Formative 2: Eigenvalues & Eigenvectors

$$A = \begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix}$$

$$\det(A - \lambda \underline{I}) = 0$$

(A - matrix, λ is an eigenvalue, I - identity matrix)

Therefore:

$$\det\left[\begin{pmatrix} 4 & 8 & -1 & 2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] = 0$$

$$\therefore \det\left[\begin{pmatrix} 4 & 8 & -1 & 2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}\right] = 0$$

$$\det\begin{pmatrix} 4-\lambda & 8 & -1 & -2 \\ -2 & -9-\lambda & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ -1 & -13 & -14 & -13-\lambda \end{pmatrix} = 0$$

Matrices of Minors

$$+\,4-\lambda\left(\det\begin{pmatrix} -9-\lambda & -2 & -4 \\ 10 & 5-\lambda & -10 \\ -13 & -14 & -13-\lambda \end{pmatrix}\right)$$

$$-\,8\left(\det\begin{pmatrix} -2 & -2 & -4 \\ 0 & 5-\lambda & -10 \\ -1 & -14 & -13-\lambda \end{pmatrix}\right)$$

$$\cancel{+(-1)\left(\det\begin{pmatrix} 9-\lambda & -2 & -4 \\ 10 & 5-\lambda & -10 \\ -13 & -14 & -13-\lambda \end{pmatrix}\right)}$$

$$+(-1)\left(\det\begin{pmatrix} -2 & -9-\lambda & -4 \\ 0 & 10 & -10 \\ -1 & -13 & -13-\lambda \end{pmatrix}\right)$$

$$-(+2)\left(\det\begin{pmatrix} -2 & -9-\lambda & -2 \\ 0 & 10 & 5-\lambda \\ -1 & -13 & -14 \end{pmatrix}\right)$$

$$4-\lambda\left(\det\begin{pmatrix}9-\lambda & -2 & -4\\ 10 & 5-\lambda & -10\\ -13 & -14 & -13-\lambda\end{pmatrix}\right) =$$

$$4-\lambda\left(9-\lambda\begin{vmatrix}5-\lambda & -10\\ -14 & -13-\lambda\end{vmatrix} - (-2)\begin{vmatrix}10 & -10\\ -13 & -13-\lambda\end{vmatrix}\right.$$

$$\left.+(-4)\begin{vmatrix}10 & 5-\lambda\\ -13 & -14\end{vmatrix}\right)$$

$$= 4-\lambda(9-\lambda((5-\lambda)(-13-\lambda))-140) + 2(-130-10\lambda$$
$$-130) \quad -4(-140 + 13(5-\lambda))$$

$$= 4-\lambda(\cancel{\lambda}9-\lambda((\lambda^2+8\lambda-65)-140) + 2(-10\lambda$$
$$-260) \quad -4(-140 \cancel{+65 \pm 13} + 65 - 13\lambda)$$

$$= 4-\lambda(9-\lambda(\lambda^2+8\lambda-205) \cancel{+2(-520+}$$
$$+(-20\lambda - 520) - (\cancel{-300} - \cancel{52\lambda}) (52\lambda - 300)$$

$$= 4-\lambda((-\lambda^3 - 17\lambda^2 + 133\lambda + 1845) + (-20\lambda \cancel{+} -$$
$$520) - (-52\lambda - 300)$$

$$= 4-\lambda(-\lambda^3 - 17\lambda^2 + 165\lambda + 1625)$$
$$= -4\lambda^3 - 68\lambda^2 + 660\lambda + 6500 + \lambda^4$$
$$+ 17\lambda^3 - 165\lambda^2 - 1625\lambda$$

$$\cancel{\lambda^4 - 4\lambda^3} \quad \cancel{\lambda^4 + 13\lambda^3 - 88\lambda} \; 230$$
$$\lambda^4 + 13\lambda^3 - 233\lambda^2 - 965\lambda + 6500$$

$$-8\left(\det\begin{pmatrix} -2 & -2 & -4 \\ 0 & 5-\lambda & -10 \\ -1 & -14 & -13-\lambda \end{pmatrix}\right) =$$

