Order of Growth of Functions

Analysis and Design of Algorithms

Research Group on Artificial Life – Grupo de investigación en vida artificial – (Alife)
Computer and System Department
Engineering School
Universidad Nacional de Colombia

Agenda

1. Asymptotic notation
2. Common functions
3. Examples

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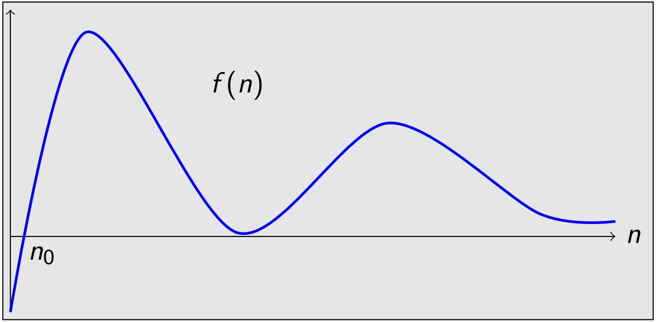
Outline

Asymptotically No Negative Functions

Definition

$f(n)$ is asymptotically no negative if there exist $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $0 \leq f(n)$.

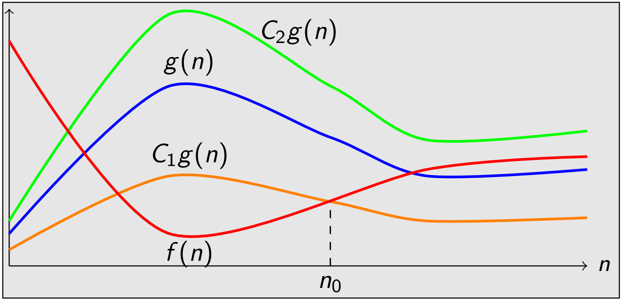


Theta Θ

Definition

$$\Theta(g(n)) = \{f : \mathbb{N} \to \mathbb{R}^* : (\exists C_1, C_2 \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N})$$
$$(\forall n \geq n_0)\big(0 \leq C_1 g(n) \leq f(n) \leq C_2 g(n)\big)\}$$

$$C_1 \leq \lim_{n\to\infty} \frac{f(n)}{g(n)} \leq C_2$$



“$f(n) = \Theta\big(g(n)\big)$” $\equiv$ “$f(n) \in \Theta\big(g(n)\big)$”

g is asymptotically tight bound for f
or
f is of the exact order of g

- Every member of $\Theta\big(g(n)\big)$ is asymptotically no negative.
- The function $g(n)$ must itself asymptotically no negative, or else $\Theta\big(g(n)\big) = \varnothing$.

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Example

Lets show that

$$\frac{1}{2}n^2 - 3n = \Theta\left(n^2\right)$$

We have to find C_1, C_2 and n_0 such that

$$C_1 n^2 \leq \frac{1}{2}n^2 - 3n \leq C_2 n^2$$

For all $n \geq n_0$, Dividing by n^2 yields

$$C_1 \leq \frac{1}{2} - \frac{3}{n} \leq C_2$$

We have that $\frac{3}{n}$ is a decreasing sequence

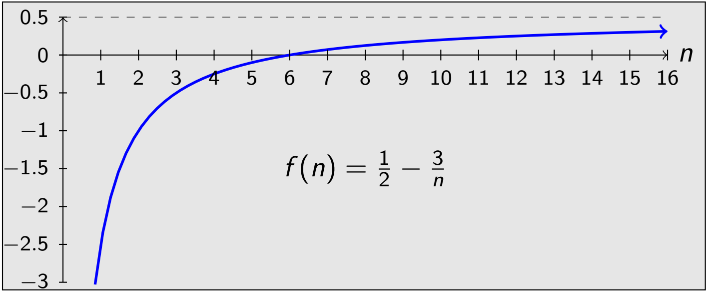
$$3, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \frac{1}{2}, \frac{3}{7}, \frac{3}{8}, \frac{1}{3}, \ldots$$

and then $\frac{1}{2} - \frac{3}{n}$ is an increasing sequence

$$-\frac{5}{2}, -1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{10}, 0, \frac{1}{14}, \ldots$$

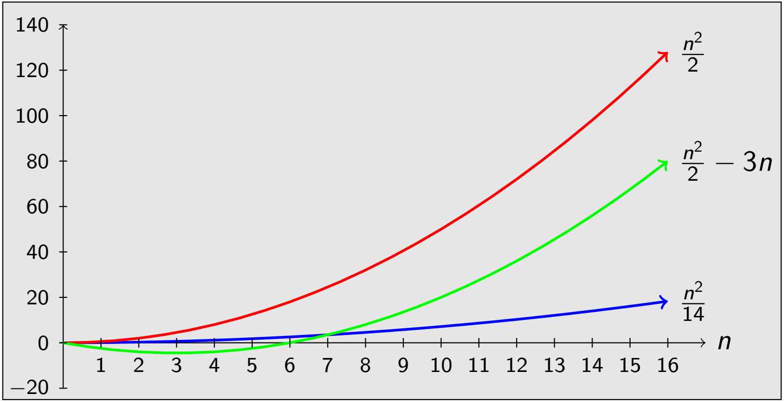
that is upper bounded by $\frac{1}{2}$.

