Part A : Forward Euler method

Step 1 : One strain increment

An unique strain increment $\Delta\varepsilon = [0.003, 0.004]^T$ is applied to a non-stressed steel plate.

At first, the stress increment is explicitly derived by the forward Euler method from the strain increment, as following:

- Compute trial stress increment $\Delta\sigma_{trial} = \mathrm{D}\Delta\varepsilon$, D being the constitutive matrix, function of Young's modulus and Poisson ratio of the plate.
- If the trial stress (initial stress + trial stress increment) check the failure condition, the yielding factor α is computed, allowing to compute the yield stresses for the same load direction. Similarly, the strain increment is divided in its elastic (until failure condition verified) and plastic parts.
- From its plastic part, the stress σ^1 is computed as the projection of the trial stress on the tangent of the yield criterion at the yield stress computed previously.
- The criterion value associated with stress σ^1 is then computed, to observe the "distance" between the numerical approximation and actual stress update.

For this first step, where no sub-increments are implemented in the stress computation, the following values were obtained:

$$\sigma_{trial} = [600, 800]^T\ Mpa$$

$$\sigma^1 = [440.306, 400.766]^T\ Mpa$$

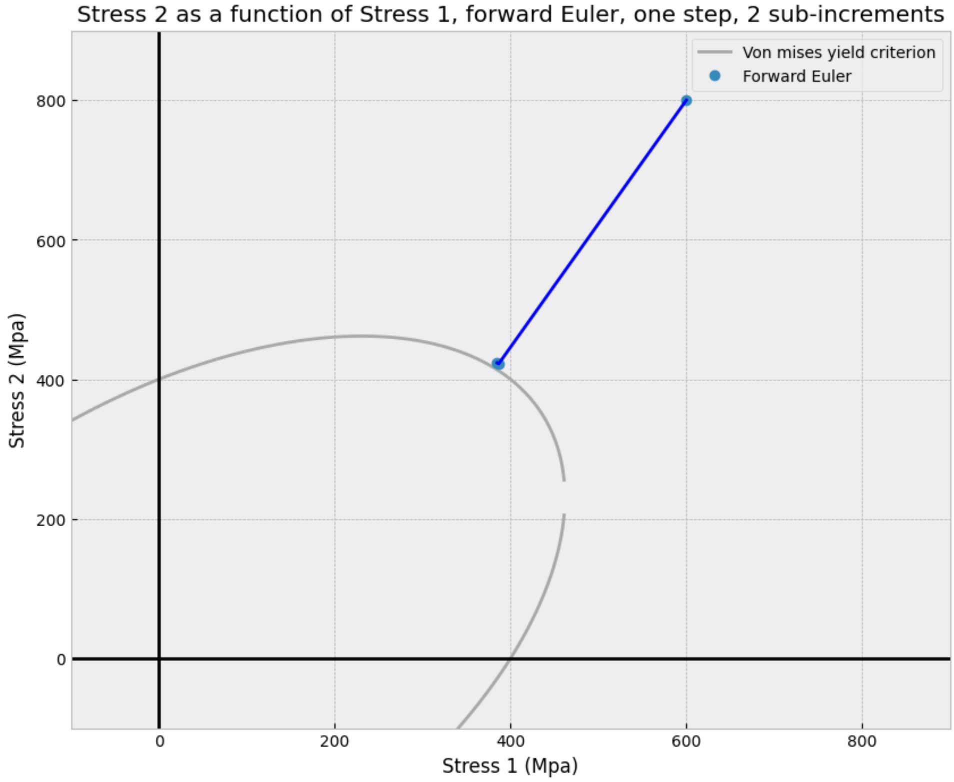
$$\text{Von mises criterion value } f(\sigma^1) = 18023$$

Step 2 : Sub-increments introduction, 2 sub-increments

The elastic part of the strain increment is divided in two sub-increments. For each sub-increment i, a new $\sigma^{1,i}_{trial}$ is computed, and projected on the tangent of the $\sigma^{1,i-1}$ stress obtained at the previous sub-increment. Therefore, an increased number of sub-increments allows to fit the curvature of the yield criterion, reducing the distance of the numerical stress value.

The first trial stress is kept the same as in step 1. The plastic strain increment is divided by the number of sub-increments, and associated stress sub-increments are computed with the tangent of the previous sub-increment stress.

In a stress x – stress y graph, the forward Euler method for one strain increment with 2 sub-increments is as following:



With σ_{trial} in the top right corner, and the computed stress update closer to the yield criterion.

