# TOPOLOGY BETWEEN THE WAVES VI: QUANTUM TQFTS AND THE VOLUME CONJECTURE

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This workshop aims at bringing together PhD students and young Postdocs to spend one week studying a topic in low-dimensional topology. This is the sixth edition of BTW. Details about previous editions may be found at . This year's edition aims at understanding quantum TQFT's and the volume conjecture.

**Topic.** There are two well-established directions to build (2+1)-dimensional TQFTs: starting from manifolds presented by surgery or from triangulated manifolds. These constructions require different kinds of category as input, *modular tensor categories* and *spherical fusion categories*. The results of such constructions are usually known as *Reshetikhin–Turaev TQFTs (RT)* and *Turaev–Viro TQFTs (TV)*, respectively. The quantum invariants of 3-manifolds arising from these TQFT's are known to be related: the square of the norm of the RT invariant associated with a category $\mathcal{C}$ is equal to the TV invariant associated with the Drinfeld center $\mathcal{Z}(\mathcal{C})$. Our first goal will be to understand this relation and to visualize it in some examples (categories arising from the quantum group associated with the Lie algebra $\mathfrak{sl}_2$).

The techniques for computing the quantum invariants in the RT case have been developed more. On the other hand, the fact that TV invariants are computed from triangulations helps tackling particular topological questions. For instance, as triangulations are related with the hyperbolic volumes of some 3-manifolds, this has led to *the Chen–Yang volume conjecture* that states how TV invariants should carry the information of the hyperbolic volume. Some results were obtained in the direction of the conjecture and they often rely on techniques for RT invariants and the RT–TV correspondence. A second goal of the workshop will be to understand the conjecture and to study recent advances including some arising from the relation between RT and TV.

**Location.** For the first time, the workshop will take place in Fréjus in La Villa Clythia, a CAES CNRS center in the south of France. Fréjus is a small seaside town on the French Riviera, not to be confused with the Fréjus tunnel in the Alps. More details about accessibility will be provided later.

**Funding.** We will provide accommodation and meals for 30 participants, but we will not cover travel expenses. If you encounter any problem with the funding of your train tickets, please contact the organizers.

**Organization.** This document provides the list of talks for the 6th edition of Topology between the waves. Once you picked a talk, it is important that you speak with other participants about your choices (notations, prototypical examples, sources, etc.) to ensure consistency along the week. Do not hesitate to contact organizers if you have questions (titles of sections contains first name(s) of organizer(s) to help you direct your complaints). A schedule will be provided once all the talks are taken.

## 1. Quantum invariants of 3-manifolds

The main reference for this section is [2]; we ask speakers to stick to the notations and conventions there as much as possible. Other references include [11, 10, 8].

**Talk 1** **Jones–Wenzl projectors and the Turaev–Viro invariant**

*Abstract:* The goal of this talk is to introduce the Jones–Wenzl projectors and define the Turaev–Viro invariant, following [2, sections 6.1 to 6.4] and [10, section 3.2]. The talk should include: (1) a definition of the SU(2)-skein module of a 3-manifold via the Kauffman bracket; (2) a definition of the Jones–Wenzl projectors, stressing the connection with Chebyshev polynomials and their universal property (see also [8, chapter 13], in particular Lemma 13.2); (3) the fusion rule [2, figure 6.7], with sketch of proof if time permits; (4) a definition of admissible triples, the Turaev–Viro invariant and Pachner moves, and the statement of invariance [10, section 3.2].

**Talk 2** **The Kirby colour and the Reshetikhin–Turaev invariant**

*Abstract:* The goal of this talk is to define the Reshetikhin–Turaev invariant following [2, sections 6.6 to 6.8]; see also [10, section 3.3] for a quick overview. The talk should include a definition of the Kirby colour and some of its properties (with proof if time permits), a review of surgery (Lickorish–Wallace theorem, Kirby moves and Fenn–Kourke moves, without proof), the definition of the Reshetikhin–Turaev invariant, a proof of invariance, its behaviour under orientation reversal and disjoint union, and finally a comment on the SO(3) (versus SU(2)) variant (see e.g. [8, p. 142]).

