FINANCE WEB APP

ASIAN OPTION

Optimized Lower Bound (General Models)

THIS section follows the work of Fusai and Kyriakou (2016) "General Optimized Lower and Upper Bounds for Discrete and Continuous Arithmetic Asian Options". The authors derive a lower bound to the price of an arithmetic Asian options with discrete or continuous average in a general model setting. The method is applicable to a various range of stochastic dynamic models: exponential Lévy models, stochastic volatility models and constant elasticity of variance diffusion. The user can select one of the cited models and obtain the price of an Asian Option.

Background information

In this section we describe the calculation of an optimal lower bound formula for the price of the arithmetic Asian call option with fixed or floating strike price. Price results for put-type options can then be obtained via standard put-call parity. The original idea for the derivation of the bound in the case of a geometric Brownian motion process is given in Curran (1994) and Rogers and Shi (1995). Fusai and Kyriakou (2016) generalize the idea, using a Fourier transform method, to Lévy, stochastic volatility models belonging to the affine class and the CEV diffusion. Next is presented the framework for the discrete average, in the case of Lévy models.

Lévy models

Assume that the price of the underlying asset S is observed at the equally spaced discrete times $t_0 \equiv 0, t_1 \equiv \Delta, \dots, t_j \equiv \Delta j, \dots, t_N \equiv \Delta N = T$, where T is a fixed time horizon. We assume a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} \equiv (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ where $\mathbb{P}$ is the risk neutral probability measure. Assume that $X \equiv \ln S$ is represented by a Lévy process of the general form

$$dX_t = mdt + \sigma dW_t + dL_t^x,$$

whith $m \in \mathbb{R}$ the deterministic drift, $\sigma \geq 0$ the constant diffusion parameter, W a standard Brownian motion and L^x a purely discontinuous random process. Consider the log-increments of the underlying

$$\ln S_{\Delta j} - \ln S_{\Delta(j-1)} = X_{\Delta j} - X_{\Delta(j-1)} \equiv Z_j^{\Delta}, \quad (1)$$

so that the price of the underlying asset at t_j is

$$S_{\Delta j} = S_0 e^{Z_1^{\Delta}+\dots+Z_j^{\Delta}}$$

for $j = 1, \dots, N$. At the level of risk neutral modelling, exponential Lévy asset price models allow to generate implied volatility smiles and skews similar to the ones observed in market prices. Under such model assumptions, the increments (1) are independent and identically distributed. By the celebrated Lévy–Khintchine formula, the characteristic function of Z_j^{Δ} has the form

$$\mathbb{E}[\exp(iuZ_j^{\Delta})] \equiv \exp(\psi_{\Delta}(u))$$

$$\equiv \exp(ium\Delta + \varphi_{\Delta}(u)) \quad (2)$$

for all j, where $\varphi_{\Delta}(u) \equiv \ln \mathbb{E}[\exp(iu(Z_j^{\Delta} - m\Delta))]$. Further, we choose

$$m \equiv r - q - \frac{1}{\Delta}\varphi_{\Delta}(-i) = r - q - \varphi_1(-i), \quad (3)$$

where $r \geq 0$ and $q \geq 0$ denote respectively the constant instantaneous risk-free interest rate and dividend yield, to ensure that the discounted asset price is a martingale under the probability measure $\mathbb{P}$, e.g., Schoutens (2003) , which is necessary for the risk neutral pricing of derivatives. For our purposes, it is also necessary to specify the asset price dynamics under the measure $\overline{\mathbb{P}}$ where the underlying itself represents the numéraire Geman, El Karoui, and Rochet (1995). From the numéraire change formula, the characteristic function of Z_j^{Δ} under the measure $\overline{\mathbb{P}}$ has the form

$$\overline{\mathbb{E}}[\exp(iuZ_j^{\Delta})] = \mathbb{E}[\exp(-r\Delta + i(u-i)Z_j^{\Delta})]$$

$$= \exp(-r\Delta + \psi_{\Delta}(u-i)). \quad (4)$$

Discrete average

In light of the lack of analytical tractability of the law of the discrete arithmetic average of the asset prices $\frac{1}{N+1}\sum_{j=0}^{N} S_{\Delta j}$, we look for close proxies with known distributional properties. More specifically, such a proxy is given by

$$Y_{\Delta N} \equiv \frac{1}{N+1}\sum_{j=0}^{N} X_{\Delta j}, \quad (5)$$

where $X = \ln S$ in the case of the Lévy. Important to our arithmetic Asian option pricing framework is knowledge of the distribution law of the new average (5). Below, we derive key results under the Lévy.

