Some definitions & values:

$$\Delta_\zeta = \frac{H^2}{8\pi^2 \varepsilon M_p^2} \qquad K_* = 0.05\,\mathrm{Mpc}^{-1}$$

$$\Delta_\zeta = 2.4\times10^{-9}$$

$$n_s - 1 = -2\varepsilon - \eta$$

$$\varepsilon \equiv -\frac{\dot H}{H^2}\ ;\ \eta = \frac{\dot\varepsilon}{\varepsilon H}\ ;\ \eta_n = \frac{\dot\eta_{n-1}}{\eta_{n-1} H}$$

$$\alpha_s = -2\varepsilon\eta - \eta\eta_2$$

$$r = 16\varepsilon$$

$$(2\pi)^3 \delta^3(k+k') P_\zeta \equiv \langle \zeta(k,t)\zeta(-k,t)\rangle$$

$$P_\zeta = \frac{1}{2k^3}\frac{H^4}{\dot\phi^2} = \frac{1}{4k^3}\frac{H^2}{\varepsilon}\Big|_{k=aH}$$

$$\sim \frac{\pi}{2}(-\tau)^{2\nu}\,|H_\nu^{(1)}(-k\tau)|^2$$

$$\Delta_\zeta \equiv \frac{k^3}{2\pi^2} P_\zeta(k)$$

for a given $|\vec k|$, how many measurements?

$$|\vec k| = \sqrt{k_1^2 + k_2^2 + k_3^2} \Rightarrow \text{grows like } k^3$$

for every $\vec k$, or every $\vec n$, get a sample of

$$\text{Vol tot} \sim \frac{4\pi k^3}{3} \qquad \Delta\vec n = \frac{4\pi}{3}\left(|\vec n_1|^3 - |\vec n_2|^3\right)$$

if thickness $=1$

$$\Delta\vec n = \frac{4\pi}{3}\left(n^3 - (n-1)^3\right)$$
$$= 4/3\,\pi\,[n^3 - (n^3 - 3n^2 + 3n - 1)]$$
$$\Delta\vec n = 4/3\,\pi\,(3n^2 - 3n + 1)$$

$\Rightarrow$ for every shell, we add $\sim n^2$ measurements

$\Rightarrow$ for every k, we get $\dfrac{4\pi(n^2 - n + 1/3)}{2}$ independent samples of $P_\zeta(|\vec n|)$

Here, we divide by 2 since ζ_{-k} and ζ_k are not independent (and are actually the complex conjugate of each other).
This gives us a measurement of $P_\zeta(k)$ at a few fiducial $|\vec k|$'s (or $|\vec n|$'s).

Relation btw modes ζ_k & C_ℓ's in the CMB:

$$a_{\ell m} = 4\pi(-i)^\ell \int \frac{d^3k}{(2\pi)^3}\, \Delta_{T\ell}(k)\, \zeta_{\vec k}\, Y_{\ell m}(\hat k)$$

Transfer fct; spherical harmonic

$$\Delta_{T\ell} = \int_0^{\tau_0} d\tau\, \underbrace{S_T(k,\tau)}_{\text{sources along line of sight}}\, \underbrace{P_{T\ell}(k|\tau_0 - \tau|)}_{\text{projections (bessel fcts)}}$$

$\hookrightarrow$ Inside the horizon, Newtonian gauge, think of ϕ & ζ interchangeably.

$\hookrightarrow$ Important: when they re-enter the horizon, all the modes are in phase and their amplitude for a fixed $|\vec k|$ is randomly distributed. If the fluctuations are Gaussian, then each $\vec k$ w/ fixed $|\vec k|$ is drawn from a Gaussian distribution with mean $P_\zeta(|\vec k|)$ & variance given by a fct of $P_\zeta(|\vec k|)$ (?)

If we have a single fiducial measurement of $P_\zeta(k)$ for a given k_* then we can Taylor expand the potential $V(\phi)$ around ϕ_*, the point where ϕ was when k_* exited the horizon.

$$V(\Delta\phi) = V(\phi_*) + \partial_\phi V|_{\phi_*}\Delta\phi + \frac{\partial^2_\phi V|_{\phi_*}}{2}\Delta\phi^2 + \dots + \frac{1}{4!}\partial^4_\phi V|_{\phi_*}\Delta\phi^4$$

with: $\partial_\phi V = \frac{1}{2}H(-6 + 2\varepsilon - \eta)\dot\phi = \frac{1}{2}M_p H^2\sqrt{2\varepsilon}(-6 + 2\varepsilon - \eta) = M_p H^2\sqrt{2\varepsilon}\,[-3 + \varepsilon - \frac{1}{2}\eta]$

$$\partial^2_\phi V = -\frac{1}{4}H^2\left(8\varepsilon^2 - 2\varepsilon(12 + 5\eta) + \eta(6 + \eta + 2\eta_2)\right)$$
$$\sim -\frac{1}{4}H^2\{-24\varepsilon + 6\eta + 2\eta\eta_2\}$$

$$\partial^3_\phi V = \frac{H\dot\phi}{M_p^2}\left\{2\varepsilon^2 - 3\varepsilon\left(2 + \frac{3}{2}\eta\right) + \frac{\eta}{4}(18 + 6\eta + 7\eta_2) - \frac{\eta\eta_2}{4\varepsilon}(3 + \eta + \eta_2 + \eta_3)\right)$$

Note that $\varepsilon = \dfrac{\dot\phi^2}{2m_p^2 H^2} \Rightarrow \dfrac{\dot\phi^2}{2} = \varepsilon m_p^2 H^2$

$$3M_p^2 H^2 = \frac{\dot\phi^2}{2} + V = \varepsilon M_p^2 H^2 + V \Rightarrow H^2 M_p^2 (3 - \varepsilon) = V \Rightarrow H^2 M_p^2 = V/(3-\varepsilon)$$

$$\Rightarrow d_0 = \frac{V}{3m_p^2 H^2} = \frac{3-\varepsilon}{3} = 1 - \varepsilon/3$$