The right hand inequality can be made to hold for $n \geq 1$ by choosing $C_2 \geq \frac{1}{2}$. Likewise, the left hand inequality can be made to hold for $n \geq 7$ by choosing $C_1 \leq \frac{1}{14}$.



Thus, by choosing $C_1 = \frac{1}{14}$, $C_2 = \frac{1}{2}$ and $n_0 = 7$ then we can verify that

$$\frac{1}{2}n^2 - 3n = \Theta\left(n^2\right)$$

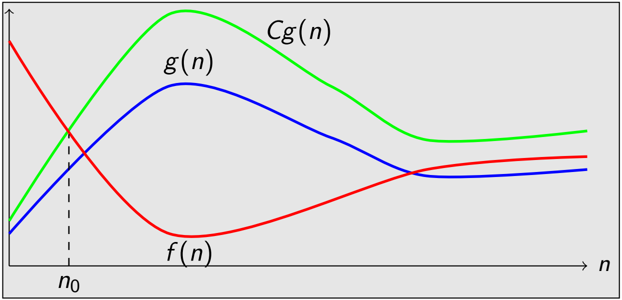


Big $\mathcal{O}$ (Omicron)

Definition

$$\mathcal{O}(g(n)) = \{f : \mathbb{N} \to \mathbb{R}^* : (\exists C \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N}) (\forall n \geq n_0)(0 \leq f(n) \leq Cg(n))\}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq C$$



"$f(n) = \mathcal{O}(g(n))$" $\equiv$ "$f(n) \in \mathcal{O}(g(n))$"

f is asymptotically upper bound for g
or
g is an asymptotic upper bound for f

- $\mathcal{O}(g(n))$ is pronounced "big-oh of $g(n)$".

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Example

Lets show that if $a, b > 0$ then

$$an + b = \mathcal{O}(n)$$

We have to find C and n_0 such that

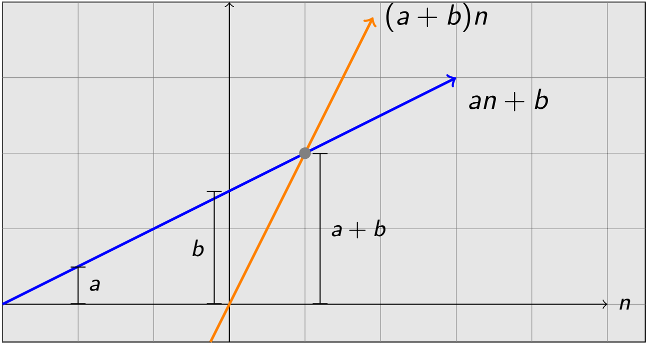
$$an + b \leq Cn$$

For all $n \geq n_0$, dividing by n yields

$$0 \leq a + \frac{b}{n} \leq C$$

The inequality can be made to hold for $n \geq 1$ by choosing $C \geq a + b$. Thus by choosing $C \geq a + b$ and $n_0 = 1$ then we can verify that $0 \leq an + b \leq (a + b)n$, that is to say

$$an + b = \mathcal{O}(n)$$

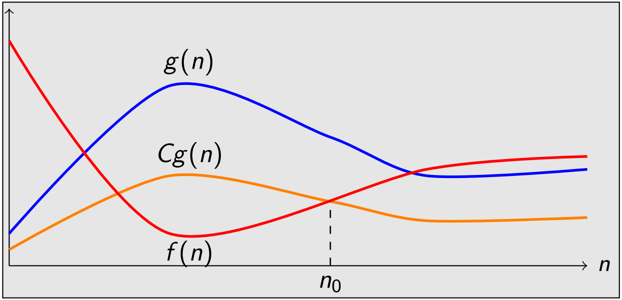


Big Omega Ω

Definition

$$\Omega\big(g(n)\big) = \big\{f : \mathbb{N} \to \mathbb{R}^* : (\exists C \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N}) \\ (\forall n \geq n_0)\big(0 \leq Cg(n) \leq f(n)\big)\big\}$$

$$C \leq \lim_{n\to\infty} \frac{f(n)}{g(n)}$$



$$\text{“}f(n) = \Omega\big(g(n)\big)\text{”} \equiv \text{“}f(n) \in \Omega\big(g(n)\big)\text{”}$$

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Lets show that

$$5n^3 - 5n^2 - 2n - 3 = \Omega\left(n^2\right)$$

We have to find C and n_0 such that

$$0 \leq Cn^2 \leq 5n^3 - 5n^2 - 2n - 3$$

if $n \geq n_0$.

