Taller 2 AED

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Taller #2 AED

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Distancias, Al. vecs y matrices aleatorias

1. $\bar{x} = \begin{bmatrix} 6 \\ 8 \\ 2 \end{bmatrix}$

$S_n = \begin{bmatrix} 7.5 & 5 & -1.75 \\ 5 & 10 & 1.5 \\ -1.75 & 1.5 & 2.5 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 0.57 & -0.4 \\ 0.57 & 1 & 0.3 \\ -0.4 & 0.3 & 1 \end{bmatrix}$

2. b. i. $long(x) = \sqrt{5^2 + 1^2 + 3^2} = \sqrt{35}$

$long(y) = \sqrt{(-1)^2 + 3^2 + 1^2} = \sqrt{11}$

ii. $\theta = \cos^{-1}\left(\frac{x'y}{\sqrt{35}\sqrt{11}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{385}}\right) = 87.08°$

iii. $P_x(y) = \left(\frac{y'x}{x'x}\right)x = \left(\frac{[-1\ 3\ 1]\begin{bmatrix}5\\1\\3\end{bmatrix}}{[5\ 1\ 3]\begin{bmatrix}5\\1\\3\end{bmatrix}}\right)\begin{bmatrix}5\\1\\3\end{bmatrix} = \left(\frac{1}{35}\right)\begin{bmatrix}5\\1\\3\end{bmatrix}$

3. $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$

a. A sí es simétrica.

b. vals. propios : $e = 5$ y $e = 10$.

Dado q' ambos son positivos, A es def. pos.

c. $\lambda_1 = 5$, $\lambda_2 = 10$

$e_1 = \begin{bmatrix} -0.89 \\ 0.45 \end{bmatrix}$ $e_2 = \begin{bmatrix} -0.45 \\ -0.89 \end{bmatrix}$

d. $A = 5\begin{bmatrix} -0.89 \\ 0.45 \end{bmatrix}[-0.89\ \ 0.45] + 10\begin{bmatrix} -0.45 \\ -0.89 \end{bmatrix}[-0.45\ \ -0.89]$

$A = \begin{bmatrix} -0.89 & -0.45 \\ 0.45 & -0.89 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} -0.89 & 0.45 \\ -0.45 & -0.89 \end{bmatrix}$

e. $A^{-1} = \begin{bmatrix} -0.89 & -0.45 \\ 0.45 & -0.89 \end{bmatrix}\begin{bmatrix} 1/5 & 0 \\ 0 & 1/10 \end{bmatrix}\begin{bmatrix} -0.89 & 0.45 \\ -0.45 & -0.89 \end{bmatrix}$

$= \frac{1}{50}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$ $\lambda_1 = 1/5$ $\lambda_2 = 1/10$

$e_1 = \begin{bmatrix} -0.89 \\ 0.45 \end{bmatrix}$ $e_2 = \begin{bmatrix} -0.45 \\ -0.89 \end{bmatrix}$

4. $V^{1/2} \rho V^{1/2} = \Sigma \rightarrow$ var-cov poblac.

$= \begin{bmatrix} \sqrt{s_{11}} & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & & & \sqrt{s_{pp}} \end{bmatrix}\begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & & \\ \vdots & & \ddots & \\ \rho_{p1} & & & 1 \end{bmatrix}\begin{bmatrix} \sqrt{s_{11}} & 0 & \cdots & 0 \\ 0 & \ddots & & \\ 0 & & & \sqrt{s_{pp}} \end{bmatrix} = \begin{bmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{n1} & \cdots & s_{nn} \end{bmatrix}$

$= \begin{bmatrix} 1 & \frac{s_{12}}{\sqrt{s_{11}}\sqrt{s_{22}}} & \cdots & \frac{s_{1p}}{\sqrt{s_{11}}\sqrt{s_{pp}}} \\ \frac{s_{p1}}{\sqrt{s_{11}}\sqrt{s_{pp}}} & \ddots & & 1 \end{bmatrix}$

$= \begin{bmatrix} \sqrt{s_{11}} & \frac{s_{12}}{\sqrt{s_{22}}} & \cdots & \frac{s_{1p}}{\sqrt{s_{pp}}} \\ \vdots & & \ddots & \\ \frac{s_{p1}}{\sqrt{s_{11}}} & & & \sqrt{s_{pp}} \end{bmatrix}\begin{bmatrix} \sqrt{s_{11}} & \cdots 0 & 0 \\ 0 & \ddots & \\ 0 & & \sqrt{s_{pp}} \end{bmatrix}$