$$d_1 = \frac{M_p}{V}\partial_\phi V = \frac{M_p^2 H^2}{V}\sqrt{2\varepsilon}\,[-3 + \varepsilon - \tfrac{1}{2}\eta] = \sqrt{2\varepsilon}/(3-\varepsilon)\,[-3 + \varepsilon - \tfrac{1}{2}\eta]$$
$$= \frac{\sqrt{2\varepsilon}}{3}\left[1 + \frac{\varepsilon}{3} + \frac{\varepsilon^2}{9}\right][-3 + \varepsilon - \tfrac{1}{2}\eta]$$
$$= \frac{\sqrt{2\varepsilon}}{3}\left[-3 - \varepsilon + \varepsilon - \tfrac{1}{2}\eta - \tfrac{1}{6}\varepsilon\eta + \tfrac{\varepsilon^2}{3} - \tfrac{\varepsilon^2}{3}\right]$$
$$= \frac{\sqrt{2\varepsilon}}{3}\left[-3 - \tfrac{1}{2}\eta - \tfrac{1}{6}\varepsilon\eta\right] = -\sqrt{2\varepsilon}\left[1 + \tfrac{1}{6}\eta + \tfrac{1}{18}\eta\varepsilon\right]$$

$$d_2 = \frac{M_p^2}{V}\partial^2_\phi V = -\frac{1}{4}\frac{M_p^2 H^2}{V}\left(8\varepsilon^2 - 2\varepsilon(12 + 5\eta) + \eta(6 + \eta + 2\eta_2)\right)$$
$$= -\frac{1}{4}\frac{1}{(3-\varepsilon)}\left[8\varepsilon^2 - 2\varepsilon(12 + 5\eta) + \eta(6 + \eta + 2\eta_2)\right]$$
$$= -\frac{1}{12}(1 + \varepsilon/3)(8\varepsilon^2 - 24\varepsilon - 10\varepsilon\eta + 6\eta + \eta^2 + 2\eta\eta_2)$$

$+8\varepsilon^2 - \varepsilon^2 8 - 10\varepsilon^2\eta - 10\varepsilon\eta + \eta^2 + \varepsilon\eta$

$$= -\frac{1}{12}\left[-24\varepsilon + 6\eta + 2\eta\eta_2 - \frac{10\varepsilon^2}{3}\eta - 8\eta\varepsilon + \eta^2 + \frac{\eta^2\varepsilon}{3} + \frac{2\eta\eta_2\varepsilon}{3}\right]$$
$$= 2\varepsilon - \frac{1}{2}\eta - \frac{1}{6}\eta\eta_2 + \frac{2}{3}\eta\varepsilon + \frac{5\varepsilon^2}{18}\eta - \frac{1}{36}\eta^2\varepsilon - \frac{1}{18}\eta\eta_2\varepsilon$$
$$\sim 2\varepsilon - \frac{1}{2}\eta - \frac{1}{6}\eta\eta_2$$

$$d_3 = \frac{M_p^3}{V}\partial^3_\phi V = \frac{M_p}{V}\frac{H\dot\phi}{4\varepsilon}\left\{8\varepsilon^3 - 24\varepsilon^2 - 18\varepsilon^2\eta + \varepsilon\eta 18 + 6\varepsilon\eta^2 + 7\varepsilon\eta\eta_2 - 3\eta\eta_2 - \eta^2\eta_2 - \eta\eta_2^2 - \eta\eta_2\eta_3\right\}$$
$$= \frac{M_p^2 H^2}{V}\frac{\sqrt{2\varepsilon}}{4\varepsilon}\left\{-24\varepsilon^2 + 18\varepsilon\eta - 3\eta\eta_2 + \dots\right\}$$
$$= \frac{1}{2}\frac{1}{\sqrt{2\varepsilon}}\frac{1}{(3-\varepsilon)}\left\{-24\varepsilon^2 + 18\varepsilon\eta - 3\eta\eta_2 + 7\varepsilon\eta\eta_2 - \eta^2\eta_2 - \eta\eta_2^2 + 8\varepsilon^3 - 18\varepsilon^2\eta + 6\varepsilon\eta^2 - \eta\eta_2\eta_3\right\}$$
$$= \frac{1}{2}\frac{1}{\sqrt{2\varepsilon}}\frac{1}{3}\left(1 - \frac{\varepsilon}{3} + \frac{\varepsilon^2}{9}\right)\{\ \}$$
$$= \frac{1}{6}\frac{1}{\sqrt{2\varepsilon}}\left\{-24\varepsilon^2 + 18\varepsilon\eta - 3\eta\eta_2 + 7\varepsilon\eta\eta_2 - \eta^2\eta_2 - \eta\eta_2^2 + \varepsilon\eta\eta_2 + 8\varepsilon^3 - 6\varepsilon^2\eta + 8\varepsilon^3 - 18\varepsilon^2\eta + 6\varepsilon\eta^2 - \eta\eta_2\eta_3\right\}$$
$$= \frac{1}{6}\frac{1}{\sqrt{2\varepsilon}}\left\{-24\varepsilon^2 + 18\varepsilon\eta - 3\eta\eta_2 + 8\varepsilon\eta\eta_2 - \eta^2\eta_2 - \eta\eta_2^2 + 16\varepsilon^3 - 24\varepsilon^2\eta + 6\varepsilon\eta^2 - \eta\eta_2\eta_3\right\}$$
$$= \frac{1}{2}\frac{1}{\sqrt{2\varepsilon}}\left\{-8\varepsilon^2 + 6\varepsilon\eta - \eta\eta_2 + \frac{8}{3}\varepsilon\eta\eta_2 - \frac{1}{3}\eta^2\eta_2 - \frac{\eta\eta_2^2}{3} + \frac{16}{3}\varepsilon^3 - 8\varepsilon^2\eta + 2\varepsilon\eta^2 - \frac{1}{3}\eta\eta_2\eta_3\right\}$$

$$d_4 = \frac{M_p^4}{V}\partial^4_\phi V = \frac{M_p^2 H^2}{V}\Big[12\varepsilon^2 - 18\varepsilon\eta + \frac{9}{4}\eta^2 + 6\eta\eta_2 - 4\varepsilon^3 + 14\varepsilon^2\eta - \frac{39}{4}\varepsilon\eta^2 + \frac{3}{4}\eta^3 - 8\varepsilon\eta\eta_2 + \frac{35}{8}\eta^2\eta_2 + \frac{9}{4}\eta\eta_2^2 + \frac{9}{4}\eta\eta_2\eta_3$$
$$+ \frac{1}{\varepsilon}\left(\frac{3}{8}\eta^2\eta_2 - \frac{3}{4}\eta\eta_2^2 - \frac{3}{4}\eta\eta_2\eta_3 + \frac{\eta^3\eta_2}{8} - \frac{3}{8}\eta^2\eta_2^2 - \frac{1}{4}\eta\eta_2^3 - \frac{3}{4}\eta\eta_2^2\eta_3 - \frac{1}{8}\eta^2\eta_2\eta_3 - \frac{1}{4}\eta\eta_2\eta_3^2 - \frac{1}{4}\eta\eta_2\eta_3\eta_4\right)\Big]$$