$$-8\left(-2\begin{vmatrix} 5-\lambda & -10 \\ -14 & -13-\lambda \end{vmatrix} - (-2)\begin{vmatrix} 0 & -10 \\ -1 & -13-\lambda \end{vmatrix} + (-4)\begin{vmatrix} 0 & 5-\lambda \\ -1 & -14 \end{vmatrix} \right)$$

$$= -8(-2((5-\lambda)(-13-\lambda)) - 140) + 2(-10) - 4(5-\lambda)$$

$$= -8(-2(\lambda^2 + 8\lambda - 65 - 140) - 20 \cancel{-80} + 4\lambda - 20)$$

$$= \cancel{-8(-2\lambda^2 + 16\lambda}$$

$$= -8(-2\lambda^2 - 16\lambda + 410 - 20 + 4\lambda - 2$$

$$= -8(-2\lambda^2 - 12\lambda + 370)$$

$$= 16\lambda^2 + 96\lambda - 2960$$

$$-1\left(\det\begin{pmatrix}-2 & -9-\lambda & -4\\ 0 & 10 & -10 \\ -1 & -13 & -13-\lambda\end{pmatrix}\right) =$$

$$-1\left(-2\begin{vmatrix}10 & -10\\ -13 & -13-\lambda\end{vmatrix} - (-9-\lambda)\begin{vmatrix}0 & -10\\ -1 & -13-\lambda\end{vmatrix}\right)$$

$$+ (-4)\begin{vmatrix}0 & 10\\ -1 & -13\end{vmatrix}$$

$$= -1\left(-2\left(10(-13-\lambda)-130\right) + (\lambda+9)(-10)\right.$$
$$\left. -4(10)\right)$$

$$= -1\left(-2(-130-10\lambda-130) \cancel{+\lambda+90}\right.$$
$$\left.\cancel{+} -\lambda - 90 - 40\right)$$

$$= -1(260 + \cancel{2}4\lambda + 260 - \lambda - 90 - 40)$$

$$= -1(\cancel{390}\ 10\lambda + 390) = -10\lambda - 390$$

$$= -1\left(-2(-130-10\lambda-130)+(-10\lambda-90)\right.$$
$$\left.+ -40\right)$$

$$= -1(260+20\lambda+260 + -10\lambda - 40$$
$$-90)$$

$$= -1(10\lambda+390) = -10\lambda-390$$

$$2\left(\det\begin{pmatrix} -2 & -9-\lambda & -2 \\ 0 & 10 & 5-\lambda \\ -1 & -13 & -14 \end{pmatrix}\right) =$$

$$2\left(-2\begin{vmatrix} 10 & 5-\lambda \\ -13 & -14 \end{vmatrix} - (-9-\lambda)\begin{vmatrix} 0 & 5-\lambda \\ -1 & -14 \end{vmatrix}\right.$$

$$\left. + (-2)\begin{vmatrix} 0 & 10 \\ -1 & -13 \end{vmatrix}\right)$$

$$= 2\left(-2(-140 + 13(5-\lambda)) + (\lambda+9)(5-\lambda)\right.$$
$$\left. - 2(10)\right)$$

$$= 2\left(-2(-140 + 65 - 13\lambda) + (-\lambda^2 - 4\lambda + 45\right.$$
$$\left. -20\right)$$

$$= 2\left(150 + 26\lambda - \lambda^2 - 4\lambda + 45\right.$$
$$\left. -20\right)$$

$$= 2(-\lambda^2 + 22\lambda + 175)$$
$$= -2\lambda^2 + 44\lambda + 350$$

Remember $\det \begin{pmatrix} 4-\lambda & 8 & -1 & -2 \\ -2 & -9-\lambda & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ -1 & -13 & -14 & -13-\lambda \end{pmatrix} =$

$$4-\lambda \left(\det \begin{pmatrix} -9-\lambda & -2 & -4 \\ 10 & 5-\lambda & -10 \\ -13 & -14 & -13-\lambda \end{pmatrix} \right)$$

