Series

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1 Definition

Given some (infinite) sequence $(a_n)_n$ we can consider its **infinite series**, denoted:

$$\sum_{n=1}^{\infty} a_n$$

We say the **series** converges iff the sequence of partial sums of the sequence converges.

$$S_n = \sum_{i=1}^{n} a_i$$

If S_n converges we say the series is **convergent**, else we say the series is **divergent**.

2 Tests

There is no general process for checking if a series converges, but we can apply tests which provide definitive statements for some series.

2.1 Nth Term Test

If a series converges then its corresponding sequence must be null.

2.1.1 Proof

Suppose S_n converges to some L then by the shifted sequence rule, S_{n+1} also converges to the same L. By the algebra of limits theorem:

$$(S_{n+1} - S_n)_n$$

converges to $L - L = 0$, but

$$(S_{n+1} - S_n)_n = (a_n)_n \implies a_n \text{ converges to } 0$$

2.2 Integral Test

Let the sequence $(a_n)_n$ be defined by $a_n = f(n)$ then if the following are true for f:

1. f is defined on the interval $I = [1, \infty)$
2. f is monotonically decresing on I
3. f is positive on I

Then the integral test states that:

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\, dx \text{ exists}$$

2.3 Comparison Test

Given two sequences a_n and b_n, for which

1. $a_n, b_n > 0 \ \forall n$
2. $a_n \leq b_n \ \forall n$

The comparison test states that:

$$\sum_{n=1}^{\infty} b_n \text{ converges } \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

And

$$\sum_{n=1}^{\infty} a_n \text{ diverges } \implies \sum_{n=1}^{\infty} b_n \text{ diverges}$$

2.3.1 Proof

Consider the sequence of partial sums: $S_n = \sum_{k=1}^{n} a_k,\ R_n = \sum_{k=1}^{n} b_k$. Given the convergence of R_n we know that it must have an upper bound B.

$$\implies S_n \leq R_n < B\ \forall n$$
$$\implies S_n < B\ \forall n$$

And we also have from (1) that S_n is monotone increasing. Hence since S_n is monotone increasing and bounded above, it must converge by the MCT.

2.4 Ratio Test

Let

$$L = \frac{|a_{n+1}|}{|a_n|}$$

Then the ratio test asserts that if:

1. $L < 1 \implies a_n$ converges absolutely
2. $L > 1 \implies$ the series is divergent
3. And if $L = 1$ or the limit does not exist, then the test is inconclusive

The proof of this statement follows via construction of an appropriate geometric series and the comparison test.

2.5 Root Test

Let

$$L = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then similar to the ratio test, the root test asserts that:

1. $L < 1 \implies a_n$ converges absolutely
2. $L > 1 \implies$ the series is divergent
3. And if $L = 1$ or the limit does not exist, then the test is inconclusive

Note that if $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists then by the properties of the limit superior $L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$. We can also use this test to define the RoC as:

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

This works as the RoC makes no claims on the convergence at the boundaries, we can also substitute in ∞ for the limit superior if $(|a_n|^{\frac{1}{n}})_n$ does not have a convergent subsequence.

2.5.1 Proof

Suppose $L < 1$, then by the properties of the limit superior:

$$\exists N \in \mathbb{N},\ k\ s.t.\ |a_n|^{\frac{1}{n}} < k < 1,\ \forall n \geq N$$

Since $f(x) = x^n$ is strictly increasing on the interval $[0, \infty)$:

$$|a_n| < k^n < 1,\ \forall n \geq N$$

Summing from N to ∞:

$$\sum_{n=N}^{\infty} |a_n| < \sum_{n=N}^{\infty} k^n$$

But $\sum_{n=N}^{\infty} k^n$ is a convergent geometric series, hence $\sum_{n=N}^{\infty} |a_n|$ converges by the comparison test.

$$\implies \sum_{n=1}^{\infty} |a_n| \text{ converges}$$

Hence if property 1. above is satisfied, a_n not only converges, but converges **absolutely**. A similar argument can be applied to prove 3.