INTEGRALES

CURVAS

$\bar{\lambda}: I \subset \mathbb{R} \to \mathbb{R}^3 \;/\; \bar{\lambda}(t) = (x(t), y(t), z(t))$

$\left.\begin{array}{l} \bar{\lambda} \in C^1 \wedge \bar{\lambda}' = \bar{T} \neq \bar{0} \quad \text{regular} \\ \bar{\lambda} \text{ inyectiva } Int(I) \end{array}\right\} C \text{ simple}$

$\text{long}(C) = \int_I \|\bar{\lambda}'(t)\| \, dt$

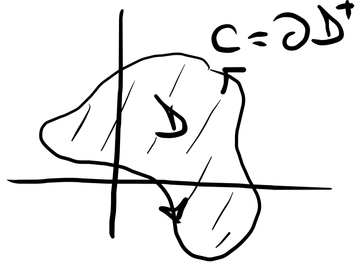
$f: A \subset \mathbb{R}^3 \to \mathbb{R} \;/\; f \in C^0 \;,\; C \subset A$

$\text{masa}(C) = \int_C f \, d\lambda = \int_I f(\bar{\lambda}) \cdot \underbrace{\|\bar{\lambda}'(t)\| \, dt}_{d\lambda}$

$\bar{f}: A \subseteq \mathbb{R}^n \to \mathbb{R}^n \;/\; \bar{f} \in C^0 \;\wedge\; C \subset A$

$\text{trabajo de } \bar{f} \text{ en } C_{\bar{\lambda}} = \int_C \bar{f} \, d\bar{\lambda} = \int_I \bar{f}(\bar{\lambda}) \cdot \underbrace{\bar{\lambda}'(t) \, dt}_{d\bar{\lambda}}$

Si C es cerrada simple

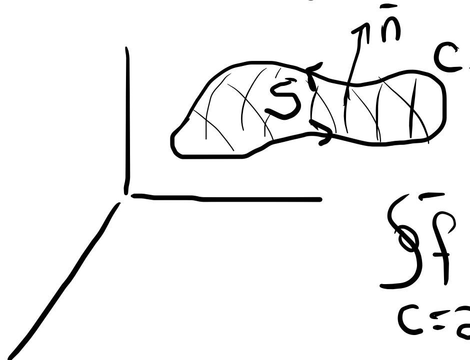


Teorema del rotor en $\mathbb{R}^2$

$$\oint_{C=\partial D^+} \bar{f} \, d\bar{\lambda} = \iint_D \text{rot} \bar{f} \, dx\,dy$$

$\bar{f}: A \subseteq \mathbb{R}^3 \to \mathbb{R}^3 \;/\; \bar{f} \in C^1 \;,\; S \subset A$

C es cerrada simple



Teorema del rotor en $\mathbb{R}^3$

$$\oint_{C=\partial S^+_n} \bar{f} \, d\bar{\lambda} = \iint_S \text{rot} \bar{f} \, d\bar{\sigma}$$

SUPERFICIES

$\bar{\sigma}: D \subset \mathbb{R}^2 \to \mathbb{R}^3 \;/\; \bar{\sigma}(u,v) = (x(u,v), y(u,v), z(u,v))$

$\left.\begin{array}{l} \bar{\sigma} \in C^1 \wedge \bar{\sigma}'_u \times \bar{\sigma}'_v = \bar{N} \neq \bar{0} \quad \text{regular} \\ \bar{\sigma} \text{ inyectiva } Int(D) \end{array}\right\} S \text{ simple}$

$\text{área}(S) = \iint_D \|\bar{\sigma}'_u \times \bar{\sigma}'_v\| \, du\,dv = \;\longrightarrow\; = \iint_D \frac{\|\bar{\nabla} g\|}{|g'_z|} \, dx\,dy$

$f: A \subset \mathbb{R}^3 \;/\; f \in C^0 \;,\; S \subset A$

$\text{masa}(S) = \iint_S f \, d\sigma = \iint_D f(\bar{\sigma}) \cdot \underbrace{\|\bar{\sigma}'_u \times \bar{\sigma}'_v\| \, du\,dv}_{d\sigma} = \iint_D f(x,y,z(x,y)) \cdot \overbrace{\frac{\|\bar{\nabla} g\|}{|g'_z|} \, dx\,dy}^{d\sigma}$

