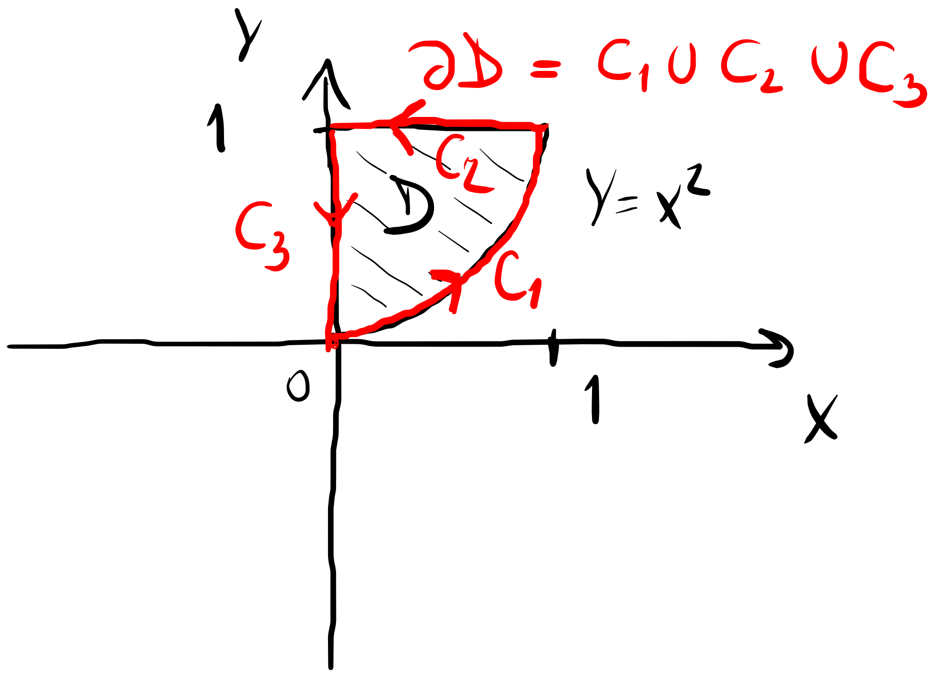
11) Calcule la circulación de $\bar{f}(x,y) = (y, -x)$ a lo largo de la frontera de la región definida por $x^2 \le y \le 1 \wedge 0 \le x \le 1$, en sentido positivo. Observe que en este ejemplo la circulación no resulta nula, aún con camino cerrado.

$$D = \{(x,y) \in \mathbb{R}^2 / 0 \le x \le 1,\ x^2 \le y \le 1\}$$



$C_1: \quad \bar{\lambda}_1(t) = (t, t^2) \qquad t \in [0,1]$

$C_2: \quad \bar{\lambda}_2(t) = (1-t, 1) \qquad t \in [0,1]$

$C_3: \quad \bar{\lambda}_3(t) = (0, 1-t) \qquad t \in [0,1]$

$$\int_{C_1} \bar{f}\, d\bar{\lambda}_1 = \int_{t=0}^{1} \bar{f}(\bar{\lambda}_1) \cdot \underbrace{\bar{\lambda}_1'\, dt}_{d\bar{\lambda}} = \int_{t=0}^{1} \bar{f}(t, t^2) \cdot (1, 2t)\, dt =$$

$$= \int_0^1 (t^2, -t) \cdot (1, 2t)\, dt = \int_0^1 (\underbrace{t^2 - 2t^2}_{-t^2})\, dt = \left[-\frac{t^3}{3}\right]_0^1 = -\frac{1}{3}$$

$$\int_{C_2} \bar{f}\, d\bar{\lambda}_2 = \int_0^1 (1, t-1) \cdot (-1, 0)\, dt = \int_0^1 (-1)\, dt = \left[-t\right]_0^1 = -1$$

$$\int_{C_3} \bar{f}\, d\bar{\lambda}_3 = \int_0^1 (1-t, 0) \cdot (0, -1)\, dt = \int_0^1 0\, dt = 0$$

$$\int_{\partial D^+} \bar{f}\, d\bar{\lambda} = \int_{C_1} \bar{f}\, d\bar{\lambda}_1 + \int_{C_2} \bar{f}\, d\bar{\lambda}_2 + \int_{C_3} \bar{f}\, d\bar{\lambda}_3 = -\frac{1}{3} - 1 + 0 = \boxed{-\frac{4}{3}}$$

