Some definitions & values:

$$\Delta_\zeta = \frac{H^2}{8\pi^2 \varepsilon M_p^2} \qquad K_\phi = 0.05\,\mathrm{Mpc}^{-1}$$

$$\Delta_\zeta = 2.4\times10^{-9}$$

$$n_s - 1 = -2\varepsilon - \eta \qquad \varepsilon \equiv -\frac{\dot H}{H^2}\ ;\ \eta = \frac{\dot\varepsilon}{\varepsilon H}\ ;\ \eta_n = \frac{\dot\eta_{n-1}}{\eta_{n-1} H}$$

$$\alpha_s = -2\varepsilon\eta - \eta\eta_2$$

$$r = 16\varepsilon$$

$$(2\pi)^3 \delta^3(k+k') P_\zeta \equiv \langle \zeta(k,t)\,\zeta(-k,t)\rangle$$

$$P_\zeta = \frac{1}{2K^3}\frac{H^4}{\dot\phi^2} = \frac{1}{4K^3}\frac{H^2}{\varepsilon}\Big|_{K=aH}$$

$$\sim \frac{\pi}{2}(-\tau)^{2\nu}\,|H_\nu^{(1)}(-K\tau)|^2$$

$$\Delta_\zeta \equiv \frac{K^3}{2\pi^2} P_\zeta(K)$$

for a given $|\vec K|$, how many measurements?

$$|\vec K| = \sqrt{K_1^2 + K_2^2 + K_3^2} \Rightarrow \text{grows like } K^3$$

for every $\vec K$, or every $\vec n$, get a sample of

$$\text{Vol tot} \sim \frac{4\pi K^3}{3} \qquad \Delta\vec n = \frac{4\pi}{3}\left(|\vec n_1|^3 - |\vec n_2|^3\right)$$

if thickness = 1

$$\Delta\vec n = \frac{4\pi}{3}\left(n^3 - (n-1)^3\right)$$
$$= 4/3\,\pi\,[n^3 - (n^3 - 3n^2 + 3n - 1)]$$
$$\Delta\vec n = 4/3\,\pi\,(3n^2 - 3n + 1)$$

$\Rightarrow$ for every shell, we add $\sim n^2$ measurements

$\Rightarrow$ for every $K$, we get $\frac{4\pi(n^2 - n + 1/3)}{2}$ independent samples of $P_\zeta(|\vec n|)$

Here, we divide by 2 since $\zeta_{-K}$ and $\zeta_K$ are not independent (and are actually the complex conjugate of each other).
This gives us a measurement of $P_\zeta(K)$ at a few fiducial $|\vec K|$'s (or $|\vec n|$'s).

Relation btw modes $\zeta_K$ & $C_\ell$'s in the CMB:

$$a_{\ell m} = 4\pi(-i)^\ell \int \frac{d^3k}{(2\pi)^3}\, \underbrace{\Delta_{T\ell}(k)}_{\text{Transfer fct}}\, \zeta_{\vec k}\, \underbrace{Y_{\ell m}(\hat k)}_{\text{spherical harmonic}}$$

$$\Delta_{T\ell} = \int_0^{\tau_0} d\tau\, \underbrace{S_T(k,\tau)}_{\text{Sources along line of sight}}\, \underbrace{P_{T\ell}(k|\tau_0 - \tau|)}_{\text{Projections (bessel fcts)}}$$

$\hookrightarrow$ Inside the horizon, Newtonian gauge, think of $\phi$ & $\zeta$ interchangeably.

$\hookrightarrow$ Important: when they re-enter the horizon, all the modes are in phase and their amplitude for a fixed $|\vec K|$ is randomly distributed. If the fluctuations are Gaussian, then each $\vec K$ w/ fixed $|\vec K|$ is drawn from a Gaussian distribution with mean $P_\zeta(|\vec k|)$ & variance given by a fct of $P_\zeta(|\vec k|)$ (?)

If we have a single fiducial measurement of $P_\zeta(K)$ for a given $K_\phi$ then we can Taylor expand the potential $V(\phi)$ around $\phi_\phi$, the point where $\phi$ was when $K_\phi$ exited the horizon

$$V(\Delta\phi) = V(\phi_*) + \partial_\phi V|_{\phi_*}\Delta\phi + \frac{\partial_\phi^2 V|_{\phi_*}}{2}\Delta\phi^2 + \dots + \frac{1}{4!}\partial_\phi^4 V|_{\phi_*}\Delta\phi^4$$

with: $\partial_\phi V = \frac{1}{2}H(-6 + 2\varepsilon - \eta)\dot\phi = \frac{1}{2}M_p H^2\sqrt{2\varepsilon}(-6 + 2\varepsilon - \eta) = M_p H^2 \sqrt{2\varepsilon}[-3 + \varepsilon - \frac{1}{2}\eta]$

$$\partial_\phi^2 V = -\frac{1}{4}H^2(8\varepsilon^2 - 2\varepsilon(12 + 5\eta) + \eta(6 + \eta + 2\eta_2))$$
$$\sim -\frac{1}{4}H^2\{-24\varepsilon + 6\eta + 2\eta\eta_2\}$$
$$\partial_\phi^3 V = \frac{H\dot\phi}{M_p^2}\left\{2\varepsilon^2 - 3\varepsilon(2 + \frac{3}{2}\eta) + \frac{\eta}{4}(18 + 6\eta + 7\eta_2) - \frac{\eta\eta_2}{4\varepsilon}(3 + \eta + \eta_2 + \eta_3)\right\}$$