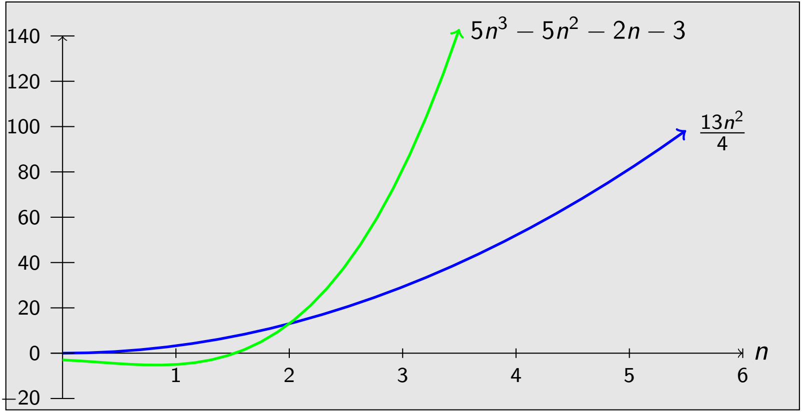
For all $n \geq n_0$, Dividing by n^2 yields

$$0 \leq C \leq 5n - 5 - \frac{2}{n} - \frac{3}{n^2} = 5n - \left(5 + \frac{2}{n} + \frac{3}{n^2}\right)$$

We have that $5 + \frac{2}{n} + \frac{3}{n^2}$ is a decreasing sequence that takes its maximum value 10 when $n = 1$; and therefore $5n - \left(5 + \frac{2}{n} + \frac{3}{n^2}\right)$ is an increasing sequence such that is non-negative for $n \geq 2$, whence if $n_0 = 2$ then it is lower bounded by $\frac{13}{4}$.

Thus, by choosing $C = \frac{13}{4}$ and $n_0 = 2$ then we can verify that

$$5n^3 - 5n^2 - 2n - 3 = \Omega\left(n^2\right)$$

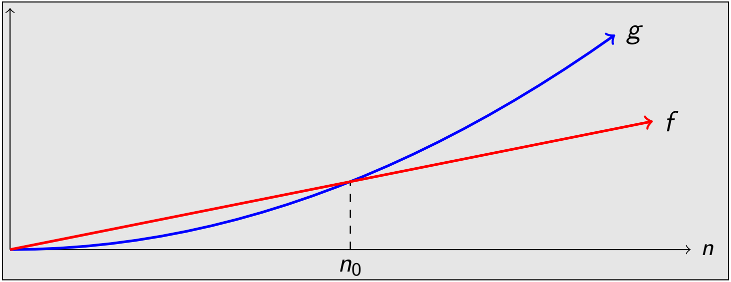


Little o

Definition

$$o\big(g(n)\big) = \big\{f : \mathbb{N} \to \mathbb{R}^* : (\forall C \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N})$$
$$(\forall n \geq n_0)\big(0 \leq f(n) < Cg(n)\big)\big\}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$



"$f(n) = o(g(n))$" $\equiv$ "$f(n) \in o(g(n))$"

f is asymptotically smaller than g

- $o(g(n))$ is pronounced "little-oh of $g(n)$".
- $o(g(n))$ is the set of functions that grow slower than g.

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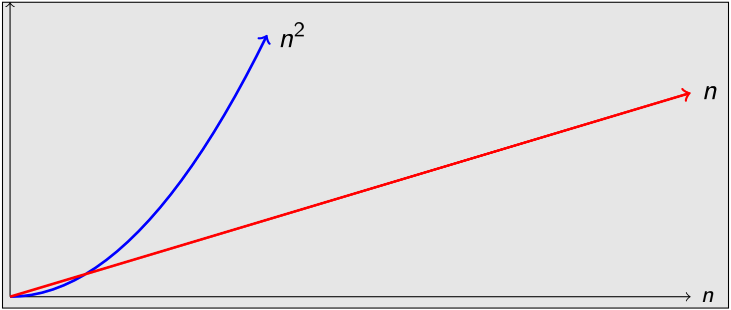
Example

Lets show that

$$n = o\left(n^2\right)$$

we only have to show

$$\lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$$



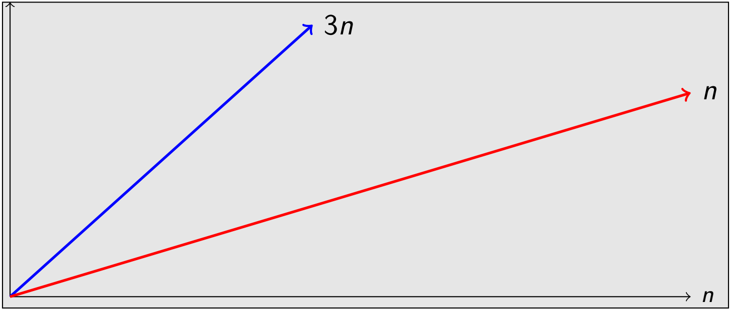
Example

Lets show that

$$n \notin o(3n)$$

we have

$$\lim_{n\to\infty} \frac{n}{3n} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} \neq 0$$

