FINANCE WEB APP
IMPLIED DISTRIBUTION

Illustration

This section regards a method implemented by Shimko (1993) “Bounds of probability". This paper explains how European option prices can be used to determine the probability that future index values will lie in a given range, enabling the user to design better speculative strategies and calculate better hedge ratios. The fact that probability distributions can be recovered from option prices was first discovered by Breeden and Litzenberger (1978). In this case Shimko (1993) explains how the original Breeden and Litzenberger (1978) approach can be implemented nonparametrically to recover risk neutral implied probability of future index values. The entire probability distribution gives a complete set of information of underlying, instead of Black-Scholes, it can provide a measure of the skewness of a distribution and it's kurtosis. For example market may place relatively greater probability on downward price movement than upward movement is known as negative skew. When traders speak of "trading the skew" they are usually referring to predicting the slope of the implied volatility curve, and choosing an option position that profits if their forecast materialises. The use of the word "skew" is statistically correct, since a negatively sloped implied volatility curve gives rise to a negatively skewed probability distribution index. The skewness implied by the Black-Scholes model is uniformly small and positive.

Background information

The inverse problem consisting of the identification of a risk-neutral distribution $q_{t+\tau}$ implied by option prices was first addressed in a seminal paper by Breeden and Litzenberger (1978). These authors show that the risk-neutral density $q_{t+\tau}$ is recovered from option prices as

$$q_{t+\tau}(x) = e^{r\tau} \left.\frac{\partial^2 c_t(K,\tau)}{\partial K^2}\right|_{K=x}. \tag{1}$$

Implementing this formula requires the knowledge of option prices for a continuum of strikes. Of course this is not possible in practice and infinitely many density functions are compatible to any given set of option prices over a finite range of strikes. However, some basic constraints have to be satisfied when constructing a risk-neutral density. For example, a well-defined risk neutral density is non-negative, integrates to one, and prices exactly all calls and puts. In Shimko (1993) is proposed to approximate the implied volatility curve as function of the strikes, $\sigma(K)$ to recover a continuous formula for the call prices. Once, we have this interpolation function, we can replace in the Black-Scholes formula and recover the probability distribution through the first two derivatives of the pricing formula wrt to the strike price:

$$\frac{\partial c(K,\sigma(K))}{\partial K} = \frac{\partial c}{\partial K} + \frac{\partial c}{\partial \sigma}\frac{\partial \sigma(K)}{\partial K}, \tag{2}$$

$$\begin{aligned}\frac{\partial^2 c(K,\sigma(K))}{\partial K^2} &= \frac{\partial^2 c}{\partial K^2} + 2\frac{\partial^2 c}{\partial\sigma\partial K}\frac{\partial\sigma(K)}{\partial K} \\ &+ \frac{\partial c}{\partial\sigma}\frac{\partial^2\sigma(K)}{\partial K^2} + \frac{\partial^2 c}{\partial\sigma^2}\left(\frac{\partial\sigma(K)}{\partial K}\right)^2\end{aligned} \tag{3}$$

In these expressions we can compute:

1. $\frac{\partial^2 c}{\partial K^2}$, $\frac{\partial^2 c}{\partial\sigma\partial K}$, $\frac{\partial c}{\partial\sigma}$ and $\frac{\partial^2 c}{\partial\sigma^2}$ using the Black-Scholes formula.
2. $\frac{\partial\sigma(K)}{\partial K}$ and $\frac{\partial^2\sigma(K)}{\partial K^2}$ from the interpolating function of the volatility curve.

Clearly, the estimate of the risk neutral probability distribution is very sensitive to the changes in the volatility so we should look for an interpolating function that is as smooth as possible, but which still fits the observed option values closely enough.

The main disadvantage of this approach is that it does not recover the tails of the risk-neutral density function outside the range of available strike prices.

Therefore some extrapolating procedure has to be adopted (**Step 3**).

- **Step 1**: Use a quadratic interpolation for the implied volatility

$$\sigma(K) = a + bK + cK^2$$

with the coefficients that can be estimated by OLS.

- **Step 2**: Exploiting the BS formula, we have $(\tau = T - t)$.

1. $$e^{-r\tau}\frac{\partial c(K,\sigma(K))}{\partial K} = 1 - \mathcal{N}(d_2) + K\phi(d_2)(b + 2cK) \tag{4}$$

2.

$$e^{-r\tau}\frac{\partial^2 c(K,\sigma(K))}{\partial K^2} = -n\left(d_2\right)\left(d_{2K} - (b+2cK)(1-d_2 d_{2K}) - 2cK\right) \tag{5}$$

where $d_{1,2} = (\ln(SD/KB) \pm 0.5\sigma^2(K))/\sigma(K)$ and $d_{1K} = -\frac{1}{K\sigma(K)} + (1 - \frac{d_1}{\sigma(K)})(b+2cK)$, $d_{2K} = d_{1K} - (b+2cK)$.

