Music of the Sphere: Connecting the Evolution of the Universe on the Largest Scales to Inflation

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I. DEFINITIONS AND CONSTANTS

Inflation provides a way of seeding perturbations to the metric outside the Hubble horizon (well) before the time of BBN. It is common to quantify these perturbations with the gauge-invariant curvature perturbation of the metric ζ, which is defined by:

$$-\zeta = \psi + H\frac{\delta\rho}{\rho}\,, \tag{1}$$

where ψ is the trace part of the spatial scalar metric perturbations, i.e. writing the most general spatial perturbation to a 4-d metric by $\delta g_{ij} = 2(\psi\delta_{ij} - E_{ij})$ with $\nabla^2 E = 0$, then ψ contains the trace of the perturbation. We also define ρ and $\delta\rho$ to be the mean energy density and the linear energy density perturbation, respectively, as well as the H, the Hubble parameter. H and ρ are related through the first Friedmann equation,

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{\rm Pl}}\rho = \frac{1}{3M_{\rm Pl}}\left(\frac{\dot{\varphi}^2}{2} + \frac{1}{2}\frac{\nabla^2\varphi}{a^2} + V\right)\,, \tag{2}$$

where a is the scale factor and, in the third equality, we have related ρ to the field content of the Universe assuming a single scalar field, φ, is dominating its energy density, as is the case in single field inflation. The 3 terms in the l.h.s. of (2) represent, in order of appearance, the kinetic energy density of the field, its gradient energy density, and its potential energy.

For inflation to occur, the field φ must be dominated by its homogeneous mode, hence at the background level one has $\frac{1}{2}\frac{\nabla^2\varphi}{a^2} = 0$, and its potential energy must dominate over its kinetic energy density. Hence $\dot{\varphi}^2 \ll V$ during inflation. Moreover, during single-field inflation, there is only one physical scalar perturbation degree of freedom representing both scalar metric fluctuations and φ fluctuations. Therefore, through an appropriate gauge choice, it is possible to gauge away all linear fluctuations in φ and be left with only ζ as the scalar perturbation. This gauge choice is commonly called the ζ-gauge, or comoving gauge.

During inflation, H remains almost constant, since $V \gg \dot{\varphi}^2$ and φ is almost static. Deviations from $H = $cst can be quantified by a hierarchy of slow-roll parameters:

$$\epsilon \equiv -\frac{\dot{H}}{H^2}\,; \quad \eta = \frac{\dot{\epsilon}}{\epsilon H}\,; \quad \eta_n = \frac{\dot{\eta}_{n-1}}{\eta_{n-1}N} \quad \text{for } n > 1\,. \tag{3}$$

Inflation requires $\epsilon < 1$, and obtaining at least 60 e-folds requires $\eta < 1$ as well. In the case of single-field inflation, which we will assume in the rest of these notes, the first slow-roll parameter ϵ can also be re-written as:

$$\epsilon = \frac{1}{2}\frac{\dot{\varphi}^2}{M_{\rm Pl}^2 H^2}\,, \tag{4}$$

a formula that will be very useful in the following. Moreover, it will prove useful to use this to re-write H as a function of ϵ and V only:

$$M_{\rm Pl}^2 H^2 = \frac{V}{3-\epsilon} \tag{5}$$

The main observable predicted by inflation is the power spectrum of curvature fluctuations, $\mathcal{P}_\zeta(k)$, defined by:

$$\langle \zeta_k \zeta_k' \,\rangle = (2\pi)^{(3)}\delta^3(k+k')\mathcal{P}_\zeta(k) \tag{6}$$

Explicitly, $\mathcal{P}_\zeta(k)$ can be calculated to be

$$\mathcal{P}_\zeta(k) = \frac{1}{4k^3}\frac{H^2}{\epsilon}\bigg|_{k=aH}\,. \tag{7}$$

Quantities in the above equation are evaluated at horizon exit for every mode. Since each mode ζ_k remains constant outside of the Hubble horizon (see, e.g. Weinberg's proof in astro-ph/0302326), it is sufficient to calculate the amplitude of each mode at horizon exit and fix it from there until horizon re-entry, much after inflation. If we do not want to perform the evaluation at horizon crossing, it is also possible to write an expression involving the explicit k-dependence, through a Hankel function of the first kind:

$$\mathcal{P}_\zeta(k) \sim \frac{\pi}{2}(-\tau)^{2\nu}\left|H_\nu^{(1)}(-k\tau)\right|^2 . \tag{8}$$

