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Convex Hull

A special thanks to

David Wu

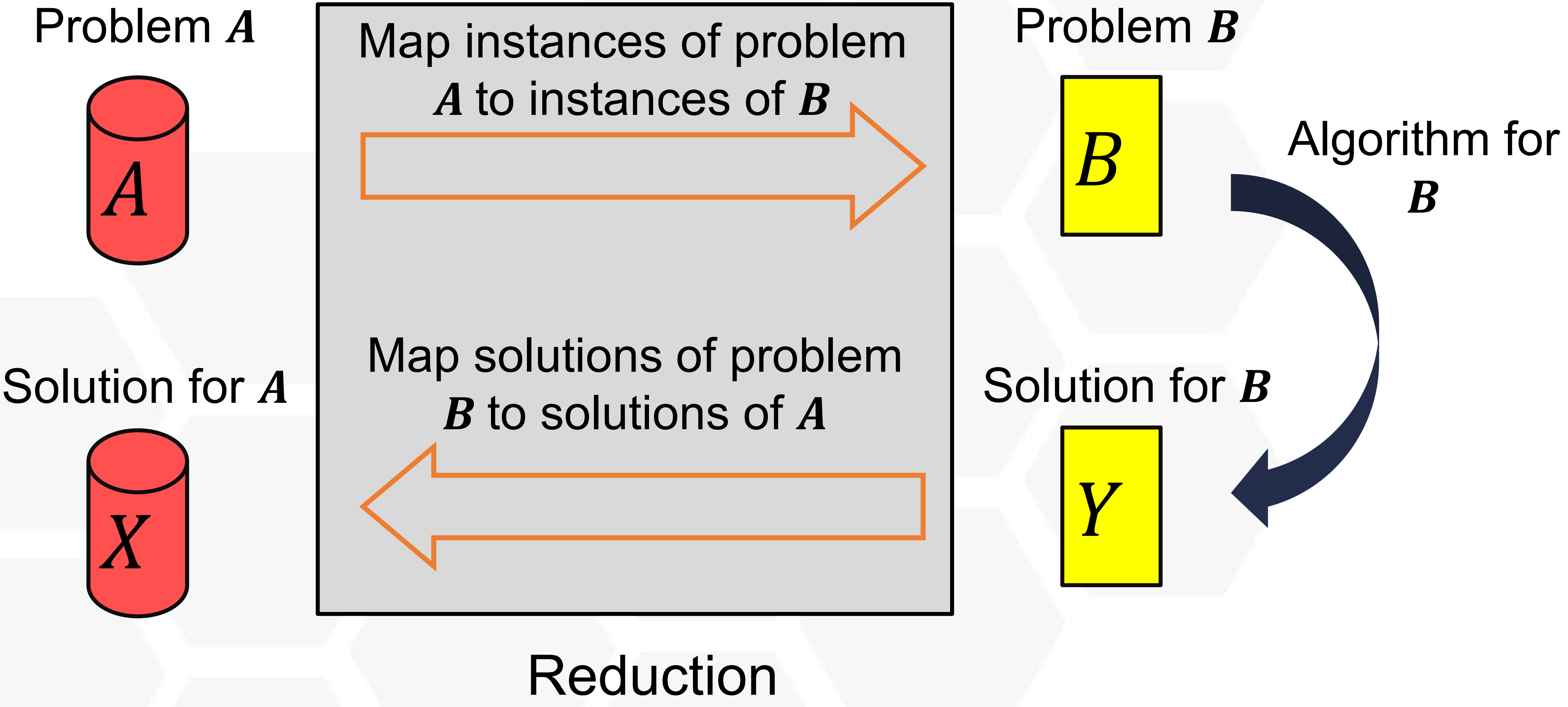
Robbie Hott

and the UVA CS Dept

Outline

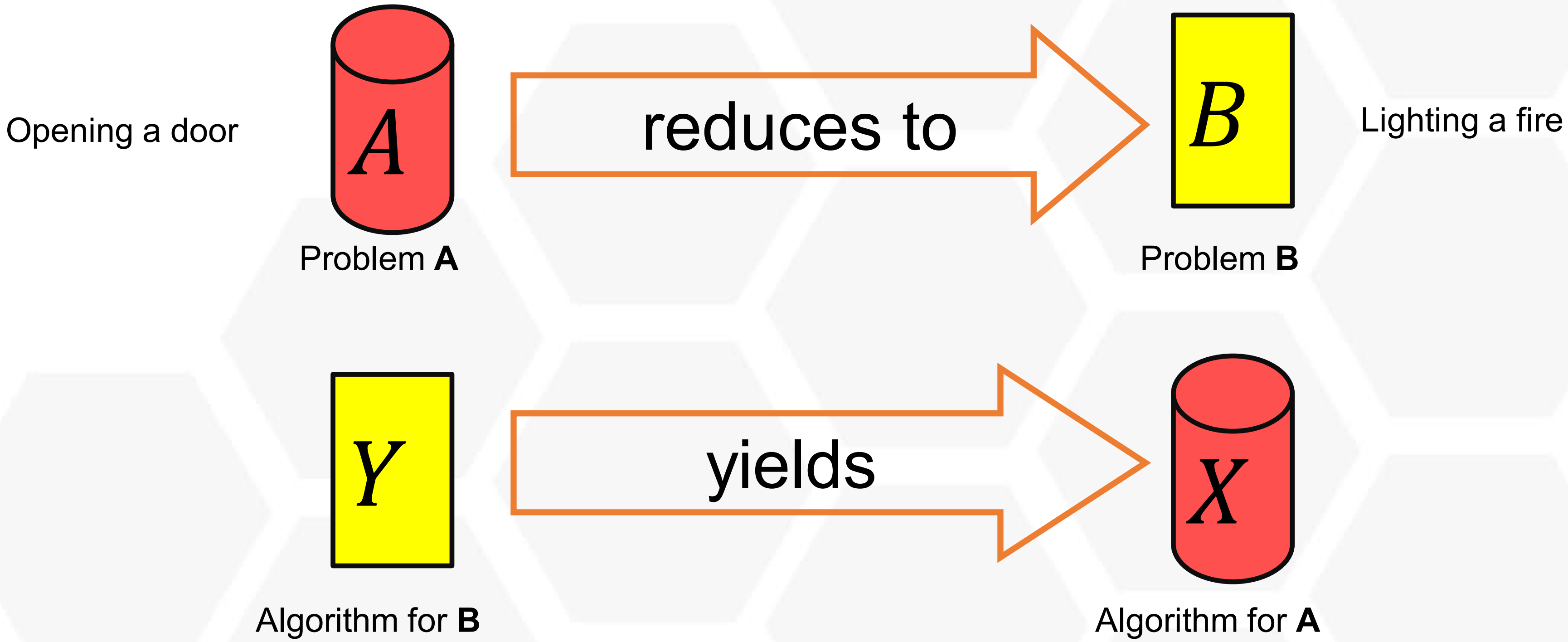
- Reductions and lower bounds
- Convex hull
  - Applications
  - Graham’s algorithm (Graham scan)
  - Jarvis’ algorithm (Jarvis march)
  - Chan’s algorithm

Reductions



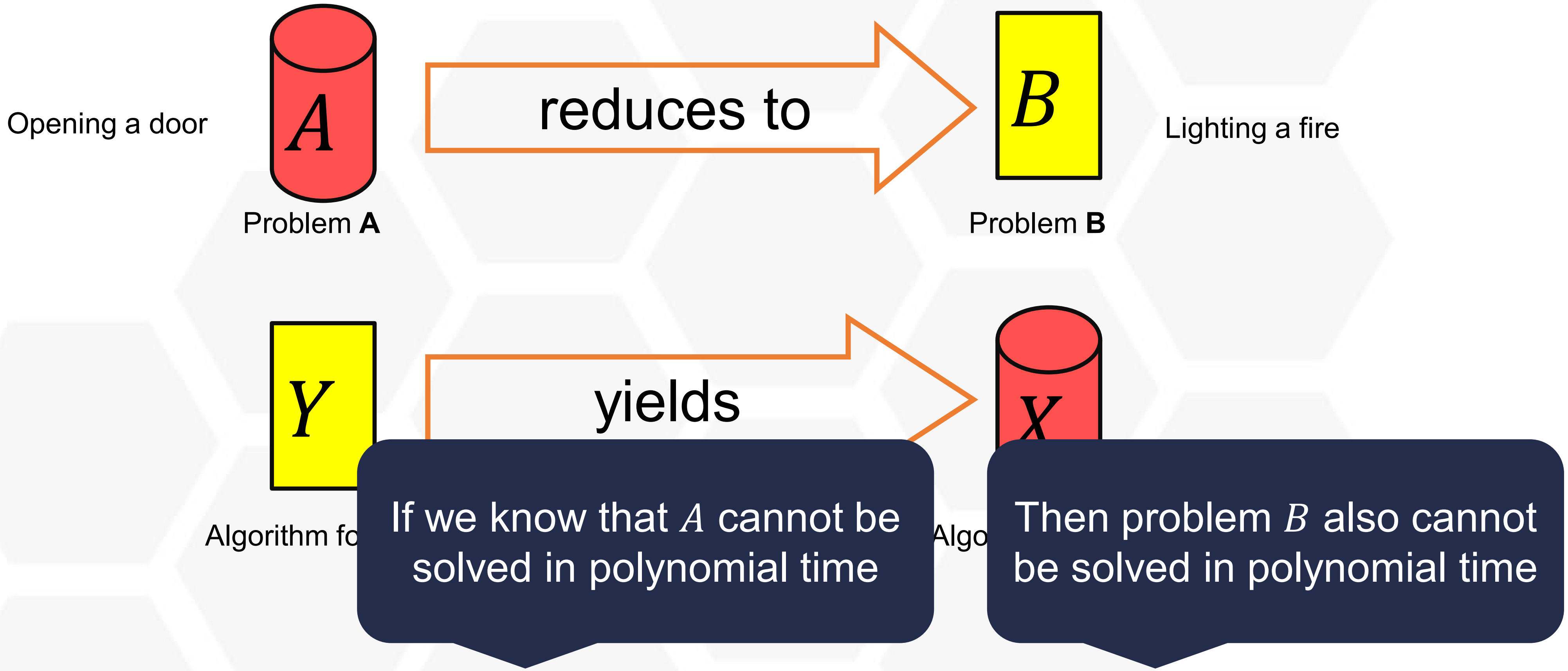
$A \leq B$: there is a reduction from A to B

Understanding Reductions



Implication: A is <u>no more difficult</u> than B
(denoted $A \leq B$)

Worst-Case Lower Bounds via Reductions



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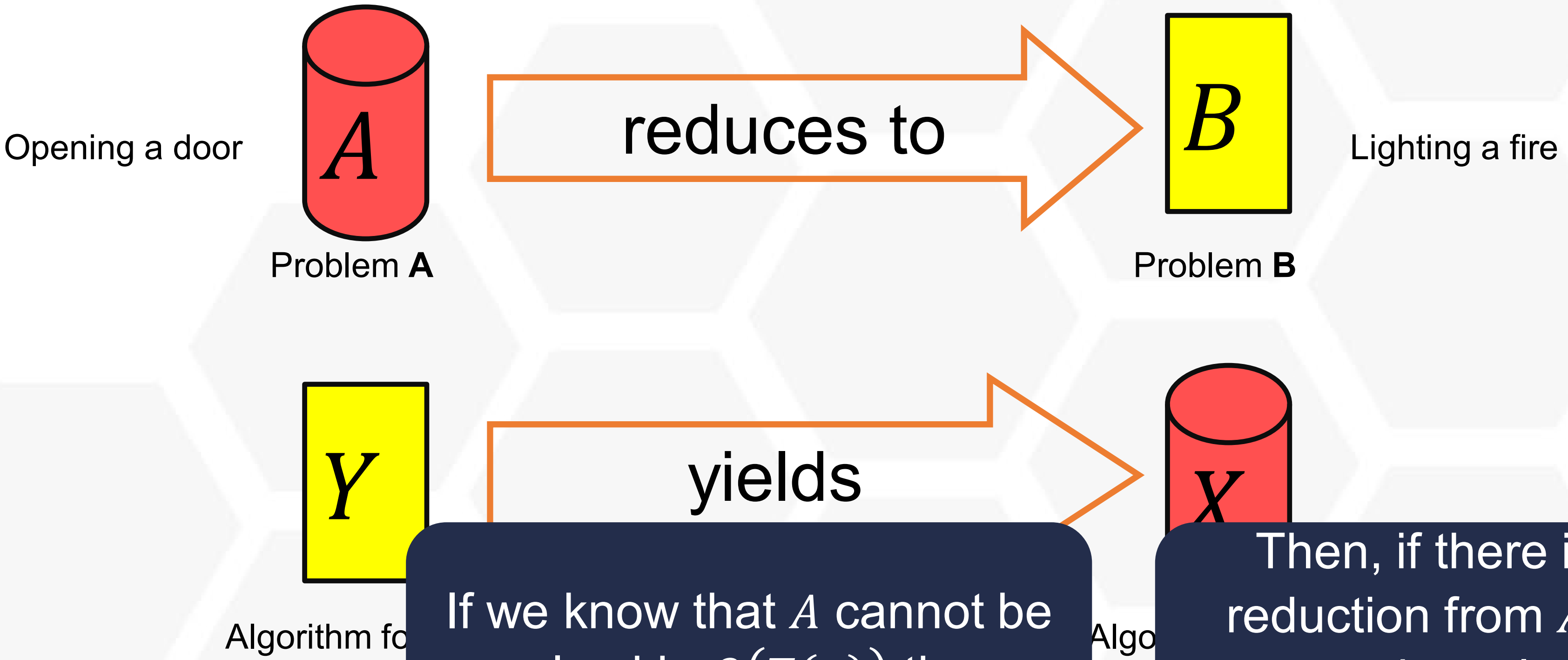
Worst-Case Lower Bounds via Reductions



If we know that A cannot be solved in $O(T(n))$ time

Then, if there is a $O(T(n))$ reduction from A to B, then B <u>cannot</u> be solved in $O(T(n))$ time

Implication: A is <u>no more difficult</u> than B (denoted $A \leq B$)

The Convex Hull Problem

Problem: find the smallest convex polygon that bounds a shape (or more generally, a collection of points)

Example application: collision detection in computer graphics; also useful for solving other problems, especially in computational geometry (e.g., furthest pair of points)

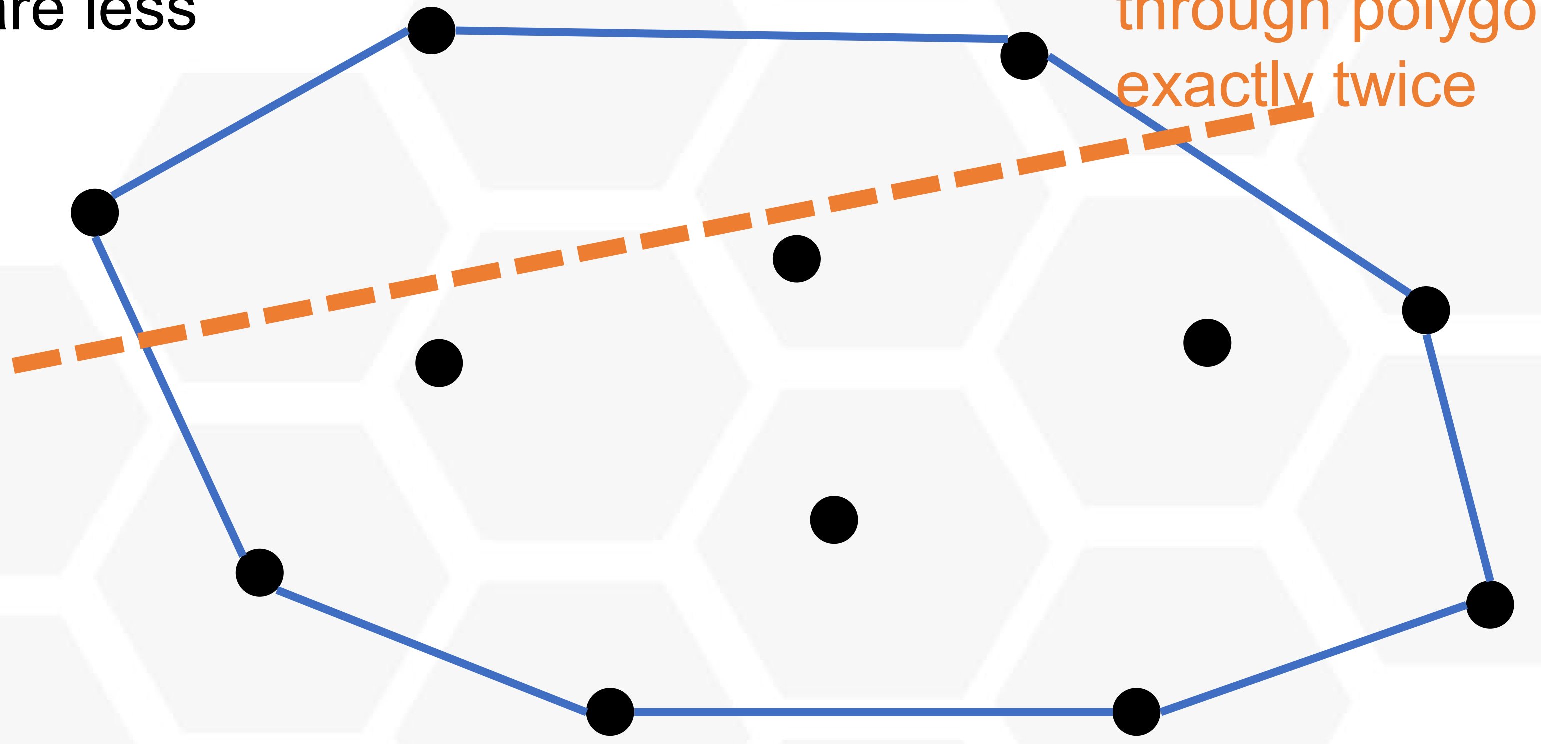
Convex Hull

Data Science Applications

- Model to represent data comprised of convex combinations.
  - E.G. Spectral data mixing (and unmixing)
- Data fusion models: convex combinations
- Model used in convex optimization

The Convex Hull Problem

Convex polygon: all interior angles are less than $180°$



Equivalently: line drawn through polygon will intersect exactly twice

Problem: given a set of n points, find the smallest convex polygon such that every point is either on the boundary or the interior of the polygon

The Convex Hull Problem

Convex polygon: all interior angles are less than $180°$

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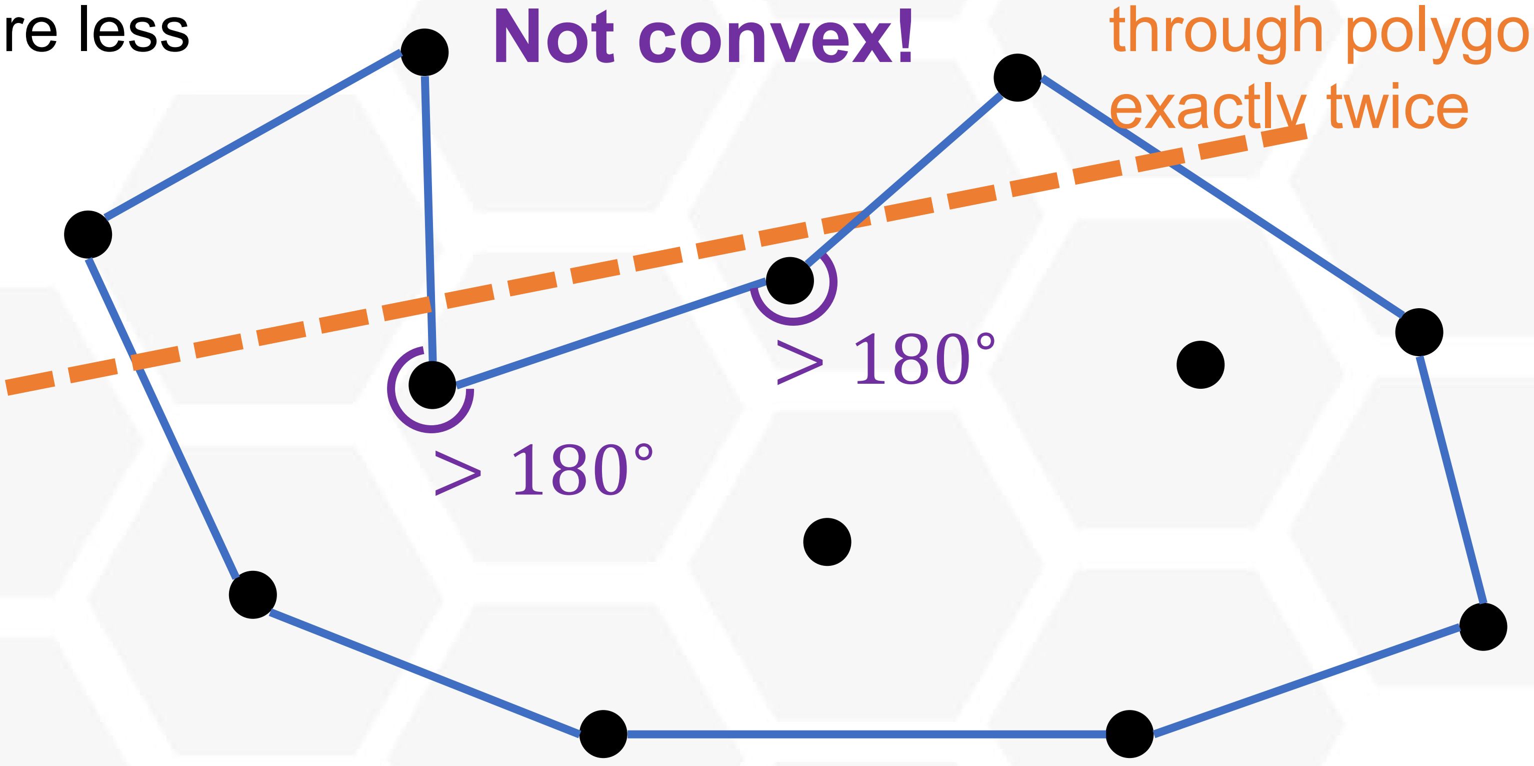

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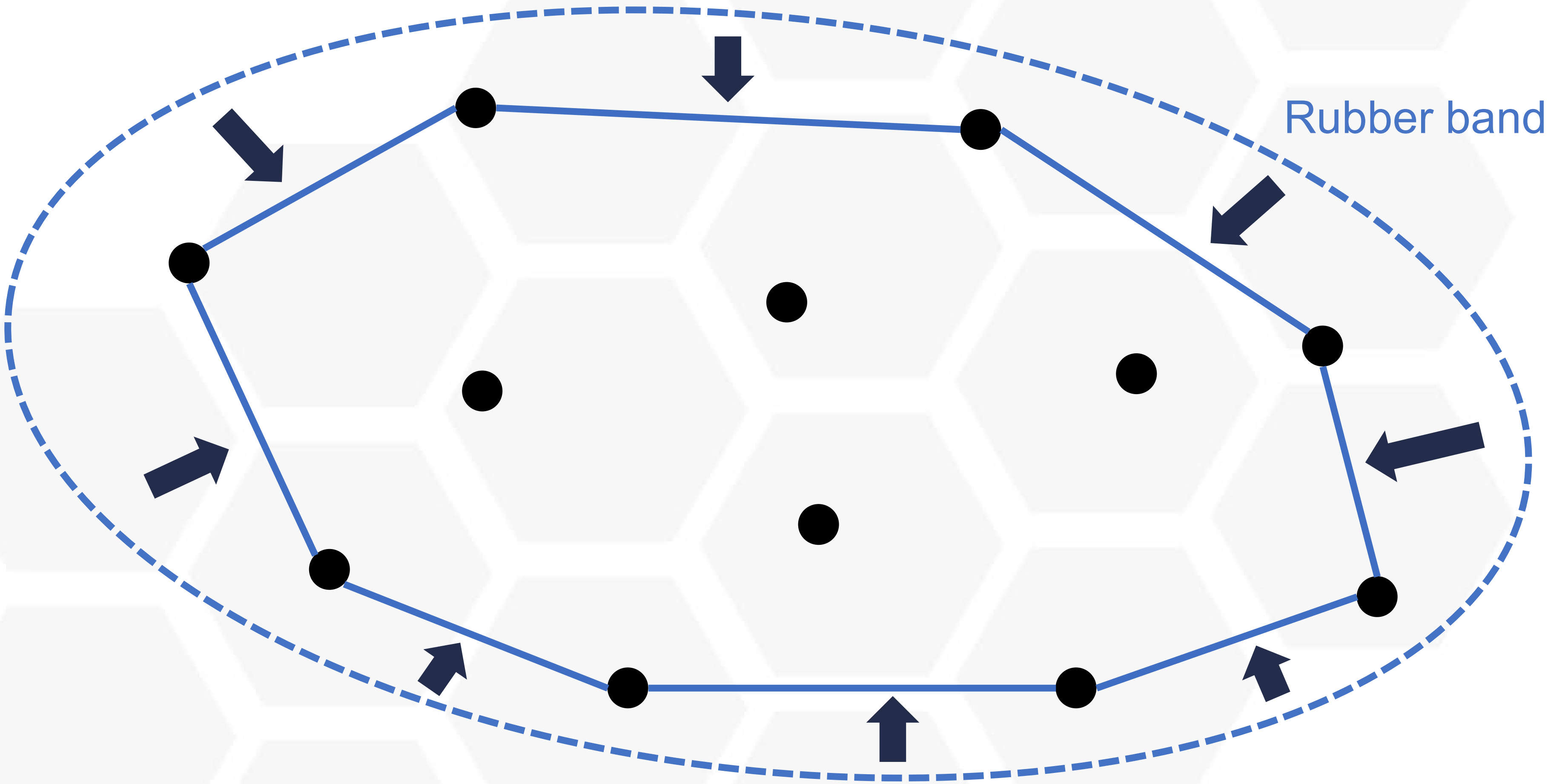
The Convex Hull Problem



Rubber band analogy: imagine the points are nails sticking out of a board and wrapping a rubber band to encompass the nails; convex hull is resulting shape

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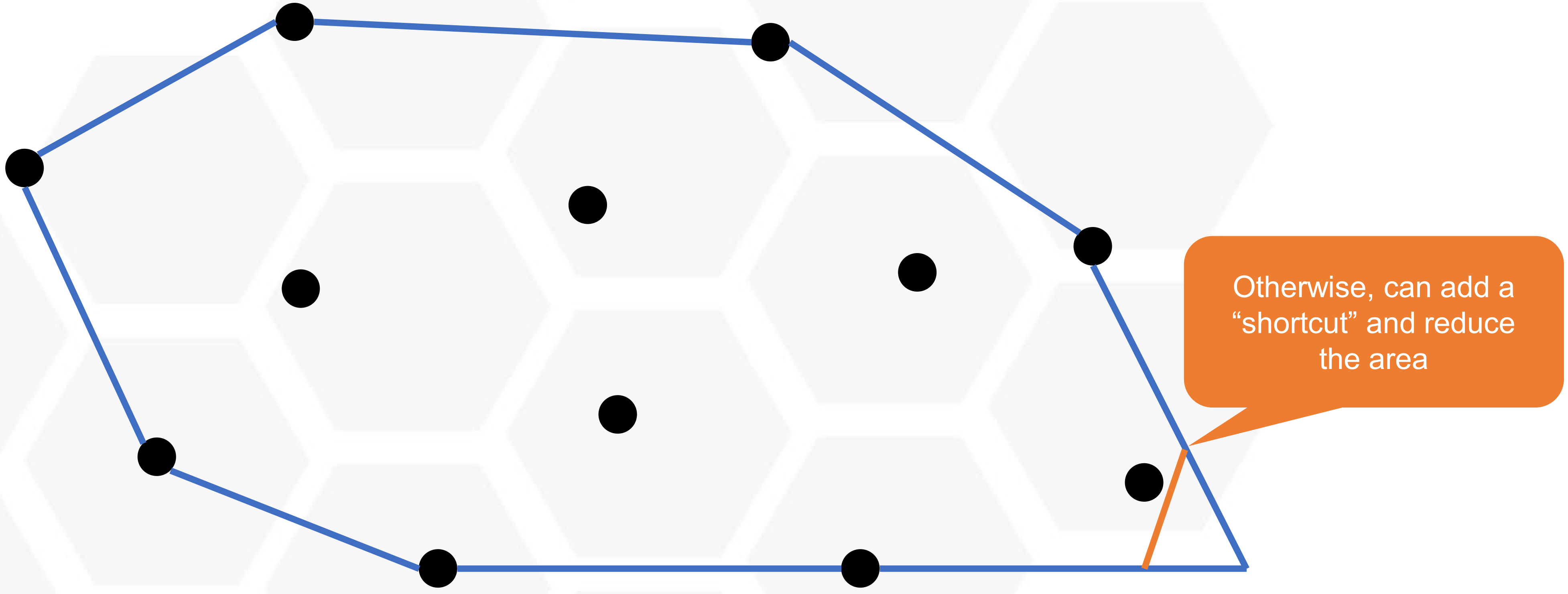
The Convex Hull Problem