$$= \frac{M_p^2 H^2}{M_p^2 H^2(3-\varepsilon)}\left[12\varepsilon^2 - 18\varepsilon\eta + \frac{9}{4}\eta^2 + 6\eta\eta_2 - 4\varepsilon^3 + 14\varepsilon^2\eta - \frac{39}{4}\varepsilon\eta^2\right.$$
$$= \frac{1}{3}\left[1 - \frac{\varepsilon}{3} + \frac{\varepsilon^2}{9}\right]\left[12\varepsilon^2 - 18\varepsilon\eta + \frac{9}{4}\eta^2 + 6\eta\eta_2 - 4\varepsilon^3 + 14\varepsilon^2\eta - \frac{39}{4}\varepsilon\eta^2 + \frac{3}{4}\eta^3 - 8\varepsilon\eta\eta_2 + \frac{35}{8}\eta^2\eta_2 + \frac{9}{4}\eta\eta_2^2 + \frac{9}{4}\eta\eta_2\eta_3 + \frac{1}{\varepsilon}\left[\frac{3}{8}\eta^2\eta_2 - \frac{3}{4}\eta\eta_2^2 - \frac{3}{4}\eta\eta_2\eta_3 + \dots\right]\right]$$
$$= \frac{1}{3}\left[12\varepsilon^2 - 18\varepsilon\eta + \frac{9}{4}\eta^2 + 6\eta\eta_2 - 8\varepsilon\eta\eta_2 + \frac{35}{8}\eta^2\eta_2 + \frac{9}{4}\eta\eta_2^2 + \frac{9}{4}\eta\eta_2\eta_3 + \frac{1}{\varepsilon}[\ ] + \frac{1}{8}\eta^2\eta_2 - \frac{1}{4}\eta\eta_2^2 - \frac{1}{4}\eta\eta_2\eta_3 - \frac{1}{8}\eta^2\eta_2^2 - \frac{1}{12}\eta\eta_2^3\right.$$
$$\left. - \frac{1}{4}\eta\eta_2^2\eta_3 - \frac{1}{24}\eta^2\eta_2\eta_3 - \frac{1}{12}\eta\eta_2\eta_3^2 - \frac{1}{12}\eta\eta_2\eta_3\eta_4\right.$$
$$= 4\varepsilon^2 - 6\varepsilon\eta + \frac{3}{4}\eta^2 + \frac{4}{3}\eta\eta_2 + \frac{1}{3}\left[8\varepsilon\eta\eta_2 + \frac{9}{2}\eta^2\eta_2 + 2\eta\eta_2^2 + 2\eta\eta_2\eta_3 + \frac{1}{\varepsilon}\{\ \} - \frac{1}{8}\eta^2\eta_2^2 - \frac{1}{12}\eta\eta_2^3 - \frac{1}{4}\eta\eta_2^2\eta_3 - \frac{1}{24}\eta^2\eta_2\eta_3 - \frac{1}{12}\eta\eta_2\eta_3^2 - \frac{1}{12}\eta\eta_2\eta_3\eta_4\right]$$
$$+ \frac{3}{4}\varepsilon\eta\eta_2^2 + \frac{3}{4}\varepsilon\eta\eta_2\eta_3$$

see 1403.4585

Then re-write the potential as:

$$V(\phi) \simeq V(\phi_*)\left[1 + d_1\frac{\Delta\phi}{M_p} + \frac{1}{2}d_2\frac{\Delta\phi^2}{M_p^2} + \frac{1}{3!}d_3\frac{\Delta\phi^3}{M_p^3} + \frac{1}{4!}d_4\frac{\Delta\phi^4}{M_p^4}\right]$$

with the amplitude of the modes of ϕ we directly probe the power spectrum

$$P_\zeta = \frac{1}{4k^3}\frac{H^2}{\varepsilon}\Big|_{k=aH}$$

And we know $H^2 = \dfrac{V}{M_p^2(3-\varepsilon)} \Rightarrow P_\zeta = \dfrac{1}{4k^3}\dfrac{V}{M_p^2(3-\varepsilon)\varepsilon}\Big|_{k=aH}$

$$\boxed{\dot\phi = \sqrt{2\varepsilon}\,M_p H}$$

choose + root since $\phi \to -\phi$ is not physically relevant.

$$\partial_t\left(\frac{\dot\varepsilon}{\sqrt{2\varepsilon}M_p H}\right) = \partial_t\left(\sqrt{\frac{\varepsilon}{2}}\frac{1}{M_p}\frac{\dot\varepsilon}{\varepsilon H}\right) = \partial_t\left(\sqrt{\frac{\varepsilon}{2}}\frac{1}{M_p}\eta\right)$$
$$= \frac{\eta}{M_p}\frac{1}{2\sqrt2}\frac{\dot\varepsilon}{\sqrt\varepsilon} + \frac{\dot\eta}{M_p}\sqrt{\frac{\varepsilon}{2}} = \frac{\eta H}{M_p\sqrt\varepsilon}\left[\frac{\dot\varepsilon}{H 2\sqrt2} + \frac{\dot\eta\,\varepsilon}{\eta H\sqrt2}\right] = \frac{\eta H}{\sqrt\varepsilon M_p}\left[\frac{\dot\varepsilon}{H2\sqrt2} + \frac{\eta_2\varepsilon}{\sqrt2}\right]$$

$$\varepsilon(t) = \varepsilon_* + \partial_\phi\varepsilon\,\Delta\phi + \mathcal{O}(\Delta\phi^2)$$
$$= \varepsilon_* + \frac{\dot\varepsilon}{\dot\phi}\Delta\phi + \partial_\phi\left(\frac{\dot\varepsilon}{\dot\phi}\right)\frac{\Delta\phi^2}{2} + \mathcal{O}(\Delta\phi^3)$$
$$= \varepsilon_* + \frac{\dot\varepsilon}{\sqrt{2\varepsilon}H}\frac{\Delta\phi}{M_p} + \partial_t\left(\frac{\dot\varepsilon}{\dot\phi}\right)\frac{1}{\dot\phi}\frac{\Delta\phi^2}{2} + \mathcal{O}(\Delta\phi^3)$$
$$= \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\frac{\dot\varepsilon}{\varepsilon H}\frac{\Delta\phi}{M_p} + \frac{\ddot\varepsilon}{\dot\phi^2}\frac{\Delta\phi^2}{2} - \frac{\dot\varepsilon\ddot\phi}{\dot\phi^3}\frac{\Delta\phi^2}{2} + \mathcal{O}(\Delta\phi^3)$$
$$= \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\,\eta\frac{\Delta\phi}{M_p} + \frac{\ddot\varepsilon}{2\varepsilon M_p^2 H^2}\frac{\Delta\phi^2}{2} - \frac{\dot\varepsilon}{\dot\phi^2}\frac{\ddot\phi}{\dot\phi}\frac{\Delta\phi^2}{2}$$
$$+ \frac{\cancel{H^2}\eta\cancel{\varepsilon}(\eta_2 + \eta - \varepsilon)}{2\cancel{\varepsilon H^2}}\frac{\Delta\phi^2}{2M_p^2}$$
$$+ \frac{\eta(\eta_2 + \eta - \varepsilon)}{4}\frac{\Delta\phi^2}{M_p^2}$$