Note that $\varepsilon = \frac{\dot\phi^2}{2M_p^2 H^2} \Rightarrow \frac{\dot\phi^2}{2} = \varepsilon M_p^2 H^2$

$$3M_p^2 H^2 = \frac{\dot\phi^2}{2} + V = \varepsilon M_p^2 H^2 + V \Rightarrow H^2 M_p^2(3 - \varepsilon) = V \Rightarrow H^2 M_p^2 = V/(3-\varepsilon)$$

$$\Rightarrow d_0 = \frac{V}{3M_p^2H^2} = \frac{3-\varepsilon}{3} = 1 - \varepsilon/3$$

$$d_1 = \frac{M_p}{V}\partial_\phi V = \frac{M_p^2 H^2}{V}\sqrt{2\varepsilon}[-3 + \varepsilon - \tfrac{1}{2}\eta] = \sqrt{2\varepsilon}/(3-\varepsilon)[-3 + \varepsilon - \tfrac{1}{2}\eta]$$
$$= \frac{\sqrt{2\varepsilon}}{3}[1 + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon^2}{9}][-3 + \varepsilon - \tfrac{1}{2}\eta]$$
$$= \frac{\sqrt{2\varepsilon}}{3}[-3 - \varepsilon + \varepsilon - \tfrac{1}{2}\eta - \tfrac{1}{6}\varepsilon\eta + \tfrac{\varepsilon^2}{3} - \tfrac{\varepsilon^2}{3}]$$
$$= \frac{\sqrt{2\varepsilon}}{3}[-3 - \tfrac{1}{2}\eta - \tfrac{1}{6}\varepsilon\eta] = -\sqrt{2\varepsilon}[1 + \tfrac{1}{6}\eta + \tfrac{1}{18}\eta\varepsilon]$$

$$d_2 = \frac{M_p^2}{V}\partial_\phi^2 V = -\frac{1}{4}\frac{M_p^2 H^2}{V}(8\varepsilon^2 - 2\varepsilon(12 + 5\eta) + \eta(6 + \eta + 2\eta_2))$$
$$= -\frac{1}{4}\frac{1}{(3-\varepsilon)}[8\varepsilon^2 - 2\varepsilon(12 + 5\eta) + \eta(6 + \eta + 2\eta_2)]$$
$$= -\frac{1}{12}(1 + \varepsilon/3)(8\varepsilon^2 - 24\varepsilon - 10\varepsilon\eta + 6\eta + \eta^2 + 2\eta\eta_2)$$

$+8\varepsilon^2 - \varepsilon^2 8 - 10\varepsilon^2\eta - 10\varepsilon\eta + \eta^2 + \varepsilon\eta\eta_2$

$$= -\frac{1}{12}[-24\varepsilon + 6\eta + 2\eta\eta_2 - \frac{10\varepsilon^2}{3}\eta - 8\eta\varepsilon + \eta^2 + \frac{\eta^2\varepsilon}{3} + \frac{2\eta\eta_2\varepsilon}{3}]$$
$$= 2\varepsilon - \frac{1}{2}\eta - \frac{1}{6}\eta\eta_2 + \frac{2}{3}\eta\varepsilon + \frac{5\varepsilon^2}{18}\eta - \frac{1}{36}\eta^2\varepsilon - \frac{1}{18}\eta\eta_2\varepsilon$$
$$\sim 2\varepsilon - \frac{1}{2}\eta - \frac{1}{6}\eta\eta_2$$