Here, τ is the conformal time, related to the cosmic time t by $dt = a d\tau$, and ν is given by:

$$\nu = \frac{3}{2} + \epsilon + \frac{1}{2}\eta \tag{9}$$

to leading order in slow-roll parameters. The dimensionless form of the power spectrum is given by

$$\Delta_\xi^2 \equiv \frac{k^3}{2\pi^2}\mathcal{P}_\zeta(k) . \tag{10}$$

Other useful observables to define are the tilt of the power spectrum, defined by:

$$n_s - 1 \equiv \frac{d\ln\Delta_\zeta^2}{d\ln k} = -\epsilon - 2\eta , \tag{11}$$

where the last equality holds to first order in slow-roll parameters, and the running, defined by (to leading order in slow roll)

$$\alpha_s = -2\eta\epsilon - \eta\eta_2 . \tag{12}$$

Finally, the tensor-to-scalar ratio is given by:

$$r = 16\epsilon . \tag{13}$$

The amplitude of the power spectrum is usually given at a pivot scale, denoted by k_*. For $k_* = 0.05 Mpc^{-1}$, we have $\Delta_\zeta^2 = 2.4 \times 10^{-9}$ from the latest Planck data.

II. RELATION TO THE CMB

The Fourier modes of ζ are related to observable coefficients of spherical harmonics on the last scattering surface through:

$$a_{lm} = 4\pi(-i)^l \int \frac{\mathrm{d}^3k}{(2\pi)^3}\Delta_{T(l)}(k)\zeta_{\vec{k}}Y_{l,m}(\hat{k}) , \tag{14}$$

where the $Y_{l,m}(\hat{k})$s are the spherical harmonics and $\Delta_{T(l)}(k)$ is a transfer function that transforms the primordial, 3 dimensional $\zeta_{\vec{k}}$ at horizon re-entry into the spectrum of metric fluctuations as observed today projected onto the 2d sphere, and can be written as

$$\Delta_{T(l)}(k) = \int_0^{\tau_0} d\tau S_T(k,\tau)P_{T,l}(k\left|\tau_0 - \tau\right|) . \tag{15}$$

This can be understood as an integral over the line of sight over a physical source term (S_T) and a geometric projection $P_{T,l}$ which can be written as a combination of Bessel functions.

Note that, inside the horizon after inflation, it is more convenient to use the Newtonian gauge. Once modes have re-entered the horizon, one can think of ζ and the gravitational potential ϕ (i.e. the metric perturbation in Newtonian gauge) interchangeably.

Moreover, upon horizon re-entry, all the modes $\zeta_{\vec{k}}$ with a fixed $|\vec{k}|$ are in phase and have an amplitude that is randomly distributed. If the fluctuations are Gaussian, the modes $\zeta_{\vec{k}}$ with a fixed $|\vec{k}|$ are drawn from a Gaussian distribution with mean 0 and variance given by $P_\zeta(k)$.

The reconstruction presented in music.pdf allows to recover ζ_k (or equivalently $|\zeta_k|^2$ which is a sample of $\mathcal{P}_\zeta(k)$) for a small, low value range of k-modes.

III. MEASUREMENTS OF $|\zeta_k|^2$

Our goal in the following will be to reconstruct the potential of the inflaton in a model-independent fashion, by using only the samples of the specific ζ_k we get from reconstructing the fluctuations of the gravitational potential in the 3d Universe. Since in k-space we now look at a volume, the number for modes will therefore grow as k^3, as opposed to the usual l^2 when the Universe is projected onto a 2d surface at last scattering. It is useful to see how many samples we get for each additional surface k-shell we add to the considered volume. That is, for every additional shell in $\vec{n}$ space of unit thickness we add the number of additional samples of the norm of ζ_k is

$$\begin{aligned} \delta Vol_{\vec{n}} &= \frac{4\pi}{3}\left(n_2^3 - n_1^3\right) &(16)\\ &= \frac{4\pi}{3}\left(n_2^3 - (n_2-1)^3\right) &(17)\\ &= 4\pi(n^2 - n + 1)\,. &(18)\end{aligned}$$