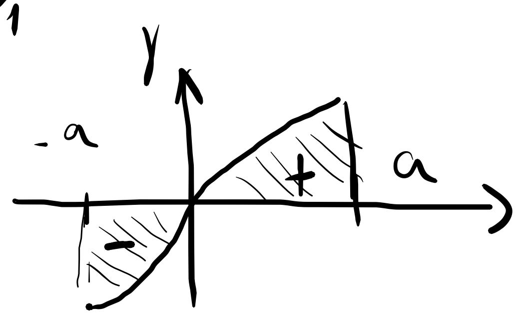
12) Calcule la circulación de $\bar{f}(x,y,z) = (x-y,\, x,\, yz)$ a lo largo de la curva intersección de $z = x - y^2$ con $y = x^2$ desde $(1,1,0)$ hasta $(-1,1,-2)$.

$$C \begin{cases} z = x - y^2 \\ y = x^2 \end{cases} \qquad \bar{\lambda}_{(t)} = \left(t,\, t^2,\, t - t^4\right)$$

$$\bar{\lambda} \begin{cases} x = t \\ y = t^2 \\ z = t - t^4 \end{cases} \qquad \underset{\bar{\lambda}_{(1)}}{(1,1,0)} \longrightarrow \underset{\bar{\lambda}_{(-1)}}{(-1,1,-2)} \qquad t = 1 \longrightarrow t = -1$$

$$\int_C \bar{f}\, d\bar{\lambda} = \int_{t=1}^{-1} \bar{f}(t, t^2, t - t^4) \cdot (1, 2t, 1 - 4t^3)\, dt = \int_1^{-1} \Big(t - t^2,\, t,\, \overbrace{t^2(t - t^4)}^{t^3 - t^6}\Big) \cdot (1, 2t, 1 - 4t^3)\, dt$$

$$= \int_1^{-1} \left(\cancel{t} - t^2 + 2t^2 + \cancel{t^3} - t^6 - 4t^6 + \cancel{4t^9}\right) dt = \left[-\frac{5t^7}{7} + \frac{t^3}{3} \right]_1^{-1} = \frac{5}{7} - \frac{1}{3} + \frac{5}{7} - \frac{1}{3} =$$



$$= \frac{30 - 14}{21} = \boxed{\frac{16}{21}}$$

13) Calcule el trabajo de $\bar{f}(x,y,z) = 3x\,\bar{\imath} - xz\,\bar{\jmath} + yz\,\bar{k}$ a lo largo de la curva de ecuación $\bar{X} = (t-1, t^2, 2t)$ con $t \in [1,3]$. ¿Cuáles son los puntos inicial y final del recorrido? ¿puede asegurar el mismo resultado si manteniendo dichos puntos se utiliza otra curva?.

$$\bar{f}(x,y,z) = (3x\,,\,-xz\,,\,yz)$$

$$\bar{\lambda}(t) = (t-1,\, t^2,\, 2t) \qquad t \in [1,3] \qquad \bar{\lambda}(1) = (0,1,2) \longrightarrow \bar{\lambda}(3) = (2,9,6)$$

$$\int_C \bar{f}\, d\bar{\lambda} = \int_1^3 \bar{f}(t-1, t^2, 2t) \cdot (1, 2t, 2)\, dt = \int_1^3 (3t-3\,,\, \underbrace{(1-t)\,2t}_{2t-2t^2}\,,\, 2t^3) \cdot (1, 2t, 2)\, dt =$$