$$\eta_2 = \frac{\dot\eta}{H\eta} = \frac{1}{H\eta}\partial_t\left(\frac{\dot\varepsilon}{H\varepsilon}\right)$$
$$= \frac{1}{H\eta}\left[\frac{\ddot\varepsilon}{H\varepsilon} - \frac{\dot\varepsilon}{(H\varepsilon)^2}(\dot H\varepsilon + H\dot\varepsilon)\right]$$
$$= \frac{\ddot\varepsilon}{H^2\eta\varepsilon} - \frac{\dot H\dot\varepsilon\cancel{\varepsilon}}{H\eta H^2\varepsilon^{\cancel{2}}} - \frac{\cancel{H}\dot\varepsilon^2}{H\eta H^{\cancel{2}}\varepsilon^2}$$
$$= \frac{\ddot\varepsilon}{H^2\eta\varepsilon} + \frac{\varepsilon}{2}\eta - \frac{\eta^2}{2}$$
$$\eta_2 = \frac{\ddot\varepsilon}{H^2\eta\varepsilon} + \varepsilon - \eta \quad \boxed{\Rightarrow(\eta_2 + \eta - \varepsilon)H^2\eta\varepsilon = \ddot\varepsilon}$$

Second order

$$\frac{\ddot\varepsilon}{\dot\phi^2} = \frac{\ddot\varepsilon}{2\varepsilon H^2 M_p^2} = \frac{1}{2H}\frac{\eta}{M_p^2} \quad \Big/\!\!\Big/ \quad \frac{\ddot\phi}{\dot\phi} = \frac{\partial_t(\sqrt{2\varepsilon}HM_p)}{\sqrt{2\varepsilon}HM_p} = \frac{\sqrt2 M_p}{\sqrt2 M_p}\frac{\frac{\dot\varepsilon H}{2\sqrt\varepsilon} + \dot H\sqrt\varepsilon}{\sqrt\varepsilon H} = \frac{\dot\varepsilon}{2\varepsilon} + \frac{\dot H}{H} = \frac{H\dot\varepsilon}{2\varepsilon H} - H\varepsilon = \frac{H\eta}{2} - H\varepsilon$$

$$\Rightarrow \frac{\dot\varepsilon}{\dot\phi^2}\frac{\ddot\phi}{\dot\phi} = \frac{1}{2\cancel{H}}\frac{\eta}{M_p^2}\cancel{H}\left(\frac{\eta}{2} - \varepsilon\right) = \left(\frac{\eta^2}{4} - \frac{\varepsilon\eta}{2}\right)\frac{1}{M_p^2}$$

$$\Rightarrow \varepsilon = \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\,\eta\frac{\Delta\phi}{M_p} + \left[\frac{\eta\eta_2}{4} + \frac{\eta^2}{4} - \frac{\cancel{\eta\varepsilon}}{4} - \frac{\eta^2}{8} + \frac{\cancel{\varepsilon\eta}}{4}\right]\frac{\Delta\phi^2}{M_p^2} + \mathcal{O}(\Delta\phi^3/M_p^3)$$

$$\varepsilon = \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\,\eta\frac{\Delta\phi}{M_p} + \left[\frac{\eta\eta_2}{4} + \frac{\eta^2}{8}\right]\frac{\Delta\phi^2}{M_p^2} + \frac{\partial_\phi}{3!}\left(\frac{\eta\eta_2}{2} + \frac{\eta^2}{4}\right)\frac{\Delta\phi^3}{M_p^2} + \mathcal{O}(\Delta\phi^4/M_p^4)$$

third order $\rightarrow$

$$\frac{1}{\dot\phi}\partial_t\left(\frac{\eta\eta_2}{2} + \frac{\eta^2}{4}\right) = \frac{1}{\sqrt{2\varepsilon}HM_p}\left(\frac{\eta\dot\eta_2}{2} + \frac{\eta_2\dot\eta}{2} + \frac{\eta\dot\eta}{2}\right)$$
$$= \frac{1}{\sqrt{2\varepsilon}M_p}\left(\frac{\eta\eta_2}{2}\frac{\dot\eta_2}{H\eta_2} + \frac{\eta_2\eta}{2}\frac{\dot\eta}{H\eta} + \frac{\eta^2}{2}\frac{\dot\eta}{\eta H}\right)$$
$$= \frac{1}{\sqrt{2\varepsilon}M_p}\left(\frac{\eta\eta_2\eta_3}{2} + \frac{\eta_2^2\eta}{2} + \frac{\eta^2\eta_2}{2}\right)$$

$$\Rightarrow \varepsilon = \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\,\eta\frac{\Delta\phi}{M_p} + \left[\frac{\eta\eta_2}{4} + \frac{\eta^2}{8}\right]\frac{\Delta\phi^2}{M_p^2} + \frac{1}{6}\frac{1}{\sqrt{2\varepsilon}}\left[\frac{\eta\eta_2\eta_3}{2} + \frac{\eta\eta_2^2}{2} + \frac{\eta^2\eta_2}{2}\right]\frac{\Delta\phi^3}{M_p^3} + \mathcal{O}(\Delta\phi^4/M_p^4)$$

fourth order $\rightarrow$

$$\frac{\partial_t}{\dot\phi}\left(\frac{1}{\sqrt{2\varepsilon}}\left[\frac{\eta\eta_2\eta_3}{2} + \frac{\eta\eta_2^2}{2} + \frac{\eta^2\eta_2}{2}\right]\frac{1}{M_p^3}\right)$$

$$\frac{1}{\sqrt{2\varepsilon}M_p H}\left[\frac{-1}{2\sqrt2}\frac{\dot\varepsilon}{\varepsilon^{3/2}}[\ ] + \frac{1}{\sqrt{2\varepsilon}}\ \frac{1}{2}\{\dot\eta\eta_2\eta_3 + \eta\dot\eta_2\eta_3 + \eta\eta_2\dot\eta_3 + \dot\eta\eta_2^2 + 2\eta\eta_2\dot\eta_2 + 2\eta\dot\eta\eta_2 + \eta^2\dot\eta_2\}\right]$$

