Pricing Financial Derivatives
Project for Evaluation 3 - March 7, 2017

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Exercise 1: Pricing and Hedging Asian options

Consider a stock whose price under the risk neutral probability $\tilde{\mathbf{P}}$ follows the Black and Scholes model:

$$dS_t = S_t(rdt + \sigma dW_t), \qquad t \in [0,T] \text{ where: } S_0 > 0, r > 0 \tag{1}$$

Consider a fixed-strike Asian call option whose payoff is given by:

$$H = \max(\frac{1}{T}\int_0^T S_u du - K, 0) \tag{2}$$

Consider the process:

$$Y_t = \int_0^t S_u du \qquad \text{that satisfies: } dY_t = S_t dt \tag{3}$$

Then S_t and Y_t are two Itô processes, and the pair (S_t, Y_t) has the Markov property. Therefore there exists a function $v(t,x,y)$ such that the price of the option at time $t \in [0,T]$ is of the form:

$$V_t = v(t, S_t, Y_t) \tag{4}$$

1. (i) Show that the function $v(t,x,y)$ satisfies the following partial differential equation for $0 \leq t < T$, $x \geq 0$ and $y \in \mathbf{R}$:

$$\partial_t v(t,x,y) + rx\partial_x v(t,x,y) + x\partial_y v(t,x,y) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} v(t,x,y) = rv(t,x,y) \tag{5}$$

(ii)Check the following boundary conditions:

$$v(t,0,y) = e^{-r(T-t)} \max\left(\frac{y}{T} - K\right) \qquad v(T,x,y) = \max\left(\frac{y}{T} - K\right) \tag{6}$$

(i) We know that S_t and Y_t are Itô's processes, in order to obtain the PDE of the form (5) we can apply Itô's formula to the discounted portfolio value, with:

$$dR_t = rR_t dt \longrightarrow R_t = e^{rt} \tag{7}$$

We obtain the discounted stock price:

$$\tilde{V}_t = \frac{V_t}{R_t} = \frac{v(t, S_t, Y_t)}{e^{rt}} \tag{8}$$

And so;

$$f(t, x, y) := \tilde{V}_t = e^{-rt} v(t, x = S_t, y = Y_t) \tag{9}$$

Recall Itô's formula:

$$df(t, x, y) = \partial_x f(t, x, y) dx + \partial_y f(t, x, y) dy + \partial_t f(t, x, y) dt + \frac{1}{2} \partial^2_{xx} (dx)^2 \tag{10}$$

Firstly, we compute the differentials $dx, dy, (dx)^2$ in Itô's formula (10). From the definition of the Y_t process (3), and from the stock price equation (1) we gt:

$$\begin{aligned}
dy &= dY_t = S_t dt \\
dx &= dS_t = S_t(rdt + \sigma dW_t) \\
(dx)^2 &= (dS_t)^2 = S_t^2 (rdt + \sigma dW_t)^2 = S_t^2 (r^2 dt^2 + \sigma^2 dW_t^2 + 2r\sigma dt dW_t) = S_t^2 \sigma^2 dt
\end{aligned}$$

Where we have used the fact that W_t is a Brownian motion under $\tilde{\mathrm{P}}$ such that from Itô's calculus rules: $(dt)^2 = 0, (dW_t)^2 = dt$ and $dt dW_t = 0$.

Secondly, we compute the partial derivatives of $f(t, x, y)$ defined in (9):

$$\begin{aligned}
\frac{\partial f(t, x, y)}{\partial t} &= -re^{-rt} v(t, x, y) + e^{-rt} \frac{\partial v(t, x, y)}{\partial t} \\
\frac{\partial f(t, x, y)}{\partial y} &= e^{-rt} \frac{\partial v(t, x, y)}{\partial y} \\
\frac{\partial f(t, x, y)}{\partial x} &= e^{-rt} \frac{\partial v(t, x, y)}{\partial x} \\
\frac{\partial^2 f(t, x, y)}{\partial x^2} &= e^{-rt} \frac{\partial^2 v(t, x, y)}{\partial x^2}
\end{aligned}$$