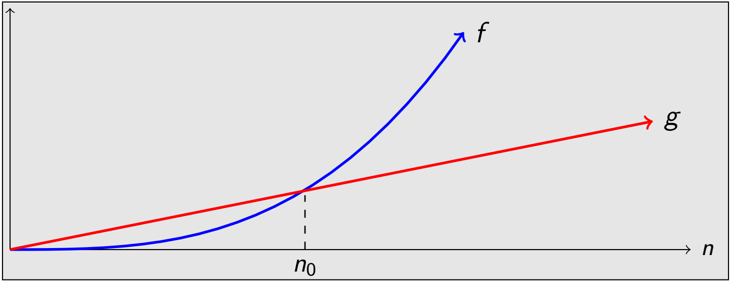


Little omega ω

Definition

$$\omega(g(n)) = \{f : \mathbb{N} \to \mathbb{R}^* : (\forall C \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N}) \\ (\forall n \geq n_0)\big(0 \leq Cg(n) < f(n)\big)\}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$



“$f(n) = \omega(g(n))$” $\equiv$ “$f(n) \in \omega(g(n))$”

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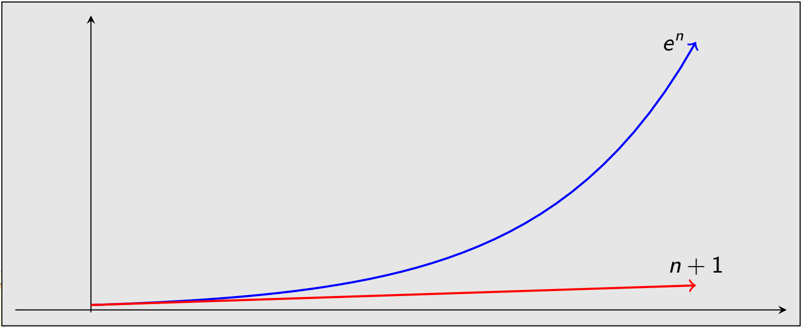
Example

Lets show that

$$e^n = \omega(n+1)$$

we only have to show

$$\lim_{n\to\infty} \frac{e^n}{n+1} = \lim_{n\to\infty} e^n = \infty$$



Analogy with the comparison of two real numbers

Asymptotic notation	Real numbers
$f(n) \in \mathcal{O}(g(n))$	$f \leq g$
$f(n) \in \Omega(g(n))$	$f \geq g$
$f(n) \in \Theta(g(n))$	$f = g$
$f(n) \in o(g(n))$	$f < g$
$f(n) \in \omega(g(n))$	$f > g$

Trichotomy does not hold!

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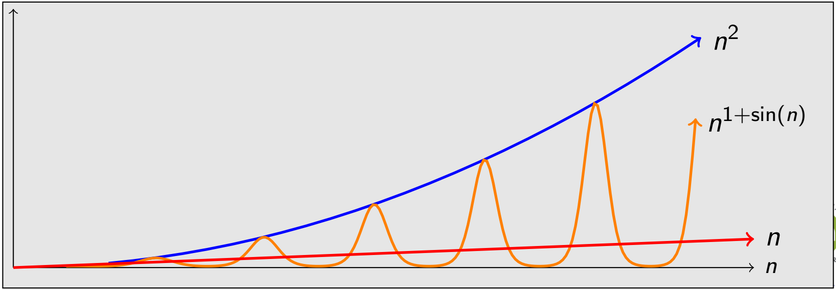
Not all functions are asymptotically comparable
Trichotomy does not hold

Example

Following functions are asymptotically non-negative

- $f(n) = n$
- $g(n) = n^{1+\sin(n)}$

but, they are not comparable because $1 + \sin(n) \in [0, 2]$, the function g varies between 1 and n^2, when $n \to \infty$.



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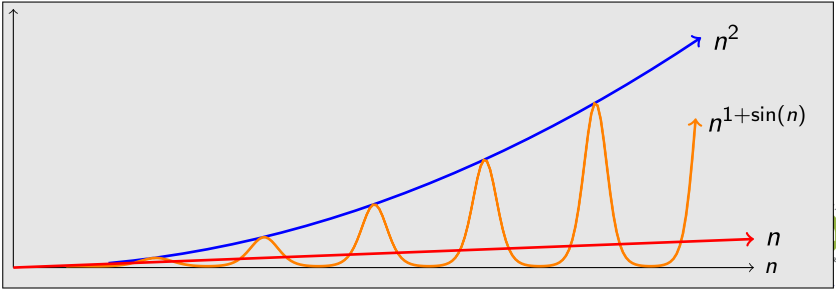