(i) Define

$$\eta_j(u,v) = \begin{cases} v\left(1-\frac{j}{N+1}\right), & k \vee n < j \le N \\ u+v\left(1-\frac{j}{N+1}\right), & k \wedge n < j \le k \vee n \\ 2u+v\left(1-\frac{j}{N+1}\right), & 0 < j \le k \wedge n \end{cases} \tag{6}$$

and

$$\phi_{k,n,N}(u,v) = \mathbb{E}[\exp\{iu(X_{\Delta k}+X_{\Delta n})+ivY_{\Delta N}\}]$$

under the risk neutral measure.

Under the assumption of increments Z_j^{Δ} satisfying (2) under the risk neutral measure, define

$$\Psi_{h,\Delta}(u,v) = \sum_{j=h+1}^{N} \psi_\Delta(\eta_j(u,v)),$$

where $h = 0, k \wedge n, k \vee n$. Then,

$$\phi_{k,n,N}(u,v) = \exp\{i(2u+v)X_0 + \Psi_{0,\Delta}(u,v)\}. \tag{7}$$

(ii) Define

$$\bar{\eta}_j(u,v) = \begin{cases} -2u - v\frac{j}{N+1}, & k \vee n < j \le N \\ -u - v\frac{j}{N+1}, & k \wedge n < j \le k \vee n \\ -v\frac{j}{N+1}, & 0 < j \le k \wedge n \end{cases} \tag{8}$$

and

$$\bar{\phi}_{k,n,N}(u,v) = \bar{\mathbb{E}}[\exp\{iu(X_{\Delta k}+X_{\Delta n}-2X_{\Delta N})+iv(Y_{\Delta N}-X_{\Delta N})\}]$$

under the measure $\bar{\mathbb{P}}$.

Under the assumption of increments Z_j^{Δ} satisfying (4) under the measure $\bar{\mathbb{P}}$, define

$$\bar{\Psi}_{h,\Delta}(u,v) = \exp\left\{-r\Delta(N-h) + \sum_{j=h+1}^{N} \psi_\Delta(\bar{\eta}_j(u,v) - i)\right\},$$

where $h = 0, k \wedge n, k \vee n$. Then,

$$\bar{\phi}_{k,n,N}(u,v) = \bar{\Psi}_{0,\Delta}(u,v). \tag{9}$$

Lower bounds for discrete Asian options

In the case of the discrete average, the payoff of the arithmetic Asian call option with time to maturity T has general form

$$P_{\Delta N} \equiv \left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - \bar{K}S_{\Delta N} - K\right)^{+}$$

$$= \left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - \bar{K}S_{\Delta N} - K\right)\mathbf{1}_A \tag{10}$$

consisting of the fixed strike price $K \ge 0$ and coefficient $\bar{K} \ge 0$ for floating strike options, with

$$A \equiv \left\{\omega : \frac{1}{N+1}\sum_{k=0}^{N} S_{\Delta k} > \bar{K}S_{\Delta N} + K\right\}. \tag{11}$$