$$d_3 = \frac{M_p^3}{V}\partial_\phi^3 V = \frac{M_p}{V}\frac{H\dot\phi}{4\varepsilon}\left\{8\varepsilon^3 - 24\varepsilon^2 - 18\varepsilon^2\eta + \varepsilon\eta 18 + 6\varepsilon\eta^2 + 7\varepsilon\eta\eta_2 - 3\eta\eta_2 - \eta^2\eta_2 - \eta\eta_2^2 - \eta\eta_2\eta_3\right\}$$
$$= \frac{M_p^2 H^2\sqrt{2\varepsilon}}{V\,4\varepsilon}\{-24\varepsilon^2 + 18\varepsilon\eta - 3\eta\eta_2 + \dots\}$$
$$= \frac{1}{2}\frac{1}{\sqrt{2\varepsilon}}\frac{1}{(3-\varepsilon)}\{-24\varepsilon^2 + 18\varepsilon\eta - 3\eta\eta_2 + 7\varepsilon\eta\eta_2 - \eta^2\eta_2 - \eta\eta_2^2 + 8\varepsilon^3 - 18\varepsilon^2\eta + 6\varepsilon\eta^2 - \eta\eta_2\eta_3\}$$
$$= \frac{1}{2}\frac{1}{\sqrt{2\varepsilon}}\frac{1}{3}(1 - \tfrac{\varepsilon}{3} + \tfrac{\varepsilon^2}{9})\{\ \}$$
$$= \frac{1}{6}\frac{1}{\sqrt{2\varepsilon}}\{-24\varepsilon^2 + 18\varepsilon\eta - 3\eta\eta_2 + 7\varepsilon\eta\eta_2 - \eta^2\eta_2 - \eta\eta_2^2 + \varepsilon\eta\eta_2 + 8\varepsilon^3 - 6\varepsilon^2\eta + 8\varepsilon^3 - 18\varepsilon^2\eta + 6\varepsilon\eta^2 - \eta\eta_2\eta_3\}$$
$$= \frac{1}{6}\frac{1}{\sqrt{2\varepsilon}}\{-24\varepsilon^2 + 18\varepsilon\eta - 3\eta\eta_2 + 8\varepsilon\eta\eta_2 - \eta^2\eta_2 - \eta\eta_2^2 + 16\varepsilon^3 - 24\varepsilon^2\eta + 6\varepsilon\eta^2 - \eta\eta_2\eta_3\}$$
$$= \frac{1}{2}\frac{1}{\sqrt{2\varepsilon}}\{-8\varepsilon^2 + 6\varepsilon\eta - \eta\eta_2 + \tfrac{8}{3}\varepsilon\eta\eta_2 - \tfrac{1}{3}\eta^2\eta_2 - \tfrac{\eta\eta_2^2}{3} + \tfrac{16}{3}\varepsilon^3 - 8\varepsilon^2\eta + 2\varepsilon\eta^2 - \tfrac{1}{3}\eta\eta_2\eta_3\}$$

$d_3$ & $d_4$ similar... see 1403.4585

Then re-write the potential as:

$$V(\phi) \simeq V(\phi_*)\left[1 + d_1\frac{\Delta\phi}{M_p} + \frac{1}{2}d_2\frac{\Delta\phi^2}{M_p^2} + \frac{1}{3!}d_3\frac{\Delta\phi^3}{M_p^3} + \frac{1}{4!}d_4\frac{\Delta\phi^4}{M_p^4}\right]$$

with the amplitude of the modes of $\phi$ we directly probe the power spectrum

$$P_\zeta = \frac{1}{4K^3}\frac{H^2}{\varepsilon}\Big|_{K=aH}$$

And we know $H^2 = \frac{V}{M_p^2(3-\varepsilon)} \Rightarrow P_\zeta = \frac{1}{4K^3}\frac{V}{M_p^2(3-\varepsilon)\varepsilon}\Big|_{K=aH}$

$$\boxed{\dot\phi = \sqrt{2\varepsilon}\,M_p H}$$

choose + root since $\phi \to -\phi$ is not physically relevant.

$$\varepsilon(t) = \varepsilon_* + \partial_\phi\varepsilon\,\Delta\phi + \mathcal{O}(\Delta\phi^2)$$
$$= \varepsilon_* + \frac{\dot\varepsilon}{\dot\phi}\Delta\phi + \partial_\phi\left(\frac{\dot\varepsilon}{\dot\phi}\right)\frac{\Delta\phi^2}{2} + \mathcal{O}(\Delta\phi^3)$$
$$= \varepsilon_* + \frac{\dot\varepsilon}{\sqrt{2\varepsilon}H}\frac{\Delta\phi}{M_p} + \partial_t\left(\frac{\dot\varepsilon}{\dot\phi}\right)\frac{1}{\dot\phi}\frac{\Delta\phi^2}{2} + \mathcal{O}(\Delta\phi^3)$$
$$= \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\frac{\dot\varepsilon}{\varepsilon H}\frac{\Delta\phi}{M_p} + \frac{\ddot\varepsilon}{\dot\phi^2}\frac{\Delta\phi^2}{2} - \frac{\dot\varepsilon\ddot\phi}{\dot\phi^3}\frac{\Delta\phi^2}{2} + \mathcal{O}(\Delta\phi^3)$$
$$= \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\eta\frac{\Delta\phi}{M_p} + \frac{\ddot\varepsilon}{2\varepsilon M_p^2 H^2}\frac{\Delta\phi^2}{2} - \frac{\dot\varepsilon}{\dot\phi^2}\frac{\ddot\phi}{\dot\phi}\frac{\Delta\phi^2}{2}$$
$$+ \frac{H^2\eta\varepsilon(\eta_2 + \eta - \varepsilon)}{2\varepsilon H^2}\frac{\Delta\phi^2}{2M_p^2}$$
$$+ \frac{\eta(\eta_2 + \eta - \varepsilon)}{4}\frac{\Delta\phi^2}{M_p^2}$$