Observation: every point on the convex hull is one of the input points

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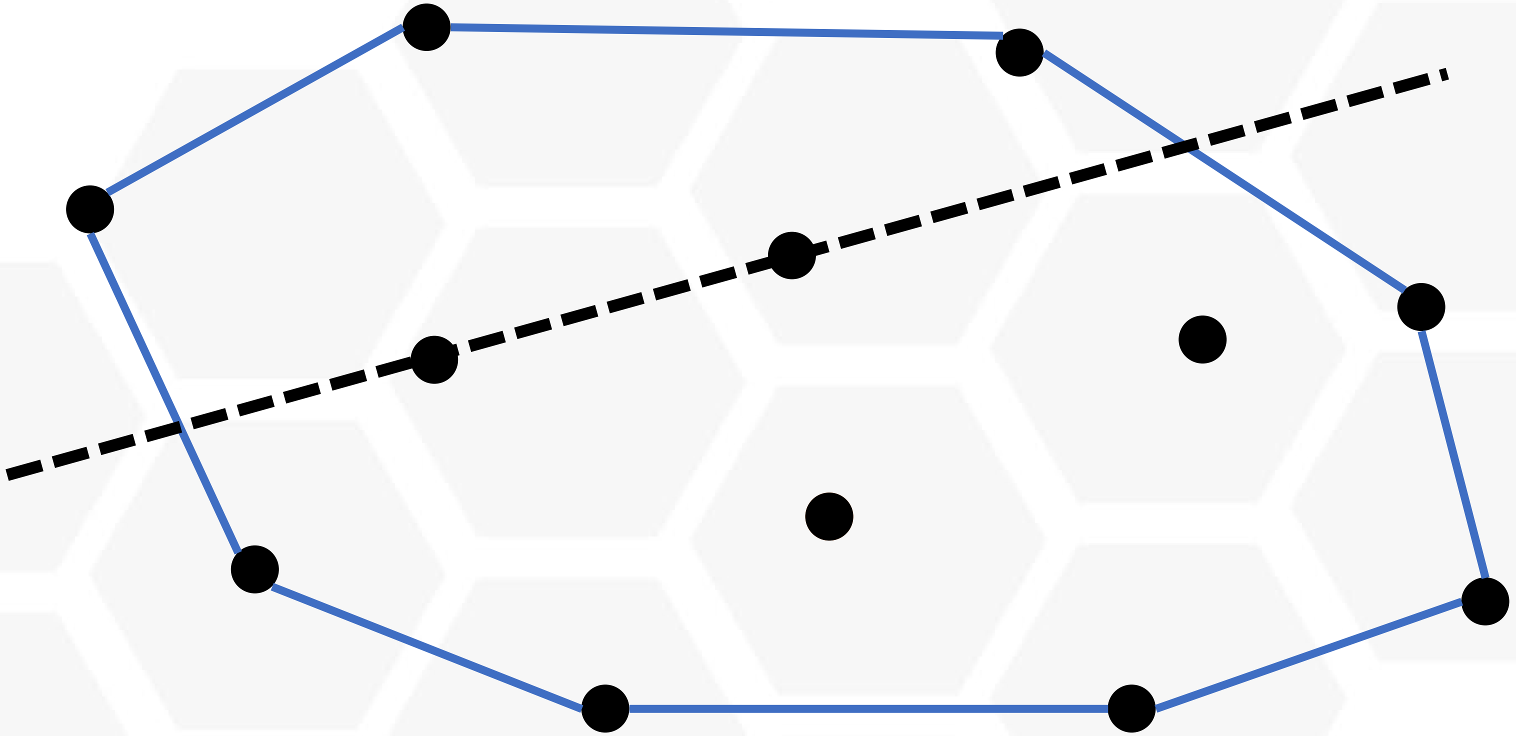
A Brute Force Approach



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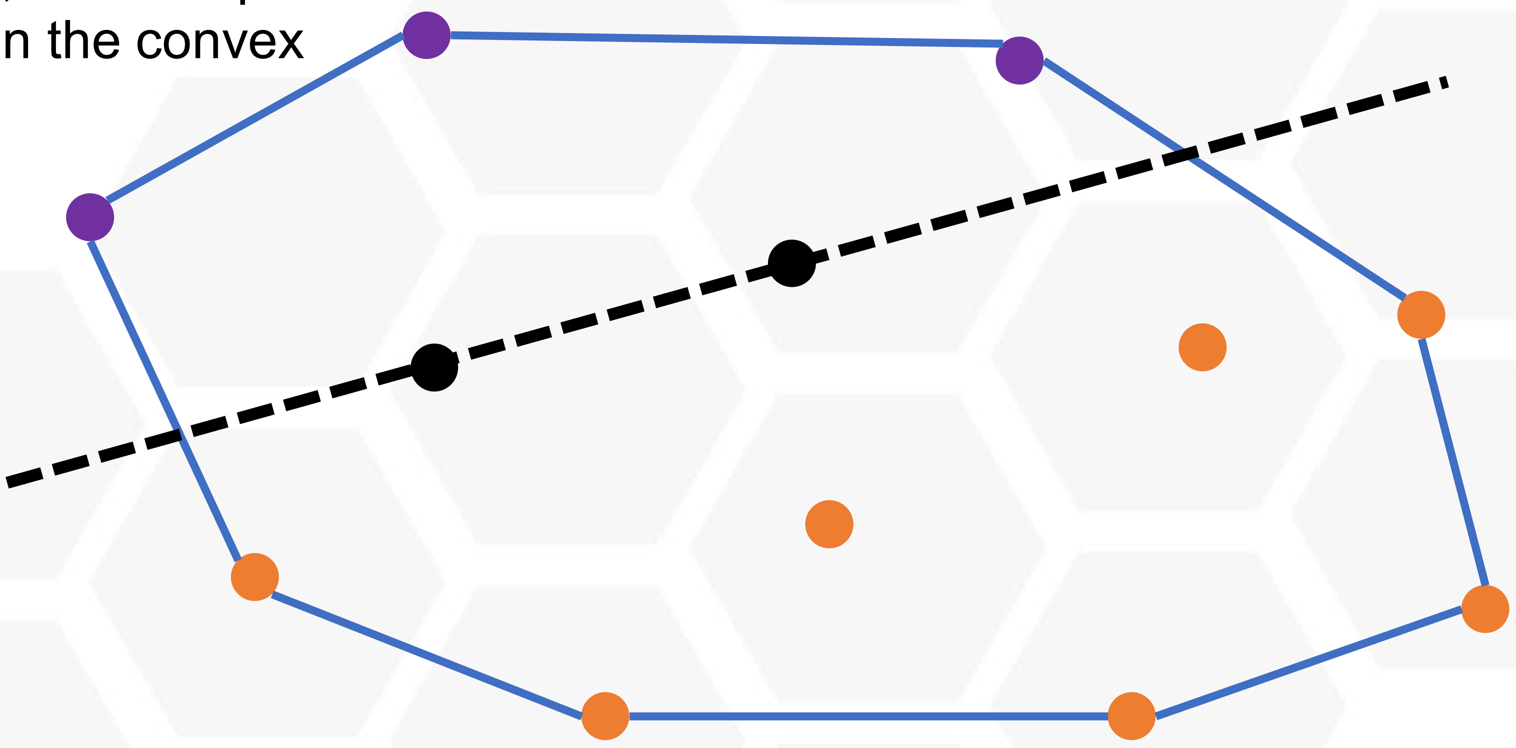
A Brute Force Approach

Observation: if there are points on <u>both</u> sides of the line, then the pair cannot be an edge in the convex hull



Run-time: $O(n^3)$

Brute force approach: for every pair of points, check if all other points are on the same side of the line

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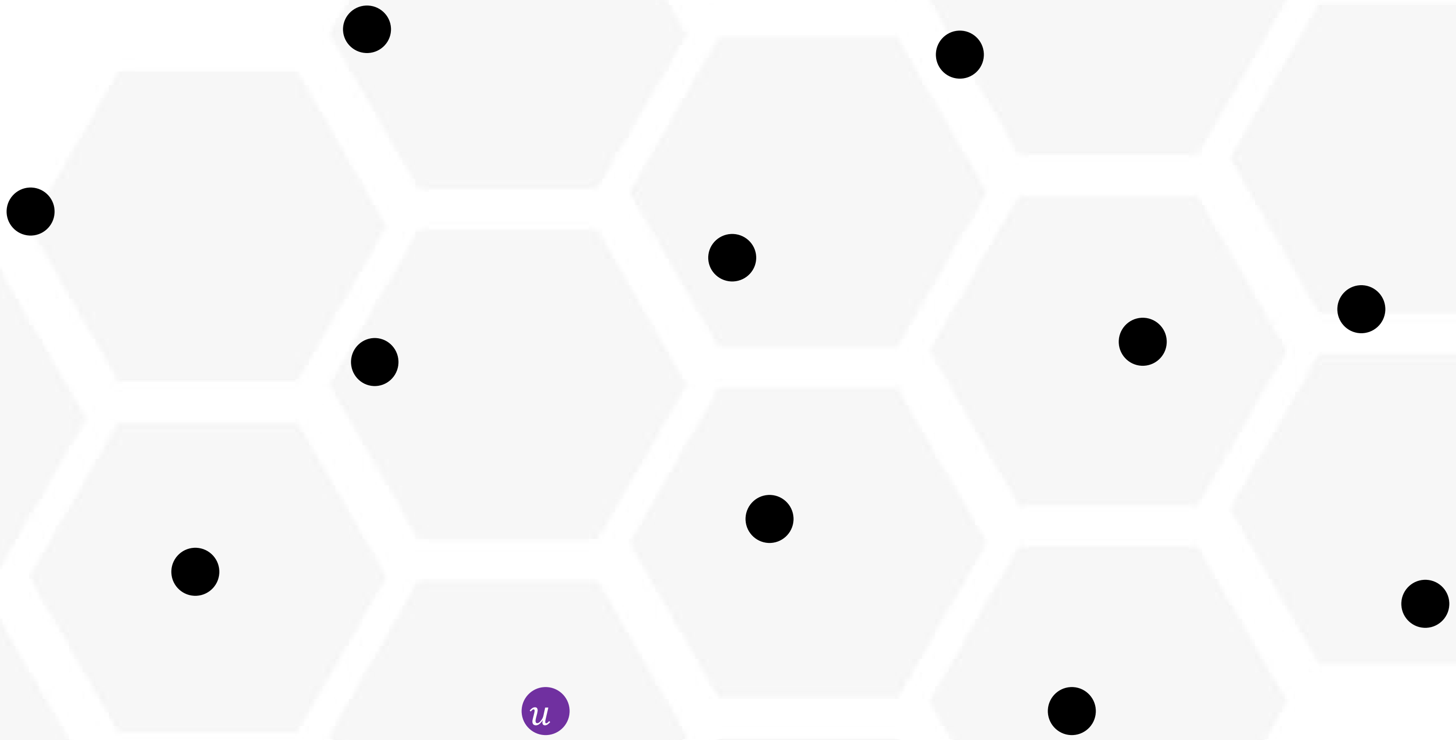
Graham's Algorithm



Observation: Extremal points must be part of the convex hull (e.g., bottom-most point, left-most point, etc.)

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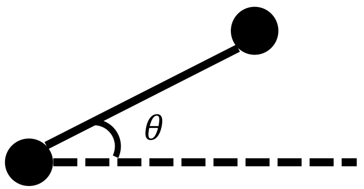
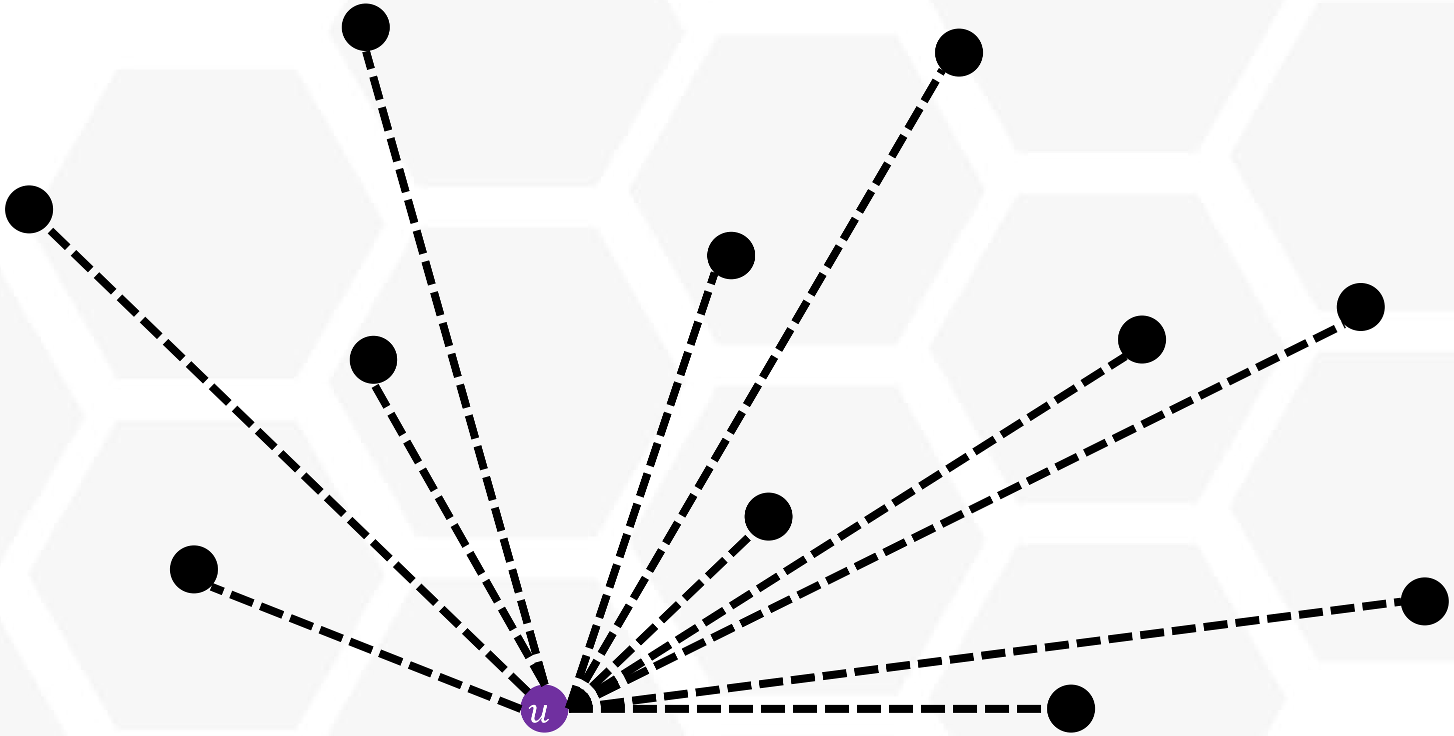
Graham’s Algorithm

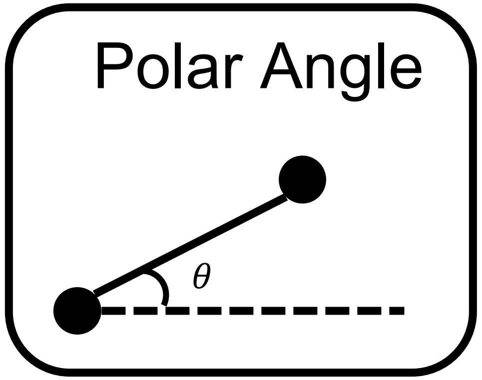
Polar Angle

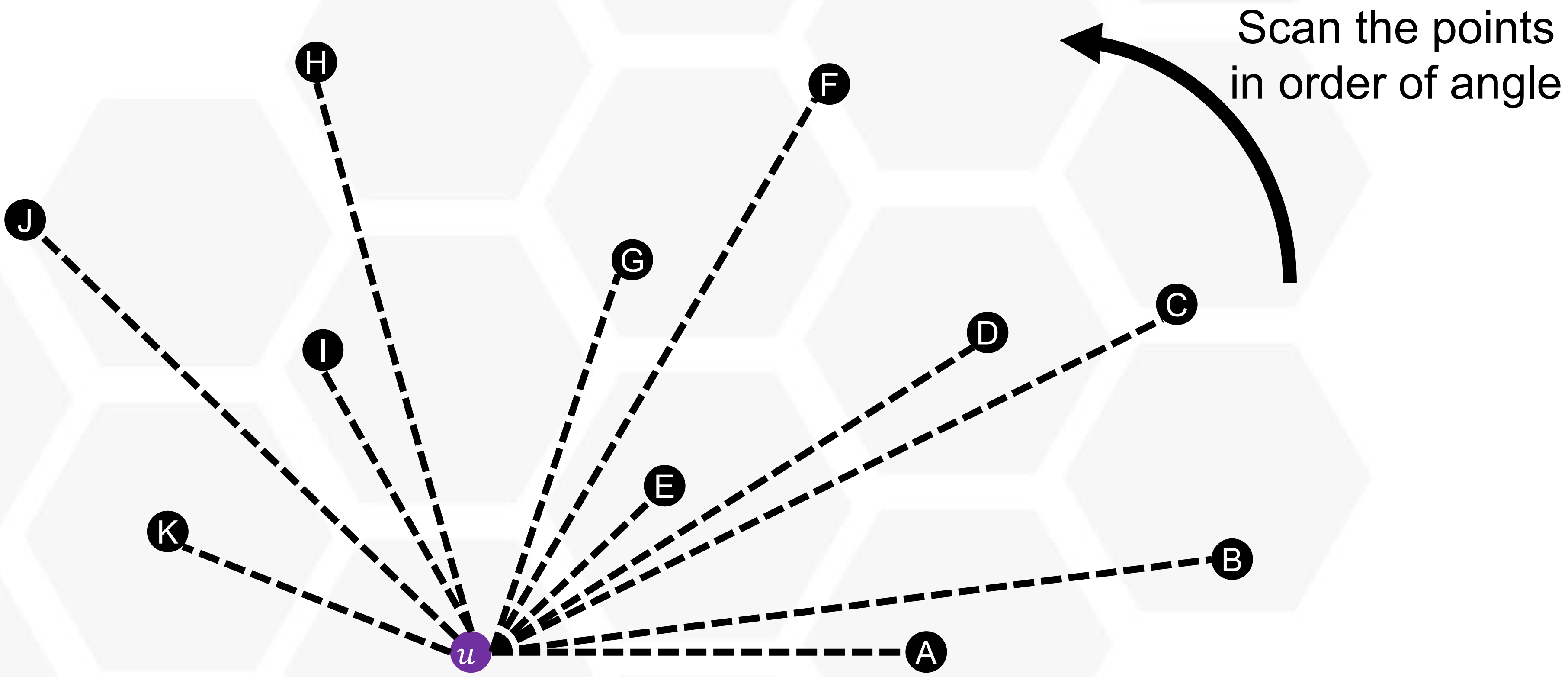




Consider the (polar) angle formed between base point u and every other point

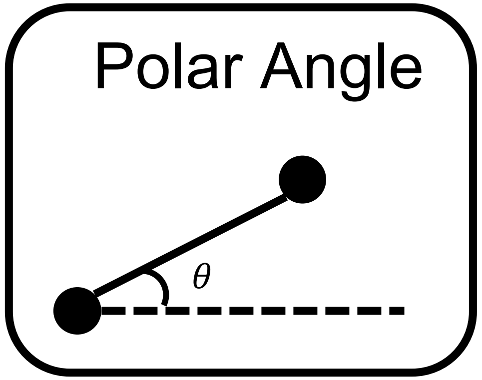
Graham's Algorithm

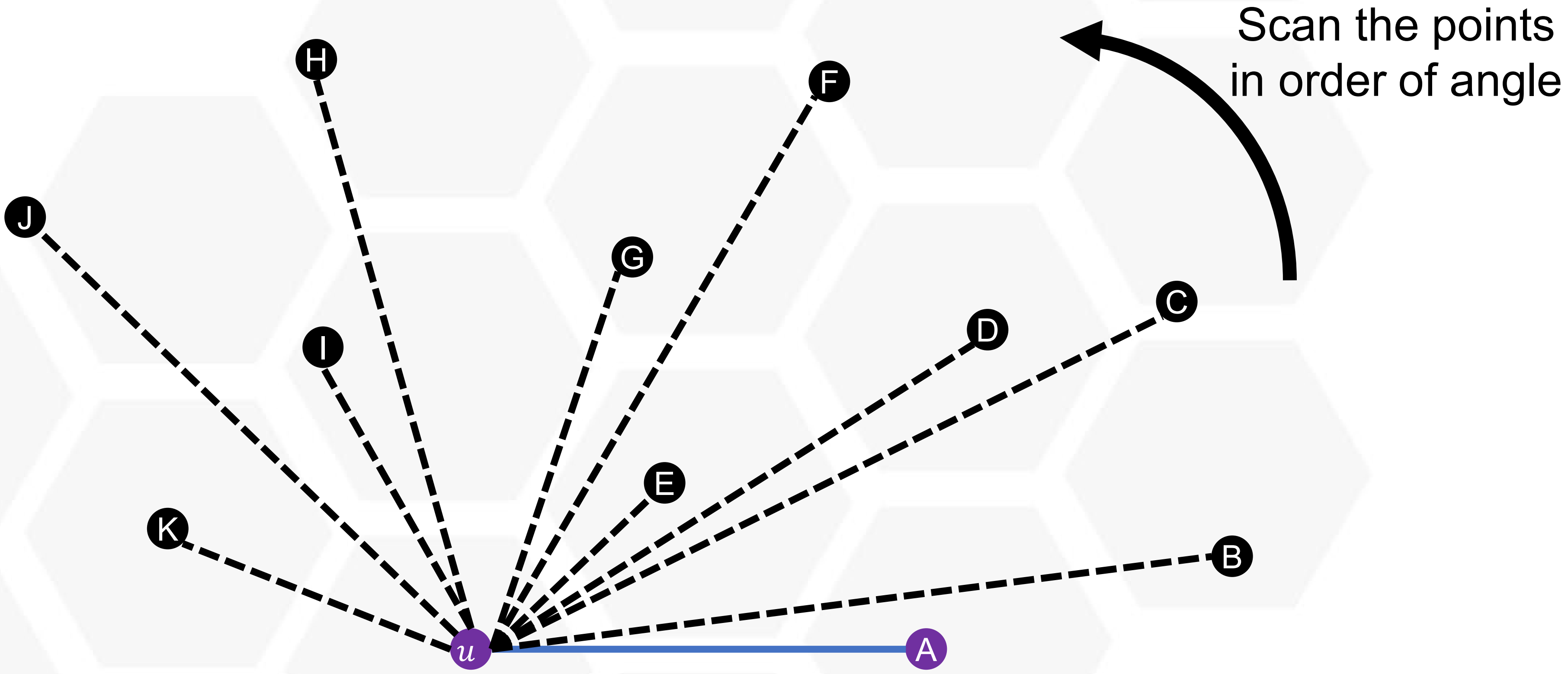




Idea: In order of angle, add points to the convex hull as long as it <u>preserves</u> convexity

Graham's Algorithm





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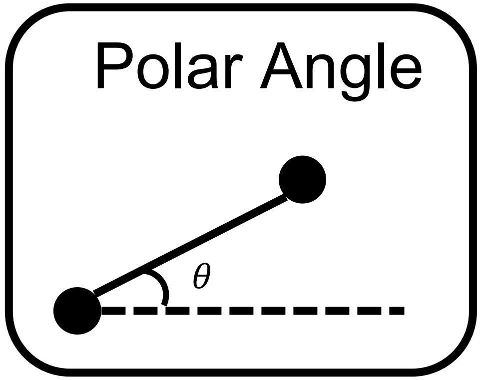

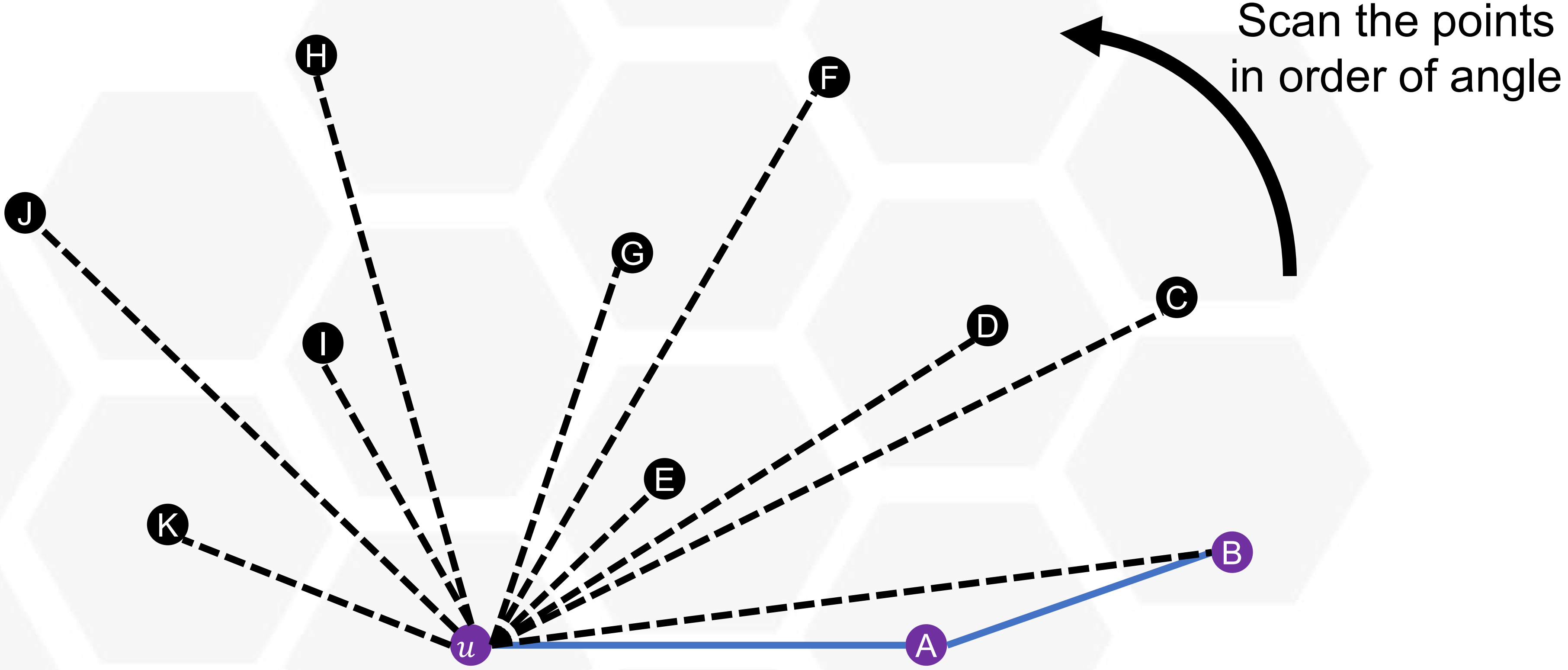

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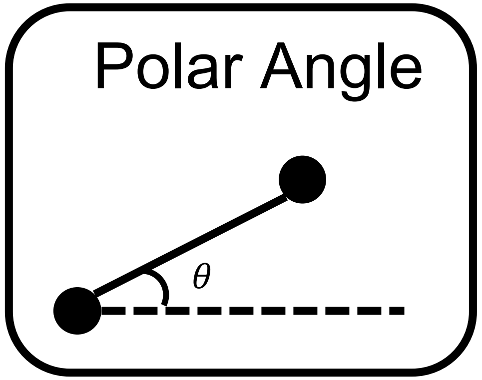

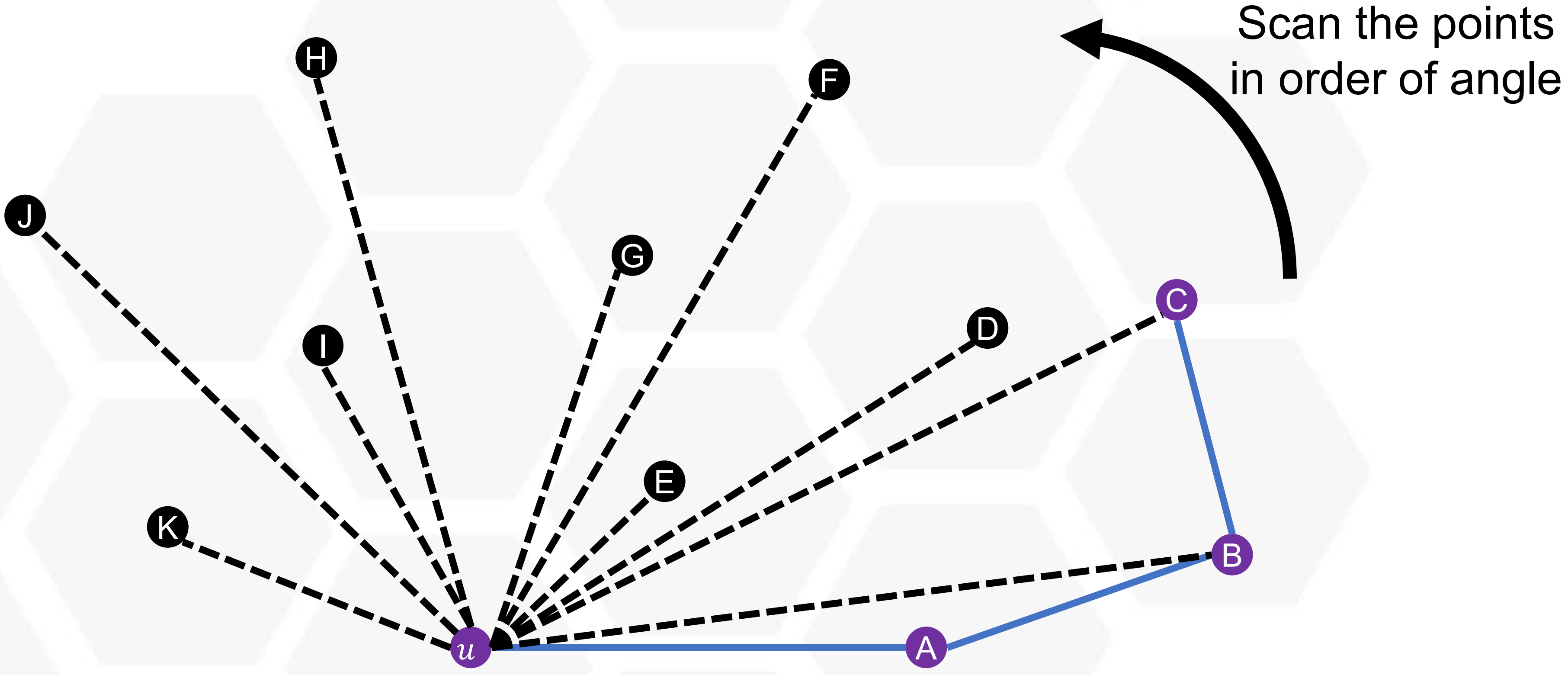

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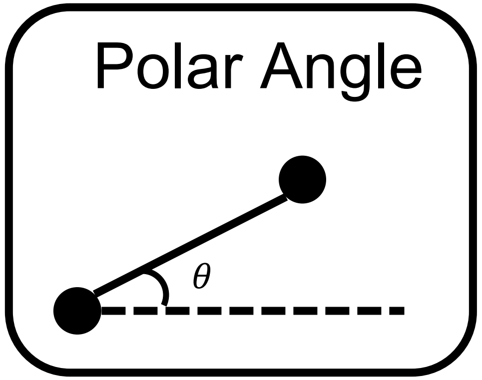

