The Proof is Trivial!

Problems collected from the tsr thread: https://www.thestudentroom.co.uk/showthread.php?t=2313384
Key:

1. * = requires only A-level knowledge
2. ** = may require a little extra (induction, L'hôpital's etc.)
3. *** = requires undergraduate knowledge.

Problem 2*

Define a function f:R â†' R using the following rule:"For any x,yâˆˆR, f(x+y)=f(x)f(y)"Show that, if f is non-zero:(a) f(0)=1
(b) f(x)=0 has no solutions
(c) f(-x)=1/f(x)
Posted by electriic_ink

Problem 3*

Show that for positive integers m, n :

$$\int_0^1 \sqrt[m]{1-x^n} - \sqrt[n]{1-x^m}\, dx = 0$$

Posted by Lord of the Flies

Problem 4

*If N is any perfect power of 2 (i.e. N=2ˆn for some integer n) then there is no number, M, formed by permutating the digits of N, which is also a perfect power of 2.
Posted by james22

Problem 5** De Moivre's Theorem

By considering the real and imaginary parts of:

$$e^{i\theta} \cdot e^{i\phi}$$

Prove that

$$\sin(\theta+\phi) = sin(\theta)cos(\phi) + sin(\phi)cos(\theta)$$

and that

$$\cos(\theta+\phi) = cos(\theta)cos(\phi) - sin(\theta)sin(\phi)$$

Show also that:

$$S_1(n) =^n \mathrm{C}_0 -^n \mathrm{C}_2 +^n \mathrm{C}_4 - ... + (-1)^m \times^n \mathrm{C}_{2m} \text{ has the value } 2^{\frac{n}{2}} cos(\frac{n\pi}{4})$$

Where

$$n - 1 \leq 2m \leq n$$

Derive another expression for:

$$S_2(n) =^n \mathrm{C}_1 -^n \mathrm{C}_3 +^n \mathrm{C}_5 - ... + (-1)^m \times^n \mathrm{C}_{2m+1}$$

Where

$$n - 1 \leq 2m + 1 \leq n$$

Posted by joostan

Problem 6**/***

Let

$$\mathbb{N}_\omega = \mathbb{N} \cup \{\omega\}$$

where we define for all $n \in \mathbb{N}_\omega$, $\omega \geq n$. For any set

$$S \subseteq \mathbb{N}_\omega$$

we write $\sqcup S$ to denote the smallest member of $\mathbb{N}_\omega$ such that $\sqcup S \geq n$ for all $n \in S$. (Hence

$$\sqcup \mathbb{N}_\omega = \omega$$

.)Let

$$f : \mathbb{N}_\omega \to \mathbb{N}_\omega$$

be a function such that:

- For all m, n,

  $$m \leq n \implies f(m) \leq f(n)$$

  .
- For all

  $$S = \{a_1, a_2, \dots\} \subseteq \mathbb{N}_\omega$$

  , we have

  $$f(\sqcup S) = \sqcup \{f(a_1), f(a_2), \dots\}$$

  .

Show that the value F given by

$$F = \sqcup \{f^n(1) | n \in \mathbb{N} \cup \{0\}\}$$

satisfies $f(F) = F$ and is the smallest member of $\mathbb{N}_\omega$ to do so.EDIT: Right I see where people are getting confused. ω is *not* a variable - it is actually an element in addition to the naturals that is defined to be greater than or equal to all of them. See it as an element representing "infinity".
Posted by

Problem 7*

Let p be a prime number. Find all triples of positive integers (a, b, p) such that

$$2^a + p^b = 19^a$$

.@ukdragon37 I am still unable to comprehend your definition of $\mathbb{N}_\omega$. Can we write

$$\mathbb{N}_\omega = \{\alpha | \alpha \in \{\mathbb{N} \cup \{\omega\}\}, \alpha \leq \omega\}$$

? And if this is the case, then is $\mathbb{N}_\omega$ not finite?
Posted by Mladenov

Problem 8*

A harder version of problem 0. Prove that for any n, it is possible to fine n consecutive integers such that none of them are prime powers.
Posted by und

Problem 9**

Show that there is no function

$$f: \mathbb{N} \to \mathbb{N}$$

satisfying $f(f(n)) = n + 2013$
Posted by Lord of the Flies

Problem 10*

:Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{17} x}{\sin^{17} x + \cos^{17} x}\, dx$$

Posted by j.alexanderh

Problem 11* (if you have never seen substitution before, **)

Find the general solution, by a suitable substitution or otherwise of this differential equation.

$$\frac{dy}{dx} = \frac{sinx(cosx + y)}{cosx - y}$$

Posted by bogstandardname

Problem 12*

I have a cube and I want to paint each side a different colour. How many different ways can I paint the cube with n colours,

$$n \in \mathbb{N}, n \geq 6$$

?
Posted by Star-girl

Problem 13*

Find the exact value of:

$$\sqrt{42 + \sqrt{42 + \sqrt{42 + ...}}}$$

As an extention, try the above with 6 instead of 42. Also try 2 and then try 8. Can you spot a pattern and/or find a way to spot more numbers that "work"?
Posted by

Problem 14**

Prove that $2^{60} - 1$ is divisible by 61.
Posted by Star-girl

Problem 15*/**

Evaluate

$$\lim_{n\to\infty} \int_1^n \frac{n}{1 + x^n} \, dx$$

What if the upper limit in the integral were fixed, say π?
*Posted by Lord of the Flies*Problem 16 **/***Let f be a function such that

$$\int_{-\infty}^{\infty} f(x) \, dx < \infty$$

.Define

$$f_{\pm}(x) = f(x + \sqrt{x^2+1}) \pm f(x - \sqrt{x^2+1})$$

and

$$\Im(f)(x) = f_{+}(x) + \frac{x}{\sqrt{x^2+1}} f_{-}(x)$$

.Show that

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{-\infty}^{\infty} \Im(f)(x)\ dx$$

.

Find

$$\int_{-\infty}^{\infty} \frac{\cos(x)\sin(\sqrt{x^2+1})}{\sqrt{x^2+1}}\ dx$$

.

Posted by jack.hadamard

Problem 17*

Let a_n be an arithmetic sequence and g_n be a geometric sequence. The first four terms of $s_n = g_n + a_n$ is 0, 0, 1 and 0.Find s_{10} .

Problem 18**

Evaluate

$$\lim_{x\to 0}(x^{x^x} - x^x)$$

Problem 19*

If $x + y + z = 0$, evaluate

$$\frac{xy + yz + zx}{x^2 + y^2 + z^2}$$

Problems 20 and 21 are below. I told you I'd spam this thread

Problem 22*

Evaluate

$$\int_1^\infty (\frac{\log x}{x})^n dx$$

Problem 23*

Let S denote the set of triples (i,j,k) such that $i+j+k=n$ .Evaluate $\sum_{(i,j,k)\in S} ijk$ (I wanted to edit this to restrict the choice of n, but actually it makes little difference. Do people need a hint?)
Posted by shamika

Problem 20 **

By using a suitable substitution, or otherwise, show that

$$\int \sqrt{r^2-x^2}\, dx = \frac{1}{2}\left(r^2 \arctan\left(\frac{x}{\sqrt{r^2-x^2}}\right) + x\sqrt{r^2-x^2}\right) + C$$

Show further that the area of a circle, A, satisfies $A=\pi r^2$
Posted by Indeterminate

Problem 21 - */**

The uniqueness theorem of anti-derivatives states that, if $f'(x)=g'(x)$, then $f(x)=g(x)+c$ .By considering the derivatives of $cos^2 x$ and $sin^2 x$, verify that $cos^2 x + sin^2 x = 1$ .By considering suitable derivatives, prove the following identities:i) $lnx^n = nlnx$ ii)

$$ln(f(x)g(x)) = lnf(x) + lng(x)$$

iii)

$$lnx = \frac{log_a x}{log_a e}$$

.Deduce that the results in parts i) and ii) hold independently of the base of the logarithm.

Spoiler:

Show

This was a question I came up with when there was talk about a user-contributed STEP paper being made by TSR members. However, I think here is a better place for it.

Posted by DJMayes

Problem 24

Evaluate

$$\prod_{n=1}^{1006} \sin\left(\frac{n\pi}{2013}\right)$$

Posted by Lord of the Flies

Problem 25**/***

1. Show that for any (potentially infinite) set A , there is no surjection

   $$f : A \to \mathcal{P}(A)$$

   , where $\mathcal{P}(A)$ is the powerset of A .

2. Hence show there is no surjection from the the set A to set of functions $A \to A$ if $|A| > 1$.

3. Determine the relationship between $\mathbb{R}$ and

   $$\mathbb{N} \to \mathbb{N}$$

   .

Posted by

Problem 26 ***

Let G_1 and G_2 be finite groups such that $|G_1|$ and $|G_2|$ are coprime and let $K \leq G_1 \times G_2$ .Set

$$H_1 = \{g \in G_1 : (g, e) \in K\}$$

and

$$H_2 = \{g \in G_2 : (e, g) \in K\}$$

Show that $K = H_1 \times H_2$
Posted by Noble.

Problem 27 **/***

(the ** rating very loosely)Let $a, b, c \in \mathbb{R}$. Determine

$$I = \iiint\limits_{x^2+y^2+z^2 \leq 1} \cos(ax + by + cz) \;\; \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

Also, show that your result is consistent with the fact the volume of the unit sphere is $4\pi/3$
Posted by Noble.

Problem 28 *

Determine whether or not the line $ax + by + r = 0$ intersects the circle, C, with equation $(x - a)^2 + (y - b)^2 = r^2$ If it does, give the coordinates.Is the line ever a tangent to C?
Posted by Indeterminate

Problem 29**

Let

$$(a_n)_{n \geq 1}$$

be a sequence of positive integers satisfying the following condition:

$$0 < a_{n+1} - a_n \leq 2001$$

for each $n \geq 1$. Then there exist infinitely many ordered pairs (p, q) of distinct positive integers such that

$$a_p \equiv 0 \pmod{a_q}$$

.
Posted by Mladenov

Problem 30*

More limits & integrals?Evaluate

$$\lim_{n\to\infty}\left(\int_0^1 \frac{dx}{1+x^n}\right)^n$$

Posted by Lord of the Flies

Problem 31*

Find all real numbers x, y and z which satisfy the simultaneous equations $x^2 - 4y + 7 = 0$, $y^2 - 6z + 14 = 0$ and $z^2 - 2x - 7 = 0$.
Posted by und

Problem 32*

Find all positive integers n such that $12n - 119$ and $75n - 539$ are both perfect squares.
Posted by und

Problem 33

Determine

$$\int\limits_b^a \frac{1}{\sqrt{(a-x)(x-b)}}dx$$

Posted by Benjy100

Problem 34*

Prove that for all primes $p > 3,\ 24|(p^2 - 1)$.
Posted by Star-girl

Problem 35*

Evaluate

$$\int_0^{\frac{\pi}{2}} \left(\frac{x}{\sin x}\right)^2 dx$$

Posted by Lord of the Flies

Problem 36: */**

A particle is projected from the top of a plane inclined at an angle ϕ to the horizontal. It is projected down the plane. Prove that; if the particle is to attain it's maximum range, the angle of projection θ from the horizontal must satisfy:

$$\theta = \frac{\pi}{4} - \frac{\phi}{2}$$

Posted by DJMayes

Problem 37*/**

Let S_r for $r = 1, 2, 3, ..., 100$ denote the sum of the infinite geometric series whose first term is $\dfrac{r-1}{r!}$ with common ratio $\dfrac{1}{r}$ .Evaluate

$$\frac{100^2}{100!} + \sum_{r=1}^{100} |(r^2 - 3r + 1)S_r|$$

Posted by Felix Felicis

Problem 38*

Show that there is no polynomial p with integer coefficients such that

$$p(a) = b,\ p(b) = c,\ p(c) = a$$

for distinct integers a, b, c

Problem 39*

Show that if p is a polynomial satisfying

$$p(x) + p'''(x) \geq p'(x) + p''(x)$$

for all x then $p(x) \geq 0$
Posted by Lord of the Flies

Problem 40*

Find all functions

$$f : \mathbb{R} \to \mathbb{R}$$

which satisfy

$$f(x^3) + f(y^3) = (x+y)(f(x^2) + f(y^2) - f(xy))$$

for all real numbers x and y .
Posted by und

Problem 41***

A (countably) infinite class of Cambridge students who believe in the Axiom of Choice are taking their finals for the Mathematical Tripos. At the start of the exam, the examiner places either a white or a black hat on each student at random. There is only one question in the exam: "What colour is your hat?" If only a finite number of students answers incorrectly then everyone becomes a Wrangler (very good), otherwise the Wooden Spoon is given to the whole class (very bad). Everyone can see the hats of everyone else besides their own, and since it's exam conditions they are not allowed to communicate. Students may not remove their hats or try to look at it in any way.What strategy could the students devise together so to avoid failure and the impending doom of unemployment? Being clever clogs from Cambridge, memorising infinitely large amounts of data is no problem to the students.Note: A student suggested that each person just guess at random. However this has already been ruled out by the cleverer students who realised that if they were really unlucky, infinite of them could guess wrong.
Posted by

Problem 42*/**

Show that

$$\sum_{r=0}^{88} \frac{1}{\cos r \cdot \cos(r+1)} = \frac{\cos 1}{\sin^2 1}$$

(angles measured in degrees)
Posted by Felix Felicis

Problem 43

/*Evaluate

$$\lim_{m\to\infty}(\lim_{n\to\infty}\cos^{2n}(m!\pi x))$$

(To do it rigorously you obviously need analysis, but don't bother if you've done no Analysis)
Posted by shamika

Problem 44***

Let $P(n)$ be a polynomial with coefficients in $\mathbb{Z}$. Suppose that $\deg(P) = p$, where p is a prime number. Suppose also that $P(n)$ is irreducible over $\mathbb{Z}$. Then there exists a prime number q such that q does not divide $P(n)$ for any integer n .

Problem 45*

Let p be a prime number, $p > 3$. Given that the equation $p^k + p^l + p^m = n^2$ has an integer solution, then $p \equiv -1 \pmod 8$.
Posted by Mladenov

Problem 46

This is a little Mechanics problem that I come up with (you can use a calculator or you can leave your answers exact) :A car, $2.5m$ in length, is travelling along a road at a constant speed of $14ms^{-1}$. The car takes up the whole of one half of a road of width $7m$. You want to cross the road at a constant speed of $2.5ms^{-1}$. You will cross the road at an angle θ degrees to the shortest path to the other side of the road, where theta is positive to the right of this line and negative to the left. The car is approaching you from a distance of $40m$ to the left of this line, keeping to the opposite side of the road. This car is the only car on the road. Find all possible angles θ such that you can cross the road without being hit.
Posted by metaltron

Problem 47 ***

(came across this nice problem thanks to a friendFor $0 < p < q$, evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+p^2)(x^2+q^2)}\, dx$$

and deduce that

$$\int_{-\infty}^{\infty} \frac{x \cos x}{(x^2+p^2)(x^2+q^2)}\, dx = 0$$

Posted by Indeterminate

Problem 48*

$$f : \mathbb{R} \to \mathbb{R}$$

and

$$w : \mathbb{R} \to \mathbb{R}$$

. Prove that there exist no f, w such that $f\big(w(x)\big) = x^2$ and $w\big(f(x)\big) = x^3$, but that there exist f, w satisfying $f\big(w(x)\big) = x^2$ and $w\big(f(x)\big) = x^4$
Posted by Lord of the Flies

Problem 49**

Show that

$$7|5555^{2222}+2222^{5555}$$

Posted by

Problem 50:

An inextensible rope of length l and uniform mass per unit length lies on a rough table with one end on an edge. The co-efficient of friction between the table and the rope is Î¼. The rope receives an impulse which sets it moving off of the edge of the table at speed v. Prove that, if the rope does not fall off the table, then:

$$v^2 < \frac{lg\mu^2}{1+\mu}$$

By considering this as μ varies between zero and 1, find the maximum possible impulse that could potentially be given to the rope without it falling off of the table.
Posted by DJMayes

Problem 51**

One of my favorite number theory problems.Let $n > 1$ be an integer, and k be the number of distinct prime divisors of n . Then there exists an integer a , $1 < a < \frac{n}{k}+1$, such that $a^2 \equiv a \pmod{n}$.

Problem 52**

Find all infinite bounded sequences $a_1, a_2, ...$ of positive integers such that for each $n > 2$,

$$a_n = \frac{a_{n-1}+a_{n-2}}{(a_{n-1}, a_{n-2})}$$

.

Problem 53**

For those who have not covered much number theory.Let k be a positive integer. Then there exist infinitely many positive integers n such that $n2^k - 7$ is a perfect suqare.
Posted by Mladenov

Problem 54

*A triangle has sides of length at most 2, 3 and 4 respectively.
Determine, with proof, the maximum possible area of the triangle.

Spoiler:

Show

Come on, it's a good question!