$$= \int_1^3 (3t - 3 + 4t^2 - \cancel{4t^3} + \cancel{4t^3})\, dt = \left[\frac{3t^2}{2} - 3t + \frac{4t^3}{3}\right]_1^3 = \frac{27}{2} - 9 + 36 - \frac{3}{2} + 3 - \frac{4}{3} =$$

$$= 12 + 30 - \frac{4}{3} = \frac{126 - 4}{3} = \boxed{\frac{122}{3}}$$

21) Dada $\bar{f}(x,y)=(a\,x,\,y-a\,x)$ y la curva C de ecuación $\bar{X}=(\cos(t),b\,\text{sen}(t))\wedge t\in[0,2\pi]$, determine a y b de manera que $a+b=6$ y la circulación de $\bar{f}$ a lo largo de C sea mínima.

$$\int_C \bar{f}\,d\bar{\lambda} = \int_0^{2\pi} \bar{f}(\overset{x}{\cos(t)}, \overset{y}{b\,\text{sen}(t)}) \cdot (-\text{sen}(t),\, b\cdot\cos(t))\,dt =$$

$$= \int_0^{2\pi} (a\cdot\cos(t),\, b\,\text{sen}(t) - a\cos(t)) \cdot (-\text{sen}(t),\, b\cos(t))\,dt =$$

$$= \int_0^{2\pi} (\underbrace{-a\,\text{sen}(t)\cos(t) + b^2\,\text{sen}(t)\cos(t)}_{(-a+b^2)\,\text{sen}(t)\cos(t)} - ab\cos^2(t))\,dt =$$

$$= \cancel{(-a+b^2)\left[\frac{\text{sen}^2(t)}{2}\right]_0^{2\pi}} - ab\left[\frac{t}{2}+\frac{\text{sen}(2t)}{4}\right]_0^{2\pi} = -ab\pi = g(a,b)$$

$$a+b=6 \to b=6-a \to g(a,6-a) = \boxed{h(a) = -a(6-a)\pi} = a^2-6a$$

$$b = 6-3 = 3$$

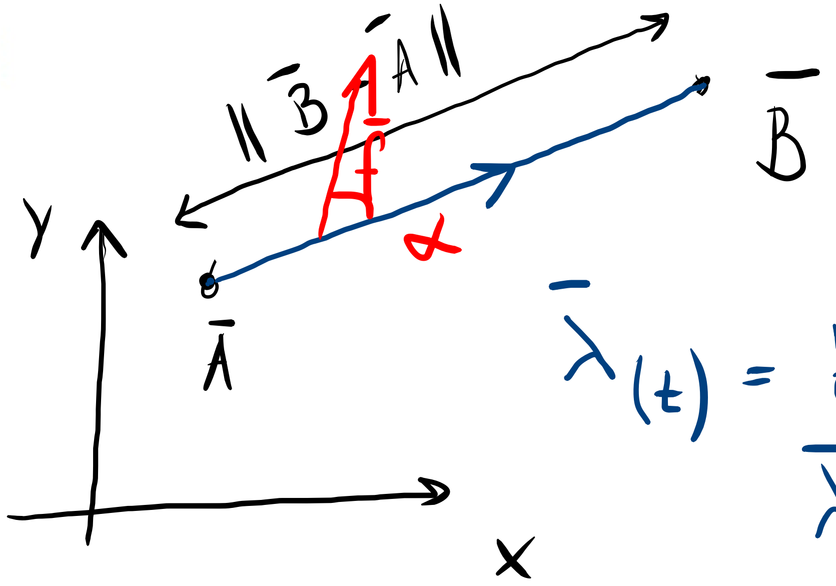
$$h'(a) = 2a-6 = 0 \to a=3 \to b=$$

$$h''(a=3) = 2 > 0 \Rightarrow h \text{ tiene mín. local en } a=3$$

$$\boxed{a=3 \text{ y } b=3}$$

09) Sea $\overline{F}$ constante en todo punto del segmento $\overline{AB}$, verifique que $\int_{\overline{AB}} \overline{F}\cdot d\overline{s} = \overline{F}\cdot(\overline{B}-\overline{A})$. Relacione este resultado con la conocida expresión del trabajo de una fuerza contante a lo largo de un camino recto.