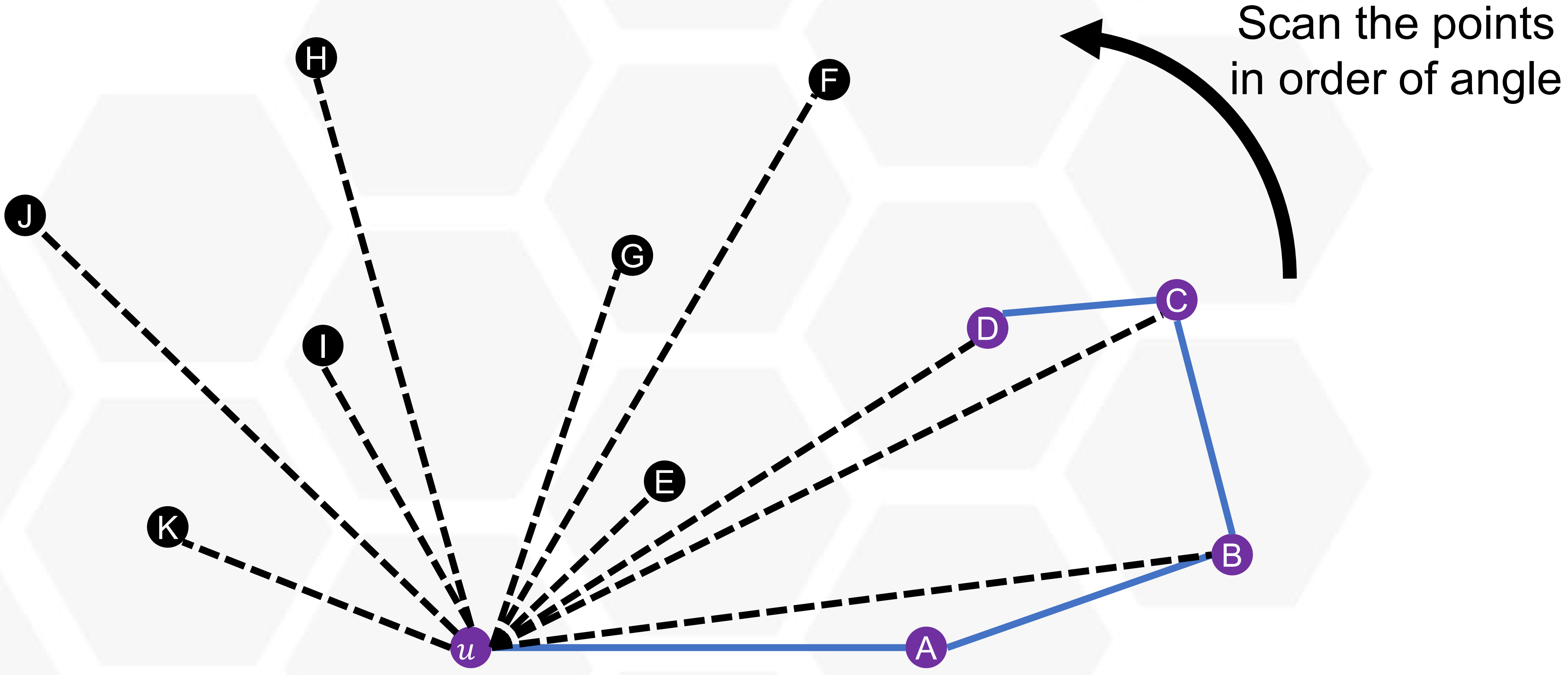

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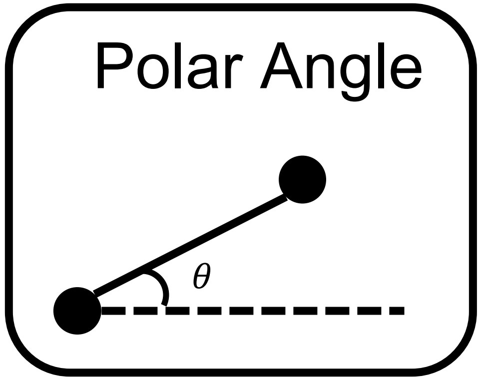

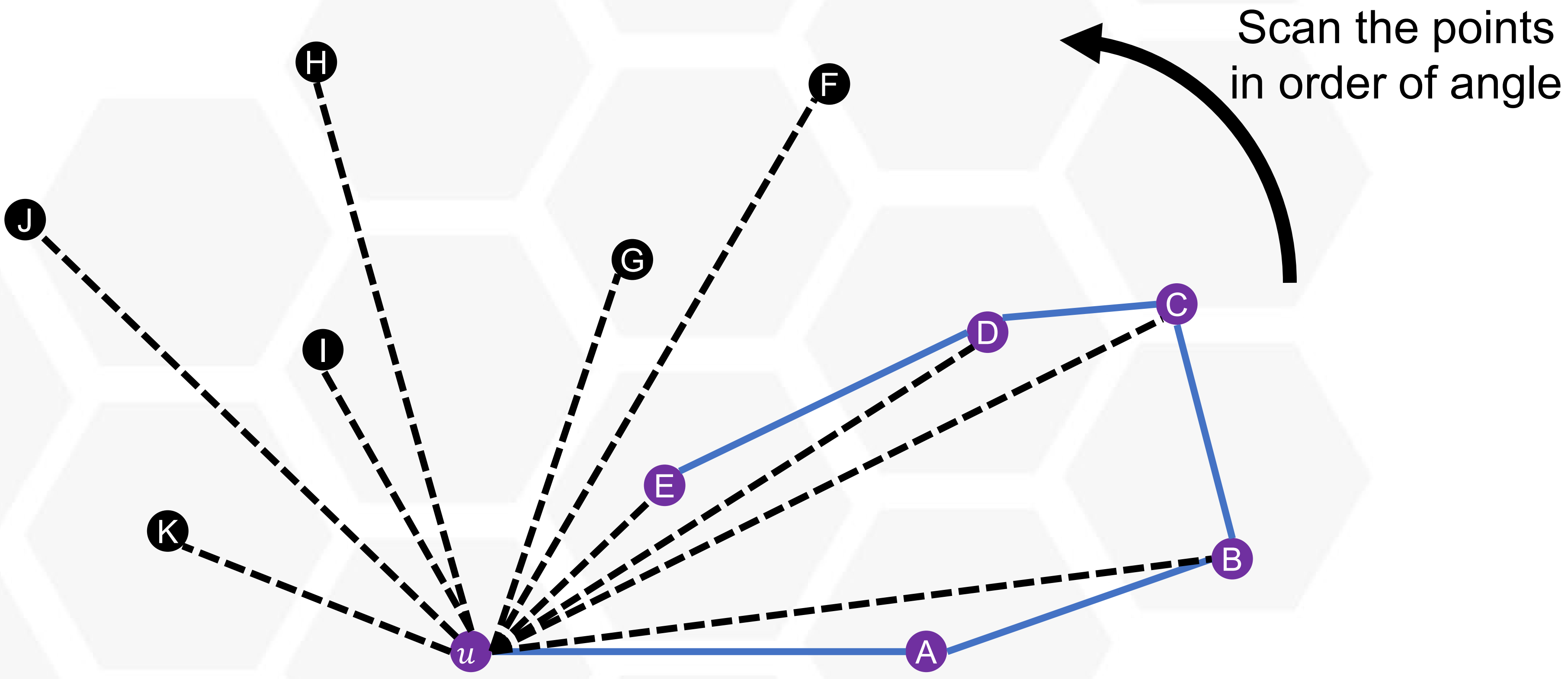

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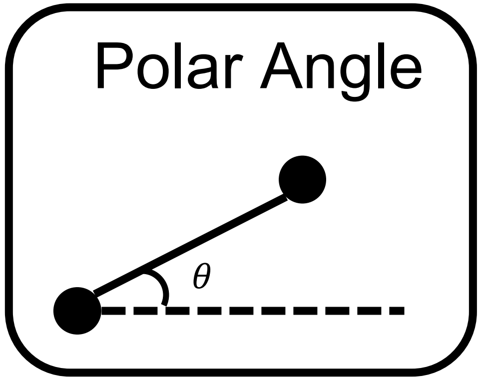
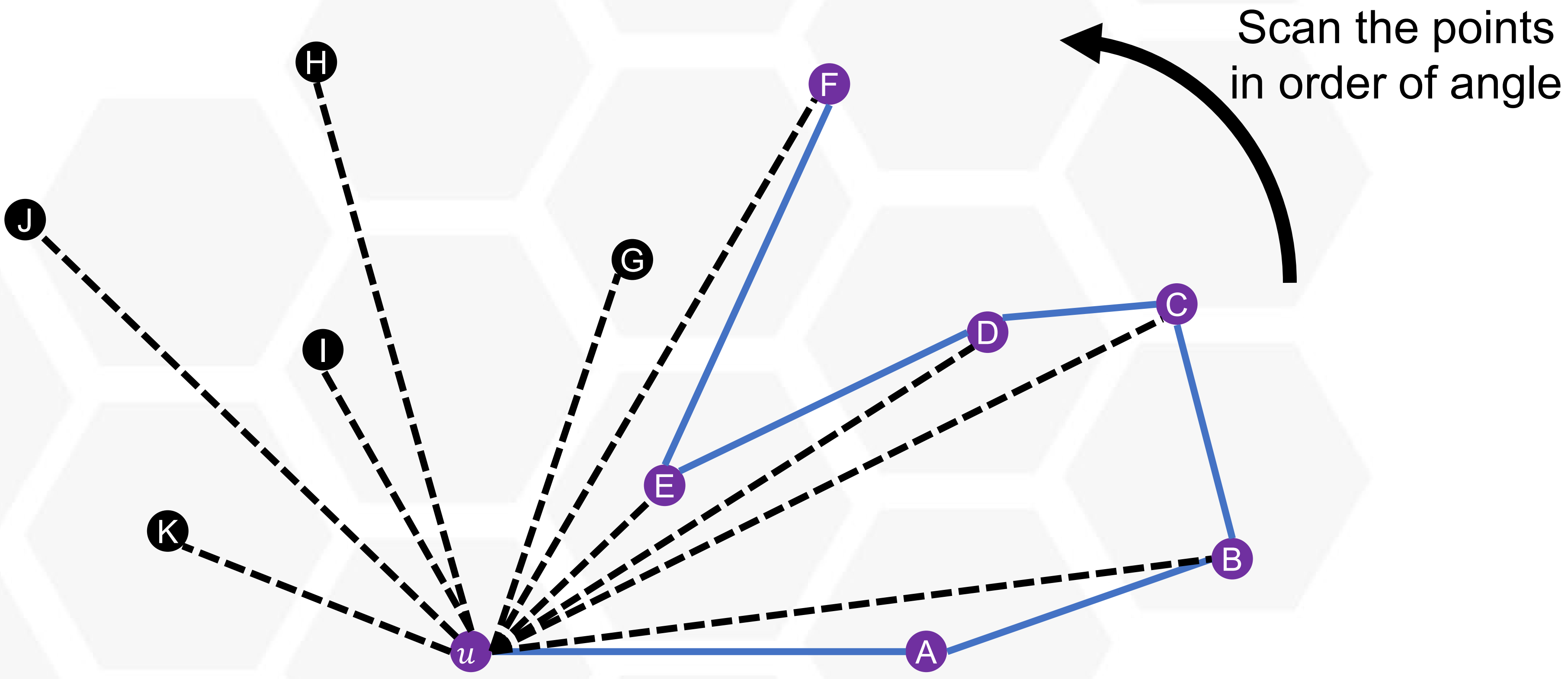
Graham's Algorithm



Not convex anymore!



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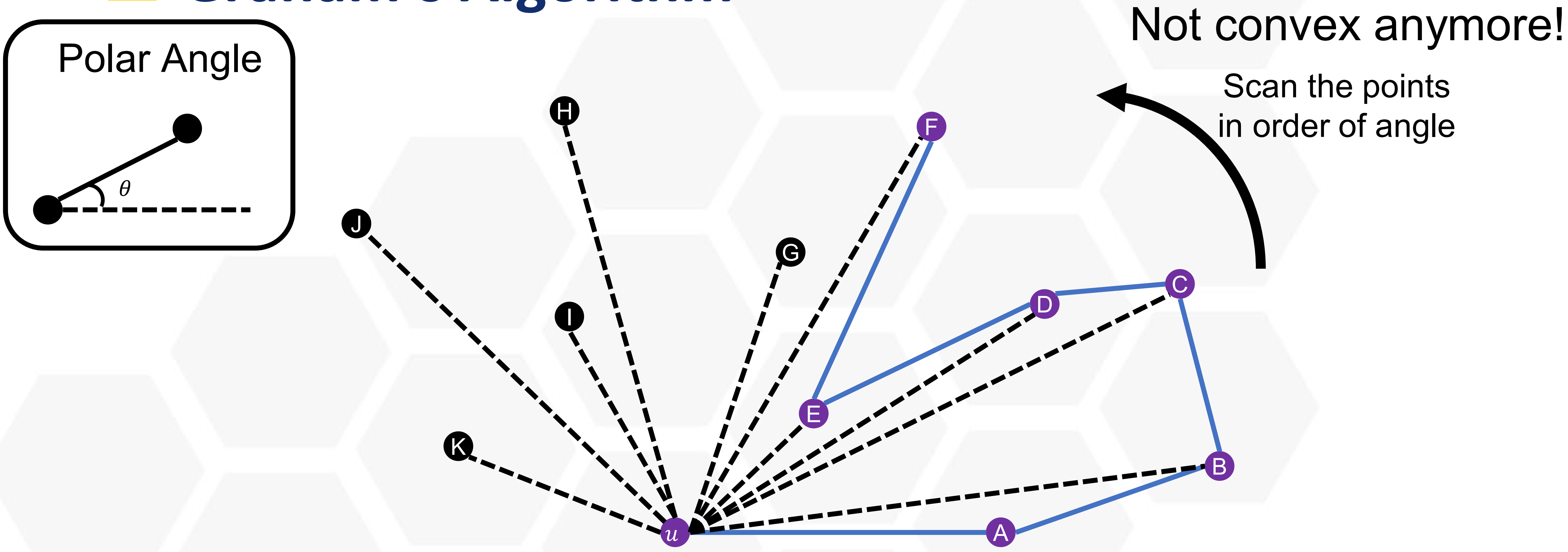

Idea: Try extending the convex hull from the previous vertex if we are unable to extend from the current one

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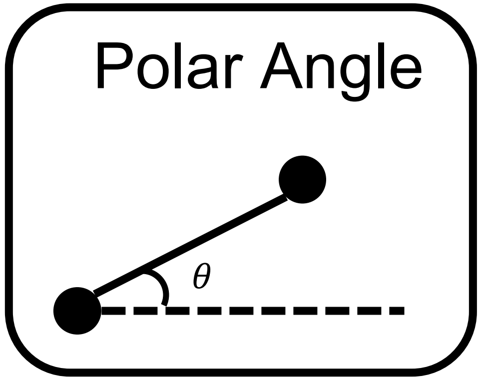
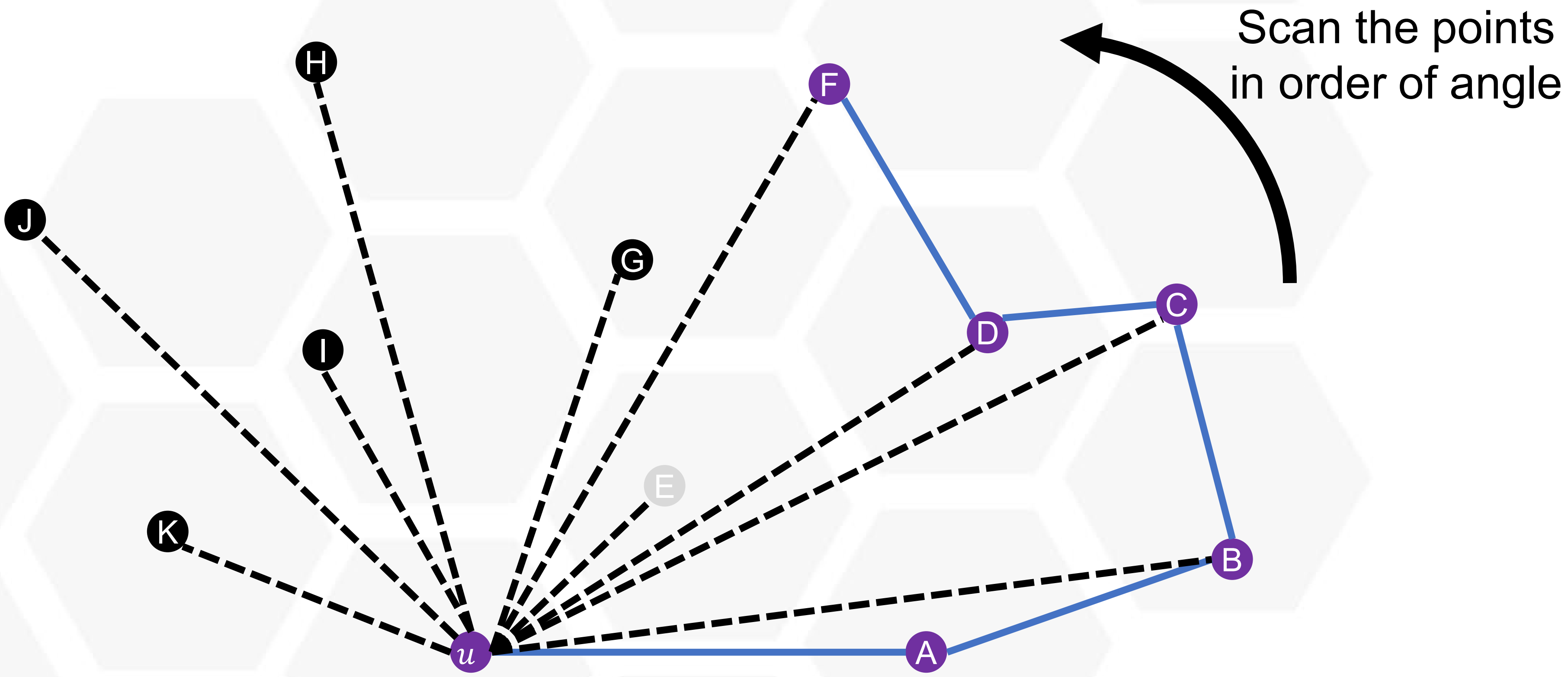
Graham’s Algorithm





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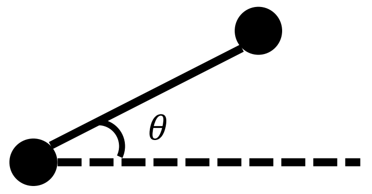
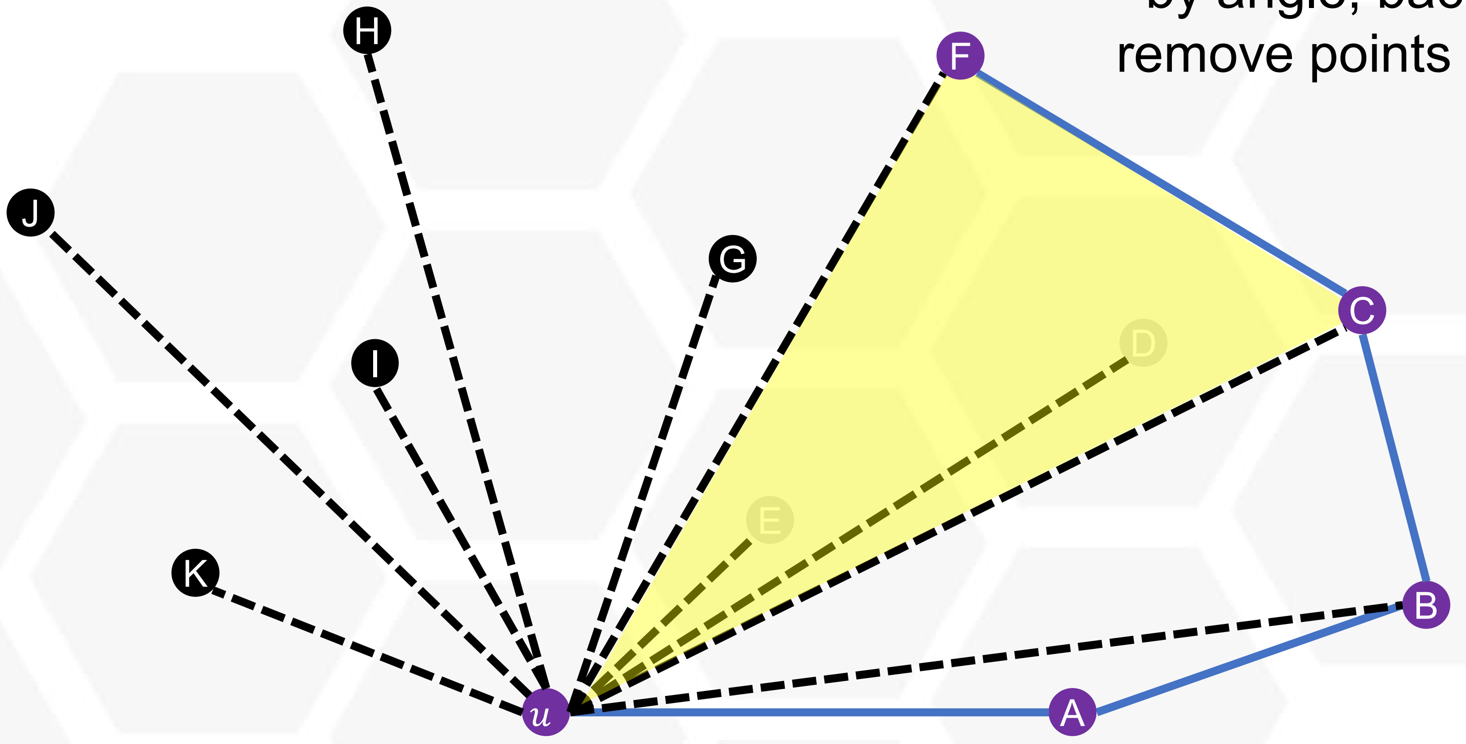
Graham's Algorithm

Polar Angle



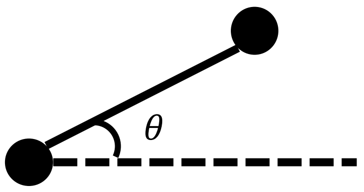


Observe: since points are sorted by angle, backtracking will <u>never</u> remove points from the convex hull

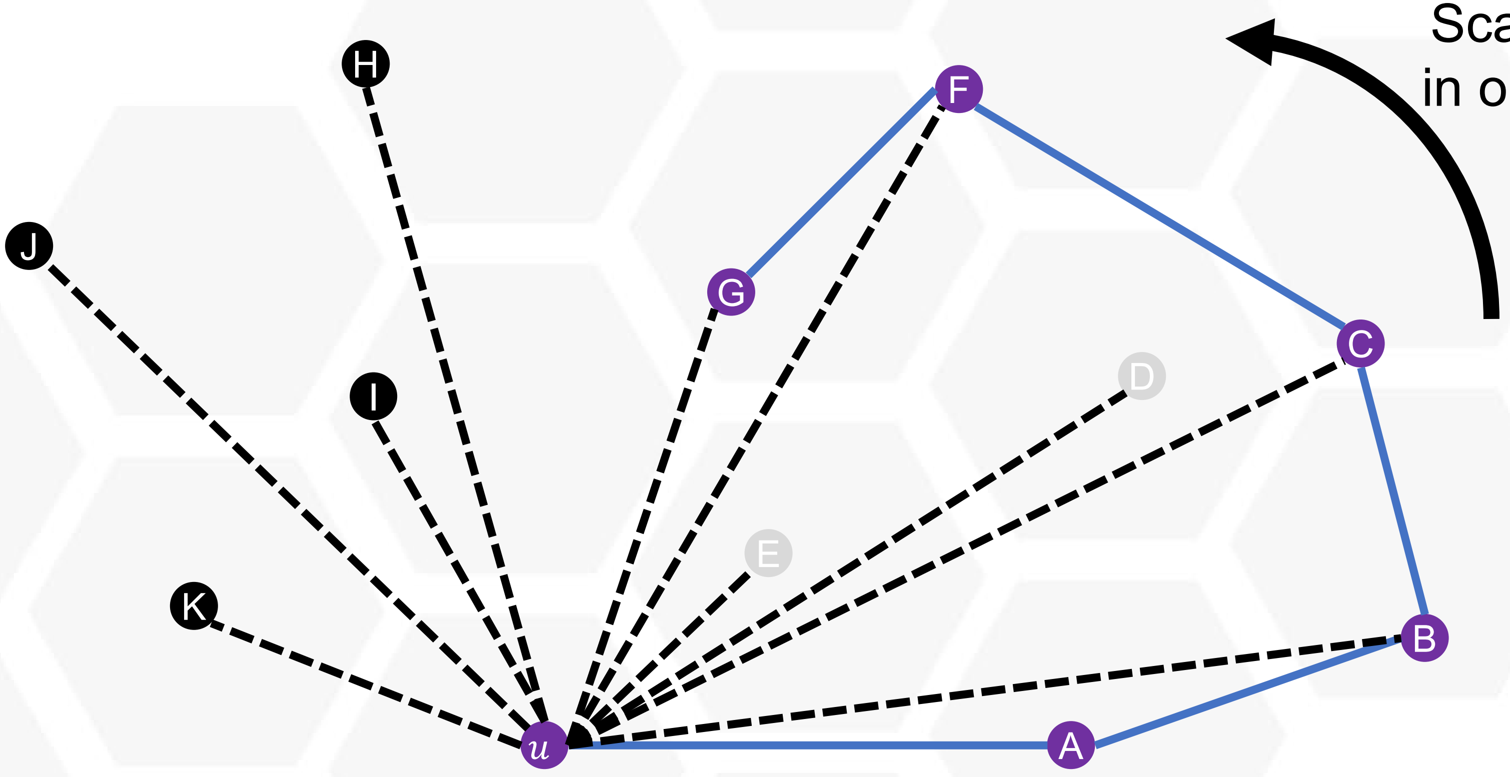
Idea: Try extending the convex hull from the previous vertex if we are unable to extend from the current one

Graham's Algorithm

Polar Angle



Scan the points
in order of angle



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Graham's Algorithm

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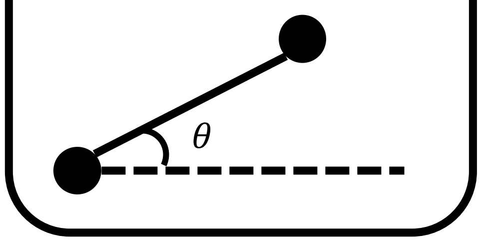

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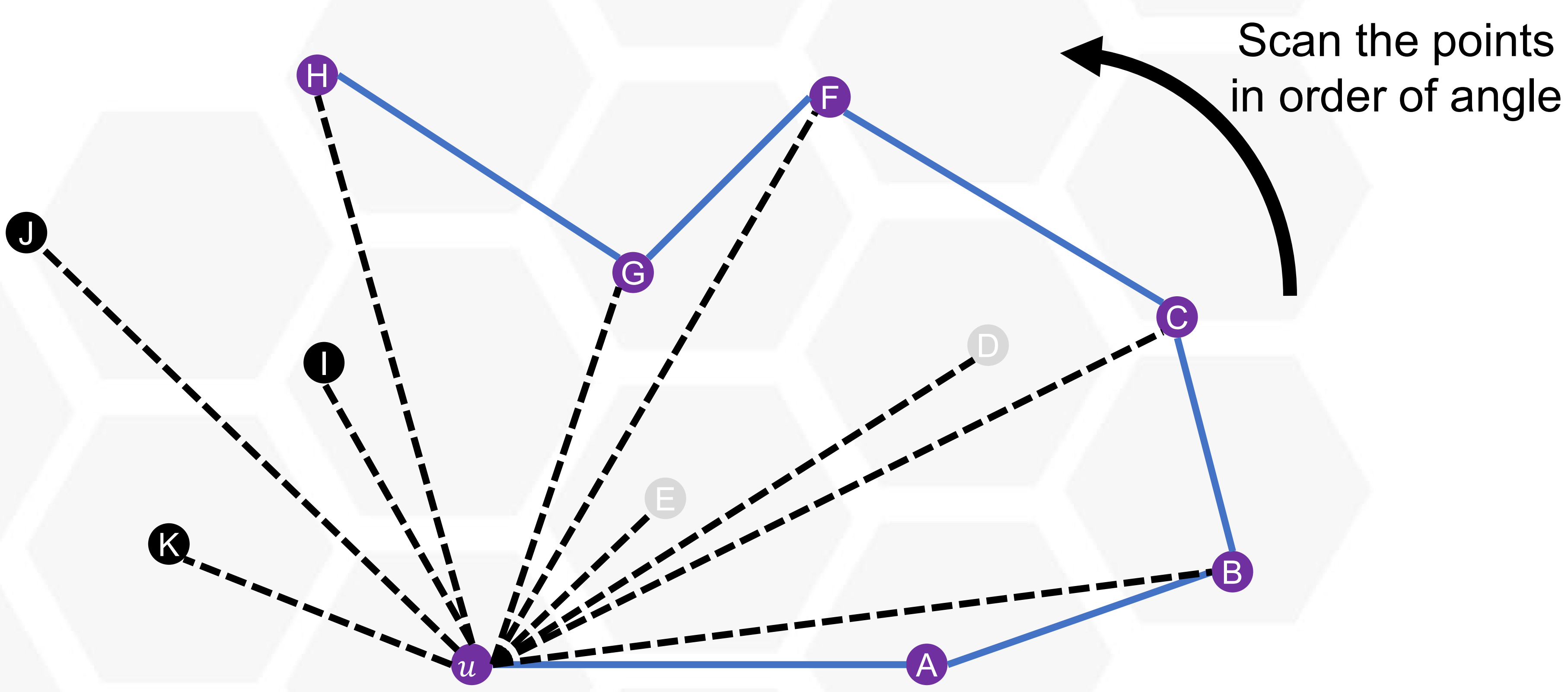

