Some definitions & values:

$$\Delta_\zeta = \frac{H^2}{8\pi^2 \varepsilon M_p^2} \qquad K_* = 0.05\,\mathrm{Mpc}^{-1}$$

$$\Delta_\zeta = 2.4\times 10^{-9}$$

$$n_s - 1 = -2\varepsilon - \eta$$

$$\varepsilon \equiv -\frac{\dot H}{H^2}\,;\quad \eta = \frac{\dot\varepsilon}{\varepsilon H}\,;\quad \eta_n = \frac{\dot\eta_{n-1}}{\eta_{n-1} H}$$

$$\alpha_s = -2\varepsilon\eta - \eta\eta_2$$

$$r = 16\varepsilon$$

$$P_\zeta = \langle \zeta(k,t)\zeta(k,t)\rangle = \frac{1}{2k^3}\frac{H^4}{\dot\phi^2} = \frac{1}{4k^3}\frac{H^2}{\varepsilon}\Big|_{k=aH}$$

$$\Delta_\zeta = \frac{k^3}{2\pi^2} P_\zeta(k) \qquad \sim \frac{\pi}{2}(-\tau)^{2\nu}\,|H_\nu^{(1)}(-k\tau)|^2$$

for a given $|\vec k|$, how many measurements?

$$|\vec k| = \sqrt{k_1^2 + k_2^2 + k_3^2} \Rightarrow \text{grows like } k^3$$

for every $\vec k$, or every $\vec n$, get a sample of

$$\mathrm{Vol}_{tot} \sim \frac{4\pi k^3}{3} \qquad \Delta\vec n = \frac{4\pi}{3}\left(|\vec n_1|^3 - |\vec n_2|^3\right)$$

if thickness $= 1$

$$\Delta\vec n = \frac{4\pi}{3}\left(n^3 - (n-1)^3\right)$$
$$= 4/3\,\pi\left[n^3 - (n^3 - 3n^2 + 3n - 1)\right]$$
$$\Delta\vec n = 4/3\,\pi\,(3n^2 - 3n + 1)$$

$\Rightarrow$ for every shell, we add $\sim n^2$ measurements

$\Rightarrow$ for every k, we get $4\pi(n^2 - n + 1/3)$ independent samples of $P_\zeta(|\vec n|)$

This gives us a measurement of $P_\zeta(k)$ at a few fiducial $|\vec k|$'s (or $|\vec n|$'s).

Relation btw modes ζ_k & C_ℓ's in the CMB:

$$a_{\ell m} = 4\pi(-i)^\ell \int \frac{d^3k}{(2\pi)^3}\, \Delta_{T\ell}(k)\, \zeta_{\vec k}\, Y_{\ell m}(\hat k)$$

$\Delta_{T\ell}$: Transfer fct; $Y_{\ell m}$: spherical harmonic

$$\hookrightarrow = \int_0^{\tau_0} d\tau\, \underbrace{S_T(k,\tau)}_{\text{sources along line of sight}}\, \underbrace{P_{T\ell}(k|\tau_0 - \tau|)}_{\text{Projections (bessel fcts)}}$$

↳ Inside the horizon, Newtonian gauge, think of ϕ & ζ interchangeably.

↳ Important: when they re-enter the horizon, all the modes are in phase and their amplitude for a fixed $|\vec k|$ is randomly distributed. If the fluctuations are Gaussian, then each $\vec k$ w/ fixed $|\vec k|$ is drawn from a Gaussian distribution with mean $P_\zeta(|\vec k|)$ & variance given by a fct of $P_\zeta(|\vec k|)$ (?)