$= \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1p} \\ \vdots & \ddots & & \\ s_{p1} & & \ddots & s_{pp} \end{bmatrix} = \Sigma$

- $\Sigma = V^{1/2}\rho V^{1/2}$

$(V^{1/2})^{-1}\Sigma(V^{1/2})^{-1} = (V^{1/2})^{-1}(V^{1/2}\rho V^{1/2})(V^{1/2})^{-1} = \rho$

5. a. $X_1 - 2X_2$

$E(X_1 - 2X_2) = E[X_2] - 2E[X_2]$

$var(X_1 - 2X_2) = var(X_1) + 4\,var(X_2) - 4\,cov(X_1, X_2)$

$= S_{11} + 4S_{22} - 4S_{12}$

b. $-X_1 + 3X_2$

$E(-X_1 + 3X_2) = -E[X_1] + 3E[X_2]$

$var(-X_1 + 3X_2) = var(X_1) + 9\,var(X_2) - 6\,cov(X_1, X_2) = S_{11} + 9S_{22} - 6S_{12}$.

c. $X_1 + X_2 + X_3$

$E[X_1 + X_2 + X_3] = E[X_1] + E[X_2] + E[X_3]$

$Var[X_1 + X_2 + X_3] = Var(X_1) + Var(X_2) + Var(X_3) + 2Cov(X_1, X_2) + 2Cov(X_1, X_3) + 2Cov(X_2, X_3)$

$= S_{11} + S_{22} + S_{33} + 2S_{12} + 2S_{13} + 2S_{23}$

d. $X_1 + 2X_2 - X_3$

$E[X_1 + 2X_2 - X_3] = E(X_1) + 2E[X_2] - E[X_3]$

$Var(X_1 + 2X_2 - X_3) = var(X_1) + 4\,var(X_2) + var(X_3) + 4\,cov(X_1, X_2) - 2\,cov(X_1, X_3) - 4\,cov(X_2, X_3)$

$= S_{11} + 4S_{22} + S_{33} + 4S_{12} - 2S_{13} - 4S_{23}$.

e. $3X_1 - 4X_2$ $\quad X_1$ y X_2 indeps.

$E[3X_1 - 4X_2] = 3E[X_1] - 4E[X_2]$

$Var(3X_1 - 4X_2) = 9\,var(X_1) + 16\,var(X_2)$

6. c.

$d_1 = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$ $\quad d_2 \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$

$\ell(d_1) = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$

$\ell(d_2) = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\cos\theta_{12} = \frac{d_1' d_2'}{\ell(d_1)\,\ell(d_2)} = \frac{-4}{8} = \frac{-1}{2}$

sabiendo que $r_{12} = \cos\theta_{12}$ tenemos que $r_{12} = -1/2$

con lo que

$$R = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

calculando $\sqrt{s_{11}} = 4$ y $\sqrt{s_{22}} = 1$

tenemos que

$$S_n = \begin{bmatrix} 1(4^2) & -1/2(4)(1) \\ -1/2(4)(1) & 1(1)^2 \end{bmatrix} = \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}$$

d) $|S| = 16 - 4 = 12.$

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7. $|S| = (s_{11} s_{22} \cdots s_{pp})|R|$

S es la matriz de varianzas y covarianzas muestrales, R es la de coeficientes de correlación muestral, note que se mantiene la relación de

$\Sigma = V^{1/2} \rho V^{1/2}$ para las muestrales, dando que
$S = V^{1/2} R V^{1/2}$ Donde $V^{1/2}$ son las desviaciones estandar en la diagonal con los demas entradas 0.

Note tambien que $|V^{1/2} R V^{1/2}| = |V^{1/2}| R |V^{1/2}|$
entonces $|S| = |V^{1/2}||R||V^{1/2}|$ y como $|V^{1/2}|$ es diagonal
$|V^{1/2}| = \sqrt{s_{11}} \sqrt{s_{22}} \sqrt{s_{33}} \cdots \sqrt{s_{pp}}$
con lo que $|S| = (\sqrt{s_{11}} \sqrt{s_{22}} \cdots \sqrt{s_{pp}})|R| (\sqrt{s_{11}} \sqrt{s_{22}} \cdots \sqrt{s_{pp}})$
y $|S| = (s_{11} s_{22} \cdots s_{pp})|R|$ ①

