The Music of the Sphere: Inferring the Evolution of the Universe on the Largest Scale from Cosmic Microwave Background Observations and Deep Survey Data.

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1 Overview

Observations of temperature fluctuations in the CMB tell us directly about the perturbation to g_{00} within a Newtonian gauge, henceforth the potential Φ. We see this quite directly on large angular scale ($\ell \lesssim 30$) at the epoch of recombination ($a \sim 0.0009$, $t \sim 380$ kyr). At shorter wavelengths, we see it indirectly because we can derive the velocity and density perturbations from its first and second derivatives. Furthermore, if we can derive the potential at recombination and have a good enough model of the contents of the universe and the physics that controls its evolution (which we do for a "Flat ΛCDM cosmology coupled with a power law distribution of initial adiabatic, Gaussian perturbations), we can evolve these modes forward in time until and if they become nonlinear. This implies that it should be possible, in principle at least, to relate the surface potential (and perhaps its low radial derivatives which are also be accessible to measurement) seen at the CMB photosphere to the body modes seen within this surface at later time. In practice this will only be possible at long wavelengths. This paper is concerned with understanding some aspects of how to set about this task in principle.

2 Basic Assumptions

We work with an idealized problem which we specify as follows:

- We represent the big bang as a sphere of comoving radius ~ 13.8 Gpc (adopting a Hubble constant of 70 km $\mathrm{s}^{-}1$ Mpc^{-1}). This is our unit of length. The last scattering surface – the *cosmic photosphere* – is a sphere with radius ~ 280 Mpc smaller and recombination occurs over a short but finite interval of time. We start by ignoring these small differences.

- We ignore the expansion of the universe and just consider the potential at the time of recombination when all modes of interest can be considered as linear. In practice, the potential changes little after recombination although shorter-wavelength modes eventually become nonlinear.

- We consider only scalar modes, setting the tensor contribution to zero. If a significant tensor component is confidently measured, minor modifications to what follows can be included.

- We suppose that the potential within our horizon is given by a set of discrete Fourier components defined within a l box of side 2 with the fundamental wavelength being $2\pi/k = 2$, continuing down to some minimum wavelength $2n_{\rm max}^{-1}$. Shorter wavelengths contribute an effective measurement error.

- We hypothesize that the amplitude of each of these modes is drawn from a Gaussian distribution with zero mean, random phase and variance at the time of recombination given by the measured power spectrum of fluctuations. For the longer wavelength modes in which we are most interested, the variance is roughly proportional to k^{-3} and we start with this simple model

- We ignore modes. with wavelength longer than the side of the box. Their contributions can be incorporated into the lowest Fourier components in a given realization as long as we only care about observations within our horizon. Care must be taken only to include physical contributions and to avoid spurious effects that derive from the chosen gauge.

3 Fourier Modes

We approximate the potential Φ within the box in polar coordinates as a Fourier series

$$\Phi(\mathbf{x}, \ell) = \sum_{\mathbf{k}} \tilde{\Phi}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{1}$$

where $\mathbf{k} = \pi[n_1, n_2, n_3]$, with n_1, n_2, n_3 integers. As Φ is real, we know that $\tilde{\Phi}_{\mathbf{k}}(-\mathbf{k}) = \tilde{\Phi}^*_{\mathbf{k}}(\mathbf{k})$ and so we only need define the Fourier components over a hemisphere $\mathcal{H}$ in $\mathbf{k}$ space. We choose $\mathcal{H} = \{n_1 > 0\} \cup \{n_1 = 0, n_2 > 0\} \cup \{n_1 = n_2 = 0, n_3 > 0\}$. The Fourier coefficients $\tilde{\Phi}_{\mathbf{k}}$ are chosen randomly from a Gaussian distribution with zero mean and variance consistent with the assume power spectrum, specifically with $P_{\mathbf{k}} \propto < |\tilde{\Phi}_{\mathbf{k}}|^2 > \propto k^{-3}$ for long wavelengths.

Figure 1: Aitoff projections of the potential on the cosmic photosphere $\Phi(\theta,\phi;\ell)$ at $x=1$ for different angular resolutions, parametrized by ℓ. The contour levels are 0 ± 1. A fixed set of random Fourier components truncated with $n_{max}=3$ is used.