Posted by shamika

Problem 55*

The problem is concerned with constructing lines of length root n , where n is not a perfect square. Suppose you only have a ruler which can only measure integer lengths and assume you can draw perfect right-angled triangles. Show that unless n is of the form $4k - 2$, then you only ever need one right-angled triangle to construct a line of length root n , e.g. To construct $\sqrt{2}$, you create a

$$\left\{1, 1, \sqrt{2}\right\}$$

triangle, to construct $\sqrt{3}$ you create a

$$\left\{1, \sqrt{3}, 2\right\}$$

triangle.
Posted by Blazy

Problem 56

/*Evaluate

$$\int_0^{2\pi} \frac{1}{\cos x + 5}\ dx$$

Posted by Indeterminate

Problem 57 *

find

$$\int_{-1}^{1} \frac{1 + x^4 tan(x)}{1 + x^2}\ dx$$

Posted by james22

Problem 58 **/***

A double pendulum consists of a mass m_2 suspended by a rod of length l_2 from a mass m_1, which is itself suspended by a rod of length l_1 from a fixed pivot, as shown below.

Show that the equations of motion for ***small displacements*** can be written as

$$M\ddot{\Theta} = -K\Theta$$

where,

$$M = \begin{pmatrix} {l_1}^2(m_1 + m_2) & l_1 l_2 m_2 \\ l_1 l_2 m_2 & {l_2}^2 m_2 \end{pmatrix}$$

$$K = \begin{pmatrix} g l_1 (m_1 + m_2) & 0 \\ 0 & g l_2 m_2 \end{pmatrix}$$

and

$$\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

Posted by dknt

Problem 59**

Prove that there are no integers x, y for which $x^2 + 3xy - 2y^2 = 122$.
Posted by und

Problem 60**/***

Evaluate the integral

$$I = \int_{-\infty}^{\infty} sin\left(\pi^4 x^2 + \frac{1}{x^2}\right) dx$$

.
Posted by und

Problem 61**

Find all functions

$$f : \mathbb{R}^+ \to \mathbb{R}^+$$

satisfying $f(x^y) = f(x)^{f(y)}$.
Posted by Mladenov

Problem 62*

Show that if $p > m > 0$, then

$$\frac{p-m}{p+m} \leq \frac{x^2 - 2mx + p^2}{x^2 + 2mx + p^2} \leq \frac{p+m}{p-m},\ \forall x \in \mathbb{R}$$

Problem 63*

Prove

$$\forall x, y, z, w,\ x^2 + y^2 + z^2 + w^2 + 1 \geq x + y + z + w$$

Posted by Star-girl

Problem 64 */**(?)

Prove that the following are irrational:(a) $\sqrt{2}+\sqrt{3}$;
(b) the real root of $x^3-4x+7=0$; and
(c) $\log_2 3$ Spoiler:

Show

This was on a first year Cambridge paper. Standards these days

Posted by shamika

Problem 65 **/***

A cone of base radius R, height H, mass M and homogeneous mass density is orientated as shown below.

The angle the cone makes to the xy plane is . The cone is then attached to an axis along its outer mantle in the direction of $\mathbf{\hat{u}}$ .Calculate the moment of inertia of the cone, with respect to this axis.
Posted by dknt

Problem 66*

For any fixed number q , let

$$A_n = \frac{q^n}{n!}\int_0^\pi \left[x\left(\pi - x\right)\right]^n sinxdx$$

.Show that

$$A_n = \left(4n-2\right)qA_{n-1} - \left(q\pi\right)^2 A_{n-2}$$

and use this result to deduce that A_n is an integer for all n if q and $q\pi$ are integers.Show also that $A_n \to 0$ as $n \to \infty$. Deduce that π is irrational.
Posted by und

Problem 67*

A lense maker wishes to measure the radius of a hemispherical lense by placing a 1 meter ruler on top of the lense and causing it oscillate with a small amplitude. Assuming the ruler is much larger than the radius of the lense, show, using appropriate approximations that:

$$r = \frac{1}{3g}\left(\frac{\pi}{T}\right)^2$$

T is the time period. r radius.
Posted by bananarama2

Problem 68**

Evaluate

$$\int_0^1 \int_0^1 ... \int_0^1 [x_1 + x_2 + ... + x_n] dx_1 dx_2 ... dx_n$$

.
Of course, $[x]$ is the largest integer not greater than x .
Posted by Mladenov

Problem 69

***For $x \neq 0$ evaluate

$$\sum_{r=1}^{\infty} \frac{1}{r^2 + x^2}$$

Posted by Indeterminate

Problem 70 */**

Consider the set $\{1, 2, 3, ..., N\}$. Two sets A and B are chosen independently and equally likely among all the subsets of $\{1, 2, 3, ..., N\}$ .Find $P(A \subseteq B)$ .I hope it's not already been done.
Posted by TheJ0ker

Problem 71**

Find all $x, y \in \mathbb{C}$ such that $x^y = y^x$ .EDIT: The question is asking for a way to generate solutions of that kind. To be precise, define the pair of parametric functions $x, y : S \to \mathbb{C}$ such that

$$x(a)^{y(a)} = y(a)^{x(a)}$$

for all a of some set S (for example, $\mathbb{R}$ or $\mathbb{C}$), and the functions must be able to generate all the possible solutions in $\mathbb{C}$. You shouldn't need to use any function definitions beyond A-level.
Posted by

Problem 72: *

Bit of an effort: - But hey ho.
Prove by considering:

$$I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\ dx$$

that $\pi < \frac{22}{7}$ Find

$$I = \int \sqrt{1+e^x}\ dx$$

Posted by joostan

(You'll probably all find this easy, but perhaps its a good one to make you think about initial approach. Plus i made it myself!)

Problem 73*

i) Find the number of ordered sets of positive integers

$$x_1, x_2, x_3, ..., x_n$$

such that

$$\sum_{i=1}^{n} x_i - \prod_{i=1}^{n} x_i = n - 1$$

and $x_i \leq m$ where m is a positive integer such that $m \geq n$. Justify your answer.
ii) Extend your result to include all ordered sets of non-negative integers.
Posted by Jkn

Problem 74**

Let a, b, c be positive real numbers. Then we have the following inequality:

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} + \frac{2(ab+bc+ca)}{(a+b)(b+c)(c+a)} \geq \frac{3(a+b+c)}{2(ab+bc+ca)}$$

.

Spoiler:

Show

It seems more difficult, than it actually is.

*Posted by Mladenov*problem 74 (fml!) so thought "if you can't beat them, join them")

Problem 75

Show that

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{3}{2}$$

for positive real numbers a,b,c such that $abc = 1$.
Posted by Jkn

Problem 76*/**

For positive real numbers a,b,c, prove that

$$(a^2 + b^2)^2 \geq (a+b+c)(a+b-c)(b+c-a)(c+a-b)$$

Posted by Jkn

Problem 77*

Long John Silverman has captured a treasure map from AdamMcBones.
Adam has buried the treasure at the point (x,y) with integer co-ordinates (not necessarily positive).
He has indicated on themap the values of $x^2 + y$ and $x + y^2$, and these numbers are distinct.Prove that Long John has to dig only in one place to find the treasure.
Posted by Jkn

Problem 75

, notice that by plugging in a couple coefficients into the inequality and taking the exact same approach as in Mladenov's solution we can easily generalise the result:

$$\sum_{a,b,c} \frac{1}{b(a\lambda + b\mu)} \geq \left(x^2 + y^2 + z^2\right)^2 \left(\lambda \sum_{\text{cyc}} x^3 y + \mu \sum_{\text{cyc}} x^2 y^2\right)^{-1}$$

Then using the same tricks to show that

$$\lambda(x^2 + y^2 + z^2)^2 \geq 3\lambda \sum_{\text{cyc}} x^3 y$$

and

$$\mu(x^2 + y^2 + z^2)^2 \geq 3\mu \sum_{\text{cyc}} x^2 y^2$$

we get:

$$\frac{1}{b(a\lambda + b\mu)} + \frac{1}{c(b\lambda + c\mu)} + \frac{1}{a(c\lambda + a\mu)} \geq \frac{3}{\lambda + \mu}$$

For those how dislike inequalities with a passion:

Problem 78

(a classic!)Evaluate

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\cdots}}}}$$

Problem 79

This is lolz-maths, but whatever. Let's see who can do it the fastest!If

$$f(x) = 6x^7 e^{x^2} \sin^2(x^{1000})$$

find

$$f^{(2013)}(0) - f^{(2012)}(0)$$

*Posted by Lord of the Flies*Problem 76 is wrong in this form (I thought $c \leq a+b$).Edit: We have to bound c ; I can try to find the best possible c .**Solution 79**As $x \to 0$, from Taylor's series we have

$$\sin(x) = x + \mathcal{O}(x^3)$$

,

$$e^x = 1 + x + \frac{1}{2}x + \frac{1}{6}x^3 + \mathcal{O}(x^4)$$

.
Thus,

$$6x^7 e^{x^2} \sin(x^{1000}) = 6x^7(x^{1000} + \mathcal{O}(x^{2000}))^2(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \mathcal{O}(x^8))$$

.
Hence

$$f^{(2013)}(0) - f^{(2012)}(0) = 2013!$$

Problem 80

Let $(a_n)_{n\geq 1}$ be an increasing sequence of positive integers. Suppose also that

$$\lim_{n\to\infty} \frac{a_n}{n} = 0$$

. Then, the sequence

$$b_n = \frac{n}{a_n}$$

contains all positive integers.
Posted by Mladenov

Problem 81

Find the smallest positive integer n such that n can be uniquely expressed as the sum of two distinct cubes in two distinct ways. Justify your answer.
Posted by Jkn

Problem 82**

We have not done any combinatorics.Let $n \in \mathbb{Z}^+$ and $A \subset Z$ are such that:
i) each element $a \in A$ can be represented as $a = b + c$, where $b, c \in A$;
ii) if

$$a_1, a_2, .., a_k \in A$$

and $k \leq n$, then

$$\sum_{0\leq i\leq k} a_i \neq 0$$

.
Then, we have $card(A) \geq 2n + 2$.

Problem 83**

Let x , a , and b be positive integers such that $x^{a+b} = a^b b$. Then $a = x$ and $b = x^x$.
Posted by Mladenov

Problem 84**

Let S be a set of n elements. Denote by $\mathcal{P}(S)$ the set of all subsets of S .
Suppose

$$f : \mathcal{P}(S) \to \mathbb{Z}^+$$

satisfies:
i) for every subset A of S , we have $f(A) = f(S - A)$;
ii) for every two subsets A and B of S , we have

$$\max(f(A), f(B)) \geq f(A \cup B)$$

.
Then, the number of positive integers m such that there exists $A \subseteq S$ with $f(A) = m$ is less than n .
Posted by Mladenov

Problem 85*

How many common terms have the following arithmetic progressions:
$2, 7, 12, 17, ..$ and $2, 5, 8, 11, ..$, if they have the same number of terms - 121 ?

Problem 86*

Solve in integers:
$1 + x + x^2 = y^n$, where $n \geq 2$ is even number.
*Posted by Mladenov*problem 86. You have missed solutions and $(1, 1)$ is not a solution, for example.

Problem 87*

A group of boys and girls went to a dance party. It is known that for every pair of boys, there are exactly two girls who danced with both of them; and for every pair of girls there are exactly two boys who danced with both of them. Prove

that the numbers of girls and boys are equal.
Posted by Mladenov

Problem 88*

$P(N)$ is defined as the largest product that can be made from positive integers that add up to N.i.e.

$$P(N) = max(a_1a_2...a_n) : N = a_1 + a_2 + ... + a_n$$

.Prove the Goldbach Conjecture

Spoiler:

Show

lol jk find $P(N)$

Problem 89*

Find the smallest prime number N such that the following is true:The largest prime factor of N-1 is A;
The largest prime factor of A-1 is B;
The largest prime factor of B-1 is 7.
Posted by Jkn

Problem 90*

Find the sum of all values of a such that the equation

$$(x^2 - x + a + 1)^2 = 4a(5x^2 - x + 1)$$

has 3 distinct real solutions.
Posted by Jkn

Problem 91*

Solve in integers $x^5 - x^3 = y^3z$, where y, z are prime numbers.

Problem 92*

Define $a_1 = 1$, and for $n > 1$,

$$a_n = n(a_1 + .. + a_{n-1})$$

. Then for every $n \equiv 0 \pmod 2$, a_n is divisible by $n!$.
Also, find all $n \equiv 1 \pmod 2$ such that

$$a_n \equiv 0 \pmod{n!}$$

.
Posted by Mladenov

Oh, sorry, I hadn't seen their solution.I will now see if I can find any interesting problems.

Edit: Ah, found one. The only problem I managed to get in the BMO2Problem 93*

Prove there are an infinite number of positive integers m and n, such that $m|n^2 + 1$ and $n|m^2 + 1$
Posted by Zephyr1011

Problem 94**

One really nice functional equation.Find all strictly increasing functions

$$f : \mathbb{R} \to \mathbb{R}$$

such that $f(x + f(y)) = f(x + y) + 1$, for all real numbers x and y .

Problem 95**

warm-upLet a, b, c be positive real numbers. Then,

$$\sum_{cyc} \frac{a^2b^2(b - c)}{a + b} \geq 0$$

.Problem 93 has been already
Posted.
Posted by Mladenov

Problem 96*

Let x, y and a be positive integers such that $x^3+ax^2=y^2$. Given that $1 \leq x \leq b$ where b is a positive integer, find, in terms of a and b, the number of possible pairs (x, y) that satisfy the equation.
Posted by Jkn

Problem 97**

Find all continuous functions, defined for every $x \in \mathbb{R}$, which satisfy $f(f(x)) = f(x) + x$.
Posted by Mladenov

Problem 98**

Let $(a_n)_{n\geq 0}$ be a sequence such that $a_0 > 0$ and

$$a_{n+1} = a_n + \frac{1}{a_n}$$

.
Then, the sequence $(a_n)_{n\geq 0}$ is divergent.
Also, find:

$$\lim_{n\to\infty} \frac{a_n}{\sqrt{n}}$$

.
*Posted by Mladenov*problem 97 was quite different (it took me 4 pages A4 to solve itLet me suggest another lovely functional equation. It seems as though you are the only one who enjoys doing functional eqs.

Problem 99**

Find all continuous functions $f(x)$, defined for all $x > 1$, which satisfy $f(xy) = xf(y) + yf(x)$, for all $x, y > 1$.

Nope. As the expression, $a_n = m\sqrt{n}$ is not correct, the subsequent equalities in your solution are also incorrect. And finally, your conclusion is not true.

I second!
To summarise: breathtaking and elegant.
I absolutely enjoyed the last two chapters on inversive and projective geometry.
Posted by Mladenov

Problem 100*/**

Rigorously prove that the area of a rectangle of side lengths p and q is pq (for real numbers p, q).
Posted by Jkn

Problem 101**

Prove that

$$(\frac{27(x^2+y^2+z^2)}{4})^{\frac{1}{4}} \geq \sum_{cyc} \frac{x}{\sqrt{x+y}} \geq (\frac{27(xy+yz+zx)}{4})^{\frac{1}{4}}$$

for positive real numbers x, y and z.

Spoiler:

Show

Posted by Jkn

Problem 102*

Mladenov's functional eq. reminded me of this one.For $0 \leq (x, y) \leq 1,\ f$ is a continuous function satisfying $xf(y) + yf(x) \leq 1$. Prove that:

$$\int_0^1 f(x)\, dx \leq \frac{\pi}{4}$$

Problem 103*

(there is a slick *** solution though)

$$\int_0^{\infty} \arctan\left(\frac{x(e-1)}{ex^2+1}\right) \frac{dx}{x}$$

Functional equations are lovely! Here is a not-so-easy one:

Problem 104**

Find all continuous $f : [-1,1] \to \mathbb{R}$ such that $2xf(x) = f(2x^2 - 1)$ (this one is popular on TSR so many of you may have seen it)
Posted by Lord of the Flies

Problem 105*

(technically, if we are rigorous, this is a ***)

$$\int_0^\infty \left\{ x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots \right\} \left\{ 1 + \left(\frac{x}{2}\right)^2 + \left(\frac{x^2}{2\cdot 4}\right)^2 + \left(\frac{x^3}{2\cdot 4\cdot 6}\right)^2 + \cdots \right\} dx$$

(the curly braces don't mean anything in particular - I just thought they looked nice)
Posted by Lord of the Flies

Problem 106

Prove the formula

$$\sum_{k=0}^{n} \prod_{\mu=1}^{a} (k+\mu) = \frac{1}{a+1} \prod_{\mu=1}^{a+1} (n+\mu)$$

Problem 107

Evaluate the series

$$\sum_{n=0}^{\infty} \frac{(n+\mu)!}{2^n n! \mu!}$$

Posted by FireGarden

Problem 108*

Find all pairs of integers (p,q) such that the the roots of the equation,

$$(px - q)^2 + (qx - p)^2 = x$$

, are integers.