The time-0 value of this option, P_0, satisfies

$$P_0 \equiv e^{-rT}\mathbb{E}\left[\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - \bar{K}S_{\Delta N} - K\right)\mathbf{1}_A\right] \ge$$

$$\mathrm{LB}_0 \equiv e^{-rT}\mathbb{E}\left[\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - \bar{K}S_{\Delta N} - K\right)\mathbf{1}_{A'}\right]$$

for any event set $A' \subset \Omega$ as $\frac{1}{N+1}\sum_{k=0}^{N} S_{\Delta k} \le \bar{K}S_{\Delta N} + K$ for $\omega \in A' \setminus A$. Therefore, the value of the option with fixed strike (i.e., $K > 0, \bar{K} = 0$), $P_{\text{fix},0}$, or floating strike (i.e., $K = 0, \bar{K} > 0$), $P_{\text{fl},0}$, satisfies respectively

$$P_{\text{fix},0} \ge \mathrm{LB}_{\text{fix},0} = e^{-rT}\mathbb{E}\left[\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - K\right)\mathbf{1}_{A'}\right], \tag{12}$$

$$P_{\text{fl},0} \ge \mathrm{LB}_{\text{fl},0} = e^{-rT}\mathbb{E}\left[S_{\Delta N}\left(\frac{\sum_{k=0}^{N} S_{\Delta k}S_{\Delta N}^{-1}}{N+1} - \bar{K}\right)\mathbf{1}_{A'}\right]$$

$$= S_0\bar{\mathbb{E}}\left[\left(\frac{\sum_{k=0}^{N} S_{\Delta k}S_{\Delta N}^{-1}}{N+1} - \bar{K}\right)\mathbf{1}_{A'}\right], \tag{13}$$

where the last equality in (13) follows by a change to the $\bar{\mathbb{P}}$ measure.

Thus, the choice of an event set A' gives us a lower bound for the option price. The idea is that the chosen event set A' relates as closely as possible to the true event set A, so that the distance between the lower bound and the true option price is minimized, while at the same time makes the problem more analytically tractable compared to the original A. In what follows, we explain how the event set A' is determined optimally in case of Lévy models. In consistency with (12) and (13), we define a parameter m taking value 0 (1) in the case of the fixed (floating) strike option.

The case of Lévy models

Given the form $A = \{\omega : \frac{1}{N+1}\sum_{k=0}^{N} S_{\Delta k}S_{\Delta N}^{-m} > K\}$ and that

$$\frac{1}{N+1}\sum_{k=0}^{N} S_{\Delta k}S_{\Delta N}^{-m} \ge \left(\prod_{k=0}^{N} S_{\Delta k}S_{\Delta N}^{-m}\right)^{1/(N+1)} \tag{14}$$

we choose

$$A' \equiv \left\{\omega : \left(\prod_{k=0}^{N} S_{\Delta k} S_{\Delta N}^{-m}\right)^{1/(N+1)} > \exp(\lambda)\right\}$$

$$\equiv \left\{\omega : \frac{1}{N+1} \sum_{k=0}^{N} \ln S_{\Delta k} - m \ln S_{\Delta N} > \lambda\right\} \tag{15}$$

in the lower bounds (12) and (13) where λ is a real parameter. The choice of A' based on the (log-)geometric average was originally applied in in Curran (1994) and Rogers and Shi (1995) in the log-normal model and turned out to be a very accurate choice due to the high correlation between the arithmetic and (log) geometric averages. The fact that the (log) geometric average also has favourable distributional properties under the Lévy models, as opposed to the arithmetic average, further motivates its choice.

Lower bound optimization and transform representations for discrete Asian option

Due to the high correlation between the two types of average, i.e., $\frac{1}{N+1}\sum_{k=0}^{N} S_{\Delta k} S_{\Delta N}^{-m}$ and $\frac{1}{N+1}\sum_{k=0}^{N} \ln(S_{\Delta k} S_{\Delta N}^{-m})$ for Lévy models, by replacing A by A' and additionally optimizing the parameter λ we minimize the error in the lower bound. Next, we determine the value of parameter λ which maximizes the lower bounds (12) and (13). Consider the random variables $Y_{\Delta N} = \frac{1}{N+1}\sum_{k=0}^{N} X_{\Delta k}$ and $\bar{Y}_{\Delta N}$, where $X = \ln S$, $\bar{Y}_{\Delta N} \equiv Y_{\Delta N} - X_{\Delta N}$. Then, the optimal lower bounds for fixed and floating strike options are given by

