Pricing FInancial Derivatives

Project 2

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Exercise 2

We are given a Stochastic Differential Equation:

$$dX_t = a(m - X_t)dt + \sigma dB_t \tag{1}$$
$$X_0 = x \tag{2}$$

where a>0, σ >0, m$\geq$ 0

$$f(X_t, t) = e^{at}X_t \tag{3}$$
$$\Rightarrow^{From\ Itô} \quad f(X_t, t) = aX_t e^{at}dt + e^{at}dX_t \tag{4}$$

Substituting dX_t with (1) we get:

$$f(X_t, t) = aX_t e^{at}dt + ae^{at}mdt - ae^{at}X_t dt + e^{at}\sigma dB_t \tag{5}$$
$$\Rightarrow = ame^{at}dt + e^{at}\sigma dB_t \tag{6}$$

Now integrating over $[0, t]$:

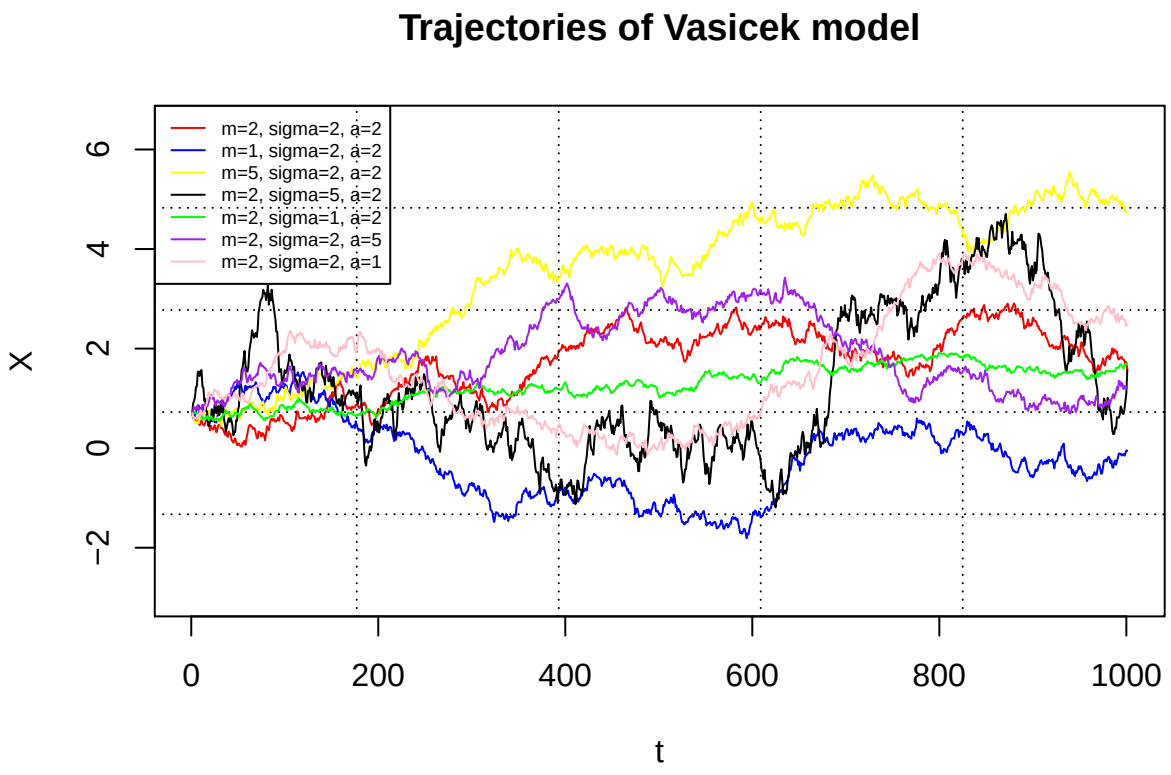
$$f(X_t, t) = e^{at}X_t = X_0 + \int_0^t ame^{as}ds + \int_0^t e^{as}\sigma dB_s \tag{7}$$
$$\Rightarrow X_t = xe^{-at} + e^{-at}[me^{as}]_0^t + e^{-at}\sigma \int_0^t e^{as}dB_s \tag{8}$$
$$X_t = xe^{-at} + m - me^{at} + e^{-at}\sigma \int_0^t e^{as}dB_s \tag{9}$$
$$X_t = m + (x - m)e^{-at} + \sigma e^{-at} \int_0^t e^{as}dB_s \tag{10}$$

$$E[X_t] = xe^{-at} + m(1 - e^{-at}) \tag{11}$$

$$Var[X_t] = E[(X_t - E[X_t])^2] \tag{12}$$
$$= E\left[\int_0^t \sigma^2 e^{2(s-t)a} dB_s\right] \tag{13}$$
$$= \sigma^2 e^{-2at} E\left[\int_0^t e^{2sa} dB_s\right] \tag{14}$$
$$= \sigma^2 e^{-2at} \left[\frac{e^{2sa}}{2a}\right]_0^t \tag{15}$$
$$= \frac{\sigma^2}{2a} e^{-at}(e^{at} - 1) \tag{16}$$
$$= \frac{\sigma^2}{2a}(1 - e^{2at}) \tag{17}$$

As $t \to \infty,\ xe^{-at} \to 0,\ (1 - e^{-at}) \to 1$

$$\Rightarrow lim_{t\to\infty} E[X_t] = m \tag{18}$$



Appendix

```
library(ggplot2)


N <- 1000
T <- 1
x <- 0.7
dt <- 1 / N

Vasicek_function <- function(m, sigma, a){
  dX <- numeric ( N +1)
  dX[1] <- x
  Z <- rnorm ( N, 0, 1 )
  for ( i in 1: N ){
    dX [i +1] <- dX [ i ] + a*(m -  dX [ i ]) * dt + sigma * sqrt ( dt ) * Z [ i ]
  dX <- ts (dX , start =0 , deltat =1 / N )
  return(dX)
  }
}

V<-matrix(0,7,N+1)
m<-c(2,1,5,2,2,2,2)
sigma<-c(2,2,2,5,1,2,2)
a<-c(2,2,2,2,2,5,1)

for (i in 1:nrow(V)){
  V[i,]<-Vasicek_function(m[i],sigma[i],a[i])
}
V
color<-c("red","blue","yellow","black","green","purple","pink")

plot(V[1,], main= "Trajectories of Vasicek model",
     col="red", ylim=c(-3,6.5), xlab="t", ylab="X",type = "l")
grid(5,5,col="black")

for (i in 2:nrow(V)){
lines(V[i,], col= color[i])
}

legend("topleft",
       c("m=2, sigma=2, a=2","m=1, sigma=2, a=2","m=5, sigma=2, a=2",
         "m=2, sigma=5, a=2","m=2, sigma=1, a=2","m=2, sigma=2, a=5","m=2, sigma=2, a=1"),
       col=c("red","blue","yellow","black",
               "green","purple","pink"),
       lty=c(1,1),
       cex=0.58)
```