**Talk 3** **Turaev–Viro is the square of Reshetikhin–Turaev**

*Abstract:* The goal of this talk is to explain why the Turaev–Viro invariant is the square of the Reshetikhin–Turaev invariant, following Roberts' approach [10]. The talk should include a definition of the chain-mail invariant $CH(M)$, the proof that $CH(M) = TV(M)$ and the proof the $CH(M) = |RT(M)|^2$; see [10, sections 3.1 to 3.4]. If time permits, you can explain the alternative proof of invariance (section 3.5).

**Talk 4** **The Reshetikhin–Turaev TQFT**

*Abstract:* The goal of this talk is to explain how to turn the Reshetikhin–Turaev invariant into a TQFT via the universal construction. Explain the statement following [2, section 7], including Lagrangian submanifolds, Maslov index and extended manifolds; see [11] for more details. Explain the universal construction following [2, section 2]. Finally, give a proof of [2, Theorem 7.9].

## 2. Categorical approach to the WRT and TV TQFT

The goal of this section is to be able to understand the statement [12, Theorem 17.1]

$\boxed{\textbf{Talk 5}}$ **Monoidal categories and Hopf algebras**

*Abstract:* You may use [12] for details and [7, Sections 2.1 and 2.2] for a light reference. Introduce monoidal categories [12, Section 1.2], without dwelling too much on the details (you may skip unitors, triangle coherence... etc.). Introduce rigid categories [12, Section 1.6]. Recall the definition of Hopf algebras, and explain why modules over bialgebras form a monoidal category and why modules over Hopf algebras form a rigid category [12, Section 6]. Introduce pivotal categories. You may want to introduce quickly the pictorial representation of morphisms [12, Section 2.1]. Introduce the quantum group $u_q sl(2)$ and its Hopf algebra structure.

$\boxed{\textbf{Talk 6}}$ **Braidings**

*Abstract:* You may use [12, Chapter 3] for details and [7, Sections 2.3, 2.4 and 6.1] for a light reference. Introduce braidings. Explain what is a braided bialgebra (also called quasitriangular in [12]), and why the category of modules over such an algebra is braided [7, Section 4]. Continue the $u_q sl(2)$ example, relate it to the Kauffman bracket. If time allows, explain the notion of Drinfeld center.

$\boxed{\textbf{Talk 7}}$ **Modular categories and 3-manifold invariants**

*Abstract:* Explain what is a modular category [7, Section 6.3], and how to obtain an invariant of 3-manifolds from such a category [7, Section 6.4].

$\boxed{\textbf{Talk 8}}$ **TQFT's and the universal construction**

*Abstract:* You may use [7, Sections 6.2 and 6.3], [3, Section 3] and [13], keeping in mind that the third reference is master thesis and may contain imprecisions, but is also short and self-contained. After giving the necessary definitions, explain the universal construction [3, Prop. 3.5]. If time allows, explain without proofs why the universal construction does not give a TQFT and how one may solve this problem [3, Section 7].

$\boxed{\textbf{Talk 9}}$ **Turaev Viro TQFT's and the relation with the WRT TQFT**

*Abstract:* Wave your hands to explain how a spherical fusion category (recall the definition) yields a TV TQFT. Explain that the Drinfeld center of a spherical fusion category is a non-degenerate modular category [11, Section 5.3]. State Theorem 17.1.

## 3. Hyperbolic geometry

In this section, we study the basics of hyperbolic geometry, following [9].

A hyperbolic manifold is a complete Riemannian manifold with sectional curvature constantly equal to -1. The concept started to gained interest in the second half of the 20th century: First, the Mostow rigidity theorem showed that if two hyperbolic manifolds are homeomorphic (or even simply homotopy equivalent) they are isometric. Secondly, Thurston showed that there is a plethora of hyperbolic manifolds in dimension 3. The former result allows us to use hyperbolic geometry in the study of topology; for example, to prove that two manifolds are not homeomorphic we can simply find a hyperbolic metric on both of them and show that they have different volumes (or injectivity radius, diameter…). The latter shows that a lot of 3-manifolds are hyperbolic, which makes this technique extremely effective in 3-dimensional topology.

**Talk 10** **Introduction to hyperbolic geometry**

*Abstract:* Start by defining hyperbolic spaces (specifically, the Poincaré disk model of Section 2.1.6 [9]) and hyperbolic manifolds (Sections 3.1.1-3.1.3). Briefly say what is a hyperbolic polyhedron and especially what is an ideal polyhedron, also explaining what it means for something to have points "at infinity" (Sections 3.2.1 and 3.2.2). Then, state Mostow's rigidity theorem (Section 13.3), and give a (very vague) statement of the Geometrization theorem of Perelman. Lastly, introduce the volume conjectures, first in the formulation of Kashaev and then in the formulation of Chen-Yang, which we will focus on in Section 4.