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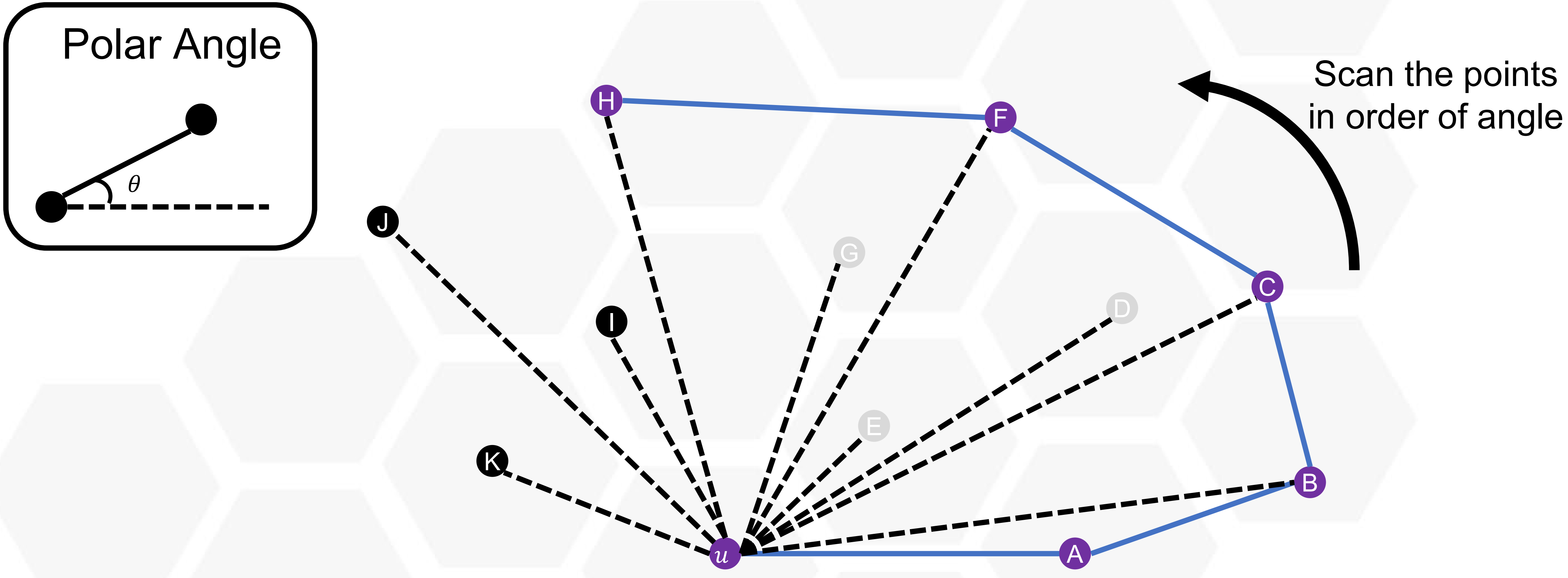
Graham's Algorithm



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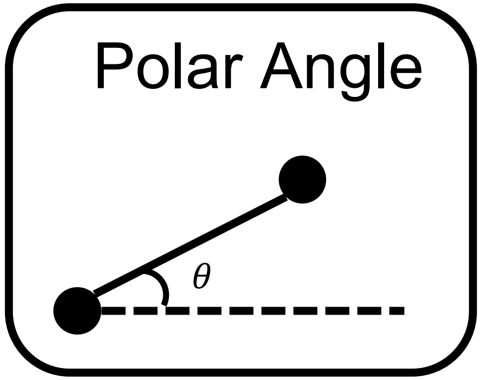

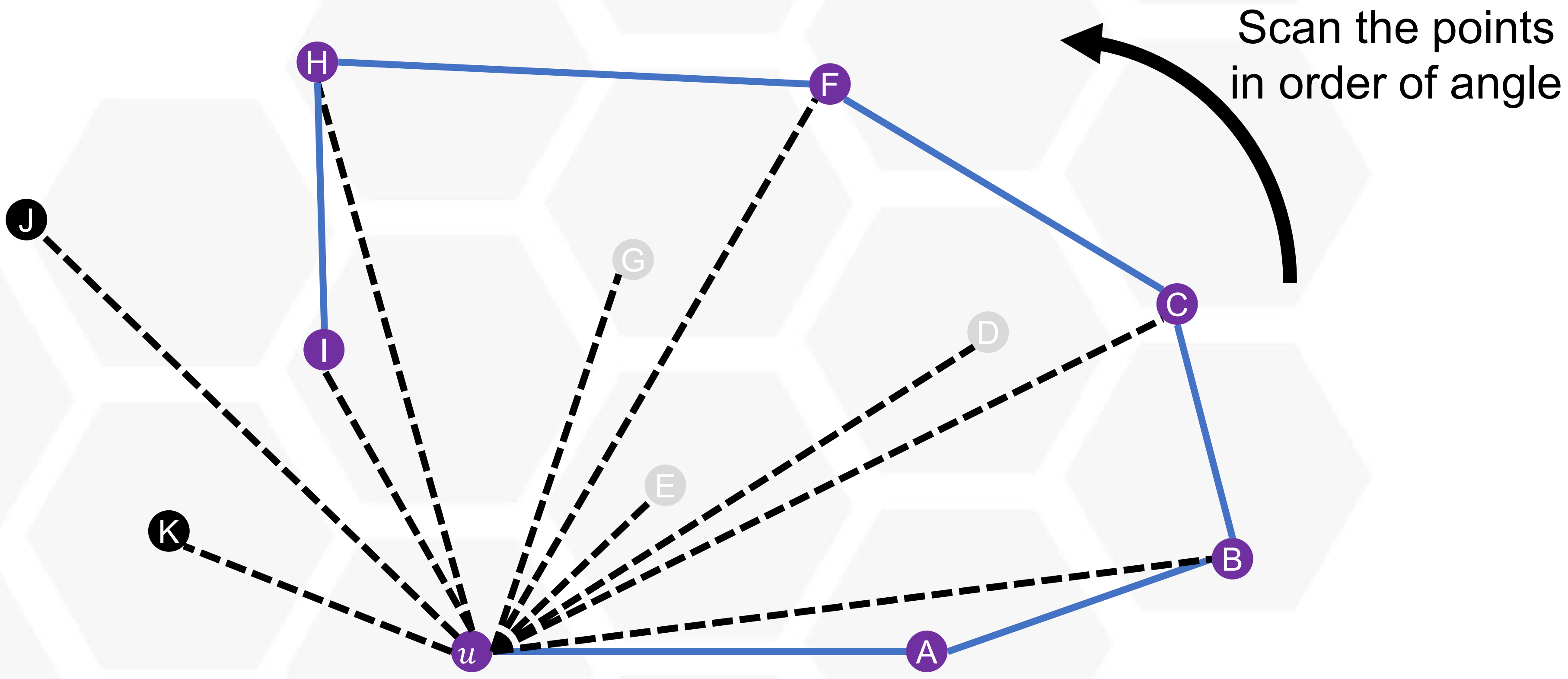

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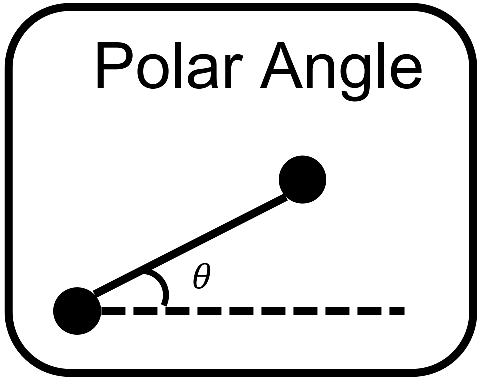

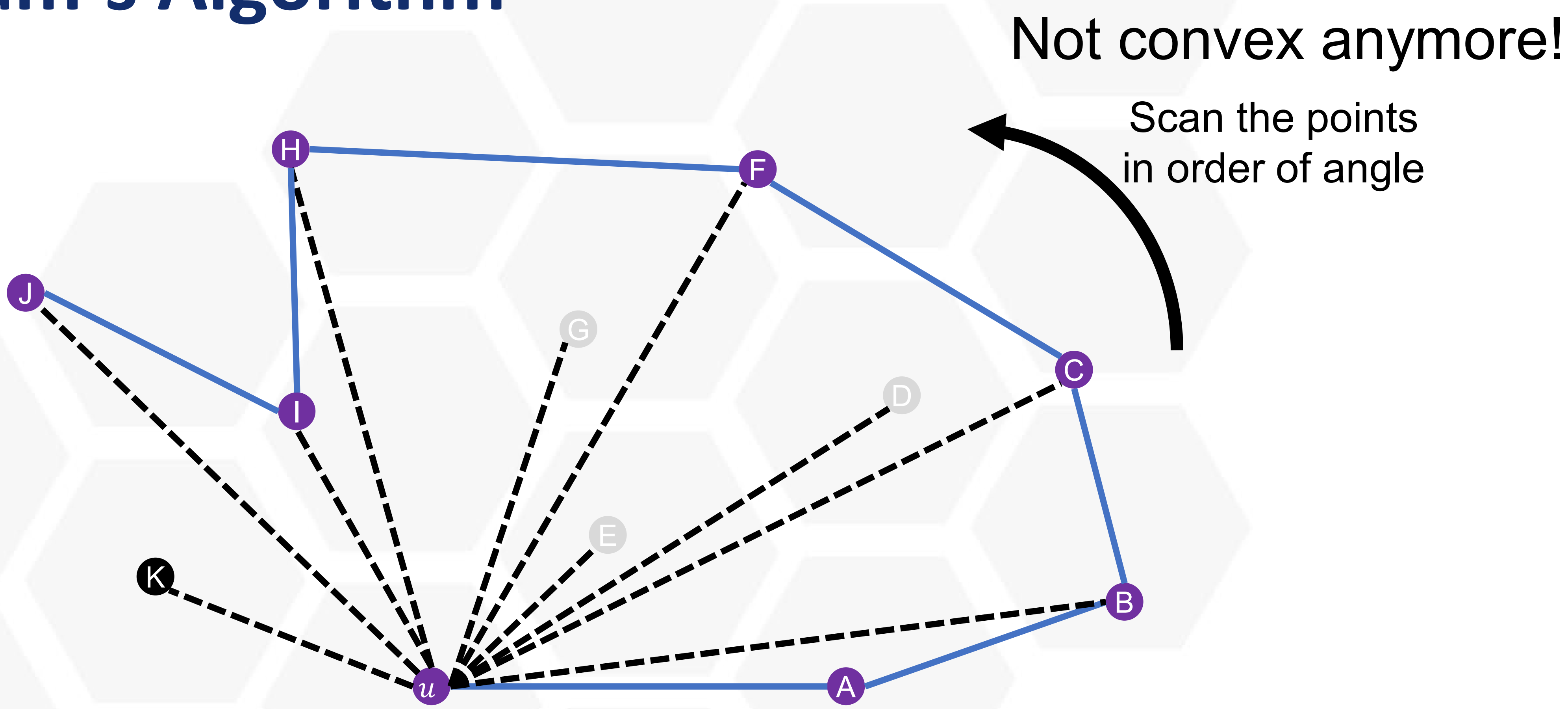

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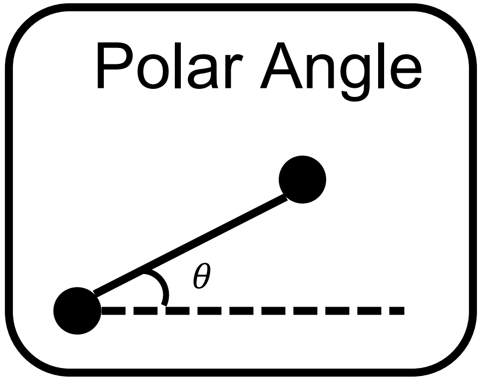
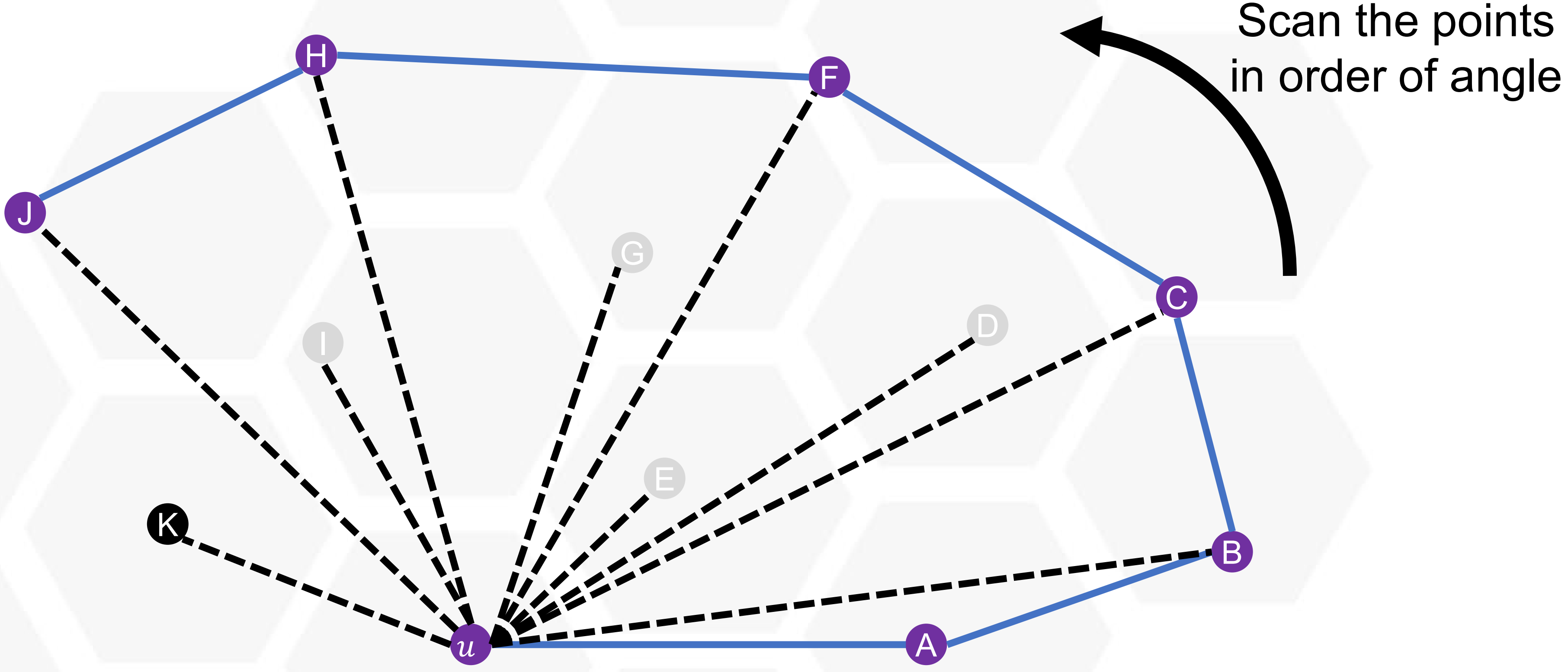
Graham's Algorithm





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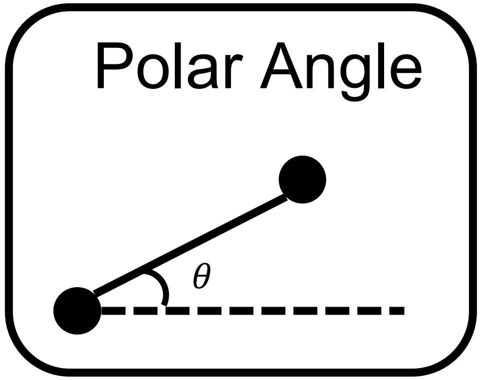

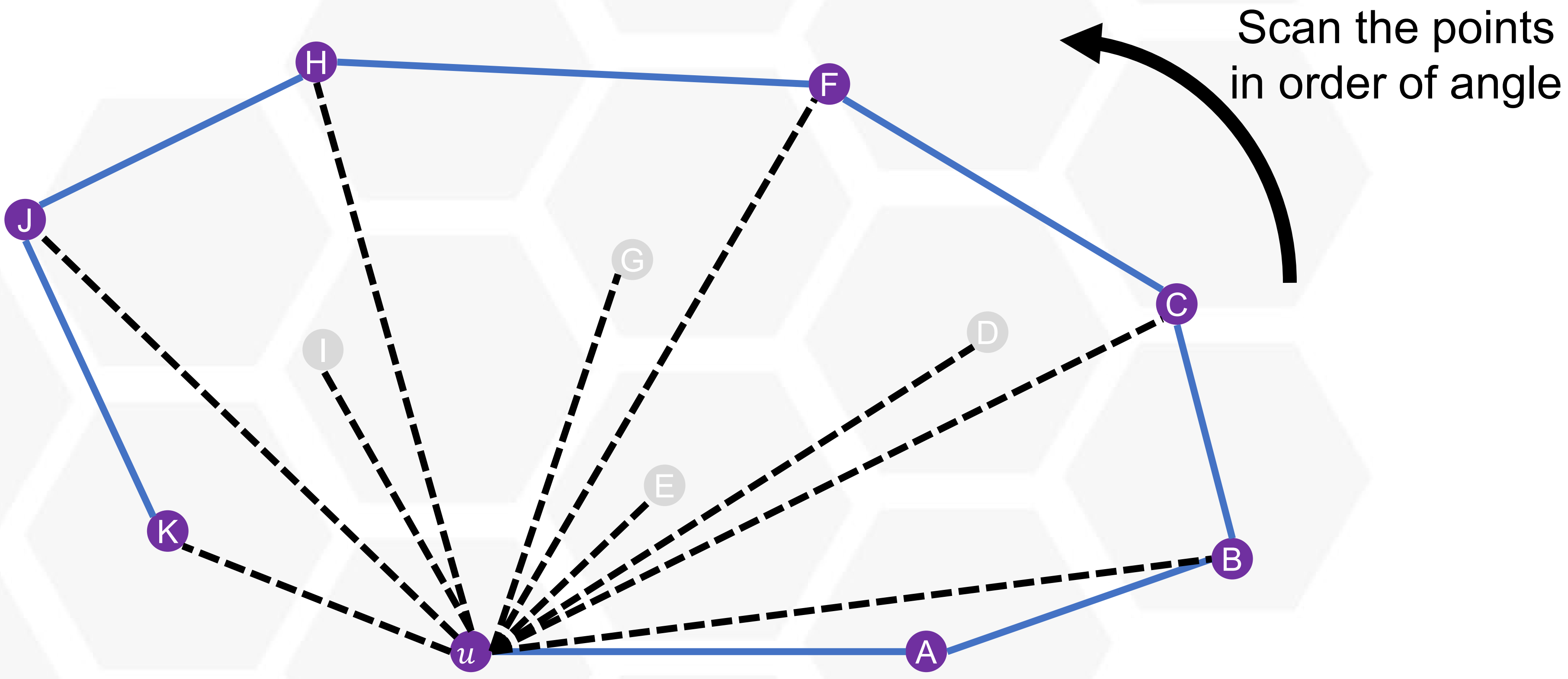

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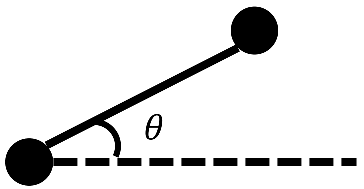
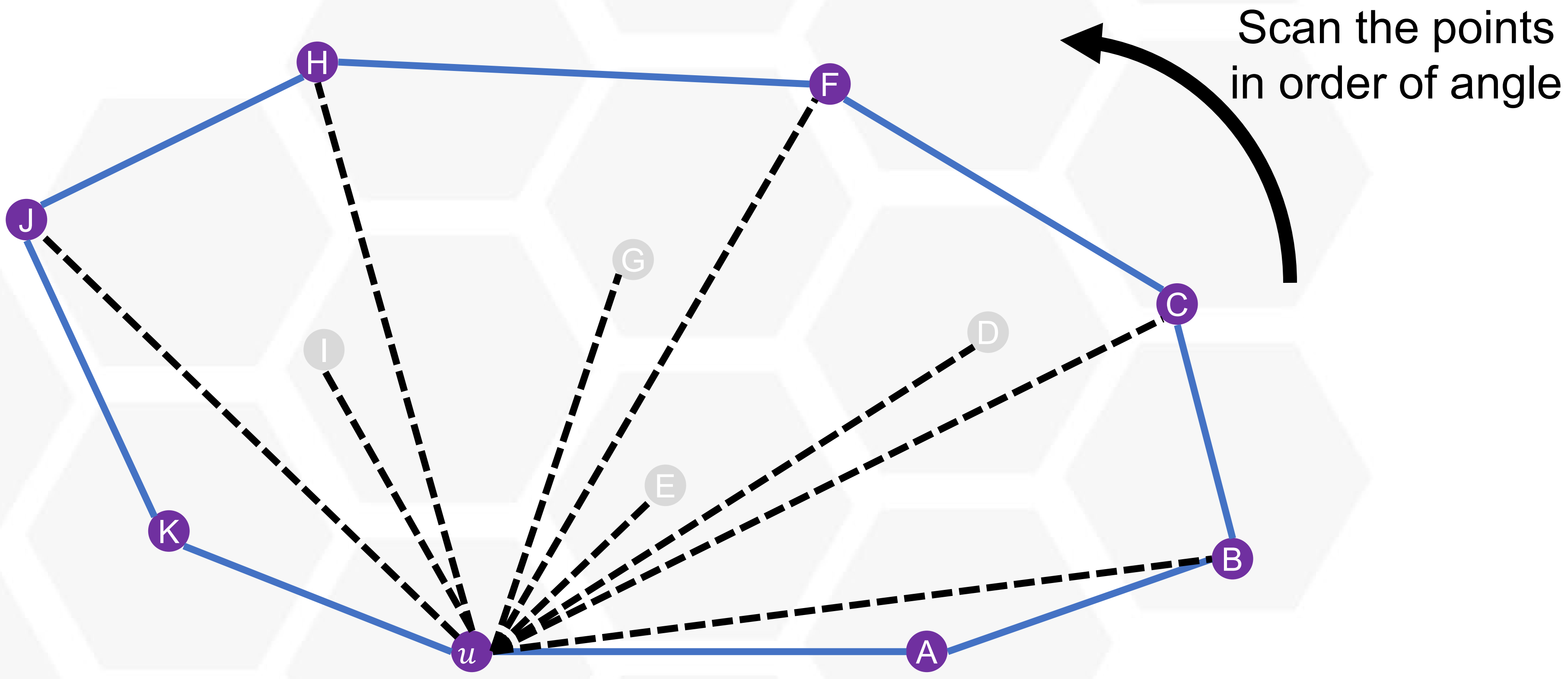
Graham's Algorithm

Polar Angle





Idea: In order of angle, add points to the convex hull as long as it <u>preserves</u> convexity

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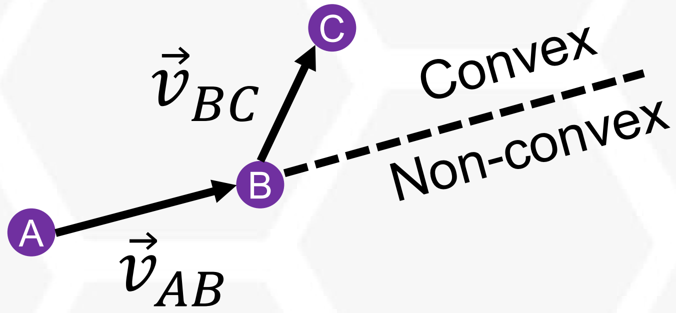
Graham's Algorithm

1. Let p_1 be the point with the smallest y-coordinate (and smallest x-coordinate if multiple points have the same minimum-y coordinate)
2. Add p_1 to the convex hull C (represented as an ordered list)
3. Sort all of the points based on their angle relative to p_1
4. For each of the points p_i in sorted order:
    - Try adding p_i to the convex hull C
    - If adding p_i makes C non-convex, then remove the last component of C and repeat this check

How to implement this?



Imagine driving from $A \rightarrow B$

- $B \rightarrow C$ is convex if need to take a "left turn" to reach C
- $B \rightarrow C$ is non-convex if need to take a "non-left turn"

Decide "left turn" vs. "right turn" by computing the sign of the (vector) cross product between $\vec{v}_{AB}$ and $\vec{v}_{BC}$

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Which data structure to use?

Need to be able to insert elements and remove in order of most-recent insertion

Can implement both operations in constant-time using a stack

Graham's Algorithm

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Correctness?

See Cormen 33.3

Running Time of Graham's Algorithm

1. Let p_1 be the point with the smallest y-coordinate (and smallest x-coordinate if multiple points have the same minimum-y coordinate) — $O(n)$
2. Add p_1 to the convex hull C (represented as **a stack**) — $O(1)$
3. Sort all of the points based on their angle relative to p_1 — $O(n \log n)$
4. For each of the points p_i in sorted order:
   - Try adding p_i to the convex hull C — $O(1)$
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Running time: $O(n \log n)$

Graham's Algorithm

1. Let p_1 be the point with the smallest y-coordinate (and smallest x-coordinate if multiple points have the same minimum-y coordinate) $O(n)$
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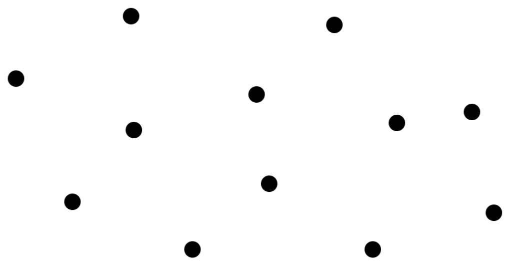
4. For each of the points p_i in sorted order:
   - Try adding p_i to the convex hull C $O(1)$
   - If adding p_i makes C non-convex, then remove the last component of C and repeat this check

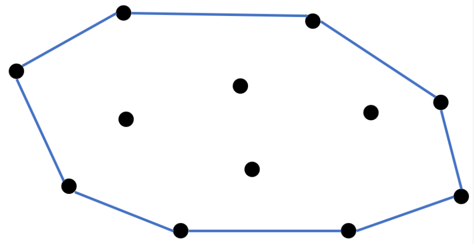
We have essentially reduced the problem of computing a convex hull to the problem of sorting!

Convex Hull to Sorting Reduction

convex hull



convex hull



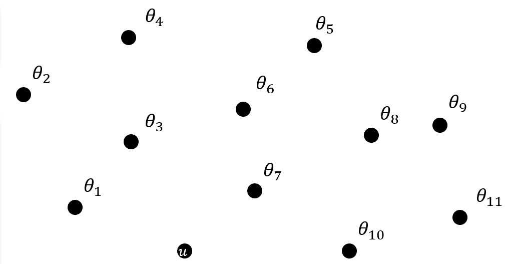
Map instances of problem $\boldsymbol{A}$ to instances of $\boldsymbol{B}$

$O(n)$

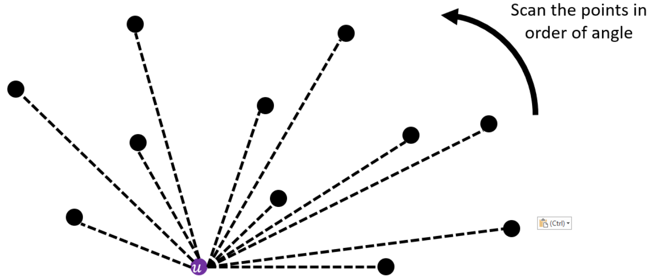
Map solutions of problem $\boldsymbol{B}$ to solutions of $\boldsymbol{A}$

$O(n)$

sorting



points sorted by angle



convex hull $\leq$ sorting

convex hull can be reduced to sorting in $O(n)$ time

Running Time of Graham's Algorithm

1. Let p_1 be the point with the smallest y-coordinate (and smallest x-coordinate if multiple points have the same minimum-y coordinate) $O(n)$
2. Add p_1 to the convex hull C (represented as a stack) $O(1)$
3. Sort all of the points based on their angle relative to p_1 $O(n \log n)$
4. For each of the points p_i in sorted order:
   - Try adding p_i to the convex hull C $O(1)$
   - If adding p_i makes C non-convex, then remove the last component of C and repeat this check

$O(n \log n)$

Running time of Graham's algorithm: same as best sorting algorithm

Can we do better (without going through sorting)?

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Running Time of Graham's Algorithm

1. Let p_1 be the point with the smallest y-coordinate (and smallest x-coordinate if multiple points have the same minimum-y coordinate) $O(n)$
2. Add p_1 to the convex hull C (represented as a stack) $O(1)$
3. Sort all of the points based on their angle relative to p_1 $O(n \log n)$
4. For each of the points p_i in sorted order:
   - Try adding p_i to the convex hull C $O(1)$
   - If adding p_i makes C non-convex, then remove the last component of C and repeat this check

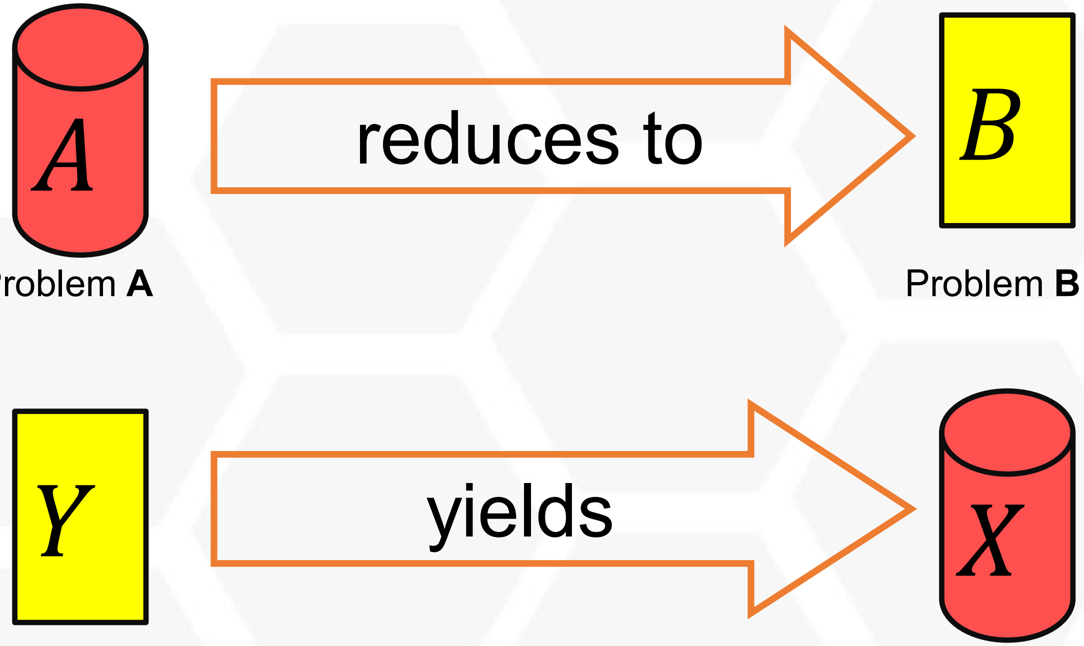
Trivial lower bound: $\Omega(n)$

m: same as best sorting algorithm

Can we do better (without going through sorting)?

