FLOWSORT: A flow-based sorting method with limiting or central profiles

Article *in* Top · February 2008

Impact Factor: 0.83 · DOI: 10.1007/s11750-007-0036-x · Source: RePEc

2 authors, including:

Philippe Nemery
2 SAP Software Systems Belgium

Top
DOI 10.1007/s11750-007-0036-x

ORIGINAL PAPER

$\mathcal{F}$low$\mathcal{S}$ort: a flow-based sorting method with limiting or central profiles

Philippe Nemery · Claude Lamboray

Received: 25 January 2007 / Accepted: 26 October 2007

Abstract Based on the ranking methodology of PROMETHEE, a new sorting method ($\mathcal{F}$low$\mathcal{S}$ort) is proposed for assigning actions to completely ordered categories defined either by limiting profiles or by central profiles. The $\mathcal{F}$low$\mathcal{S}$ort assignment rules are based on the relative position of an action with respect to the reference profiles, in terms of the incoming, leaving, and/or net flows. For a better understanding of the issues involved, a graphical representation is given. An explicit relationship between the assignments obtained when working either with limiting or central profiles is formalized. Finally, an empirical comparison with Electre-Tri is made to compare the resulting assignments.

Keywords Multicriteria decision aid · Sorting · Preference relation

Mathematics Subject Classification (2000) 91B08 · 91B06 · 91B50

1 Introduction

In multicriteria decision aid, one usually distinguishes between the "problematic" of choosing, ranking, and sorting (Roy and Bouyssou 1993). The latter consists in assigning a set of decision actions evaluated with respect to several criteria to predefined, completely ordered homogeneous classes. These classes are also called categories. A crucial issue in multicriteria decision aid is to design assignment rules that explicitly take into account the decision maker's preferences. Several approaches

P. Nemery (✉) · C. Lamboray
Ecole Polytechnique, Service de Mathématiques de la Gestion—CoDE,
Université Libre de Bruxelles, Boulevard du Triomphe CP 210-01, 1050 Brussels, Belgium
e-mail: pnemeryd@ulb.ac.be

C. Lamboray
e-mail: claude.lamboray@ulb.ac.be

have been proposed, and let us mention, among others, outranking approaches (with, for example, Electre-Tri Roy and Bouyssou 1993; Yu 1992), multi-attribute theory (with, for example, UTADIS Zopounidis and Doumpos 2002a), or rough sets (Greco et al. 2001). Often the use of these methods has been facilitated by ways of inferring the parameters involved (Devaud et al. 1980; Dias and Mousseau 2006; Doumpos and Zopounidis 2004a, 2004b; Huenaerts and Nemery 2007; Ngo The and Mousseau 2002). For a more exhaustive survey on sorting methods, we refer the reader to Araz and Ozkarahan (2007) and Zopounidis and Doumpos (2002b).

A category is either delimited by two boundaries, in which case we speak about limiting profiles (like, for instance, in Electre-Tri), or it can be represented by centroids, in which case we speak about central profiles (such as in the model proposed by Doumpos and Zopounidis 2004b; Figueira et al. 2004). Nevertheless, no method handles both types of profiles.

In this paper, we propose an integrated method that allows defining a category either by a lower and an upper limiting profile, such as in Electre-Tri, or by one central profile, such as in the model proposed by Doumpos and Zopounidis.

Another particularity of our method is that, unlike in Electre-Tri, the assignment of an action depends on a global comparison with all the profiles simultaneously. In fact, we address the sorting problematic by means of a ranking method. The ranking method is applied on the data-set consisting of the actions to assign and the reference profiles. The category will then be deduced from the relative position of the action with respect to the references profiles in the ranking obtained. One advantage of such an approach is that a decision maker who is familiar with a particular ranking method can easily understand the corresponding sorting method. We will illustrate this idea with the well-known PROMETHEE method (Brans and Vincke 1985).

More particularly, preference degrees are computed between the action to sort and the reference profiles that determine the categories. Three different assignment rules are then defined based on the incoming and leaving flows of the action to sort.

The paper is organized as follows. First, we introduce some notation and definitions and clearly state the assumptions on which the model is built. Section 3 is devoted to presenting the assignments rules. Section 4 presents an empirical comparison with Electre-Tri. The paper ends with conclusions and some further directions of research. The proofs of the propositions can be found in the Appendix.

2 Notation and conditions

Let us denote by $\mathcal{A}$ the set of n actions to be sorted: $\mathcal{A} = \{a_1, \ldots, a_n\}$. Each action is evaluated on q criteria g_j $(j = 1, \ldots, q)$. In what follows, we suppose, without any loss of generality, that these q criteria have to be maximized.

