01) Calcule el área de las siguientes regiones planas mediante integrales dobles; se recomienda no aplicar propiedades de simetría, plantee los límites para toda la región.

a) ~~$D=\{(x,y)\in\Re^2 \ / \ y\ge 2x^2+1 \ \wedge \ x+y\le 4\}$~~.

b) ~~D : definida por $x^2 \le y \le \sqrt{2-x^2}$~~.

c) D : dominio del campo $\bar{f}(x,y)=(\underbrace{\ln(x+y-2)}_{f_1},\underbrace{\sqrt{y-2x+2}}_{f_2},\underbrace{(2x+2-y-x^2)^{-1/4}}_{f_3})$. ←

D_{f_1} : $x+y-2>0 \rightarrow x+y>2$

D_{f_2} : $y-2x+2\geq 0 \rightarrow y\geq 2x-2$

D_{f_3} : $2x+2-y-x^2>0$

$\hookrightarrow \ y<-x^2+2x+2$

$D_{\bar f}=D_{f_1}\cap D_{f_2}\cap D_{f_3}$

$$\text{área}(D)=\iint_D dx\,dy=\int_{x=0}^{x=\frac{4}{3}}\int_{y=2-x}^{y=-x^2+2x+2} 1\,dy\,dx+\int_{x=\frac{4}{3}}^{x=2}\int_{y=2x-2}^{y=-x^2+2x+2} dy\,dx=$$

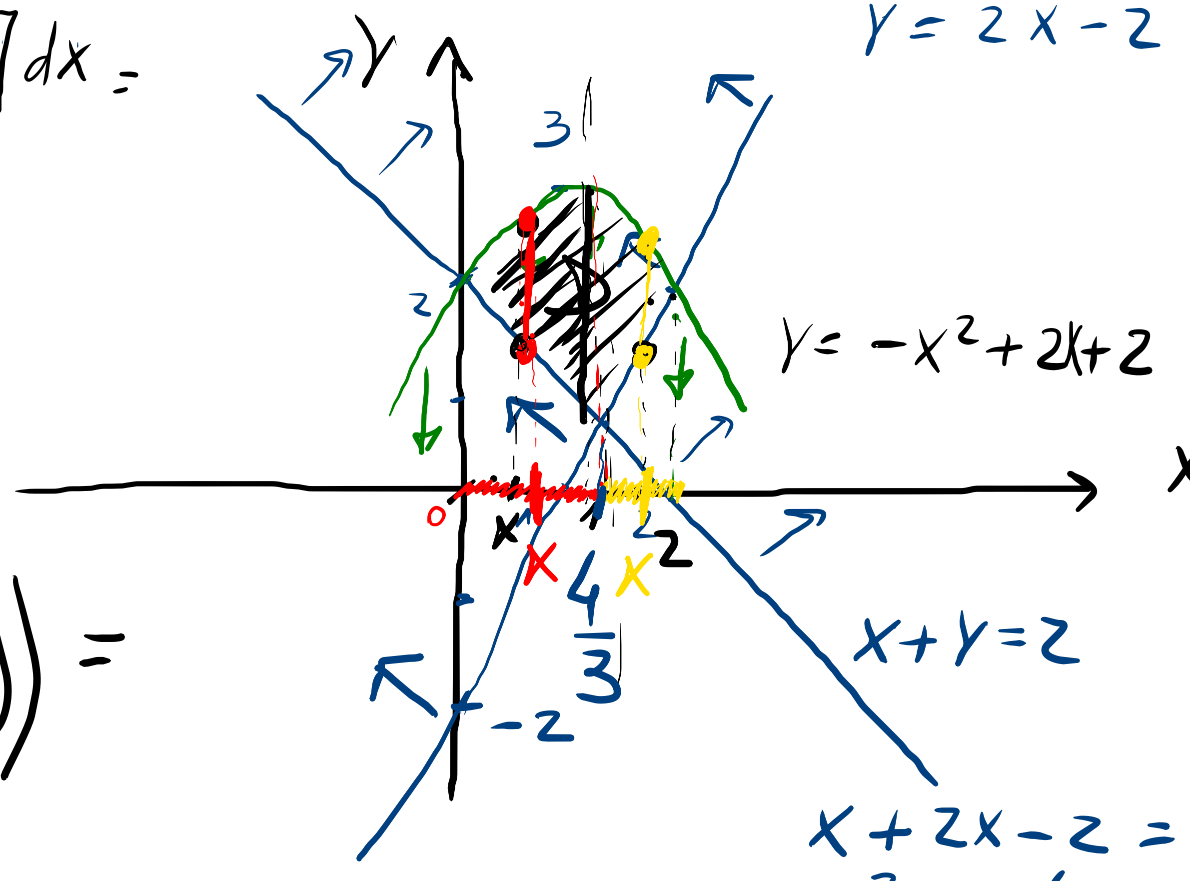
$$=\int_0^{4/3}\left[-x^2+2x+2-(2-x)\right]dx+\int_{4/3}^{2}\left[-x^2+2x+2-(2x-2)\right]dx=$$