Therefore, for every additional shell, we add $2\pi(n^2 - n + 1)$ independent measurements, where we divide by two to only consider the half shell, since ζ_k and ζ_{-k} are related to each other through complex conjugation and are therefore not independent. Therefore the procedure from Music.pdf gives us a number of fiducial measurements of $|\zeta(k)|^2$ over a range of $|\vec{k}|$ (or $|\vec{n}|$) values.

IV. RECONSTRUCTING THE POTENTIAL OF THE INFLATON

Only assuming single field inflation, from the measurements of $|\zeta_k|^2$ it is possible to reconstruct locally the shape on the inflaton potential, in a model-independent way. To see how this is possible, we can re-write the expression of the $|\zeta_k|^2$ in terms of $V(\varphi)$ and slow-roll parameters, using (5):

$$|\zeta_k|^2 = \mathcal{P}_\zeta(k) = \frac{1}{4k^3}\frac{V}{M_{\rm Pl}^2\,\epsilon(3-\epsilon)}\bigg|_{k=aH}\,. \tag{19}$$

Given a single fiducial measurement of $\mathcal{P}_\zeta(k)$ at a fiducial value of k_*, we can find an expression for $V(\varphi)$ in the local neighbourhood of $V(\varphi)$ as a function of $V(\varphi_*)$ and slow-roll parameters at φ_*, where φ_* is the value of the background inflaton field at the moment when the mode with wavenumber k_* exited the horizon. To do this, we simply Taylor expand V around φ_*:

$$\begin{aligned} V(\varphi_*+\Delta\varphi) &= V(\varphi_*) + \partial_\varphi V(\varphi)\big|_{\varphi_*}\Delta\varphi + \partial_\varphi^2 V(\varphi)\big|_{\varphi_*}\frac{\Delta\varphi^2}{2} + \partial_\varphi^3 V(\varphi)\big|_{\varphi_*}\frac{\Delta\varphi^3}{3!} + \partial_\varphi^4 V(\varphi)\big|_{\varphi_*}\frac{\Delta\varphi^4}{4!} + \ldots\,; &(20)\\ &= V(\varphi_*)\left[1 + \frac{M_{\rm Pl}}{V(\varphi_*)}\,\partial_\varphi V(\varphi)\big|_{\varphi_*}\frac{\Delta\varphi}{M_{\rm Pl}} + \frac{1}{2}\frac{M_{\rm Pl}^2}{V(\varphi_*)}\,\partial_\varphi^2 V(\varphi)\big|_{\varphi_*}\frac{\Delta\varphi^2}{M_{\rm Pl}^2} + \frac{1}{3!}\frac{M_{\rm Pl}^3}{V(\varphi_*)}\,\partial_\varphi^3 V(\varphi)\big|_{\varphi_*}\frac{\Delta\varphi^3}{M_{\rm Pl}^3}\right.\\ &\qquad\left. + \frac{1}{4!}\frac{M_{\rm Pl}^4}{V(\varphi_*)}\,\partial_\varphi^4 V(\varphi)\big|_{\varphi_*}\frac{\Delta\varphi^4}{M_{\rm Pl}^4} + \ldots\right]\,; &(21)\\ &\equiv V(\varphi_*)\left[1 + d_1\frac{\Delta\varphi}{M_{\rm Pl}} + \frac{1}{2}d_2\frac{\Delta\varphi^2}{M_{\rm Pl}^2} + \frac{1}{3!}d_3\frac{\Delta\varphi^3}{M_{\rm Pl}^3} + \frac{1}{4!}d_4\frac{\Delta\varphi^4}{M_{\rm Pl}^4} + \ldots\right]\,. &(22)\end{aligned}$$

where we have included only relevant and marginal operators in the expansion (up to the fourth derivative of V). Higher derivatives represent irrelevant operators and are neglected here. Since we are only performing a local expansion around a point of the potential for which the range of the inflaton is small and are not considering the full range of the inflaton during inflation, we do not expect a breakdown of perturbation theory; i.e., we consider sub-Planckian excursions of the field around φ_*, $\Delta\varphi < M_{\rm Pl}$, so that the tower of higher order operators suppressed by at least 5 powers of the Planck mass do not become relevant.