For this second step, the following value were obtained:

$$\text{Stress, first sub-increment } \sigma^{1,1} = [386.563, 422.263]^T \ Mpa$$

$$\text{Stress, second sub-increment } \sigma^{1,2} = [384.752, 423.651]^T \ Mpa$$

$$\sigma^1 = \sigma^{1,2} = [384.752, 423.651]^T \ Mpa$$

$$\text{Von mises criterion value } f(\sigma^1) = 4513$$

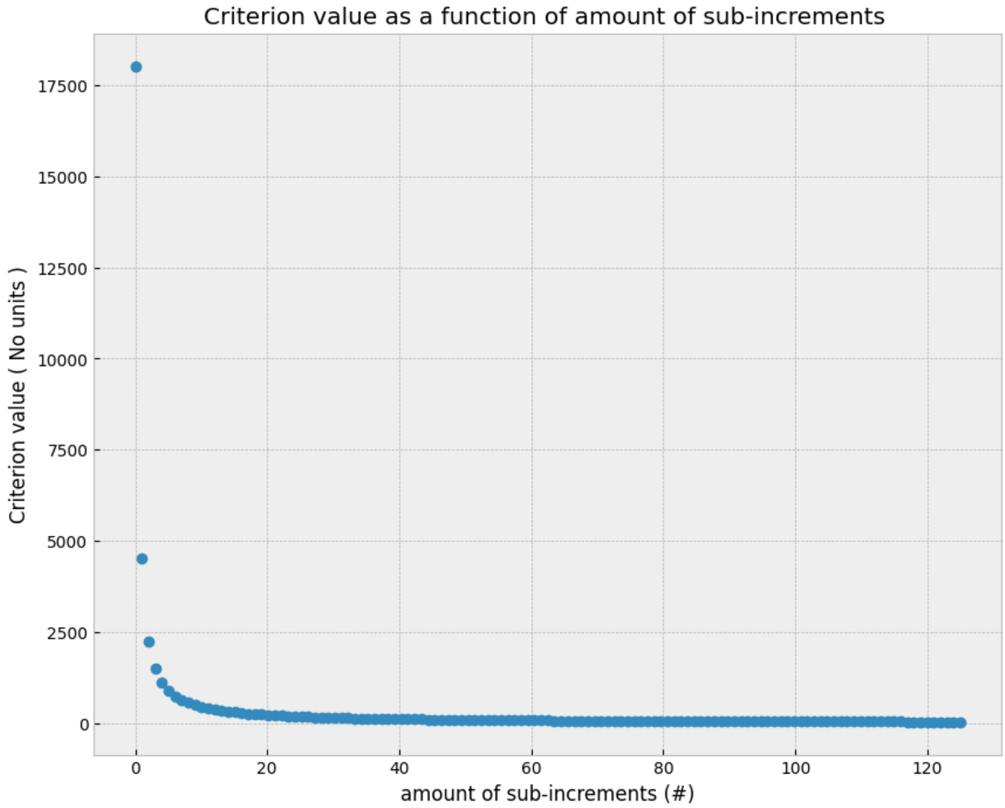
<u>Step 3 :</u> Minimal number of sub-increments.

We saw in the first two steps that increasing the number of sub-increments reduce the value of the yield criterion, and bring the computed stress update closer to the actual stress update.

Therefore, we want to increase the number of sub-increments in order to have the criterion value under a given threshold, here:

$$f(\sigma^1) \leq f(\sigma_{trial}) \cdot 10^{-4} = 36$$

We chose an iterative approach to the problem, calculating $f(\sigma^1)$ for increasing number of sub-increments.



A minimum of 125 sub-increments was observed in order to obtain a stress update respecting the yield criterion value threshold.

In a stress x – stress y graph, the forward Euler method for one strain increment with 125 sub-increments is as following:

The following values were obtained:

$$\sigma^1 = \sigma^{1,125} = [375.259, 420.913]^T \ Mpa$$

$$\text{Von mises criterion value } f(\sigma^1) = 35.75$$

Steps results comparison:

Step	Number of sub-increments	Stress update [Mpa]	Criterion value
1	1	$440.306, 400.766$	18023
2	2	$384.752, 423.651$	4513
3	125	$375.259, 420.913$	35.75

Firstly, we can observe that the stress sub-increments computations for a large amount of strain sub-increments lie much closer to the yield criterion than for the previous steps. This is expected, as the projection is closer to the previous sub-increment stress, reducing the error.

Also, the stresses varies in a non-negligeable amount compared to the step 1: -15 % for σ_I and + 5 % for σ_2.