Putting together the partial derivatives and the differentials into (10) and for $v := v(t, x, y)$ we get:

$$d\tilde{V}(t, x, y) = e^{-rt} \frac{\partial v}{\partial x} S_t (rdt + \sigma dW_t) + e^{-rt} \frac{\partial v}{\partial y} S_t dt + (-re^{-rt} v + e^{-rt} \frac{\partial v}{\partial t}) dt + \frac{1}{2} (\sigma S_t)^2 e^{-rt} \frac{\partial^2 v}{\partial x^2} dt \tag{11}$$

$$= e^{-rt} \frac{\partial v}{\partial x} S_t \sigma dW_t + e^{-rt} \left(rS_t \frac{\partial v}{\partial x} + S_t \frac{\partial v}{\partial y} - rv + \frac{\partial v}{\partial t} + \frac{1}{2} (\sigma S_t)^2 \frac{\partial^2 v}{\partial x^2} \right) dt \tag{12}$$

The second step is to apply the self-financing portfolio condition to the discounted portfolio $\tilde{V}_t$. That is:

$$d\tilde{V}_t = b_t d\tilde{S}_t = b_t \tilde{S}_t \sigma dW_t \tag{13}$$

Where $\tilde{S}_t$ is the discounted stock price under the risk-neutral probability. Therefore for the discounted portfolio to meet the self-financing requirement, the coefficient of the dt term in equation (12) and (13) have to be equal, being null for this last equation. Followingly;

$$e^{-rt} \left(rS_t \frac{\partial v}{\partial x} + S_t \frac{\partial v}{\partial y} - rv + \frac{\partial v}{\partial t} + \frac{1}{2} (\sigma S_t)^2 \frac{\partial^2 v}{\partial x^2} \right) = 0$$

$$rS_t\frac{\partial v(t,x,y)}{\partial x}+S_t\frac{\partial v(t,x,y)}{\partial y}+\frac{\partial v(t,x,y)}{\partial t}+\frac{1}{2}(\sigma S_t)^2\frac{\partial^2 v(t,x,y)}{\partial x^2}=rv(t,x,y) \tag{14}$$

(ii) To check the boundary conditions we recall the theorem that states that the Black-Scholes model is complete: under the risk neutral probability $\tilde{P}$, for every positivie variable H F_T-measurable there is an admissible self-financing portfolio such that its value at time T is $V_T = H$. Therefore for an Asian call with exercise price K and maturity T the final condition of the call is just its payoff at time T:

$$v(T,x,y)=H=\max(\frac{1}{T}\int_0^T S_u du - K, 0)=\max(\frac{y}{T}-K,0) \tag{15}$$

An Asian option is an option whose payoff depends on the average price of the underlying asset over a certain period of time so even if at a given time $S_t = 0$ the payoff still depends on the path taking up to time t and therefore, the value of the portfolio at the boundary $S_t = 0$ is:

$$v(t,0,y)=e^{-r(T-t)}\max(\frac{y}{T}-K,0) \tag{16}$$

2. i) Using the pricing formula below and the expression for S_t, find a closed expression for $v(t,x,y)$ when $K = 0$.

$$V_t = e^{-r(T-t)}\tilde{E}(H|F_t) \tag{17}$$

We first compute the expectation under the risk neutral probability of the payoff H (2) for $K = 0$:

$$\begin{aligned}
\tilde{E}[H|F_t] &= \tilde{E}\Big[\Big(\frac{1}{T}\int_0^T S_u du\Big)^+|F_t\Big] = \frac{1}{T}\tilde{E}\Big[\Big(\int_0^t S_u du\Big)^+ + \Big(\int_t^T S_u du\Big)^+|F_t\Big] \\
&= \frac{1}{T}\tilde{E}\Big[Y_t + \Big(\int_t^T S_u du\Big)^+|F_t\Big] = \frac{1}{T}\Big(Y_t + \tilde{E}\Big[\Big(\int_t^T S_u du\Big)^+|F_t\Big]\Big) \\
&= \frac{1}{T}\Big(Y_t + S_t\tilde{E}\Big[\Big(\int_t^T \frac{S_u}{S_t} du\Big)^+|F_t\Big]\Big) = \frac{1}{T}\Big(Y_t + S_t\tilde{E}\Big[\Big(\int_0^{T-t} \frac{S_u}{S_t} du\Big)^+|F_t\Big]\Big) \\
&= \frac{1}{T}\Big(Y_t + S_t\tilde{E}\Big[\int_0^{T-t}\frac{S_u}{S_0}du\Big]\Big) = \frac{1}{T}\Big(Y_t + S_t\int_0^{T-t}\frac{\tilde{E}[S_u]}{S_0}du\Big)
\end{aligned}$$