Spoiler:

Show

Mladenov isn't allowed to solve it and I'm sure he knows why

————————Btw, can people make sure they put 'assumed knowledge' ratings on problems please.
Posted by Jkn

Problem 109*/**

$$f(x) = sin(x) - \int_0^x (x - t)f(t)dt$$

Find f exactly
Posted by FireGarden

Problem 110*

Prove that

$$(\sum_{k=1}^{\infty} a_k b_k)^2 \leq \sum_{k=1}^{\infty} {a_k}^2 \sum_{k=1}^{\infty} {a_k}^2$$

for all real numbers a_k, b_k with using Cauchy-Schwartz.

Problem 111*

(another problem I've adapted from something else. Oh how I love generalisations...)Let a,b and λ be positive integers such that $a\lambda = b^3$. Find the minimum value of $a + b$ for the different cases that arise according to the value of λ given that λ has 4 *proper factors* (i.e. factors not including itself and 1.)
Posted by Jkn

Problem 112***

difficult..Evaluate

$$\int_0^{\frac{\pi}{2}} x \ln(\tan x) dx$$

.

Problem 113***

I cannot resist posting number theory!Let p be a prime number and m, n - positive integers. We have

$$\sum_{i \equiv 0 \pmod p} (-1)^i \binom{n}{i} i^m \equiv 0 \pmod{p^{\left[\frac{n-m-1}{p-1}\right]}}$$

.

It is given on the third round, right?
It is too easy for the fourth round; I have solved negligible number of questions of our third round.

Well. I am gonna ace it now.
Posted by Mladenov

Problem 114

*Find

$$\int \sqrt{tanx}.dx$$

Posted by Jkn

Problem 115**

Evaluate

$$I = \int \left(\int_{-x}^{x} sec(t)\ dt \right) sin(x)\ dx$$

Posted by FireGarden

Problem 117**

Find all continuous functions defined for each $x \in \mathbb{R}$, which satisfy

$$f(x^p + y^q) = f(x)^p + f(y)^q$$

, where p and q are positive integers at least one of which is greater than 1 .By the way, very elegant solution to problem 112.I noticed discussion regarding mathematical books. I do not know whether or not Spivak's book is a must, but Rudin's Principles of Mathematical Analysis definitely is. For algebra - look at Lang's book - his approach is rigorous and modern, yet he does not give many examples, not to mention his proofs. In other words, his books will teach you how to run if you know how to walk. Another good reference which comes to my mind is Waerden's Algebra - it is good but a bit out of date. For topology - Munkres is good, but I prefer Kelley's approach.

I shall check it out!
Posted by Mladenov

Problem 118**

Let n be a positive integer. Let $S_1, S_2, ..., S_n$ be subsets of $\{1, 2, ..., n\}$ such that for any $1 \leq k \leq n$ the union of any k of the subsets contains at least k elements. Prove that there is a permutation

$$(a_1, a_2, ..., a_n)$$

of $(1, 2, ..., n)$ such that $a_i \in S_i$.
Posted by Mladenov

Problem 119***

$$\sum_{n=0}^{\infty} \binom{2n}{n}^{-1}$$

Posted by Lord of the Flies

Problem 120***

Evaluate:

$$\int_{-\infty}^{\infty} \frac{\cos x - 1}{x^2(x^2+a)} dx$$

, where $a > 1$.
Posted by Mladenov

Problem 121***

It is not that awful.Evaluate

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{sign}(x)\text{sign}(y) e^{-\frac{x^2+y^2}{2}} \sin(xy)\, dx\, dy$$

Problem 122***

This is tough, unless one has studied analytic number theory.Let

$$\sum_{n \equiv 1 \pmod 2} (-1)^{\frac{n-1}{2}} \frac{\log n}{\sqrt{n}}$$

, and

$$\sum_{n \equiv 1 \pmod 2} (-1)^{\frac{n-1}{2}} \frac{1}{\sqrt{n}}$$

. Find an explicit form for

$$\left(\sum_{n \equiv 1 \pmod 2} (-1)^{\frac{n-1}{2}} \frac{\log n}{\sqrt{n}} \right) \left(\sum_{n \equiv 1 \pmod 2} (-1)^{\frac{n-1}{2}} \frac{1}{\sqrt{n}} \right)^{-1}$$

.
Posted by Mladenov

Problem 123***

Prove that $x^8 \equiv 16$ (Mod p) is solvable for every prime p.
Posted by FireGarden

Problem 124

Evaluate

$$\tau_p = \sum_{k=0}^{p-1} \left(\frac{k}{p}\right) e^{\frac{2\pi i k}{p}}$$

Where

$$\left(\frac{k}{p}\right)$$

is the legendre symbol.
Posted by FireGarden

Problem 125**

Let $a_1, a_2, ..., a_n$ be n students. Some of these students know each other. Show that we can split these students into two groups such that if given student knows m students from his or her group, then he or she knows at least m students from the other group.

Problem 126***

Let

$$f : [0, 1] \to \mathbb{R}$$

be a continuous non-decreasing function. Show that

$$\frac{1}{2}\int_0^1 f(x)dx \leq \int_0^1 xf(x)dx \leq \int_{\frac{1}{2}}^1 f(x)dx$$

.
*Posted by Mladenov*Problem 127 *Two cards are chosen (randomly, without replacement) from a standard deck (52). Find the probability that**i)** both cards are aces given that at least one ace is chosen**ii)** both cards are aces given that the ace of spades is chosenand comment on your answers.
*Posted by jack.hadamard*problem 128, try to prove the following result:

Problem 130*

$$\frac{n}{2n+1}\frac{((2n)!!)^2}{((2n-1)!!)^2(2n+1)} < \left(\int_0^\infty e^{-x^2}\,dx\right)^2 < \frac{n}{2n-1}\frac{(2n-3)!!)^2(2n-1)}{((2n-2)!!)^2}\left(\frac{\pi}{2}\right)^2$$

.
Then note that e^{-x^2} is even.
Posted by Mladenov

Problem 129 **

Show that:y1=(1/k)*integral [0,x] f(x')sinh{k(x-x')} dx'Is a particular solution to the second order differential equation:y"-(k^2)y=f(x)
Posted by

Problem 131 (*)

Find the sum of squares of all real roots of the polynomial :

$$f(x) = x^5 - 7x^3 + 2x^2 - 30x + 6$$

If you find it easy, please wait for somebody else to post an answer!
Posted by metaltron

Problem 132*

$$\int_0^\infty \frac{\ln x}{(x^2+1)^2}\,dx$$

Problem 133*

$$\int_0^\infty \frac{\ln x}{x^2+\alpha^2}\,dx \ \ (\alpha > 0)$$

Problem 134*

$$\int_0^\infty \left(\frac{1}{1+x^2}\right)\left(\frac{x^y-x^z}{(1+x^y)(1+x^z)}\right)\,dx \ \ (y,z \in \mathbb{R})$$

Problem 135**

Find all n such that: $1! + 2! + 3! + \cdots + n!$ is a perfect square.Find all m such that:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

is an integer.
Posted by Lord of the Flies

Problem 136*

Let

$$f : \mathbb{R} \to \mathbb{R}$$

be a strictly increasing invertible function such that for all $x \in \mathbb{R}$ we have $f(x) + f^{-1}(x) = e^x - 1$. Prove that f has at most one fixed point.

Problem 137**

Find all continuous functions

$$f : \mathbb{R} \to \mathbb{R}$$

which satisfy $f(f(x)) - 2f(x) + x = 0$ for all $x \in \mathbb{R}$.

Problem 138***

Let $P(X)$ be an irreducible polynomial over $\mathbb{Z}[X]$. Show that $P(XY)$ is irreducible over $\mathbb{Z}[X, Y]$.
Posted by Mladenov

Problem 139 / **

Given an integer-valued polynomial, can you always find some values such that their sum is divisible by $n \in \mathbb{N}$?
Posted by jack.hadamard

Solution 139 Fix n . We work in $\mathbb{Z}/n\mathbb{Z}$ .Let $P(m)$ be an integer-valued polynomial. Clearly, there are infinitely many $m_i \in \mathbb{Z}$, $i \in \mathbb{Z}^+$ such that $P(m)$ is constant over the set

$$\{m_i | i \in \mathbb{Z}^+\}$$

. Hence, we can always choose n elements from the set

$$\{m_i | i \in \mathbb{Z}^+\}$$

, and

$$\sum_{i=1}^{n} P(m_i) = 0$$

.

I had found Lang's Undergraduate Algebra quite easy to comprehend, and thus I decided to buy his Algebra.Yep, I am finishing high school this year.

Problem 140*

Evaluate

$$\sum_{v=1}^{\infty} \frac{1}{2^v v^2}$$

.

Problem 141**

Let $f \in C^\infty$ for $x > 0$, and

$$0 \leq (-1)^n f^{(n)}(x) \leq e^{-x}$$

for all $x > 0$ and

$$n \in \mathbb{Z}^+ \cup \{0\}$$

. Find f .Remark: $f^{(n)}(x)$ is the n th derivative of f .
Posted by Mladenov

Problem 142*

Evaluate

$$\sum_{v=1}^{\infty} \frac{x^v \sin v\alpha}{v}$$

, ($|x| < 1$).
Hence, find

$$\sum_{v=1}^{\infty} \frac{\sin vx}{v}$$

, ($0 < x < 2\pi$). Can you evaluate

$$\sum_{v=1}^{\infty} \frac{\sin vx}{v^3}$$

, ($0 \leq x \leq 2\pi$)?

Problem 143*

Prove that

$$\frac{1}{n} \sum_{v=1}^{n} \left(1 + \frac{1}{2v}\right)^{2v} \leq \left(1 + \frac{1}{n+1}\right)^{n+1}$$

.

Problem 144*

There are several castles in one country and three roads lead from every castle. A knight leaves his castle. Traveling around the country he leaves every new castle via the road that is either to the right or to the left of the one by which he arrived. According to The Rule the knight never takes the same direction (right or left) twice in a row. Prove that some day he will return to his own castle.
Posted by Mladenov

Problem 145**

Let $\mathcal{P}_2(x)$ be the set of all polynomials of degree at most 2.Find

$$q(x) \in \mathcal{P}_2(x)$$

such that

$$\int_0^1 p(x)q(x)\ dx = p(\frac{1}{2})$$

for *all*

$$p(x) \in \mathcal{P}_2(x)$$

Posted by FireGarden

Problem 146**

Evaluate

$$\sum_{p\in P}^{\infty} \frac{1}{p}$$

Posted by Jkn

Problem 147 / ***

Assume s is a set containing numbers. Let $\sigma(s)$ and $\pi(s)$ denote the sum and product, respectively, of all the elements in the set s .Let S be the set $\{1, 2, ..., n\}$.**i)** Find an expression for

$$\sum_{s\in\mathcal{P}(S)} \frac{1}{\pi(s)}$$

.**ii)** Prove that

$$\sum_{s\in\mathcal{P}(S), s\neq\varnothing} \frac{\sigma(s)}{\pi(s)} = n(n+2) - (n+1)\sum_{k=1}^{n} \frac{1}{k}$$

.Note that $\pi(\varnothing) = 1$.
Posted by jack.hadamard

Problem 148

**Evaluate

$$\int_{-\infty}^{\infty} \frac{sin^2 x}{x^2} dx$$

Posted by Jkn

Problem 149 */**

Part 1) Take

$$f(x) = \pi x - x^2, 0 \leq x \leq \pi$$

. Find its periodic extension (ie, calculate its Fourier Series).Part 2) Solve the Basel Problem.Edit: The hope are for literally anyone who comes across this to give it a try to see some particularly nice mathematics. I just realised, this does need more thought than simply applying the definitions to construct a fourier series, since upon first glance you can't apply it directly (as f is not explicitly defined over an interval of the form -L to L).. if you get stuck here, that's when this spoiler will be useful

Spoiler:

Show

f(x) here is an even function, and as such you don't even need to calculate any of the b_k , as obviously an even function has no odd parts! Now for the a_n , by the definition of the range x can take, you find limits of -L to L to be impossible.. well, since f is even, the integral can be doubled, and instead go from 0 to L. (where L will be $\pi/2$)

Posted by FireGarden

Problem 150

*i) Prove, from first principles, the relation between the relativistic mass and the rest mass of an object.

Spoiler:

Show

$$m_{rel.} = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

ii) Hence derive Einstein's mass-energy equivalence equation.

Spoiler:

Show $E = mc^2$ **Note:** I have put the equations in spoiler-brackets so as not to lead you into thinking that algebraic manipulation is satisfactory and also not to annoy undergraduates who may be more accustomed to seeing each formula is far more sophisticated formsAlso, I like to think the aim of this thread is to share ideas so please justify and derive fully (e.g. please don't put "using the Lorentz factor" or anything like that)

Hmm, do you think that would help launch someone into an academic career more than the UK PhD then?Out of interest, how significant a contribution must be made to warrant a PhD? Because it seems unlikely that everyone with a PhD has been able to really "discover" something new (or perhaps I am being incredibly naive).Ahhh that all sounds awesome
Posted by Jkn

Problem 151**

Prove that $\zeta(3)$ is irrational.

Spoiler:

Show

Here, $\zeta(3)$ represents a specific value of the "Riemann Zeta Function", often referred to as Apery's Constant.So

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Posted by Jkn

Problem 152

**Given that

$$\sum_{i=1}^{n} x_i = S$$

and

$$max(\prod_{i=1}^{n} x_i) = P$$

for real numbers x_i such that $x_i > 0 \ \forall i$, find a condition on S for which P is irrational.

Spoiler:

Show

Posted by Jkn]**Note:** To find a complete condition on P would require solving one of "the great unsolved problems" in mathematics. For this problem, you must find all conditions on P that are you can and then prove this assertion from first principles

Posted by Jkn

For anyone taking Edexcel Statistics 4...

Problem 153

*/**Prove that a t-distribution with degree of freedom ν is asymptotically equivalent a the Normal distribution for sample size n such that $n \to \infty$, stating the parameters of the Normal distribution to which it is equivalent.Hint:

Spoiler:

Show

Would be awesome if someone derived this from definitions, but if not:It can be shown that

$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left(\frac{1+\frac{x^2}{\nu}}{}\right)^{-\frac{\nu+1}{2}}$$

, where $f(x)$ denotes the probability density function of a t-distribution with degree of freedom ν Note: You may not use standard values/results for the gamma function. First principles por favor

Spoiler:

Show

Also... Is this so-called hint just an over complication?

*Posted by Jkn*Problem 154 / *Let

$$a_1 < a_2 < a_3 < ...$$

be positive, non-consecutive integers. Let

$$s_m = a_1 + a_2 + a_3 + ... + a_m$$

for $m \in \mathbb{N}$ .Prove that, for all $n \in \mathbb{N}$, the interval $[s_n, s_{n+1})$ contains a perfect square.
Posted by jack.hadamard

Problem 155

*Find (without calculus) a fifth degree polynomial $p(x)$ such that $p(x) + 1$ is divisible by $(x-1)^3$ and $p(x) - 1$ is divisible by $(x+1)^3$.

Spoiler:

Show

Not too bad but is very reminiscent of a creation STEP question, but without being permitted (let alone led into) the simple differentiation approach. So is going to be good STEP preparation!

Problem 156

*/**/***Evaluate

$$\int_0^\pi \ln(1 - 2a\cos(x) + a^2)dx$$

, where a is a real number, in the case such that $|a| > 1$.

Problem 157

*/**/***Evaluate

$$\int_0^1 \frac{x-1}{\ln(x)} dx$$

Problem 158

*/**/***Evaluate

$$\int_0^\infty \frac{\ln(1+x^2)}{1+x^2} dx$$

and finally, because there's no point in me simply typing my solution up (especially since no-one seems too interested anyway)...

Problem 159

**Evaluate

$$\int_{-\infty}^\infty \left(\frac{\sin(x)}{x}\right)^n dx$$

, where n is a positive integer, in terms of a combination of a finite number of elementary functions.
*Posted by Jkn*Problem 158 but the rest I did differently. I used differentiation under the integral sign for 156 which pops the solution out almost instantly (yours looks a little complicated!) Really like the elegance in 157And I don't see what's going on in 159. Where does the first line come from? And how does it link to the second and third? The differentiation thing doesn't seem to make sense (though I'm sure it does), could you clarify it for meAnd yeah, as said above, it's far too easy and boring to do it that wayPlease **post** a load of really nice integrals! (None of that crap involving 2013s and stuff where you're never going to come across a similar thingThis one is really nice:

Problem 160

*/**/***Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}dx$$

Posted by Jkn

Problem 161

/*Let k be an integer greater than 1. Suppose $a_0 > 0$ and

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$ Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}$$

Posted by Jkn

Problem 162

*****If I drop a ball off the Eiffel tower facing east. Where does it land? (Neglect air resistance)
Posted by bananarama2

Problem 163**

Let $f \in C^1$. Find

$$\lim_{n\to\infty} n\left(\int_0^1 f(x)dx - \frac{1}{n}\sum_{i=1}^{n} f(\frac{i}{n})\right)$$

.