$$\bar f(x,y) = (f_1, f_2)$$



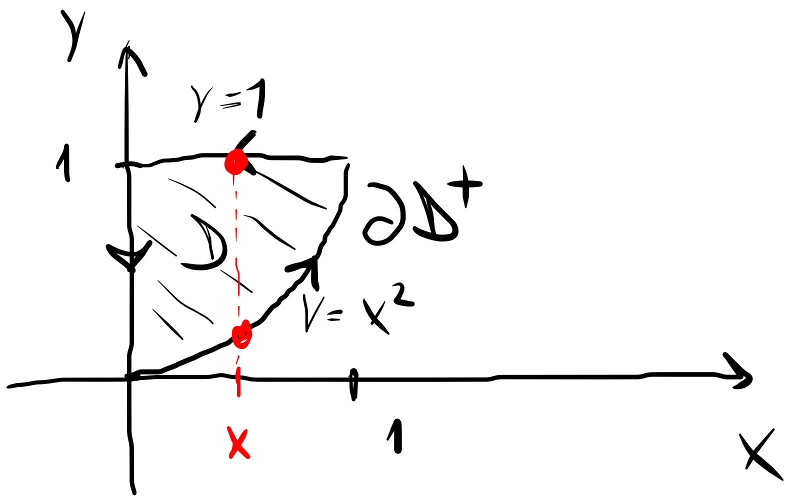
$$\bar\lambda(t) = t\cdot\bar B + (1-t)\cdot\bar A \qquad t\in[0,1]$$

$$\bar\lambda(0) = \bar A \qquad \bar\lambda(1) = \bar B$$

$$\int_C \bar f\, d\bar\lambda = \int_{t=0}^{1} (\overbrace{f_1, f_2}^{\bar f})\cdot(\bar B - \bar A)\, dt = \bar f\cdot(\bar B - \bar A)\underbrace{\int_0^1 dt}_{=1} = \boxed{\bar f\cdot(\bar B - \bar A)}$$

$$\bar f\cdot(\bar B - \bar A) = \|\bar f\|\cdot\|\bar B - \bar A\|\cdot\cos(\alpha) =$$ TRABAJO DE $\bar f$ DESDE $\bar A$ HASTA $\bar B$

11) Calcule la circulación de $\bar{f}(x,y) = (y,-x)$ a lo largo de la frontera de la región definida por $x^2 \le y \le 1 \wedge 0 \le x \le 1$, en sentido positivo. Observe que en este ejemplo la circulación no resulta nula, aún con camino cerrado.



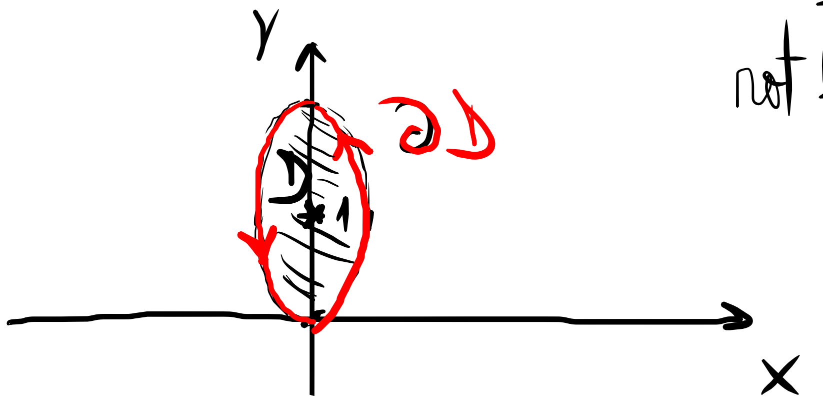
$$\bar{f} = (f_1, f_2) = (y, -x)$$

$$\operatorname{rot} \bar{f} = f'_{2\,x} - f'_{1\,y} = -1 - 1 = -2$$

$$\int_{\partial D^+} \bar{f}\, d\bar{\lambda} = \iint_D \operatorname{rot} \bar{f}\, dx\, dy = \int_{x=0}^{1} \int_{y=x^2}^{1} (-2)\, dy\, dx = -2 \int_0^1 (1 - x^2)\, dx =$$

$$= -2 \left[x - \frac{x^3}{3} \right]_0^1 = -2 \left(1 - \frac{1}{3} \right) = \boxed{-\frac{4}{3}}$$