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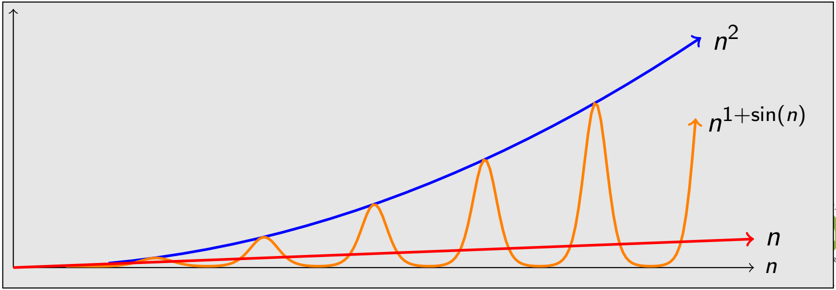

Properties

$$\Theta(f(n)) = \mathcal{O}(f(n)) \cap \Omega(f(n))$$

The running time of an algorithm is $\Theta(f(n))$

if and only if

1. Its worst-case running time is $\mathcal{O}(f(n))$, and
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Properties (conti.)

Given f, g and h asymptotically no negative functions, we have:

Transitivity of $\mathcal{O}, \Omega, \Theta$ $f(n) \in \Delta(g(n))$ and $g(n) \in \Delta(h(n))$ then $f(n) \in \Delta(h(n))$, for $\Delta \in \{\mathcal{O}, \Omega, \Theta\}$.

Reflexivity of $\mathcal{O}, \Omega, \Theta$ $f(n) \in \Delta(f(n))$, for $\Delta \in \{\mathcal{O}, \Omega, \Theta\}$.

Symmetry of Θ $f(n) \in \Theta(g(n)) \iff g(n) \in \Theta(f(n))$.

Anti-symmetry of $\mathcal{O}, \Omega$ $\forall f(n) \notin \Theta(g(n))$, $f(n) \in \Delta(g(n)) \implies g(n) \notin \Delta(f(n))$, for $\Delta \in \{\mathcal{O}, \Omega\}$.

Transpose Symmetry

$$f(n) \in \mathcal{O}(g(n)) \iff g(n) \in \Omega(f(n))$$

$$f(n) \in o(g(n)) \iff g(n) \in \omega(f(n))$$

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Properties (conti.)

- $f \leq g \iff f(n) \in \mathcal{O}(g(n))$ **order relation**
  - reflexive
  - anti-symmetric
  - transitive
- $f \geq g \iff f(n) \in \Omega(g(n))$ **order relation**
- $f = g \iff f(n) \in \Theta(g(n))$ **equivalence relation**

Properties (conti.)

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$$f(n) \in o(g(n)) \Longrightarrow f(n) \in O(g(n))$$

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Asymptotic notation two variables

Definition

$$\mathcal{O}\big(g(m,n)\big) = \Big\{f : \mathbb{N}\times\mathbb{N} \to \mathbb{R}^* : (\exists C \in \mathbb{R}^+)(\exists m_0, n_0 \in \mathbb{N}) \\ (\forall m \geq m_0)(\forall n \geq n_0)\big(f(m,n) \leq Cg(m,n)\big)\Big\}$$

Outline

Monotonicity

f is monotonically increasing if: $\forall x, y \in \mathbb{R}, x < y \implies f(x) \leq f(y)$

f is monotonically decreasing if: $\forall x, y \in \mathbb{R}, x < y \implies f(x) \geq f(y)$

f is strictly increasing if: $\forall x, y \in \mathbb{R}, x < y \implies f(x) < f(y)$

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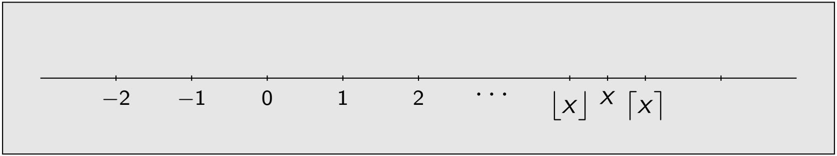
Floors and Ceilings

Definition

$\lfloor x \rfloor$ floor of x: The greatest integer less than or equal to x.