Worst-Case Lower Bounds via Reductions

Opening a door



Lighting a fire

Problem **A**

Problem **B**

Algorithm for **B**

Algorithm for **A**

Implication: A is no more difficult than B
(denoted $A \leq B$)

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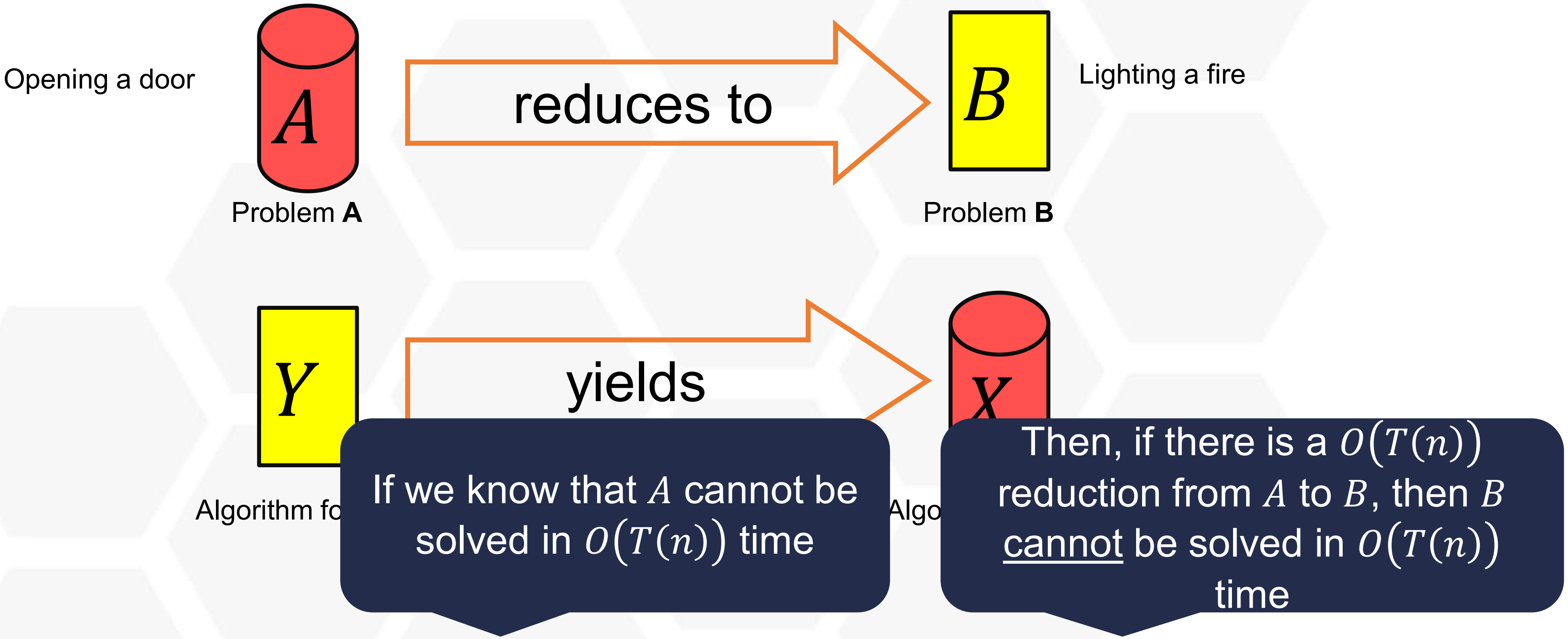
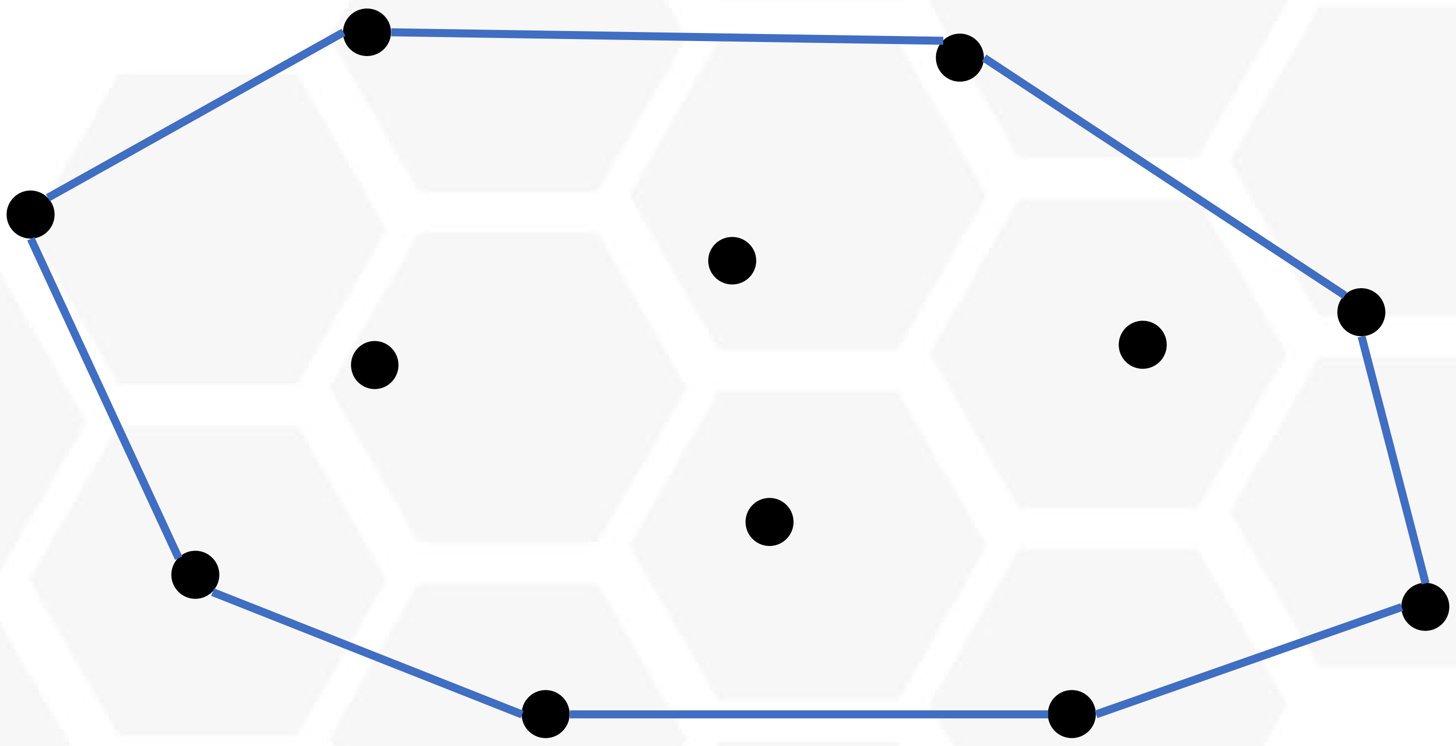
Worst-Case Lower Bounds via Reductions



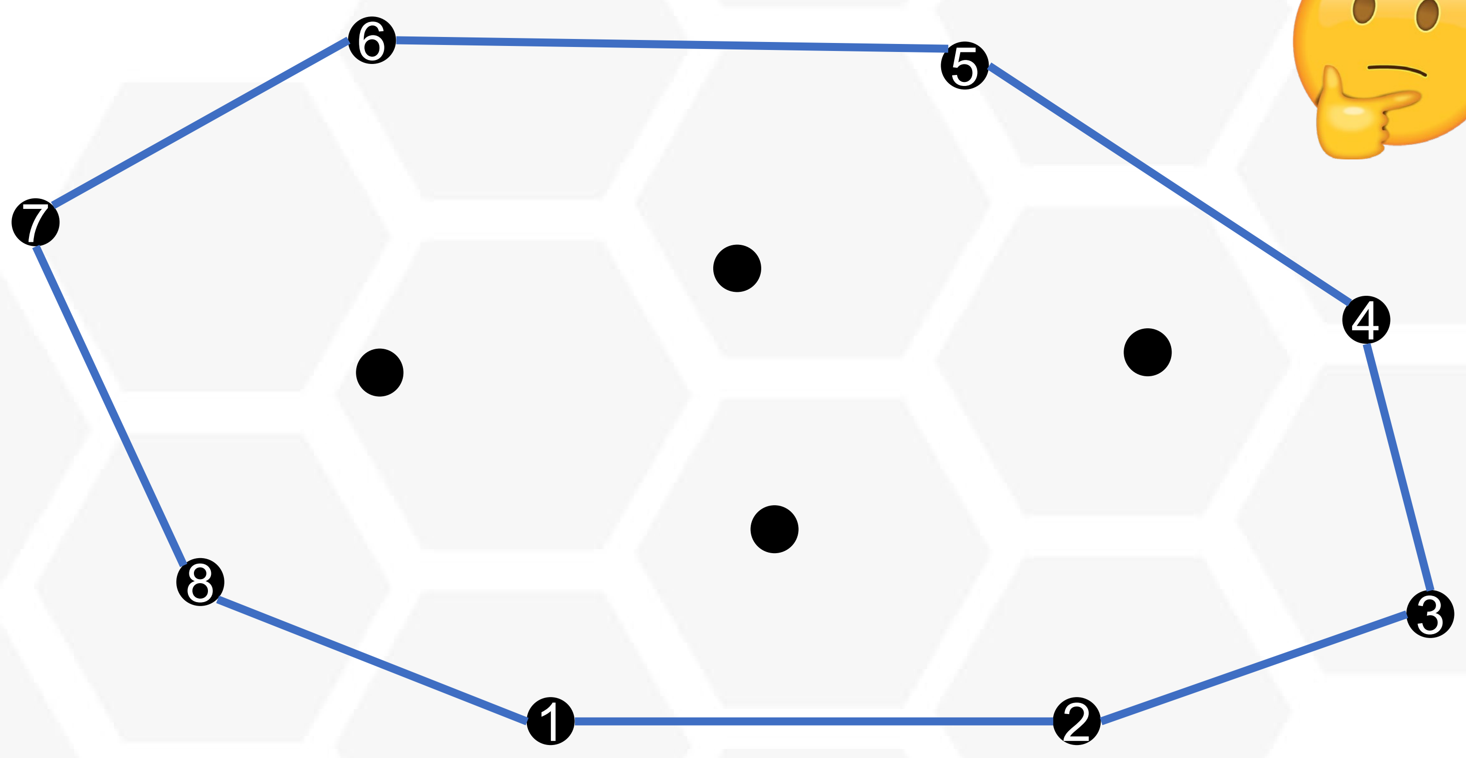
Implication: A is no more difficult than B
(denoted $A \leq B$)

Sorting to Convex Hull Reduction



Observe: convex hull consists of a subset of points in a prescribed ord

Sorting to Convex Hull Reduction



Can we use this to sort?

Observe: convex hull consists of a subset of points in a prescribed ord

Sorting to Convex Hull Reduction

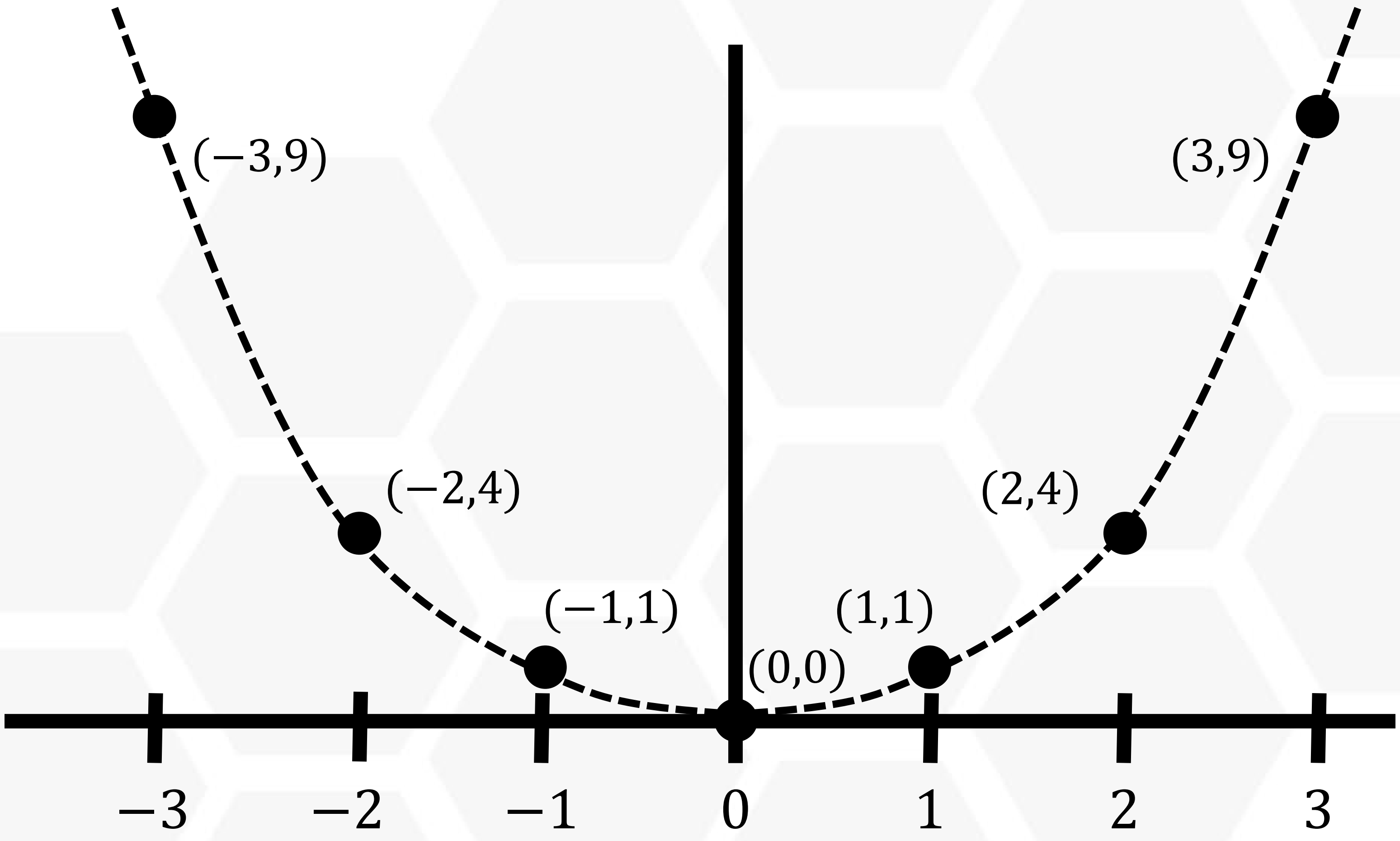
Can we use this to sort?

To get full sorted ordering, convex hull should contain <u>all</u> of the points (i.e., values in the set)

Want order of points in convex hull to be the order of elements in sorted order

Observe: convex hull consists of a subset of points in a prescribed ord

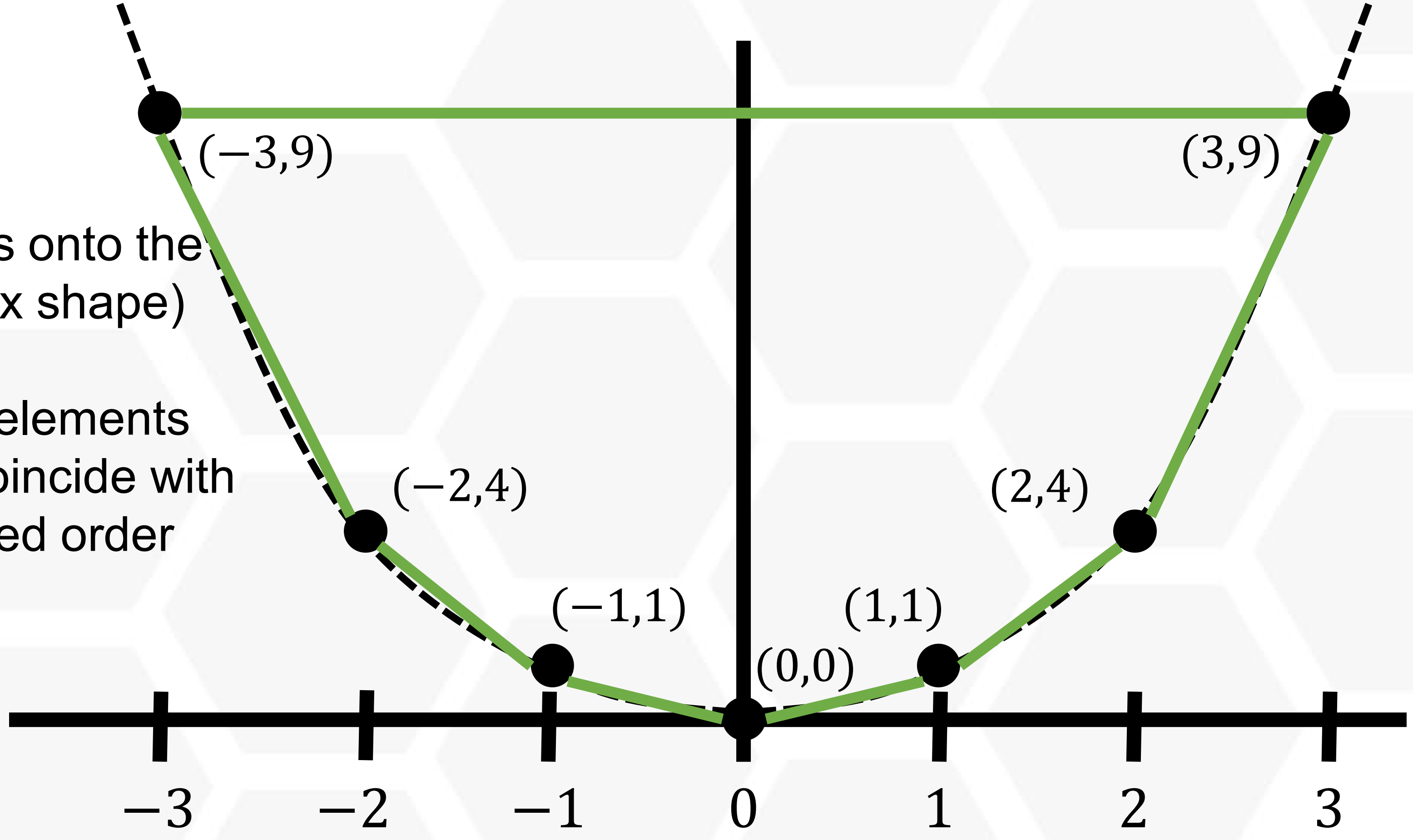
Sorting to Convex Hull Reduction



$x \mapsto (x, x^2)$

Goal: need a way to map list of (numeric) values onto a convex hull instance

Sorting to Convex Hull Reduction



Idea: Map points onto the parabola (convex shape)

Claim: order of elements in convex hull coincide with elements in sorted order

$x \mapsto (x, x^2)$

Goal: need a way to map list of (numeric) values onto a convex hull instance

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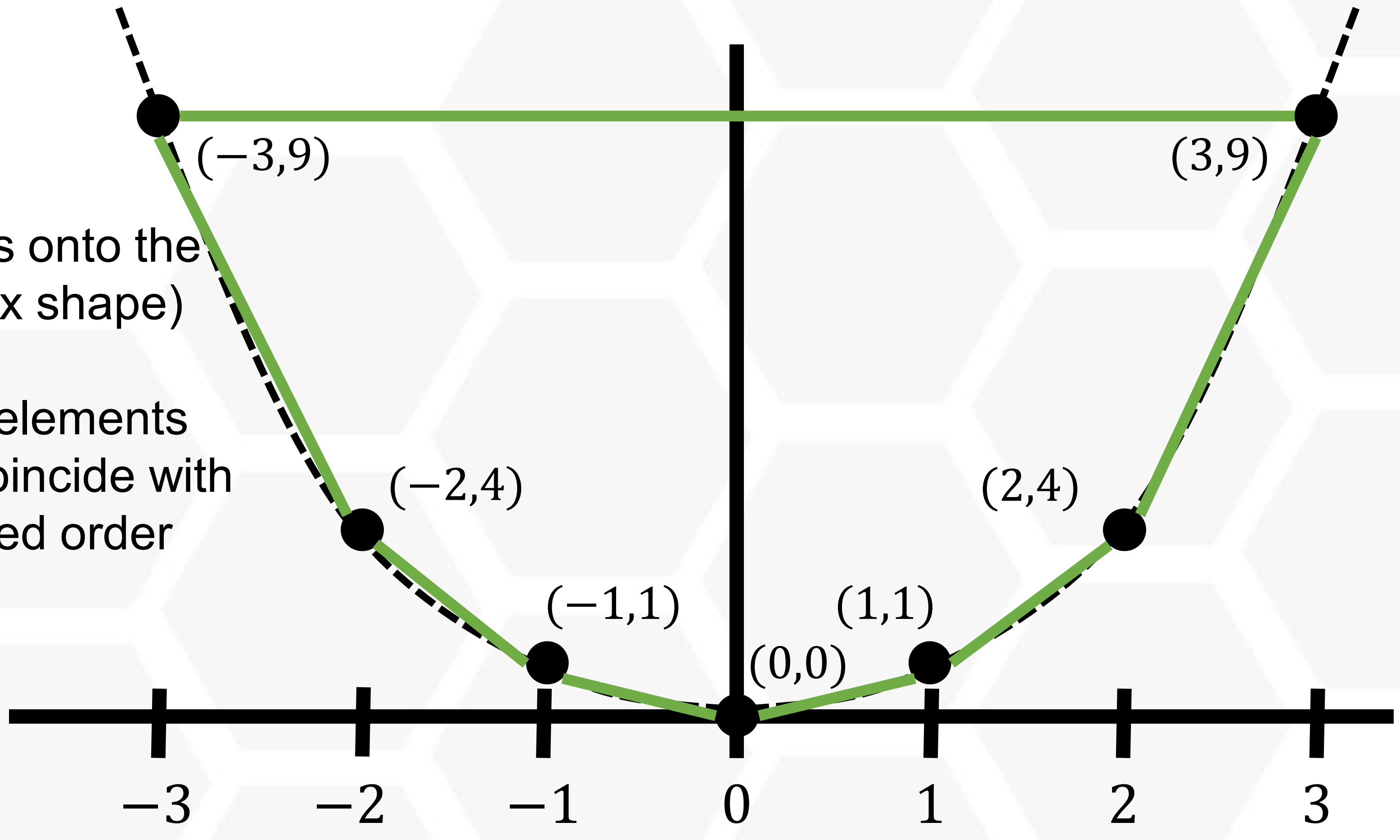
Sorting to Convex Hull Reduction

Idea: Map points onto the parabola (convex shape)

Claim: order of elements in convex hull coincide with elements in sorted order



Conclusion: If we can solve convex hull, then we can sort numeric values

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Convex Hull to Sorting Reduction

sorting

$-2 \quad 1 \quad -3 \quad 0 \quad 2 \quad 3 \quad -1$

Map instances of problem $\boldsymbol{A}$ to instances of $\boldsymbol{B}$

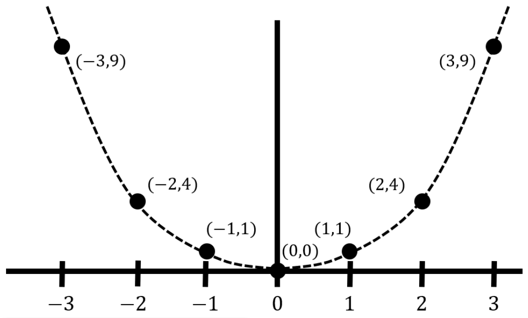
$O(n)$

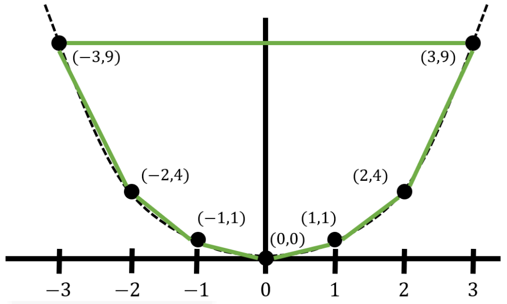
Map solutions of problem $\boldsymbol{B}$ to solutions of $\boldsymbol{A}$

$O(n)$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

convex hull

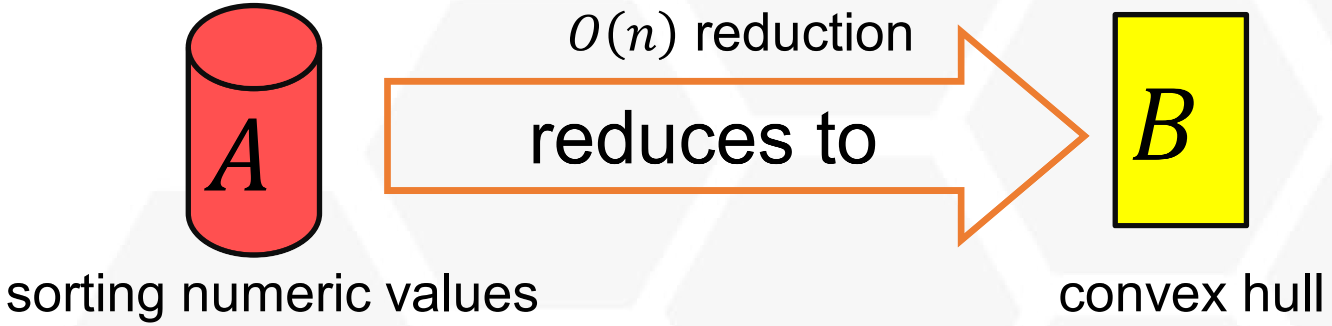




sorting numeric values $\leq$ convex hull

sorting numeric values can be reduced to convex hull in $O(n)$ time

Lower Bound for Convex Hull



Conclusion: a lower bound for sorting translates into one for convex hull

Our lower bound for sorting: $\Omega(n \log n)$ for comparison sorts

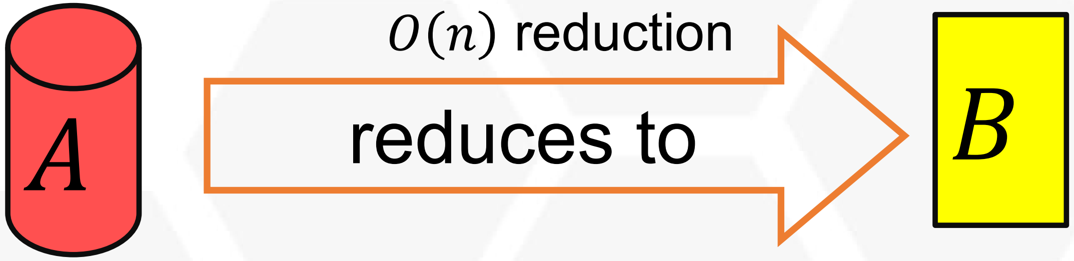
Our reduction is not a comparison sort algorithm, so cannot directly appeal to it

$\Omega(n \log n)$ lower bound for sorting also holds in an "algebraic decision tree model"

(i.e., decisions can be an algebraic function of inputs)

Implies $\Omega(n \log n)$ lower bound for computing convex hull in this model

Lower Bound for Convex Hull



sorting numeric values

convex hull

Conclusion: a lower bound for sorting translates into one for convex hull

Our lower bound for sorting: $\Omega(n \log n)$ for comparison sorts

Our reduction is not a

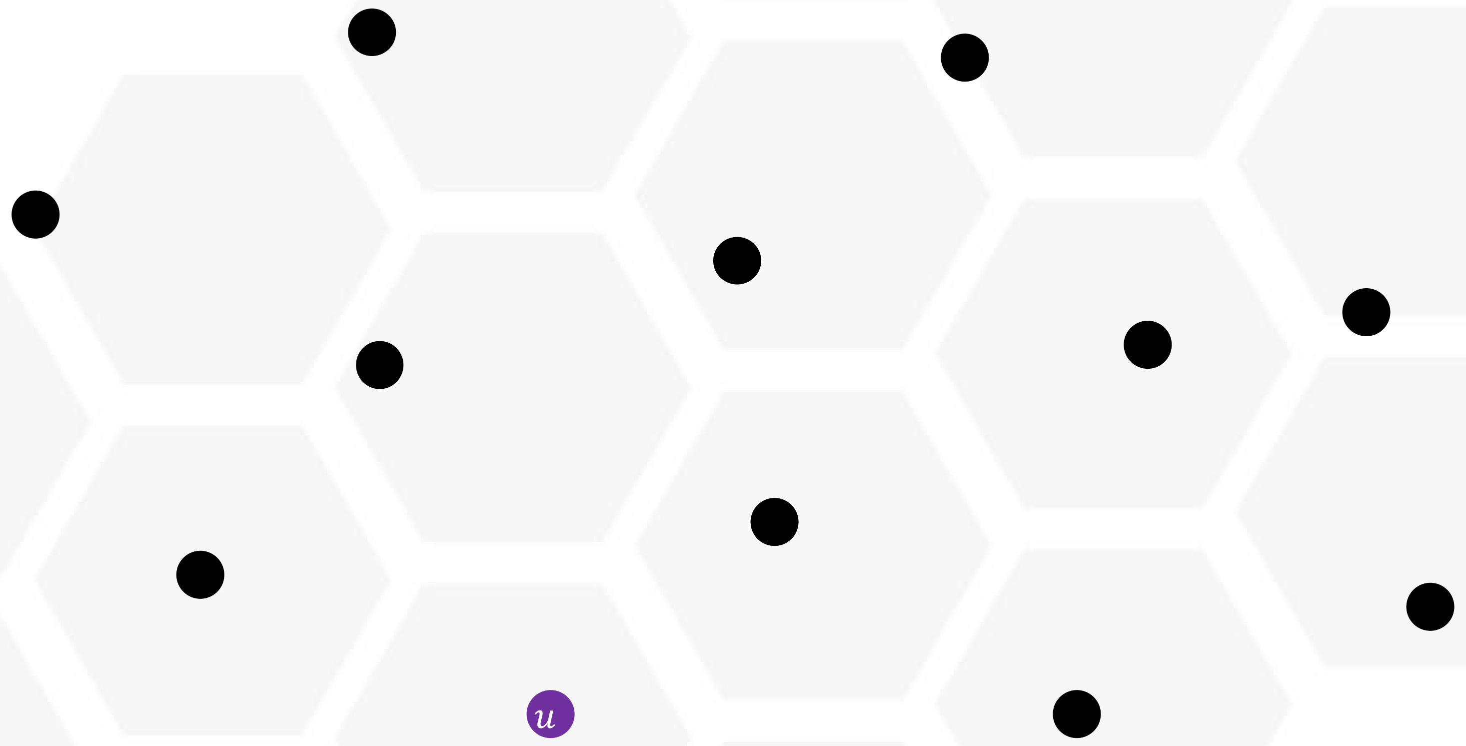
In fact, this lower bound holds even for algorithms that just identify the set of points on the convex hull (and not necessarily their order)!

