Sequences

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August 27, 2019

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1 Definition

Formally we can define a sequence $(a_n)_n$ as a mapping from the natural numbers to the reals or more generally the complex numbers. Informally a sequence is an ordered collection of numbers.

We make common use of the notation:

$$(\overbrace{a_n}^{\text{Denotes the sequence function}})_{\underbrace{n \in S}_{\text{domain of the function}}}$$

The domain of the function a_n is almost always the natural numbers which is abbreviated to just n, as above. It is also common to forgo the brackets and domain subscript and refer to the sequence simply as a_n.

2 Convergence

We say a sequence $(a_n)_n$ **converges** to a limit L iff:

$$\forall \epsilon \in \mathbb{R} > 0 \; \exists N \in \mathbb{N} \; s.t. \; \forall n \geq N, \; |a_n - L| < \epsilon$$

3 Algebra of Limits

Suppose $(a_n)_n$ and $(b_n)_n$ are convergent sequences with limit a and b respectively, then:

1. $(|a_n|)_n$ converges with limit $|a|$
2. $(ka_n)_n$ converges with limit ka
3. $(a_n + b_n)_n$ converges with limit $a + b$
4. $(a_n b_n)_n$ converges with limit ab
5. If $b_n \neq 0 \; \forall n$ and $b \neq 0$ then $(\frac{a_n}{b_n})_n$ converges to $\frac{a}{b}$
6. If $b_n \neq 0 \; \forall n$ and $b \neq 0$ then $(\frac{1}{b})_n$ converges to $\frac{1}{b}$

4 Sandwich Theorem

Suppose $(a_n)_n$ and $(b_n)_n$ are convergent sequences with the same limit L. The sandwich theorem states that:

$$\exists k \in \mathbb{N} \; s.t. \; b_n \leq c_n \leq a_n \forall n \geq k \implies c_n \to L$$

5 Monotone Convergence Theorem

The **MCT** states that if a sequence a_n is bounded above and monotonically increasing ($a_{n+1} \geq a_n \; \forall n$), then it is convergent. It can be shown that the limit is the supremum of the sequence.

5.1 Proof

Suppose a_n is monotone increasing and bounded with supremum L. Since L is the supremum:

$$\forall \epsilon > 0 \; \exists N \; s.t. \; a_N > L - \epsilon$$

Now since the sequence is monotone increasing:

$$\forall \epsilon > 0 \; \exists N \; s.t. \; \forall n \geq N, \; a_n > L - \epsilon$$

Rearranging the last part:

$$\forall \epsilon > 0 \; \exists N \; s.t. \; \forall n \geq N, \; |a_n - L| < \epsilon$$

Which is the definition of convergence.