Graph Colouring

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1 Definition

We define a k-colouring of an undirected graph $G(V, E)$ as a function $f : V \to \mathbb{Z}_k = \{1...k\}$ such that:

$$(u, v) \in E \implies f(u) \neq f(v)$$

Ie no neighbouring vertices have the same colour. Additionally, if G admits a k-colouring, we say G is **k-colourable**.

2 Chromatic Number

We define $\chi(G)$ as the smallest k such that G is k-colourable. Clearly such a k must always exist as a graph on n vertices is always n-colourable (assign all the vertices different colours).

However determining $\chi(G)$ for an arbitrary graph is nontrivial and is in fact NP-complete.

2.1 Bounds

Using a greedy algorithm we can deduce that:

$$\chi(G) \leq \max_{v \in V} (deg(v)) + 1$$

And if G is connected, simple and is not a complete or odd cycle graph, then Brook's Theorem offers the slight improvement of:

$$\chi(G) \leq \max_{v \in V} (deg(v))$$

Clearly if G has **k-clique** then:

$$\chi(G) > k$$

3 Chromatic Polynomial

Given some simple graph G we define the **Chromatic Function** $P_G : \mathbb{Z}^+ \to \mathbb{Z}^+$ as the number of ways we can colour G using k or fewer colours. Hence if $k < \chi(G)$ then $P_G(k) = 0$.

3.1 Theorem

The Chromatic Function of a simple graph G is polynomial.

Proof:

We prove the following statement: If $e = (v, w)$ is some edge in G then $P_G(k) = P_{G-e}(k) - P_{G/e}(k)$, where G/e is the graph produced by contracting e. Let $G' = G - e$. Now, the number of k-colourings of G' in which v and w are different colours is equal to $P_G(k)$ since the vertices are adjacent in G.

The number of k-colouring of G' in which v and w are the same colour is equal to the number of k-colourings of G/e. Combining these statements gives:

$$\begin{aligned} P_{G'}(k) = & P_{G-e}(k) = P_G(k) + P_{G/e} \\ \implies & \quad P_G(k) = P_{G-e}(k) - P_{G/e}(k) \end{aligned}$$

We can apply this procedure to the two new graphs $G - e$ and G/e, and so on until we eventually end up with only **null graphs** whose chromatic number is k^n for N_n, a polynomial in k. Hence since polynomials are closed under addition we can conclude that $P_G(k)$ is polynomial also.