$$\partial_t\left(\frac{\dot\varepsilon}{\sqrt{2\varepsilon}M_p H}\right) = \partial_t\left(\sqrt{\frac{\varepsilon}{2}}\frac{1}{M_p}\frac{\dot\varepsilon}{\varepsilon H}\right) = \partial_t\left(\sqrt{\frac{\varepsilon}{2}}\frac{1}{M_p}\eta\right)$$
$$= \frac{\eta}{M_p}\frac{1}{2\sqrt2}\frac{\dot\varepsilon}{\sqrt\varepsilon} + \frac{\dot\eta}{M_p}\sqrt{\frac{\varepsilon}{2}} = \frac{\eta H}{M_p\sqrt\varepsilon}\left[\frac{\dot\varepsilon}{H2\sqrt2} + \frac{\dot\eta}{\eta H}\frac{\varepsilon}{\sqrt2}\right] = \frac{\eta H}{\sqrt\varepsilon M_p}\left[\frac{\dot\varepsilon}{H2\sqrt2} + \frac{\eta_2\varepsilon}{\sqrt2}\right]$$

$$\eta_2 = \frac{\dot\eta}{H\eta} = \frac{1}{H\eta}\partial_t\left(\frac{\dot\varepsilon}{H\varepsilon}\right)$$
$$= \frac{1}{H\eta}\left[\frac{\ddot\varepsilon}{H\varepsilon} - \frac{\dot\varepsilon}{(H\varepsilon)^2}(\dot H\varepsilon + H\dot\varepsilon)\right]$$
$$= \frac{\ddot\varepsilon}{H^2\eta\varepsilon} - \frac{\dot H\dot\varepsilon\varepsilon}{H\eta H^2\varepsilon^2} - \frac{H\dot\varepsilon^2}{H\eta H^2\varepsilon^2}$$
$$= \frac{\ddot\varepsilon}{H^2\eta\varepsilon} + \frac{\varepsilon}{2}\eta - \frac{\eta^2}{2}$$
$$\eta_2 = \frac{\ddot\varepsilon}{H^2\eta\varepsilon} + \varepsilon - \eta \quad \boxed{\Rightarrow (\eta_2 + \eta - \varepsilon)H^2\eta\varepsilon = \ddot\varepsilon}$$

Second order

$$\frac{\ddot\varepsilon}{\dot\phi^2} = \frac{\ddot\varepsilon}{2\varepsilon H^2 M_p^2} = \frac{1}{2H}\frac{\eta}{M_p^2} \quad // \quad \frac{\ddot\phi}{\dot\phi} = \frac{\partial_t(\sqrt{2\varepsilon}HM_p)}{\sqrt{2\varepsilon}HM_p} = \frac{\sqrt2 M_p}{\sqrt2 M_p}\frac{\frac{\dot\varepsilon H}{2\sqrt\varepsilon} + \dot H\sqrt\varepsilon}{\sqrt\varepsilon H} = \frac{\dot\varepsilon}{2\varepsilon} + \frac{\dot H}{H} = \frac{H\dot\varepsilon}{2\varepsilon H} - H\varepsilon = \frac{H\eta}{2} - H\varepsilon$$

$$\Rightarrow \frac{\dot\varepsilon}{\dot\phi^2}\frac{\ddot\phi}{\dot\phi} = \frac{1}{2H}\frac{\eta}{M_p^2}H\left(\frac{\eta}{2} - \varepsilon\right) = \left(\frac{\eta^2}{4} - \frac{\varepsilon\eta}{2}\right)\frac{1}{M_p^2}$$

$$\Rightarrow \varepsilon = \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\eta\frac{\Delta\phi}{M_p} + \left[\frac{\eta\eta_2}{4} + \frac{\eta^2}{4} - \frac{\eta\varepsilon}{4} - \frac{\eta^2}{8} + \frac{\varepsilon\eta}{4}\right]\frac{\Delta\phi^2}{M_p^2} + \mathcal{O}(\Delta\phi^3/M_p^3)$$

$$\varepsilon = \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\eta\frac{\Delta\phi}{M_p} + \left[\frac{\eta\eta_2}{4} + \frac{\eta^2}{8}\right]\frac{\Delta\phi^2}{M_p^2} + \frac{\partial_\phi}{3!}\left(\frac{\eta\eta_2}{2} + \frac{\eta^2}{4}\right)\frac{\Delta\phi^3}{M_p^2} + \mathcal{O}(\Delta\phi^4/M_p^4)$$

third order $\to$ $\frac{1}{\dot\phi}\partial_t\left(\frac{\eta\eta_2}{2} + \frac{\eta^2}{4}\right) = \frac{1}{\sqrt{2\varepsilon}HM_p}\left(\frac{\dot\eta\eta_2}{2} + \frac{\eta\dot\eta_2}{2} + \frac{\eta\dot\eta}{2}\right)$

$$= \frac{1}{\sqrt{2\varepsilon}M_p}\left(\frac{\eta\eta_2}{2}\frac{\dot\eta_2}{H\eta_2} + \frac{\eta_2\eta}{2}\frac{\dot\eta}{H\eta} + \frac{\eta^2}{2}\frac{\dot\eta}{\eta H}\right)$$
$$= \frac{1}{\sqrt{2\varepsilon}M_p}\left(\frac{\eta\eta_2\eta_3}{2} + \frac{\eta_2^2\eta}{2} + \frac{\eta^2\eta_2}{2}\right)$$