$\Omega(n \log n)$ lower bound for … n tree model”

(i.e., decisions can b…

Implies $\Omega(n \log n)$ lower bound for computing convex hull in this model

Jarvis' Algorithm (Gift Wrapping Method)



Idea: Start with extremal point and "wrap" points in counter-clockwise fashion

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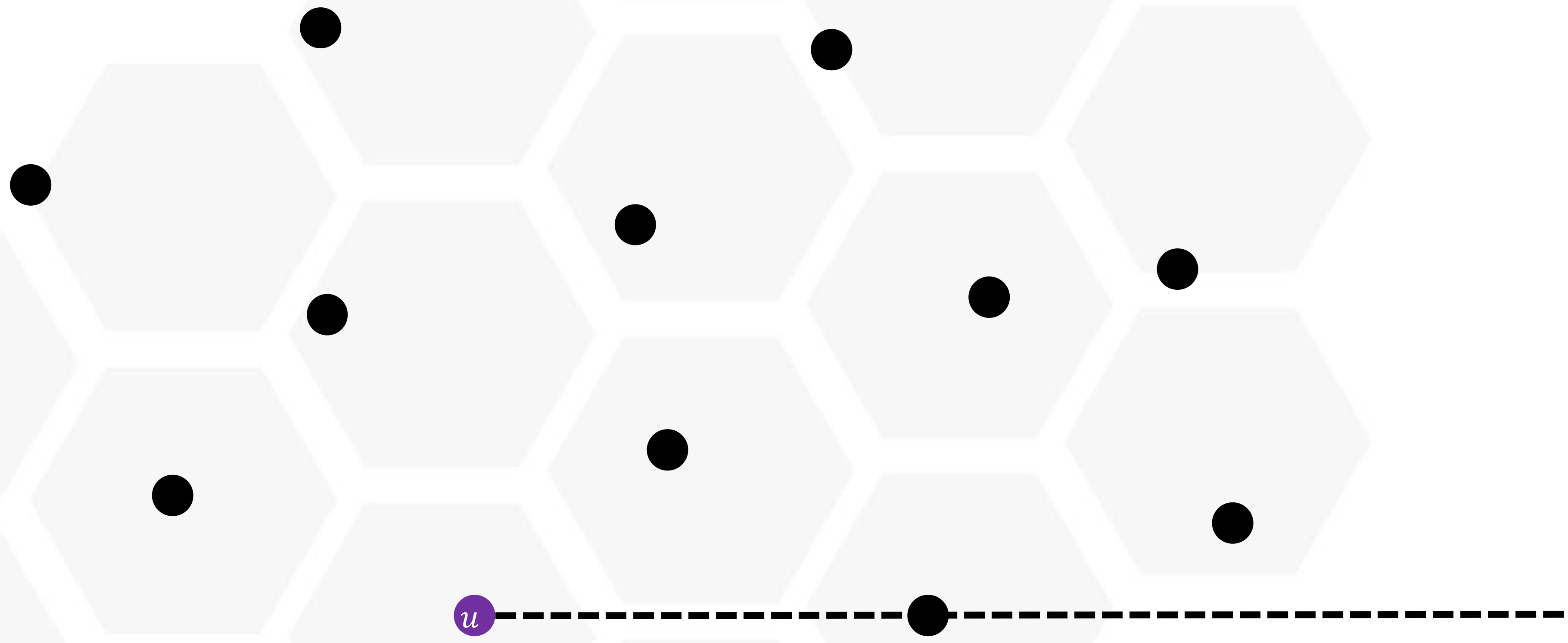
Jarvis' Algorithm (Gift Wrapping Method)



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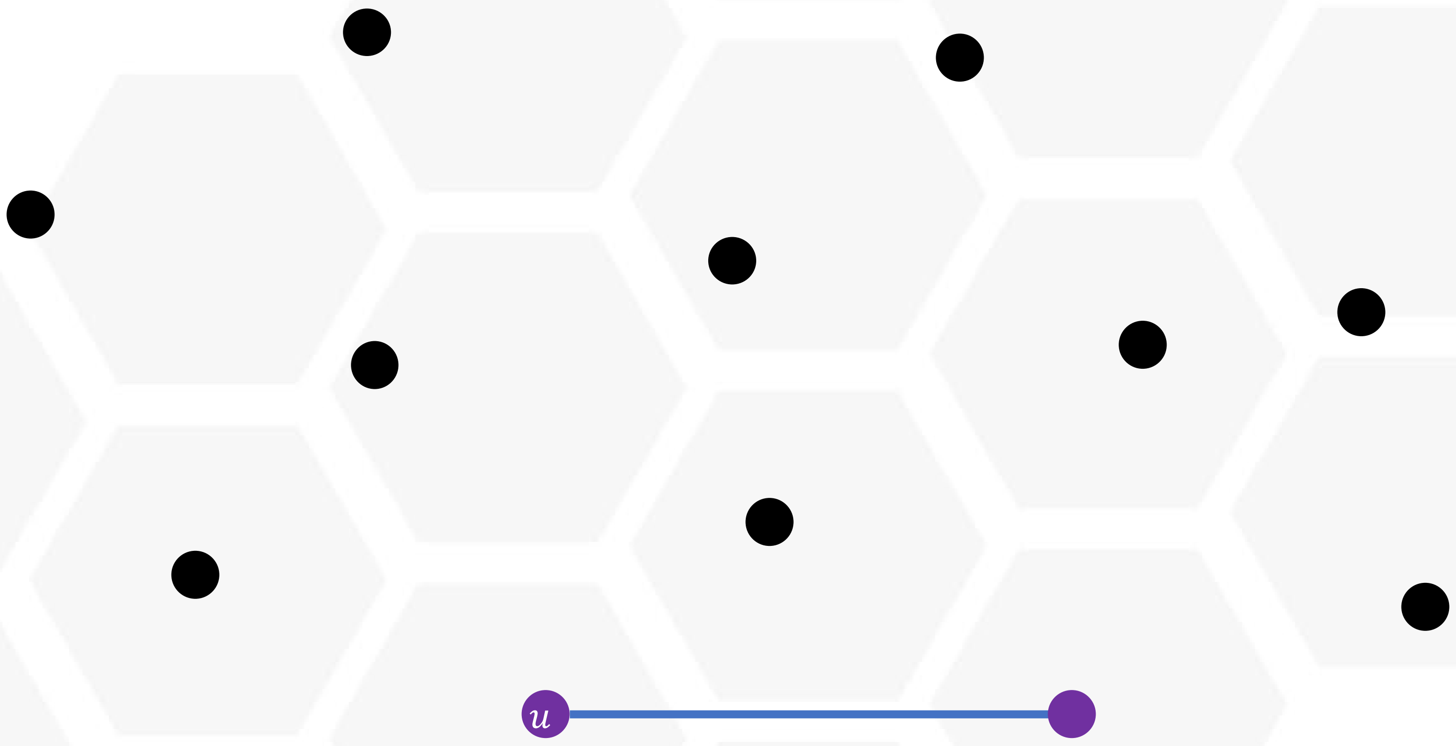

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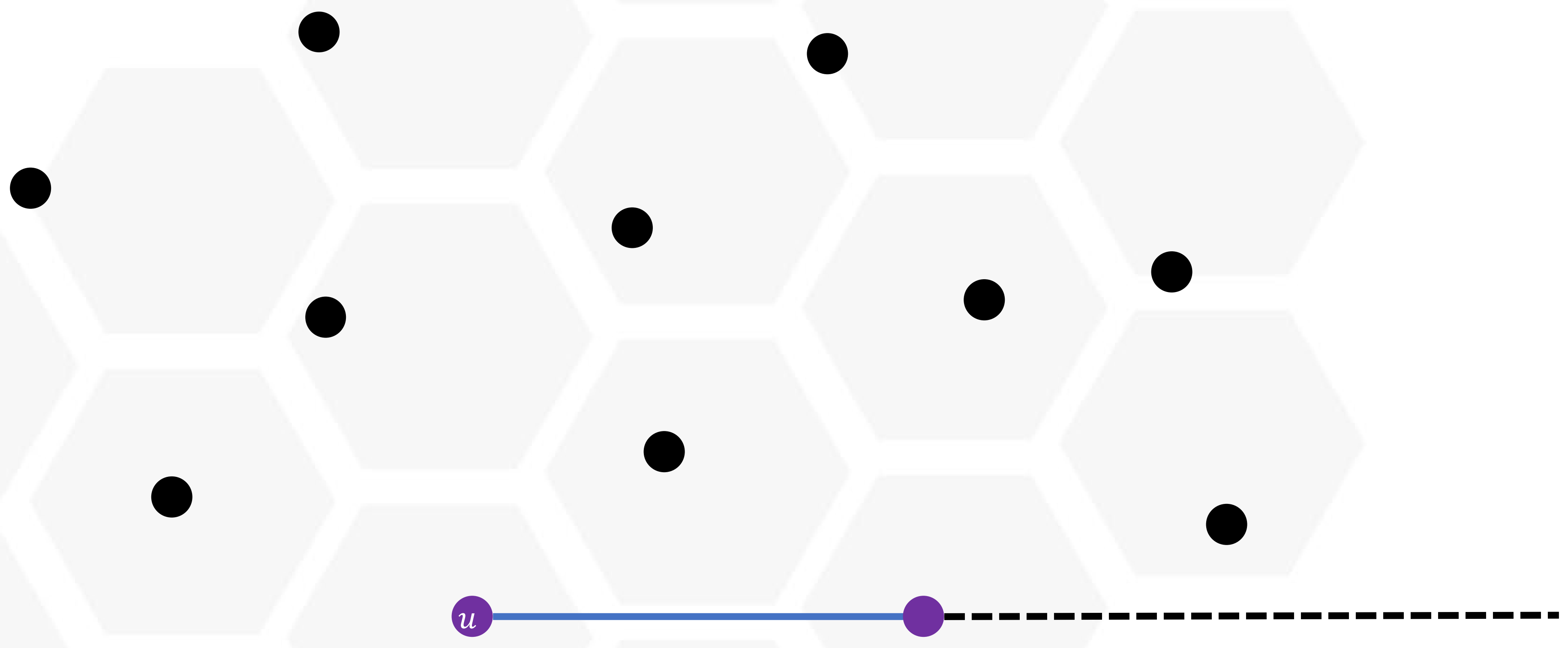

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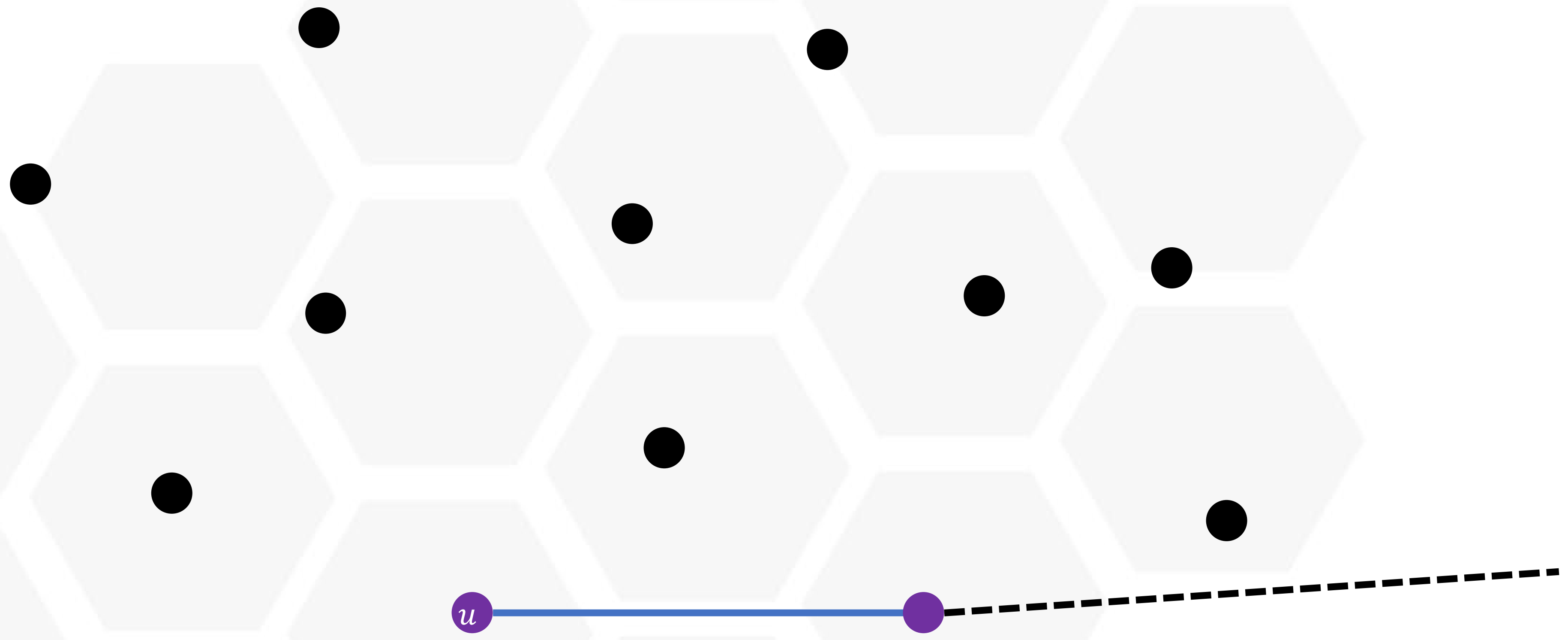

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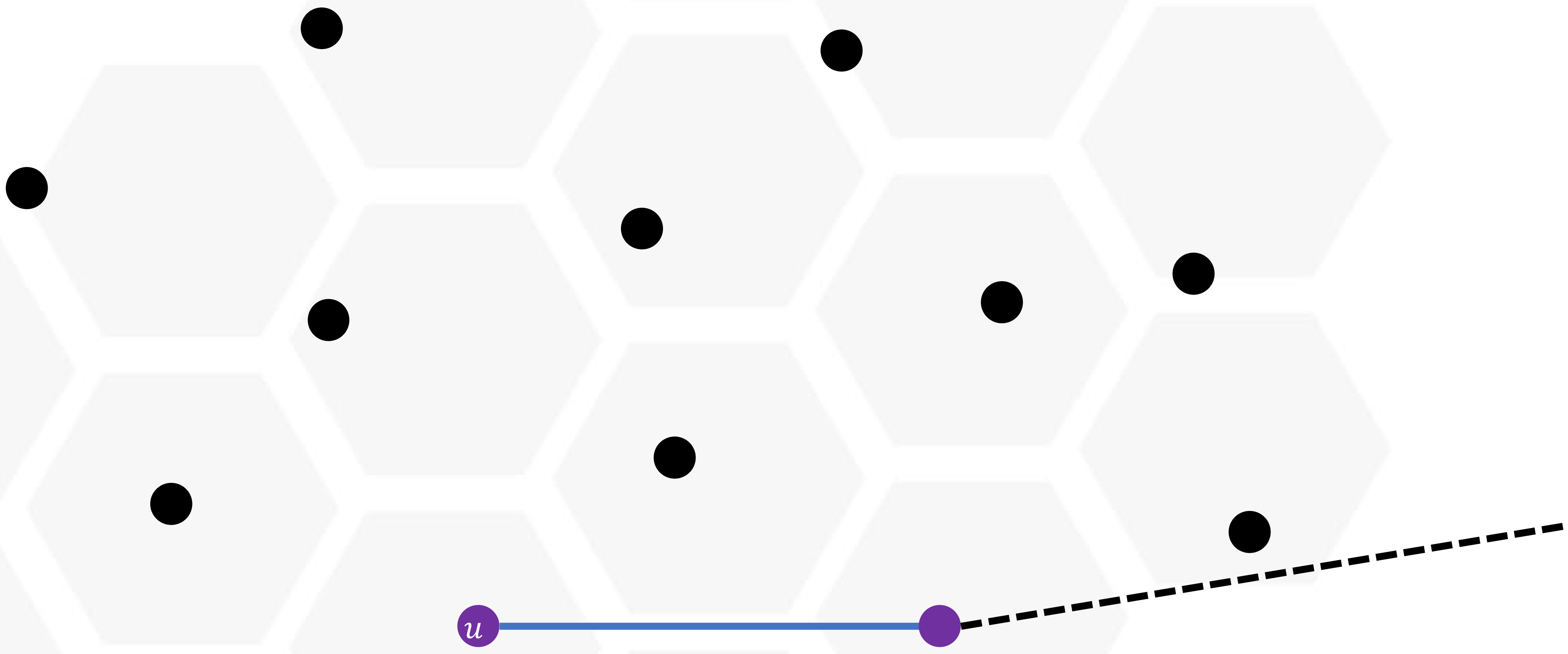

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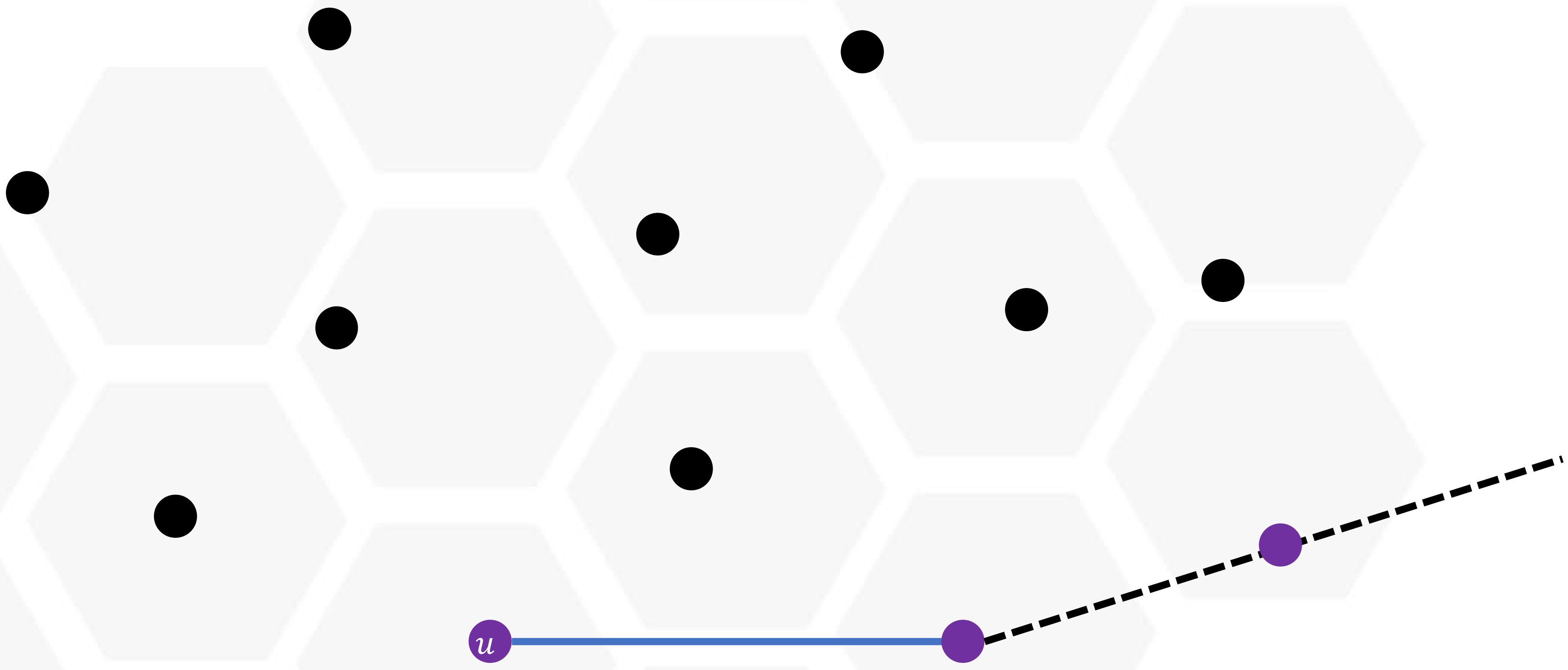

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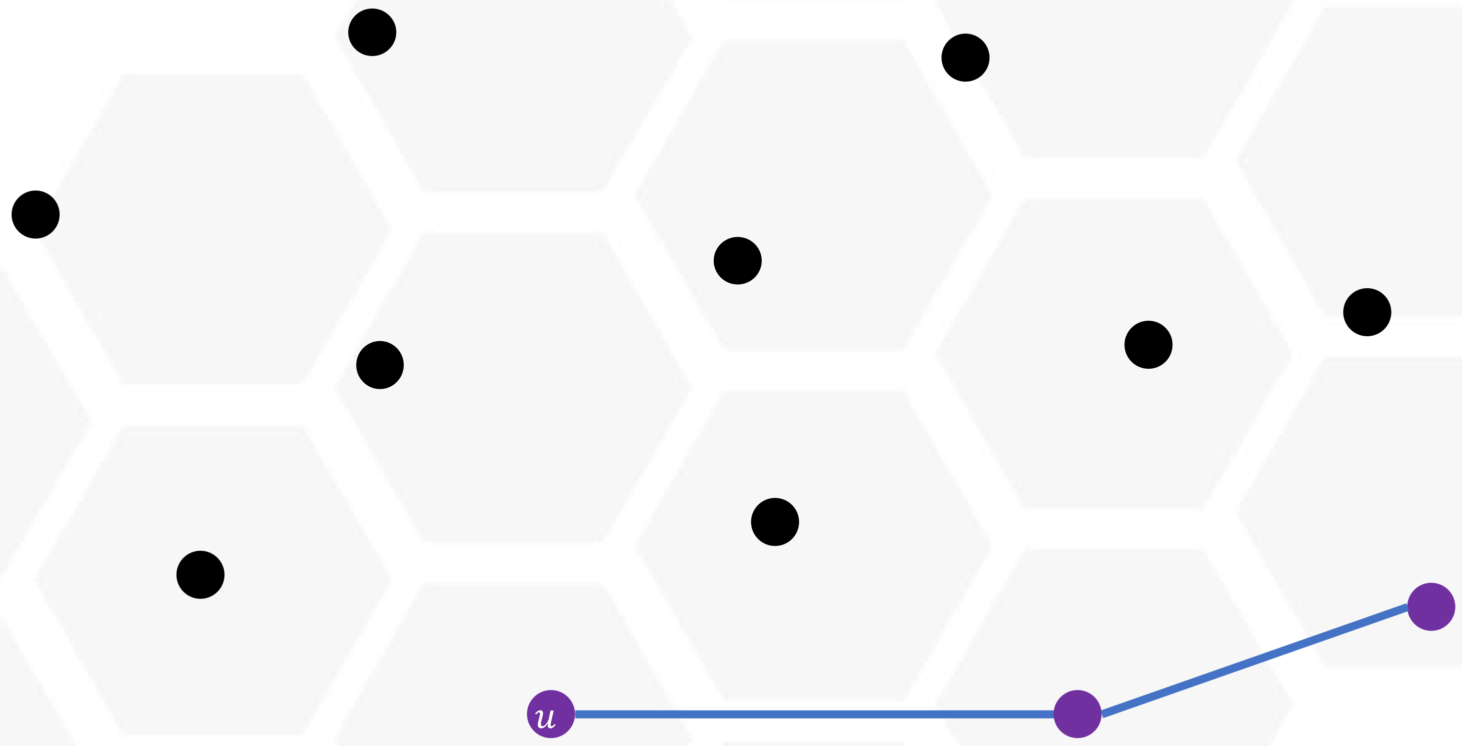

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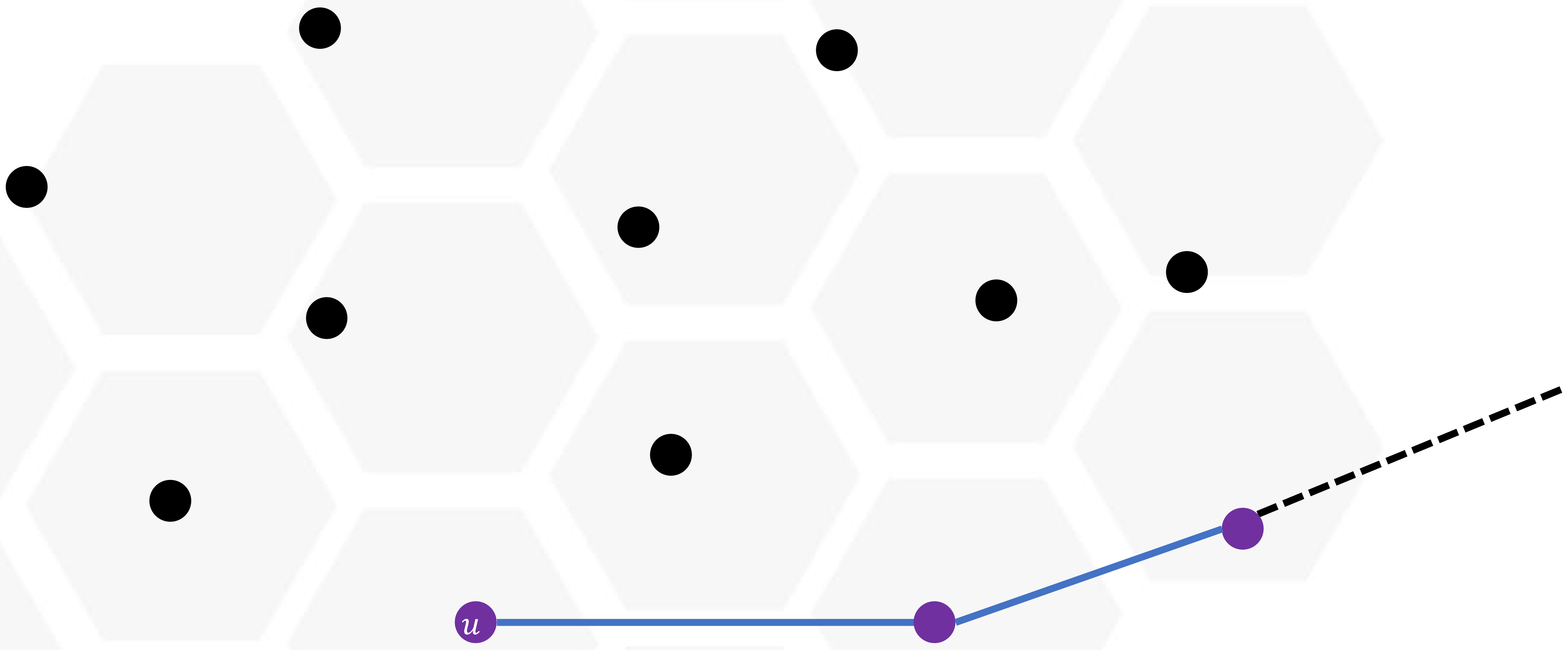

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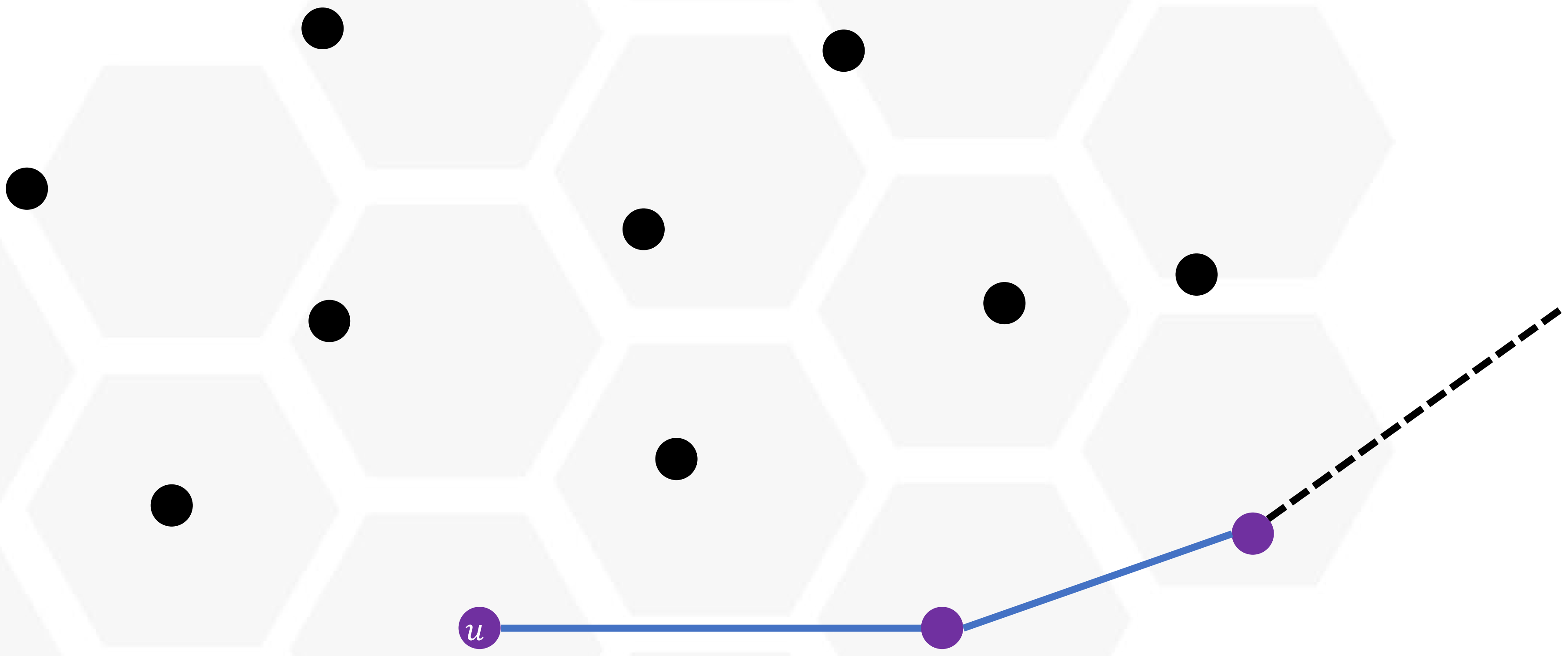