Therefore, increasing the amount of sub-increments is important to a trustworthy result.

Finally, it is observed that the yield criterion value decrease rapidly with the increase of amount of sub-increments, approaching rapidly the limit of 0. Consequently, passed a certain amount, a very large amount of sub-increments will not improve further the stress update value, reaching a limit.

Part B: Backward Euler method

The problem is now solved with the Backward Euler method. The principal difference with the Forward Euler method is that there is no need to find the intersection with the yield function, but the solution is found by using iterations with an iterative approach.

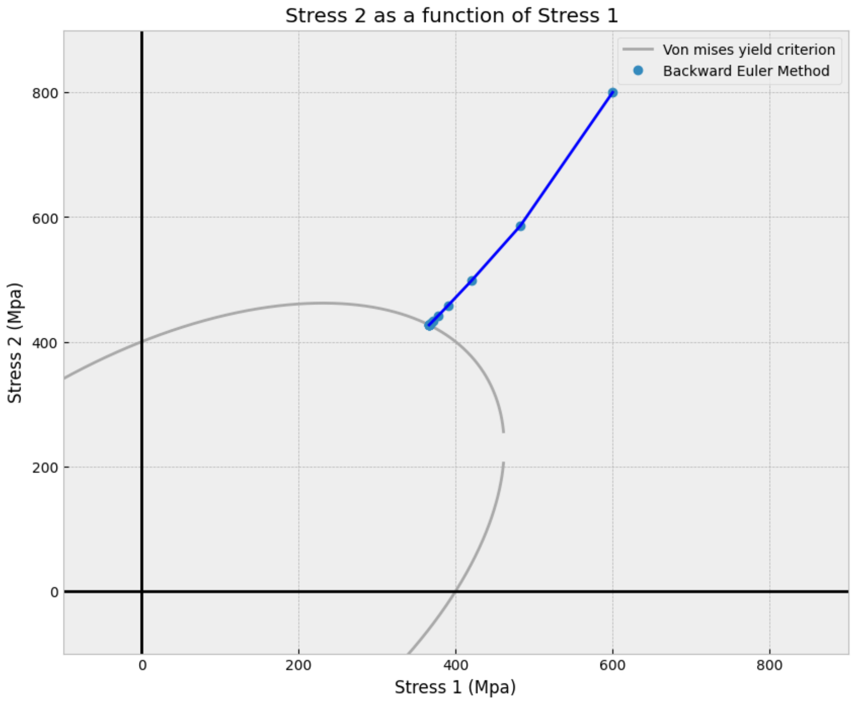
For the first iteration, the stress $\sigma^{n,0} = \sigma^{trial}$ (same value as for the Forward Euler Method) and the stress increment $\Delta\lambda^{n,0} = 0$.

At each iteration, the stress at the previous increment, as well as the value of the yield function for the previous stress and the previous stress increment are given. To perform the next iteration, $\sigma^{n,i}$ needs to be calculated. From the obtained value, the yield function evaluated at $\sigma^{n,i}$ is computed*. If the convergence $\frac{f(\sigma^{n,i})}{f(\sigma^{trial})} < error = 0.0001$ is reached, the stress in the steel plate subjected to one strain increment is found.

*To calculate $\sigma^{n,i}$, the strain increment $\Delta\lambda^{n,i}$ needs to be defined. This is calculated using the Newton-Raphson method: $\Delta\lambda^{n,i} = \Delta\lambda^{n,i-1} - \frac{f(\Delta\lambda^{n,i-1})}{f\prime(\Delta\lambda^{n,i-1})}$, where $f(\Delta\lambda^{n,i-1})$ is calculated as $f(\sigma^{n,i-1})$ and $f'(\Delta\lambda^{n,i-1}) = \frac{\partial f}{\partial \sigma} * \frac{\partial \sigma}{\partial \lambda}$. $\frac{\partial \sigma}{\partial \lambda}$ is evaluated at $\sigma^{n,0}$, and so as $\frac{\partial f}{\partial \sigma}$ for the equilibrium of the chain derivative.

Using this iterative approach, 11 iterations are needed to reach the stress in the steel plate for one strain increment. The stresses in the steel plate are:

$$\sigma = [366.21, 426.90]^T \text{ MPa and } f = 16.41$$



The stress resulting from the strain increment are similar in both models, but with a tiny difference. This difference may become from the incrementation on the neighbouring of the yield function for the Forward Euler Method, as the error increases with the curvature of the yield surface.