$\lceil x \rceil$ ceiling of x: The smallest integer greater than or equal to x.



$$\forall x \in \mathbb{R}, \qquad x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$$

$$\forall n \in \mathbb{Z}, \qquad \lfloor n \rfloor = n = \lceil n \rceil \text{ and } \lfloor n/2 \rfloor + \lceil n/2 \rceil = n$$

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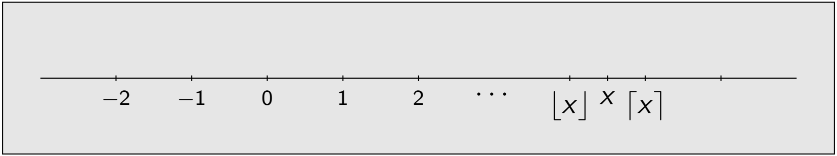

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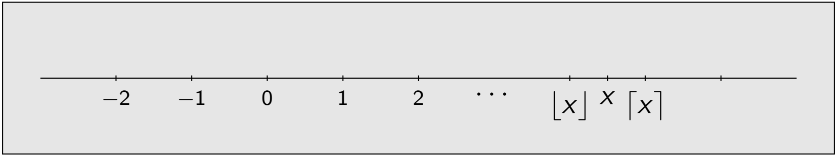

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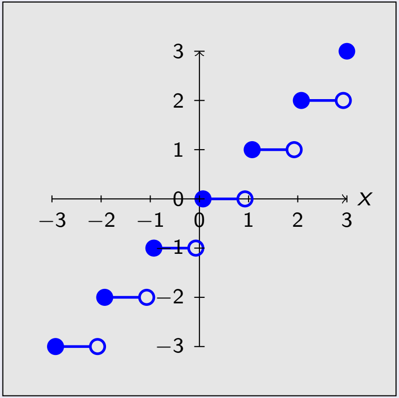

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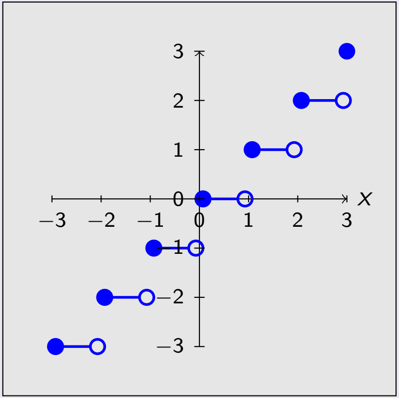

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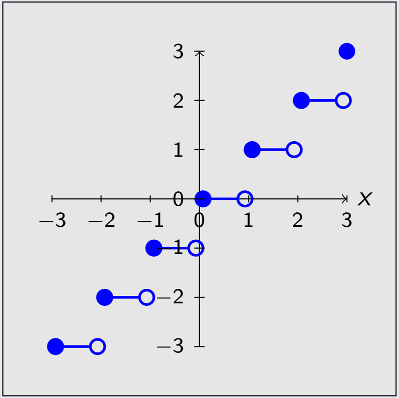

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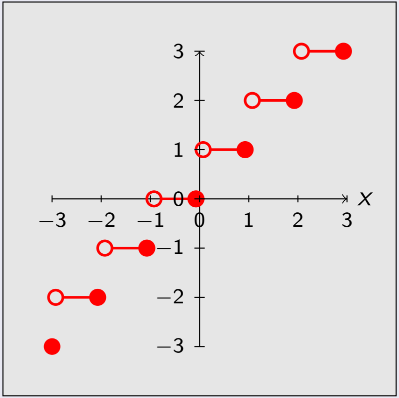

Properties

$\forall x \in \mathbb{R}$ and $n, m \in \mathbb{Z}^+$

$$\begin{aligned}
\lfloor \lfloor x/n \rfloor /m \rfloor &= \lfloor x/nm \rfloor \\
\lceil \lceil x/n \rceil /m \rceil &= \lceil x/nm \rceil \\
\lfloor n/m \rfloor &\leq (n + (m-1))/m \\
\lceil n/m \rceil &\geq (n - (m-1))/m
\end{aligned}$$

$\lfloor x \rfloor$ and $\lceil x \rceil$ are monotonically increasing.

Modular arithmetic

For every integer a and any possible positive integer n,

$$a \bmod n$$

is the **remainder** (or **residue**) of the quotient a/n

$$a \bmod n = a - \lfloor a/n \rfloor n$$

Congruency or equivalence mod n

If $(a \bmod n) = (b \bmod n)$ we write

$$a \equiv b \pmod{n}$$

and we say that a is **equivalent** to b module n or that a is **congruent** to b module n.

In other words $a \equiv b \pmod{n}$ if a and b have the same remainder when they are divided by n.

Also $a \equiv b \pmod{n}$ if and only if n is a divisor of $b - a$

$\equiv \pmod{n}$

defines a equivalence relation in $\mathbb{Z}$ and produces a partitioned set called $\mathbb{Z}_n = \mathbb{Z}_{/n} = \{0, 1, 2, \ldots, n-1\}$ in which can be defined arithmetic operations

$$a + b \pmod{n}$$

$$a * b \pmod{n}$$

Example $\equiv \pmod{4}$

$$\mathbb{Z}_{/4} = \{[0], [1], [2], [3]\} = \{0, 1, 2, 3\}$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$
-12	-11	-10	-9
-8	-7	-6	-5
-4	-3	-2	-1
0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
$\vdots$	$\vdots$	$\vdots$	$\vdots$
[0]	[1]	[2]	[3]

$$4 + 1 \pmod{4} = 1$$

$$5 \times 2 \pmod{4} = 2$$

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Polynomials

Given a no negative integer d, a **polynomial in n of degree** d is a function $p(n)$ of the form:

$$p(n) = \sum_{i=0}^{d} a_i n^i$$

Where $a_0, a_1, a_2, \ldots, a_d$ are the **coefficients** and $a_d \neq 0$, a_d is called the **main coefficient** and a_0 is called the **independent term**.

