Topic 12.0: Analysis of Algorithms

Learning Goals:

- Describe why Computer Scientists care about the major “steps” in an algorithm over raw measurements like CPU time.
- Express the complexity of a basic algorithm using big-O notation.
- Compare and contrast the running times of linear versus binary search using big-O notation.
- Compare and contrast the running times of insertion, selection, merge, and quick sort using big-O notation

Searching Algorithms

- Previously we had seen two different searching algorithms: linear search and binary search.
- We know that the two algorithms have different performance (speed). How could we compare them?
    - We could implement them and time them, but this is dependent on the implementation details, computer system we are using, and test data.
    - Or we could **analyze the algorithms.**

Comparing Relative Speed

- The difference between the two algorithms is **huge**!
- If you double the list size:
    - The linear search **doubles** the iterations
    - The binary search **adds only 1** iteration!

List Size	Linear iterations	Binary iterations
10	10	4
20	20	5
1000	1000	10
1,000,000	1,000,000	20
1,000,000,000	1,000,000,000	30

Analyzing Speed

- To analyse the speed of an algorithm
    - Count the number of steps or operations needed
    - Look at it as a function of the size of the problem

    - You can look at either the **average** number of steps, or the **maximum** number of steps

- For any kind of search, the size of the problem (size of the input) is the size of the list/array – **call it n**

Analyzing Speed: Linear

- For a linear search:
- The loop will be executed an average of $\frac{n}{2}$ times, or a maximum of n times
- The number of operations (time) needed for each iteration will be some small constant amount – call it c
- So the total average number of operations (time) is $t(n) = c \times \frac{n}{2}$ and the maximum time is $t(n) = c \times n$
- $t(n)$ means the total average number of operations (total time) it takes to complete the given algorithm over n elements

Analyzing Speed: Binary

- For a binary search:
  - each iteration will take some small constant amount of time c
  - Each iteration will cut the size of the list in half
  - The **maximum number of iterations is related to the maximum number of times you can cut n in half**
  - $ceil\left(log_2 n\right)$
- For n **= 15** it will search 15, 7, 3, and 1 elements
  - $ceil\left(log_2 15\right)$ = 3.90689... ~maximum 4 iterations

Analyzing Speed: Binary

- For a binary search:
  - The average number of operations (the average case) is exactly 1 less than this (it takes some mathematics to prove that - out of the scope of this course)
  - So the average time is:
    - $t\left(n\right) = c \times \left(\left(log_2 n\right) - 1\right)$
  - and the maximum time is
    - $t\left(n\right) = c \times log_2 n$

Comparing Algorithms

- In summary:
    - Linear search: $t\left(n\right) = c \times \frac{n}{2}$ or n
    - Binary search: $t\left(n\right) = c \times log_2 n - 1$ or $log_2 n$

- **Constants don't really matter, so you can ignore c's, or ½ or 2 or -1**

Comparing Algorithms: Takeaways

- The really important thing about an algorithm is: as **n** grows, how does the number of operations (time) grow?

- That's determined only by how **n** (size of input) affects the number of operations

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- The really important thing about an algorithm is: as **n** grows, how does the number of operations (time) grow?

- That's determined only by how **n** (size of input) affects the number of operations
- The linear search is "**O(n)**"
  - If **n** doubles, the **time** doubles

- The binary search is "**O(log n)**"
  - If **n** doubles, the time goes up by a small increment (**logn**)

Big “O” Notation?

- Big “O” Notation means **Order of the Function**
  - i.e. How the function grows with respect to **n**
  - (also known as Landau's Symbol)
  - i.e.i.e When **n** gets bigger, what happens to our running time/algorithm?
- Often called “worst-case” complexity
  - As the needs of our algorithm increase, how does the performance of said algorithm hold?

