Interpreting logistic-regression coefficients as odds ratios

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In logistic regression, we are fitting a model to predict binary outcomes y_i, conditional upon regressors $\mathbf{x}_i = (x_{i1}, \ldots, x_{ip})$. This involves assuming that the probability of "success" for the ith observation is

$$\Pr(y_i = 1 \mid \mathbf{x}_i) = p_i = \frac{\exp(\beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip})}{1 + \exp(\beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip})} \, . \tag{1}$$

Recall that odds are just a different way of expressing probabilities:

$$(\text{Odds that } y_i \text{ is } 1) = O_i = \frac{p_i}{1 - p_i} \, .$$

If you churn through the algebra and re-express the logistic-regression equation (1) in terms of odds, you will see that the log-odds of success are being modeled as a linear function of the predictors:

$$\log O_i = \log \left(\frac{p_i}{1 - p_i} \right) = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} \, .$$

For the sake of simplicity, image a data set with only a single regressor x_i that can take the values 0 or 1 (a dummy variable). Perhaps, for example, x_i denotes whether someone received the new treatment (as opposed to the control) in a clinical trial.

For this hypothetical case, let's consider the ratio of two quantities: the odds of success for person i with $x_i = 1$, versus the odds of success for person

j with $x_j = 0$. Denote this ratio by R_{ij}. We can write this as

$$\begin{aligned}
R_{ij} &= \frac{O_i}{O_j} \\
&= \frac{\exp\{\log(O_i)\}}{\exp\{\log(O_j)\}} \\
&= \frac{\exp\{\beta_0 + \beta_1 \cdot 1\}}{\exp\{\beta_0 + \beta_1 \cdot 0\}} \\
&= \exp\{\beta_0 + \beta_1 - \beta_0 - 0\} \\
&= \exp(\beta_1)\,.
\end{aligned}$$

Therefore, we can interpret the quantity e^{β_1} as an *odds ratio*. Since $R_{ij} = O_i/O_j$, we can also write this as:

$$O_i = e^{\beta_1} \cdot O_j\,.$$

In words: if we start with $x = 0$ and move to $x = 1$, our odds of success ($y = 1$) will change by a multiplicative factor of e^{β_1}.