03) Calcule la circulación de $\bar f(x,y) = (\overbrace{x^2+y^2}^{f_1}, \overbrace{3xy+\ln(y^2+1)}^{f_2})$ a lo largo de la frontera de la región definida por $4x^2+(y-1)^2 \le 1$ recorrida en sentido positivo.



$$\operatorname{rot}\bar f = 3y - 2y = y$$

$$\frac{x^2}{\frac{1}{4}} + (y-1)^2 \le 1$$

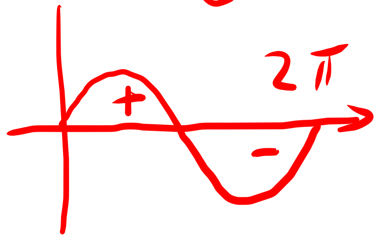
$$\oint_{\partial D^+} \bar f\, d\bar\lambda = \iint_D \overbrace{\operatorname{rot}\bar f}^{y}\, dx\,dy =$$

$$x = \frac{1}{2}\rho\cos(\varphi)$$
$$y = 1\cdot\rho\,\operatorname{sen}(\varphi) + 1$$

$$\left|\frac{\partial(x,y)}{\partial(\rho,\varphi)}\right| = \rho\cdot\frac{1}{2}\cdot 1$$

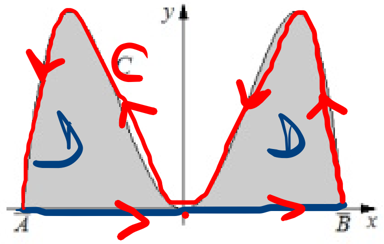
$$\int_{\rho=0}^{1}\int_{\varphi=0}^{2\pi} (\rho\,\operatorname{sen}(\varphi)+1)\,\frac{\rho}{2}\,d\varphi\,d\rho =$$

$$= \int_0^1\int_0^{2\pi} \frac{\rho^2}{2}\cdot\operatorname{sen}(\varphi)\,d\varphi\,d\rho + \int_0^1\int_0^{2\pi}\frac{\rho}{2}\,d\varphi\,d\rho = \frac{1}{2}\int_0^1 \rho^2\,d\rho\,\underbrace{\int_0^{2\pi}\operatorname{sen}(\varphi)\,d\varphi}_{=0} + \frac{\cancel{2}\pi}{\cancel{2}}\left[\frac{\rho^2}{2}\right]_0^1 =$$



$$= \boxed{\frac{\pi}{2}}$$

05) La región plana D sombreada en la figura tiene como frontera el segmento $\overline{AB}$ y el arco de curva C de ecuación $y=x^2-x^4$. Dado $\bar{f}=(P,Q)\in C^1$ con matriz jacobiana $D\bar{f}(x,y)=\begin{pmatrix} P'_x(x,y) & 3x-1 \\ 3x+2 & Q'_y(x,y)\end{pmatrix}$, calcule la circulación de $\bar{f}$ desde $\bar{A}$ hasta $\bar{B}$ a lo largo de C sabiendo que a lo largo del segmento resulta $\int_{\overline{AB}} \bar{f}\cdot d\bar{s} = 17$.