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- Let **n** be the size of the input
- Let **T(n)** be the function that defines the number of operations (or the amount of space) required by an algorithm on the input **n**

- Then $T(n) = O(f(n))$ if for positive constants c and n_0 ,

$T(n) <= c \times f(n)$ when $n >= n_0$

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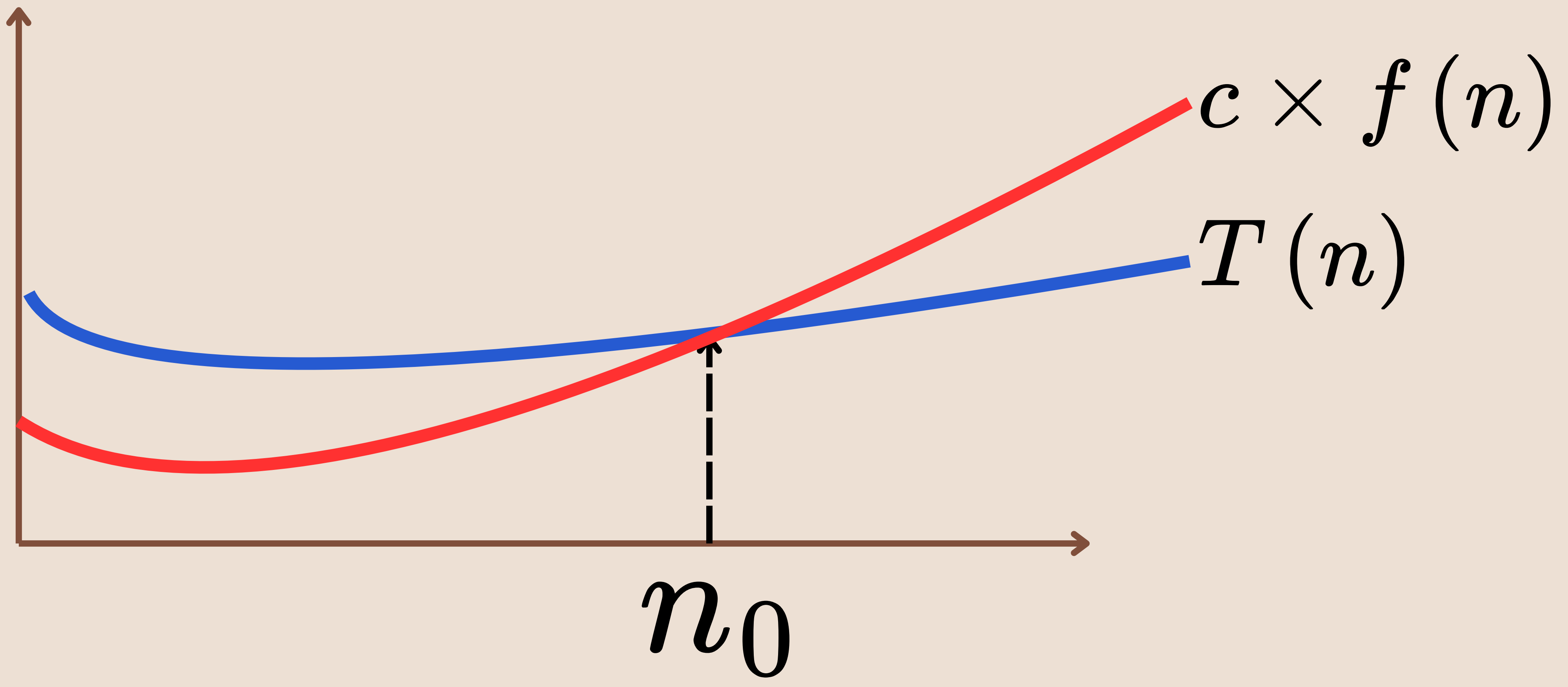
$T(n) <= c \times f(n)$ when $n >= n_0$