Similarly, we can expand ϵ appearing in (19) around the fiducial value ϵ_* at the same φ_*,

$$\epsilon = \epsilon_* + \Gamma_1\frac{\Delta\varphi}{M_{\rm Pl}} + \frac{\Gamma_2}{2}\frac{\Delta\varphi^2}{M_{\rm Pl}^2} + \frac{\Gamma_3}{3!}\frac{\Delta\varphi^3}{M_{\rm Pl}^3} + \frac{\Gamma_4}{4!}\frac{\Delta\varphi^4}{M_{\rm Pl}^4}\,. \tag{23}$$

Every derivative of V (more specifically, the d_i parameters) can be expressed in terms of slow-roll parameters at φ_* alone, and similarly for all the derivatives of ϵ. Therefore, we find a fitting function in terms of the slow-roll parameters. The explicit expressions for the Γ_i and d_i were obtained in the calculation notes accompanying this file. The Γ_i parameters are given by:

$$\Gamma_1 = \sqrt{\frac{\epsilon}{2}}\eta\,; \tag{24}$$

$$\Gamma_2 = \frac{\eta\eta_2}{2} + \frac{\eta^2}{4}\,; \tag{25}$$

$$\Gamma_3 = \frac{1}{2\sqrt{2\epsilon}}\left[\eta_3\eta_2\eta + \eta^2\eta_2 + \eta_2^2\eta\right]\,; \tag{26}$$

$$\Gamma_4 = \frac{\eta\eta_2}{4\epsilon}\left[\eta_4\eta_3 + \eta_3^2 + 3\eta_3\eta_2 + \eta_2^2 + \frac{3}{2}\eta\eta_2 - \frac{1}{2}\eta^2 + \frac{1}{2}\eta\eta_3\right]\,. \tag{27}$$

The d_i parameters are given by:

$$\begin{aligned}
d_1 &= \sqrt{2\epsilon}(3-\epsilon)\left[-3+\epsilon-\frac{1}{2}\eta\right]\,,\\
&\approx -\sqrt{2\epsilon}\left(1+\frac{1}{6}\epsilon+\frac{1}{18}\eta\epsilon\right)\,;
\end{aligned} \tag{28}$$

$$\begin{aligned}
d_2 &= -\frac{1}{12}\frac{1}{(3-\epsilon)}\left(-24\epsilon+6\eta+2\eta\eta_2+8\epsilon^2-10\epsilon\eta+\eta^2\right)\,,\\
&\approx 2\epsilon-\frac{1}{2}\eta-\frac{1}{6}\eta\eta_2\,;
\end{aligned} \tag{29}$$

$$\begin{aligned}
d_3 &= \frac{1}{2\sqrt{2\epsilon}}\frac{1}{(3-\epsilon)}\left[-24\epsilon^2+18\epsilon\eta+7\epsilon\eta\eta_2+8\epsilon^3-18\epsilon^2\eta+6\epsilon\eta^2-\eta\eta_2\left(3+\eta+\eta_2+\eta_3\right)\right]\,,\\
&\approx \frac{8\epsilon^2-6\epsilon\eta-\frac{8}{3}\epsilon\eta\eta_2+\eta\eta_2\left(1+\frac{\eta+\eta_2+\eta_3}{3}\right)}{2\sqrt{2\epsilon}}
\end{aligned} \tag{30}$$