Si $z = z(x,y)$

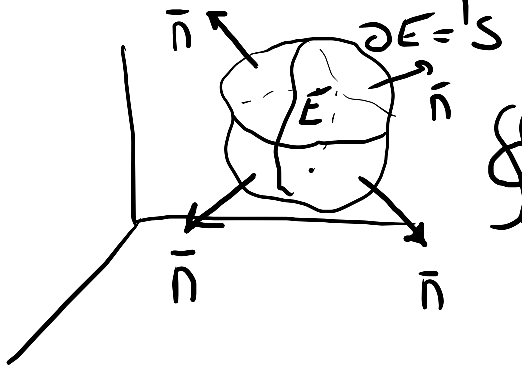
$S = N_0 \, g(x,y,z)$

$g(x,y,z(x,y)) = 0$

$\bar{f}: A \subset \mathbb{R}^3 \to \mathbb{R}^3 \;/\; \bar{f} \in C^0 \;\wedge\; S \subset A$

$\text{flujo de } \bar{f} \text{ a través } S_{\bar{\sigma}} = \iint_S \bar{f} \, d\bar{\sigma} = \iint_D \bar{f}(\bar{\sigma}) \cdot \underbrace{(\bar{\sigma}'_u \times \bar{\sigma}'_v) \, du\,dv}_{d\bar{\sigma}} \;\longrightarrow\; = \iint_D \bar{f}(x,y,z(x,y)) \cdot \frac{\bar{\nabla} g}{|g'_z|} \, dx\,dy$

Si S cerrada simple



Teorema de la divergencia.

$$\oiint_{S=\partial E^+} \bar{f} \, d\bar{\sigma} = \iiint_E \text{div} \bar{f} \, dx\,dy\,dz$$

05) Calcule el área de las siguientes superficies:

a) Trozo de superficie cilíndrica $z = 2x^2$ con $y \le x$, $z \le 6$, 1° octante.

$$S: \underset{(x,y)}{z} = 2x^2 \longrightarrow S = N_0\, g_{(x,y,z)} \longrightarrow g(x,y,z) = 2x^2 - z$$

$$\bar{\nabla} g = (4x, 0, \underbrace{-1}_{g'_z})$$

$$\|\bar{\nabla} g\| = \sqrt{16x^2+1}$$

$$\text{área}(S) = \iint_{S_{xy}} \frac{\|\bar{\nabla} g\|}{|g'_z|}\, dx\, dy =$$

$$= \int_0^{\sqrt{3}} \int_0^{x} \frac{\sqrt{16x^2+1}}{|-1|}\, dy\, dx =$$

$$= \int_0^{\sqrt{3}} \sqrt{16x^2+1}\; x\; dx = \left[\frac{(16x^2+1)^{3/2}}{48} \right]_0^{\sqrt{3}} = \frac{343-1}{48} = \frac{\cancel{342}^{\cancel{171}\ 57}}{\cancel{48}_{\cancel{24}\ 8}}$$

$$u = 16x^2+1$$

$$du = 32x\, dx \rightarrow \int \sqrt{u}\, \frac{du}{32} = \frac{u^{3/2}}{3/2 \cdot 32} = \frac{u^{3/2}}{48}$$

$$y \le x \longrightarrow 0 \le y \le x$$

$$z \le 6 \rightarrow 2x^2 \le 6 \rightarrow x^2 \le 3$$

$$|x| \le \sqrt{3}$$

$$-\sqrt{3} \le x \le \sqrt{3}$$

$$0 \le x \le \sqrt{3}$$

$$x \ge 0$$

$$y \ge 0$$

$$z \ge 0$$

h) Trozo de plano tangente a $z = x + \ln(xy)$ en $(1,1,z_0)$ con $x^2+y^2 \le 9$.

$f(a,b)$ $f'_x(a,b)$ $f'_y(a,b)$

$S: \; z = x + \ln(xy) \rightarrow$ plano tangente: $z = z_0 + z'_{x\,(1,1)}(x-1) + z'_{y\,(1,1)}(y-1)$