$$\mathrm{LB}^*_{\mathrm{fix},0} = e^{-rT}\mathbb{E}\left[\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - K\right)\mathbf{1}_{\{Y_{\Delta N} > \lambda^*\}}\right],$$

$$\mathrm{LB}^*_{\mathrm{fl},0} = S_0\bar{\mathbb{E}}\left[\left(\frac{\sum_{k=0}^{N} S_{\Delta k} S_{\Delta N}^{-1}}{N+1} - \bar{K}\right)\mathbf{1}_{\{\bar{Y}_{\Delta N} > \lambda^*\}}\right],$$

where

$$\lambda^* = \_argmax_{\lambda}\mathrm{LB}_0(\lambda) \tag{16}$$

satisfies the optimality conditions

$$\mathbb{E}\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1}\middle| Y_{\Delta N} = \lambda^*\right) = K \text{ and}$$

$$\bar{\mathbb{E}}\left(\frac{\sum_{k=0}^{N} S_{\Delta k} S_{\Delta N}^{-1}}{N+1}\middle| \bar{Y}_{\Delta N} = \lambda^*\right) = \bar{K} \tag{17}$$

respectively, for a fixed and a floating strike option under Lévy, ASV and CEV models.

We consider the case of the fixed strike option (the floating strike case is proved similarly). From (12) and the definitions of A' given in (15)

$$\mathbb{E}\left[\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - K\right)\mathbf{1}_{\{Y_{\Delta N} > \lambda\}}\right]$$

$$= \frac{1}{N+1}\sum_{k=0}^{N} \mathbb{E}[\mathbb{E}[S_{\Delta k} - K | Y_{\Delta N}]\mathbf{1}_{\{Y_{\Delta N} > \lambda\}}].$$

Differentiating w.r.t. λ and interchanging with the expectation yields

$$\frac{1}{N+1}\sum_{k=0}^{N} \mathbb{E}\left[\mathbb{E}\left[S_{\Delta k} - K | Y_{\Delta N}\right] \frac{d}{d\lambda}\mathbf{1}_{\{Y_{\Delta N} > \lambda\}}\right]$$

$$= \frac{-1}{N+1}\sum_{k=0}^{N} \mathbb{E}(S_{\Delta k} - K | Y_{\Delta N} = \lambda) f_N(\lambda), \tag{18}$$

where the last equality follows from $d\mathbf{1}_{\{Y_{\Delta N} > \lambda\}}/d\lambda = -\delta(Y_{\Delta N} - \lambda)$ with δ representing the Dirac delta function and f_N the density function of $Y_{\Delta N}$ under the risk neutral measure. Then (17) follows from (18) by setting this equal to zero.

We now proceed to derive the Fourier transform representations of the lower bounds (12) and (13) with respect to λ, which can then be inverted numerically to recover the lower bounds.

(i) Suppose $X = \ln S$. The Fourier transform of the lower bound (12) w.r.t. λ is

$$\Phi(u;\delta) \equiv$$

$$\int_{\mathbb{R}} e^{iu\lambda + \delta\lambda}\left\{\frac{e^{-rT}}{N+1}\sum_{k=0}^{N} \mathbb{E}[(e^{X_{\Delta k}} - K)\mathbf{1}_{\{Y_{\Delta N} > \lambda\}}]\right\} d\lambda$$

$$= \frac{e^{-rT}}{iu+\delta}\left\{\frac{1}{N+1}\sum_{k=0}^{N} \phi_{k,k,N}(-i/2, u - i\delta) - K\phi_{N,N,N}(0, u - i\delta)\right\}, \tag{19}$$

where constant $\delta > 0$ ensures integrability and ϕ is given in (14) for Lévy model. The Fourier transform of the lower bound (13) w.r.t. λ is