We denote the categories to which the actions are assigned by $C_1, C_2, \ldots, C_K$. Furthermore, we suppose that the categories are completely ordered as follows: $C_1 \rhd \cdots \rhd C_l \rhd \cdots \rhd C_K$, where $C_h \rhd C_l$ with $h < l$ denotes that the category C_h is "preferred to" the category C_l.

Each category is defined either by one central profile or by two reference profiles. First we shall consider the case where the categories are defined by limiting profiles.

Consequently, a category C_h is defined by an upper and lower profile noted as r_h and r_{h+1}, respectively. At the same time, r_h is the lower profile of C_{h-1}, and r_{h+1} is the upper profile of C_{h+1}.

The set of limiting profiles is denoted as $\mathcal{R} = \{r_1, \ldots, r_{K+1}\}$. Furthermore, we suppose that the performances of all the actions in $\mathcal{A}$ are between the worst r_{K+1} and best r_1 limiting profiles. Figure 3 illustrates an example of some limiting profiles defining 4 categories.

On the other hand, instead of using two limiting profiles for the definition of each category, the decision maker may choose to define these by one central profile (often called the centroid of the category). In such a situation, for K completely ordered categories, there are K centroids denoted by $\tilde{\mathcal{R}} = \{\tilde{r}_1, \ldots, \tilde{r}_K\}$, where $\tilde{r}_j$ is the centroid of category $\tilde{C}_j$. Figure 3 illustrates an example where centroids are used for defining 4 categories.

We shall further note the reference profiles by $\mathcal{R}^* = \{r_1^*, r_2^*, \ldots\}$ when no distinction has to be made between a set of limiting profiles and a set of centroids. Moreover, let us define for any action a_i to be assigned the following set: $\mathcal{R}_i^* = \mathcal{R}^* \cup \{a_i\}$, where a_i is the action to be assigned.

Since the reference profiles define ordered categories, we shall assume that two consecutive reference profiles dominate each other. This is formulated by the following condition:

Condition 1 $\forall r_h^*, r_l^* \in \mathcal{R}^*$ *such that* $h < l$: $g_j(r_h^*) \geq g_j(r_l^*)\ \forall j \in \{1, \ldots, q\}$.

We suppose that a preference degree $\pi(x, y)$, which evaluates the preference strength of action x over an action y according to the preferences of the decision maker, can be computed for all the actions x, y of $\mathcal{R}_i^*$. We suppose that, for all $x, y \in \mathcal{R}_i^*$, the following conditions hold:

Condition 2 $0 \leq \pi(x, y) \leq 1$.

Condition 3 $\pi(x, y) + \pi(y, x) \leq 1$.

Condition 4 $\pi(x, x) = 0$.

Condition 5 $\forall x', y' \in \mathcal{R}_i^*$, *if* $\forall\ j \in \{1, \ldots, q\} : g_j(x) - g_j(y) \leq g_j(x') - g_j(y')$, *then* $\pi(x, y) \leq \pi(x', y')$.

The preference degrees can, for instance, be obtained as in the Promethee methodology (Brans and Vincke 1985).

Since the reference profiles define ordered categories, we will impose that a reference profile of a lower (better) category is "preferred", according to the decision maker, to the reference profiles of a higher (worse) category. Formally, we have that:

Condition 6 $\forall r_h^*, r_l^* \in \mathcal{R}^*$ *such that* $h < l$: $\pi(r_h^*, r_l^*) > 0$ *and* $\pi(r_l^*, r_h^*) = 0$.

On the basis of these preference degrees, the positive (leaving), negative (incoming), and net flows of each action x of $\mathcal{R}_i^*$ are computed as follows (Brans and Vincke 1985):

$$\phi^+_{\mathcal{R}_i^*}(x) = \frac{1}{|\mathcal{R}_i^*| - 1} \sum_{y \in \mathcal{R}_i^*} \pi(x, y), \tag{1}$$

$$\phi^-_{\mathcal{R}_i^*}(x) = \frac{1}{|\mathcal{R}_i^*| - 1} \sum_{y \in \mathcal{R}_i^*} \pi(y, x), \tag{2}$$

$$\phi_{\mathcal{R}_i^*}(x) = \phi^+_{\mathcal{R}_i^*}(x) - \phi^-_{\mathcal{R}_i^*}(x). \tag{3}$$