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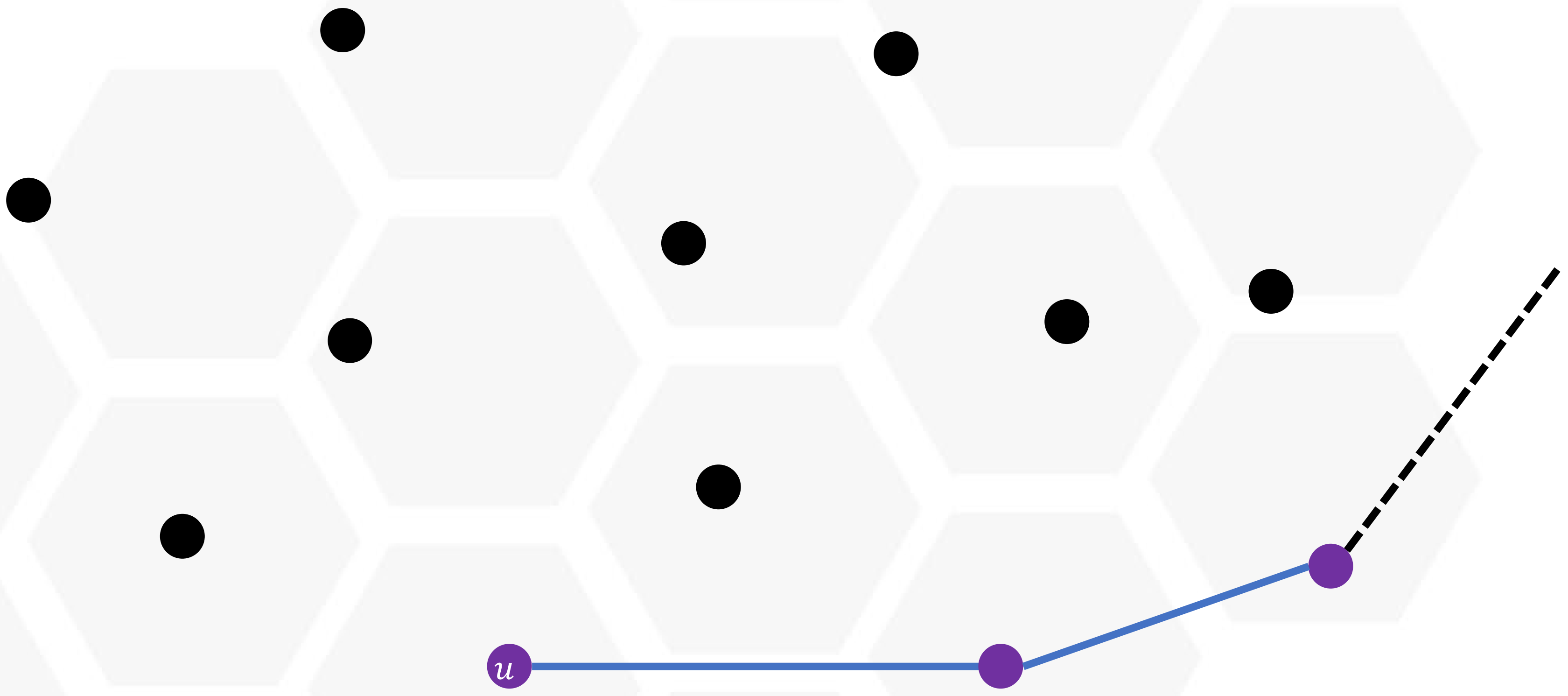

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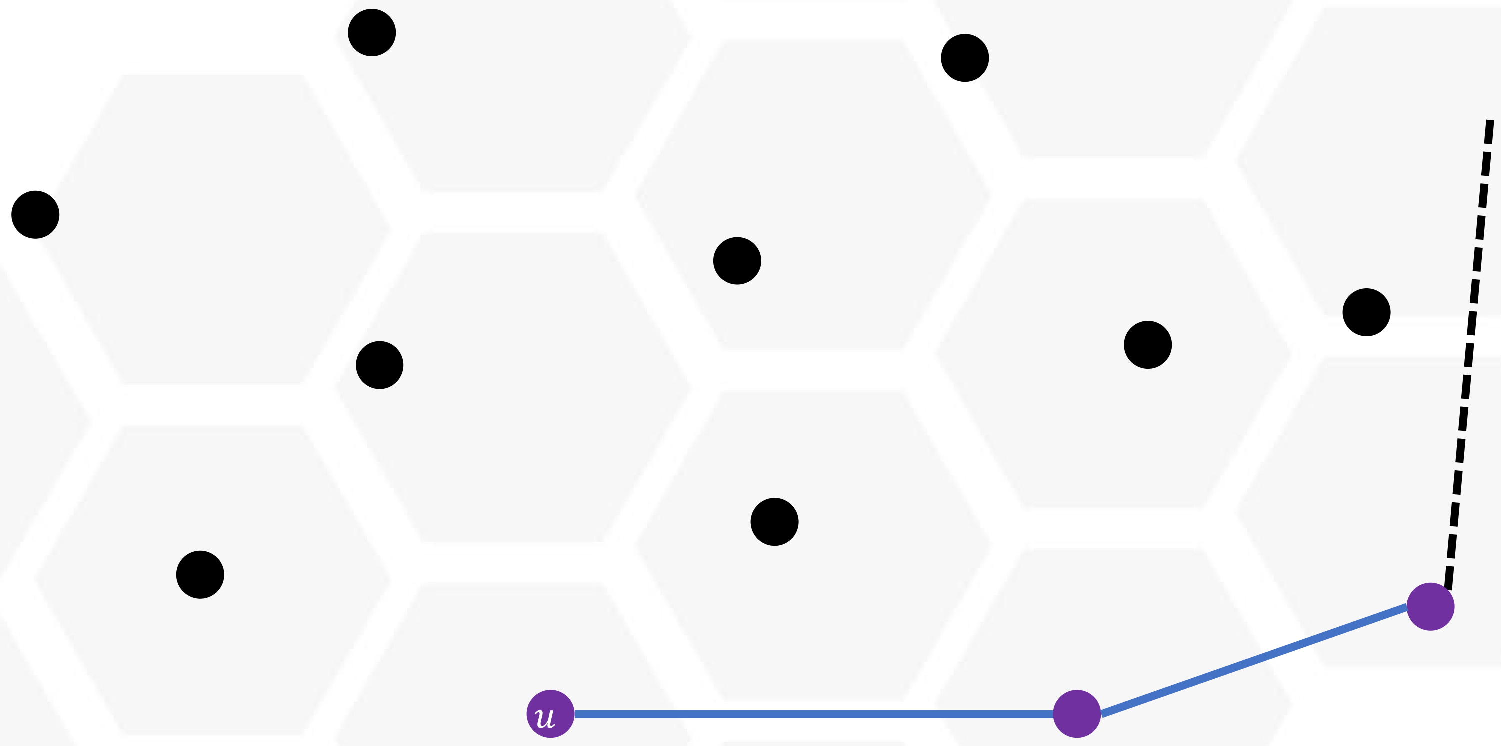

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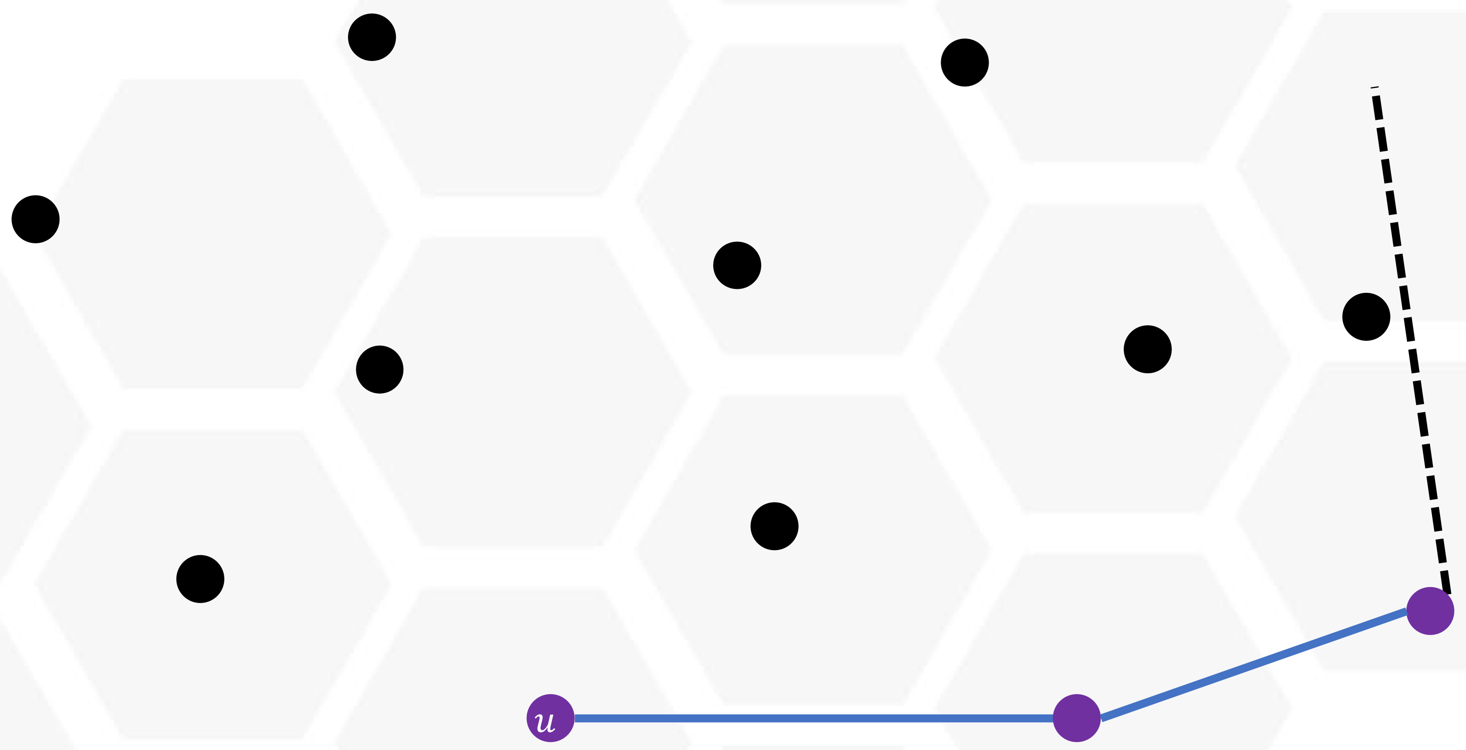

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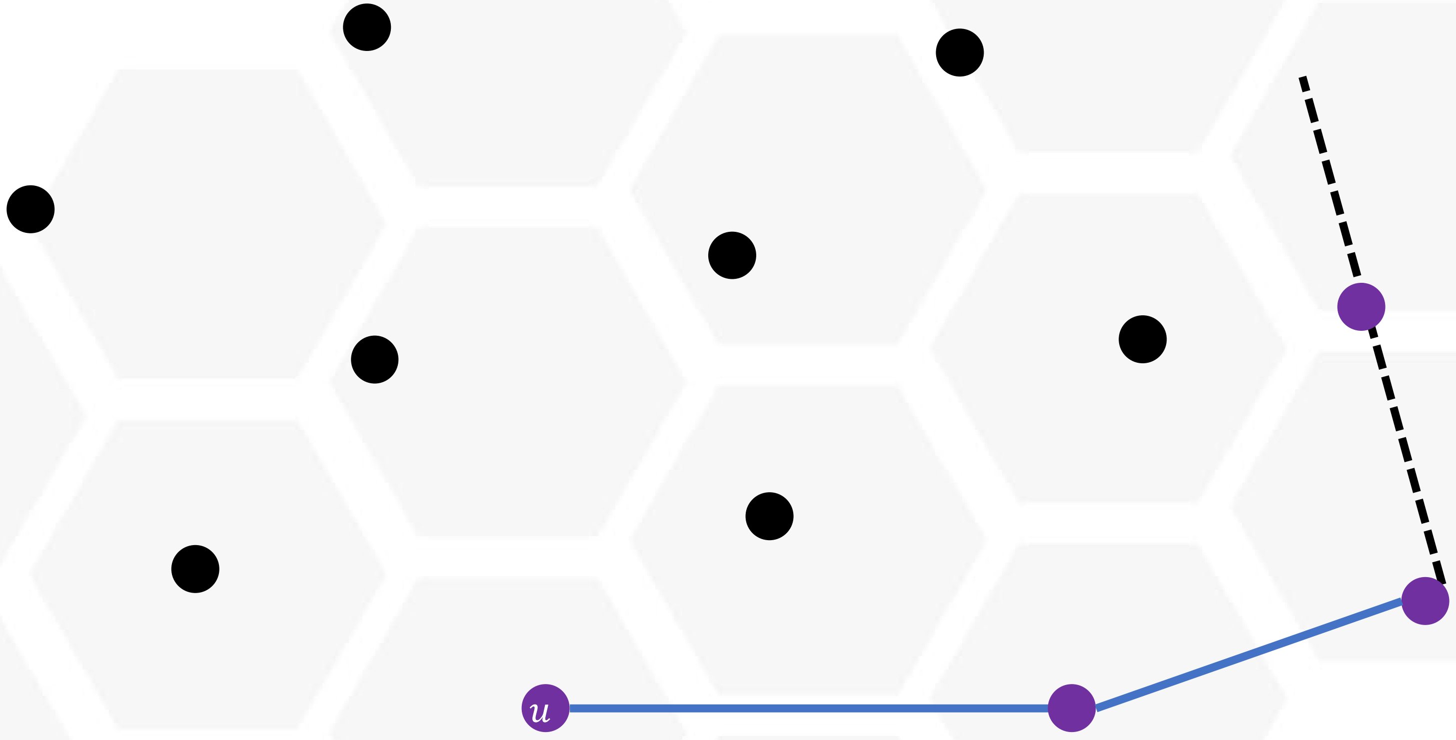

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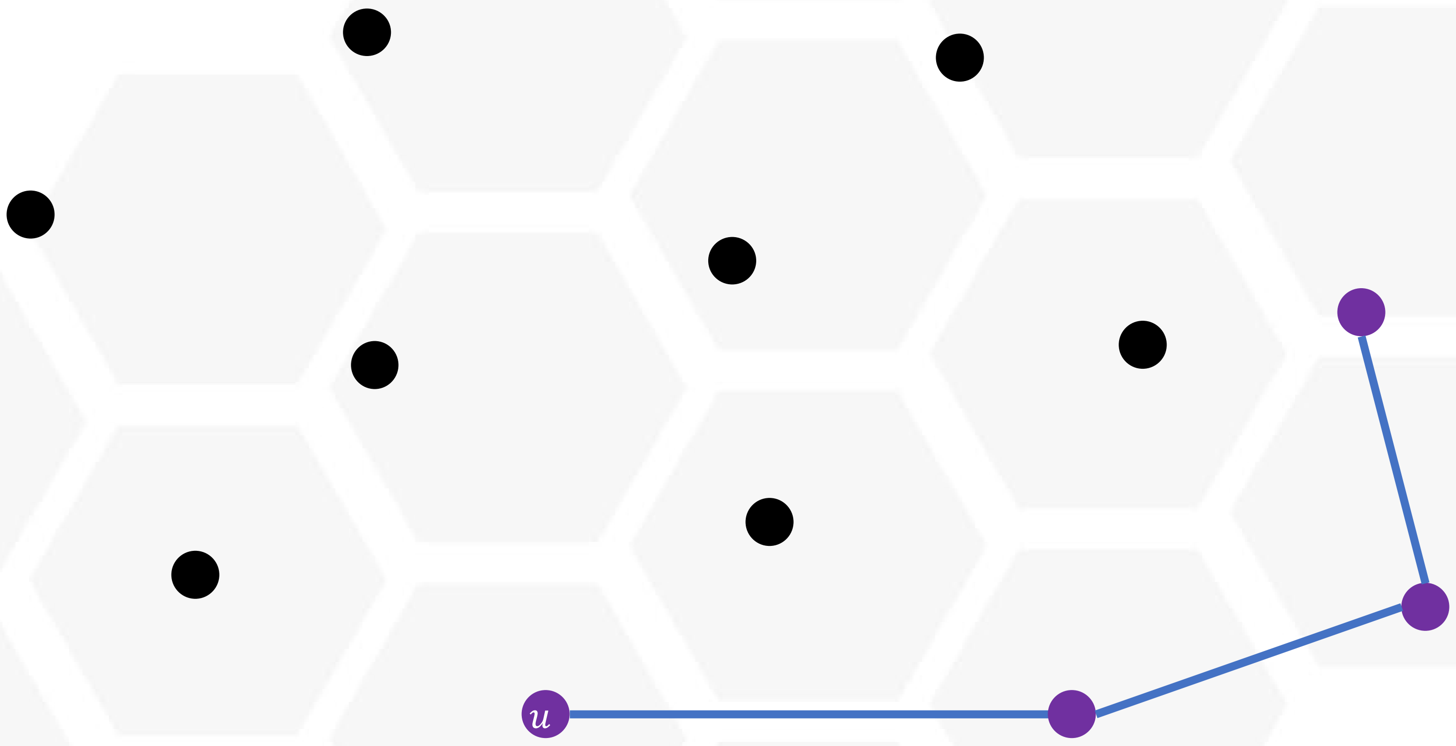

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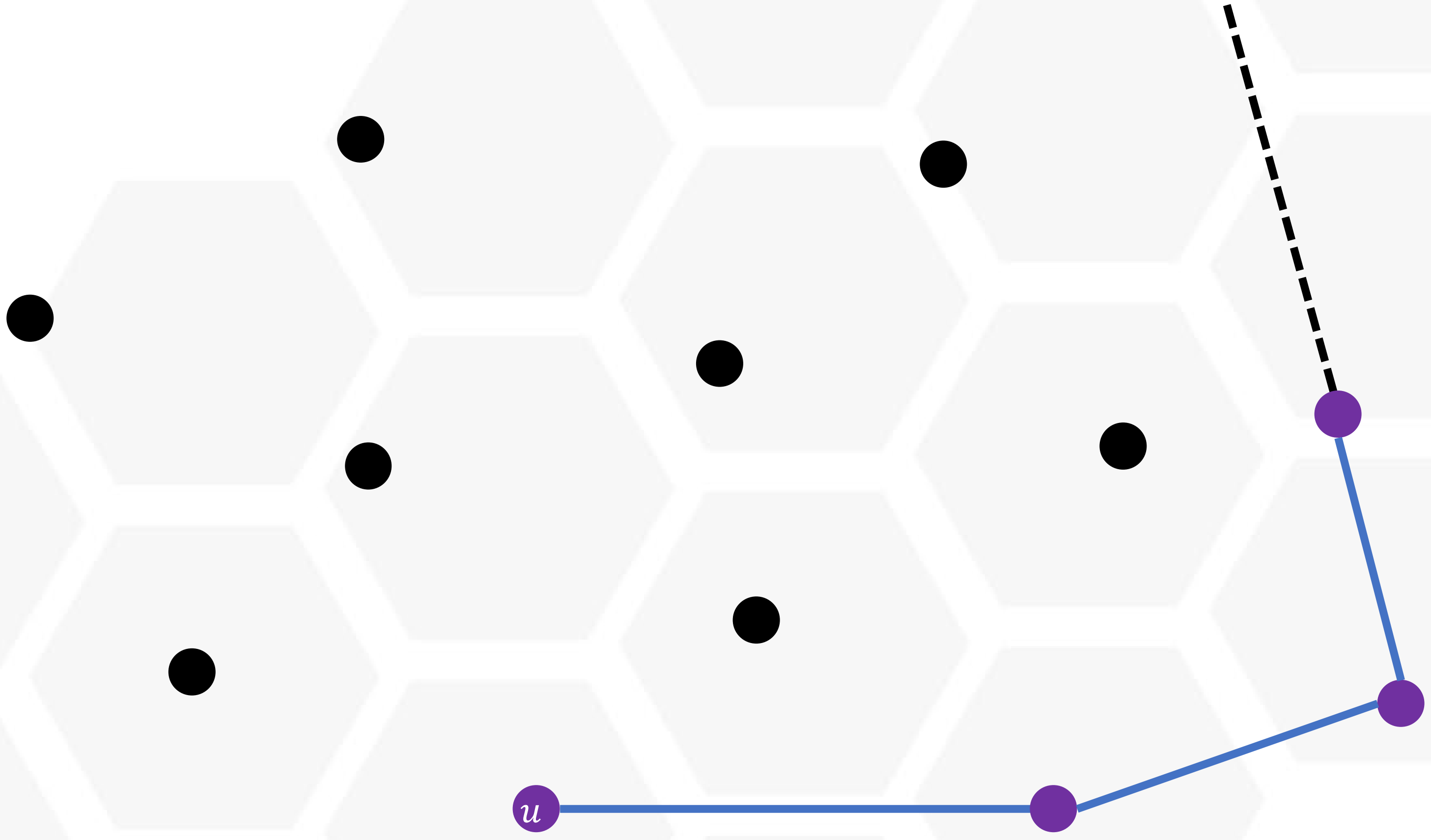

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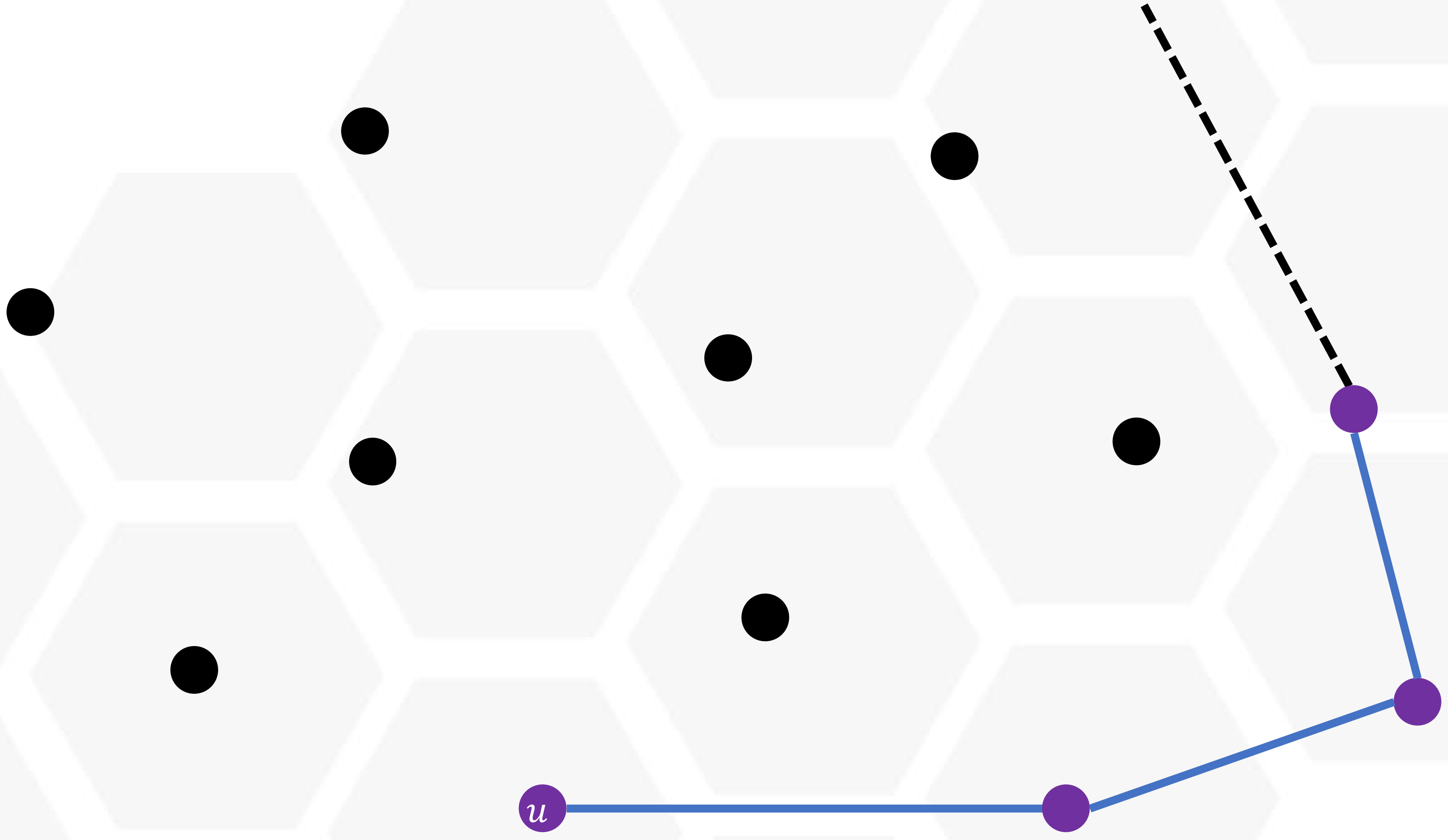

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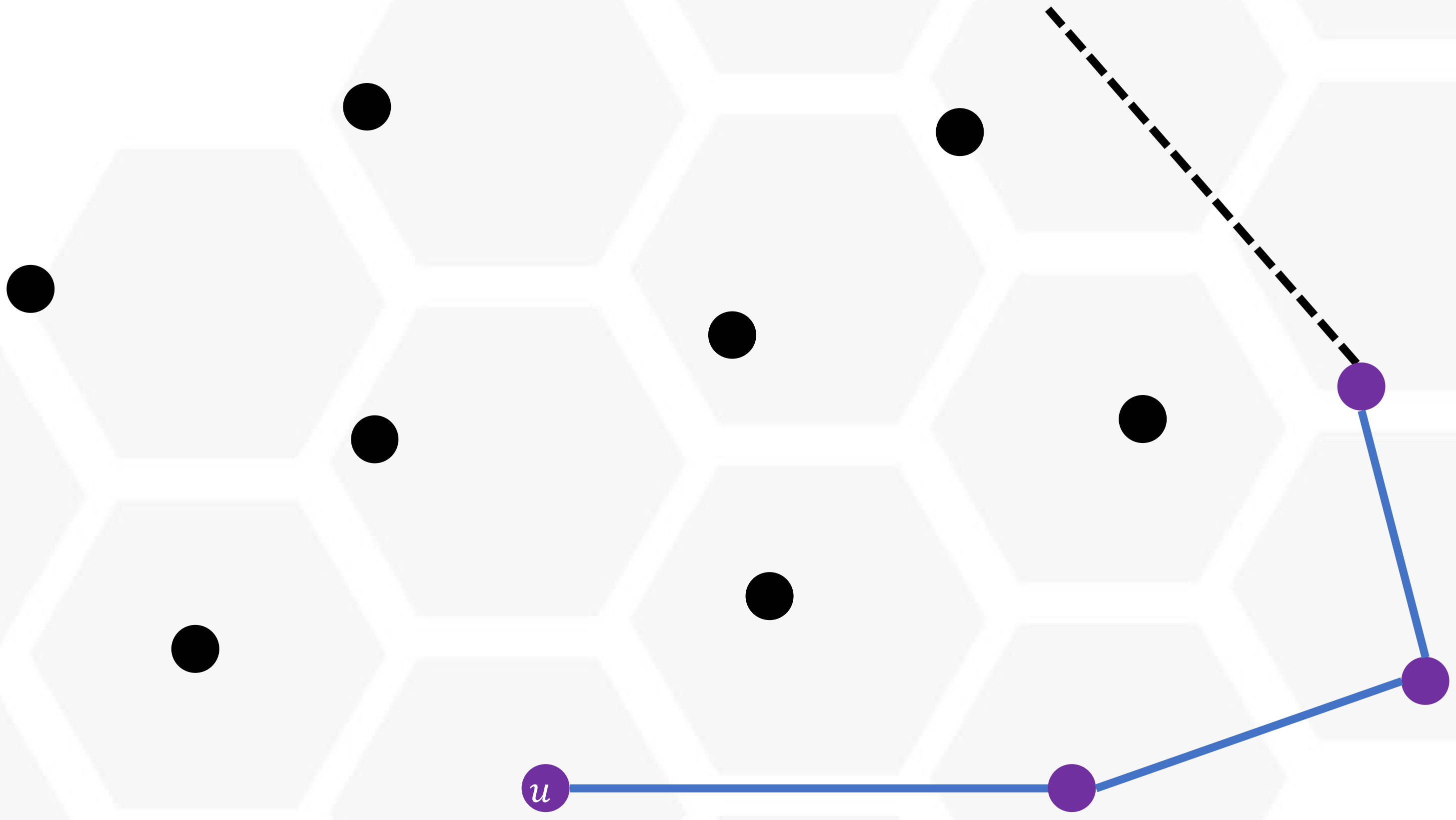

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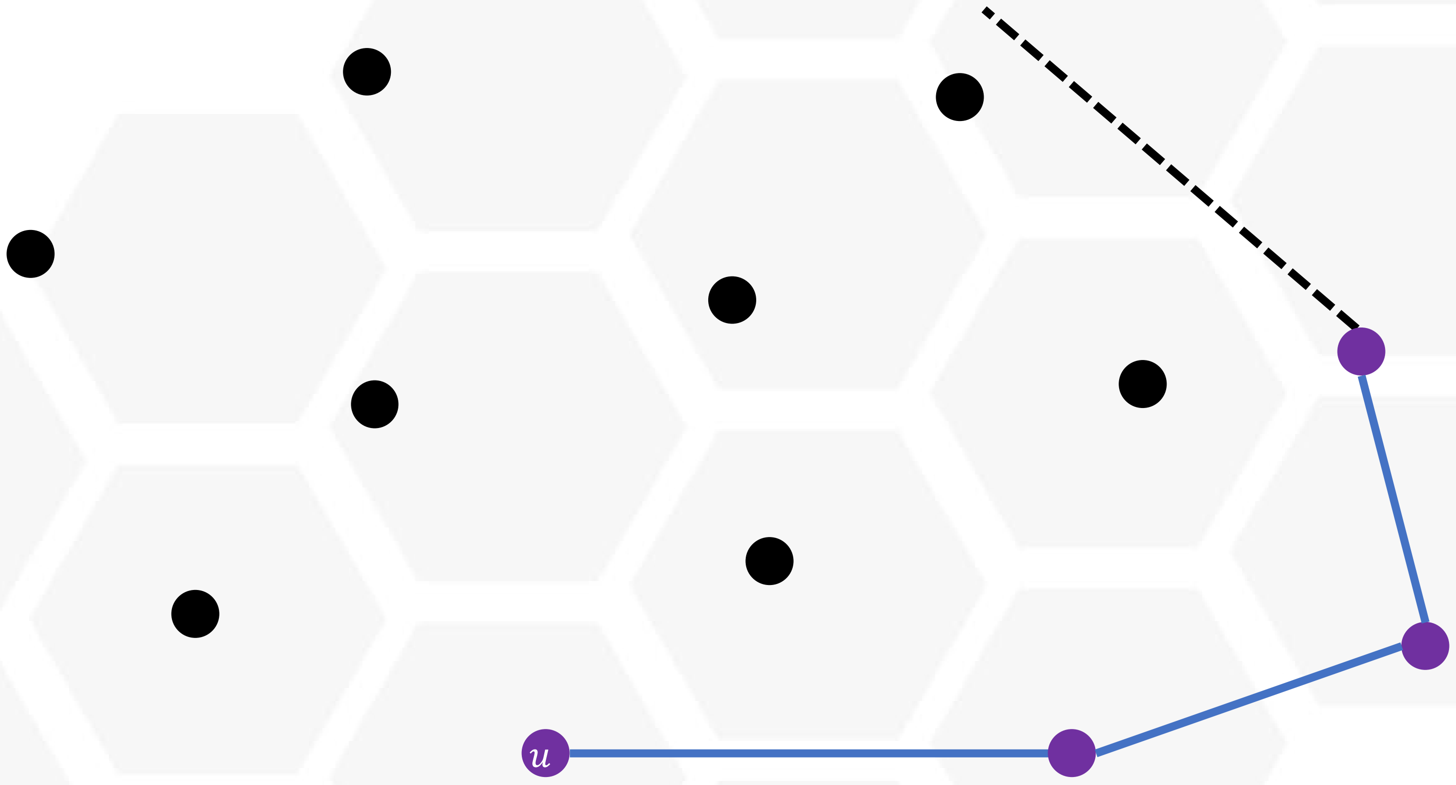

