $\frac{1}{C}$	$V_c = \frac{q}{C}$	Force F	Voltage V
Mass M	$F_m = m\ddot{x}$	Inductor L	$V_L = L\ddot{q}$	Displacement x	Charge q
Damper R	$F_d = R\dot{x}$	Resistor R	$F_R = R\dot{q}$	Velocity $v = \dot{x}$	Current $i = \dot{q}$

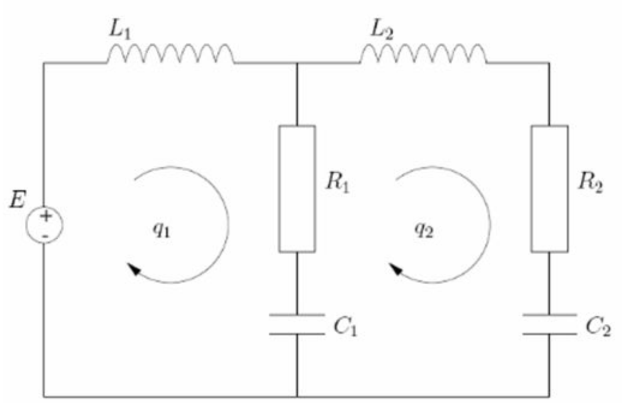
(a) From the figure in the exercise note that mass M_2, spring k_2, and damper R_2 have the same velocity. Hence the equivalent inductor L_2, capacitor $\frac{1}{C_2}$, and resistor R_2 should be in series (since the current flowing through these elements is equal!).

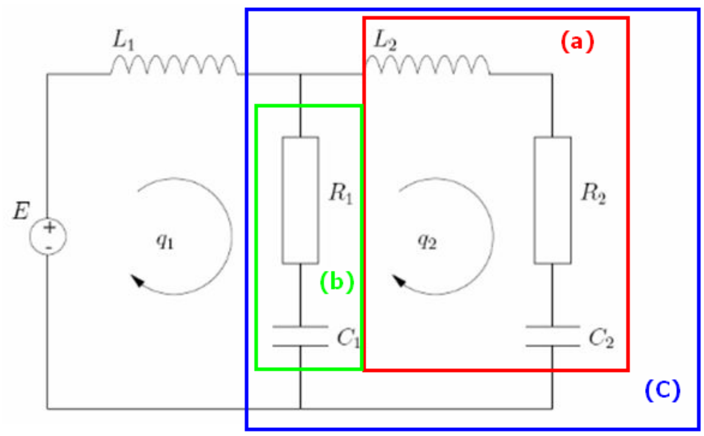
(b) Furthermore we note that spring k_1 and damper R_1 have the same velocity. Hence capacitor $\frac{1}{C_1}$ and resistor R_2 should be in series.

(c) The electrical equivalents described in (a) and (b) should be in parallel. To see why, fix spring k_1 and damper R_1 at the top end (i.e., mass M_1 is fixed): it then immediately follows that the forces in the upper part and lower part should be equal, hence the voltage in the electrical equivalent should be equal: parallel circuit.

(d) The total velocity of the components in (a) and (b) (i.e., $\dot{q}_1 + \dot{q}_2$) should be the same as the velocity of mass M_1, which implies that inductor L_1 and voltage source E should be in series with subsystem (c).

(e) The following figures shows the equivalent circuit (top) and illustrate the steps above (bottom)





Exercise 5. (Book exercise 2.1)

Consider the linear ordinary differential equation[1]

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = u. \tag{2}$$

1. Show that by choosing a state space representation with $x_1 = y$, the dynamics can be written as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{3}$$

with

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & \ldots & 0 & 0 \end{pmatrix}.$$

This canonical form is called the *chain of integrators* form.

2. There exist other canonical forms as well. So see this, define $x_1 = \frac{dy^{n-1}}{dt}, \ldots, x_{n-1} = \frac{dy}{dt}, x_n = y$ and write the system (2) in the form (3) using this set of state variables.

SOLUTION

1. The system in equation (2) can be rewritten as:

$$\frac{d^n y}{dt^n} = -a_1 \frac{d^{n-1} y}{dt^{n-1}} - a_2 \frac{d^{n-2} y}{dt^{n-2}} - \ldots - a_n y + u.$$

Let

$$x = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}^T = \begin{pmatrix} y & \frac{dy}{dt} & \ldots & \frac{d^{n-1}y}{dt^{n-1}} \end{pmatrix}^T.$$

Then the state space representation becomes

$$\begin{aligned} \dot{x} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ -a_1 x_n - a_2 x_{n-1} - \ldots - a_n x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u \\ &= \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u = Ax + Bu \\ y &= \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = Cx. \end{aligned}$$

[1] Equation (2.7) in the book.

2. The state space representation becomes

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} -a_1 x_1 - a_2 x_2 - \ldots - a_n x_n \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} u \\
&= \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} u = Ax + Bu \\
y &= \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = Cx.
\end{aligned}
$$

Observe that the system is now in *reachable canonical form.*