2. Demues. que $f'(z)$ no existe en:

a. $f(z) = \operatorname{Re} z \rightarrow f'(z) = \lim\limits_{\Delta z \to z_0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim\limits_{\Delta z \to z_0} \dfrac{\operatorname{Re}(z + \Delta z) - \operatorname{Re}(z)}{\Delta z}$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\operatorname{Re}(x + iy + \Delta x + i\Delta y) - \operatorname{Re}(x + iy)}{\Delta z} = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\cancel{x} + \Delta x - \cancel{x}}{\Delta x + i\Delta y}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x}{\Delta x + i\Delta y} = \begin{cases} \lim\limits_{\Delta x \to 0} \dfrac{\Delta x}{\Delta x} = 1, & \text{si tomamos } (\Delta x, 0) \\ \lim\limits_{\Delta y \to 0} \dfrac{0}{i\Delta y} = 0, & \text{si tomamos } (0, \Delta y) \end{cases}$$

[El límite esta tomando valores 1 y 0, esto contradice la unicidad del límite. Por lo tanto $f(z) = \operatorname{Re}(z)$ no es derivable en ningún punto.]

b. $f(z) = \operatorname{Im} z \rightarrow f'(z) = \lim_{\Delta z \to z_0} \dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to z_0} \dfrac{\operatorname{Im}(z_0+\Delta z) - \operatorname{Im}(z_0)}{\Delta z}$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\operatorname{Im}(x+iy+\Delta x+i\Delta y) - \operatorname{Im}(x+iy)}{\Delta z} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\cancel{iy} + i\Delta y - \cancel{iy}}{\Delta x + i\Delta y}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{i\Delta y}{\Delta x + i\Delta y} = \begin{cases} \lim\limits_{\Delta x \to 0} \dfrac{0}{\Delta x + 0} = 0, & \text{si } (\Delta x, 0) \\ \lim\limits_{\Delta y \to 0} \dfrac{i\Delta y}{0 + i\Delta y} = 1, & \text{si } (0, \Delta y) \end{cases}$$

[Se contradice la unicidad del límite, pues está tomando los valores 0 y 1. Por lo tanto $f(z) = \operatorname{Im} z$ no es derivable en ningún punto.]

Ejercicios

1. Calcular $f(z)=\frac{(1+z^2)^4}{z^2}$, $z\neq 0$ $\longrightarrow$ Haciendo uso de la regla de la cadena:

$$f(z)=\left(\frac{(1+z^2)2}{z}\right)^2 , \quad \frac{d}{dz}(f(g))=\frac{d}{dg}(f(g))\cdot\frac{d}{dz}(g)$$

$$f'(z)=\frac{d}{dg}(g^2)\cdot\frac{d}{dz}\left(\frac{(1+z^2)^2}{z}\right) = 2g\cdot\frac{2(1+z^2)\cdot 2z\cdot z-(1+z^2)^2\cdot 1}{z^2} , \quad g=\frac{(1+z^2)^2}{z}$$

$$f'(z)=2\cdot\frac{(1+z^2)^2}{z}\cdot\frac{2(1+z^2)\cdot 2z^2-(1+z^2)^2}{z^2} = 2\cdot\frac{(1+z^2)^2}{z}\cdot\frac{4z^2(1+z^2)-(1+2z^2+z^4)}{z^2}$$

$$f'(z)=\frac{(2+4z^2+2z^4)}{z}\cdot\frac{(4z^2+4z^4)-1-2z^2-z^4}{z^2} = \frac{2+4z^2+2z^4}{z}\cdot\frac{2z^2+3z^4-1}{z^3}$$

$$f'(z)=\frac{\cancel{4z^2}+\cancel{6z^4}-2+\cancel{8z^4}+\cancel{12z^6}-\cancel{4z^2}+\cancel{4z^6}+\cancel{6z^8}-\cancel{2z^4}}{z^3} = \frac{6z^8+16z^6+12z^4-2}{z^3}$$

$$\boxed{f'(z)=6z^5+16z^3+12z-\frac{2}{z^3}}$$