$$\Rightarrow \varepsilon = \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\eta\frac{\Delta\phi}{M_p} + \left[\frac{\eta\eta_2}{4} + \frac{\eta^2}{8}\right]\frac{\Delta\phi^2}{M_p^2} + \frac{1}{6}\frac{1}{\sqrt{2\varepsilon}}\left[\frac{\eta\eta_2\eta_3}{2} + \frac{\eta\eta_2^2}{2} + \frac{\eta^2\eta_2}{2}\right]\frac{\Delta\phi^3}{M_p^3} + \mathcal{O}(\Delta\phi^4/M_p^4)$$

fourth order $\to$ $\frac{\partial_t}{\dot\phi}\left(\frac{1}{\sqrt{2\varepsilon}}\left[\frac{\eta\eta_2\eta_3}{2} + \frac{\eta\eta_2^2}{2} + \frac{\eta^2\eta_2}{2}\right]\frac{1}{M_p^3}\right)$

$$\frac{1}{\sqrt{2\varepsilon}M_pH}\left[\frac{-1}{2\sqrt2}\frac{\dot\varepsilon}{\varepsilon^{3/2}}[\ ] + \frac{1}{\sqrt{2\varepsilon}}\frac{1}{2}\{\dot\eta\eta_2\eta_3 + \eta\dot\eta_2\eta_3 + \eta\eta_2\dot\eta_3 + \dot\eta\eta_2^2 + 2\eta\eta_2\dot\eta_2 + 2\eta\dot\eta\eta_2 + \eta^2\dot\eta_2\}\right]$$

$$\frac{1}{M_p}\left[\frac{-1}{2\cdot2\varepsilon}\eta[\ ] + \frac{1}{2\varepsilon}\left(\frac{1}{2}\{\eta\eta_2^2\eta_3 + \eta\eta_2\eta_3^2 + \eta\eta_2\eta_3\eta_4 + \eta\eta_2^3 + 2\eta\eta_2^2\eta_3 + 2\eta^2\eta_2^2 + \eta^2\eta_2\eta_3\}\right)\right]$$

$$\frac{1}{M_p}\left[-\frac{1}{4\varepsilon}\eta[\ ] + \frac{1}{4\varepsilon}\left(\eta\eta_2\eta_3\left\{3\eta_2 + \eta_3 + \eta_4 + \frac{\eta_2^2}{\eta_3} + \frac{2\eta\eta_2}{\eta_3} + \eta\right\}\right)\right]$$

$$\frac{1}{4\varepsilon M_p}\left[-\frac{\eta^2\eta_2}{2}[\eta_3 + \eta_2 + \eta] + \eta\eta_2\eta_3\left\{3\eta_2 + \eta_3 + \eta_4 + \frac{\eta_2^2}{\eta_3} + \frac{2\eta\eta_2}{\eta_3} + \eta\right\}\right]$$

$$\left[-\frac{1}{2}\eta\eta_2\eta_3\left[\eta + \frac{\eta\eta_2}{\eta_3} + \frac{\eta^2}{\eta_3}\right] + \eta\eta_2\eta_3\left\{3\eta_2 + \eta_3 + \eta_4 + \frac{\eta_2^2}{\eta_3} + \frac{2\eta\eta_2}{\eta_3} + \eta\right\}\right]$$

$$\frac{1}{M_p}\Gamma_4 \equiv \boxed{\frac{1}{4\varepsilon M_p}\left[\eta\eta_2\eta_3\left\{\frac{1}{2}\eta + 3\eta_2 + \eta_3 + \eta_4 + \frac{\eta_2^2}{\eta_3} + \frac{3}{2}\frac{\eta\eta_2}{\eta_3} - \frac{1}{2}\frac{\eta^2}{\eta_3}\right\}\right]}$$

$$\Rightarrow \boxed{\varepsilon = \varepsilon_* + \sqrt{\frac{\varepsilon}{2}}\eta\frac{\Delta\phi}{M_p} + \left[\frac{\eta\eta_2}{4} + \frac{\eta^2}{8}\right]\frac{\Delta\phi^2}{M_p^2} + \frac{1}{6}\frac{1}{\sqrt{2\varepsilon}}\left[\frac{\eta\eta_2\eta_3}{2} + \frac{\eta\eta_2^2}{2} + \frac{\varepsilon\eta^2}{4} + \frac{\eta_2\eta\varepsilon}{4}\right]\frac{\Delta\phi^3}{M_p^3} + \Gamma_4\frac{\Delta\phi^4}{M_p^4} + \mathcal{O}\left(\frac{\Delta\phi^5}{M_p^5}\right)}$$

$\hookrightarrow$ all $\varepsilon, \eta, \eta_2, \eta_3$ are evaluated at $*$-time

We also need to transform $K$ values into distances in $\Delta\phi$ along the potential...

$$K = aH \sim e^{Ht}H \qquad H = \frac{1}{M_p}\sqrt{\frac{V}{(3-\varepsilon)}}$$

$$\text{or} \to \ln a = \int dt\,H = \int d\phi\,\frac{H}{\dot\phi} = \int d\phi\,\frac{H}{\sqrt{2\varepsilon}HM_p} = \int d\phi\,\frac{1}{\sqrt{2\varepsilon}M_p}$$

$$\Rightarrow K = \exp\left(\int d\phi/M_p\,(2\varepsilon)^{-1/2}\right)\frac{1}{M_p}\sqrt{\frac{V}{3-\varepsilon}}$$

This is probably not the easiest way to do this.
Instead of evaluating the power spectrum at horizon crossing, maybe it is easier to look at the full K-dependent power spectrum.
But then, how do we relate $P_\zeta(K)$ to $\Delta\phi$ & the potential?
I'm not sure this would work...