$$=\int_0^{4/3}(-x^2+3x)\,dx+\int_{4/3}^{2}(-x^2+4)\,dx=$$

$$=\left[-\frac{x^3}{3}+\frac{3x^2}{2}\right]_0^{\frac{4}{3}}+\left[-\frac{x^3}{3}+4x\right]_{4/3}^{2}=$$

$$=-\cancel{\frac{64}{81}}+\cancel{\frac{8}{3}}+\left(-\cancel{\frac{8}{3}}+8-\left(-\cancel{\frac{64}{81}}+\frac{16}{3}\right)\right)=$$

$$=8-\frac{16}{3}=\frac{24-16}{3}=\boxed{\frac{8}{3}}$$



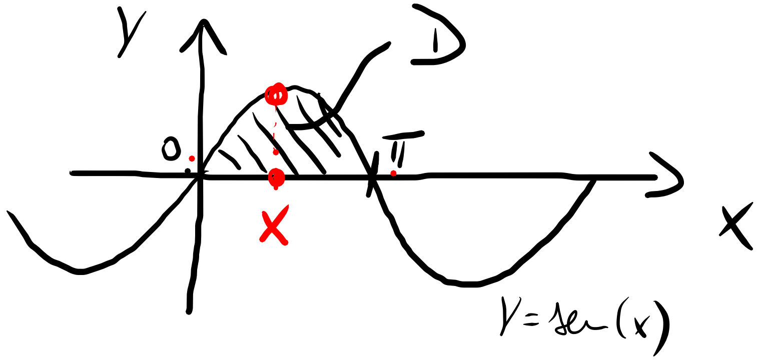
$x+2x-2=2$

$3x=4$

$x=\frac{4}{3}$

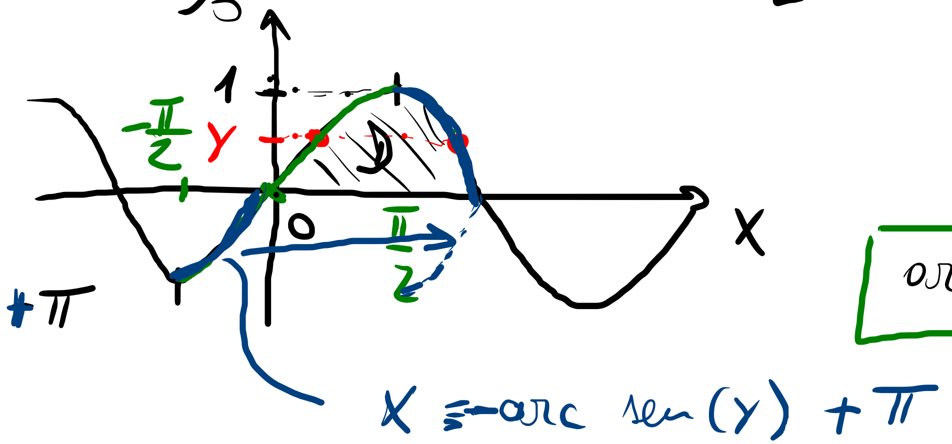
02) Calcule las siguientes integrales en ambos órdenes de integración y verifique que los resultados coinciden.

a) $\iint_D dx\,dy$, D definido por: $0 \le y \le \text{sen}(x)$, $0 \le x \le \pi$.



$$\iint_D dx\,dy = \int_{x=0}^{x=\pi}\int_{y=0}^{y=\text{sen}(x)} 1\,dy\,dx =$$

$$= \int_0^{\pi}\left[y\right]_0^{\text{sen}(x)} dx = \int_0^{\pi} \text{sen}(x)\,dx = \left[-\cos(x)\right]_0^{\pi} = 1-(-1) = \boxed{2}$$