Problem 164**

Evaluate

$$I = \int_0^\infty \frac{\arctan x}{x^\alpha} dx$$

, where

$$\alpha \in (1, 2)$$

.

Problem 165***

Evaluate

$$\int_0^2 \frac{x^4}{(x^2+1)(x(2-x)^3)^{\frac{1}{4}}} dx$$

.

Problem 166***

Evaluate

$$\int_0^\infty \frac{x \ln x}{(1+x^2)(1+x^3)^2} dx$$

.

Problem 167***

(one that is interesting, and not trivial)Evaluate

$$\sum_n \frac{1}{n^2}$$

, over the set of all biquadrate free positive integers.
If you are interested, find

$$\sum_n \frac{(-1)^{\frac{n-1}{2}}}{n}$$

, over the set of all odd cube free positive integers.
Posted by Mladenov

Problem 168*

$$\int_0^{\frac{\pi}{2}} \ln \sin x \ln \cos x \, dx$$

Posted by Lord of the Flies

Problem 169

*/**/*** (Looks friendly… at first)Evaluate

$$\int_0^{\frac{\pi}{2}} x \cot(x) dx$$

Problem 170

***Prove that

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1} d\theta = \frac{x+y}{2xy} \prod_{n=1}^{\infty} \left(1 + \frac{xy}{n(x+y+n)}\right)^{-1}$$

Posted by Jkn

Problem 172*

Is

$$\prod_{r=0}^{100} r!$$

a perfect square? If not, could we remove one of the factorials to make it a perfect square?

Problem 173*

Let

$$P(x) = 24x^{24} + \sum_{j=1}^{23}(24 - j)(x^{24-j} + x^{24+j})$$

Let

$$z_1, z_2, ..., z_r$$

be the distinct zeros of $P(x)$ and let

$$z_k^2 = a_k + ib_k,\ \ k = 1, 2, ..., r$$

and $i = \sqrt{-1}$ and $a,\ b \in \mathbb{R}$ Let

$$\sum_{k=1}^{r} |b_k| = m + n\sqrt{p}$$

where m, n, p are integers and p is not divisible by the square of any primes.
Calculate $m + n + p$
Posted by Felix Felicis

Problem 174

*Evaluate

$$\int_0^{\infty} \frac{\ln(x)}{1 + x^2} dx$$

Problem 175

*Evaluate

$$\int_0^1 \sin(\arccos(x)) dx$$

Problem 176

*Evaluate

$$\int_0^1 x(1-x)^{99}dx$$

Problem 177

*Evaluate

$$\int \sqrt{\csc(x)-\sin(x)}dx$$

Problem 178

*Evaluate

$$\int_0^2 x^5\sqrt{1+x^3}dx$$

Problem 179

*Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sin(4x)}{\sin(x)}dx$$

Problem 180

*Evaluate

$$\int \left(\cos(x)\ln(x)+\frac{\sin(x)}{x}\right)dx$$

Problem 181

*Evaluate

$$\int \frac{1}{\sqrt{x}-1}dx$$

Problem 182

*Evaluate

$$\int \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}}dx$$

Problem 183

*Evaluate

$$\int \frac{1}{\sqrt{x}(\sqrt[4]{x}+1)^{10}}dx$$

...and finally, this one was tie breaker between the top two (how it took them a whole minute I have no idea! You can do this **** in your head

Problem 184

*Evaluate

$$\int \frac{x^4}{1-x^2}dx$$

Posted by Jkn

Problem 185*

Find all

$$f:\ \mathbb{R} \to \mathbb{R}$$

such that:

(i) $f(x) > 0$ (ii) $f(x) \geq x+1$ (iii) $f(x)f(-x) = 1$ (iv) $f(2x) = f^2(x)$

Problem 186*

A nice problem. Prove/disprove that there exists

$$f:\ \mathbb{N} \to \mathbb{N}$$

such that: $f^{f(n)}(n) = n+1$ f^k means f applied k times in this case.

Well, to those who were unhappy about the lack of (*)'s, you now have *loads.*
*Posted by Lord of the Flies*Problem 187 **/***

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+x+1} dx$$

Problem 188

$$\int_{0}^{\infty} x e^{-x^3} dx$$

Posted by Benjy100

Problem 189*

Not hard but it's quirky

$$\int x e^{e^{x^2}+x^2}\ dx$$

Posted by Felix Felicis

Problem 190

*Evaluate

$$\int_0^{\frac{1}{2}} (\arcsin(x))^2 dx$$

Posted by Jkn

Problem 191*

Determine all real numbers z which satisfy this equation, and remarking as to why some roots are disregarded.

$$\sqrt{\sqrt{3-z}-\sqrt{z+1}} > \frac{1}{2}$$

Posted by Zakee

Problem 192*

f is a twice-differentiable function with continuous derivatives, and satisfies the following conditions over (a, b) :

$$\text{(i) } f(x) > 0 \qquad \text{(ii) } f''(x) + f(x) > 0$$

Additionally,

$$\text{(iii) } f(a) = f(b) = 0$$

Show that $b - a > \pi$
Posted by Lord of the Flies

Problem 193 / ****

Prove that $x^2 + y^2 + z^4 = p^2$ has no integer solutions, with $xyz \neq 0$, for a prime $p \equiv 7\ (8)$.
Posted by jack.hadamard

Problem 194/ ****

Let a number x be known as a 'Zakee number' if $\sum$ divisors of $x = 2x + 1$ Find any such number and if found demonstrate the pattern that exists between this number and the array of numbers which conform to this sequence of numbers.
Posted by Zakee

Problem 195

*/**Prove that there exists infinitely many arithmetic progressions of 3 distinct perfect squares.

Problem 196

/*Prove that there exists no arithmetic progressions of 4 distinct perfect squares.

Problem 197

*/**For a non-negative integer n, is $5^{5^{n+1}} + 5^{5^n} + 1$ ever prime? Prove your assertions.

Problem 198

**Let the real numbers a,b,c,d satisfy the relations $a + b + c + d = 6$ and $a^2 + b^2 + c^2 + d^2 = 12$ .Prove that

$$36 \leq 4(a^3 + b^3 + c^3 + d^3) - (a^4 + b^4 + c^4 + d^4) \leq 48$$

Problem 199

***Prove that

$$\int_0^1 t^x(1-t)^y dt = \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}$$

Hint:

Spoiler:

Show

Convolution

Problem 200

*/***Evaluate

$$\int_0^1 \ln(t)t^2 dt$$

Posted by Jkn

Problem 201: *

A stick of unit length is bent at a point along its length, with each such point being equally likely. The stick is then placed on a horizontal surface so that it forms a right-angled triangle, with the shorter edge of the stick being perpendicular to the ground. The angle of elevation of the longer edge of the stick with the ground is α . Prove that the expected value of α is given by: $\dfrac{4-\pi}{2}$
Posted by DJMayes

Problem 202:

*
Just a bit of fun. Really not very difficult, but it looks nice
Find:

$$\int \frac{1}{x\ln(x)\ln(\ln(x))}\ dx$$

And hence evaluate:

$$\int_{e^e}^{e^{e^e}} \frac{1}{x\ln(x)\ln(\ln(x))}\,dx$$

Posted by joostan

Problem 202**

Find all polynomials $P(x)$ such that, for all $n \in \mathbb{Z}^+$, there exists at least one integer m such that $P(m) = 2^n$.

Problem 203**

For any positive integer n set

$$A_n = \{j | 1 \leq j \leq n, \gcd(j, n) = 1\}$$

. Find all n such that

$$P(x) = \sum_{j \in A_n} x^{j-1}$$

is irreducible over $\mathbb{Z}[X]$.

Problem 204**

Let p be a prime number,

$$0 \leq a_1 < \cdots < a_m < p$$

and

$$0 \leq b_1 < \cdots < b_n$$

be arbitrary integers. Let k be the number of different reminders of $a_j + b_i$, $1 \leq j \leq m$ and $1 \leq i \leq n$ modulo p .
Prove that $m + n > p$ implies $k = p$ and $m + n \leq p$ - $k \geq m + n - 1$.
Posted by Mladenov

Problem 205/ **

At the age of three, Zakee begins to learn how to count, and is faced with a problem in his counting book. He believes it is just a trivial problem, after all, he is only three years of age, and so looks at it and tries to solve it. What was the outcome of his attempts? You decide (see below):Sequence of Real numbers:

$$x_0, x_1, \ldots, x_n, \ldots$$

accede to these mathematical conditions

$$1 = x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots$$

Additionally, let

$$y_1, y_2, \ldots, y_n, \ldots$$

be defined as:

$$y_n = \sum_{k=1}^{n} \frac{1 - \frac{x_{k-1}}{x_k}}{\sqrt{x_k}}$$

Prove that: $0 \leq y_n < 2$ FOR ALL n Edit: Question amended due to erratum.
Posted by Zakee

Well, I see what kind of problems people prefer.

Problem 206*

Evaluate

$$\int_0^{\pi} \frac{\ln(1 + a\cos x)}{\cos x}\, dx$$

, where $|a| < 1$.

Problem 207*

*Let α_i , $i \in \{1, \cdots, k\}$ be numbers such that for any two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$, which satisfy the relation

$$b_n = a_n + \sum_{i=1}^{k} \alpha_i a_{n-i}$$

, for all $n > k$, from the convergence of $(b_n)_{n\geq 1}$ follows the convergence of $(a_n)_{n\geq 1}$. Prove that all the roots of the polynomial

$$x^k + \sum_{i=1}^{k} \alpha_i x^{k-i}$$

have absolute values which are less than 1 .

Problem 208***

Let

$$f : [a, b] \to \mathbb{R}$$

be C^2 , and suppose that

$$0 < h < \frac{b-a}{2}$$

. Then,

$$\int_a^b |f'(x)|^p dx \leq 2^p h^{p-1}(b-a) \int_a^b |f''(x)|^p dx + \frac{2^{2p}}{h^p} \int_a^b |f(x)|^p dx$$

$(p > 1)$.
Posted by Mladenov

Problem 209*

Evaluate

$$\int_0^\infty \frac{A_1 \cos a_1 x + \cdots + A_k \cos a_k x}{x} dx$$

when $a_i > 0$ for all

$$i \in \{1, 2, \cdots, k\}$$

and $A_1 + \cdots + A_k = 0$.

Problem 210**

Find

$$\int_0^1 \frac{(1-x^\alpha)(1-x^\beta)}{(1-x)\ln x}\, dx$$

, for $\alpha, \beta > -1$ and $\alpha + \beta > -1$.
*Posted by Mladenov*problem 198, for I have one which is a hell of an arithmetic and I am not quite enthusiastic to type it in latex.

Problem 211**

Evaluate

$$\int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh bx}\, dx$$

, where $b > |a|$.

Problem 212**

Evaluate

$$\int_0^{\infty} \frac{2 - 2\cos x - x \sin x}{x^4}\, dx$$

Posted by Mladenov

Problem 213 **/***

By considering an appropriate contour integral of

$$f(z) = \frac{\sec z}{z^5}$$

and its residues, show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5} = \frac{5\pi^5}{1536}$$

Posted by dknt

A bit of a joke...

Problem 214

**Evaluate

$$\int_0^\infty \frac{3\sin(x) - x\cos(x) - 2x}{x^5}\ dx$$

Posted by Jkn

Problem 21â€‹5

**Prove that

$$\int_0^\infty \int_0^x \frac{\sin(t)}{t}\ dt\ dx$$

does not converge.

Problem 216

*/**Find a series solution for

$$\int_0^1 x^{-x}\ dx$$

(it's fair to say this is going to blow people's minds!)

Well the decision maths syllabus is horrendous! The introduce you to beautiful problems like that of the travelling salesman and things like Game Theory and the ruin them by emphasising memorisation of algorithms instead of any of the maths :'(Exams are basically an exercise is how well you can emulate a computer! :/That's right bud! 7 Pure (****ing only. . .), 5 Mechanics, 4 Statistics and 2 Decision. This is Edexcel btw but represents the typical setupWhat on earth! Firstly, why did they give you an offer anyway without any interview? What things had you done to put in your application that impressed them? Secondly, if they wanted you anyway, why didn't they just give you an offer for the same year? Perhaps they thought you were too young? Thirdly,
was the original offer unconditional or based on STEP I and II?
Posted by Jkn

Problem 217

*Simplify

$$\sum_{r=1}^{n} \sin(rx)$$

Hint:

Spoiler:

Show

There are several accessible methods. One of which involves considering that

$$\sin(x) = Im\left(\cos(x) + i\sin(x)\right)$$

Posted by Jkn

Problem 218 *

Using the operations addition and multiplication as often as required, and using multiplcation only 3 times (no other operations allowed) write (a+bi)(c+di) in the form e+fi
Posted by james22

Problem 219

*Prove that, when evaluating

$$\int_a^b u(x)v'(x)\ dx$$

using integration by parts (the 'parts' being $u(x)$ and $v'(x)$), we may disregard an arbitrary constant when integrating $v'(x)$ .So that, for example, if $v'(x) = 3x^2 + 1$, writing $v(x) = x^3 + x$ will consistently yield the correct answer.
Posted by Jkn

Problem 221

**Let x, y, z be sides of a triangle. Show that

$$\frac{(x+z-y)^4}{x(x+y-z)} + \frac{(x+y-z)^4}{y(y+z-x)} + \frac{(y+z-x)^4}{z(z+x-y)} \geq xy + yz + zx$$

.

Problem 222**

Find all

$$f : \mathbb{R} \to \mathbb{R}$$

which satisfy

$$f(x^2) + f(xy) = f(x)f(y) + yf(x) + xf(x+y)$$

.C'mon problem 219 is trivial, $C(u(b) - u(a))$ cancels as it is equal to

$$\int_a^b Cu'(x)dx$$

Posted by Mladenov

Problem 223

*A spherical ball is dropped from the Eiffel Tower from rest. Give a convincing argument as to why Olber's Paradox gives strong evidence to support the idea that the observable universe has existed for a finite period of time. Note that, for such an argument to be convincing, it must rely solely on assumptions that are supported by strong evidence. In this question you are required to quote primary data sources to support **all** assumptions made.
Posted by Jkn

I've decided to give you a taste of what I do~~**Problem 224**~~

*/**:Whenever we say "function" we mean total function.Denote for every $n \in \mathbb{N}_0$ the set

$$\underline{n} = \{m \in \mathbb{N}_0 | 0 < m \leq n\}$$

.The mystery concept of a **strange product**

$$\underline{m} \otimes \underline{n}$$

for two such sets $\underline{m}$ and $\underline{n}$ satisfies the following property:

There exist functions

$$\pi_1 : \underline{m} \otimes \underline{n} \to \underline{m}$$

and

$$\pi_2 : \underline{m} \otimes \underline{n} \to \underline{n}$$

such that:For each $k \in \mathbb{N}_0$ and each pair of functions

$$f : \underline{k} \to \underline{m}$$

and

$$g : \underline{k} \to \underline{n}$$

there exists a **unique** function

$$h : \underline{k} \to \underline{m} \otimes \underline{n}$$

such that for all $x \in \underline{k}$ we have $f(x) = \pi_1(h(x))$ and $g(x) = \pi_2(h(x))$

Show that regardless of what the exact definition for $\otimes$ is, if it satisfies the above property then for all sets X and (non-zero) natural n ,

$$|X \otimes \underline{n}| = n$$

iff $|X| = 1$ (in other words $X = \underline{1}$).EDIT1: made it clearer, but question is still the same.EDIT2: Hmmm I just realised that the elegant solution is far beyond *-level, and the inelegant one may be quite long.EDIT3: Actually for the above reason I'm going to withdraw it as a formal problem for this thread, but it's open to anyone who wants to try it.
Posted by

Problem 225

*Let ρ_0 be a density such that, if the density of the universe were greater than ρ_0 , the gravitational forces would be strong enough to halt the expansion of the universe (i.e. the universe would be closed).Assuming that the universe is both isotropic and homogenous, prove that

$$\rho_0 \approx \frac{3H_0^2}{8\pi G}$$

, where H_0 is Hubble's constant and G is the universal gravitational constant.
Posted by Jkn

Problem 227 **/***

Solve $y'' + \alpha y = u(x)$, for *any* function u(x).
Posted by FireGarden

Problem 228**

Calculate

$$\sum_{n=4}^{\infty} \frac{1}{nH_nH_{n-1}}$$

, where

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

.Hint:

Spoiler:

Show

The answer's a rational number.

Posted by henpen

Problem 229 *

Find $\int_1^e \ln(1 + lnx)\ dx$ giving your answer as a summation.

Spoiler:

Show

Note I think that it can be solved with just A-Level knowledge, I can't check my solution

Posted by Pterodactyl

Problem 230

How many triangles are there?

I think more mathematicians should be interested in cognitive psychology.
Posted by jack.hadamard

Problem 231* (*** if you don't look at the spoiler)

Prove the annulus

$$A = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$$

is homeomorphic to the cylinder

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 \leq z \leq 1\}$$

Spoiler:

Show

A glimpse into the geometric side of topology. A homeomorphism between two topological spaces says they are topologically identical, and consists of a pair of *continuous* functions

$$f : A \mapsto C, g : C \mapsto A$$

, such that f and g are inverses of each other (more precisely, f(g(x)) = identity of C, g(f(x)) = identity of A). The question then is now simply to find such functions.