$x=-1$ $x=1$

$C: \; y = x^2 - x^4 \rightarrow x^2 - x^4 = 0 \rightarrow x^2(1-x^2) = 0$

$x = \pm 1$

$\operatorname{rot} \bar{f} = Q'_x - P'_y = 3\cancel{x} + 2 - 3\cancel{x} + 1 = 3$

$$\oint_{\partial D^+} \bar{f}\, d\bar{\lambda} = \iint_D \operatorname{rot} \bar{f}\, dx\, dy = 3 \int_{x=-1}^{1} \int_{y=0}^{x^2-x^4} dy\, dx = 3 \int_{-1}^{1} (x^2 - x^4)\, dx = 3\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} =$$

$$= 3\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5}\right) = \cancel{3}^{1}\, \frac{10-6}{\cancel{15}_{5}} = \frac{4}{5}$$

$$\underbrace{\oint_{\partial D^+} \bar{f}\, d\bar{\lambda}}_{\frac{4}{5}} = \underbrace{\int_{\substack{\bar{A}\to\bar{B} \\ \overline{AB}}} \bar{f}\, d\bar{\lambda}}_{17} + \underbrace{\int_{\substack{C \\ \bar{B}\to\bar{A}}} \bar{f}\, d\bar{\lambda}}_{¿?}$$

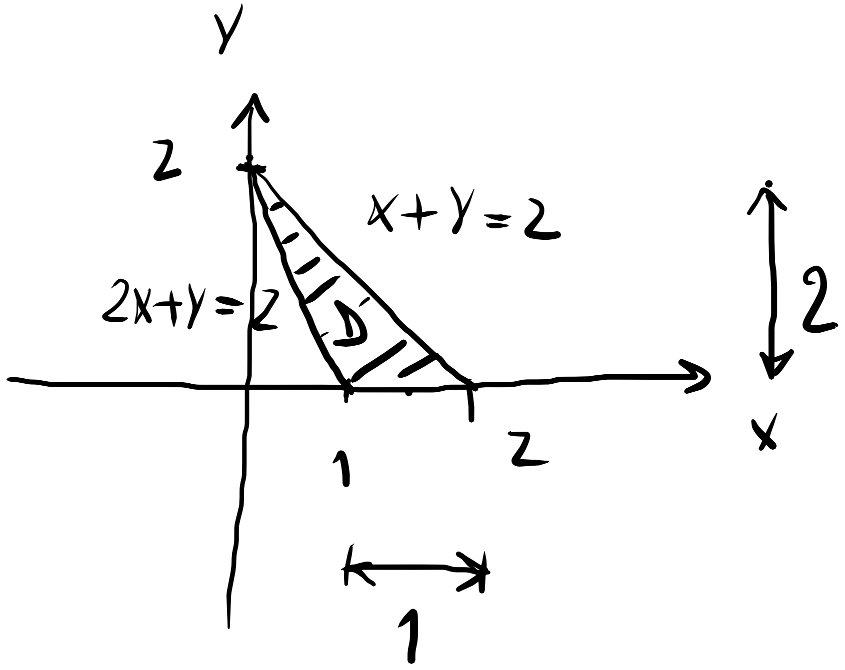
$$\rightarrow \int_{\substack{C \\ \bar{B}\to\bar{A}}} \bar{f}\, d\bar{\lambda} = \frac{4}{5} - 17 \Rightarrow \int_{\substack{C \\ \bar{A}\to\bar{B}}} \bar{f}\, d\bar{\lambda} = \boxed{17 - \frac{4}{5}} = \boxed{\frac{81}{5}}$$

06) Calcule la circulación en sentido positivo de $\bar{f} \in C^1$ a lo largo de la frontera de la región plana definida por $x+y \le 2$, $2x+y \ge 2$, 1° cuadrante, siendo:

a) $\bar{f}(x,y) = (\underbrace{2y-g(x)}_{f_1}, \underbrace{5x-h(y)}_{f_2})$. ~~b) $\bar{f}(x,y) = (2y+g(x-y), 2x-g(x-y))$.~~