$y = \text{sen}(x)$

$\boxed{\text{arc sen}(y) = x}$

$$\iint_D dx\,dy = \int_{y=0}^{y=1}\int_{x=\text{arc sen}(y)}^{x=-\text{arc sen}(y)+\pi} dx\,dy =$$

$$= \int_0^1 \left(-2\,\text{arc sen}(y) + \pi\right) dy = \left[-2\left[y\,\text{asen}(y) + \sqrt{1-y^2}\right] + \pi\right]_0^1 = -\cancel{2}\cdot\frac{\pi}{\cancel{2}} + \cancel{\pi} - (-2)$$

$$= \boxed{2}$$

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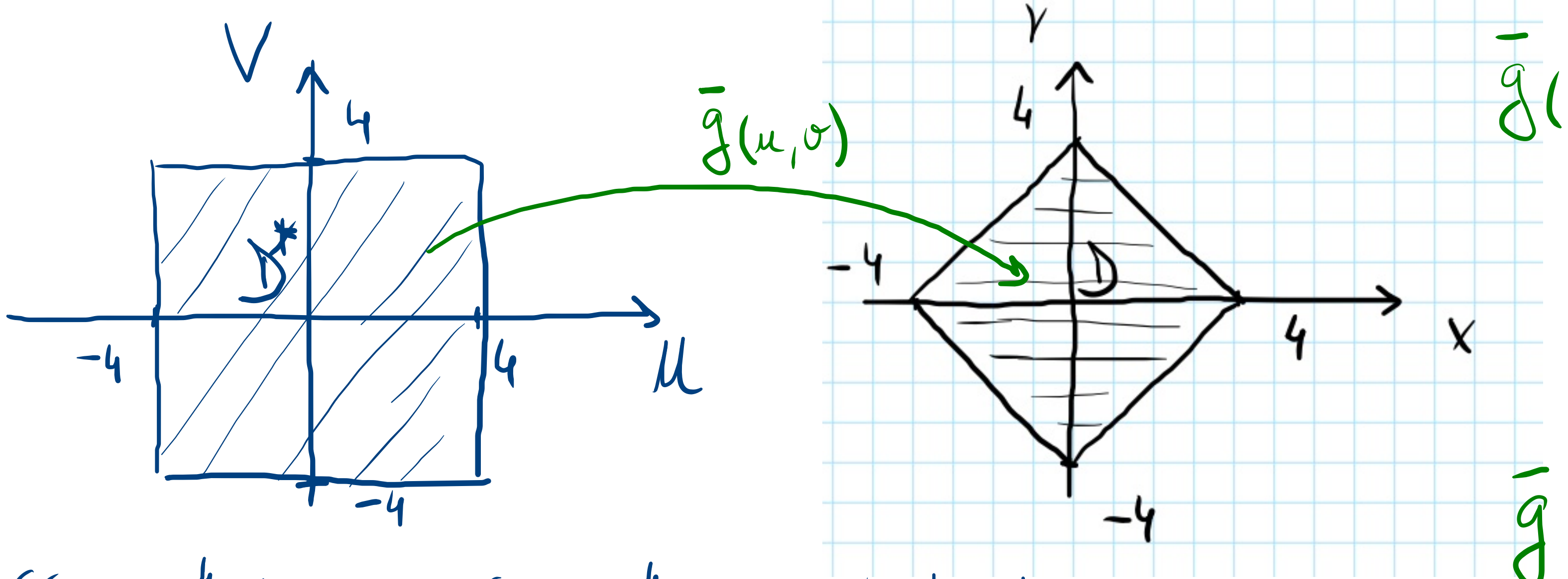
$$\int \text{sen}^{-1}\left(\frac{x}{a}\right) dx = x\,\text{sen}^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2}$$

c) $\iint_D (x-y)^4\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2 \,/\, |x|+|y|\le 4\}$, aplicando una transformación lineal apropiada.