Now, putting together
1- the $V(\phi)$ Taylor expansion
2- the $\varepsilon$ expansion
3- the conversion from $K$ to $\Delta\phi/M_p$

we get a fitting formula in terms of 4 parameters (or 3 if we neglect $\eta_3$) for the samples of the power spectrum. With these best-fit values, we can reconstruct the local shape of the inflationary potential that the Hubble volume we live in went through. We can also find local values of $n_s$ & $\alpha_s$ over those few e-folds.
I guess since the parameters $\varepsilon, \eta, \eta_2, \eta_3$ depend on each other, once the best-fit model is found we can find how consistent the parameters are with each other... eg test if $\eta = \frac{\dot\varepsilon}{\varepsilon H}$

$$\sigma_{P_j} = \langle P^2(k)\rangle = \left\langle \int d^3x\, \zeta_R(x) e^{-ikx} \int d^3y\, \zeta_n(y) e^{-iky} \right\rangle$$

$$= \left\langle \int d^3x\, \zeta_R(x) e^{-ikx} \int d^3r\, \zeta_R(r+x)\, e^{-ik(r+x)} \right\rangle$$

$$\int d^3_x d^3_r\, e^{-i2kx} \underbrace{\langle \zeta_R(x)\zeta_R(r-x)\rangle}_{\sigma_{\zeta_R}(r)} e^{-ikr}$$

$$= \left\langle \int d^3x \int d^3p\, R(p)R(p+x) e^{-ikx} \int d^3y \int d^3q\, R(q)R(q+y) e^{-iky} \right\rangle$$

$$= \int d^3x\, d^3p\, d^3y\, d^3q\, \langle R(p)R(p+x)R(q)R(q+y)\rangle\, e^{-ik(x+y)}$$

$$\frac{H^2 M_p^2}{V} = \frac{1}{(3-\varepsilon)}$$

$$\partial_\phi V = \partial_\phi\left\{ H^2 M_p^2 (3-\varepsilon)\right\} = \frac{\partial_t}{\dot\phi}\left\{ H^2 M_p^2 (3-\varepsilon)\right\} = \frac{2H}{\dot\phi}\dot H M_p^2 (3-\varepsilon) + \frac{H^2 M_p^2}{\dot\phi}(-\dot\varepsilon)$$

$$= \frac{2HM_p^2}{\dot\phi}\dot H(3-\varepsilon) - \frac{HM_p^2}{\dot\phi} H^2 \varepsilon \frac{\dot\varepsilon}{\varepsilon H}$$

$$= \frac{M_p 2}{\sqrt{2\varepsilon}}\frac{\dot H}{H^2}(3-\varepsilon)H^2 - \frac{M_p}{\sqrt{2\varepsilon}} H^2 \varepsilon\eta$$

$$= M_p \sqrt{\frac{\varepsilon}{2}} H^2\left(-2(3-\varepsilon) - \eta\right)$$

$$= M_p H^2 \sqrt{\frac{\varepsilon}{2}}\left(-6 + 2\varepsilon - \eta\right) \checkmark$$

$$\frac{1}{2} H\dot\phi(-6+2\varepsilon-\eta)$$

$$\frac{1}{2} H\sqrt{2\varepsilon}\, H M_p(-6+2\varepsilon-\eta)$$

$$\frac{H^2 M_p}{\sqrt{2\varepsilon}}(-6\varepsilon+2\varepsilon^2-\eta\varepsilon)$$

$$\frac{H^2 M_p}{\sqrt{2\varepsilon}}$$

$$\Rightarrow \partial_\phi^2 V = \partial_\phi\left[M_p H^2 \sqrt{\frac{\varepsilon}{2}}(-6+2\varepsilon-\eta)\right]$$

$$= \frac{\partial_t}{\dot\phi}\left[M_p H^2 \sqrt{\frac{\varepsilon}{2}}(-6+2\varepsilon-\eta)\right] = \frac{\cancel{M_p}}{\sqrt{2\varepsilon}\, \cancel{M_p} H}\left\{2H\dot H\sqrt{\frac{\varepsilon}{2}}(-6+2\varepsilon-\eta) + \frac{1}{2}\frac{H^2\dot\varepsilon}{\sqrt{2\varepsilon}}(\ ) + H^2\sqrt{\frac{\varepsilon}{2}}(2\dot\varepsilon - \dot\eta)\right\}$$