Our initial goal is to relate surface information on the photosphere to the underlying $\tilde{\Phi}_{\mathbf{k}}$.[1] The approach that we will follow is constructive. Φ can be expanded exactly as an infinite sum of Legendre polynomials which we truncate at a finite value of ℓ, starting with $\ell=1$.

$$\Phi(\mathbf{x},\ell)=\sum_{\ell'=0}^{\ell}(2\ell'+1)\sum_{\mathbf{k}}^{\mathcal{H}}j_{\ell'}(kx)P_{\ell'}(\hat{\mathbf{k}}\cdot\hat{\mathbf{x}})\Re[i^{\ell'}\tilde{\Phi}_{\mathbf{k}}], \tag{2}$$

As ℓ is increased the effective resolution of Φ, in radius and angle, improves. It is convenient to treat ℓ as a continuous variable by the device of summing up to $\lfloor\ell\rfloor$ and then adding the next term in the sum multiplied by $(\ell-\lfloor\ell\rfloor)$. Although the sum in k-space is formally infinite, In practice, we only need a limited range of Fourier modes, $(n_1^2+n_2^2+n_3^2)^{1/2}\leq n_{\rm max}\sim\ell/3$, to provide an accurate representation of the degraded potential on the cosmic photosphere (Fig. 1). An immediate implication is that an accurate Φ map on the sphere, degraded to resolution ℓ provides $(\ell+1)^2$ real numbers which can be used to solve for $\sim 4\pi n_{\rm max}^3/3$ real Fourier coefficients suggesting that there is enough information in a $\ell=9$ map to solve for ~ 100 Fourier components up to $n_{\rm max}\sim 3$ independent of the priors on their actual values. When we include the priors, we can proceed to finer scale. Also, when we reduce the radius of the sphere, the value of $n_{\rm max}$ probed by a given ℓ scales $\propto x^{-1}$ (Fig. 2).

Figure 2: Aitoff projections of the potential on a sphere with $x=0.5$ with the same Fourier series as in Fig. 1. The expansion up to $\ell=4$ is a very good approximation to the full potential.

4 Nesting of Equipotential Lines on the Sphere

Analyze the potential on the cosmic photosphere ($x=1$) in a way that will generalize to the 3D potential in the sphere. The $\ell=0$ term is just a constant and, although necessary for the expansion we have just described, can be ignored here. The $\ell=1$ surface potential $\Phi(\ell,\theta,\phi)$ develops a single simple minimum (henceforth designated as L_0) and single simple maximum (henceforth designated by H_0). These are the global minimum and maximum; additional higher and lower minima and maxima will be designated L_i and H_i respectively in the order in which they appear.

[1]This is sometimes called *holography*, though it is not the original meaning of the word.

4.1 Classification of saddles

Now gradually increase ℓ. One contour develops a simple *saddle*[2] and an extremum simultaneously, becoming a *separatrix*. This contour divides the sphere into an *inside* containing L_0 and *outside* containing H_0. When the new extremum is a L lying outside the contour, the separatrix/saddle combination has the shape of an infinity symbol and called a *lemniscate*, denoted by X; when it is a H lying inside the contour, the separatrix/saddle combination has the shape of a pinched annulus and is called a *limaçon*, denoted by K (Fig. 3). As ℓ is further increased, the two stationary points separate and the smaller loop grows in size.[3] Further increasing ℓ will lead to the *creation* of new separatrices. These may be located between two existing separatrices or out of a contour encircling a L or a H. Occasionally, the inverse process - *annihilation* will occur. If we designate the total number of maxima, minima, saddles, lemniscates and limaçons by N_H, N_L, N_S, N_X, N_K, respectively, then clearly $N_S = N_X + N_K = N_H + N_L - 2$.

Figure 3: Two elementary types of saddle as characterized by the shape of the separatrix – the equipotential passing through the saddle. The lemniscate, or X-saddle, is accompanied by two maxima H , as here, or two minima, L. The X may be created or annihilated with an extremum by growing or shrinking either of the closed loops that passes through it. The limaçon, or K-saddle, is accompanied by a H as here (or a L) and a L as here (or a H). It can be create or annihilated only with one of the two accompanying extrema, in this case the H, so long as the contour does not expand and pass around the back of the sphere.

4.2 Tree representation

The nesting of all of the separatrices suffices to describe the nesting of all of the equipotentials. It is convenient to represent this nesting using a type of *directed graph* or *tree* representation. A simple example of a nesting is shown in Fig. 4a and the associated tree is presented in Fig. 4b. The tree comprises *nodes*, each with three *branches* each corresponding to a separatrix and *leaves* at the end of a branch corresponding to local extrema. There is only one path connecting any two leaves and no circuits, consistent with the usual definition of a tree. However, our tree has some novel features including growth. We start with $\ell = 1$ and a single branch from the L (the *root*) to the H (the *apex*). We let the vertical height of the branch be the potential difference between the two extrema. as we increase ℓ, new node-leaf (saddle-extremum) pairs appear and we locate them with ordinates corresponding to their respective potentials. The X- and K-saddles are easily distinguished. The pre-existing nodes and leaves will also move and occasionally they may disappear; they may also pass through each other on a branch when their heights coincide. The horizontal separation of

[2]We are interested in *non-degenerate* critical points and exclude *degenerate* extrema like *monkey saddles* which are structurally unstable and unfold into simple extrema under a small perturbation.