$$\frac{1}{M_p}\left[\frac{-1}{2\cdot2\varepsilon}\eta[\ ] + \frac{1}{2\varepsilon}\left(\frac{1}{2}\{\eta\eta_2^2\eta_3 + \eta\eta_2\eta_3^2 + \eta\eta_2\eta_3\eta_4 + \eta\eta_2^3 + 2\eta\eta_2^2\eta_3 + 2\eta^2\eta_2^2 + \eta^2\eta_2\eta_3\}\right)\right]$$

$$\frac{1}{M_p}\left[-\frac{1}{4\varepsilon}\eta[\ ] + \frac{1}{4\varepsilon}\left(\eta\eta_2\eta_3\left\{3\eta_2 + \eta_3 + \eta_4 + \frac{\eta_2^2}{\eta_3} + \frac{2\eta\eta_2}{\eta_3} + \eta\right\}\right)\right]$$

$$\frac{1}{4\varepsilon M_p}\left[-\frac{\eta^2\eta_2}{2}[\eta_3 + \eta_2 + \eta] + \eta\eta_2\eta_3\left\{3\eta_2 + \eta_3 + \eta_4 + \frac{\eta_2^2}{\eta_3} + \frac{2\eta\eta_2}{\eta_3} + \eta\right\}\right]$$
$$\left[-\frac{1}{2}\eta\eta_2\eta_3\left[\eta + \frac{\eta\eta_2}{\eta_3} + \frac{\eta^2}{\eta_3}\right] + \eta\eta_2\eta_3\left\{3\eta_2 + \eta_3 + \eta_4 + \frac{\eta_2^2}{\eta_3} + \frac{2\eta\eta_2}{\eta_3} + \eta\right\}\right]$$

$$\frac{1}{M_p}\Gamma_4 \equiv \boxed{\frac{1}{4\varepsilon M_p}\left[\eta\eta_2\eta_3\left\{\frac{1}{2}\eta + 3\eta_2 + \eta_3 + \eta_4 + \frac{\eta_2^2}{\eta_3} + \frac{3}{2}\frac{\eta\eta_2}{\eta_3} - \frac{1}{2}\frac{\eta^2}{\eta_3}\right.\right.}$$

$$\Rightarrow \boxed{\varepsilon = \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\,\eta\frac{\Delta\phi}{M_p} + \left[\frac{\eta\eta_2}{4} + \frac{\eta^2}{8}\right]\frac{\Delta\phi^2}{M_p^2} + \frac{1}{6}\frac{1}{\sqrt{2\varepsilon}}\left[\frac{\eta\eta_2\eta_3}{2} + \frac{\eta\eta_2^2}{2} + \frac{\varepsilon\eta^2}{4} + \frac{\eta_2\eta\varepsilon}{4}\right]\frac{\Delta\phi^3}{M_p^3} + \Gamma_4\frac{\Delta\phi^4}{M_p^4} + \mathcal{O}\left(\frac{\Delta\phi^5}{M_p^5}\right)}$$

$\hookrightarrow$ all $\varepsilon, \eta, \eta_2, \eta_3$ are evaluated at $*$-time

We also need to transform k values into distances in $\Delta\phi$ along the potential...

$$k = aH \sim e^{Ht}H \qquad H = \frac{1}{M_p}\sqrt{\frac{V}{(3-\varepsilon)}}$$

$$\text{or} \rightarrow \ln a = \int dt\, H = \int d\phi\,\frac{H}{\dot\phi} = \int d\phi\,\frac{\cancel{H}}{\sqrt{2\varepsilon}\cancel{H}M_p} = \int d\phi\,\frac{1}{\sqrt{2\varepsilon}M_p}$$

$$\Rightarrow k = \exp\left(\int d\phi/M_p\,(2\varepsilon)^{-1/2}\right)\frac{1}{M_p}\sqrt{\frac{V}{3-\varepsilon}} = \exp\left(\int\frac{d\phi}{M_p}\frac{1}{\sqrt2}\frac{1}{\sqrt{\varepsilon_* + \Gamma_1\Delta\phi + \Gamma_2\Delta\phi^2 + \Gamma_3\Delta\phi^3 + \Gamma_4\Delta\phi^4}}\right)\frac{1}{M_p} \times \sqrt{\frac{V}{3-\varepsilon}}$$

This is probably not the easiest way to do this.
Instead of evaluating the power spectrum at horizon crossing, maybe it is easier to look at the full k-dependent power spectrum.
But then, how do we relate $P_\zeta(k)$ to $\Delta\phi$ & the potential?
I'm not sure this would work...

Now, putting together
1- the $V(\phi)$ Taylor expansion
2- the ε expansion
3- the conversion from k to $\Delta\phi/M_p$

we get a fitting formula in terms of 4 parameters (or 3 if we neglect η_3) for the samples of the power spectrum. With these best-fit values, we can reconstruct the local shape of the inflationary potential that the Hubble volume we live in went through. We can also find local values of n_s & α_s over those few e-folds.
I guess since the parameters $\varepsilon, \eta, \eta_2, \eta_3$ depend on each other, once the best-fit model is found we can find how consistent the parameters are with each other... eg test if $\eta = \dfrac{\dot\varepsilon}{\varepsilon H}$

$$\frac{H^2 M_p^2}{V} = \frac{1}{(3-\varepsilon)}$$

$$\partial_\phi V = \partial_\phi\left\{ H^2 M_p^2 (3-\varepsilon)\right\} = \frac{\partial_t}{\dot\phi}\left\{ H^2 M_p^2(3-\varepsilon)\right\} = \frac{2H}{\dot\phi}\dot H M_p^2(3-\varepsilon) + \frac{H^2 M_p^2}{\dot\phi}(-\dot\varepsilon)$$

$$= \frac{2HM_p^2}{\dot\phi}\dot H(3-\varepsilon) - \frac{HM_p^2}{\dot\phi}H^2\varepsilon\frac{\dot\varepsilon}{\varepsilon H}$$

$$\frac{1}{2}H\dot\phi(-6+2\varepsilon-\eta)$$

$$= \frac{M_p 2}{\sqrt{2\varepsilon}}\frac{\dot H}{H^2}(3-\varepsilon)H^2 - \frac{M_p}{\sqrt{2\varepsilon}}H^2\varepsilon\eta$$

