Intermediate Value Theorem

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1 Definition

The Intermediate Value Theorem (IVT) states the following:

Let $f : [a, b] \to \mathbb{R}$ be some function continuous on the interval $[a, b]$, with $f(a) = A$ and $f(b) = B$. Let $C \in \mathbb{R}$ be such that $min\{A,\ B\} < C < max\{A,\ B\}$, then:

$$\exists c \in [a, b]\ s.t.\ f(c) = C$$

And:

$$S = \{x : x \in [a, b], f(z) \leq C\ \forall z \in [a, x]\}$$
$$c = \sup\{S\}$$

2 Proof

Suppose WLOG that $f(a) < f(b)$. Note since S is bounded above by b and nonempty, it must have a supremum by the completeness property of the reals, hence c is defined.

2.1 Case 1

Now suppose $f(c) < C \implies C - f(c) = \epsilon > 0$. Now since f is continuous on $[a, b]$:

$$\exists \delta > 0 \ s.t. \ x \in (c - \delta, c + \delta) \implies |f(c) - f(x)| < \epsilon$$

Now consider any :

$$c^* \in (c, c + \delta)$$

Note since $|f(c^*) - f(x)| < \epsilon \implies f(c^*) < f(c) + \epsilon = C$ there is no c^* such that $f(c^*) > C$, hence all c^* are in S, but this is a contradiction as their memebership in the set would mean

$$c \neq sup\{S\}$$

So we can conclude that our assumption: $f(c) < C$ is false.

2.2 Case 2

Alternatively, suppose $f(c) > C \implies f(c) - C = \epsilon > 0$. By continuity:

$$\exists \delta > 0 \ s.t. \ x \in (c - \delta, c + \delta) \implies |f(c) - f(x)| < \epsilon$$

Now choose $c' = c - \frac{\epsilon}{2}$. We know by the membership predicates of S that c cannot be in S. Suppose:

$$c' \notin S \implies \forall x \geq c', \ x \notin C \implies c' \text{ is an upper bound of } S$$

But that would contradict the definition of c being the least upper bound of S, hence we must have $c' \in S$.

$$c' \in S \implies f(c') \leq C$$

But we know from continuity: $f(c) - f(c') \leq \epsilon$

$$\implies f(c) - f(c') \leq f(c) - C \tag{1}$$
$$\implies C \leq f(c') \tag{2}$$