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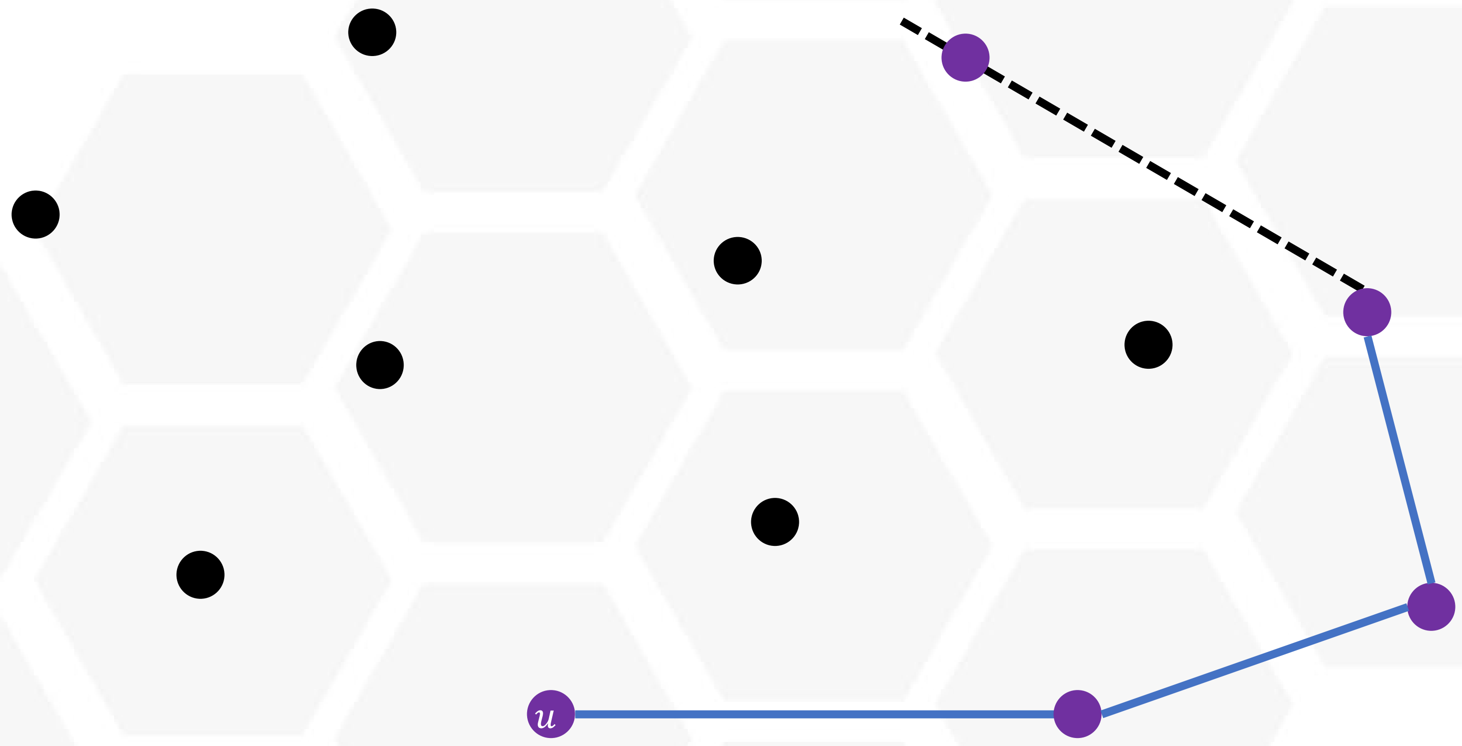

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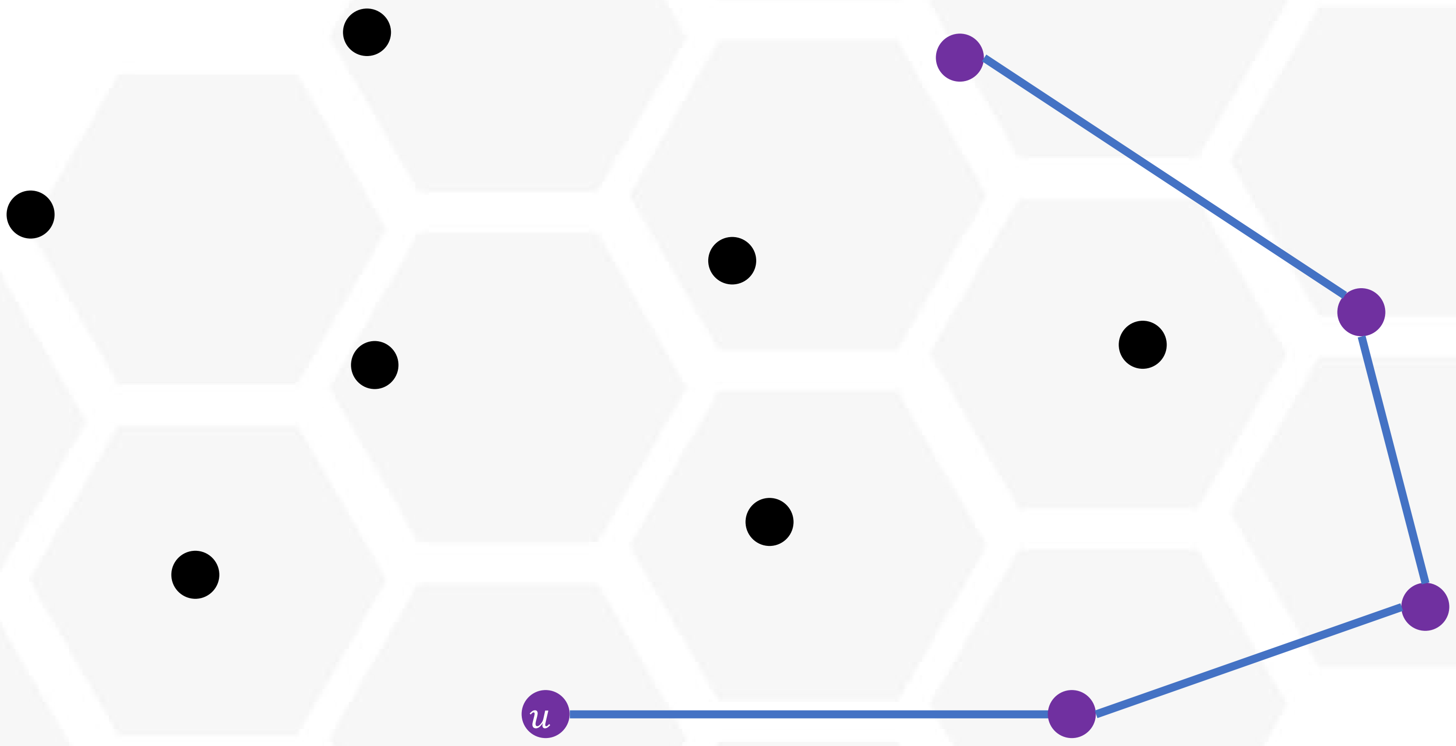

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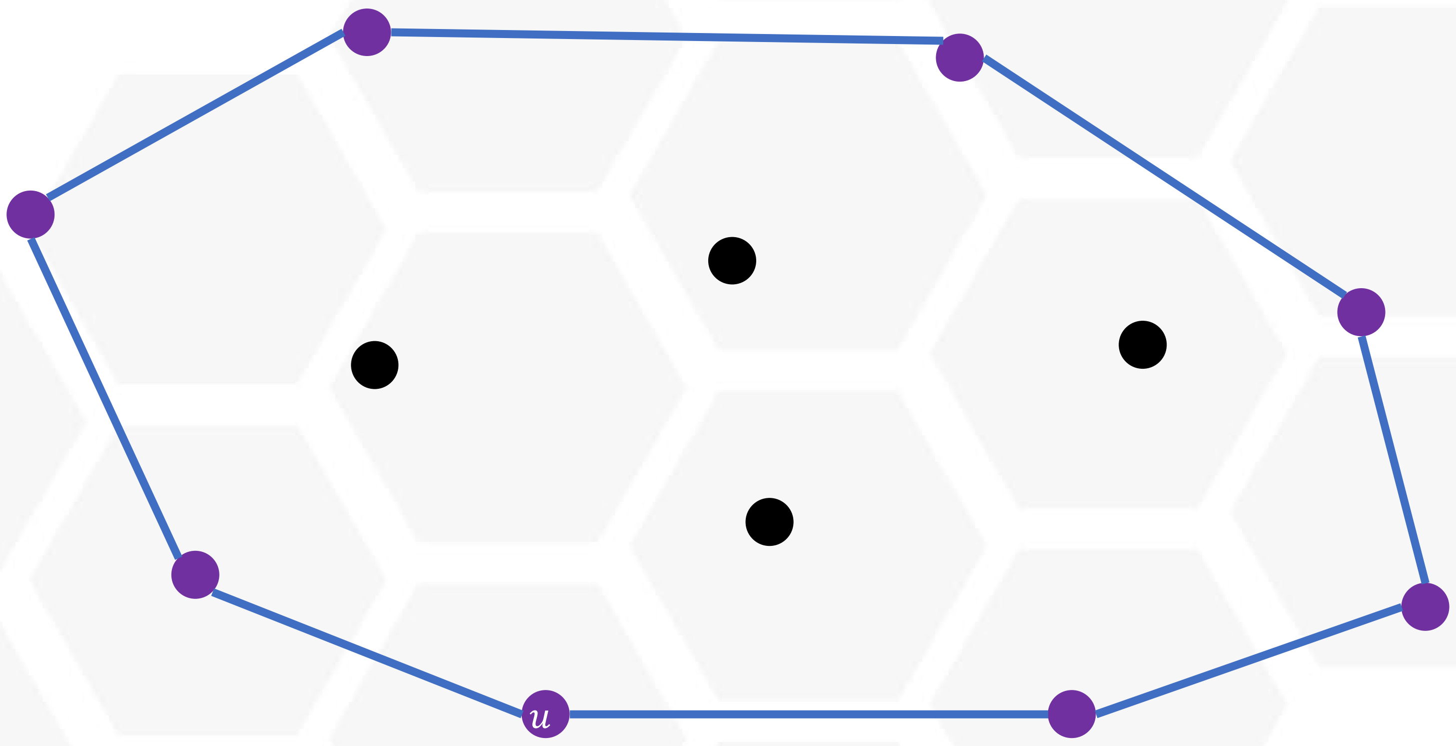

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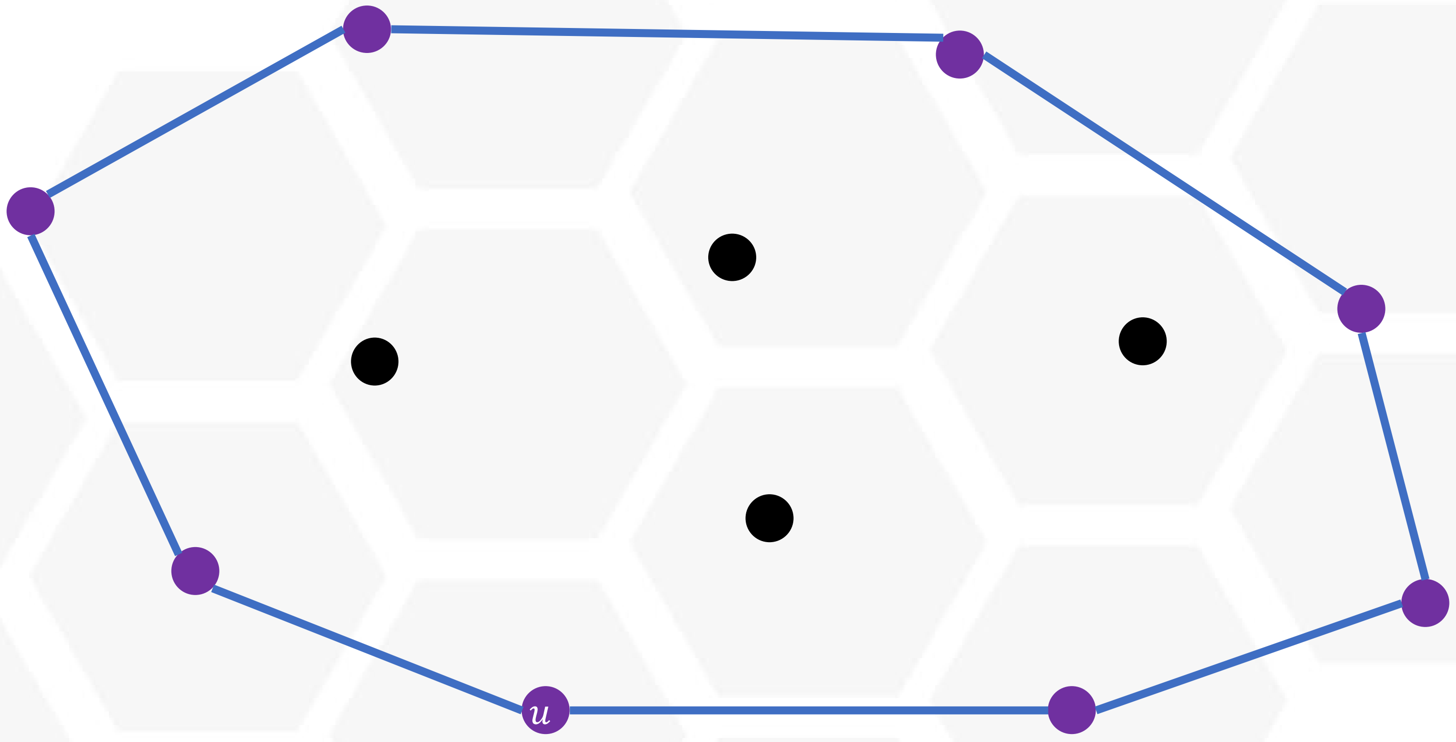
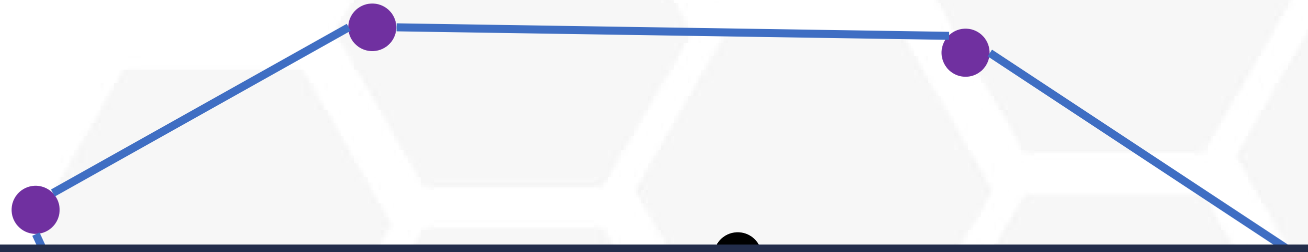


Can find the "next" point using a linear scan (i.e., point with <u>largest</u> angle)
Number of iterations: number of points on convex hull
Run time: $O(nh)$ where h is the number of points on the convex hull

Jarvis' Algorithm (Gift Wrapping Method)



Output-dependent running time (similar to Ford-Fulkerson)

- Can be better than Graham's Algorithm when $h \ll \log n$
- **Worst case:** $h = n$, so $O(n^2)$

Can find the "next" point using a linear scan

Number of iterations: number of points on convex hull

Run time: $O(nh)$ where h is the number of points on the convex hull

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Chan’s Algorithm

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Chan’s Algorithm

Divide into smaller subsets

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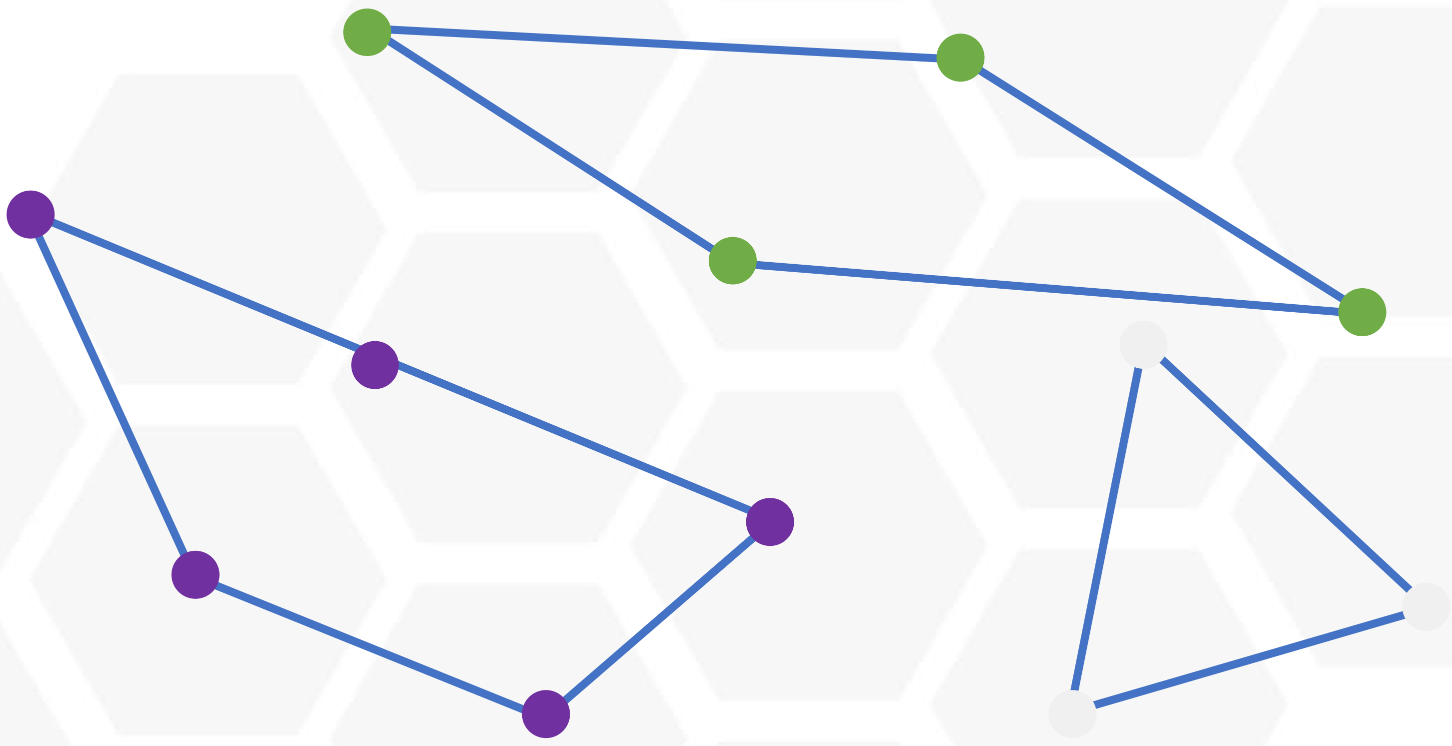
Chan's Algorithm



Use Graham's Algorithm to **conquer** the smaller subsets

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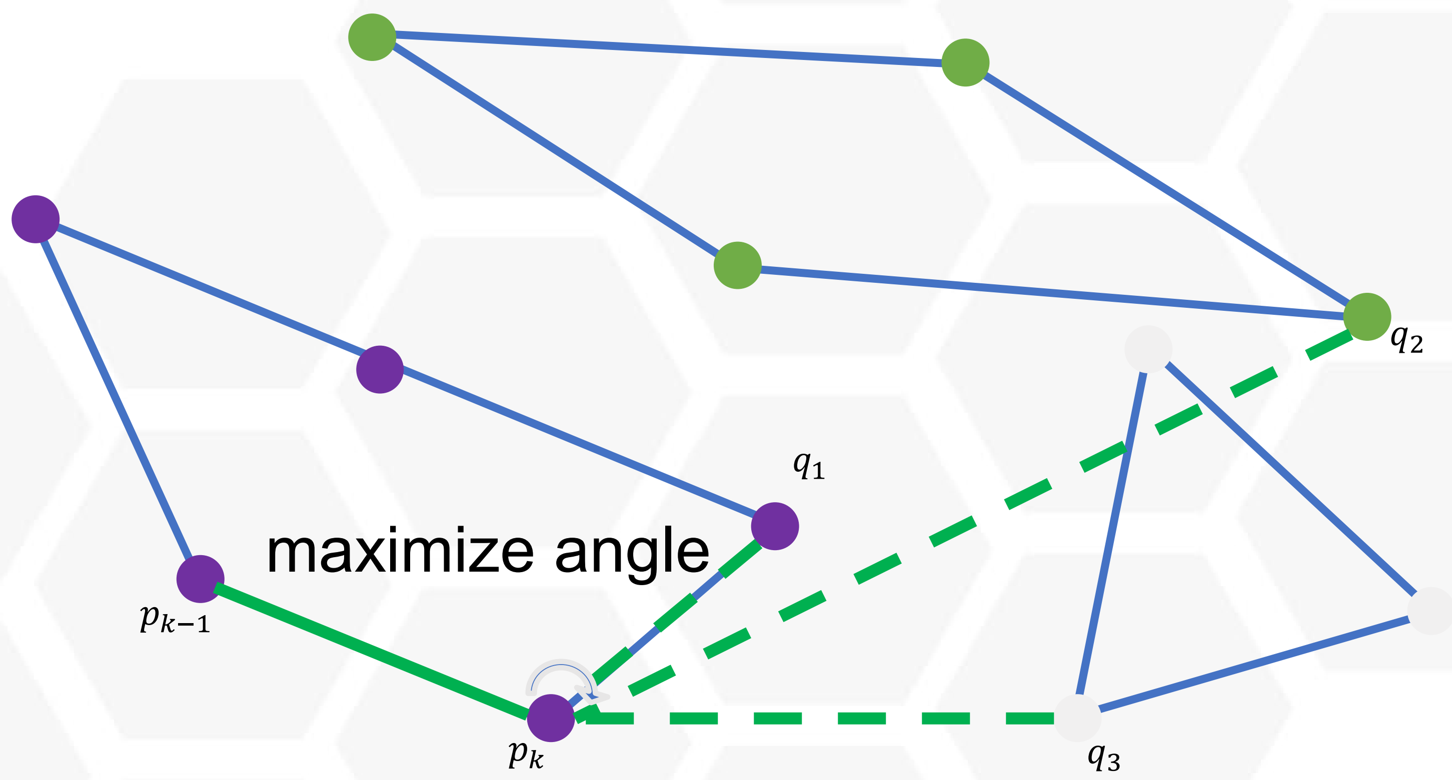
Chan's Algorithm



Use Jarvis' Algorithm to **combine** the solutions to the smaller subsets

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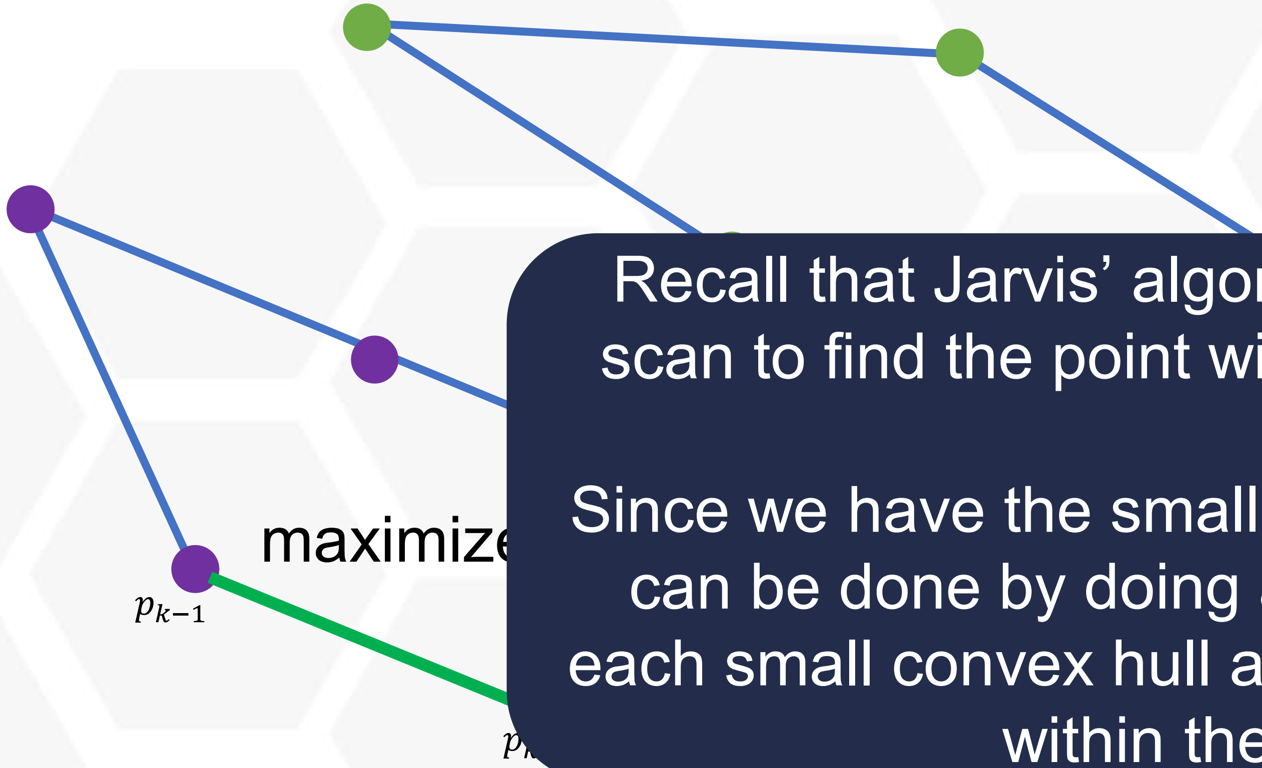
Chan's Algorithm



Recall that Jarvis' algorithm does a linear scan to find the point with maximum angle

Since we have the smaller convex hulls, this can be done by doing a linear scan over each small convex hull and binary searching within the hull

Use Jarvis' Algorithm to **combine** the solutions to the smaller subsets

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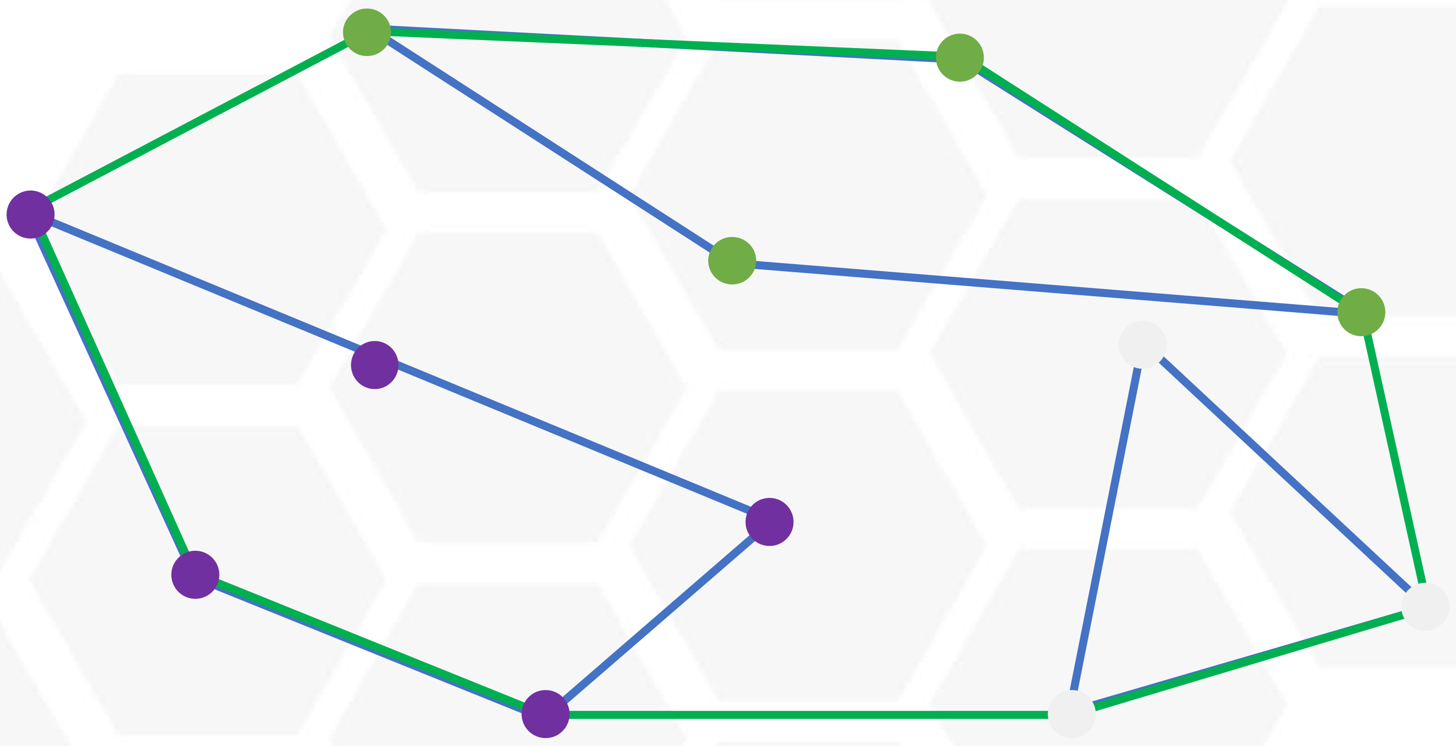
Chan's Algorithm



Use Jarvis' Algorithm to **combine** the solutions to the smaller subsets

Running time: $O(n \log h)$ – optimal!