$$-8 \left(\det \begin{pmatrix} -2 & -2 & -4 \\ 0 & 5-\lambda & -10 \\ -1 & -14 & -13-\lambda \end{pmatrix} \right)$$

$$+(-1) \left(\det \begin{pmatrix} -2 & -9-\lambda & -4 \\ 0 & 10 & -10 \\ -1 & -13 & -13-\lambda \end{pmatrix} \right)$$

$$-(-2) \left(\det \begin{pmatrix} -2 & -9-\lambda & -2 \\ 0 & 10 & 5-\lambda \\ -1 & -13 & -14 \end{pmatrix} \right)$$

$\therefore \det|A| = \lambda^4 + 13\lambda^3 - 233\lambda^2 - 965\lambda + 6500$

$+ 16\lambda^2 + 96\lambda - 2960$

$- 10\lambda - 390$

$+ 2\lambda^2 + 44\lambda + 350$

$= \lambda^4 + \cancel{13\lambda^3} \cancel{- 215\lambda^2} -$

$= \lambda^4 + 13\lambda^3 - 215\lambda^2 - 835\lambda + 3500$

Remember:
$\det(A - \lambda I) = 0$

$\therefore \lambda^4 + 13\lambda^3 - 215\lambda^2 - 835\lambda + 3500 = 0$

Trial & Error
does ~~to~~ $\lambda = 1$?

Check by substitution
$1^4 + 13(1)^3 - 215(1)^2 - 835(1) + 3500 = 0$
$1 + 13 - 215 - 835 + 3500$
$= 2464$

so no.

is $\lambda = 2$?

$(2)^4 + 13(2)^3 - 215(2)^2 - 835(2) + 3500$
$= 16 + 104 - 860 - 1670 + 3500$
$= 1090$

Using the graph method, $\lambda \approx 2 \cdot \lambda \approx 2.675$

$\therefore \; (A - \lambda I) = \cancel{4 - 2.675} \quad 8 \quad -1 \quad -2$

$-2 \quad -9-2.675$

Using the graph method $\lambda_1 = 2.675$
$\lambda_2 = -21.125 \quad \lambda_3 = -5.604 \quad \lambda_4 = 11.054$
substituting λ into matrix $(A - \lambda I)$,

we get

$$\begin{pmatrix} 4-2.675 & 8 & -1 & -2 \\ -2 & -9-2.675 & -2 & -4 \\ 0 & 10 & 5-2.675 & -10 \\ -1 & -13 & -14 & -13-2.675 \end{pmatrix}$$

$$= \begin{pmatrix} 1.325 & 8 & -1 & -2 \\ -2 & 11.675 & -2 & -4 \\ 0 & 10 & 2.325 & -10 \\ -1 & -13 & -14 & -15.675 \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix}$$

divide R1 by 1.325, then multiply by row 2 and add to R2

$$\begin{pmatrix} 1 & 6.036 & -0.754 & -1.509 \\ 0 & 0.397 & -3.509 & -7.018 \\ 0 & 10 & 2.325 & -10 \\ -1 & -13 & -14 & -15.675 \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Add R1 to R4, then multipy R2 by $\frac{1}{0.397}$

$$\begin{pmatrix} 1 & 6.036 & -0.754 & -1.509 \\ 0 & 1 & -8.834 & -17.669 \\ 0 & 10 & 2.325 & -10 \\ 0 & -6.964 & -14.754 & -17.184 \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Multiply R2 by 10 and add to R3,
Multipy R2 by 6.964 and add to R4

$$\begin{pmatrix} 1 & 6.036 & -0.754 & -1.509 & \vline & 0 \\ 0 & 1 & -8.834 & -17.669 & \vline & 0 \\ 0 & 0 & 90.669 & 166.688 & \vline & 0 \\ 0 & 0 & -76.278 & -140.231 & \vline & 0 \end{pmatrix} = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}$$

