

# An Accumulated Sequence

Laurence Warne

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## 1 Question

Define the sequence  $(a_n)_n = 1$ . Now consider taking the first 4 elements of the sequence:

$$1 \ 1 \ 1 \ 1$$

Now throw away the first element to get:

$$1 \ 1 \ 1$$

Set the new second element to the sum of the first element and old second element, and similarly set the new third element to the sum of the old third element and new second element and so on to get:

$$1 \ 2 \ 3$$

If we then apply the same process to this new sequence we get:

$$2 \ 5$$

And so if we apply the process until we are left with one element we get 5. This can easily be generalised to starting with  $n$  lots of ones, and applying the process  $n - 1$  times to be left with one element,  $f(n)$ . Can we find a closed form for  $f(n)$ ?

## 2 Recursions

If we stack each pass of the process on top of one another, we obtain the diagram:

1	1	1	1
	1	2	3
		2	5
			5

Note starting with more 1s in the first row simply extends the diagram to the right and does not change the triangle left of the new column. It can be seen from our definition above, each element in the diagram (above the first row) is equal to the sum of the element above it and the element immediately to the left of it, save for the elements we are interested in, which are equal solely to the elements above them.

However, we can incorporate these elements into the relation by making the following edit to our diagram:

1	1	1	1
0	1	2	3
	0	2	5
		0	5

Let  $n$  span accross the columns, and  $k$  span accross the rows (like Pascal's triangle), ie so that  $f(3, 1) = 3$  and consider the diagram again. We obtain:  
Formally:

$$\begin{aligned}
 f(n, 0) &= 1 & \forall n \\
 f(n, k = n + 1) &= 0 & \forall n > 0 \\
 f(n, k) &= f(n - 1, k) + f(n, k - 1) & 1 \leq k \leq n
 \end{aligned}$$

We can reason from the above recursion that each number in the diagram can be written as some sum of the leftmost nonzero elements on each row, added to some sum of elements in the first row ( $k = 0$ ), which also happen to be equal to the leftmost element of the first row ( $f(0, 0) = 1$ ). These are the numbers we are interested in. ie:

$$f(n) = \sum_{r=1}^{n-1} A_r * f(r)$$

Where  $A_r$  is some constant to be determined. Now we can consider how each  $f(r)$  contributes to  $f(n)$  via the following diagram:

$f(r) \rightarrow$	$f(r) + B_1 \rightarrow \downarrow$	$f(r) + C_1 \rightarrow \downarrow$	$f(r) + D_1 \rightarrow \downarrow$
0	$f(r+1)$	$f(r) + B_2 \rightarrow$	$2f(r) + C_2 \rightarrow \downarrow$
	0	$f(r+2)$	$2f(r) + B_3 \rightarrow$

Where  $C, B, D$  are constants consisting of sums of other  $f(s)$ . Note we do not arrow contributions to  $f(r+1)$  as these are the source of the contributions and we need to keep these "atomic". We can observe from this diagram that:

$$A_r = f(n-r)$$

This also works for the base row, as we can just rewrite  $1 = f(1)$  (thus all constants are 0), and hence:

$$f(n) = \sum_{r=1}^{n-1} f(r) * f(n-r)$$

### 3 To Closed Form

First of all lets let  $f(0) = 0$  which will make things a little simpler, now let's define the generating function:

$$F(x) = \sum_{n=0} f(n)x^n$$

And now sum our recursion over all  $n$  where it's valid ( $n > 2$ ):

$$\sum_{n=2} f(n)x^n = \sum_{n=2} \sum_{r=0}^n f(r) * f(n-r)x^n$$

Simplifying the LHS and using the definition of multiplication in the ring of formal power series on the RHS:

$$F(x) - x = \sum_{n=0} f(n)x^n \sum_{n=0} f(n)x^n$$

Simplifying and collecting terms:

$$F(x)^2 - F(x) + x = 0$$

Quadratic formula (neglecting + sign, this will give us  $f(0) = 1$ ):

$$F(x) = \frac{1 - \sqrt{1 - 4x}}{2}$$

Attempt to extract  $x$  coefficients using binomial expansion:

$$F(x) = \frac{-1}{2} \left( \frac{\frac{1}{2}}{1!} (-4x) + \frac{\frac{1}{2} \cdot \frac{-1}{2}}{2!} (-4x)^2 + \dots \right)$$

After some arduous simplifications, we find the general  $x$  coefficient is given by:

$$\frac{(2n-2)!}{n! * (n-1)!}$$

Which is the  $(n-1)$ th Catalan number, ie  $f(n) = C_{n-1}$ .