$$= \frac{1}{\sqrt{2\varepsilon}\, H}\left\{\sqrt{2\varepsilon}\, H^3(-\varepsilon)(-6+2\varepsilon-\eta) + \frac{H^3}{2\sqrt{2\varepsilon}}\varepsilon\eta(-6+2\varepsilon-\eta) + H^3\sqrt{\frac{\varepsilon}{2}}(2\varepsilon\eta - \eta\eta_2)\right\}$$

$$= H^2(6\varepsilon - 2\varepsilon^2 + \eta\varepsilon) + \frac{H^2}{4\cancel{\varepsilon}}(-6\eta + 2\varepsilon\eta - \eta^2) + \frac{H^2}{2}(2\varepsilon\eta - \eta\eta_2)$$

$$= H^2\left(6\varepsilon - \frac{3}{2}\eta - 2\varepsilon^2 + \varepsilon\eta + \frac{1}{2}\varepsilon\eta + \varepsilon\eta - \frac{1}{4}\eta^2 - \frac{\eta\eta_2}{2}\right)$$

$$= H^2\left(6\varepsilon - \frac{3}{2}\eta - 2\varepsilon^2 + \frac{5}{2}\varepsilon\eta - \frac{1}{4}\eta^2 - \frac{\eta\eta_2}{2}\right)$$

$$= -\frac{H^2}{4}\left\{-24\varepsilon + 6\eta + 8\varepsilon^2 - 10\varepsilon\eta + \eta^2 + 2\eta\eta_2\right\} \checkmark$$

$$\Rightarrow \partial_\phi^3 V = \partial_\phi\left\{\frac{-H^2}{4}\left[-24\varepsilon + 6\eta + 8\varepsilon^2 - 10\varepsilon\eta + \eta^2 + 2\eta\eta_2\right]\right\}$$

$$= \frac{1}{4\dot\phi}\left\{-2H\dot H[\ ] - H^2\{-24\dot\varepsilon + 6\dot\eta + 16\varepsilon\dot\varepsilon - 10\varepsilon\dot\eta - 10\dot\varepsilon\eta + 2\eta\dot\eta + 2\dot\eta\eta_2 + 2\eta\dot\eta_2\}\right\}$$

$$= \frac{1}{4\sqrt{2\varepsilon}M_p}\left\{-2H^2\frac{\dot H}{H^2}[\ ] - H^2[-24\eta\varepsilon + 6\eta\eta_2 + 16\varepsilon^2\eta - 10\varepsilon\eta\eta_2 - 10\eta^2\varepsilon + 2\eta^2\eta_2 + 2\eta\eta_2^2 + 2\eta\eta_2\eta_3]\right\}$$

$$\frac{H\dot\phi}{4\varepsilon} = \frac{H^2 M_p\sqrt{2\varepsilon}}{4\varepsilon} = \frac{H^2 M_p}{2\sqrt{2\varepsilon}}$$

$$M_p^2\partial_\phi^3 V = \frac{H^2 M_p}{4\sqrt{2\varepsilon}}\left\{2\varepsilon[\ ] + 24\eta\varepsilon - 6\eta\eta_2 - 16\varepsilon^2\eta + 10\varepsilon\eta\eta_2 + 10\eta^2\varepsilon - 2\eta\eta_2(\eta+\eta_2+\eta_3)\right\}$$

$$= \frac{H^2 M_p}{2\sqrt{2\varepsilon}}\left\{\varepsilon[\ ] + 12\eta\varepsilon - \eta\eta_2 - 8\varepsilon^2\eta + 5\varepsilon\eta\eta_2 + 5\eta^2\varepsilon - \eta\eta_2(\eta+\eta_2+\eta_3)\right\}$$

$$= \left\{-24\varepsilon^2 + 6\varepsilon\eta + 8\varepsilon^3 - 10\varepsilon^2\eta + \varepsilon\eta^2 + 2\eta\eta_2\varepsilon + 12\varepsilon\eta - 8\varepsilon^2\eta + 5\varepsilon\eta\eta_2 + 5\varepsilon\eta^2 - \eta\eta_2(3+\eta+\eta_2+\eta_3)\right\}$$

$$= \left\{-24\varepsilon^2 + 18\varepsilon\eta + 8\varepsilon^3 - 18\varepsilon^2\eta + 6\varepsilon\eta^2 + 7\varepsilon\eta\eta_2 - \eta\eta_2(3+\eta+\eta_2+\eta_3)\right\}$$

$$\Rightarrow \partial_\phi^4 V M_p^2 = \frac{\partial_t}{\dot\phi}\left[\frac{H^2 M_p}{2\sqrt{2\varepsilon}}\left\{-24\varepsilon^2 + 18\varepsilon\eta + 8\varepsilon^3 - 18\varepsilon^2\eta + 6\varepsilon\eta^2 + 7\varepsilon\eta\eta_2 - \eta\eta_2(3+\eta+\eta_2+\eta_3)\right\}\right]$$