$R_3/90.669$, $R_4 + 76.278 R_3 \to R_4$, $R_2 - 8.834 R_3 \to R_2$

~~1 6.036 -0.754 -1.509~~

~~0 1 -~~

$R_3/90.669$

$$\begin{array}{cccc|c|c} 1 & 6.036 & -0.754 & -1.509 & 0 & v_1 \\ 0 & 1 & -8.834 & -17.669 & 0 & v_2 \\ 0 & 0 & 1 & 1.838 & 0 & v_3 \\ 0 & 0 & -76.278 & -140.231 & 0 & v_4 \end{array}$$

$R_4 + 76.272 R_3 \rightarrow R_4$, $R_2 + 8.834 R_3 \rightarrow R_2$,

$R_1 + 0.754 R_3 \rightarrow R_1$

$$\begin{pmatrix} 1 & 6.036 & 0 & -0.122 \\ 0 & 1 & 0 & -1.428 \\ 0 & 0 & 1 & 1.838 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$v_4 = t$

from row 3: $v_3 + 1.838 v_4 = 0$

$v_3 = -1.834 v_4$

$\therefore v_3 = -1.838t$

from row 2: $v_2 - 1.428 v_4 = 0$

$v_2 = 1.428 v_4$

$\therefore v_2 = 1.428t$

from row 1 $v_1 + 6.036 v_2 - 0.122 v_4 = 0$

substituting $v_1 + 6.036(1.428 v_4) - 0.122 v_4 = 0$

$\therefore v_1 + 8.619 v_4 - 0.122 v_4 = 0$

$v_1 + 8.497 v_4 = 0$

$\therefore v_1 = -8.497 v_4$

$\therefore v_1 = -8.497t$

$v_1 = -8.497t$
$v_2 = 1.428t$
$v_3 = -1.838t$
$v_4 = t$

$\therefore$ the ~~set~~ vector =

$$t \begin{bmatrix} -8.497 \\ 1.488 \\ -1.838 \\ 1 \end{bmatrix}$$

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Using $\lambda_4 = 11.054$

and substituting into $A - \lambda I$ we have;

$$\begin{pmatrix} -7.054 & 8 & -1 & -2 \\ -2 & -20.054 & -2 & -4 \\ 0 & 10 & -6.054 & -10 \\ -1 & 13 & -14 & -24.054 \end{pmatrix}$$

Using Gaussian Elimination

$$\left(\begin{array}{cccc|c} -7.054 & 8 & -1 & -2 & 0 \\ -2 & -20.054 & -2 & -4 & 0 \\ 0 & 10 & -6.054 & -10 & 0 \\ -1 & -13 & -14 & -24.054 & 0 \end{array}\right)$$

Row 1 $\div$ -7.054 = Row 1

$$\left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ -2 & -20.054 & -2 & -2 & 0 \\ 0 & 10 & -6.054 & -10 & 0 \\ -1 & 13 & -14 & -24.054 & 0 \end{array}\right)$$

Row 2 $-$ (-2 $\times$ Row 1) = Row 2

$$\left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ 0 & -22.322 & -1.716 & -3.433 & 0 \\ 0 & 10 & -6.054 & -10 & 0 \\ -1 & -13 & -14 & -24.054 & 0 \end{array}\right)$$

Row 4 $-$ ($-1 \cdot$ Row 1) $=$ Row 4

$$\left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ 0 & -22.322 & -1.716 & -3.433 & 0 \\ 0 & 10 & -6.054 & -10 & 0 \\ 0 & -14.134 & -13.858 & -23.771 & 0 \end{array}\right)$$

Row 2 $\div$ -22.322 $=$ Row 2

$$\left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ 0 & 1 & 0.077 & 0.154 & 0 \\ 0 & 10 & -6.054 & -10 & 0 \\ 0 & -14.134 & -13.858 & -23.771 & 0 \end{array}\right)$$

Row 3 $-$ ($10 \cdot$ Row 2) $=$ Row 3 first, then
Row 3 $\div$ -6.823 $=$ Row 3

$$\left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ 0 & 1 & 0.077 & 0.154 & 0 \\ 0 & 0 & 1 & 1.691 & 0 \\ 0 & -14.134 & -13.858 & -23.771 & 0 \end{array}\right)$$