Properties

- A polynomial $p(n)$ es asymptotically positive if and only if $a_d > 0$.
- If $p(n)$, of degree d is asymptotically positive, we have $p(n) = \Theta(n^d)$.
- $\forall a \in \mathbb{R}$, $a > 0$, n^a es monotonically increasing.
- $\forall a \in \mathbb{R}$, $a < 0$, n^a es monotonically decreasing.
- A function $f(n)$ is **polynomially bounded** if $f(n) = \mathcal{O}(n^d)$ for some constant d.

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Exponentials

For all reals $a > 0$, m and n, we have the following identities:

- $a^0 = 1$
- $a^1 = a$
- $a^{-1} = 1/a$
- $(a^m)^n = a^{mn}$
- $(a^m)^n = (a^n)^m$
- $a^m a^n = a^{m+n}$
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- $\frac{d}{dx}(a^u) = a^u \cdot \ln a \frac{du}{dx}$

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- If $a > 1$, for all $n \in \mathbb{Z}^+$,

$$a^n \text{ is monotonically increasing}$$

- If $0 < a < 1$, for all $n \in \mathbb{Z}^+$,

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Logarithms

Notations:

- $\lg n = \log_2 n$
- $\ln n = \log_e n$
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Logarithm function will only apply to next term in the formula:

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For $x \in \mathbb{R}$, if $|x| < 1$ then:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{i=1}^{\infty} \frac{(-1)^{i-1} x^i}{i}$$

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A function $f(n)$ is **polylogaritmically** bounded if

$$f(n) = \mathcal{O}(\lg^k n) \text{ for some constant } k$$

We have the following relation between polynomials and polylogarithms:

$$n^d = 2^{\lg n^d} = 2^{d(\lg n)} = \left(2^d\right)^{\lg n}$$

$$\lim_{n\to\infty} \frac{\lg^k n}{n^d} = \lim_{n\to\infty} \frac{\lg^k n}{(2^d)^{\lg n}} = \lim_{x\to\infty} \frac{x^k}{(2^d)^x} = 0, \quad \text{as} \quad x = \lg n \to \infty$$

then $\lg^k n = o\left(n^d\right)$, for any constant $d > 0$.

Factorials

Given $n \in \mathbb{N}$, factorial of n is defined as:

Definition (No recursive)

$$n! = \begin{cases} 1, & \text{if } n = 0; \\ \prod_{i=1}^{n} i, & \text{if } n > 0. \end{cases}$$

Definition (Recursive)

$$n! = \begin{cases} 1, & \text{if } n = 0; \\ n \cdot (n-1)!, & \text{if } n > 0. \end{cases}$$

Weak upper bound

$$\boxed{n! \leq n^n}$$

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Stirling's approximation

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left[1 + \Theta\left(\frac{1}{n}\right)\right]$$

then

$$\begin{aligned} n! &= o\left(n^n\right) \\ n! &= \omega\left(a^n\right) \\ \lg(n!) &= \Theta(n \lg n) \\ n! &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n}, \qquad \text{where} \qquad \frac{1}{12n+1} \leq \alpha_n \leq \frac{1}{12n} \end{aligned}$$

Functional iteration

Definition

Given a function $f(n)$ the i-th functional iteration of f is defined as:

$$f^i = \begin{cases} I, & \text{if } i = 0; \\ f \circ f^{(i-1)}, & \text{if } i > 0. \end{cases}$$

with I the identity function.

For a particular n, we have:

$$f^i(n) = \begin{cases} n, & \text{if } i = 0; \\ f\left(f^{(i-1)}(n)\right), & \text{if } i > 0. \end{cases}$$

Examples

1. $f(n) = 2n$ then $f^{(i)}(n) = 2^i n$
2. $f(n) = n^2$ then:

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\begin{aligned}
f^{(2)}(n) &= \left(n^2\right)^2 = \left(n^2\right)\left(n^2\right) = n^{2*2} = n^4 \\
f^{(3)}(n) &= \left(n^{2*2}\right)^2 = n^{2*2*2} = n^8 \\
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The iterated logarithm of n, denoted $\lg^* n$ ("log star of n") is defined as:

$$\lg^* n = \min\left\{ i \geq 0 : \lg^{(i)} n \leq 1 \right\}$$

$\lg^* n$, is a very slowly growing function

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$$\lg^* 1 = 0$$
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Summary