If we have a single fiducial measurement of $P_\zeta(k)$ for a given k_* then we can Taylor expand the potential $V(\phi)$ around ϕ_*, the point where ϕ was when k_* exited the horizon

$$V(\Delta\phi) = V(\phi_*) + \partial_\phi V|_{\phi_*}\Delta\phi + \frac{\partial_\phi^2 V|_{\phi_*}}{2}\Delta\phi^2 + \ldots + \frac{1}{4!}\partial_\phi^4 V|_{\phi_*}\Delta\phi^4$$

with: $\partial_\phi V = \frac{1}{2}H(-6 + 2\varepsilon - \eta)\dot\phi = \frac{1}{2}M_p H^2\sqrt{2\varepsilon}(-6 + 2\varepsilon - \eta) = M_p H^2 \sqrt{2\varepsilon}\left[-3 + \varepsilon - \frac{1}{2}\eta\right]$

$$\partial_\phi^2 V = -\frac{1}{4}H^2\left(8\varepsilon^2 - 2\varepsilon(12 + 5\eta) + \eta(6 + \eta + 2\eta_2)\right)$$
$$\sim -\frac{1}{4}H^2\{-24\varepsilon + 6\eta + 2\eta\eta_2\}$$
$$\partial_\phi^3 V = \frac{H\dot\phi}{M_p^2}\left\{2\varepsilon^2 - 3\varepsilon\left(2 + \frac{3}{2}\eta\right) + \frac{\eta}{4}(18 + 6\eta + 7\eta_2) - \frac{\eta\eta_2}{4\varepsilon}(3 + \eta + \eta_2 + \eta_3)\right)$$

Note that $\varepsilon = \frac{\dot\phi^2}{2M_p^2 H^2} \Rightarrow \frac{\dot\phi^2}{2} = \varepsilon M_p^2 H^2$

$$3M_p^2 H^2 = \frac{\dot\phi^2}{2} + V = \varepsilon M_p^2 H^2 + V \Rightarrow H^2 M_p^2 (3 - \varepsilon) = V \Rightarrow H^2 M_p^2 = V/(3-\varepsilon)$$

$$\Rightarrow d_0 = \frac{V}{3M_p^2 H^2} = \frac{3-\varepsilon}{3} = 1 - \varepsilon/3$$

$$d_1 = \frac{M_p \partial_\phi V}{V} = \frac{M_p^2 H^2}{V}\sqrt{2\varepsilon}\left[-3 + \varepsilon - \frac{1}{2}\eta\right] = \sqrt{2\varepsilon}/(3-\varepsilon)\left[-3 + \varepsilon - \frac{1}{2}\eta\right]$$
$$= \frac{\sqrt{2\varepsilon}}{3}\left[1 + \frac{\varepsilon}{3} + \frac{\varepsilon^2}{9}\right]\left[-3 + \varepsilon - \frac{1}{2}\eta\right]$$
$$= \frac{\sqrt{2\varepsilon}}{3}\left[-3 - \varepsilon + \varepsilon - \frac{1}{2}\eta - \frac{1}{6}\varepsilon\eta + \frac{\varepsilon^2}{3} - \frac{\varepsilon^2}{3}\right]$$
$$= \frac{\sqrt{2\varepsilon}}{3}\left[-3 - \frac{1}{2}\eta - \frac{1}{6}\varepsilon\eta\right] = -\sqrt{2\varepsilon}\left[1 + \frac{1}{6}\eta + \frac{1}{18}\eta\varepsilon\right]$$

$$d_2 = \frac{M_p^2}{V}\partial_\phi^2 V = -\frac{1}{4}\frac{M_p^2 H^2}{V}\left(8\varepsilon^2 - 2\varepsilon(12 + 5\eta) + \eta(6 + \eta + 2\eta_2)\right)$$
$$= -\frac{1}{4}\frac{1}{(3-\varepsilon)}\left[8\varepsilon^2 - 2\varepsilon(12 + 5\eta) + \eta(6 + \eta + 2\eta_2)\right]$$
$$= -\frac{1}{12}(1 + \varepsilon/3)(8\varepsilon^2 - 24\varepsilon - 10\varepsilon\eta + 6\eta + \eta^2 + 2\eta\eta_2)$$
$$= -\frac{1}{12}\left[-24\varepsilon + 6\eta + 2\eta\eta_2 - 8\varepsilon^2 + \frac{10}{3}\varepsilon^2\eta + 2\varepsilon\eta + \frac{\eta^2\varepsilon}{3} + \frac{2\eta\varepsilon\eta_2}{3}\right]$$
$$= 2\varepsilon - \frac{1}{2}\eta - \frac{1}{6}\eta\eta_2 + \frac{4}{3}\varepsilon^2 - \frac{5}{18}\varepsilon^2\eta - \frac{1}{6}\varepsilon\eta - \frac{1}{36}\eta^2\varepsilon - \frac{1}{18}\eta\varepsilon\eta_2$$
$$\sim 2\varepsilon - \frac{1}{2}\eta - \frac{1}{6}\eta\eta_2$$

d_3 & d_4 similar...