$$\frac{1}{2}H\sqrt{2\varepsilon}HM_p(-6+2\varepsilon-\eta)$$

$$= M_p\sqrt{\frac{\varepsilon}{2}}H^2(-2(3-\varepsilon)-\eta)$$

$$\frac{H^2M_p}{\sqrt{2\varepsilon}}(-6\varepsilon+2\varepsilon^2-\eta\varepsilon)$$

$$= M_p H^2\sqrt{\frac{\varepsilon}{2}}(-6+2\varepsilon-\eta) \checkmark$$

$$\frac{H^2M_p}{\sqrt{2\varepsilon}}$$

$$\Rightarrow \partial_\phi^2 V = \partial_\phi\left[M_p H^2\sqrt{\frac{\varepsilon}{2}}(-6+2\varepsilon-\eta)\right]$$

$$= \frac{\partial_t}{\dot\phi}\left[M_pH^2\sqrt{\frac{\varepsilon}{2}}(-6+2\varepsilon-\eta)\right] = \frac{\cancel{M_p}}{\sqrt{2\varepsilon}\,\cancel{M_p}H}\left\{2H\dot H\sqrt{\frac{\varepsilon}{2}}(-6+2\varepsilon-\eta) + \frac{1}{2}\frac{H^2\dot\varepsilon}{\sqrt{2\varepsilon}}(\;)\right.$$
$$\left. + H^2\sqrt{\frac{\varepsilon}{2}}(2\dot\varepsilon-\dot\eta)\right\}$$

$$= \frac{1}{\sqrt{2\varepsilon}H}\left\{\sqrt{2\varepsilon}H^3(-\varepsilon)(-6+2\varepsilon-\eta) + \frac{H^3}{2\sqrt{2\varepsilon}}\varepsilon\eta(-6+2\varepsilon-\eta) + H^3\sqrt{\frac{\varepsilon}{2}}(2\varepsilon\eta-\eta\eta_2)\right\}$$

$$= H^2(6\varepsilon-2\varepsilon^2+\eta\varepsilon) + \frac{H^2}{4\varepsilon}\varepsilon(-6\eta+2\varepsilon\eta-\eta^2) + \frac{H^2}{2}(2\varepsilon\eta-\eta\eta_2)$$

$$= H^2\left(6\varepsilon-\frac{3}{2}\eta-2\varepsilon^2+\varepsilon\eta+\frac{1}{2}\varepsilon\eta+\varepsilon\eta-\frac{1}{4}\eta^2-\frac{\eta\eta_2}{2}\right)$$

$$= H^2\left(6\varepsilon-\frac{3}{2}\eta-2\varepsilon^2+\frac{5}{2}\varepsilon\eta-\frac{1}{4}\eta^2-\frac{\eta\eta_2}{2}\right)$$

$$= -\frac{H^2}{4}\left\{-24\varepsilon+6\eta+8\varepsilon^2-10\varepsilon\eta+\eta^2+2\eta\eta_2\right\} \checkmark$$

$$\Rightarrow \partial_\phi^3 V = \partial_\phi\left\{\frac{-H^2}{4}\left[-24\varepsilon+6\eta+8\varepsilon^2-10\varepsilon\eta+\eta^2+2\eta\eta_2\right]\right\}$$

$$= \frac{1}{4\dot\phi}\left\{-2H\dot H[\;] - H^2\{-24\dot\varepsilon+6\dot\eta+16\varepsilon\dot\varepsilon-10\varepsilon\dot\eta-10\dot\varepsilon\eta+2\eta\dot\eta+2\dot\eta\eta_2+2\eta\dot\eta_2\}\right\}$$

$$= \frac{1}{4\sqrt{2\varepsilon}M_p}\left\{-2H^2\frac{\dot H}{H^2}[\;] - H^2\left[-24\eta\varepsilon+6\eta\eta_2+16\varepsilon^2\eta-10\varepsilon\eta\eta_2-10\eta^2\varepsilon+2\eta^2\eta_2+2\eta\eta_2^2+2\eta\eta_2\eta_3\right]\right\}$$

$$\frac{H\dot\phi}{4\varepsilon} = \frac{H^2M_p\sqrt{2\varepsilon}}{4\varepsilon} = \frac{H^2M_p}{2\sqrt{2\varepsilon}}$$

$$M_p^2\partial_\phi^3 V = \frac{H^2M_p}{4\sqrt{2\varepsilon}}\left\{2\varepsilon[\;]+24\eta\varepsilon-6\eta\eta_2-16\varepsilon^2\eta+10\varepsilon\eta\eta_2+10\eta^2\varepsilon-2\eta\eta_2(\eta+\eta_2+\eta_3)\right\}$$

$$= \frac{H^2M_p}{2\sqrt{2\varepsilon}}\left\{\varepsilon[\;]+12\eta\varepsilon-\eta\eta_2-8\varepsilon^2\eta+5\varepsilon\eta\eta_2+5\eta^2\varepsilon-\eta\eta_2(\eta+\eta_2+\eta_3)\right\}$$

$$= \left\{-24\varepsilon^2+6\varepsilon\eta+8\varepsilon^3-10\varepsilon^2\eta+\varepsilon\eta^2+2\eta\eta_2\varepsilon+12\varepsilon\eta-8\varepsilon^2\eta+5\varepsilon\eta\eta_2+5\varepsilon\eta^2-\eta\eta_2(3+\eta+\eta_2+\eta_3)\right\}$$

$$= \left\{-24\varepsilon^2+18\varepsilon\eta+8\varepsilon^3-18\varepsilon^2\eta+6\varepsilon\eta^2+7\varepsilon\eta\eta_2 \quad -\eta\eta_2(3+\eta+\eta_2+\eta_3)\right\}$$

$$\Rightarrow \partial_\phi^4 V M_p^2 = \frac{\partial_t}{\dot\phi}\left[\frac{H^2M_p}{2\sqrt{2\varepsilon}}\left\{-24\varepsilon^2+18\varepsilon\eta+8\varepsilon^3-18\varepsilon^2\eta+6\varepsilon\eta^2+7\varepsilon\eta\eta_2-\eta\eta_2(3+\eta+\eta_2+\eta_3)\right\}\right]$$