$$\begin{aligned}
d_4 = \frac{1}{(3-\epsilon)}\Bigg[&12\epsilon^2-18\epsilon\eta+\frac{9}{4}\eta^2+6\eta\eta_2-4\epsilon^3+14\epsilon^2\eta-\frac{39}{4}\epsilon\eta^2+\frac{3}{4}\eta^3-8\epsilon\eta\eta_2+\frac{35}{8}\eta^2\eta_2+\frac{9}{4}\eta\eta_2^2+\frac{9}{4}\eta\eta_2\eta_3\\
&+\frac{1}{\epsilon}\left(\frac{3}{8}\eta^2\eta_2-\frac{3}{4}\eta\eta_2^2-\frac{3}{4}\eta\eta_2\eta_3+\frac{\eta^3\eta_2}{8}-\frac{3}{8}\eta^2\eta_2^2-\frac{1}{4}\eta\eta_2^3-\frac{3}{4}\eta\eta_2^2\eta_3-\frac{1}{8}\eta^2\eta_2\eta_3-\frac{1}{4}\eta\eta_2\eta_3^2-\frac{1}{4}\eta\eta_2\eta_3\eta_4\right)\Bigg]\,,
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\approx 4\epsilon^2-6\epsilon\eta+\frac{3}{4}\eta^2+\frac{4}{3}\eta\eta_2+\frac{\eta\eta_2}{3\epsilon}\left(\frac{3}{8}\eta-\frac{3}{4}\eta_2-\frac{3}{4}\eta_3+\frac{\eta^2}{8}-\frac{3}{8}\eta\eta_2-\frac{1}{4}\eta_2^2-\frac{3}{4}\eta_2\eta_3-\frac{1}{8}\eta\eta_3-\frac{1}{4}\eta_3^2-\frac{1}{4}\eta_3\eta_4\right)\\
&\quad+\frac{\eta\eta_2}{3}\left[8\epsilon+\frac{9}{2}\eta+2\eta_2+2\eta_3+\frac{3}{4}\epsilon\eta_2+\frac{3}{4}\epsilon\eta_3-\frac{1}{8}\eta\eta_2-\frac{1}{12}\eta_2^2-\frac{1}{4}\eta_2\eta_3-\frac{1}{24}\eta\eta_3-\frac{1}{12}\eta_3^2-\frac{1}{12}\eta_3\eta_4\right]\,.
\end{aligned} \tag{32}$$

In the above expressions, equations (28)-(32), all slow-roll parameters are evaluated at the time when the mode k_* exits the Hubble horizon and the background field has value φ_*, so they should actually read $(\epsilon_*, \eta_*, (\eta_2)_*, (\eta_3)_*, (\eta_4)_*)$ but we have omitted the $*$ for the sake of simplifying the notation. Moreover, assuming that the slow-roll approximation holds, it is safe to assume that the last slow-roll parameter $(\eta_4)_*$ is negligible, and so we are left with a 5-parameters model of the local $\mathcal{P}_\zeta(k)$ around k_*,

$$\vec{\eta} \equiv \left(V(\varphi_*),\, \epsilon_*,\, \eta_*,\, (\eta_2)_*,\, (\eta_3)_*\right)\,. \tag{33}$$

We also expect a hierarchy between the different parameters, i.e., $1 > (\epsilon_*,\, \eta_*) > (\eta_2)_* > (\eta_3)_*$.

We are now left with the problem of converting measurements at different k for $|\zeta_k|$ into corresponding measurements at different $\Delta\varphi$. To do so, we use the identity $k = aH$, and the identity

$$\ln a \equiv \int \mathrm{d}t\, H(t) \;=\; \int \mathrm{d}\varphi \frac{H}{\dot{\varphi}} \;=\; \int \mathrm{d}\varphi \frac{1}{\sqrt{2\epsilon}M_{\rm Pl}}\,. \tag{34}$$

This leads to

$$k = \exp\left\{\int \frac{\mathrm{d}\varphi}{M_{\rm Pl}}(2\epsilon)^{-1/2}\right\}\frac{1}{M_{\rm Pl}}\sqrt{\frac{V}{3-\epsilon}}, \tag{35}$$