Then re-write the potential as:

$$V(\phi) \simeq V(\phi_*)\left[1 + d_1\frac{\Delta\phi}{M_p} + \frac{1}{2}d_2\frac{\Delta\phi^2}{M_p^2} + \frac{1}{3!}d_3\frac{\Delta\phi^3}{M_p^3} + \frac{1}{4!}d_4\frac{\Delta\phi^4}{M_p^4}\right]$$

What we probe directly with modes of ϕ perturbations is the power spectrum

$$P_\zeta = \frac{1}{4k^3}\frac{H^2}{\varepsilon}\Big|_{k=aH}$$

And we know $H^2 = \frac{V}{M_p^2(3-\varepsilon)} \Rightarrow P_\zeta = \frac{1}{4k^3}\frac{V}{M_p^2(3-\varepsilon)\varepsilon}\Big|_{k=aH}$

$$\boxed{\dot\phi = \sqrt{2\varepsilon}\,M_p H}$$

choose + root since $\phi \to -\phi$ is not physically relevant.

$$\varepsilon(t) = \varepsilon_* + \partial_\phi\varepsilon\,\Delta\phi + \mathcal{O}(\Delta\phi^2)$$
$$= \varepsilon_* + \frac{\dot\varepsilon}{\dot\phi}\Delta\phi + \partial_\phi\left(\frac{\dot\varepsilon}{\dot\phi}\right)\frac{\Delta\phi^2}{2} + \mathcal{O}(\Delta\phi^3)$$

$$\partial_t\left(\frac{\dot\varepsilon}{\sqrt{2\varepsilon}M_p H}\right) = \partial_t\left(\sqrt{\frac{\varepsilon}{2}}\frac{1}{M_p}\frac{\dot\varepsilon}{\varepsilon H}\right) = \partial_t\left(\sqrt{\frac{\varepsilon}{2}}\frac{1}{M_p}\eta\right)$$
$$= \frac{\eta}{M_p}\frac{1}{2\sqrt2}\frac{\dot\varepsilon}{\sqrt\varepsilon} + \frac{\dot\eta}{M_p}\sqrt{\frac{\varepsilon}{2}} = \frac{\eta}{M_p}\frac{H}{\sqrt\varepsilon}\left[\frac{\dot\varepsilon}{H2\sqrt2} + \frac{\dot\eta}{\eta H}\frac{\varepsilon}{\sqrt2}\right] = \frac{\eta H}{\sqrt\varepsilon M_p}\left[\frac{\dot\varepsilon}{H2\sqrt2} + \frac{\eta_2\varepsilon}{\sqrt2}\right]$$

$$= \varepsilon_* + \frac{\dot\varepsilon}{\sqrt{2\varepsilon}H}\frac{\Delta\phi}{M_p} + \partial_t\left(\frac{\dot\varepsilon}{\dot\phi}\right)\frac{1}{\dot\phi}\frac{\Delta\phi^2}{2} + \mathcal{O}(\Delta\phi^3)$$
$$= \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\frac{\dot\varepsilon}{\varepsilon H}\frac{\Delta\phi}{M_p} + \frac{\ddot\varepsilon}{\dot\phi^2}\frac{\Delta\phi^2}{2} - \frac{\dot\varepsilon\ddot\phi}{\dot\phi^3}\frac{\Delta\phi^2}{2} + \mathcal{O}(\Delta\phi^3)$$
$$= \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\eta\frac{\Delta\phi}{M_p} + \frac{\ddot\varepsilon}{2\varepsilon M_p^2 H^2}\frac{\Delta\phi^2}{2} - \frac{\dot\varepsilon}{\dot\phi^2}\frac{\ddot\phi}{\dot\phi}\frac{\Delta\phi^2}{2}$$