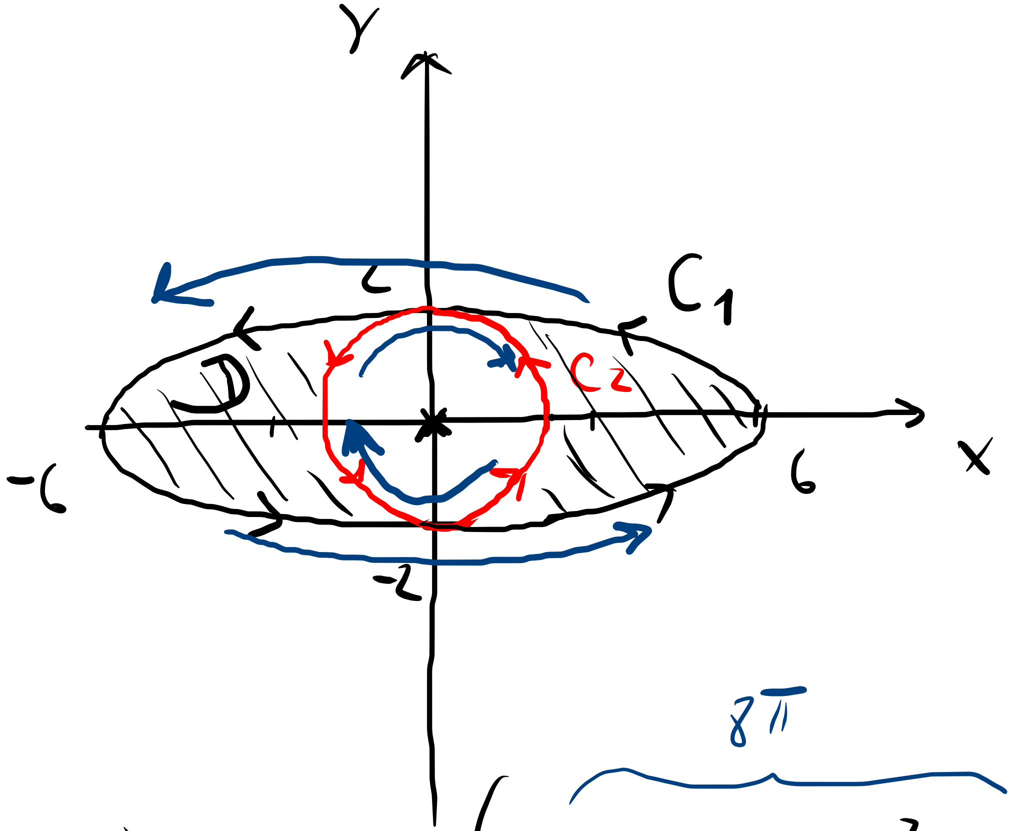
$$\text{rot}\,\bar{f} = 5 - 2 = 3$$

$$\int_{\partial D^+} \bar{f}\, d\lambda = \iint_D \underbrace{\text{rot}\,\bar{f}}_{3}\, dx\, dy = 3 \iint_D dx\, dy = 3 \cdot \text{área}(D) = 3 \cdot \frac{1 \cdot 2}{2} = \boxed{3}$$



08) Sea $\bar{f} = (P,Q) \in C^1$ en $\Re^2 - \{\bar{0}\}$ tal que $Q'_x - P'_y \equiv 5$, dadas las curvas C_1: $x^2 + 9y^2 = 36$ y C_2: $x^2 + y^2 = 4$, calcule $\oint_{C_2^+} \bar{f} \cdot d\bar{s}$ sabiendo que $\oint_{C_1^+} \bar{f} \cdot d\bar{s} = 7\pi$.