$$= \frac{1}{M_{\rm Pl}}\exp\left\{\int \frac{\mathrm{d}\Delta\varphi}{\sqrt{2}M_{\rm Pl}}\frac{1}{\sqrt{\epsilon_* + \Gamma_1\frac{\Delta\varphi}{M_{\rm Pl}} + \Gamma_2\frac{\Delta\varphi^2}{2M_{\rm Pl}^2} + \Gamma_3\frac{\Delta\varphi^3}{3!M_{\rm Pl}^3} + \Gamma_4\frac{\Delta\varphi^4}{4!M_{\rm Pl}^4}}}\right\}$$
$$\times\sqrt{\frac{V(\varphi_*)\left[1 + d_1\frac{\Delta\varphi}{M_{\rm Pl}} + \frac{1}{2}d_2\frac{\Delta\varphi^2}{M_{\rm Pl}^2} + \frac{1}{3!}d_3\frac{\Delta\varphi^3}{M_{\rm Pl}^3} + \frac{1}{4!}d_4\frac{\Delta\varphi^4}{M_{\rm Pl}^4}\right]}{3 - \epsilon_* + \Gamma_1\frac{\Delta\varphi}{M_{\rm Pl}} + \Gamma_2\frac{\Delta\varphi^2}{2M_{\rm Pl}^2} + \Gamma_3\frac{\Delta\varphi^3}{3!M_{\rm Pl}^3} + \Gamma_4\frac{\Delta\varphi^4}{4!M_{\rm Pl}^4}}}\,. \tag{36}$$

The solution to the integral can be given in terms of elliptical functions. For every k this has to be inverted and a value of $\Delta\varphi$ has to be found, but this cannot be done analytically. At this point I think that for every choice of $\vec{\eta}$, the solution will have to be found numerically.

V. SLOW ROLL PARAMETERS INFERENCE

We want to infer the probability of a given vector of slow-roll parameters $\vec{\eta}$ given the observations, $\mathcal{P}(\vec{\eta}|a_{lm})$. To find an expression for this distribution, we start from

$$\mathcal{P}(\vec{\eta}|f_n, a_{lm}) = \frac{\mathcal{P}(f_n,\, a_{lm}|\vec{\eta})\mathcal{P}(\vec{\eta})}{\mathcal{P}(f_n,\, a_{lm})} = \frac{\mathcal{P}(a_{lm}|f_n,\, \vec{\eta})\mathcal{P}(f_n|\vec{\eta})\mathcal{P}(\vec{\eta})}{\mathcal{P}(f_n,\, a_{lm})} = \frac{\mathcal{P}(a_{lm}|f_n)\mathcal{P}(f_n|\vec{\eta})\mathcal{P}(\vec{\eta})}{\mathcal{P}(f_n,\, a_{lm})} \tag{37}$$

where in the last equality we used that $a_{l,m}$ and $\vec{\eta}$ are conditionally independent given f_n, therefore $\mathcal{P}(a_{lm}|f_n,\, \vec{\eta}) = \mathcal{P}(a_{lm}|f_n)$. Rewriting the denominator, we obtain:

$$\mathcal{P}(\vec{\eta}|f_n, a_{lm}) = \frac{\mathcal{P}(a_{lm}|f_n)\mathcal{P}(f_n|\vec{\eta})\mathcal{P}(\vec{\eta})}{\mathcal{P}(f_n|a_{lm})\mathcal{P}(a_{lm})} \tag{38}$$

The corresponding Bayesian network representation of the joint distribution $\mathcal{P}(a_{lm},\, f_n,\, \vec{\eta})$ is given by

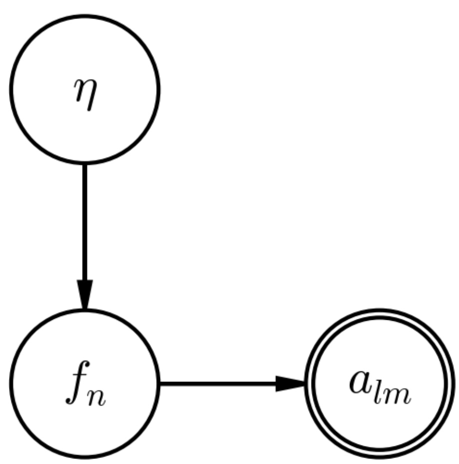

Figure 1: Probabilistic graphical model for the joint distribution $\mathcal{P}(a_{lm},\, f_n,\, \vec{\eta})$.