$$\bar{\Phi}(u;\delta) \equiv$$

$$\int_{\mathbb{R}} e^{iu\lambda + \delta\lambda}\left\{\frac{e^{X_0}}{N+1}\sum_{k=0}^{N} \bar{\mathbb{E}}[(e^{X_{\Delta k} - X_{\Delta N}} - \bar{K})\mathbf{1}_{\{Y_{\Delta N} - X_{\Delta N} > \lambda\}}]\right\} d\lambda$$

$$= \frac{e^{X_0}}{iu+\delta}\left\{\frac{1}{N+1}\sum_{k=0}^{N} \bar{\phi}_{k,k,N}(-i/2, u - i\delta) - \bar{K}\bar{\phi}_{N,N,N}(0, u - i\delta)\right\}, \tag{20}$$

where $\bar{\phi}$ is given in (9) for Lévy process.

(ii) The lower bounds (12) and (13) are given in terms of the inversion formulae

$$\mathrm{LB}_{\mathrm{fix},0}(\lambda) = \frac{e^{-\delta\lambda}}{2\pi}\int_{\mathbb{R}} e^{-iu\lambda}\Phi(u;\delta)du \text{ and}$$
$$\mathrm{LB}_{\mathrm{fl},0}(\lambda) = \frac{e^{-\delta\lambda}}{2\pi}\int_{\mathbb{R}} e^{-iu\lambda}\bar{\Phi}(u;\delta)du, \tag{21}$$

where Φ (resp. $\bar{\Phi}$) is given in (19) (resp. 20) for a general Lévy model. The whole procedure also for ASV models and CEV model is explained in Fusai and Kyriakou (2016).

Form field

There are 3 panels for inputing the model and contract parameters, see Figure 1 Asian option input form. We now illustrate their features.

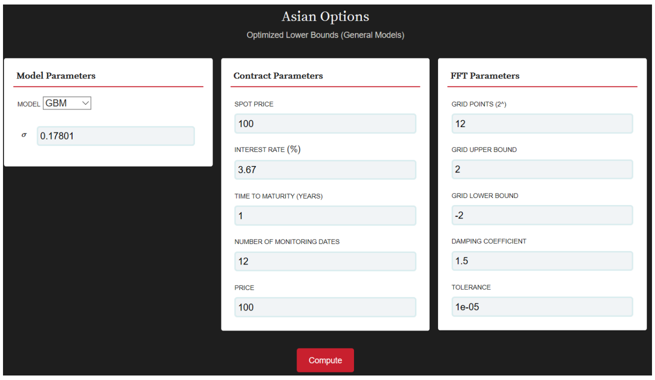

Figure 1: Asian option input form

Model Parameters

In the first panel we can select the different stochastic dynamic models as listed here below (see Table 2).

- ❏ GBM - Geometric Brownian motion.
- ❏ VG - Variance gamma process.
- ❏ Heston - Heston model
- ❏ NIG - Normal inverse Gaussian
- ❏ CGMY - CGMY model
- ❏ Meixner - Meixner model
- ❏ MJD - Merton's jump-diffusion model
- ❏ DEJD - Double exponential jump-diffusion model
- ❏ CEV - Constant elasticity of variance model
- ❏ OU - Ornstein-Uhlenbeck Model

Contract Parameters

In this panel we can set the characteristics of the contract to be priced, such as

- ❏ Spot price: initial price of underlying asset.
- ❏ Interest rate (%): annual risk free rate continuously compounded
- ❏ Time to maturity (years): time to expiry (in years) of the Asian option.
- ❏ Number of monitoring dates: specifies how often the price is monitored from time 0 to maturity time T.
- ❏ Strike price.

FFT Parameters

In this panel we set the parameters for performing the numerical inversion of the FFT transform, such as

- ❏ Number of grid points.
- ❏ Grid upper bound.
- ❏ Grid lower bound.
- ❏ Damping coefficient.
- ❏ Tolerance.