$$\text{rot}\,\bar{f} = 5$$



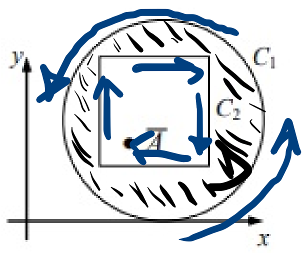
$$C_1 : \frac{x^2}{36} + \frac{y^2}{4} = 1 \rightarrow \int_{C_1^+} \bar{f}\, d\bar{s} = 7\pi$$

$$C_2 : x^2 + y^2 = 4$$

$$\int_{\partial D^+} \bar{f}\, d\bar{\lambda} = \iint_D \underbrace{\text{rot}\,\bar{f}}_{5}\, dx\, dy = 5 \cdot \iint_D dx\, dy = 5 \text{ área}(D) = 5 \cdot \left(\underbrace{6 \cdot 2 \cdot \pi - 2^2 \pi}_{8\pi} \right) = 40\pi$$

$$\underbrace{\int_{\partial D^+} \bar{f}\, d\bar{\lambda}}_{40\pi} = \underbrace{\int_{C_1^+} \bar{f}\, d\bar{\lambda}}_{7\pi} + \int_{C_2^-} \bar{f}\, d\bar{\lambda} \rightarrow \int_{C_2^-} \bar{f}\, d\bar{\lambda} = 40\pi - 7\pi = 33\pi \rightarrow \boxed{\int_{C_2^+} \bar{f}\, d\bar{\lambda} = -33\pi}$$

09) Dado $\bar{f} : \Re^2 - \{\bar{A}\} \to \Re^2 \,/\, \bar{f} = (P,Q)$; suponga matriz jacobiana continua con $Q'_x - P'_y \equiv 6$.
Calcule $\oint_{C_1^+} \bar{f} \cdot d\bar{s}$ sabiendo que $\oint_{C_2^+} \bar{f} \cdot d\bar{s} = 12$, C_1 es una circunferencia de radio 8, C_2 es un cuadrado de lado 5.



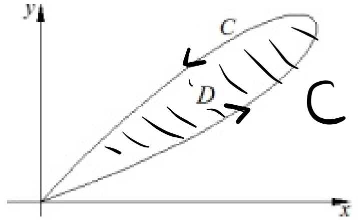
$$\operatorname{rot} \bar{f} = 6$$

$$\int_{\partial D^+} \bar{f}\, d\bar{\lambda} = \iint_D \operatorname{rot} \bar{f}\, dx\, dy = 6 \text{ área}(D) = 6 \cdot \left(\pi \cdot 8^2 - 25\right)$$

$$\underbrace{\int_{\partial D^+} \bar{f}\, d\bar{\lambda}}_{6(64\pi - 25)} = \int_{C_1^+} \bar{f}\, d\bar{\lambda} + \underbrace{\int_{C_2^-} \bar{f}\, d\bar{\lambda}}_{-12} \to \int_{C_1^+} \bar{f}\, d\bar{\lambda} = \boxed{6\left(64\pi - 25\right) + 12} = \boxed{384\pi - 138}$$

02) Calcule el área de la región plana D de la figura, sabiendo que su curva frontera C admite la ecuación vectorial:

$$\bar{X} = (\underbrace{u-u^2}_{x}, \underbrace{u-u^4}_{y}) \quad \text{con } 0 \le u \le 1$$



$C = \partial D^+$

$$\bar{f}(x,y) = (-y, 0)$$

$$\iint_D \operatorname{rot} \bar{f}\, dx\, dy = \int_{\partial D^+} \bar{f}\, d\bar{\lambda}$$

si $\operatorname{rot} \bar{f} = 1$ ⟶ $\boxed{\bar{f}(x,y) = (0, x)}$

$$\iint_D \operatorname{rot}\, dx\, dy = \iint_D dx\, dy = \text{área}(D)$$

$$\text{área}(D) = \int_C \bar{f}\, d\bar{\lambda} = \int_{u=0}^{1} (0, u-u^2) \cdot (1-2u, 1-4u^3)\, du = \int_0^1 (u - u^2 - 4u^4 + 4u^5)\, du =$$

$$= \left[\frac{u^2}{2} - \frac{u^3}{3} - \frac{4}{5} u^5 + 2\frac{u^6}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} - \frac{4}{5} + \frac{2}{3} = \frac{15+10-24}{30} = \boxed{\frac{1}{30}}$$