From the differential equation for the stock price in (1) we get:

$$S_u = S_0 e^{(r-\sigma^2/2)u-\sigma W_u}$$

Where $W_u \sim \mathcal{N}(0,u)$. Therefore, the expectation of the stock price is:

$$\tilde{E}[S_u] = S_0 e^{rt} \tag{18}$$

If we substitute this into the calculation for $\tilde{E}[H|F_t]$ we get:

$$\begin{aligned}
\tilde{E}[H|F_t] &= \frac{1}{T}\Big(Y_t + S_t\int_0^{T-t}\frac{\tilde{E}[S_u]}{S_0}du\Big) = \frac{1}{T}\Big(Y_t + S_t\int_0^{T-t}\frac{S_0 e^{ru}}{S_0}du\Big) \\
&= \frac{1}{T}\Big(Y_t + S_t\int_0^{T-t} e^{ru}du\Big) = \frac{1}{T}\Big(Y_t + \frac{S_t}{r}e^{r(T-t)}\Big)
\end{aligned}$$

So the pricing formula is:

$$\boxed{V_t = e^{-r(T-t)}\tilde{E}(H|F_t) = \frac{e^{-r(T-t)}}{T}\Big(Y_t + \frac{S_t}{r}e^{r(T-t)}\Big)} \tag{19}$$

(ii) Check that the function obtained satisfies the PDE in 1.
We proceed to compute the partial derivatives of $V_t = v(t,x,t)$ with the formula presented in (19):

$$\partial_t V_t = r\frac{e^{-r(T-t}}{T}\Big(Y_t + \frac{S_t}{r}(e^{r(T-t)} - 1)\Big) + \frac{e^{-r(T-t}}{T}\Big(\frac{-rS_t}{r}e^{r(T-t)}\Big) = rV_t - \frac{S_t}{T}$$
$$\partial_x V_t = \frac{e^{-r(T-t}}{T}\Big(\frac{1}{r}(e^{r(T-t)} - 1)\Big)$$
$$\partial_y V_t = \frac{e^{-r(T-t}}{T}$$

Recall the PDE (5) and insert the partial derivatives computed:

$$\partial_t V_t + rS_t\partial_x V_t + S_t\partial_y V_t = rV_t - \frac{S_t}{T} + rS_t\Big(\frac{e^{-r(T-t)}}{rT}(e^{r(T-t)} - 1)\Big) + S_t\Big(\frac{e^{-r(T-t)}}{T}\Big)$$
$$= rV_t - \frac{S_t}{T} + \frac{S_t}{T} - \frac{S_t}{T}e^{-r(T-t)} + \frac{S_t}{T}e^{-r(T-t)} = rV_t$$

So we just proved that for $v(t,x,y) = V_t$ the PDE holds:

$$\partial_t V_t + rS_t\partial_x V_t + S_t\partial_y V_t = rV_t$$

3. Compute the hedging strategy b_t of the formula obtained in 2 .
To compute the hedging strategy for a self financing portfolio with dR_t from (7) and dS_t from (1) we recall the condition already introduced in equation (13), so we can find b_t by requesting the dW_t terms being equal in (12) and (13):

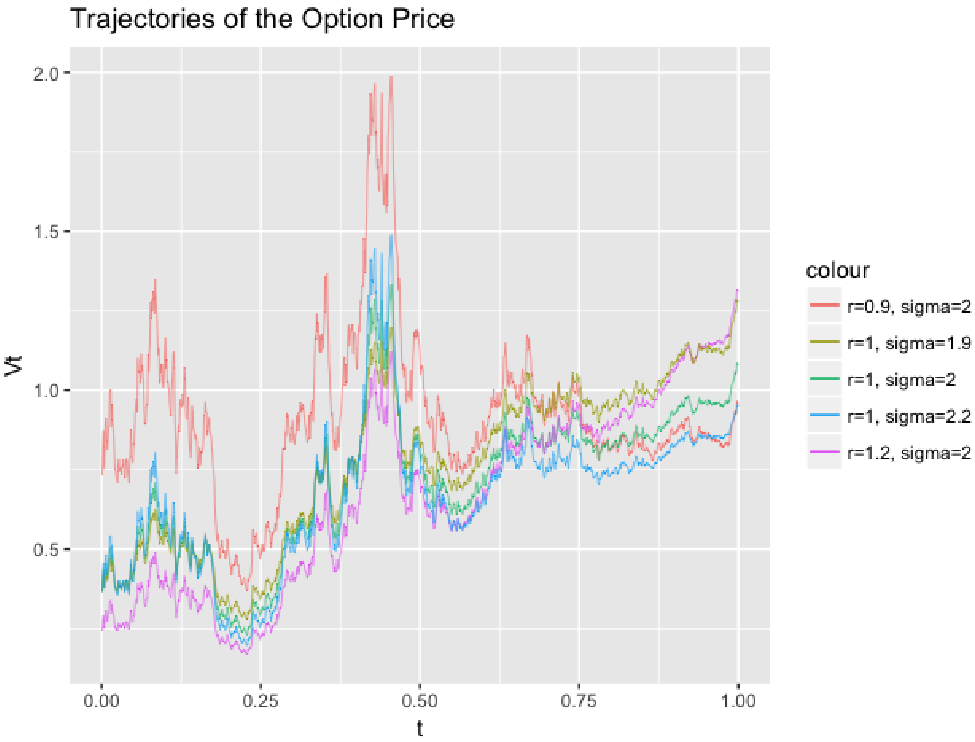
$$b_t\tilde{S}_t\sigma dW_t = e^{-rt}S_t\sigma\frac{\partial v}{\partial x}dW_t$$

