# Homework 3
# (due on Friday 7 June)

Version: May 24, 2024

Chapters covered: from 1 to 13.

- Upload your **own individual** solutions on Brightspace until **Friday 7 June, 23:59**.
- It can be either a (phone) scan or typeset with Latex, but it should be in any case a **single pdf properly oriented** (correct side up).
- Your submission should be properly written. It should not look like a draft but like something that is **pleasant to read for the TA**. In particular, you should **avoid scratching**. In case you do so for some reason, do it in a clean way, so that it is still easy to read your answer and follow your arguments and computation.
- For each problem you need to **justify your result**, in a clear way. If you provide a numerical result without explanations, then your answer will be ignored.

If you fail to satisfy any of the above conditions you will be penalized. In particular **late submissions, unreadable or poorly formatted submissions will not be graded at all**.
There is a total of 10 points, including 0.25 points per exercise for style.

**P1 [2.25pts]** Let $X$, $Y$, and $Z$ be independent random variables, exponentially distributed with rate parameters $\lambda$, $\mu$, and $\nu$, respectively.

(a) Find $\mathbb{P}(X<Y)$.

(b) Find the distribution of $\min(Y,Z)$.

(c) Find $\mathbb{P}(X<Y<Z)$.

**P2 [3.25pts]** Let $X$ and $Y$ be independent random variables, each uniformly distributed on the interval $(0,1)$. Set $W=XY$ and $Z=X/Y$. Find the joint pdf of $W$ and $Z$.
*Remark:* You will probably consider a function $g$ such that $(X,Y)=g(W,Z)$. You are allowed to give the range of $g$ without justification. You might want to do a sketch of the the region $R$.

**P3 [3.25pts]** For $x>0$ and $n\in\mathbb{N}$ set

$$A_{x,n} := \sum_{\substack{k\in\mathbb{N}:\\ |k-\frac{1}{2}n|\le\frac{1}{2}x\sqrt{n}}} \binom{n}{k}, \quad \text{and} \quad B_{x,n} := \sum_{\substack{k\in\mathbb{N}:\\ |k-n|\le x\sqrt{n}}} \frac{n^k}{k!}.$$

Show that, for any fixed $x>0$, one has $(e^n A_{x,n})/(2^n B_{x,n})\to 1$, as $n\to\infty$.
*Hint:* You might want to use that the sum of $n$ independent Poisson random variables with parameters $\lambda_1,\dots,\lambda_n$ is Poisson distributed of parameter $\lambda_1+\cdots+\lambda_n$. You might also want to use a well chosen limit theorem.

**P4 [1.25 pts]** Let $X$ be a Poisson distributed random variable with parameter 1. Show that $\mathbb{P}(X\ge t)\le e^{t-1}/t^t$ for $t\ge 1$.
*Hint:* You might want to first show that $\mathbb{P}(X\ge t)\le e^{-\theta t}e^{e^{\theta}-1}$ for $\theta\ge 0$.
*Remark:* This bound can be used to show the following threshold phenomena: Let $M_n$ be the maximum of $n$ i.i.d. Poisson distributed random variables of parameter 1. For any fixed $a\in\mathbb{R}$, as $n\to\infty$, we have that

$$\mathbb{P}(M_n\ge(1+a)\log n/\log\log n)\to\begin{cases}1 & \text{if } a<0,\\ 0 & \text{if } a>0.\end{cases}$$

If you want an extra challenge, you can try to prove this as well (but it is not required, and it will not be graded).