Posted by FireGarden

Had a nice afternoon avoiding STEP and indulging in some IMO problems! These questions are from a paper sat by Timonthy Gowers and Imre Leader who got Gold (42/42) and Silver (37/42) respectively (where each question is work 7). Also note that, whilst most IMO problems nowadays are restricted to those who are comfortable with the techniques taught, these particular problems do not require such knowledge and so should be just as accessible to someone who has been preparing for a year as they would to someone who hasn't (though there are many intern ate solutions to each). Enjoy!

Problem 232

*/**Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, \cdots, n\}$. Each of these subsets has a smallest member.
Let $F(n, r)$ denote the arithmetic mean of these smallest numbers.Prove that

$$F(n, r) = \frac{n+1}{r+1}$$

.

Problem 233

*/**Determine that maximum value of $m^3 + n^3$, where m and n are integers satisfying

$$m, n \in \{1, 2, \cdots, 2013\}$$

and $(n^2 - mn - m^2)^2 = 1$.

Edit: Another fun problem (not from IMO):Problem 234

*Find all possible n-tuples of reals $x_1, x_2, \cdots, x_n$ such that $\prod_{i=1}^{n} x_i = 1$ and

$$\prod_{i=1}^{k} x_i - \prod_{i=k+1}^{n} x_i = 1$$

for all $1 \leq k \leq n - 1$

Problem 235

*Prove that

$$1 + \frac{1}{1!\sqrt{2!}} + \frac{1}{2\sqrt{2!}\sqrt[3]{3!}} + \cdots + \frac{1}{(n-1)\sqrt[n-1]{(n-1)!}\sqrt[n]{n!}} > \frac{2(n^2+n-1)}{n(n+1)}$$

where n is a natural number greater than 1.
Posted by Jkn

Problem 236 *

Find

$$\int (sin(ln(x)) + cos(ln(x)))\, dx$$

Problem 237 */**

$$\int_0^\infty \frac{x^{29}}{(5x^2+49)^{17}}\ dx$$

For the last one, there is an easier way than partial fractions.
Posted by james22

problem 238 *

Using methods encountered at A-level, find the mean value of f(x) over an interval [0,xa].
Posted by hecandothatfromran

Problem 239

*Define the symbol π (which we can assume to be a new idea) to be equal to the circumference of a circle divided by its diameter. We can assume, without proof, that π is a constant.Prove that the area of a circle is equal to πr^2 , where r is the radius.Hence evaluate $\sin\pi$.**Note:** You'd think that this would go without saying but I will remind people that implicitly suggesting that you have proved something and not actually writing a proof does not constitute a mathematical proof. Note also that theorems cannot be applied unless you offer justification that the object that you are proving is not a necessary condition for that theorem to hold (as this would form a circular argument).
Posted by Jkn

Problem 240 * / **

Find all $n \in \mathbb{N}$ for which

$$\underbrace{2^{2^{2^{\cdot^{\cdot^{2}}}}}}_{n} - 3$$

is a perfect cube.
Posted by jack.hadamard

Problem 241 * / **

i) Find

$$\sin\left(\frac{\pi}{5}\right)$$

.ii) Prove that

$$\pi\phi > 5$$

, where ϕ is the golden ratio.

I made this problem up and it is not straightforward for a single star.
Posted by jack.hadamard

Problem 242 * and basic continuity knowhow

Let

$$f : (-\phi/2, \phi/2) \to R$$

be a continuous function satisfying

$$2xf(x) = f(2x^2 - 1)\forall x$$

in the domain. Determine all possible f.HINT:

Spoiler:

Show

The domain is deliberately misleading

Posted by TheMagicMan

Problem 243*

Find and prove formulae for the sum of the first n odd, and first n even numbers.
Posted by FireGarden

Problem 244**

Prove that for any integer $a \geq 4$ there exist infinitely many positive square-free integers n such that $a^n \equiv 1 \pmod{n}$.
Posted by Mladenov

Problem 247**

Evaluate

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{kn} \left(\prod_{i=kn+1}^{(k+1)n} i \right)^{\frac{1}{n}}$$

.
Posted by Mladenov

Problem 248**

Let a be an arbitrary integer, p - prime number, pick random divisor $n > 1$ of $p-1$, k - integer which is not divisible by n , and

$$U_{a,p} = \sum_{1 \leq x \leq p-1} e^{2k\pi i \frac{ind(x)}{n}} e^{2\pi i \frac{ax}{p}}$$

.If $\gcd(a,p) = 1$, show $U_{a,p} = \pm\sqrt{p}$.
Next, prove that

$$e^{-2k\pi i \frac{ind(a)}{n}} = \frac{U_{a,p}}{U_{1,p}}$$

.Now, let $p \equiv 1 \pmod{4}$ and

$$S = \sum_{1 \leq x \leq p-2} e^{2\pi i \frac{ind(x^2+x)}{4}}$$

. Show that $p = A^2 + B^2$, where A and B satisfy $S = A + iB$ .We next let $\gcd(a,p) = 1$ and define x_s to be an arbitrary reduced residue system modulo p such that

$$ind x_s \equiv s \pmod{n}$$

. Set

$$S_1 = \sum_{x_s} e^{2\pi i \frac{a x_s}{p}}$$

. Show that

$$|S_1 + \frac{1}{n}| < \left(1 - \frac{1}{n}\right)\sqrt{p}$$

.

Let n and m be integers, $n \geqslant 3$, $m \geqslant 2$, $\gcd(a, m) = 1$ (a is defined as above). Further, let R_m be an arbitrary complete residue system modulo m , and R'_m - reduced.Denote

$$S_{a,m} = \sum_{\eta} e^{2\pi i \frac{a\eta^n}{m}}$$

and

$$S'_{a,m} = \sum_{\xi} e^{2\pi i \frac{a\xi^n}{m}}$$

, where $\eta \in R_m$, $\xi \in R'_m$ .Let $\delta = \gcd(n, p-1)$ Show that

$$|S_{a,p}| \leqslant (\delta - 1)\sqrt{p}$$

.Let $\delta = 1$, and $s \in \mathbb{Z}$,

$$2 \leqslant s \leqslant n$$

. Prove that $S_{a,p^s} = p^{s-1}$, and that $S'_{a,p^s} = 0$ .In the case $s \geqslant n + 1$ show that

$$S_{a,p^s} = p^{n-1} S_{a,p^{s-n}}$$

and $S'_{a,p^s} = 0$ .Prove that

$$|S_{a,m}| < Cm^{1-\frac{1}{n}}$$

, where C is independent of m .

Problem 249**

Let p be a prime number. Show that for all $k \in \mathbb{Z}$, there exists integer n such that

$$\left(\frac{n}{p}\right) = \left(\frac{n+k}{p}\right)$$

.

Posted by Mladenov

Problem 250 *

Paint every point in the plane one of three colours.**i)** Are there two points of the same colour exactly 1 (cm) apart?

This time, paint every point one of six colours.**ii)** Prove that there exist two points of the same colour with one of three possibilities:**a)** they are 1 (cm) apart; **b)** $\sqrt{3}$ (cm) apart; **c)** 2 cm apart;

Posted by jack.hadamard

Problem 251**

Let A and B be two disjoint finite non-empty sets in the plane such that every segment joining two points in the same set contains a point from the other set. Show that all the points of the set $A \cup B$ lie on a single line.

Posted by Mladenov

Problem 252***

Let G be a group and a smooth manifold. Suppose that the map $(a, b) \mapsto ab$ is smooth. Show that $a \mapsto a^{-1}$ is smooth and, hence, that G is a Lie group. Is the corresponding statement about topological groups true?

Posted by Mladenov

Problem 254**

Let n be a fixed positive integer. Suppose also that

$$f : \mathbb{C} \to \mathbb{R}$$

is a function such that, for any points P_i , $i \in \{1, 2, \cdots, n\}$ which are vertices of a regular n -gon, we have

$$\sum_{1 \le i \le n} f(P_i) = 0$$

. Show that $f \equiv 0$ over $\mathbb{C}$.
Posted by Mladenov

Problem 255

what are the largest basins of attraction for the roots of $y = x^3 - 2x$?(this might be a simple one for most, though)
Posted by Hasufel

Problem 256***

Let E be a σ -locally compact space, R - equivalence relation on E such that the set C determined by R in $E \times E$ is closed. Show that E/R is separable. Does the claim hold true if E is not σ -compact?
Posted by Mladenov

Problem 257 */**

How many subsets of

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

are there such that the sum of the smallest and largest element is 11 ?
Posted by jack.hadamard

Problem 258*

Prove that the segments are perpendicular, no matter what the central quadrilateral is.

Problem 259**

Prove that the set of points at which the smaller circles touch all lie on a cirle (dashed).

Problem 260***

$$\int_{-\infty}^{\infty} x^{-1} \tan x \cos \tan x \, dx$$

Posted by Lord of the Flies

Problem 261 ***

The Brocard-Ramanujan diophantine equation is $n! + 1 = m^2$ where $n, m \in \mathbb{N}$.**i)** Find three pairs (n, m) which are solutions to the equation.**ii)** Prove that if $p \equiv \pm 3 \ (8)$ is a prime, then no solutions of the forms $(p-2, m)$ and $(p-3, m)$ exist.
Posted by jack.hadamard

Problem 262*/**/***

(can be done with only basic knowledge, but is far harder)You have a fair coin and start with a score of 0. Whenever you throw a heads you +1 to your score, whenever you get a tails you -1.i) What is the average number of times you will hit 0 if you throw the coin 10 times? What about 100?
ii) What is the average number of flips before you reach 0 again?
Posted by james22

Problem 263

(Haven't actually done this but looks like fun -sorry)Let a, b, c, d be integers with $a > b > c > d > 0$ .Suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c)$$

.Prove that $ab + cd$ is not prime.
Posted by Jkn

Problem 264**

Find all $x : \sin(x) = 13$
Posted by FireGarden

Problem 265**

Is the following matrix singular?

$$\begin{pmatrix} 54401 & 57668 & 15982 & 103790 \\ 33223 & 26563 & 23165 & 71489 \\ 36799 & 37189 & 16596 & 46152 \\ 21689 & 55538 & 79922 & 51277 \end{pmatrix}$$

Posted by FireGarden

Problem 266***

Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan(x)^{\sqrt{2}}}\ dx$$

Posted by FireGarden

Problem 267**

Evaluate

$$\int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}\ dx$$

Posted by FireGarden

Problem 268

***Evaluate

$$\int_0^\infty \frac{\sin(2\tan^{-1}(t))}{(1+t^2)(e^{\pi t}+1)}\ dt$$

Edit: Btw, had another look at the dreaded

$$\frac{1}{2}(tanxcos(tanx))$$

integral and have made a few breakthroughs. I feel as though I might be nearing a solution but I'm not quite there yet! Has anyone else done anything substantial?
Posted by Jkn

Problem 269

***Using the previous problem as a hint,Express ζ in the form

$$\zeta(s) = f(s) + \int_0^\infty g(s,t)\ dt$$

where f and g are combinations of 'elementary' functions and $s \in \mathbb{C} \setminus \{1\}$.
Posted by Jkn

Problem 270

what is:

$$\int \frac{1}{x^4(1+x^2)^{1/2}} dx$$

?
Posted by Hasufel

Problem 271

***Prove that

$$\int_0^\infty e^{-x}\log(x)\ dx = \Gamma'(1) = \psi(1) = -\gamma$$

.Note that I require you to prove any non-* theorems/results that you use in your proof.

Problem 272

*Prove that

$$\gamma = -4\int_0^\infty e^{-x^2} x \log(x)\ dx = -\int_0^1 \log\log\left(\frac{1}{x}\right)\ dx = \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{xe^x}\right)\ dx$$

$$= \int_0^1 \left(\frac{1}{\log(x)} + \frac{1}{1-x}\right)\ dx = \int_0^\infty \frac{1}{x}\left(\frac{1}{1+x^k} - e^{-x}\right)\ dx$$

Problem 273

***Evaluate

$$\int_0^1 \int_0^1 \frac{x-1}{(1-xy)\log(xy)}\ dx$$

Prove all non-* theorems, as before.

Problem 274

***Evaluate

$$\int_0^\infty e^{-x^2} \log(x)\ dx$$

Problem 275

***Evaluate

$$\int_0^\infty \frac{1}{x^2}\left(\frac{\log(1+x)}{(\log(x))^2 + \pi^2}\right)\ dx$$

Posted by Jkn

Problem 276

(simple one, really)if $u+v=50$ which choice of both u and v make $u \times v$ as large as posilble?
Posted by Hasufel

PROBLEM 277

Derive (pardon the pun) a series representation, excluding the constant of integration, for:

$$\int x^n (In(x))^m dx$$

(Hint: one way is to incorporate derivatives into your answer)
Posted by Hasufel

Problem 278

***Evaluate

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}}$$

for $x \in \mathbb{R}$ .Comment also on what happens when $x \in \mathbb{C}$ and $|x| \to \infty$.

Problem 279

*/**Evalute

$$\lim_{x\to 0^+} (\ln x)^3 \left(\arctan(\ln(x+x^2)) + \frac{\pi}{2}\right) + (\ln x)^2$$

Problem 280

*Prove that

$$\sinh^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1}$$

for $|x| \leq 1$.

Courtesy of “peter12345”:

Problem 281

*Find an infinite series representation for

$$\cos \frac{\pi}{x}$$

and

$$\sin \frac{\pi}{x}$$

involving non-trancendental numbers (ignoring, of course, the possible transcendence of x). A transcendental number is a number that cannot be expressed as the root of a polynomial equation with integer coefficients.

Problem 282

**Prove or disprove the statement that

$$\int_0^1 \frac{\sqrt{1+x^2} - \frac{1}{2}x^2 - 1 - 3x^4}{(\sin x - x)^2}\, dx$$

is convergent. If the integral is convergent, what value does it converge to?

Problem 283

***Prove that

$$\sum_{n\geq 1}^{\Re} \frac{1}{n} = \gamma$$

, where $\Re$ denotes a 'Ramanujan Summation' (a technique used by Srinivasa Ramanujan to assign meaningful values to divergent series).Hence, or otherwise, comment humorously upon the reaction of English professors that he sent letters to that included such series (without specification that they were 'Ramanujan Summations', as they are now known).

Problem 284

*/**/***Evaluate

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{(1+q^{n+1})(1+q^{n+2})}$$

Problem 285

/*Find, with proof,

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} x^n$$

, where we define the half derivative to be the unique operator H such that

$$H^2 f(x) = Df(x)$$

where D denotes the differential operator.Verify also, by applying this operation twice, that your relation is consistent with

$$\frac{d}{dx} x^n$$

.

Problem 286

***Evaluate

$$\prod_{n=1}^{9} \Gamma\left(\frac{n}{10}\right)$$

Can the result be easily generalised in any interesting way? If so, provide proof of your assertions.
Posted by Jkn

Problem 287

*/***Prove that

$$2(ab+bc+ca)^4 = a^4(b-c)^4 + b^4(c-a)^4 + c^4(a-b)^4$$

when $a+b+c=0$ .Another problem that came from an interesting place and has a rather insane generalisation. As usual, the tall tale will follow a correct proof!

Sounds a bit brutal, I may take a peak later in the summer.Oh right, yeah I've used similar things before. They tended to help with boring slogs (spanish vocal, stats definitions, physics definitions, etc..), I used several iPhone apps.Hmm, I doubt that! I still have {groups}, matrices, a lot of vector stuff, applications of s. relativity to solving problems, all of probability, etc.. from 1A that I haven't really scratched the surface on!

Nice stuff man.Ah, that's what I got!