$$= \frac{\cancel{M_p}}{2\sqrt{2}}\frac{1}{H\cancel{M_p}\sqrt{2\varepsilon}}\left[-\frac{1}{2}\frac{\dot\varepsilon}{\varepsilon^{3/2}}H^2\{\ \} + \frac{2H\dot H}{\sqrt\varepsilon}\{\ \} + \frac{H^3}{\sqrt\varepsilon}\left\{-48\frac{\varepsilon^2\dot\varepsilon}{\varepsilon H} + 18\frac{\varepsilon\dot\eta}{\eta H}\eta + 18\frac{\dot\varepsilon\eta}{\varepsilon H}\varepsilon + 24\varepsilon^3\eta - 36\varepsilon^2\eta^2 - 18\varepsilon^2\eta\eta_2 \right.\right.$$
$$+ 6\varepsilon\eta^3 + 12\varepsilon\eta^2\eta_2 + 7\varepsilon\eta^2\eta_2 + 7\varepsilon\eta\eta_2^2 + 7\varepsilon\eta\eta_2\eta_3 - \eta\eta_2^2(3+\eta+\eta_2+\eta_3)$$
$$\left.\left. - \eta\eta_2\eta_3(3+\eta+\eta_2+\eta_3) - \eta\eta_2(\eta\eta_2 + \eta_2\eta_3 + \eta_3\eta_4)\right\}\right]$$

$$= \frac{H^2}{4\varepsilon}\left[-\frac{1}{2}\eta\{\ \} - 2\varepsilon\{\ \} - 48\varepsilon^2\eta + 18\varepsilon\eta\eta_2 + 18\varepsilon\eta^2 + 24\varepsilon^3\eta - 36\varepsilon^2\eta^2 - 18\varepsilon^2\eta\eta_2 + 6\varepsilon\eta^3 + 12\varepsilon\eta^2\eta_2 + 7\varepsilon\eta^2\eta_2 \right.$$
$$+ 7\varepsilon\eta\eta_2^2 + 7\varepsilon\eta\eta_2\eta_3 - (\eta\eta_2^2 + \eta\eta_2\eta_3)(3+\eta+\eta_2+\eta_3)$$
$$\left. - \eta\eta_2(\eta\eta_2 + \eta_2\eta_3 + \eta_3\eta_4)\right]$$

$$\times\frac{1}{4\varepsilon}$$

+3 −9/4 −1 +9/4 −3/4 −7/8 +1/8 −4 +12 −9/2 +3/2 +1/2 +1/2

$$= H^2\left[-\frac{1}{8}\left\{-24\varepsilon\eta + 18\eta^2 + 8\varepsilon^2\eta - 18\varepsilon\eta^2 + 6\eta^3 + 7\eta^2\eta_2 - \frac{\eta^2\eta_2}{\varepsilon}(3+\eta+\eta_2+\eta_3)\right\} + 8\varepsilon^3 - 24\varepsilon^2 + 7\varepsilon\eta\eta_2 - 3\eta\eta_2 - \eta\eta_2^2 - \eta\eta_2\eta_3\right.$$
$$- \frac{1}{2}\left\{+18\varepsilon\eta - 18\varepsilon^2\eta + 6\varepsilon\eta^2 - \eta^2\eta_2\right.$$

−9 +9 −3 +1/2

$$-12\varepsilon\eta + \frac{9}{2}\eta^2 + 6\varepsilon^2\eta - 9\varepsilon\eta^2 + \frac{3}{2}\eta^3 + \frac{19}{4}\eta^2\eta_2$$

$$-\frac{9}{2}\varepsilon\eta\eta_2 + \frac{9}{2}\eta\eta_2 - \frac{3\eta\eta_2^2}{4\varepsilon} - \frac{\eta\eta_2\eta_3}{4\varepsilon} + \frac{7}{4}\eta\eta_2^2 + \frac{7}{4}\eta\eta_2\eta_3 \left[\begin{matrix}-\eta^2\eta_2^2 - \eta\eta_2^3 - \eta\eta_2^2\eta_3 - \eta^2\eta_2\eta_3 - \eta\eta_2^2\eta_3 - \eta\eta_2\eta_3^2 \\ -\eta^2\eta_2^2 \qquad -\eta\eta_2^2\eta_3 \qquad -\eta\eta_2\eta_3\eta_4\end{matrix}\right]$$

$$= H^2\left[-18\varepsilon\eta + \frac{9}{2}\eta^2 + 14\varepsilon^2\eta - \frac{39}{4}\varepsilon\eta^2 + \frac{3}{4}\eta^3 + \frac{35}{8}\eta^2\eta_2 + \frac{\eta^2\eta_2}{8\varepsilon}(3+\eta+\eta_2+\eta_3) - 4\varepsilon^3 + 12\varepsilon^2 - 8\varepsilon\eta\eta_2 + 6\eta\eta_2 + \frac{9}{2}\eta\eta_2^2 + \frac{9}{4}\eta\eta_1\eta_2 - \frac{3}{4}\frac{\eta\eta_2^2}{\varepsilon} - \frac{\eta\eta_2\eta_3}{4\varepsilon} - \frac{1}{4\varepsilon}[\ ]\right.$$