The quantity we are after is $\mathcal{P}(\vec{\eta}|a_{lm})$, which is marginalized over the potential map (the f_ns). We therefore proceed as follows:

$$\mathcal{P}(\vec{\eta}|a_{lm}) = \int \mathrm{d}f_n \, \mathcal{P}(\vec{\eta}|f_n, a_{lm}) \mathcal{P}(f_n|a_{lm}) \,, \quad (39)$$

$$= \frac{\mathcal{P}(\vec{\eta})}{\mathcal{P}(a_{lm})} \int \mathrm{d}f_n \mathcal{P}(a_{lm}|f_n) \,.\mathcal{P}(f_n|\vec{\eta}) \quad (40)$$

We have:

$$\mathcal{P}(a_{lm}|f_n) = \frac{1}{\sqrt{(2\pi)^p \mathrm{Det}(C_a)}} \exp\left[-\frac{1}{2} \left(a_{lm} - \mathbf{R} f_n\right)^{\mathrm{T}} C_a^{-1} \left(a_{lm} - \mathbf{R} f_n\right) \right] \,, \quad (41)$$

where $p = l_{max}^2 + 2l_{max} + 1$ is the total number of independent measurements in a_{lm} up to the harmonic l_{max}. Assuming the CMB fluctuations are Gaussian, the f_ns are described by a Gaussian distribution with variance given by the inflationary power spectrum:

$$\mathcal{P}(f_n|\vec{\eta}) = \sqrt{\frac{\mathrm{Det}(C_f^{-1})}{(2\pi)^{n_{tot}}}} \exp\left[f_n^{\mathrm{T}} C_f^{-1} f_n \right] \,, \quad (42)$$

with C_f^{-1} a diagonal matrix given by:

$$C_f^{-1} = \begin{pmatrix} 1/\sigma_{n_1}^1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1/\sigma_{n_2}^2 & 0 & & \dots & & \\ \dots & & & & & & \\ 0 & \dots & & & 1/\sigma_{n_i}^2 & & \\ \dots & & & & & \dots & \\ 0 & & & & & & 1/\sigma_{n_{tot}}^2 \end{pmatrix} \,, \quad (43)$$

where the σ_{n_i} only depend on the norm of the $\vec{n}_i$ vector to which they correspond (or equivalently on the norm or the corresponding Fourier mode $|\vec{k}|$). More specifically, they are given in terms of the power spectrum by:

$$\sigma_k^2 = \mathcal{P}_\zeta(k)$$
$$= \frac{1}{4k^3} \frac{V(\varphi_*) \left[1 + d_1 \frac{\Delta\varphi}{M_{\mathrm{Pl}}} + \frac{1}{2} d_2 \frac{\Delta\varphi^2}{M_{\mathrm{Pl}}^2} + \frac{1}{3!} d_3 \frac{\Delta\varphi^3}{M_{\mathrm{Pl}}^3} + \frac{1}{4!} d_4 \frac{\Delta\varphi^4}{M_{\mathrm{Pl}}^4} \right]}{M_{\mathrm{Pl}}^2 \left(\epsilon_* + \Gamma_1 \frac{\Delta\varphi}{M_{\mathrm{Pl}}} + \frac{\Gamma_2}{2} \frac{\Delta\varphi^2}{M_{\mathrm{Pl}}^2} + \frac{\Gamma_3}{3!} \frac{\Delta\varphi^3}{M_{\mathrm{Pl}}^3} + \frac{\Gamma_4}{4!} \frac{\Delta\varphi^4}{M_{\mathrm{Pl}}^4}) \right) \left(3 - \epsilon_* + \Gamma_1 \frac{\Delta\varphi}{M_{\mathrm{Pl}}} + \frac{\Gamma_2}{2} \frac{\Delta\varphi^2}{M_{\mathrm{Pl}}^2} + \frac{\Gamma_3}{3!} \frac{\Delta\varphi^3}{M_{\mathrm{Pl}}^3} + \frac{\Gamma_4}{4!} \frac{\Delta\varphi^4}{M_{\mathrm{Pl}}^4} \right)} \,. \quad (44)$$