$$\left.\begin{array}{lll} x\ge 0\,, & y\ge 0 & x+y\le 4 \\ x<0\,, & y<0 & -x-y\le 4 \to x+y\ge -4 \end{array}\right\} \quad -4\le \underbrace{x+y}_{u}\le 4 \qquad -4\le u\le 4$$

$$\left.\begin{array}{lll} x\ge 0\,, & y<0 & x-y\le 4 \\ x<0\,, & y\ge 0 & -x+y\le 4 \to x-y\ge -4 \end{array}\right\} \quad -4\le \underbrace{x-y}_{v}\le 4 \qquad -4\le v\le 4$$



$$\bar{g}(u,v) = \left(X_{(u,v)}\,,\,Y_{(u,v)}\right)$$

$$\begin{cases} u = x+y \\ v = x-y \end{cases}$$

$$\bar{g}\begin{cases} \dfrac{u+v}{2} = x \\ \dfrac{u-v}{2} = y \end{cases} \to \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{4}-\frac{1}{4} = -\frac{2}{4}$$

$$= -\frac{1}{2}$$

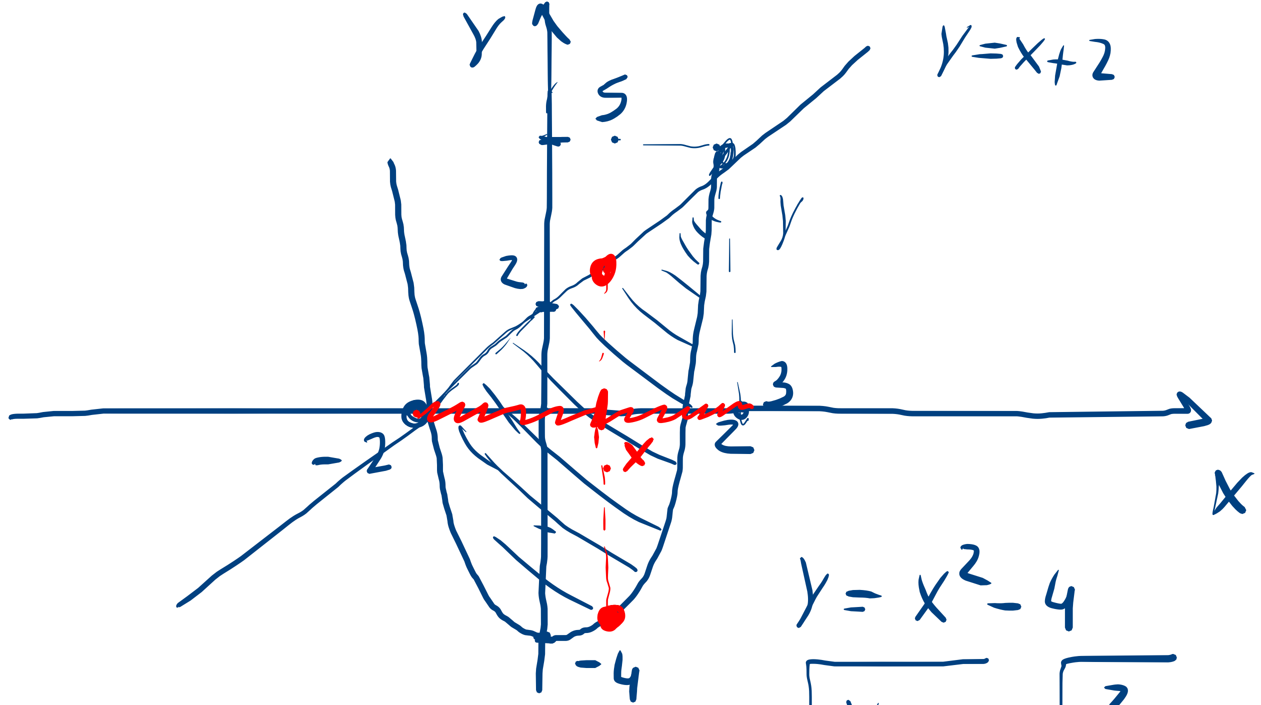
$$\iint_D (x-y)^4\,dx\,dy = \iint_{D^*} v^4 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv =$$

$$= \int_{u=-4}^{4}\int_{v=-4}^{4} v^4\cdot\left|-\frac{1}{2}\right| dv\,du = \frac{1}{2}\int_{-4}^{4}\left[\frac{v^5}{5}\right]_{-4}^{4} du = \frac{1}{2}\,\frac{(1024-(-1024))}{5}\,\left[u\right]_{-4}^{4} =$$