$z_0 = z_{(1,1)} = 1 + 0 = 1$

$z'_{x\,(1,1)} = 1 + \left.\dfrac{y}{xy}\right|_{(1,1)} = 2$

$z'_{y\,(1,1)} = \left.\dfrac{x}{xy}\right|_{(1,1)} = 1$

$z = 1 + 2(x-1) + 1(y-1)$

$z_{(x,y)} = 2x + y - 2 \rightarrow P = N_0\, g$

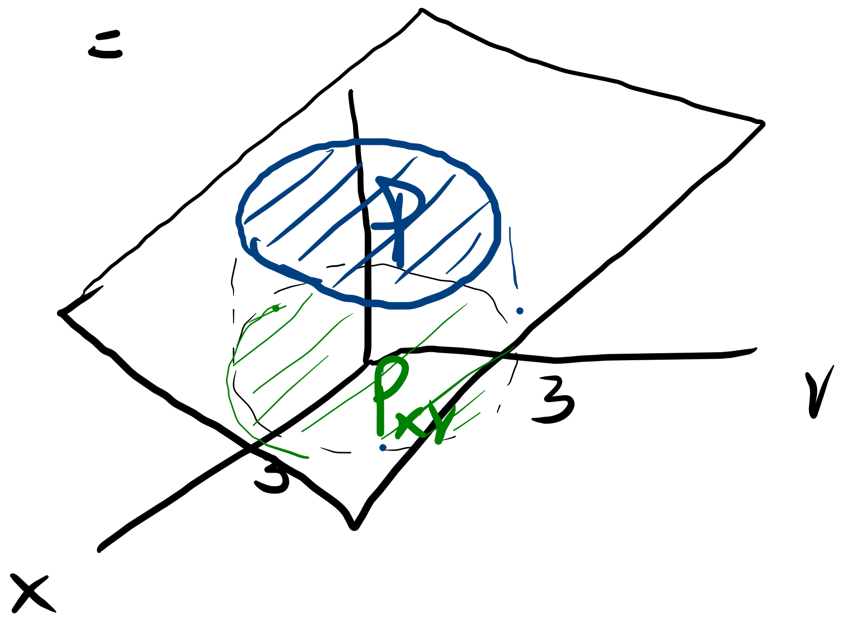
$\downarrow$ área (P)

$g(x,y,z) = z - 2x - y + 2$

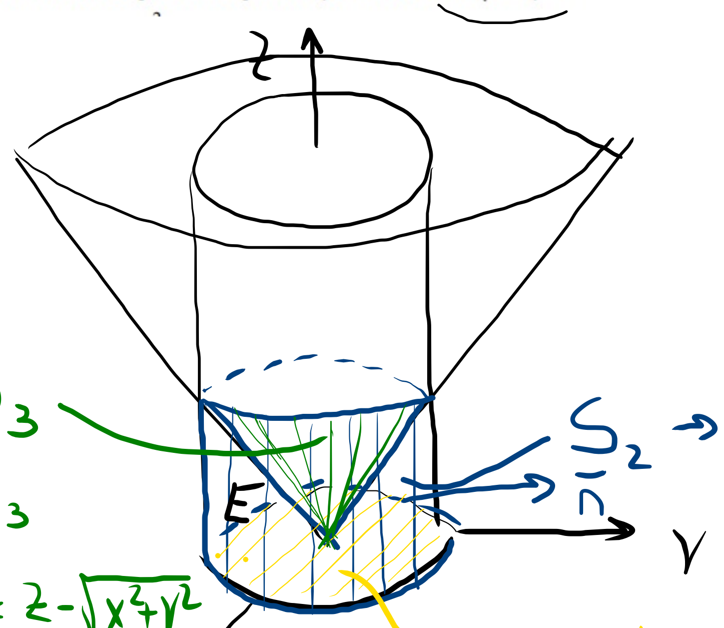
$\bar{\nabla} g = (-2, -1, 1)$, $1 = g'_z$

$$\text{área}(P) = \iint_{P_{xy}} \frac{\|\bar{\nabla} g\|}{|g'_z|}\, dx\, dy = \iint_{x^2+y^2 \le 9} \frac{\sqrt{4+1+1}}{|1|}\, dx\, dy =$$

$$= \sqrt{6}\ \text{área}(P_{xy}) = \boxed{\sqrt{6}\ \pi\, 9}$$



f) Superficie frontera del cuerpo definido por $\overbrace{x^2+y^2\le 1,\ 0\le z\le \sqrt{x^2+y^2}}^{E}$.



$$\text{área}(\partial E) = \text{área}(S_1 \cup S_2 \cup S_3)$$

∂E: borde de E

$S_3 = N_0\, g_3$

$g_3(x,y,z) = z - \sqrt{x^2+y^2}$

$$\nabla g_3 = \left(-\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}, -\frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}, 1\right)$$

$S_1 = N_0\, g_1$

$g_1(x,y,z) = z \rightarrow \nabla g_1 = (0,0,1)$

$x^2+y^2=1$

$$S_2 \rightarrow \bar r(\varphi,z) = (\overbrace{\cos(\varphi)}^{x}, \overbrace{\operatorname{sen}(\varphi)}^{y}, z)$$