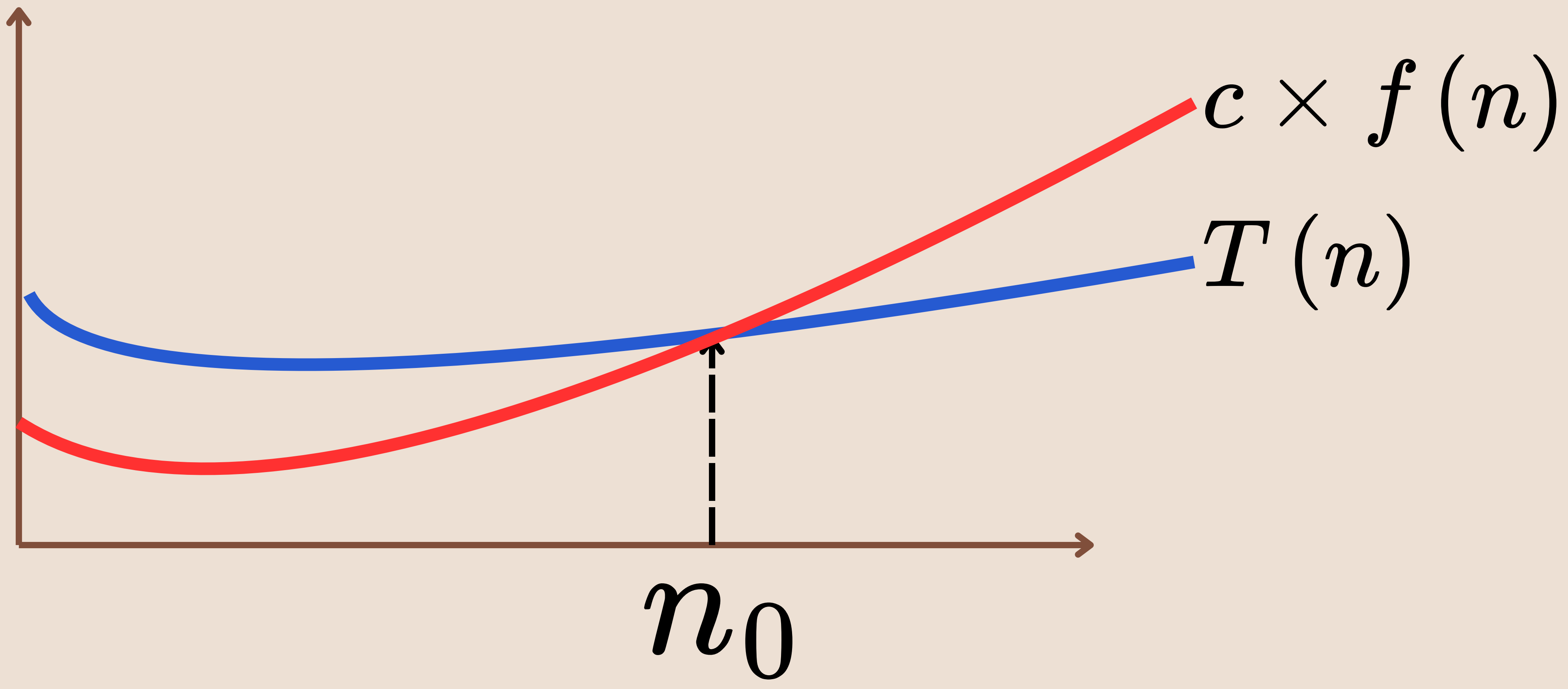
In English: When we say an algorithm is O(f(n)), we are saying that there is a constant **c** and starting **n (n0)** such that, for all larger inputs, the function will not exceed c * the f(n) function



Big “O” Notation?

The goal is to find a **function of n** that will be an upper bound on the number of operations that our algorithm uses **(f(n))**
Then we can say that the number of operations required by our algorithm **is in O(f(n))**

More on this in 2140 next year!