- **Step 3**: Outside the quoted range strike, the implied volatility curve is assumed to be constant and the implied density is taken to be lognormal.

Recovering risk neutral density: "Illustration" vs "Market application"

In the section called "Market application" the user can import an Excel file containing a series of call/put market option prices and the implied volatilities are computed by the web app. In this section instead of prices the user supplies the value of 3 of volatilities and strikes in order to manage the shape of the implied volatility profile and the impact on implied risk neutral density function.

Form Field

There are 4 different groups (Figure 1) of input parameters. Here we are going to explain the features of each one.

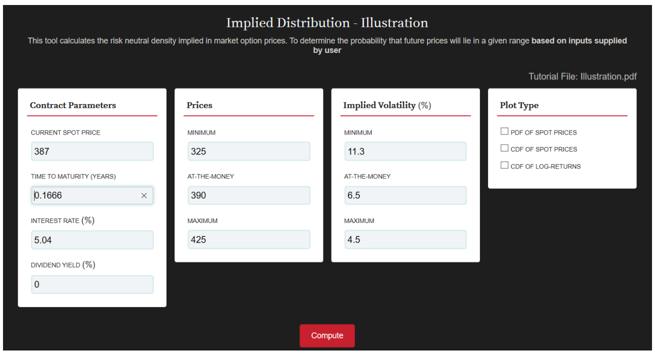

Figure 1: Input form

Contract Parameters

- Current spot price: the price of the underlying observed today on the market.
- Time to maturity (years): the time at which the option expires on annual basis.
- Dividend yield (%): the dividend yield paid by the index (annual continuously compounded).
- Interest rate (%): an appropriate risk free rate (annual continuously compounded).

Strike Prices

- Minimum: is an arbitrary out-the-money strike price (Strike minimum < Strike at-the-money).
- At-the-money: is the strike of the at-the-money option close to the spot price.
- Maximum: is an arbitrary in-the-money strike price (Strike maximum > Strike at-the-money).

Implied Volatility (%)

- Minimum: is the implied volatility corresponding to "Strike minimum" choice.
- At-the-money: is the implied volatility corresponding to "Strike at-the-money" choice.
- Maximum: is the implied volatility corresponding to "Strike maximum" choice.

Plot Type

In this panel the user can select which plot will be displayed.

- PDF of spot prices: probability distribution of underlying asset prices.
- CDF of spot prices: cumulative distribution of underlying asset prices.
- CDF of log-returns: cumulative distribution of underlying asset log-returns.

  The probability distribution of log-returns is computed by default whenever the user clicks on "Compute".

The output

The procedure implemented in this section leads the user to two main results. The first one is the implied volatility profile represented by a parabola (Figures 2 and 3). The second one is a series of graphs and moments related to the implied risk-neutral probability distribution (Figures 4 and 5). In adding we propose an example in which there is the comparison between "Skewed Case" and "Flat Case" . The "Contract Parameters" (Figure 1) are the same for each example. Table 1 displays the different strikes and volatilities supplied.

Case	Skewed Case		Flat Case	
Inputs	Strike	Vol (%)	Strike	Vol (%)
OTM	325	11.3	325	6.5
ATM	390	6.5	390	6.5
ITM	425	4.5	425	6.5

Table 1: "Skewed Case" and Flat "Case inputs"

Implied volatility Profile

The implied volatility profile is determined in **Step 1.** We allowed the implied volatility function to be a parabolic function of the exercise price ($a_0 + a_1 k + a_2 k^2$). On x-axis we have strikes (k) and on y-axis smoothed implied volatilities.

- ❑ **Skewed Case** (Figure 2): the implied volatility profile is negatively sloped, implying a negative skew distribution of future index values: the market is pricing relatively greater on a fall in the index than on a rise.

- ❑ **Flat Case** (Figure 3): the implied volatility profile is flat because the volatility is constant for each strike. It is the case of Black and Scholes model. The market is pricing equally a fall and a rise in the index.