$$= \frac{1}{\cancel{2}}\cdot\frac{2048}{5}\,\overset{4}{\cancel{8}} = \boxed{\frac{8192}{5}} = \boxed{1638{,}4}$$

c) $\int_{-4}^{0} dy \int_{-\sqrt{y+4}}^{\sqrt{y+4}} dx + \int_{0}^{5} dy \int_{y-2}^{\sqrt{y+4}} dx .$

$$y-2 \le x \le \sqrt{y+4}$$
$$0 \le y \le 5$$



$$-\sqrt{y+4} \le x \le \sqrt{y+4}$$
$$-4 \le y \le 0$$

$$y = x^2-4$$
$$\sqrt{y+4} = \sqrt{x^2}$$
$$\sqrt{y+4} = |x|$$

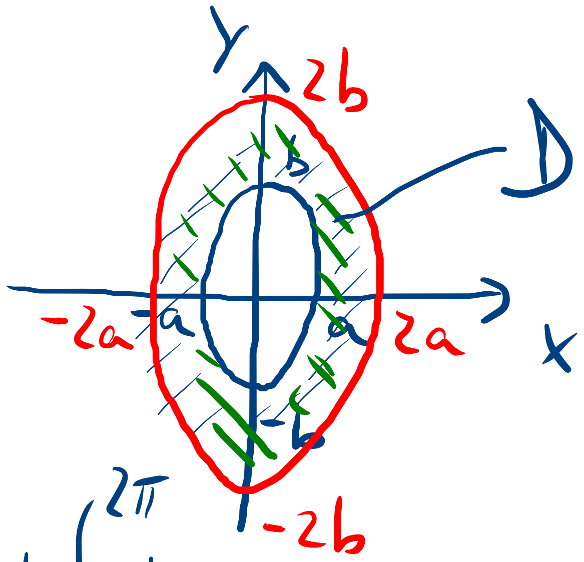
$$\int_{x=-2}^{x=3} \int_{y=x^2-4}^{y=x+2} dy\,dx = \int_{-2}^{3} (x+2-x^2+4)\,dx =$$

$$= \left[-\frac{x^3}{3}+\frac{x^2}{2}+6x\right]_{-2}^{3} = -9+\frac{9}{2}+18-\left(\frac{8}{3}+2-12\right) =$$

$$= 9+\frac{9}{2}-\frac{8}{3}+10 = \frac{114+27-16}{6} = \boxed{\frac{125}{6}}$$

b) Calcule el área de la región plana definida por $1 \le \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 4$, $a,b \in \mathbb{R}^+$ aplicando la transformación $(x,y) = (a\rho\cos(\varphi), b\rho\operatorname{sen}(\varphi))$.

$$1 \le \frac{x^2}{a^2}+\frac{y^2}{b^2} \quad \wedge \quad \frac{x^2}{4a^2}+\frac{y^2}{4b^2} \le \cancel{4}\ 1$$



$$\iint_D dx\,dy = \xrightarrow[\quad]{\substack{x = a\rho\cos(\varphi)\\ y = b\rho\operatorname{sen}(\varphi)}} \int_{\rho=1}^{2}\int_{\varphi=0}^{\varphi=2\pi} ab\rho\, d\varphi\, d\rho = ab\int_1^2 \rho\, d\rho \int_0^{2\pi} d\varphi =$$

$$\left|\frac{\partial(x,y)}{\partial(\rho,\varphi)}\right| = ab\rho$$

$$= ab\left[\frac{\rho^2}{2}\right]_1^2 2\pi = ab\,2\pi\left(2-\frac{1}{2}\right) = \boxed{3\pi ab}$$

$$\frac{\partial(x,y)}{\partial(\rho,\varphi)} = \begin{vmatrix} a\cos(\varphi) & -a\rho\operatorname{sen}(\varphi) \\ b\operatorname{sen}(\varphi) & b\rho\cos(\varphi) \end{vmatrix} = ab\rho\cos^2(\varphi) + ab\rho\operatorname{sen}^2(\varphi)$$

$$= ab\rho > 0$$

$$\frac{x^2}{a^2} = \rho^2\cos^2(\varphi)$$

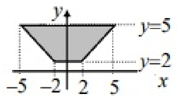
$$+ \quad \frac{y^2}{b^2} = \rho^2\operatorname{sen}^2(\varphi)$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \rho^2$$