$$= \frac{\cancel{M_p}}{2\sqrt{2}}\frac{1}{H\cancel{M_p}\sqrt{2\varepsilon}}\left[-\frac{1}{2}\frac{\dot\varepsilon}{\varepsilon^{3/2}}H^2\{\;\} + \frac{2H\dot H}{\sqrt\varepsilon}\{\;\} + \frac{H^3}{\sqrt\varepsilon}\left\{-48\frac{\varepsilon\dot\varepsilon}{\varepsilon H}+18\frac{\dot\varepsilon\eta}{\eta H}\eta+18\frac{\dot\varepsilon\eta}{\varepsilon H}\varepsilon+24\varepsilon^3\eta-36\varepsilon^2\eta^2-18\varepsilon^2\eta\eta_2\right.\right.$$
$$+6\varepsilon\eta^3+12\varepsilon\eta^2\eta_2+7\varepsilon\eta^2\eta_2+7\varepsilon\eta\eta_2^2+7\varepsilon\eta\eta_2\eta_3-\eta\eta_2^2(3+\eta+\eta_2+\eta_3)$$
$$\left.\left.-\eta\eta_2\eta_3(3+\eta+\eta_2+\eta_3)-\eta\eta_2(\eta\eta_2+\eta_2\eta_3+\eta_3\eta_4)\right\}\right]$$

$$= \frac{H^2}{4\varepsilon}\left[-\frac{1}{2}\eta\{\;\}-2\varepsilon\{\;\}-48\varepsilon^2\eta+18\varepsilon\eta\eta_2+18\varepsilon\eta^2+24\varepsilon^3\eta-36\varepsilon^2\eta^2-18\varepsilon^2\eta\eta_2+6\varepsilon\eta^3+12\varepsilon\eta^2\eta_2+7\varepsilon\eta^2\eta_2\right.$$
$$+7\varepsilon\eta\eta_2^2+7\varepsilon\eta\eta_2\eta_3-(\eta\eta_2^2+\eta\eta_2\eta_3)(3+\eta+\eta_2+\eta_3)$$
$$\left.-\eta\eta_2(\eta\eta_2+\eta_2\eta_3+\eta_3\eta_4)\right]$$

$$+3 \quad -\frac{9}{4} \quad -1 \quad +\frac{9}{4} \quad -\frac{3}{4} \quad -\frac{7}{8} \quad +\frac{1}{8}$$

$$= H^2\left[-\frac{1}{8}\left\{-24\varepsilon\eta+18\eta^2+8\varepsilon^2\eta-18\varepsilon\eta^2+6\eta^3+7\eta^2\eta_2-\frac{\eta^2\eta_2}{\varepsilon}(3+\eta+\eta_2+\eta_3)\right\}\right.$$
$$-\frac{1}{2}\left\{+18\varepsilon\eta \quad -18\varepsilon^2\eta+6\varepsilon\eta^2 \quad -\eta^2\eta_2\right.$$
$$-9 \quad +9 \quad -3 \quad +\frac{1}{2}$$
$$-12\varepsilon\eta+\frac{9}{2}\eta^2+6\varepsilon^2\eta-9\varepsilon\eta^2+\frac{3}{2}\eta^3+\frac{19}{4}\eta^2\eta_2$$

$$-4 \quad +12 \quad -9/2 \quad +3/2 \quad +1/2 \quad +\frac{1}{2}$$
$$+8\varepsilon^3 \quad -24\varepsilon^2 \quad +7\varepsilon\eta\eta_2-3\eta\eta_2-\eta\eta_2^2-\eta\eta_2\eta_3$$
$$-\frac{9}{2}\varepsilon\eta\eta_2+\frac{9}{2}\eta\eta_2-3\frac{\eta\eta_2^2}{4\varepsilon}-3\frac{\eta\eta_2\eta_3}{4\varepsilon}+\frac{7}{4}\eta\eta_2^2+\frac{7}{4}\eta\eta_2\eta_3\left[-\eta^2\eta_2^2-\eta\eta_2^3-\eta\eta_2^2\eta_3-\eta^2\eta_2\eta_3-\eta\eta_2^2\eta_3-\eta\eta_2\eta_3^2 \atop -\eta^2\eta_2^2 \quad -\eta\eta_2^2\eta_3 \quad -\eta\eta_2\eta_3\eta_4\right]$$

$$\times\frac{1}{4\varepsilon}$$

$$= H^2\left[-18\varepsilon\eta+\frac{9}{4}\eta^2+14\varepsilon^2\eta-\frac{39}{4}\varepsilon\eta^2+\frac{3}{4}\eta^3+\frac{35}{8}\eta^2\eta_2+\frac{\eta^2\eta_2}{8\varepsilon}(3+\eta+\eta_2+\eta_3)-4\varepsilon^3+12\varepsilon^2-8\varepsilon\eta\eta_2+6\eta\eta_2+\frac{9}{4}\eta\eta_2^2+\frac{9}{4}\eta\eta_2\eta_3-\frac{3}{4}\frac{\eta\eta_2^2}{\varepsilon}-\frac{3\eta\eta_2\eta_3}{4\varepsilon}-\frac{1}{4\varepsilon}[\;]\right.$$

$$\frac{1}{\varepsilon}\left[\frac{3\eta^2\eta_2}{8}-\frac{3}{4}\eta\eta_2^2-3\frac{\eta\eta_2\eta_3}{4}-\frac{\eta^2\eta_2^2}{2}-\frac{\eta\eta_2^3}{4}-3\frac{\eta\eta_2^2\eta_3}{4}-\frac{\eta^2\eta_2\eta_3}{4}-\frac{\eta\eta_2\eta_3^2}{4}-\frac{\eta\eta_2\eta_3\eta_4}{4}\right]$$
$$+\frac{\eta^3\eta_2}{8} \qquad +\frac{\eta^2\eta_2^2}{8} \qquad +\frac{\eta^2\eta_2\eta_3}{8}$$
$$-\frac{3\eta^2\eta_2^2}{8} \qquad -\frac{\eta^2\eta_2\eta_3}{8}$$

$$= H^2\left[12\varepsilon^2-18\varepsilon\eta+\frac{9}{4}\eta^2+6\eta\eta_2-4\varepsilon^3+14\varepsilon^2\eta-\frac{39}{4}\varepsilon\eta^2+\frac{3}{4}\eta^3-8\varepsilon\eta\eta_2+\frac{35}{8}\eta^2\eta_2+\frac{9}{4}\eta\eta_2^2+\frac{9}{4}\eta\eta_2\eta_3\right.$$

$$\left.+\frac{1}{\varepsilon}\left(\frac{3}{8}\eta^2\eta_2-\frac{3}{4}\eta\eta_2^2-\frac{3}{4}\eta\eta_2\eta_3+\frac{\eta^3\eta_2}{8}-\frac{3}{8}\eta^2\eta_2^2-\frac{1}{4}\eta\eta_2^3-\frac{3}{4}\eta\eta_2^2\eta_3-\frac{1}{8}\eta^2\eta_2\eta_3-\frac{1}{4}\eta\eta_2\eta_3^2-\frac{1}{4}\eta\eta_2\eta_3\eta_4\right)\right]$$

Inflationary Parameters Inference:

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

$P(a_{\ell m}|f, \vec{\eta}, H_0)$

want: $P(\vec{\eta}\,|\,a_{\ell m}, f_n, H_0) = \dfrac{P(a_{\ell m}|f_n H_0)\,P(f_n|\vec{\eta}, H_0)\,P(\eta|H_0)}{P(a_{\ell m}|H_0)}$ ← how to derive?