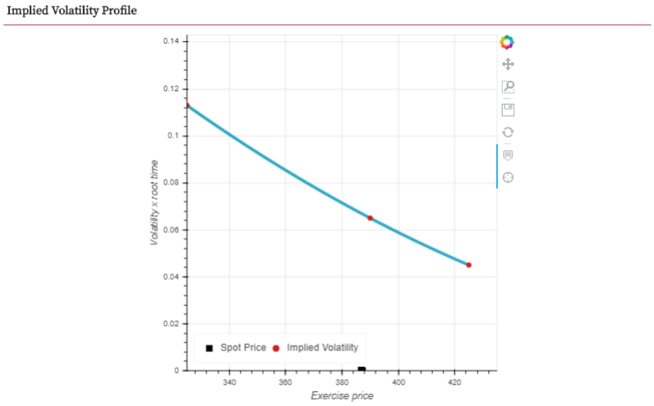

Figure 2: Implied Volatility Profile "Skewed Case"

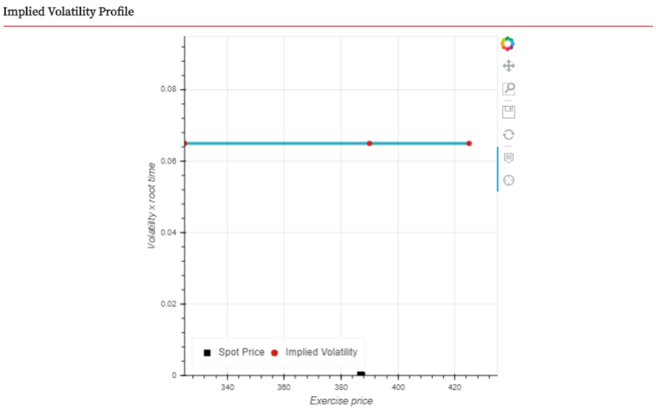

Figure 3: Implied Volatility Profile "Flat Case"

In the Table 2 there are three parameters (a_0, a_1, a_2) that control the shape of the implied volatility parabola.

- ❑ a_0 : with higher (lower) a_0 the parabola moves up (down) so higher (lower) implied volatilities.

- ❑ a_1 : positive (negative) a_1 causes the parabola to have a positive (negative) slope.

- ❑ a_2 : positive (negative) a_2 causes the parabola to be concave (convex).

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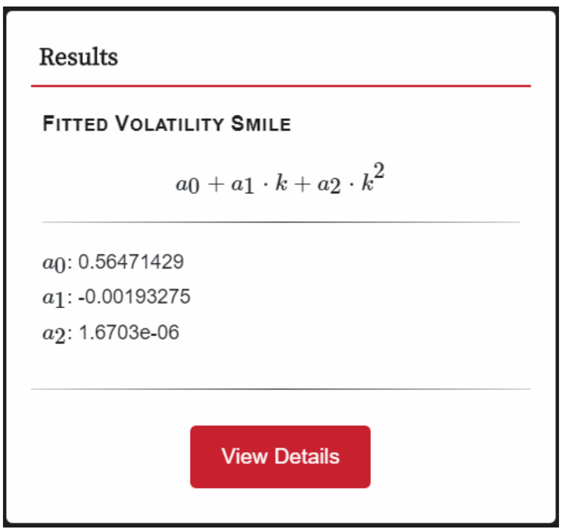

Table 2: Results "Skewed Case"

Recovering index probability distribution from option prices

The fitted implied volatilities can be used to find smoothed call/put prices (via Black-Scholes) and the smoothed call/put prices can be differentiated to find the values of the density function and cumulative distribution for each possible index value at a certain maturity (**Step 2**). The benchmark lognormal (normal) distribution is determined with the same mean and variance as the implied index (log-returns) distribution.

- ❑ Log-returns (Figure 4): on the left (right) side the blue line is the recovered CDF (PDF) of log-returns and the red line is the benchmark log-normal CDF (PDF).

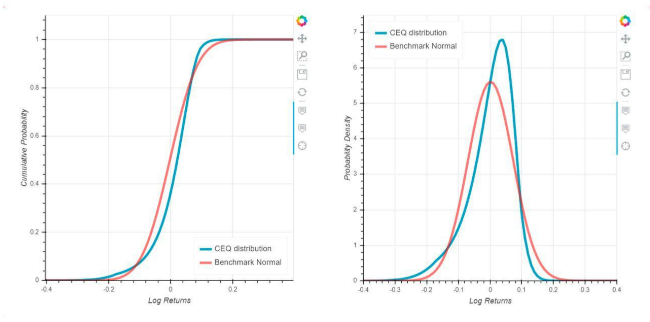

Figure 4: Implied recovered log returns distributions "Skewed Case"

- ❑ Index values (Figure 5): on the left(right) side the blue line is the recovered PDF(CDF) Shimko and the red line is the benchmark normal PDF(CDF).

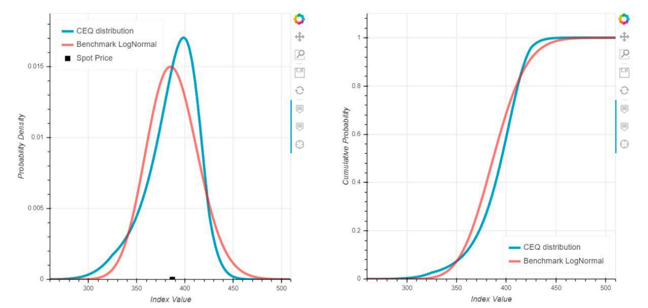

Figure 5: Implied recovered index values distributions "Skewed Case"

Table of moments

There is the possibility for the user to click on "View details" (Table 2). The output is a pop-up table (Tables 3 and 4) containing the moments of each distribution.