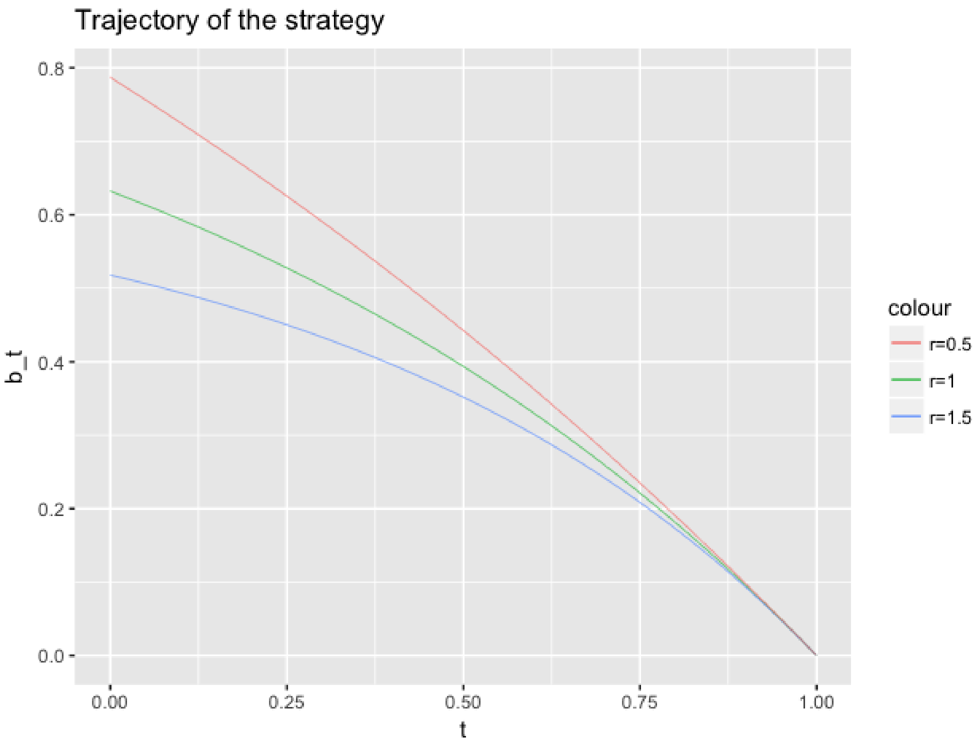
And so;

$$\boxed{b_t = \frac{\partial v}{\partial x} = \frac{e^{-r(T-t)}}{T}\Big(\frac{e^{r(T-t)} - 1}{r}\Big) = \frac{1 - e^{-r(T-t)}}{rT}}$$

4. Choosing a value for σ and r, simulate one trajectory of the option price V_t and the strategy b_t on the interval $[0,1]$.
Using;

$$V_t = \frac{e^{-r(T-t)}}{T}\Big(Y_t + \frac{S_t}{r}e^{r(T-t)}\Big) \qquad\qquad b_t = \frac{1 - e^{-r(T-t)}}{rT}$$


Trajectories of the Option Price
2.0
1.5
1.0
0.5
Vt
0.00
0.25
0.50
0.75
1.00
t
colour
r=0.9, sigma=2
r=1, sigma=1.9
r=1, sigma=2
r=1, sigma=2.2
r=1.2, sigma=2


Trajectory of the strategy
0.8
0.6
0.4
0.2
0.0
b_t
0.00
0.25
0.50
0.75
1.00
t
colour
r=0.5
r=1
r=1.5