(1) $\rightarrow P(a_{\ell m}|f, H_0) \rightarrow$ what Roger calculates

$\propto \exp\left\{-\frac{1}{2}(a_{\ell m} - Rf_n)^T C_{f_n}^{-1}(a_{\ell m} - Rf_n)\right\}$

(2) $\rightarrow P(\eta|H_0) \rightarrow$

$$\begin{cases} V(\varphi_*) \rightarrow \text{anything btw } 10^{11.5} - 10^{15}\,\text{GeV for } H_0 \\ \varepsilon_* \in \{\varepsilon_\xi, 1\} \quad \text{where } \varepsilon_\xi = \frac{H_0}{2\pi}\frac{\tilde{\xi}}{\varphi_0} \;\&\; \tilde{\xi} \text{ is a Weiner process} \\ \qquad\qquad\qquad\qquad \varepsilon_0 \sim 10^{-6} \\ \eta_* \in \{-1, 1\} \\ (\eta_2)_* \in \{-1, 1\}, \text{ but probably less than } \eta_* \\ |(\eta_3)_*| < |(\eta_3)_*| \end{cases}$$

$$P_k \sim \frac{H^2}{8\pi^2 M_p^2 \varepsilon} \sim \frac{10^{2n}\,\text{GeV}^2}{8\pi^2 (2.4\times 10^{18})^2 \text{GeV}^2\, 10^{-6}}$$

$$\sim \frac{10^{2n}\,\cancel{\text{GeV}^2}}{\underbrace{72 \cdot 4}_{10^2}\; 10^{30}\,\cancel{\text{GeV}^2}} \sim \frac{10^{2n}}{10^{32}} = 10^{-9}$$

$10^{2n} = 10^{23}$

$n = 11.5$

$\rightarrow \dfrac{10^{2n}}{10^{38}} \sim 10^{-9} \rightarrow 10^{2n} \sim 10^{29} \Rightarrow n = 14.5$

CONFUSION!

$$\frac{P(f, a_{\ell m} \cap \eta)}{P(a_{\ell m}, f)} \overset{?}{=} \frac{P(\eta, f \cap a_{\ell m})}{P(a_{\ell m})} \quad \text{NO}$$

$$P(\eta|f_k) = \frac{P(f_n|\eta)P(\eta)}{P(f_n)}$$

$$P(\eta|f_k, a_{\ell m}) = \frac{P(f, a_{\ell m}|\eta)\,P(\eta)}{P(a_{\ell m}, f)} = \frac{P(a_{\ell m}|f, \eta)P(f|\eta)P(\eta)}{P(a_{\ell m}, f)}$$

?!

$P(\eta|f, a_{\ell m}, H_0) \overset{?}{=} P(f, a_{\ell m}|\eta H_0)\,P(\eta|H_0)/P(f, a_{\ell m}) \rightarrow \dfrac{P(\eta \cap f \cap a_{\ell m} \cap H_0)}{P(f \cap a_{\ell m} \cap H_0)} = \dfrac{P(f, a_{\ell m} \cap \eta \cap H_0)}{P(f, a_{\ell m})}$

$$P(\eta|f, a_{\ell m}, H_0) = \frac{P(f, a_{\ell m}, H_0|\eta)\,P(\eta)}{P(f, a_{\ell m}, H_0)} = \frac{P(a_{\ell m}|f_n, \vec{\eta}, H_0)\,P(f_n|\vec{\eta}, H_0)\,\overbrace{P(H_0|\vec{\eta})P(\vec{\eta})}^{=P(\vec{\eta}|H_0)P(H_0)}}{\int P(f, a_{\ell m}, H_0|\eta)\,d\eta} \quad ?!$$

$$\boxed{P(\eta|f, a) = \frac{P(f, a|\eta)\,P(\eta)}{P(a, f)} = \frac{P(a|f, \eta)P(f|\eta)P(\eta)}{\int P(a, f|\eta)P(\eta)\,d\eta}}$$

doesn't the denominator also depends on f?!

(3) $\rightarrow P(f|\vec{\eta}, H_0) \rightarrow \dfrac{\exp\left\{-\frac{1}{2} f_n^T (\sigma_f^2)^{-1} f_n\right\}}{Z_f}$

$$(\sigma_f^2)^{-1} = \begin{pmatrix} 1/\sigma_1^2 & & & \\ & 1/\sigma_2^2 & & \\ & & 1/\sigma_3^2 & \\ & & & \ddots \end{pmatrix}$$

$$\sigma_i^2 = P_\zeta(k_i)$$

Marginalize over f_n:

$$P(\eta|a_{\ell m}) \propto \int df_n\, P(f_n|\eta)\,P(a_{\ell m}|f_n)\,P(\eta)$$

$$= \int df_n \exp\left\{-\frac{1}{2}(a_{\ell m} - Rf_n)^T C_{a_{\ell m}}^{-1}(a_{\ell m} - Rf_n) - \frac{1}{2} f_n^T C_f^{-1} f_n\right\} P(\eta)$$

$$= \int df_n \exp\left\{-\frac{1}{2} f_n^T R^T C_a^{-1} R f_n + a_{\ell m}^T C_a^{-1} R f_n - \frac{1}{2} a_{\ell m}^T C_a^{-1} a_{\ell m} - \frac{1}{2} f_n^T C_f^{-1} f_n\right\} P(\eta)$$

$$= \int df_n \exp\left\{-\frac{1}{2} f_n^T \underbrace{\left[R^T C_a^{-1} R + C_f^{-1}\right]}_{A} f_n + \underbrace{a_{\ell m}^T C_a^{-1} R}_{B^T} f_n\right\} \exp\left\{-\frac{1}{2} a_{\ell m}^T C_a^{-1} a_{\ell m}\right\} P(\eta)$$

$$= \frac{(2\pi)^{n/2}}{\det A^{1/2}} \exp\left\{\frac{1}{2} B^T A^{-1} B\right\} \exp\left\{-\frac{1}{2} a_{\ell m}^T C_a^{-1} a_{\ell m}\right\} P(\eta)$$