Row 4 $-$ $(-14.134 \cdot \text{Row } 2)$ = Row 4 first, then
Row 4 $-$ $(-12.771 \cdot \text{Row } 3)$ = Row 4

$$\left(\begin{array}{cccc|c} 1 & -1.134 & 0.142 & 0.284 & 0 \\ 0 & 1 & 0.077 & 0.154 & 0 \\ 0 & 0 & 1 & 1.691 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Row 2 $-$ $0.077 \cdot \text{Row } 3$ = Row 2
Row 1 $-$ $0.142 \cdot \text{Row } 3$ = Row 1

$$\left(\begin{array}{cccc|c} 1 & -1.134 & 0 & 0.044 & 0 \\ 0 & 1 & 0 & 0.044 & 0 \\ 0 & 0 & 1 & 1.691 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Row 1 $-$ $(-1.134 \cdot \text{Row } 2)$ = Row 1

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0.071 & 0 \\ 0 & 1 & 0 & 0.024 & 0 \\ 0 & 0 & 1 & 1.691 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

RREF

$$\begin{pmatrix} 1 & 0 & 0 & 0.071 \\ 0 & 1 & 0 & 0.024 \\ 0 & 0 & 1 & 1.691 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

let $V_4 = t$

from Row 1; $V_1 + 0.071t = 0$
from Row 2; $V_2 + 0.024t = 0$
from Row 3; $V_3 + 1.691t = 0$
from Row 4; $V_4 = V_4 = t$

$$\therefore \quad \vec{V}_1 = -0.071t$$
$$V_2 = -0.024t$$
$$V_3 = -1.691t$$
$$V_4 = V_4 = t$$

$$\therefore \vec{V} = \begin{bmatrix} -0.071t \\ -0.024t \\ -1.691t \\ t \end{bmatrix}$$

$$\vec{V} = t\begin{bmatrix} -0.071 \\ -0.024 \\ -1.691 \\ 1 \end{bmatrix}$$ for $\lambda = 11.054$

Date:

Using gaussian Elimination

$\lambda_1 = -21.125$, for the matrix

$$A = \begin{bmatrix} 4 & 5 & -1 & 2 \\ -2 & -9 & -2 & -10 \\ -1 & -13 & -14 & -13 \\ 0 & 10 & -8 & -10 \end{bmatrix}$$

so since $\lambda = -21.125$, then

$$A - \lambda_1 = A + 21.125 I,$$

$$A - \lambda I = \begin{bmatrix} 25.125 & 5 & -1 & 2 \\ -2 & 12.125 & -2 & -10 \\ -1 & -13 & 7.125 & -13 \\ 0 & 10 & -8 & 11.125 \end{bmatrix}$$

$$(A - \lambda I) \cdot \vec{v} = \vec{0},$$

$$= \left[\begin{array}{cccc|c} 25.125 & 5 & -1 & 2 & 0 \\ -2 & 12.125 & -2 & -10 & 0 \\ -1 & -13 & 7.125 & -13 & 0 \\ 0 & 10 & -8 & 11.125 & 0 \end{array}\right]$$

the eigen vector T_1 is

Date:

$$\vec{V}_1 = \begin{bmatrix} 0.0524 \\ -0.3379 \\ -0.9078 \\ -0.2429 \end{bmatrix}$$

EIGENVALUE IMPORTANCE

$\lambda_1 = 2.675$

$\lambda_2 = -21.125$

$\lambda_3 = -5.604$

$\lambda_4 = 11.054$

~~For~~ Formula =

Importance is calculated by ~~div~~ adding the absolute values and dividing by the total number of λ's

total = 40.458

λ_1 importance $= \dfrac{2.675}{40.458} = 6.61\%$

$\lambda_2 = \dfrac{21.125}{40.458} = 52.2\%$

$\lambda_3 = \dfrac{5.604}{40.458} = 13.9\%$

$\lambda_4 = \dfrac{11.054}{40.458} = 27.3\%$