Looks similar to the series I derived using IBP => recurrence relation (functional equation) => solve. Probably the same but my notebooks/ridiculous piles of paper are not beside me atm!
Posted by Jkn

Would anyone benefit from small hints to some of the problems I've set recently? A few of them follow trivially from theorems that can be quoted (once found). Also, on the problems where I have said that you must prove all theorems used, do feel that you can 'look them up' so long as they seem like the kind of thing that you would be expected to memorize in a course (the so-called 'book work').Yet another that requires a simple yet rather advanced 'trick':

Problem 288

***Evaluate

$$\int_0^\infty \frac{\gamma x + \log\Gamma(1+x)}{x^{5/2}}\ dx$$

Problem 289

*/**Let x, y and z denote positive integers (which can be extended without any loss of generality from the case whereby they denote non-zero integers).Find the general solution to the equation

$$x^2 + y^2 = z^2$$

(the 'Pythagorean Triples').Hence show that

$$x^4 + y^4 \neq z^4$$

('Fermat's Last Theorem' in the case n=4).Can you generalize the result to other classes of exponents in the set of natural numbers? How many other ways can you find to prove the above result? (bonus).
Posted by Jkn

Problem 290*/**

Suppose a, b are fixed points of $f(x)$ (that is, f(a)=a, f(b)=b)). Prove

$$\int_a^b f(x) + f^{-1}(x)\ dx = b^2 - a^2$$

Posted by FireGarden

Problem 291*

Prove the above statement that sines and cosines are orthogonal under the inner product given. This is to show the integral of $\sin(mx)\cos(nx)$, $\cos(mx)\cos(nx)$, or $\sin(mx)\sin(nx)$, are zero if and only if $m \neq n$
Posted by FireGarden

Problem 292***

Group summary:

Spoiler:

Show

I define a “group” to be a set X, and an operation +, such that the following four axioms are true:
1) a+b is in X for every a,b in X
2) there is an element, e (the identity), such that a+e = a = e+a for all a in X
3) every element a in X has a corresponding element z in X with a+z = z+a = e
4) (a+b)+c = a+(b+c) for all a,b,c in X.So the set “the integers, with +” is a group:
1) a+b is an integer for all integers a,b
2) such an element is 0
3) this is true: set z=-a
4) this is blindingly obviously true.The set “{1,2,3,4} with the operation *” is a group, where in the following table, the entry in the mth row and the nth column gives m*n:
1234
2143
3412
4321
(it’s clear from the table that the first axiom holds; 1 is our e; each element a has a*a=e; and it can be tediously checked that the fourth axiom holds.)Groups do not have to be “commutative” - that is, a+b isn’t necessarily b+a. However, the first example is of size 6, and it’s a pain (and not particularly helpful) to write out the table.A subgroup H of G is a subset of G such that H is a group. Hence, of course, it must contain the same identity.

Prove Lagrange’s Theorem: if H is a subgroup of G, then |H| divides |G|.
Hint:

Spoiler:

Show

Consider the cosets of H in G - that is, for a given $a \in G$, consider the set $\{ah, h \in H\}$. How many of these sets are there as we vary a? What size are they?

Hence prove that if g is in G, then $o(g)$ the order of g (that is, the minimum n such that g^n is the identity) divides |G|.
Corollary: Prove Fermat’s Little Theorem: that if $(a, p) = 1$ then

$$a^{p-1} \equiv 1 \pmod{p}$$

.
Posted by Smaug123

Problem 293

Prove that

$$\int_1^\infty \ln\left(\frac{\ln(x)}{x}\right)\left(\frac{\ln(x)}{x}\right)^n dx = \frac{\Gamma(n+1)}{(n-1)^{n+1}}\left(\psi(n+1) - \frac{n+1}{n-1} - \ln(n-1)\right)$$

Sorry if it's a bit mundane.
Posted by henpen

Problem 294**/***

Prove

$$\frac{\zeta^2(2)}{2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^4 + m^2 n^2}$$

I still can't really do any of Jkn's integrals, but I'll keep trying.

Spoiler:

Show

Not necessarily a hint, but

Problemette 294:

Prove

$$\frac{\pi}{m}\frac{1}{e^{2\pi m} - 1} + \frac{\pi}{2m} - \frac{1}{2m^2} = \sum_{n=1}^{\infty}\frac{1}{m^2 + n^2}$$

Posted by henpen

Problem 295***

Suppose we have a closed contour γ in the complex plane. Then it defines an enclosed area. We shall define

$$S = \frac{1}{2i} \int_\gamma \bar{z} \, dz$$

To be the *signed* area enclosed by γ Show that S is real, and that it recovers πr^2 for a circle, show both signs can occur, and explain the significance of the sign.
Posted by FireGarden

Problem 296***

Consider a smooth manifold X with an open cover $n < \infty$ of sets $\{X_k\}_1^n$, which are contractible. Assume also that

$$\pi_0(X_k \cap X_l) \leq m$$

, for all k, l . Find an upper bound to the first Betti number of X .

At least, I have picked a nice substitute.
Posted by Mladenov

Problem 297***

Calculate the volume of the ellipsoid defined by

$$V : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

Where a, b, c are positive constants.
Posted by FireGarden

Problem 298

*Evaluate

$$\Re \int_0^\pi \underbrace{e^{e^{\cdots ix}}}_{n} \, dx$$

for $n \geq 2$ where the underbrace represents the number of times e appears in the exponentiation.
Posted by Jkn

Problem 299*

: prove this), so the volume of the stretched hypersphere is

$$\frac{\pi^{n/2}}{\Gamma\left(1+\frac{n}{2}\right)}\prod_{k=1}^{n} r_k$$

This is probably what you meant or had in mind, but I wrote it here for explicitness.
Posted by henpen

OMG! This follows as a corollary from what I just typed up!So satisfying to rediscover something! Do you know when my result was established and by who?**Edit:** How did you get the result for a the generalized ellipse btw? I can't find stuff about it anywhere online (did you read it somewhere and, if so, is the original derivation the same as mine?)

Problem 300

*/***Working within the base-10 number system, find all 9-digit numbers containing all digits except zero (a zeroless Pandigital number) such that the first n digits form a number that is divisible by n for all n.*[For example, 123456789 is of the correct form. We also note that 1 is divisible by 1, 12 is divisible by 2, 123 is divisible by 3 though, as 1234 is not divisible by 4, this number fails to meet the requirements.]*Can you generalise interestingly to number systems with other bases and/or in any other way? (bonus-only)
Posted by Jkn

Problem 301**

Prove, given

$$a, c > 0; \quad ac - bd \neq 0$$

$$\arctan\left(\frac{ad+bc}{ac-bd}\right) = \arctan\left(\frac{b}{a}\right) + \arctan\left(\frac{d}{c}\right)$$

Posted by FireGarden

Problem 302**

Let $F_0(x) = lnx$. For $n \geq 0$ and $x > 0$, let

$$F_{n+1}(x) = \int_0^x F_n(t)\ dt$$

.Evaluate:

$$\lim_{n\to\infty} \frac{n! F_n(1)}{\ln n}$$

.

.

Posted by MW24595

Problem 303**

Give a justification for the claim $1^\infty > 2$

Posted by FireGarden

Problem 304*

Does there exist an integer, k, such that using only at most k coin flips, a number in the set {1,2,3} can be randomly picked with each being equally likely?

*Posted by james22*problem 1').

Problem 305

*Prove that the sequence defined by $y_0 = 1$,

$$y_{n+1} = \frac{1}{2}\left(3y_n + \sqrt{5y_n^2 - 4}\right)$$

for $n \geq 0$ consists only of integers.

[As a bonus, can you find any 'similar' sequences for which this is true?]

Problem 306

*Find all solutions in non-negative integers a,b to

$$\sqrt{a} + \sqrt{b} = \sqrt{2009}$$

[As a bonus, why not generalise this in the obvious way? Giving all solutions a,b,c dependent on either c having a certain property or c simply being any integer]

Problem 307

*Find the minimum possible value of $x^2 + y^2$ where x,y are real numbers such that

$$xy(x^2 - y^2) = x^2 + y^2$$

and $x \neq 0$.

Problem 308

*Find all pairs of integers (x,y) such that

$$1 + x^2y = x^2 + 2xy + 2x + y$$

Problem 309

*/**/***Find the volume of the region of points (x,y,z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2)$$

Time to see who has been paying attention to this thread (if you have been you will do this in minutes)! This one is a 'problem 4' from Putnam:

Problem 310

**Evaluate

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n}{3^m(n3^m+m3^n)}$$

Some easier ones. Please do not post a solution if they do not interest/challenge you - though whilst they are mostly easier than those above, they are still by no means 'trivial' or 'easy' and would still be considered difficult by the standards of numerous institutions/exams. Also, do not be discouraged if you cannot do them as problem-solving takes practice and is certainly something that has to be learnt in almost all cases:

Problem 311

*/**/*** (the one-* solution does exist and is not hard to find - please don't moan on about it!)Let $x_0 = 0$, $x_1 = 1$ and

$$x_{n+1} = \frac{1}{n+1}x_n + \left(1 - \frac{1}{n+1}\right)x_{n-1}$$

for $n \geq 1$ .Show that the sequence $\{x_n\}$ convergences as $n \to \infty$ and determine its limit.

Problem 312

*Evaluate

$$\frac{4}{4^2-1} + \frac{4^2}{4^4-1} + \frac{4^4}{4^8-1} + \frac{4^8}{4^{16}-1} + \cdots$$

Problem 313

*Find all real values of x,y and z such that

$$(x+1)yz = 12,\ (y+1)zx = 4,\ (z+1)xy = 4$$

[As a bonus, can you solve the problem with a more general set of constraints?]

Problem 314

*Find all positive integers n such that $n+2012$ divides n^2+2012 and $n+2013$ divides n^2+2013 .This is the kind of problem you could find really easy or rather difficult:

Problem 315

/*Determine, with proof, whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{2+\cos(2\pi \ln n)}}$$

converges.
Posted by Jkn

Problem 316*

Prove that

$$\sum_{j<k,j,k\in\mathbb{N}} \frac{1}{j^2k^2} = \frac{\pi^4}{120}$$

and find a general form for

$$\sum_{x_1<x_2<\ldots<x_n,x_1,x_2,\ldots,x_n\in\mathbb{N}} \frac{1}{\prod_{k=1}^{n} x_k^2}$$

.Note that I'm taking the set of natural numbers to not include 0.

Spoiler:

Show

Consider the roots of the Taylor series for

$$\frac{\sin(\sqrt{y})}{\sqrt{y}}$$

.

Posted by henpen

Problem 317**

Find

$$\sum_{k=-\infty}^{\infty} \frac{1}{2k+1} - \frac{2}{\pi(2k+1)^2}$$

Posted by henpen

Problem 318*

Find all 3-digit numbers such that, if you divide them by 11, you get the sum of the squares of the original numbers' digits.
Posted by MW24595

Problem 319**

$$\iint_R \frac{x-3y}{2x+y}\ dx\ dy$$

where R is the parallelogram defined by the lines:

$$y = -2x+1,\ y = -2x+4,\ y = \frac{x}{3},\ y = \frac{x-7}{3}$$

Problem 320**

Find the volume underneath the surface bounded by $z = 36 - 9x^2 - 4y^2$ and over the xy-plane
Posted by Pterodactyl

Problem 322**/***

Given

$$y(x) = \frac{\pi}{2} + \int_0^x 3t^2 \sec(y(t))\ dt$$

Find a closed form for $y(x)$
*Posted by FireGarden*Problem 326 looks somewhat doable, though I don't think I'd be quick enough to solve and post it. (Edit: Scratch that, it wasn't at all)

*Posted by ctrls*Problem 328**Using the algebra of propositions evaluate {(~p V ~q V r) ∧ (p ∧ r)} V {p ∧ (~q V r)}
Posted by Arieisit

Problem 329*

a,b,c,d are positive real numbers.5 of the set ab,ac,ad,bc,bd,cd are the numbers 2,3,4,5,6 in no particular order. Can you find what the other is?
*Posted by james22*Problem 330* (this could be done with gcse knowledge in fact)Consider the sentence:THIS IS ONE GREAT CHALLENGE IN MATHEMATICSEvery minute, the first letter of each word is moved to the other end of the word. After how many minutes will the original sentence first reappear?
Posted by Arieisit

Problem 331*

Show that if $\frac{1}{x}f(x)$ tends to a non-zero limit as $x \to 0$ then

$$\sum_{k=1}^{\infty} f\left(\frac{1}{k}\right)$$

cannot converge.(that one is roughly the same level as the most recent questions on this thread)Here is a nice one:

Problem 332*

$x, y, z > 0$ are positive integers satisfying $x^2 + y^2 + 1 = xyz$. Show that $z = 3$
Posted by Lord of the Flies

Problem 333**

Let ABCD be a fixed convex quadrilateral with BC = DA and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy BE = DF. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Show that the circumcircles of the triangles PQR , as E and F vary, have a common point other than P.
Posted from TSR Mobile
Posted by Arieisit

Problem 334**

Find all pairs (x, y) of positive integers satisfying the equation $x^2 + xy - y^2 = 1$.
Posted by Mladenov

Problem 335**/***

Let $x\sin(2x)$ be a solution of a homogeneous fourth order ODE with real constant coefficients. Determine the general solution and the ODE it solves.
Posted by FireGarden

Problem 336 (**)

Prove that for all positive a,b,c:

$$\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} \geq ab + bc + ac$$

Posted by metaltron

Problem 337**

Let C_1 and C_2 be externally tangent circles at M and the radius of C_2 is greater than the radius of C_1 . Let A be a point on C_2 which does not lie on the line joining the centres of C_1 and C_2 . Let B and C be points on C_1 such that AB and AC are tangent to C_1 . The lines BM and CM intersect C_2 again at E and F , respectively. Let D be the point of intersection of the tangent to C_2 at A and the line EF . Prove that as A varies, the locus of D is a line.
Posted by Mladenov

Problem 338*

Consider the sequence for

$$a_n = n! + 100 \ for \ n \geq 1$$

, what is the maximum value of $gcd(a_n, a_{n+1})$?
Posted by Pterodactyl

Problem 339**

Let $(a_i)_{1\le i\le n}$ be a sequence of non-negative integers. Let

$$m_k = \max_{1\le i\le k} \frac{a_{k-i+1} + \cdots + a_k}{i}$$

where $k \in \{1, 2, \cdots, n\}$. Prove that for any $m > 0$ the number of integers k such that $m_k \ge m$ is

$$\le \frac{a_1 + \cdots + a_n}{m}$$

.

Problem 340**

Find all maps

$$f : \mathbb{N} \to \mathbb{N}$$

such that for all m, n the relation $f(m) + f(n) | m + n$ holds. Now let k be an arbitrary positive integer. Find all maps

$$f : \mathbb{N} \to \mathbb{N}$$

such that $f(m) + f(n) | (m + n)^k$ for all choices of positive integers m, n .
Posted by Mladenov

Problem 341**/***

Find

$$\int_{-\infty}^{\infty} \frac{log(x^2 + 1)}{x^2 + 1}\ dx$$

I don't actually know how to solve this one, but I do have the answer.
Posted by james22

Problem 342*

Evaluate

$$\sum_{k=1}^{n} \binom{2n-k-1}{n-1}$$

Posted by Ateo

Problem 343 */**

Prove that:

$$\sum_{a_n=0}^{r} \sum_{a_{n-1}=0}^{a_n} \dots \sum_{a_1=0}^{a_2} \sum_{a_0=0}^{a_1} 1 = \binom{n+1+r}{r}$$

Posted by metaltron

Problem 344*/***

Evaluate

$$\sum_{r=0}^{\infty} \frac{1}{4r+1} - \frac{1}{4r+2}$$

.
Posted by Felix Felicis

Problem 345 **/***

Evaluate

$$\lim_{x \to 0} \frac{\displaystyle\int_0^{\tan x} \tan \sin t \; dt}{\displaystyle\int_0^{\sin x} \ln(1+\sin t) \; dt}$$

.
Posted by jack.hadamard

Problem 346 *

$$\int_a^b \arccos\left(\frac{x}{\sqrt{(a+b)\,x-ab}}\right)\,dx$$

Posted by Pterodactyl

Problem 347*

Prove that

$$gcd(2^a-1,2^b-1)=2^{gcd(a,b)}-1$$

.

Problem 348**

Prove that

$$\frac{x^2+1}{y^2-5}$$

is never an integer for $x,y\in\mathbb{Z}$.
Posted by MW24595

Problem 349 / **

Let

$$h:\mathbb{R}^{\geq 0}\to\mathbb{R}^+$$

be such that $h'(x)>2x$ for all x . Prove that

$$\sum_{k=1}^{n} h(1+F_k^2)>4F_nF_{n+1}$$

where F_n is the n -th Fibonacci.

Problem 350 / **

The Tribonacci numbers, T_n , are defined as $T_0 = 0, T_1 = T_2 = 1$ and

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n$$

for $n \geq 0$ .Find

$$\sum_{k=1}^{n} T_k^2$$

.
Posted by jack.hadamard

Problem 351 **

$$\int_0^{\infty} \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + ... \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + ... \right) \, dx$$

Problem 352 */**

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1}$$

Posted by Pterodactyl

Problem 353*/**

For a positive integer n , let S_n be the total sum of the intervals of x such that $\sin 4nx \geq \sin x$ in

$$0 \leq x \leq \frac{\pi}{2}$$

.Find

$$\lim_{n \to \infty} S_n$$

.

Problem 354*/**

Evaluate

$$\lim_{n\to\infty}\left(\int_0^{\pi}\frac{\sin^2 nx}{\sin x}\ dx-\sum_{k=1}^{n}\frac{1}{k}\right)$$

.

