# Graphs

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## 1 Introduction

We denote a graph as $G(V, E)$, where $V = \{v_1, v_2...v_n\}$ is the vertex set of the graph and $E = \{(v_a, v_b)...(v_c, v_d)\}$ as the edge set of the graph.

If $G$ is a **directed graph** then the elements of $E$ will be ordered (ie $(v_1, v_2)$ is distinct from $(v_2, v_1)$), otherwise $G$ is **undirected**.

## 2 Vertex Sequences

### 2.1 Definition

A **walk** in a graph $G(V, E)$ is a finite sequence of vertices $(v_1...v_k)$ such that each consecutive pair forms an edge of $G$.

### 2.2 Definition

A **trail** is a walk in which all the edges $e_i = (v_i, v_{i+1})$ are distinct. We can remember this by associating them with **Eulerian trails** (which much visit every edge exactly once).

### 2.3 Definition

A **path** is a trail in which a vertex occurs occurs only once in the edge sequence.

### 2.4 Definition

A **cycle** is a trail in which all the vertices are distinct.

## 3 Adjacency Matrices

Given a directed or undirected graph $G(V, E)$ on $n$ vertices, we define the **adjacency matrix**, $A$, of $G$ as an $n * n$ matrix with entries given by:

$$A_{i,j} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

This is always symmetric if $G$ is undirected, and zeroes always fill the main diagonal if $G$ is **simple**.

### 3.1 Applications

Note $(A^k)_{i,j}$ gives the number of paths of length $k$ starting at $v_i$ and ending at $v_j$. Hence using matrix associativiy we can compute this value in $O(log(k))$ or $O(1)$ time.

We can extends this to find the number of paths of length $1 \leq x \leq k$ starting at $v_i$ and ending at $v_j$, by noting this is equal to:

$$(A + A^2 + ... + A^k)_{i,j} = \Big(A(A^k - I_n)(A - I_n)^{-1}\Big)_{i,j}$$

Which can be computed in the same time complexity.