$$\eta_2 = \frac{\dot\eta}{H\eta} = \frac{1}{H\eta}\partial_t\left(\frac{\dot\varepsilon}{H\varepsilon}\right)$$
$$= \frac{1}{H\eta}\left[\frac{\ddot\varepsilon}{H\varepsilon} - \frac{\dot\varepsilon}{(H\varepsilon)^2}(\dot H\varepsilon + H\dot\varepsilon)\right]$$
$$= \frac{\ddot\varepsilon}{H^2\eta\varepsilon} - \frac{\dot H\dot\varepsilon\varepsilon}{H\eta H^2\varepsilon^2} - \frac{H\dot\varepsilon^2}{H\eta H^2\varepsilon^2}$$
$$= \frac{\ddot\varepsilon}{H^2\eta\varepsilon} + \frac{\varepsilon}{\eta}\eta - \frac{\eta^2}{\eta}$$
$$\eta_2 = \frac{\ddot\varepsilon}{H^2\eta\varepsilon} + \varepsilon - \eta \quad \boxed{\Rightarrow (\eta_2 + \eta - \varepsilon)H^2\eta\varepsilon = \ddot\varepsilon}$$

$$+ \frac{H^2\eta\varepsilon(\eta_2 + \eta - \varepsilon)}{2\varepsilon H^2}\frac{\Delta\phi^2}{2M_p^2}$$
$$+ \frac{\eta(\eta_2 + \eta - \varepsilon)}{4}\frac{\Delta\phi^2}{M_p^2}$$

Second order

$$\frac{\dot\varepsilon}{\dot\phi^2} = \frac{\dot\varepsilon}{2\varepsilon H^2 M_p^2} = \frac{1}{2H}\frac{\eta}{M_p^2} \;\Big/\!\!\Big/\; \frac{\ddot\phi}{\dot\phi} = \frac{\partial_t(\sqrt{2\varepsilon}HM_p)}{\sqrt{2\varepsilon}HM_p} = \frac{\sqrt2 M_p}{\sqrt2 M_p}\frac{\frac{\dot\varepsilon H}{2\sqrt\varepsilon} + \dot H\sqrt\varepsilon}{\sqrt\varepsilon H} = \frac{\dot\varepsilon}{2\varepsilon} + \frac{\dot H}{H} = \frac{H\dot\varepsilon}{2\varepsilon H} - H\varepsilon = \frac{H\eta}{2} - H\varepsilon$$

$$\Rightarrow \frac{\dot\varepsilon}{\dot\phi^2}\frac{\ddot\phi}{\dot\phi} = \frac{1}{2H}\frac{\eta}{M_p^2}H\left(\frac{\eta}{2} - \varepsilon\right) = \left(\frac{\eta^2}{4} - \frac{\varepsilon\eta}{2}\right)\frac{1}{M_p^2}$$

$$\Rightarrow \varepsilon = \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\eta\frac{\Delta\phi}{M_p} + \left[\frac{\eta\eta_2}{4} + \frac{\eta^2}{4} - \cancel{\frac{\eta\varepsilon}{4}} - \frac{\eta^2}{8} + \cancel{\frac{\varepsilon\eta}{4}}\right]\frac{\Delta\phi^2}{M_p^2} + \mathcal{O}(\Delta\phi^3/M_p^3)$$

$$\varepsilon = \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\eta\frac{\Delta\phi}{M_p} + \left[\frac{\eta\eta_2}{4} + \frac{\eta^2}{8}\right]\frac{\Delta\phi^2}{M_p^2} + \frac{\partial_\phi}{3!}\left(\frac{\eta\eta_2}{2} + \frac{\varepsilon\eta}{4}\right)\frac{\Delta\phi^3}{M_p^2} + \mathcal{O}(\Delta\phi^4/M_p^4)$$