*Posted by Felix Felicis*problem 350.**Solution 350**I shall use the following identities which are easily proved by induction (if time permits, I can add their proofs).We have

$$\sum_0^n T_i=\frac{1}{2}(T_{n+2}+T_n-1)$$

;

$$T_{2n}=T_{n-1}^2+T_n(T_{n+1}+T_{n-1}+T_{n-2})$$

;

$$T_{2n-1}=T_n^2+T_{n-1}^2+2T_{n-1}T_{n-2}$$

;

$$T_{2n}-T_{2n-1}=T_{2n-2}+T_{2n-3}$$

.Now it follows that

$$\frac{1}{2}(T_{2n}+T_{2n-2}-1)=-2S+2T_nT_{n+1}+2T_nT_{n-1}$$

which is

$$4S=4T_nT_{n+1}+4T_nT_{n-1}+1-T_{2n}-T_{2n-2}=4T_nT_{n+1}-(T_{n+1}-T_{n-1})^2+1$$

, where S is the sum in question.

Spoiler:

Show

This solution is motivated by the Fibonacci case.
We can generalize by replacing the square with an arbitrary positive integer; however, it would not be as easy.

Problem 355*

Let S be a set of positive integers. We say that S has the property P if no element of S is a multiple of another. Find the largest subset S of $\{1, \cdots, 2n-1\}$ having the property P .

Problem 356*

In a town there are $2n$ people and m clubs - $A_1, \cdots, A_m$. If $|A_i|$ and $|A_i \cap A_j|$ are even for all $i, j \in \{1, \cdots, m\}$, then $m \leq 2^n$.
Posted by Mladenov

Problem 357:**

Evaluate

$$\int_0^1 \frac{xlnx}{1+x^2}\, dx$$

Posted by DJMayes

Problem 358*

Find all positive integers n such that the product of all positive divisors of n is 24^{240} .
Posted by Felix Felicis

Problem 359**

In a town there are n people and m clubs - $A_1, \cdots, A_m$. If $|A_i|$ is odd for all i and $|A_i \cap A_j|$ is even for all $i, j \in \{1, \cdots, m\}$, then $m \leq n$. Is the bound attainable?

Problem 360**

Let

$$A_i \subseteq \{1, \cdots, n\}$$

, $i \in \{1, \cdots, m\}$ be such that for $i \neq j$ the inequality

$$|A_i \cap A_j| \leq 1$$

holds. Show that

$$m \leq n + 1 + \binom{n}{2}$$

.

Problem 361*

*Let a, A, B be integers, M - a positive integer. Prove that the numbers $Aa^m + Bm \pmod{M}$ form a complete residue system $\pmod{M}$ if and only if $\gcd(B, M) = 1$.
Posted by Mladenov

Problem 362*

This is actually very similar to an A level question but without the added guidance that they usually provide.A sports association is planning to construct a running track in the shape of a rectangle surmounted by a semicircle. Let the letter x represents the length of the rectangular section and r represents the radius of the semicircle.
Determine the length, x , that **maximises** the area enclosed by the track.**EDITThe perimeter of the track must be 600 metres.**I completely forgot to put that in. Sorry

Problem 363*

Given that the sum of the angles A , B and C of a triangle is π radians, show that $\sin(A) = \sin(B + C)$

Problem 364***

Determine the extreme values of $f(x,y,z,u,v,w) = \frac{1}{1+x+u} + \frac{1}{1+y+v} + \frac{1}{1+z+w}$ where $xyz = a^3$, $uvw = b^3$, and $x,y,z,u,v,w > 0$.
Posted by Arieisit

Problem 365

/*Find a closed form expression for

$$\int_0^\infty \frac{x^a}{(b+x^c)^d}\ dx$$

in terms of elementary functions where $d \in \mathbb{Z}$ .Be sure to consider the different cases that arise according to the values of a, b and c.
Posted by Jkn

Problem 366

*Find the average of the numbers

$$n \sin n^\circ$$

($n = 2, 4, 6, \cdots, 180$)
*Posted by Jkn*Problem 364 I'll provide a hint or two.

Problem 364***

Determine the extreme values of $f(x,y,z,u,v,w) = \frac{1}{1+x+u} + \frac{1}{1+y+v} + \frac{1}{1+z+w}$ where $xyz = a^3$, $uvw = b^3$, and $x,y,z,u,v,w > 0$.

Spoiler:

Show

Find all the intrinsic critical points of f on M
Calculate

$$\begin{matrix} sup \\ M \end{matrix}$$

f .
Investigate whether or not f attains the value

$$\begin{matrix} sup \\ M \end{matrix}$$

f at some point of M

Problem 367**

If $I_n =$

$$\int_0^{\frac{\pi}{4}} tan^n x \ dx, n \geq 2$$

find I_4

Problem 368**

Given that $I_m =$

$$\int (cos^m x)(sin3x)dx$$

and $J_m =$

$$\int (cos^m x)(sin2x)dx,$$

prove that

$$(m+3)I_m = mJ_{m-1} - (cos^m x)(cos3x).$$

Posted by Arieisit

Problem 369***

*(easy)*Prove that any diagonalisable matrix satisfies it's own characteristic polynomial

Problem 369***

*(hard)*This time, without the assumption of diagonalisability.The hard version is the Caley-Hamilton theorem; the easy one just a special case which is much more straightforward. Getting boring just seeing number theory and integrals.
Posted by FireGarden

Problem 370**/***

Prove that in the real number system that the **inverse of addition** is represented by $x + (-x) = 0$
Posted from TSR Mobile
Posted by Arieisit

Problem 371 ***

Prove that for any integer $M \geq 1$

$$\sum_{m=1}^{M} \frac{1}{m} \leq \prod_{p \leq M} \left(1 - \frac{1}{p}\right)^{-1}$$

where p is a prime number
Posted by Indeterminate

Problem 372***

Let A and B be subsets of the topological space X so that

$$X = A^\circ \cup B^\circ$$

, where A° is the interior of A . Show that there exist natural homomorphisms

$$\Delta : H_n(X) \to H_{n-1}(A \cap B)$$

such that

$$\cdots \xrightarrow{\Delta} H_n(A \cap B) \xrightarrow{\varphi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\Delta} H_{n-1}(A \cap B) \xrightarrow{\varphi} \cdots$$

is an exact sequence, where φ and ψ are defined by the induced homomorphisms, namely

$$\varphi = (i_*, j_*)$$

and

$$\psi = k_* - l_*$$

,

$$i_* : H_n(A \cap B) \to H_n(A)$$

,

$$j_* : H_n(A \cap B) \to H_n(B)$$

,

$$k_* : H_n(A) \to H_n(X)$$

and

$$l_* : H_n(B) \to H_n(X)$$

.
This is a well-known sequence in algebraic topology which plays crucial role in homology theory. Using the same assumptions, show that the homomorphism

$$\Delta : H_n(X) \to H_{n-1}(A \cap B)$$

is a composition of the following morphisms

$$H_n(X) \xrightarrow{j_*} H_n(X, B) \approx H_n(A, A \cap B) \xrightarrow{\delta_*} H_{n-1}(A \cap B)$$

(now, you might come up with a second proof of the result).

Problem 373***

Let X be a subset of S^n , homeomorphic to S^k ,

$$k \in \{0, \cdots, n-1\}$$

. Prove that

$$\tilde{H}_{n-k-1}(S^n - X) = \mathbb{Z}$$

and

$$\tilde{H}_i(S^n - X) = 0$$

, $i \neq n-k-1$.

Spoiler:

Show

The order of the above problems is intentional.

Posted by Mladenov

Problem 373***

Let G be a group, which acts on the set X .Suppose $x, y \in X$ have the same orbit. Prove that their stabilisers are conjugates (i.e.

$$\exists g \in G : G_y = gG_xg^{-1})$$

Problem 374***

How many ways can you colour the vertices of a square using at most three colours, which are distinct under the action of D_8 ? (i.e. colourings which cannot be obtained from another by rotations or reflections)
*Posted by FireGarden*problem 372.Basic commutative algebra.

Problem 377***

Prove that if M is an $n \times n$ matrix over a local ring, with coefficients in the maximal ideal J , then $J + M$ is invertible.

Problem 378***

If $f : A \to A$ is a surjective endomorphism of a Noetherian ring A , then f is an isomorphism.
Posted by Mladenov

Problem 379 **

How many subsets of $\{1, 2, ..., 50\}$ are there such that the sum of the elements exceeds 637 ?

Is this possible to generalise?
Posted by henpen

Problem 380*

$$\int sinxcos2x$$

dx Would this be considered as trivial?
Posted by Arieisit

Problem 381***

Let A be an nxn real matrix with the modulus of each entry strictly less than 1/n. Is $I - A$ invertable in all cases? If not, when is it invertable.
Posted by james22

Problem 382**

Let $a_1 \in (1, 2)$ and

$$a_{k+1} = a_k + \frac{k}{a_k}$$

. Prove that there are at most two terms in the sequence that add to make an integer.
Posted by henpen

Problem 383**

Ms. Janis Smith takes out an endowment policy with an insurance company which involves making a fixed payment of $\$P$ each year. At the end of n years, Janis expects to receive a payout of a sum of money which is equal to her total payments together with interest added at the rate of $\alpha\%$ per annum of the total sum in the fund.Show, by **mathematical induction** or otherwise, that the total sum in the fund at the end of the n^{th} year is

$$\$\frac{PR(R^n-1)}{R-1}$$

where

$$R=(1+\frac{\alpha}{100})$$

Posted by Arieisit

Problem 384***

It seems I incorrectly starred this one. Here's a better rewording:Prove that if

$$a,b,\frac{a^2+b^2}{1+ab}\in\mathbb{Z}$$

, then $\frac{a^2+b^2}{1+ab}$ is a perfect square.
Posted by henpen

Problem 385***

â€‹

$$0<k<1$$

Find in terms of k $\displaystyle\int_0^{2\pi}\frac{1}{1-2k\cos\theta+k^2}\,d\theta$

Posted by Flauta

Problem 386**/***

Find $\int_0^{2\pi} \frac{1}{5-3\sin x}dx$
Hints

Spoiler:

Show

Use substitution $t = \tan\frac{x}{2}$ OR
do a bit of complex analysis

Posted by Flauta

Problem 387**

Find (as a power series) the function that satisfies $f''(x) + f'(x) = f(x) + 1$.
Spoiler:

Show

http://math.uchicago.edu/~chonoles/e…tricseries.pdf, also use the Fibonacci numbers' generating function.

Posted by henpen

Problem 388***Here's another oneEvaluate

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{r^2}$$

Posted by Indeterminate

Actually I had thought up the problem with the following solution in mind.Alternative solution II to problem 389:Consider the Fourier series

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n\geq 1} \frac{(-1)^n \cos(nx)}{n^2}$$

Where setting $x = 0$ leads to the required solution.
Posted by Indeterminate

Problem 395**

If

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx.$$

Show that $I = \sqrt{\pi}$
Posted by Arieisit

Problem 396**

Hopefully I'll solve Mladenov's problems at some point, but in the hiatus here's another integral:

$$\int_0^{\frac{\pi}{2}} \frac{\log(\sec(x))}{\tan(x)} dx.$$

Problem 397**

$$\int_0^{\frac{\pi}{2}} \frac{\log(\sec(x))}{\tan^2(x)} dx.$$

Posted by henpen

Problem 398***

Find the general solution to the system of differential equations.
$y_1' = 4y_1 - y_3$

$$y_2' = 2y_1 + 2y_2 - y_3$$

$y_3' = 3y_1 + y_2$
Posted by Arieisit

Problem 399***

Let G be a group. Show that there exists a cell complex X such that $\pi_1(X,a) = G$, where a is a vertex of X . A nice application of this topological result is the existence of coproducts in the category of groups $\mathbf{G}$.

The special case $k = 2$, $a = 2$ is easy. Clearly, if $n \equiv 0 \pmod 2$, then $n = 2$. If $n \equiv 1 \pmod 2$, we consider our equation $\pmod 3$ and see that there are no solutions.
I will do the general case tomorrow.
Posted by Mladenov

Problem 400 **

Prove

$$\frac{e^{-x}+e^{-1}}{x-1} + \cosh(x) = \sum_{n\geq 0} \frac{x^n}{n!} \left\{ \frac{n!}{e} \right\},$$

where the squiggly brackets pertain to the fractional part function.
Posted by henpen

Problem 401**/***

i^i , where i is the complex unit, is a multi-valued function that quite surprisingly always takes on real values. Each value of the function can be expressed in terms of $-k$, where k can be any integer.Find the infinite sum of the values of i^i for which k is a positive integer.
Posted by Flauta

Here are some nice and easy questions for

A-level students who may find themselves lost on this thread due to us mostly considering higher level problemsProblem 402*

Find, for $\psi, \eta \neq 0$

$$I = \int e^{\eta x} \sin(\psi x) dx$$

and give a condition on x such that $I > 0$

Problem 403*

Find ψ and η such that

$$\ln(\psi + \eta) = \ln(\psi) + \ln(\eta)$$

Posted by Indeterminate

Problem 404***

Evaluate

$$\int_0^\infty \frac{\sqrt{x}\ln x}{1+x^2}dx$$

.
Posted by Mladenov

Problem 405**

Evaluate

$$\int_0^\pi \frac{x\cos(x)}{1+\sin^2(x)}dx.$$

Problem 406**

Find a closed form (i.e. no series) for

$$\sum_{k\geq 1} \frac{\zeta(2k+1)-1}{k+1}.$$

Posted by henpen

Problem 407**

A satellite is projected towards a massive body (mass m) from infinity at an angle θ to the line joining the centre of masses of the body and satellite at a speed v . What shape will the motion be? What is the minimum distance that the satellite gets to the body?

Problem 408**

A hole is made in a table and a string, length l , with two equal masses at either end is passed through this hole. The mass on the table is pulled out as far as possible so that the mass under the table is at the hole. Gravity exists and the table is smooth. When does the mass on the table reach the hole if:
1. The mass on the table is initially at rest?
2. The mass on the table is influentially moving with tangential velocity v .Find the (polar) equation of motion for the mass on the table in both cases, and solve it.
Posted by henpen

Problem 409**

Using Cauchy's formula for repeated integration (check it at http://en.wikipedia.org/wiki/Cauchy_...ed_integra
:

$$\int_d^2 \int_0^c \int_b^2 \int_0^a (2-x)\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c$$

Then generalize the same, using the formula or otherwise, for:

$$\int_z^2 \int_0^y \int_x^2 \int_0^v \cdots \int_d^2 \int_0^c \int_b^2 \int_0^a (2-x)\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c \cdots \mathrm{d}y$$

It's very difficult thats what my teacher said! I didn't get it myself though.
Posted from TSR Mobile
Posted by journeyinwards

Problem 410**

Let

$$F_n(x) = \frac{1}{n!}\int_0^x t^n e^{-t} dt$$

.Show that if M is an integer greater than 1 , then

$$e^x F_M(x) = -(x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^M}{M!}) + (e^x - 1)$$

.
Posted by Arieisit

Problem 411**

Prove that

$$\forall n \in N,$$

(a + b)^n = $\sum_{r=0}^{n} C(n,r)a^{n-r}b^r$

Posted by Arieisit

Problem 412**

Prove that

$$\sum_{n>m\geq 1} \frac{1}{n^2m^2} = \frac{\pi^4}{120}$$

(that is, sum $\frac{1}{n^2m^2}$ over all distinct integer pairs (n,m) (distinct here means $(n,m) = (m,n)$, i.e. don't double count)).

Problem 412b**

Prove similar identities for

$$\sum_{a>b>c\geq 1} \frac{1}{a^2b^2c^2}$$

and

$$\sum_{a_1>\cdots>a_n\geq 1} \frac{1}{a_0^2\cdots a_n^2}.$$

Problem 412c**

Solve 412 and 412b with the additional requirement that $n, m \equiv 1 \mod 2$.
Posted by henpen

Problem 413***:

Let $T > 0$ and define

$$G : C[0,T] \to C[0,T]$$

by

$$G(f)(t) := 1 + \int_0^t 2cos(s(f(s))^2)ds \forall t \in [0,T]$$

.Find $T > 0$ such that G is a contraction on $C[0,T]$.

If you want one that's a little less horrible, instead try this version (vastly easier):

Problem 413***:

Let $T > 0$ and define

$$H : C[0,T] \to C[0,T]$$

by

$$H(f)(t) := 1 + \int_0^t 2cos(sf(s))ds \forall t \in [0, T]$$

.Find $T > 0$ such that H is a contraction on

$$(C[0, T], \| \cdot \|_\infty)$$

.

Some definitional/notational things for those that aren't certain of them/haven't met them:

Spoiler:

Show

If $(A, \|.\|)$ is a normed vector space, then $f : A \to A$ is a contraction iff

$$\|f(x) - f(y)\| \leq \|x - y\|$$

. $C[0, T]$ denotes the set of continuous functions on $[0, T]$.
$\| \cdot \|_\infty$ is defined (on $C[0, T]$ - the same notation is used for different norms on different spaces) by

$$\|f\|_\infty = \sup_{x \in [0,T]} |f(x)|$$

.