Here, the mapping between k and $\Delta\varphi$ has to be done using equation (36). We therefore obtain that the f_n covariance matrix only depends on the $\vec{\eta}$ vector. To completely define the problem, we are left with the task of specifying a prior on $\vec{\eta}$. This can be done using the following constraints:

$$\begin{cases} V(\varphi_*) & \rightarrow & \text{fixed by } H_* \,\varepsilon\, \left[10^{11.5}\,\mathrm{GeV},\, 10^{15}\,\mathrm{GeV}\right] \\ \epsilon_* & \varepsilon & \left[10^{-6},\, 1\right[\\ \eta_* & \varepsilon & \left]-1,\, 1\right[\\ (\eta_2)_* & \varepsilon & \left[-1,\, 1\right] \\ |(\eta_3)_*| & < & |(\eta_2)_*| \end{cases} \quad (45)$$

These are very conservative bounds, drawn only from the requirement that the slow-roll approximation holds. This should be the case since the low k values that we are interested in here correspond to the first few of the 60 e-folds of inflation that should have occurred to solve e.g. the flatness and homogeneity problems. For example, $\epsilon < 1$ is equivalent to the statement the Universe is inflating, and $|\eta| \leq 1$ translate to the requirement that inflation can last for long enough. η_2 and η_3 being higher-order parameters, it is then expected that they should be somewhat smaller. It would be possible put a somewhat stronger prior on ϵ_* using the bound on r from the latest Planck results, however this bound is for a different scale from the one we are looking at here (i.e. the first acoustic peak in the power spectrum), and it varies a lot depending on the assumptions made on other parameters (e.g. the running of the power spectrum).

To get a specific range of allowed $V(\varphi_*)$, in (45) we first used the normalization of the power spectrum at the scale k_* (which I believe was called α) and the allowed range of ϵ_* to obtain a ranged of allowed H_*

$$\alpha = \frac{H_*^2}{8\pi^2 \epsilon_* M_{\mathrm{Pl}}^2} \quad \Rightarrow \quad H_* = \sqrt{2.4 \times 10^{-9}\, 8\pi^2 \epsilon_* (2.435 \times 10^{10}\mathrm{GeV})^2} \,, \quad (46)$$

where we used $\alpha = 2.4 \times 10^{-9}$ and $M_{\rm Pl} = 2.435 \times 10^{10}{\rm GeV})^2$. From there, the range of values of $V(\varphi_*)$ can be found as follows

$$H_*^2 M_{\rm Pl}^2 (3 - \epsilon_*) = V(\varphi_*) \quad \Rightarrow \quad V(\varphi_*) \quad \varepsilon \quad [10^{61}{\rm GeV}^4,\, 1.5 \times 10^{67}{\rm GeV}^4] \tag{47}$$

Performing the integral over ${\rm d}f_n$, we obtain (see handwritten notes for details):

$$\mathcal{P}(\vec{\eta}|a_{lm}) = \sqrt{\frac{{\rm Det}(C_f^{-1})}{(2\pi)^{n_{tot}}}} \frac{1}{\sqrt{(2\pi)^p {\rm Det}(C_a)}} \frac{(2\pi)^{(n_{tot}/2)}}{({\rm Det}\,\mathbf{A})^{1/2}} \exp\left\{\frac{1}{2}\mathbf{B}^{\rm T}\mathbf{A}^{-1}\mathbf{B} - \frac{1}{2}a_{lm}^{\rm T} C_a^{-1} a_{lm}\right\} \frac{\mathcal{P}(\vec{\eta})}{\mathcal{P}(a_{lm})}\,, \tag{48}$$

$$= \frac{1}{(2\pi)^{p/2}} \frac{1}{[{\rm Det}(C_f){\rm Det}(C_a){\rm Det}\,\mathbf{A}]^{1/2}} \exp\left\{\frac{1}{2}\mathbf{B}^{\rm T}\mathbf{A}^{-1}\mathbf{B} - \frac{1}{2}a_{lm}^{\rm T} C_a^{-1} a_{lm}\right\} \frac{\mathcal{P}(\vec{\eta})}{\mathcal{P}(a_{lm})}\,, \tag{49}$$

where the $\mathbf{A}$ and $\mathbf{B}$ matrices are given by:

$$\mathbf{A} = \mathbf{R}^{\rm T} C_a^{-1} \mathbf{R} + C_f^{-1} \tag{50}$$

$$\mathbf{B}^{\rm T} = a_{lm}^{\rm T} C_a^{-1} \mathbf{R} \tag{51}$$