Typical Bounds

F(n)	# operations
c	constant
$log_2 n$	logarithmic
n	linear
$nlog_2 n$	linearithmic
n^2	quadratic
n^3	cubic
2^n	exponential
$n!$	factorial

Quick Estimations

You can get a quick estimate of the order by asking one question

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F(n)	double n?
$log_2 n$	very fast, n adds c to time
n	double n, double time
$nlog_2 n$	not much slower than n
n^2	double n, 4 times the time
n^3	double n, 8 times the time

These are considered **polynomial**

- **Computable and depends on the input size**

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But what about these two?

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- Grows more quickly than any polynomial
- Considered **“not computable”** except for teeny-tiny inputs
- Have an algorithm O(n!)?
    - Just throw it out.

Running Times: Who cares?

- We do!
- How else can we make sure we are picking the right algorithm to suit our needs?
- Let’s consider:
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  - You **need** a **sorted list** so:
    - keep it sorted as you build it
    - take an existing list and sort it …
  - But remember:

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- Let’s consider:
  - Binary Search: When should we use it?
  - You **need** a **sorted list** so:
    - keep it sorted as you build it
    - take an existing list and sort it ...
  - But remember:
    - **Sorting a list (or keeping a list sorted) is even slower than a linear search**
    - **Why bother?!**

Binary Search Times

- You happen to already have a sorted list
- You plan to do **a lot** of searching but not a lot of data manipulation
- You have lots of time to sort and then see above point.

- We really have to consider the different cases and what we are going to be doing in our program. We don’t just get to pick our favorite sorting algorithm and go.

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Speaking of which… What are the sorting algorithm running times?

Ordered Insert

- Data Structure Independent Analysis
- How fast is this?
  - There is only one loop that goes through the array one element at a time
  - It might go through all the elements and put our new piece of data at the end
  - It might insert at the front
  - On average, it goes through half the elements
- **O(n)** - same as a linear search
- it is a linear search where we move elements as we go (or don’t need to move elements in the case of a linked list but still possibly **n** steps to find the position)

Insertion Sort

- Let’s consider insertion sort (the loop around the ordered insert):

```
for(int k=1; k < n; k++)
      ordInsert(k, a, a[k]);
```

- It contains one simple loop that always runs **n** times
- Inside that loop it does an ordered insertion, which is **O(n)** – it does **n** steps
- Actually, it does **1, 2, 3, 4, ..., n** steps
  - But that's, on **average, n/2,** which is **O(n)** anyway
- So we do **n** steps, **n** times: this is $O\left(n^2\right)$

Selection Sort

- Next up, let’s analyze the Selection Sort algorithm

```
for(int k=0; k<=a.length-2; k++) {
   ...small bit of work (initializing min)...
   for(int i=k+1; i<=a.length-1; i++)
       ...small bit of work (updating min if necessary)...
   ...small bit of work (swapping)...
}
```

Selection Sort: Rough Analysis

- Next up, let's analyze the Selection Sort algorithm

```
for(int k=0; k<=a.length-2; k++) {
   ...small bit of work (initializing min)...
   for(int i=k+1; i<=a.length-1; i++)
       ...small bit of work (updating min if necessary)...
   ...small bit of work (swapping)...
}
```

- There are two nested loops, both of which are **O(n)** (a.length is n here)
  - Again, the inner one is only ½ of n, on average, but that's still **O(n).** Ignore constants like ½ .
- n steps, n times:
  - So this is $O\left(n^2\right)$

Simple Sorting Algorithms

- All of our symple sorting algorithms: insertion, selection, bubble, are $O\left(n^2\right)$

- This means that even though some may be faster than others when used on the same data, they are all ***theoretically*** equally slow
- When the lists get **very** long (millions, billions, etc) **all** these sorting algorithms become impractical or impossible to use.

The Better Way to Sort: Analysis

- Improving on these simple sorts requires us removing the nested loop situation
  - This is what is making them $O\left(n^2\right)$
- We can replace the loops with the same kind of “divide and conquer” we see to make binary search faster than linear search
  - **This is how we get merge sort**