Appendix

```r
rm(list = ls())

library("ggplot2")


N<-1000
T<-1
delta<-T/N
t<-seq(0,T,delta)

#in order to be able to compare the trajectories of different option prices according
# the value of the parameters, we generate first Z~N(0,1) so that
# the only difference between the geometric BM to be their parameters
z<-rnorm(N,0,1)

#take the BM at t=0 to be 0 by taking the first element to be zero
Z<-c(0,z)



# Create a function that simulates the geometric BM
geometricBM<- function(r,sigma){
  B<-rep(0,N+1)
  for (i in 2:N+1){
    B[i]<-B[i-1] + sqrt(delta)*Z[i]
  }
  t<-seq(0,T,delta)
  S<-r * t + sigma * B

  return(exp(S))
}


#Simulate the Y_t variable that is given by the integral from 0 to t of S_t
#through Riemann sum

#difference between t_0,t_1...t_N is const= delta
#generate a vecotor of the differences between the t

computeY<-function(S){
  diff_t<-rep(delta,N)
  Y<-rep(0,N+1)
  for (i in 1:N+1){
    for (j in 1:i){
      Y[i]<-sum(Y[i], (S[j-1]*diff_t[j-1]))
    }
  }
  return(Y)
}
```

```r
#Compute the price
Vt<-function(S,Y,r){
  vector_T<-rep(T,N+1)
  vector_r<-rep(r,N+1)
  Vt<-(exp(-vector_r*(vector_T-t))/vector_T)*(Y+(S/vector_r)*(exp(vector_r*(vector_T-t)-1)))
  return(Vt)
}



r<-     c( 0.5, 1, 1.5, 1, 1)
sigma<- c( 2, 2, 2, 1.5, 2.5)

S1<-geometricBM(r=r[1],sigma=sigma[1])
Y1<-computeY(S=S1)
Vt1<-Vt(S=S1,Y=Y1,r=r[1])


S2<-geometricBM(r=r[2],sigma=sigma[2])
Y2<-computeY(S=S2)
Vt2<-Vt(S=S2,Y=Y2,r=r[2])

S3<-geometricBM(r=r[3],sigma=sigma[3])
Y3<-computeY(S=S3)
Vt3<-Vt(S=S3,Y=Y3,r=r[3])

S4<-geometricBM(r=r[4],sigma=sigma[4])
Y4<-computeY(S=S4)
Vt4<-Vt(S=S4,Y=Y4,r=r[4])

S5<-geometricBM(r=r[5],sigma=sigma[5])
Y5<-computeY(S=S5)
Vt5<-Vt(S=S5,Y=Y5,r=r[5])


ggplot() +
  geom_line(aes(x = t, y = Vt1, colour = "r=0.9, sigma=2"), size = 0.2) +
  geom_line(aes(x = t, y = Vt2, colour = "r=1, sigma=2"), size = 0.2) +
  geom_line(aes(x = t, y = Vt3, colour = "r=1.2, sigma=2"), size = 0.2) +
  geom_line(aes(x = t, y = Vt4, colour = "r=1, sigma=1.9"), size = 0.2) +
  geom_line(aes(x = t, y = Vt5, colour = "r=1, sigma=2.2"), size = 0.2) +
  labs(title = "Trajectories of the Option Price", x = "t", y = "Vt")


##########################################################
# Simulate b
##########################################################

vector_r<-rep(1,N+1)
vector_T<-rep(1,N+1)
bt1<- (1/vector_r*vector_T)-(exp(-vector_r*(vector_T-t))/vector_r*vector_T)

vector_r2<-rep(1.5,N+1)
```

```r
vector_T<-rep(1,N+1)
bt2<-(1/vector_r2*vector_T)-(exp(-vector_r2*(vector_T-t))/vector_r2*vector_T)

vector_r3<-rep(0.5,N+1)
vector_T<-rep(1,N+1)
bt3<-(1/vector_r3*vector_T)-(exp(-vector_r3*(vector_T-t))/vector_r3*vector_T)


ggplot() +
 geom_line(aes(x = t, y = bt1, colour = "r=1"), size = 0.2) +
  geom_line(aes(x = t, y = bt2, colour = "r=1.5"), size = 0.2) +
  geom_line(aes(x = t, y = bt3, colour = "r=0.5"), size = 0.2) +
  labs(title = "Trajectory of the strategy", x = "t", y = "b_t")
```