The former of these questions was an assessed question for 2nd year maths students, but the deadline is now past.
Posted by BlueSam3

Problem 419***

Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^4 + 1} \, dx$$

Posted by Flauta

Problem 420**

If

$$\sum_{k\geq 0} x_k$$

converges, does

$$\sum_{k\geq 0} x_k^{2014}$$

converge ? What about

$$\sum_{k\geq 0} x_k^{2013}?$$

Problem 421***

$$\int_{-\infty}^{\infty} \left[1 + \left(\frac{1}{x+1} + \frac{2}{x+2} + \cdots + \frac{2013}{x+2013} - x\right)^{2014}\right]^{-1} dx$$

Posted by Lord of the Flies

Problem 422

*Easy number theory one.Find all prime numbers $p < 1000$ such that for some integer n , the expression $n^2 - pn$ is a prime power.
Posted by Tarquin Digby

Problem 423***

Prove that

$$\lim_{x\to+\infty} \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = \frac{1}{2}$$

Extension:
Then show that there exists a positive constant k for every $x \in [1, \infty)$ such that:

$$|\sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2}| \leq \frac{k}{x}$$

Posted by

Problem 424 */**

Find the integral of ln(cosx) between x=pi/2 and x=0
Posted by IceKidd

Problem 425*/**/***

Evaluate

$$\int_2^3 7x\,dx$$

Posted by Lord of the Flies

Problem 427***

Evaluate

$$\int_0^{2\pi} e^{\sin(x)}\ dx$$

Posted by james22

Problem 428**(*)

:If $S \subseteq \mathbb{N}$ does there always exist a sequence (x_i) of reals such that

$$\sum_{n=1}^{\infty} x_n^{2s+1}$$

congerges if and only if $s \in S$?
Posted by Nebula

Problem 429 *

Evaluate i^i , given that it is real.
*Posted by Omghacklol*Problem 430**Prove that e is irrational.Problem 431*Prove that

$$(Area\Delta ABC)^2 = s(s-a)(s-b)(s-c)$$

where $s = \frac{1}{2}(a+b+c)$
*Posted by Jooooshy*Problem 432*Prove that a real, symmetric matrix A has real eigenvalues λ_i Problem 433**Prove that the only two prime Catalan numbers are C_2 and C_3 Also prove that $C_n = O(4^n n^{-3/2})$
Posted by Jooooshy

Problem 434*

Find the area of a triangle with angles α, β, γ drawn on a unit sphere.I've always liked this one.
Posted by henpen

Problem 436**

Prove that if G is a subgroup of S_n , then $|G||n!$ .Obviously it falls out from Lagrange's theorem directly, but is there another way?
Posted by henpen

Problem 437*

Find the term independent of x in

$$\sqrt[4]{(x^2 - \frac{6}{x^3})^{15}} + \sqrt[4]{(x^2 + \frac{6}{x^3})^{15}}$$

.
Posted from TSR Mobile
Posted by Arieisit

Problem 438***

Find

$$\int_0^\infty x^a e^{-x}\ dx$$

where $a \in \mathbb{R}$
Posted by james22

Problem 439**

Find

$$\int_0^\pi \frac{d\theta}{\sin\theta + \csc\theta}$$

Nothing particularly ground breaking, just thought the graph looked pretty cool
*Posted by Flauta*Problem 440 *Calvin and Hobbes are playing a game. They start with the equation $x^2 + x + 2014 = 0$ on the blackboard. Calvin and Hobbes play in turns, Calvin going first. Calvin's moves consist of either increasing or decreasing the coefficient of x by 1 and Hobbes' moves consist of similarly changing the constant term by 1 , with Calvin winning the moment an equation with integer solutions appears on the board. Prove that Calvin can always win.[Source: Indian National Mathematical Olympiad 2014]
Posted by 1298977

Problem 441***

Find, for

$$n \in \mathbb{Z}, m \in \mathbb{C}, \Re(m) > -1$$

,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^m(x) e^{inx} dx.$$

Ramanujan says that the result is 'well known', so it shouldn't be too difficult.
Posted by henpen

Problem 442*

Prove that

$$2^{n-1}(x_1^n + x_2^n) \geq (x_1 + x_2)^n$$

.Where x_i are positive real numbers and n is real.if this isn't sort of familiar then opening this spoiler may be ruinous... if it is then open up

Spoiler:

Show

if you assume the power-mean inequality I will murder you

Posted by jjpneed1

Problem 443**

S an infinite collection of nested subsets of $\mathbb{N}$. Must S be countable?

Problem 444**

Take a real sequence, let Z be the set of its limit points. Must Z be countable?
Posted by Lord of the Flies

This thread *dry up?* Not on my watch!

Problem 445**

Find the Dirichlet series for

$$\frac{1}{\zeta(s)}.$$

Problem 446***

Prove

$$\sum_{k=1}^{\infty} \frac{k}{e^{2\pi k}-1} = \frac{1}{24} - \frac{1}{8\pi}.$$

Problem 446***

Prove

$$\sum_{k=-\infty}^{\infty} e^{-k^2\pi} = \frac{\pi^{\frac{1}{4}}}{\Gamma\left(\frac{3}{4}\right)}.$$

Problem 447**

Prove

$$\sum_{k=1}^{\infty} \frac{H_k}{n^2 2^n} = \zeta(3) - 2\log(2)\zeta(2).$$

Problem 448**

Find

$$\sum_{k=1}^{\infty} \arctan\left(\frac{1}{k^2}\right).$$

Problem 449**

Find

$$\lim_{n\to\infty} \prod_{k=2}^{n} \left(1 - \frac{1}{k^3}\right)$$

.

Problem 450**

Find

$$\lim_{n\to\infty}\int_0^{2\pi}\cos(x)\cos(2x)\cdots\cos(nx)\ dx.$$

Problem 451***

Find

$$\int_0^{\infty}\frac{\cos(x)-e^{-x^2}}{x}\ dx.$$

Problem 452**

Prove

$$\sum_{k=1}^{\infty}(\zeta(4k)-1)=\frac{7}{8}-\frac{\pi}{4}\left(\frac{e^{2\pi}+1}{e^{2\pi}-1}\right).$$

Problem 453**

Prove

$$\sum_{k=1}^{\infty}\frac{\zeta(k)-1}{k}=1-\gamma.$$

Posted by henpen

Problem 454**/***

Consider

$$\vartheta : \mathbb{R} \to \wp(\mathbb{R})$$

where $\vartheta(x)$ finite and $x \notin \vartheta(x)$ for all x . The question is whether there exists $S \subset \mathbb{R}$ s.t.:

$$S \cap \vartheta(S) = \varnothing$$

and $|S| = \mathfrak{c}$
Posted by Lord of the Flies

Problem 455 **/***

Let

$$J_0(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(x \cos\theta)\ d\theta$$

. Show that

$$\int_0^{\infty} J_0(x) e^{-ax}\ dx = \frac{1}{\sqrt{1+a^2}}$$

, for $a > 0$

Problem 456 **/***

For $b > a > 1$ find the value of

$$\int_1^{\infty} \frac{x^{-a} - x^{-b}}{\log(x)}\ dx$$

To make it easier, you may assume both the Tonelli and Fubini theorem hold for both.
Posted by Noble.

Problem 457

**Find all automorphisms of the real numbersi.e. find all functions

$$f : \mathbb{R} \to \mathbb{R}$$

such that $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ Now do the same of the complex numbers***
Posted by james22

Problem 458***

For $u, v > 0$ show that

$$\int_{-\infty}^{\infty} \frac{u \cos vx - x \sin vx}{(u^2 + x^2)^2} dx = \frac{\pi}{2u^2 e^{uv}}.$$

Posted by Indeterminate

Problem 459**

A fair coin is tossed n times. Let un be the probability that the sequence of tosses never has two consecutive heads. Show that u_n =0.5u_(n-1)+0.25u_(n-2) .Find also u_n

Problem 460*

The radius of a circle has the exponential distribution with parameter lambda. What is the probability density function of the area of the circle.
Posted by newblood

Problem 461 **/***

Evaluate

$$\int \frac{\log(1 + x)}{x^2 + 1}\ \mathrm{d}x$$

Posted by Khallil

Problem 462 ***

A is a symmetric matrix with eigenvectors u and v and with corresponding, distinct eigenvalues Î», Î¼. Show that the eigenvectors are orthogonal.(Sorry if this question has been
Posted before, I don't want to look through 461 other questions :L, also I'm sorry if it's too easy/too hard/no one has any idea what I'm on about)
*Posted by rayquaza17*Problem 463***:Let U, V be vector spaces over a field K , $\tau : U \times V \to K$ a bilinear form. For $W \subset V$, define the τ -orthogonal complement

$$U^{\perp} := \{u \in U : \tau(u, w) = 0 \, \forall w \in W\}$$

, and similarly for $W \subset U$. Prove that, for any $W \subset U$,

$$((W^{\perp})^{\perp})^{\perp} = W^{\perp}$$

.
*Posted by BlueSam3*Problem 464 *Take the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a,b,c and d can be any digit from 0-9 inclusive. What is the probability of choosing a singular matrix when values for a,b,c and d are randomly chosen?
*Posted by interstitial*Problem 465*If all the integer numbers from 1 to 1000 inclusive were written out as words, how many letters would be used? Don't count spaces, and make sure to use the word and, e.g. 'seven hundred and forty two', which is 23 letters long.For anyone interested, this problem was taken from Project Euler which is a series of challenging mathematical problems that can be solved using clever programming skills. Check it out!https://projecteuler.net
Posted by interstitial

Problem 466**

Determine

$$\int_0^1 \frac{\ln(1 + x^2)}{x} dx$$

Posted by jjpneed1

Now how about

Problem 467**

$$\int_0^1 \frac{\ln(1+x^2)}{1+x}\,dx$$

?

Posted by jjpneed1

Problem 468***

$$\int_0^{\pi} \frac{\cos^2 x}{5+4\cos x}\;dx$$

Posted by FireGarden

Problem 471

Determine all sets of non-negative integers x, y and z that satisfy $2^x + 3^y = z^2$.

Problem 472

Evaluate

$$\int_0^{\infty} \frac{\ln(1+x)\ln(1+x^2)}{x^3} dx$$

Last ones for a while now while I sit some tasty exams, gl on that integral, good job if someone gets it out before exams finish

Posted by jjpneed1

Problem 473 **

Show that:

$$\int_0^1 x^4\,(1-x)^{20}\;\mathrm{d}x = \frac{1}{25\binom{24}{4}}$$

Spoiler:

Show

The Beta function (and by extension, the Gamma function) might be of some use here.

*Posted by Khallil*Problem 474 *I can write any (positive) number as a sum of arbitrarily many (positive) real numbers, e.g. $16 = 10 + 6 = 3.4 + 6.6 + 6$, etc. When is the product of all elements in this sum maximised?
Posted by jjpneed1

Problem 475*

$$f:\ \mathbb{R} \to \mathbb{R}$$

. Must there exist $x, y \in \mathbb{R}$ s.t.

$$f\big(x - f(y)\big) > x + yf(x)?$$

Posted by Lord of the Flies

Problem 476***

Prove that, for integers $n \geq 2$

$$\frac{e^2}{2}\left(\frac{n}{e}\right)^n \leq n! \leq \frac{ne^2}{4}\left(\frac{n}{e}\right)^n$$

Posted by Indeterminate

Problem 477***

Evaluate the following product (note that p represents a prime number).

$$\prod_p \left(1 + \frac{1}{p(p+1)}\right)$$

Posted by Indeterminate

Problem 478*

(i) Prove that if

$$|x - x_0| < min\left(\frac{\epsilon}{2(|y_0| + 1)}, 1\right)$$

and

$$|y - y_0| < \frac{\epsilon}{2(|x_0| + 1)}$$

.then

$$|xy - x_0y_0| < \epsilon$$

.*Note that: $min(a, b) = a$ if $a < b$.(ii) Prove that if $y_0 \neq 0$ and

$$|y - y_0| < min\left(\frac{|y_0|}{2}, \frac{\epsilon|y_0|^2}{2}\right)$$

.then $y \neq 0$ and

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| < \epsilon$$

.(iii) Replace the question marks with expressions involving ϵ, x_0, y_0 so that the conclusion will be true:If $y_0 \neq 0$ and $|y - y_0| <$? and $|x - x_0| <$? then $y \neq 0$ and

$$\left|\frac{x}{y} - \frac{x_0}{y_0}\right| < \epsilon$$

.
Posted by 0x2a

Problem 479 **

A die is thrown until an even number appears. What is the expected value of the sum of all the scores?
Posted by HeavisideDelts

Problem 480 **

(not much probability theory in A-Level)Let $X_1, X_2, ..., X_n$ be i.i.d r.v's with uniform distribution on $[1, 2]$ .Determine a unique value, Δ , such that

$$\lim_{n\to\infty} Pr(a < (X_1X_2...X_n)^{1/n} < b) = 1$$

iff $a < \Delta < b$
Posted by newblood

Problem 481*

Show that the series

$$\sum_{n\geq 1} \ln(\frac{n+1}{n})$$

Is divergent, even though

$$\lim_{n\to\infty} \ln(\frac{n+1}{n}) = 0$$

Posted by Arieisit

Problem 482**

$$\int_0^1 \frac{\arctan x}{x+1}\ dx$$

Problem 483***

$$\int_{-\pi}^{\pi} \frac{e^{\sin x+\cos x}\cos(\sin x)}{e^{\sin x}+e^x}\ dx$$

Posted by ThatPerson

Problem 484**

$$\int_0^4 \frac{\ln(x)}{\sqrt{4x - x^2}}\ dx$$

Posted by ThatPerson

Problem 485**

Prove that every open set G on the real line is the union of a finite or countable system of pairwise disjoint open intervals, where we regard

$$(-\infty, x), (-\infty, \infty), (y, \infty)$$

as open intervals.
Posted by 0x2a

Problem 486**

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}}\ dx$$

Posted by ThatPerson

Problem 487***

$f:\ (0,1) \to (0,1)$ satisfies $f(x) < x$. Let

$$\hat{f}(x) = \lim_{n\to\infty} \underbrace{f(f(\cdots f}_{n}(x)\cdots);$$

can $\hat{f}$ take uncountably many values?
Posted by Lord of the Flies

Problem 488**

$$\int -\left(\frac{\pi \csc\left(\pi(\frac{-1-ix}{2})\right)}{4} + \frac{\pi \csc\left(\pi(\frac{-1+ix}{2})\right)}{4}\right) dx$$

Posted by

Problem 489*/**/*** (more knowledge will give you a stronger answer)

We say that a function

$$f : \mathbb{R} \to \mathbb{R}$$

is periodic with periods S if

$$\forall x \in \mathbb{R}, a \in S, f(x+a) = f(x)$$

and

$$\nexists n \in \mathbb{N}_{>1}, y \in \mathbb{R} : ny = a$$

and y is a period of f, for each period a .How large (cardinality wise) can you make S . Can you prove that it cannot be any larger?An example to see if it makes things clearer, f(x)=sin(x) has S={pi,-pi}. You could not say that S={3pi,4pi} as these are multiples of a smaller period.Please let me know if the above is not clear, this question is easy to visualise but not so easy to write down.
Posted by james22

Problem 490**

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$$

Posted by ThatPerson

Problem 491:**

The enthusiasts amongst you may recognize this:
Let: (x_n) be a sequence of positive real numbers such that:

$$\sum_{n=1}^{\infty} n^2 x_n^2$$

converges.
Show that:

$$\sum_{n=1}^{\infty} x_n$$

is convergent. And the converse?
Posted by joostan

Problem 492**

Prove that for any pair of positive integers k and n , there exist k positive integers $m_1, m_2, ..., m_k$ (not necessarily different) such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right)...\left(1 + \frac{1}{m_k}\right)$$

Posted by ThatPerson

Problem 493***

Let $\zeta(s)$ denote the Riemann-Zeta function in the usual way, for complex $s = \sigma + it$ Prove that, for $\sigma > 1$,

$$|\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1$$

Posted by Indeterminate

Problem 494 ***

Evaluate:

$$\int_0^\infty \frac{cos(ax)}{1+x^2}dx$$

for $a \geq 0$ .Sorry if it's been asked before.
Posted by rayquaza17

Problem 495 *

Two players take it in turn to move a king that has been initially placed on one of the centre squares of a chessboard. You lose if you move the king to a square he has previously occupied (including the initial starting square). Who wins the game, the first or the second player? Is there a winning strategy?
Posted by Renzhi10122

Problem 496 **

Find all

$$f : \mathbb{Z} \to \mathbb{Z}$$

such that

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a)$$

for all integers a, b, c satisfying $a + b + c = 0$
*Posted by Renzhi10122*Problem 497 *:Find all positive integers x, y such that

$$x^y = y^{x-y}$$

*Posted by Gawain*Problem 498**Prove that if a sequence of complex numbers z_n converges to A , then

$$\lim_{n\to\infty} \frac{1}{n}(z_1 + z_2 + ... + z_n) = A$$

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Posted by Remyxomatosis

Problem 499*

*A class of $n \geq 5$ students is organised into $n+1$ teams. Each team consists of 3 students, and no team is identical. Show that there are two teams with exactly one common student.
Posted by Renzhi10122