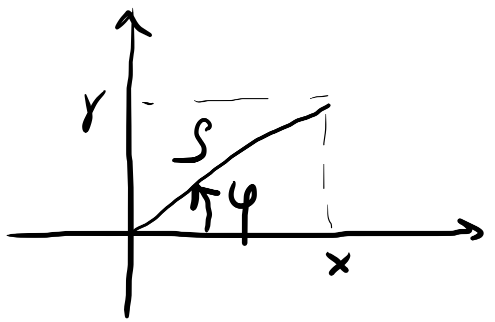
$$1 \le \rho^2 \le 4$$

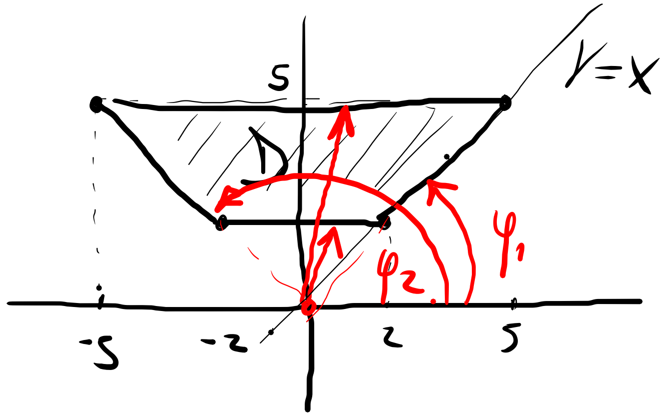
$$1 \le \rho \le 2$$

e) Siendo D la región sombreada del dibujo, **calcule** $\iint_D y\,(x^2+y^2)^{-1}dxdy$ usando coordenadas polares.



$$\begin{cases} x = \rho\cos(\varphi) \\ y = \rho\,\mathrm{sen}(\varphi) \end{cases}$$





$$\iint_D \frac{y}{x^2+y^2}\,dx\,dy = \int_{\varphi=\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_{\rho=\frac{2}{\mathrm{sen}(\varphi)}}^{5/\mathrm{sen}(\varphi)} \frac{\cancel{\rho}\,\mathrm{sen}(\varphi)}{\cancel{\rho^2}}\,\cancel{\rho}\,d\rho\,d\varphi =$$

$$= \int_{\frac{\pi}{4}}^{3/4\,\pi} \cancel{\mathrm{sen}(\varphi)}\,\frac{3}{\cancel{\mathrm{sen}(\varphi)}}\,d\varphi = 3\left(\frac{3}{4}\pi - \frac{\pi}{4}\right) = \boxed{\frac{3}{2}\pi}$$

$$\frac{\pi}{4} \le \varphi \le \frac{3\pi}{4}$$

$$2 \le y \le 5$$

$$2 \le \rho\,\mathrm{sen}(\varphi) \le 5$$

$$\frac{2}{\mathrm{sen}(\varphi)} \le \rho \le \frac{5}{\mathrm{sen}(\varphi)}$$

$$\mathrm{sen}(\varphi) > 0$$

$$D = \{(x,y)\in\mathbb{R}^2 \,/\, \underbrace{y \ge x \;\wedge\; \underbrace{y \ge -x}_{-y \le x}}_{-y \le x \le y} \;\wedge\; 2 \le x \le 5\}$$

$x = \rho\cos(\varphi)$

$y = \rho\,\mathrm{sen}(\varphi)$

$$-\frac{\cancel{\rho}\,\mathrm{sen}(\varphi)}{\mathrm{sen}(\varphi)} \le \frac{\cancel{\rho}\cos(\varphi)}{\mathrm{sen}(\varphi)} \le \frac{\cancel{\rho}\,\mathrm{sen}(\varphi)}{\mathrm{sen}(\varphi)}$$