$0 \le \varphi \le 2\pi$

$0 \le z \le 1$

$$\bar r'_\varphi \times \bar r'_z = \begin{vmatrix} i & j & k \\ -\operatorname{sen}(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos(\varphi), \operatorname{sen}(\varphi), 0) = \bar n$$

$$\|\bar r'_\varphi \times \bar r'_z\| = \sqrt{\cos^2(\varphi)+\operatorname{sen}^2(\varphi)} = 1$$

$$\text{área}(S_1) = \iint_{x^2+y^2\le 1} \frac{\|(0,0,1)\|}{|1|}\, dx\,dy = \text{área}(x^2+y^2\le 1) = \pi$$

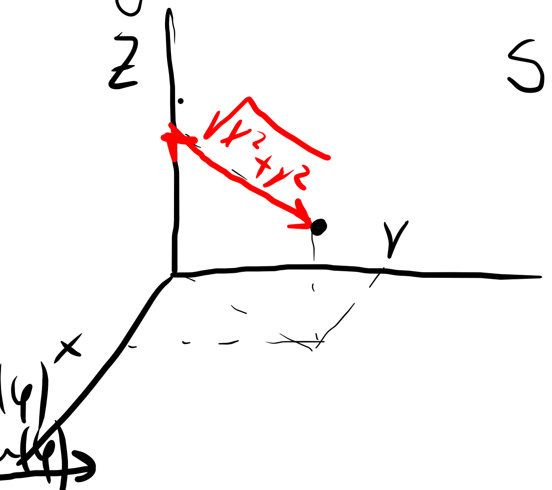
$$\text{área}(S_3) = \iint_{x^2+y^2\le 1} \frac{\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}}{|1|}\, dx\,dy = \iint_{x^2+y^2\le 1} \sqrt{\frac{2\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}}\, dx\,dy = \sqrt{2}\ \text{área}(x^2+y^2\le 1) = \sqrt{2}\,\pi$$

$$\text{área}(S_2) = \iint_{S_{\varphi z}} \|\bar r'_\varphi \times \bar r'_z\|\, d\varphi\, dz = \int_{\varphi=0}^{2\pi}\int_{z=0}^{1} 1\, dz\, d\varphi = 1\cdot 2\pi$$

$$\text{área}(S) = \pi + \sqrt{2}\,\pi + 2\pi = \boxed{(3+\sqrt{2})\,\pi}$$

06) Calcule el momento de inercia respecto del eje z de una chapa con forma de paraboloide S $z = x^2 + y^2$ con $x \geq 0 \wedge 1 \leq z \leq 4$, si la densidad superficial en cada punto de la chapa es $\delta(x,y,z) = \dfrac{k}{x^2+y^2}$ con k constante.

momento de segundo orden



$S = N_0\, g$

$g(x,y,z) = z - x^2 - y^2$

$\bar{\nabla} g = (-2x, -2y, 1)$

g'_z

$$I_z(S) = \iint_{S_{xy}} (x^2+y^2)\cdot \delta_{(x,y,z)} \cdot \frac{\|\bar{\nabla} g\|}{|g'_z|}\, dx\, dy =$$

$$= \iint_{S_{xy}} \cancel{(x^2+y^2)} \cdot \frac{k}{\cancel{x^2+y^2}} \frac{\sqrt{4x^2+4y^2+1}}{|1|}\, dx\, dy =$$

$x = \rho\cos(\varphi)$
$y = \rho\,\mathrm{sen}(\varphi)$

$$\left|\frac{\partial(x,y)}{\partial(\rho,\varphi)}\right| = \rho$$

$$= k \int_{\rho=1}^{2} \int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4\rho^2+1}\cdot \rho\, d\varphi\, d\rho =$$

$$= k\,\frac{\pi}{12}\left[(4\rho^2+1)^{3/2}\right]_1^2 = \boxed{k\,\frac{\pi}{12}\left(17^{3/2} - 5^{3/2}\right)}$$

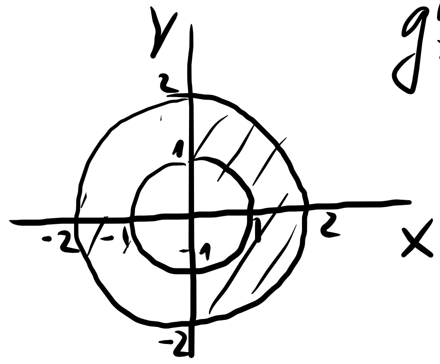
$x \geq 0$

$1 \leq z \leq 4$

$1 \leq \underbrace{x^2+y^2} \leq 4$

$1 \leq \rho^2 \leq 4$

$1 \leq \rho \leq 2$



$u = 4\rho^2+1$

$du = 8\rho\, d\rho$

$$\int \sqrt{u}\,\frac{du}{8} = \frac{u^{3/2}}{\frac{3}{2}\cdot 8} = \frac{u^{3/2}}{12}$$