8. $E[V] = \mu_V$
$E(V-\mu_V)(V-\mu_V)' = \Sigma_V$

$$\begin{aligned}
E(V-\mu_V)(V-\mu_V)' &= E(V(V-\mu_V)' - \mu_V(V-\mu_V)') \\
&= E(VV' - V\mu_V' - \mu_V V' + \mu_V\mu_V') \\
&= E(VV') - E(V\mu_V') - E(\mu_V V') + \mu_V\mu_V' \\
&= E(VV') - \mu_V\mu_V' - \mu_V\mu_V' + \mu_V\mu_V' \\
&= E(VV') - \mu_V\mu_V' = \Sigma_V
\end{aligned}$$

$\Rightarrow E(VV') = \Sigma_V + \mu_V\mu_V'.$

10.

$$\Sigma = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

i) X_1, X_2 no son indep $\operatorname{cov}(X_1, X_2) \neq 0$
b) X_2, X_3 Son indep, son normales y $\operatorname{cov}(X_1, X_2) = 0$

9. $f(x) = \frac{1}{2\pi |\Sigma|^{1/2}} e^{-\frac{(x-\mu)' \Sigma (x-\mu)}{2}}$

$\Sigma = \begin{bmatrix} 2 & \rho_{12}\sqrt{2} \\ \rho_{12}\sqrt{2} & 1 \end{bmatrix} = \begin{bmatrix} 2 & -0.8\sqrt{2} \\ -0.8\sqrt{2} & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1.13 \\ -1.13 & 1 \end{bmatrix}$

$f(x) = \frac{1}{2\pi \begin{vmatrix} 2 & -1.13 \\ -1.13 & 1 \end{vmatrix}^{1/2}} e^{-\frac{\left(\begin{bmatrix} x_1-1 \\ x_2-3 \end{bmatrix}\right)' \begin{bmatrix} 2 & -1.13 \\ -1.13 & 1 \end{bmatrix} \left(\begin{bmatrix} x_1-1 \\ x_2-3 \end{bmatrix}\right)}{2}}$

$= \frac{1}{2\pi\sqrt{2-1.28}} e^{-\frac{([x_1-1 \quad x_2-3]) \begin{bmatrix} 2x_1-2-1.13x_2+3.39 \\ -1.13x_1+1.13+x_2-3 \end{bmatrix}}{2}}$

$= \frac{1}{2\pi\sqrt{0.72}} e^{-\frac{([x_1-1 \quad x_2-3]) \begin{bmatrix} 2x_1-1.13x_2+1.39 \\ -1.13x_1+x_2-3.39 \end{bmatrix}}{2}}$

$= \frac{1}{2\pi\sqrt{0.72}} e^{-\frac{((x_1-1)(2x_1-1.13x_2+1.39)+(x_2-3)(-1.13x_1+x_2-3.39))}{2}}$

$= \frac{1}{2\pi\sqrt{0.72}} e^{-\frac{(2x_1^2-1.13x_1x_2+1.39x_1-2x_1+1.13x_2-1.39-1.13x_1x_2+x_2^2-3.39x_2+3.39x_1-3x_2+10.17)}{2}}$

$= \frac{1}{2\pi\sqrt{0.72}} e^{-\frac{(2x_1^2+x_2^2-2.26x_1x_2+2.78x_1-5.26x_2+8.78)}{2}}$

b) $(x-\mu)'\Sigma^{-1}(x-\mu) = \left(\begin{bmatrix} x_1-1 \\ x_2-3 \end{bmatrix}\right)' \begin{bmatrix} 2 & -1.13 \\ -1.13 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} x_1-1 \\ x_2-3 \end{bmatrix}\right)$

$= [x_1-1 \quad x_2-3] \begin{bmatrix} 2 & -1.13 \\ -1.13 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_1-1 \\ x_2-3 \end{bmatrix}$

$= [x_1-1 \quad x_2-3] \begin{bmatrix} 1.38 & 1.56 \\ 1.56 & 2.76 \end{bmatrix} \begin{bmatrix} x_1-1 \\ x_2-3 \end{bmatrix}$

$= [x_1-1 \quad x_2-3] \begin{bmatrix} (x_1-1)1.38+(x_2-3)1.56 \\ (x_1-1)1.56+(x_2-3)2.76 \end{bmatrix}$

$= (x_1-1)^2 1.38 + 2(x_1-1)(x_2-3)1.56 + (x_2-3)^2 2.76$