Power Series

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1 Definition

Let $(a_n)_n$ be some sequence, then a series of the form:

$$\sum_{n=1}^{\infty} a_n x^n$$

Is called a power series (in x), where $x \in \mathbb{R}$.
We make the statement that one of the following must be true:

$$S = \sum_{n=1}^{\infty} a_n x^n$$

Converges:

1. $\forall x \in \mathbb{R}$
2. Only for $x = 0$
3. $\forall |x| \leq R$, and diverges $\forall |x| > R\ whereR$ is some real. We this R **The Radius of Convergence** of the power series

1.1 Proof

Suppose S converges for some $x = r$, then we know that the sequence $(a_n r^n)_n$ is null and hence that $\exists M \ s.t. \ |a_n r^n| \leq M \ \forall n$. Now consider some $|u| < |r|$, and let $t = \frac{|u|}{|r|}$. So that $0 < t < 1$ and

$$| \ a_n u^n \ | = | \ a_n r^n \ | \ t^n \leq M t^n \ \forall n$$

Summing over infinity:

$$\implies 0 \leq \sum_{n=1}^{\infty} |a_n u^n| \leq M \sum_{n=1}^{\infty} t^n$$

Then the series on the right is a convergent geometric series by the predicate on t, and hence we can conclude by the sandwich rule that S converges absolutely for $x = u$.