Ejercicios

1) • C.R para coordenadas cartesianas: $U_x = V_y$, $U_y = -V_x$

• $z = x + iy = r(\cos\theta + i\,\text{sen}\,\theta)$

• $x = r\cos\theta$ • $y = r\,\text{sen}\,\theta$ • $r = \sqrt{x^2+y^2}$ • $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

- $\frac{dr}{dx} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$

- $\frac{dr}{dy} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{r\,\text{sen}\,\theta}{r} = \text{sen}\,\theta$

- $\frac{d\theta}{dx} = \frac{\left(-\frac{y}{x^2}\right)}{1+\frac{y^2}{x^2}} = -\frac{\frac{y}{x^2}}{\frac{x^2+y^2}{x^2}} = -\frac{yx^2}{x^2(x^2+y^2)} = -\frac{r\,\text{sen}\,\theta}{r^2} = -\frac{\text{sen}\,\theta}{r}$

- $\frac{d\theta}{dy} = \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}} = \frac{\frac{1}{x}}{\frac{x^2+y^2}{x^2}} = \frac{x}{(x^2+y^2)} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$

ahora Note que

• $\frac{du}{dx} = \frac{dr}{dx}\cdot\frac{du}{dr} + \frac{d\theta}{dx}\cdot\frac{du}{d\theta} = \cos\theta\cdot\frac{du}{dr} - \frac{\text{sen}\,\theta}{r}\cdot\frac{du}{d\theta}$

• $\frac{du}{dy} = \frac{dr}{dy}\cdot\frac{du}{dr} + \frac{d\theta}{dy}\cdot\frac{du}{d\theta} = \sin\theta\cdot\frac{du}{dr} + \frac{\cos\theta}{r}\cdot\frac{du}{d\theta}$

• $-\frac{dv}{dx} = \frac{dr}{dx}\cdot\frac{dv}{dr} + \frac{d\theta}{dx}\cdot\frac{dv}{d\theta} = -\cos\theta\cdot\frac{dv}{dr} + \frac{\text{sen}\,\theta}{r}\cdot\frac{dv}{d\theta}$

• $\frac{dv}{dy} = \frac{dr}{dy}\cdot\frac{dv}{dr} + \frac{d\theta}{dy}\cdot\frac{dv}{d\theta} = \text{sen}\,\theta\cdot\frac{dv}{dr} + \frac{\cos\theta}{r}\cdot\frac{dv}{d\theta}$

Igualando y Multiplicando

$\cos^2\theta\,\frac{du}{dr} - \cancel{\frac{\cos\theta\,\text{sen}\,\theta}{r}\,\frac{du}{d\theta}} = \cancel{\cos\theta\,\text{sen}\,\theta\,\frac{dv}{dr}} + \frac{\cos^2\theta}{r}\,\frac{dv}{d\theta}$

$\text{sen}^2\theta\,\frac{du}{dr} + \cancel{\frac{\cos\theta\,\text{sen}\,\theta}{r}\,\frac{du}{d\theta}} = \cancel{-\cos\theta\,\text{sen}\,\theta\,\frac{dv}{dr}} + \frac{\text{sen}^2\theta}{r}\,\frac{dv}{d\theta}$

$\cos^2\theta\,\frac{du}{dr} + \text{sen}^2\theta\,\frac{du}{dr} = \frac{\cos^2\theta}{r}\,\frac{dv}{d\theta} + \frac{\text{sen}^2\theta}{r}\,\frac{dv}{d\theta}$

$$(\cos^2\theta + \operatorname{sen}^2\theta)\frac{du}{dr} = \left(\frac{\cos^2\theta + \sin^2\theta}{r}\right)\frac{dv}{d\theta}$$

$$\frac{du}{dr} = \frac{1}{r}\frac{dv}{d\theta}$$

$$r\frac{du}{dr} = \frac{dv}{d\theta}$$

$$\operatorname{sen}\theta\cos\theta\frac{du}{dr} - \frac{\sin^2\theta}{r}\frac{du}{d\theta} = \operatorname{sen}^2\theta\frac{dv}{dr} + \frac{\operatorname{sen}\theta\cos\theta}{r}\frac{dv}{d\theta}$$

$$-\operatorname{sen}\theta\cos\theta\frac{du}{dr} - \frac{\cos^2\theta}{r}\frac{du}{d\theta} = \cos^2\theta\frac{dv}{dr} - \frac{\operatorname{sen}\theta\cos\theta}{r}\frac{dv}{d\theta}$$

$$-\frac{\operatorname{sen}^2\theta}{r}\frac{du}{d\theta} - \frac{\cos^2\theta}{r}\frac{du}{d\theta} = \operatorname{sen}^2\theta\frac{dv}{dr} + \cos^2\theta\frac{dv}{dr}$$

$$-\left(\frac{\operatorname{sen}^2\theta + \cos^2\theta}{r}\right)\frac{du}{d\theta} = (\operatorname{sen}^2\theta + \cos^2\theta)\frac{dv}{dr}$$

$$-\frac{1}{r}\frac{du}{d\theta} = \frac{dv}{dr}$$

$$\frac{du}{d\theta} = -r\frac{dv}{dr}$$

Por lo tanto:

$$\frac{dv}{d\theta} = r\frac{du}{dr} \qquad \frac{du}{d\theta} = -r\frac{dv}{dr}$$

2) $f(z) = \frac{1}{z}$

Note que: $\frac{1}{r} \cdot \frac{1}{\cos\theta + i\sin\theta} = \frac{1}{r} \cdot \frac{1}{\cos\theta + i\sin\theta} \cdot \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$

$= \frac{1}{r} \cdot \frac{\cos\theta - i\sin\theta}{\cos\theta^2 + \sin\theta^2} = \frac{1}{r} \cdot \frac{\cos\theta - i\sin\theta}{1}$

entonces $\frac{1}{z} = r^{-1}\cos\theta - i r^{-1}\sin\theta$

$f'(z) = (\cos\theta - i\sin\theta)(-r^{-2}\cos\theta + i r^{-2}\sin\theta) = (\cos\theta - i\sin\theta)(-r^{-2})(\cos\theta - i\sin\theta)$

$= \frac{1}{r^2} \cdot \frac{1}{(\cos\theta + i\sin\theta)} \cdot \frac{1}{(\cos\theta + i\sin\theta)} = \frac{1}{-r^2(\cos\theta + i\sin\theta)^2} = -\frac{1}{z^2}$