The output

The pricing of arithmetic Asian option doesn't admit true analytical solutions since the probability density function is not known analytically (see "Background Information"). The procedure adopted in the work mentioned above (Fusai and Kyriakou (2016)) presents an approximation of the Asian option price that is fast, accurate and flexible because it can deal with different risk neutral dynamics of the underlying stock price. The output is organized in a table (Table 1) and in a graph (Figure 2). The table reports the numerical results, whilst the graph illustrates the lower bounds as function of the shadow strike. The maximum value of the lower bound provides the numerical approximation to the Asian option price.

Asian option lower bound

In Figure 2 the blue line represents the lower bound as function of the parameter λ. The greatest lower bound is the approximation of the arithmetic Asian Option price ($\mathrm{LB}^*_{\mathrm{fix},0}$), and we call it "Optimal Shadow Lower Bound" (Table 1). The maximum value is reached in correspondence of λ^* (16) that is the value of the parameter λ which maximizes the lower bound. Given λ^* we can recover the "Optimal Strike" (Table 1), i.e. the "shadow" strike price corresponding to the "Optimal Shadow Lower Bound". We also consider a suboptimal lower bound, in which we fix the parameter λ equal to the logarithm of the option strike K.

Table of Asian Option Lower Bound

Optimal Strike	Optimal Shadow Lower Bound	Strike	Lower Bound Strike
99.9024	5.0921	100.0	5.0918

Table 1: Table of Asian option lower bound

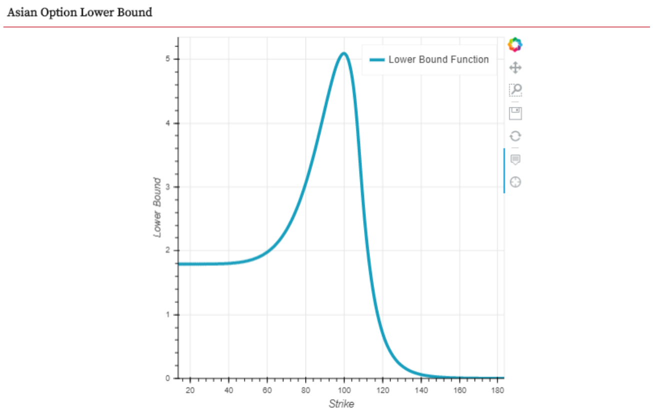

Figure 2: Asian option lower bound

Chart Tools

In each chart there are interactive tools positioned at the top right. Here we list all of them starting from the first one.

- ❑ Bokeh Logo: hyperlink to access the Bokeh site. Bokeh is the library used to create all interactive graph in the web-application.
- ❑ Pan Tool: the pan tool allows the user to pan the plot by left-dragging a mouse or dragging a finger across the plot region.
- ❑ Box Zoom: the box zoom tool allows the user to define a rectangular region to zoom the plot bounds too, by left-dragging a mouse, or dragging a finger across the plot area.
- ❑ Save: the save tool pops up a modal dialog that allows the user to save a PNG image of the plot.
- ❑ Reset: The reset tool turns off all the selected tools.
- ❑ Hoover Tool: the hover tool will generate a “tabular” tooltip where each row contains a label, and its associated value.

Table.2: Model parameter sets

Model	Parameters				
Gaussian	σ				
	0.17801				
VG	ν	θ	σ		
	0.736703	-0.136105	0.180022		
NIG	a	b	δ		
	6.1882	-3.8941	0.1622		
CGMY	C	G	M	Y	
	0.0244	0.0765	7.5515	1.2945	
MJD	σ	l	μ_x	σ_x	
	0.126349	0.174814	-0.390078	0.338796	
DEJD (Kou)	σ	l	p	η_1	η_2
	0.120381	0.330966	0.20761	9.65997	3.13868
Meixner	a	b	δ		
	0.3977	-1.494	0.3462		
Heston	α	β	γ	ρ	$\sqrt{V_0}$
	6.21	0.019	0.61	-0.7	0.101
CEV	γ	σ			
	1.5, 2.5	$0.25S_0^{1-\gamma/2}$			
OU	ϵ	κ_1	σ		
	4.282364642	5.4462283548	0.361786273		

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