third order $\to$

$$\frac{1}{\dot\phi}\partial_t\left(\frac{\eta\eta_2}{2} + \frac{\varepsilon\eta}{4}\right) = \frac{1}{\sqrt{2\varepsilon}HM_p}\left(\frac{\dot\eta\eta_2}{2} + \frac{\eta\dot\eta_2}{2} + \frac{\dot\varepsilon\eta}{4} + \frac{\varepsilon\dot\eta}{4}\right)$$
$$= \frac{1}{\sqrt{2\varepsilon}M_p}\left(\frac{\eta\eta_2}{2}\frac{\dot\eta_2}{H\eta_2} + \frac{\eta_2\eta}{2}\frac{\dot\eta}{H\eta} + \frac{\dot\varepsilon}{H\varepsilon}\frac{\varepsilon\eta}{4} + \frac{\dot\eta}{\eta H}\frac{\eta\varepsilon}{4}\right)$$
$$\frac{1}{\sqrt{2\varepsilon}M_p}\left(\frac{\eta\eta_2\eta_3}{2} + \frac{\eta_2^2\eta}{2} + \frac{\eta^2\varepsilon}{4} + \frac{\eta_2\eta\varepsilon}{4}\right)$$

$$\Rightarrow \varepsilon = \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\eta\frac{\Delta\phi}{M_p} + \left[\frac{\eta\eta_2}{4} + \frac{\eta^2}{8}\right]\frac{\Delta\phi^2}{M_p^2} + \frac{1}{6}\frac{1}{\sqrt{2\varepsilon}}\left[\frac{\eta\eta_2\eta_3}{2} + \frac{\eta\eta_2^2}{2} + \frac{\varepsilon\eta^2}{4} + \frac{\eta_2\eta\varepsilon}{4}\right]\frac{\Delta\phi^3}{M_p^3} + \mathcal{O}(\Delta\phi^4/M_p^4)$$

fourth order $\to$

$$\frac{\partial_t}{\dot\phi}\left(\frac{1}{\sqrt{2\varepsilon}}\left[\frac{\eta\eta_2\eta_3}{2} + \frac{\eta\eta_2^2}{2} + \frac{\varepsilon\eta^2}{4} + \frac{\eta_2\eta\varepsilon}{4}\right]\frac{1}{M_p^3}\right)$$

$$\frac{1}{\sqrt{2\varepsilon}M_pH}\left[\frac{-1}{2\sqrt2}\frac{\dot\varepsilon}{\varepsilon^{3/2}}[\,] + \frac{1}{\sqrt{2\varepsilon}}\left(\frac{1}{2}\{\dot\eta\eta_2\eta_3 + \eta\dot\eta_2\eta_3 + \eta\eta_2\dot\eta_3 + \dot\eta\eta_2^2 + 2\eta\eta_2\dot\eta_2\} + \frac{1}{4}\{\dot\varepsilon\eta^2 + 2\varepsilon\eta\dot\eta + \dot\varepsilon\eta\eta_2 + \varepsilon\dot\eta\eta_2 + \varepsilon\eta\dot\eta_2\}\right)\right]$$

$$\frac{1}{M_p}\left[\frac{-1}{2\cdot2\varepsilon}\eta[\,] + \frac{1}{2\varepsilon}\left(\frac{1}{2}\{\eta\eta_2^2\eta_3 + \eta\eta_2\eta_3^2 + \eta\eta_2\eta_3\eta_4 + \eta\eta_2^3 + 2\eta\eta_2^2\eta_3\} + \frac{1}{4}\{\varepsilon\eta^3 + 2\varepsilon\eta^2\eta_2 + \varepsilon\eta^2\eta_2 + \varepsilon\eta\eta_2^2 + \varepsilon\eta\eta_2\eta_3\}\right)\right]$$