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\begin{array}{ll}
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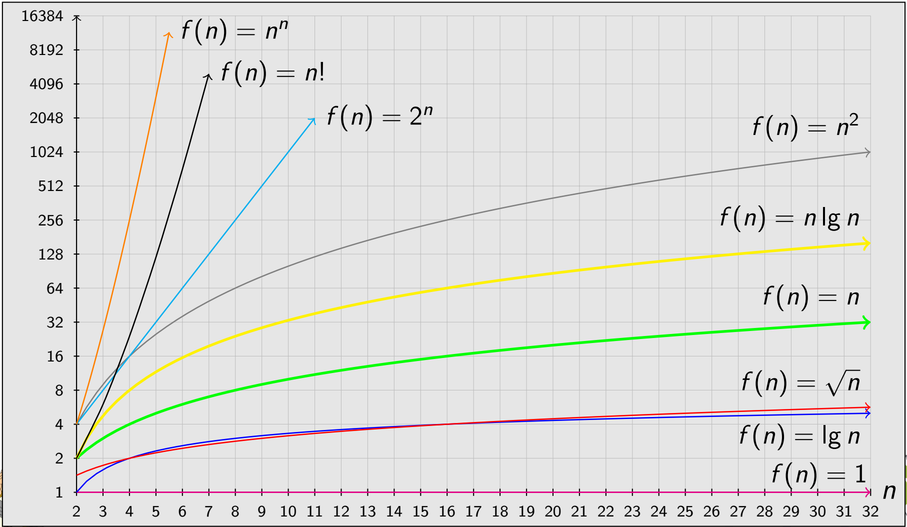
& \in \\
& o(n!) \\
& \in \\
& o(2^n) \\
& \in \\
& o(n^2) \\
& \in \\
& o(n \lg n) \\
& \in \\
& o(n) \\
& \in \\
& o(\sqrt{n}) \\
& \in \\
& o(\lg n) \\
& \in \\
& 1
\end{array}
$$

Logarithmic graph of main functions into Computer Sciences



Examples of complexities of several algorithms

- First o head in a list $\in \mathcal{O}(1)$.
- Binary search, recursive power $\in \mathcal{O}(\lg(n))$.
- Primality of a number $\in \mathcal{O}(\sqrt{n})$.
- Minimum of a list, sum of a list, counting sort $\in \mathcal{O}(n)$.
- Merge sort, Heap sort $\in \mathcal{O}(n\lg(n))$.
- Selection sort, bubble sort, sum of square matrices $\in \mathcal{O}(n^2)$.
- Multiplication of square matrices $\in \mathcal{O}(n^3)$.
- SAT, power set, tower of Hanoi $\in \mathcal{O}(2^n)$.
- TSP, Hamiltonian cycle $\in \mathcal{O}(n!)$.
- Interpretations over a n-valued logic, strings of length n with n symbols $\in \mathcal{O}(n^n)$.
- Relations of a set over itself $\in \mathcal{O}\left(2^{n^2}\right)$.

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Outline

1. Asymptotic notation
2. Common functions
3. Examples

Example

A	B	
$5n^2 + 100n$	$3n^2 + 2$	$\mathbf{A} \in \Theta(\mathbf{B})$
$\log_3\left(n^2\right)$	$\log_2\left(n^3\right)$	$\mathbf{A} \in \Theta(\mathbf{B})$
$n^{\lg 4}$	$3^{\lg n}$	$\mathbf{A} \in \omega(\mathbf{B})$
$\lg n$	$n^{1/2}$	$\mathbf{A} \in o(\mathbf{B})$

$\lg n$ denotes $\log_2 n$.

A	**B**	
$5n^2 + 100n$	$3n^2 + 2$	$\mathbf{A} \in \Theta(\mathbf{B})$

$\mathbf{A} \in \Theta(n^2), n^2 \in \Theta(\mathbf{B}) \Longrightarrow \mathbf{A} \in \Theta(B)$

$\log_3 \left(n^2\right)$	$\log_2 \left(n^3\right)$	$\mathbf{A} \in \Theta(\mathbf{B})$

$\log_b a = \log_c a / \log_c b; \mathbf{A} = 2 \lg n / \lg 3, \mathbf{B} = 3 \lg n; \mathbf{A}/\mathbf{B} = 2/(3 \lg 3)$

$n^{\lg 4}$	$3^{\lg n}$	$\mathbf{A} \in \omega(\mathbf{B})$

$a^{\log b} = b^{\log a}; B = 3^{\lg n} = n^{\lg 3}; \mathbf{A}/\mathbf{B} = n^{\lg(4/3)} \to \infty \text{ as } n \to \infty$

$\lg n$	$n^{1/2}$	$\mathbf{A} \in o(\mathbf{B})$

$\lim_{n\to\infty} \left(\log_a n / n^b\right) = 0, \text{ here } a = 2 \text{ and } b = 1/2 \Longrightarrow \mathbf{A} \in o(\mathbf{B})$

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- Prove that $5n + 3\lg n = \Theta(n)$.
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