[3]Note that had we associated the interior with H_0, a given contour would have had the opposite identification.

the nodes and leaves is adjusted to ensure that there are no crossovers; it has no quantitative significance. This description of the equipotentials is essentially topological and pays no attention to the actual location of the contours.

Figure 4: (a) Nesting of separatrices from example in Fig. 1 exhibited within a circle created by puncturing the sphere at the global maximum and stretching the surface to lie on a plane so that the circumference is identified with a single point. b) Same separatrices displayed on a sphere. (c) Tree representation of this nesting. The designation of the saddles X, K and the extrema, H, L, is redundant and can be dropped. The numbers designate the order in which the extrema appear as ℓ is increased.

4.3 Stationary points

Finding all the stationary points can be difficult. The simplest approach is to set $\nabla_2\Phi = \Phi_{ij} = \{\partial\Phi/\partial\hat{\theta}, \partial\Phi/\partial\hat{\phi}\}$ to zero, using the identity $P'_\ell(\mu) = \ell[P_{\ell-1}(\mu) - \mu P_\ell(\mu)]/(1-\mu^2)$ and differentiating the sums in Eq. 2. This defines two families of curves and the stationary points are located where they intersect. New stationary point appear when these curves first touch.[4] An alternative method, which works well, is to find local minima of $|\nabla_2\Phi|^2$.

4.4 Curvature maps

There is another way to discuss the critical points and this is to evaluate the Hessian matrix

$$H_{ij} = \Phi_{,\hat{i}\hat{j}} \tag{3}$$

where the derivatives are with respect to $\hat{\theta}$, $\hat{\phi}$. It is straightforward to compute the two real eigenvalues which represent the principle curvatures. We can then designate L-zones, where they are both negative, H-zones, where they are both positive and S-zones where they have opposite signs. (We can further subdivide these zones dependent upon which eigenvalue has the larger absolute value but will not do this here.) As can be seen in Fig. 5, H and L zones have common boundaries with S, where one eigenvalue changes sign and touch at points where both eigenvalues simultaneously pass through zero. New saddle-extremum pairs are created at saddle boundaries and then quickly separate into their respective zones. This provides an alternate means to locate new separatrices. Individual curvature zones can contain multiple or no stationary points.

[4] In the language of catastrophe theory , this is a fold catastrophe.

Figure 5: Division of sphere into curvature zones. The L-zone is denoted in blue, the H-zone in red and the S-zone in purple. Also shown are the lines where the two components of $\nabla\Phi$ vanish. There intersections correspond to stationary points.

4.5 Curve of growth

The number of extrema, measured by the number of saddles N_S increases with ℓ. The manner in which it does this is dictated by the slope of the power spectrum of potential fluctuations $P_{\mathbf{k}}$ though not it's amplitude. We have focused on the inverse cube case because this is close to the inferred initial spectrum and remains valid around recombination up to $\ell \sim 30$. (It is also the spectrum for the temperature perturbations because the gravitational redshift is the dominant effect in the *Sachs-Wolfe* regime.) General scaling arguments suggest that as long as $P(k) \propto k^{n-4}$ for fixed n then the expectation in the number of saddles counted when all $N_S(\ell)$ is given asymptotically by $K(n)\ell^2$ as $\ell \to \infty$ where K can be estimated using a large number of trials. The variance in the actual value of K for finite ℓ can also be computed. We can also apply this approach directly to the temperature fluctuations assuming the inferred fluctuation spectrum inferred by more conventional methods. This will be demonstrated in a subsequent publication but is no more than a consistency check.

4.6 Gaussianity

We have considered several ways to classify the extrema and the results can be used to check on the Gaussianity of the distribution of Fourier components. For example, if the distributions of the Fourier coefficients were systematically skewed, then we would expect differences in the numbers of L and H and the areas of the corresponding zones. More subtle tests are possible using the branching of the trees. These will be considered elsewhere and compared with the observations.

5 Nesting of Equipotential Surfaces in the Sphere

6 Relating Surface and Interior Equipotentials

7 Bayesian Approach

8 Discussion