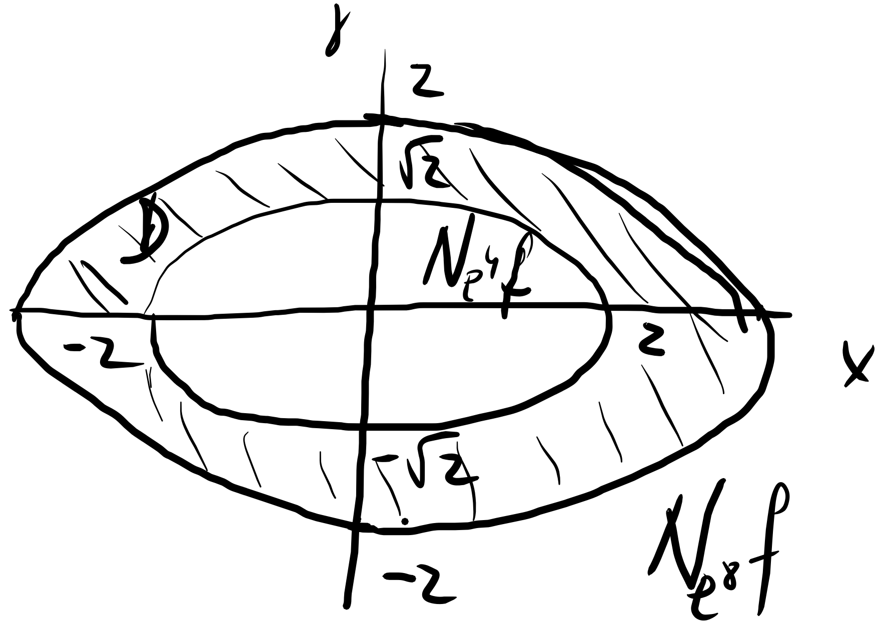
$$-1 \le \cot(\varphi) \le 1$$

$$\frac{\pi}{4} \le \varphi \le \frac{3\pi}{4}$$

07) a) Dada $f(x,y)=e^{x^2+2y^2}$, calcule el área de la región plana limitada por las curvas de nivel e^4 y e^8 de la función.

$$N_{e^4} f : \quad f(x,y) = e^4 \rightarrow e^{x^2+2y^2} = e^4 \rightarrow x^2+2y^2 = 4 \rightarrow \boxed{\frac{x^2}{4}+\frac{y^2}{2}=1}$$

$$N_{e^8} f : \quad f(x,y) = e^8 \rightarrow e^{x^2+2y^2} = e^8 \rightarrow x^2+2y^2 = 8 \rightarrow \boxed{\frac{x^2}{8}+\frac{y^2}{4}=1}$$



$$\text{área}(D) = \iint_D dx\,dy \xrightarrow[\left|\frac{\partial(x,y)}{\partial(\rho,\varphi)}\right| = 2\sqrt{2}\rho]{\substack{x = 2\rho\cos(\varphi) \\ y=\sqrt{2}\rho\,\text{sen}(\varphi)}} \int_{\rho=1}^{\rho=\sqrt{2}}\int_0^{2\pi} 2\sqrt{2}\,\rho\, d\varphi\, d\rho =$$

$$1 \leq \underbrace{\frac{x^2}{4}+\frac{y^2}{2}}_{\rho^2} \leq 2 \rightarrow 1 \leq \rho \leq \sqrt{2}$$

$$= 2\sqrt{2}\int_1^{\sqrt{2}} 2\pi\rho\, d\rho = 4\sqrt{2}\pi\left[\frac{\rho^2}{2}\right]_1^{\sqrt{2}} =$$

$$= 4\sqrt{2}\pi\left(1-\frac{1}{2}\right) = \boxed{2\sqrt{2}\pi}$$

09) Calcule $\iint_D \frac{x+4y}{x^2}\,dxdy$ con $D:\ x \geq y\,,\ x+4y \leq 4\,,\ y \geq 0$ usando coordenadas polares.

$\rho > 0$

$x \geq y \quad , \quad x + 4y \leq 4 \quad , \quad y \geq 0$

$\cancel{\rho}\cos(\varphi) \geq \cancel{\rho}\,\mathrm{sen}(\varphi) \quad , \quad \rho\cos(\varphi) + 4\rho\,\mathrm{sen}(\varphi) \leq 4 \quad\quad \rho\,\mathrm{sen}(\varphi) \geq 0$

$\dfrac{\cos(\varphi)}{\mathrm{sen}(\varphi)} \geq \dfrac{\mathrm{sen}(\varphi)}{\mathrm{sen}(\varphi)} \quad\quad 0 \leq \rho \leq \dfrac{4}{\cos(\varphi)+4\,\mathrm{sen}(\varphi)} \quad\quad \mathrm{sen}(\varphi) \geq 0$