- ❑ **Skewed Case** (Figures 6 and Table 3): the recovered implied index distribution is negatively skewed and more leptokurtic than the benchmark normal distribution. This confirm the previous observation from the implied volatility profile on skewness and kurtosis.

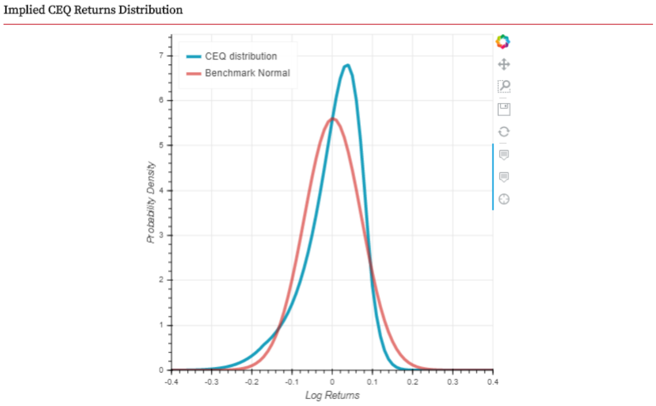

Figure 6: Implied recovered log returns distribution vs benchmark normal "Skewed Case"

Table of Moments ×

Description	Ceq	LogNormal	Ceq Return	Normal
Area	0.9969	1	0.9969	1
Mean	386.8958	386.8958	-0.0027	-0.0027
Variance	711.3289	711.3289	0.0051	0.0051
Skewness	-0.7285	0.2071	-0.9623	0.0
Kurtosis	3.6043	3.0764	4.2188	3.0

Export Table

Table 3: Table of moments "Skewed Case"

- ❑ **Flat Case** (Figure 7 and Table 4): the implied index distribution and benchmark normal overlap each other. The skewness and kurtosis are close respectively to 3 and 0. In this case the log return of underlying index follow a Normal distribution.

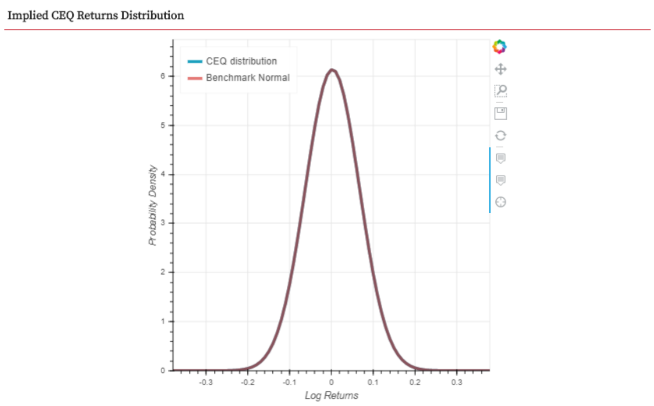

Figure 7: Implied recovered log returns distribution vs benchmark normal "Flat Case"

Table of Moments

Description	Ceq	LogNormal	Ceq Return	Normal
Area	1.0	1	1.0	1
Mean	390.2632	390.2632	0.0063	0.0063
Variance	644.8514	644.8514	0.0042	0.0042
Skewness	0.1955	0.1955	0.0	0.0
Kurtosis	3.068	3.068	3.0	3.0

Export Table

Table 4: Table of moments "Flat Case"

An additional feature is the "Export Table" (Table 4) that stores the table in an Excel file that the user can download.

Chart Tools

In each chart there are interactive tools positioned at the top right. Here are listed all of them starting from the first one.

- Bokeh Logo: hyperlink to access the bokeh site. Bokeh is the library used to create all interactive graph in the web-application.
- Pan Tool: the pan tool allows the user to pan the plot by left-dragging a mouse or dragging a finger across the plot region.
- Box Zoom: the box zoom tool allows the user to define a rectangular region to zoom the plot bounds too, by left-dragging a mouse, or dragging a finger across the plot area.
- Save: the save tool pops up a modal dialog that allows the user to save a PNG image of the plot.
- Reset: The reset tool turns off all the selected tools.
- Hoover Tool: the hover tool will generate a “tabular” tooltip where each row contains a label, and its associated value.

References

Breeden, D. T., & Litzenberger, R. H. (1978). Prices of state-contingent claims implicit in option prices. *Journal of business*, 621–651.

Shimko, D. (1993). Bounds of probability. *Risk*, *6*(4), 33–37.