$$\frac{1}{M_p}\left[-\frac{1}{4\varepsilon}\eta[\,] + \frac{1}{4\varepsilon}\left(\eta\eta_2\eta_3\left\{3\eta_2 + \eta_3 + \eta_4 + \frac{\eta_2^2}{\eta_3}\right\} + \frac{\varepsilon\eta\eta_2}{2}\left\{\frac{\eta^2}{\eta_2} + 3\eta + \eta_2 + \eta_3\right\}\right)\right]$$

$$\frac{1}{4\varepsilon M_p}\left[-\frac{\eta^2\eta_2}{2}\left[\eta_3 + \eta_2 + \frac{\varepsilon\eta}{2\eta_2} + \frac{\varepsilon}{2}\right] + \eta\eta_2\eta_3\left\{3\eta_2 + \eta_3 + \eta_4 + \frac{\eta_2^2}{\eta_3}\right\} + \frac{\varepsilon\eta\eta_2}{2}\left\{\frac{\eta^2}{\eta_2} + 3\eta + \eta_2 + \eta_3\right\}\right]$$

$-\frac{\varepsilon\eta^3}{4}$ $\quad -\frac{\eta^2\eta_2\varepsilon}{4}$ $\quad \frac{\varepsilon\eta^3}{2}$ $\quad \frac{\varepsilon}{4}\eta^2\eta_2\varepsilon$

$$\frac{1}{M_p}\Gamma_4 \equiv \boxed{\frac{1}{4\varepsilon M_p}\left[\eta\eta_2\eta_3\left\{\frac{\eta^2}{2} + 3\eta_2 + \eta_3 + \eta_4 + \frac{\eta_2^2}{\eta_3}\right\} - \frac{\eta^2\eta_2^2}{2} + \frac{\varepsilon\eta\eta_2}{2}\left\{\frac{\eta^2}{2\eta_2} + \frac{5}{2}\eta + \eta_2 + \eta_3\right\}\right]}$$

$$\Rightarrow \boxed{\varepsilon = \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\eta\frac{\Delta\phi}{M_p} + \left[\frac{\eta\eta_2}{4} + \frac{\eta^2}{8}\right]\frac{\Delta\phi^2}{M_p^2} + \frac{1}{6}\frac{1}{\sqrt{2\varepsilon}}\left[\frac{\eta\eta_2\eta_3}{2} + \frac{\eta\eta_2^2}{2} + \frac{\varepsilon\eta^2}{4} + \frac{\eta_2\eta\varepsilon}{4}\right]\frac{\Delta\phi^3}{M_p^3} + \Gamma_4\frac{\Delta\phi^4}{M_p^4} + \mathcal{O}\left(\frac{\Delta\phi^5}{M_p^5}\right)}$$

↳ all $\varepsilon, \eta, \eta_2, \eta_3$ are evaluated at $*$-time

We also need to transform k values into distances in $\Delta\phi$ along the potential...

$$k = aH \sim e^{Ht}H \qquad H = \frac{1}{M_p}\sqrt{\frac{V}{(3-\varepsilon)}}$$

$$\text{or} \to \ln a = \int dt\, H = \int d\phi\,\frac{H}{\dot\phi} = \int d\phi\,\frac{H}{\sqrt{2\varepsilon}HM_p} = \int d\phi\,\frac{1}{\sqrt{2\varepsilon}M_p}$$

$$\Rightarrow k = \exp\left(\int d\phi/M_p\,(2\varepsilon)^{-1/2}\right)\frac{1}{M_p}\sqrt{\frac{V}{3-\varepsilon}}$$

This is probably not the easiest way to do this. Instead of evaluating the power spectrum at horizon crossing, maybe it is easier to look at the full k-dependent power spectrum. But then, how do we relate $P_\zeta(k)$ to $\Delta\phi$ & the potential?

I'm not sure this would work...

Now, putting together
1- the $V(\phi)$ Taylor expansion
2- the ε expansion
3- the conversion from k to $\Delta\phi/M_p$

we get a fitting formula in terms of 4 parameters (or 3 if we neglect η_3) for the samples of the power spectrum. With these best-fit values, we can reconstruct the local shape of the inflationary potential that the Hubble volume we live in went through. We can also find local values of n_s & α_s over those few e-folds.