$\cot(\varphi) \geq 1 \quad\quad 0 \leq \varphi \leq \pi$

$\tan(\varphi) \leq 1$

$\varphi \leq \frac{\pi}{4}$

$0 < \varphi \leq \frac{\pi}{4}$

$$\iint_D \frac{x+4y}{x^2}\,dx\,dy = \int_0^{\frac{\pi}{4}} \int_0^{\frac{4}{\cos\varphi + 4\,\mathrm{sen}(\varphi)}} \frac{\cancel{\rho}\cos(\varphi) + 4\cancel{\rho}\,\mathrm{sen}(\varphi)}{\cancel{\rho^2}\cos^2(\varphi)} \cdot \cancel{\rho}\,d\rho\,d\varphi =$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cancel{\cos(\varphi) + 4\,\mathrm{sen}(\varphi)}}{\cancel{\cos^2(\varphi)}} \; \frac{4\,\sec^2(\varphi)}{\cancel{\cos(\varphi) + 4\,\mathrm{sen}(\varphi)}}\,d\varphi = 4\,\Big[\tan(\varphi)\Big]_0^{\pi/4} =$$

$$= 4\,(1-0) = \boxed{4}$$

10) En los siguientes casos se indica una integral planteada en coordenadas polares, grafique la región correspondiente en el plano xy, plantee la integral en coordenadas cartesianas y resuélvala en alguno de los dos sistemas de coordenadas.

a) $\int_0^{\pi/2} d\varphi \int_0^{2\cos(\varphi)} \rho^3\, d\rho$.

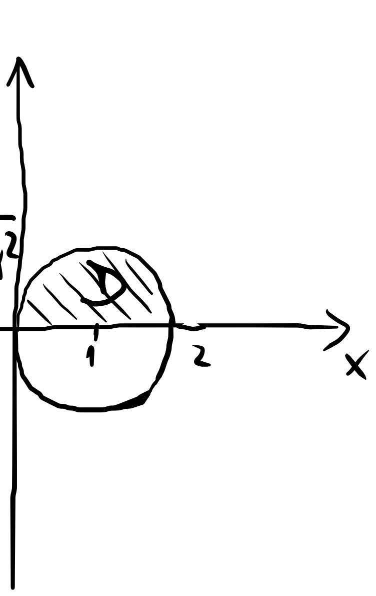
$$0 \le \varphi \le \frac{\pi}{2} \to 0 \le \arctan\left(\frac{y}{x}\right) \le \frac{\pi}{2} \to 0 \le \tan\left(\frac{y}{x}\right)$$

$$0 \le \underbrace{\rho^2}_{x^2+y^2} \le \underbrace{2\rho\cos\varphi}_{x} \to 0 \le \boxed{x^2+y^2 \le 2x} \to y^2 \le 2x-x^2 \to |y| \le \sqrt{2x-x^2}$$

$$y \le \sqrt{2x-y^2}$$

$$x^2-2x+1+y^2 \le 0+1$$

$$(x-1)^2+y^2 \le 1$$



$$\tan(\varphi) = \frac{y}{x} = \frac{\cancel{\rho}\,\text{sen}(\varphi)}{\cancel{\rho}\cos(\varphi)}$$

$$\varphi = \arctan\left(\frac{y}{x}\right)$$

$$\int_0^{\pi/2} d\varphi \int_0^{2\cos(\varphi)} \underbrace{\rho^3}_{\rho^2\cdot\rho} d\rho = \boxed{\int_{x=0}^{x=2}\int_{y=0}^{y=\sqrt{2x-x^2}} x^2+y^2\, dy\, dx}$$

$$\left|\frac{\partial(x,y)}{\partial(\rho,\varphi)}\right|$$

$$\int_0^{\pi/2} d\varphi \int_0^{2\cos(\varphi)} \rho^3\, d\rho = \int_0^{\pi/2} \left[\frac{\rho^4}{4}\right]_0^{2\cos(\varphi)} d\varphi = \int_0^{\pi/2} 4\cos^4(\varphi)\, d\varphi =$$

$$= 4\left[\frac{3}{8}\varphi + \frac{\text{sen}(2\varphi)}{4} + \frac{\text{sen}(4\varphi)}{32}\right]_0^{\pi/2} = 4\,\frac{3\pi}{16} = \boxed{\frac{3\pi}{4}}$$

TABLA 381