**Talk 11** **Thurston's gluing equations and hyperbolic Dehn filling**

*Abstract:* The most straightforward way to build a hyperbolic structure on a given 3-manifold with toroidal boundary M is to take an ideal triangulation of M and then substitute each simplex by some hyperbolic tetrahedron. In order for this to produce a complete, finite volume hyperbolic manifold, one has to choose the the tetrahedra so that their shapes satisfy a set of equations, known as Thurston's gluing equations. This construction led to the realization, in the late 70s, that a lot of 3-manifolds are hyperbolic. Moreover, by "filling in" the toroidal boundary component, one can almost always deform the hyperbolic structure on the original manifold to give a hyperbolic structure on the filled manifold. This is known as the Hyperbolic Dehn filling theorem.

In this talk we will see a detailed treatment of the gluing equations (following 14.1 of [9]); if time permits, also give the sketch of an example (the figure-eight knot, from Section 14.1.6). We will also define Dehn fillings, which formalize the concept of filling mentioned above, and state the Hyperbolic Dehn Filling theorem (Section 15.1).

## 4. Towards the volume conjecture

In this section we see recent applications of the RT–TV correspondence towards the volume conjecture.

**Talk 12** **Turaev–Viro invariants and the colored Jones polynomials**

*Abstract:*

**Talk 13** **Turaev–Viro invariants and the Gromov norm**

*Abstract:* By replacing the hyperbolic volume with the Gromov norm $\|M\|$, one obtains a generalization of the volume conjecture to 3-manifolds $M$ that are not necessarily hyperbolic ([4, Conjecture 8.1]). The goal of this talk is to present a formula which relates the TV invariants and $\|M\|$, following [4].

After briefly introducing the Gromov norm and its basic properties (Section 2.1), state and prove Theorem 1.1. The proof can be divided into 3 steps: Start by giving an analytical estimate of the 6j-symbols defining the TV invariants (Section 4). Then, use the RT–TV correspondence to study the TV invariants under the operation of cutting a 3-manifold along tori. Along with the first step, this will allow you to relate the TV invariants to $\|M\|$, when $M$ is hyperbolic or Seifert fibered (Sections 5 and 6). Lastly, for a generic 3-manifold $M$, use the Geometrization theorem (from Talk 10) to revert to the previous case (Section 7.1). As applications of Theorem 1.1, state Corollaries 7.2, 1.4 and 8.2. In particular, the last result verifies the generalized volume conjecture for a particular class of 3-manifolds.

**Talk 14** **AMU conjecture**

*Abstract:* The AMU conjecture (for Andersen–Masbaum–Ueno) states that the representations of the mapping class group defined by the RT TQFT can detect pseudo-Anosov elements. The results of Talk 13 [4] establish that the volume conjecture implies the AMU conjecture. In fact, one does not even need the full volume conjecture: if the TV invariant of a manifold fibering over $\mathbb{S}^1$ grows exponentially, then the monodromy of the fibering satisfies the AMU conjecture.

The main reference for this talk is [5]. After giving a working definitions of pseudo-Anosov mapping classes (8.4.2 in [9]) and of mapping torus, we will see a proof of the aforementioned implication [5]. In previous talks, we have seen some examples of manifolds with exponentially growing TV invariants, however, they were not necessarily mapping tori. We will see how to use the Stallings trick to obtain manifolds that fiber over $\mathbb{S}^1$, and thus get infinitely many examples of mapping classes satisfying the AMU conjecture.

**Talk 15** **Volume conjecture for graph manifolds**

*Abstract:* In this talk, we will see how to tackle the volume conjecture for graph manifolds, i.e. manifolds with simplicial volume equal to zero, following [6]. This requires studying the operator associated to a Seifert fibered cobordism.

**Talk 16** **Volume conjecture for fundamental shadow links**

*Abstract:* In this talk, we will see how to prove the volume conjecture for a wide family of hyperbolic manifolds, the exteriors of fundamental shadow links. We will follow [1]. To do this, it is necessary to prove a sharper upper bound on the growth of the 6j-symbol. Section 4 (containing the proof of the technical lemmas) can be skipped.

## References


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