The order of the flows of the reference profiles is the same as the order of the reference profiles. This is formalized in the next proposition:

Proposition 1 *The order of the flows of the reference profiles is invariant with respect to the action a_i to assign*: $\forall a_i \in \mathcal{A}$:

$$\forall h \in \{1, \dots, K+1\}, \qquad\qquad \forall h \in \{1, \dots, K\},$$

$$\begin{cases} \phi^+_{\mathcal{R}_i}(r_h) > \phi^+_{\mathcal{R}_i}(r_{h+1}), \\ \phi^-_{\mathcal{R}_i}(r_h) < \phi^-_{\mathcal{R}_i}(r_{h+1}), \\ \phi_{\mathcal{R}_i}(r_h) > \phi_{\mathcal{R}_i}(r_{h+1}); \end{cases} \qquad \begin{cases} \phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_h) > \phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h+1}), \\ \phi^-_{\tilde{\mathcal{R}}_i}(\tilde{r}_h) < \phi^-_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h+1}), \\ \phi_{\tilde{\mathcal{R}}_i}(\tilde{r}_h) > \phi_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h+1}). \end{cases}$$

In other words, although the actual flow values of the reference profiles directly depend on the action a_i, their order always respects the order of the categories. This allows us to delimit a category C_h by the flow values of r_h and r_{h+1} in the case that the categories are defined by an upper and lower limit. Alternatively, a category defined by a centroid is represented by the flows of that centroid with respect to the other centroids.

This proposition will be the basis of the $\mathcal{F}$low$\mathcal{S}$ort assignment rules.

3 Flow-based assignment procedures

3.1 Limiting profiles

We assume throughout this section that a set of limiting profiles $\mathcal{R}$ has been defined and that preference degrees have been computed between the actions of $\mathcal{R}_i$.

Two different assignment rules based on the previous considerations are defined as follows:

$$C_{\phi^+}(a_i) = C_h \quad \text{if } \phi^+_{\mathcal{R}_i}(r_h) \geq \phi^+_{\mathcal{R}_i}(a_i) > \phi^+_{\mathcal{R}_i}(r_{h+1}),$$

$$C_{\phi^-}(a_i) = C_h \quad \text{if } \phi^-_{\mathcal{R}_i}(r_h) < \phi^-_{\mathcal{R}_i}(a_i) \leq \phi^-_{\mathcal{R}_i}(r_{h+1}).$$

In other words, we evaluate the "preferred to" and the "being-preferred to" character of an action a_i with respect to the reference profiles by means of $\phi^+(a_i)$

and $\phi^-(a_i)$. In the first rule, the action is then assigned to the category C_h if the flow $\phi^+(a_i)$ is contained in the interval defined by the positive flows of the reference profiles of the category C_h. In the second rule, the action is assigned to the category C_h if the flow $\phi^-(a_i)$ is contained in the interval defined by the negative flows of the reference profiles of the category C_h.

As in the PROMETHEE methodology, two aspects are considered. On the one hand, a ranking is computed based on incoming flows, which in our model leads to the assignment C_{ϕ^+}. On the other hand, a ranking is computed based on leaving flows, which in our model leads to the assignment C_{ϕ^-}.

We can thus obtain two different assignments: $C_{\phi^+}(a_i)$ and $C_{\phi^-}(a_i)$, where either $C_{\phi^+}(a_i) \trianglerighteq C_{\phi^-}(a_i)$ or $C_{\phi^-}(a_i) \trianglerighteq C_{\phi^+}(a_i)$.[1] Let us denote by $C_b(a_i)$ the best category and by $C_w(a_i)$ the worst category obtained with these two assignment rules for a_i.

The assignment of an action depends on the comparison with all the reference profiles simultaneously and does not (as, for instance, in ELECTRE-Tri) on successive pairwise comparisons. Although, this more "global" approach may appear unconventional, it is a direct consequence of using a ranking method in a sorting context.

Furthermore, if a decision maker would impose the assignment to strictly one category, we could define a similar assignment rule using the net flows:

$$C_\phi(a_i) = C_h \quad \text{if } \phi_{\mathcal{R}_i}(r_h) \geq \phi_{\mathcal{R}_i}(a_i) > \phi_{\mathcal{R}_i}(r_{h+1}).$$

In fact, this net flow assignment rule is analogous to the PROMETHEE II ranking. This appears reasonable since the assignment obtained with the net flow rule is consistent with the two assignments obtained with the positive and negative flow rules. More formally, we have:

Proposition 2

$$\forall a_i \in \mathcal{A}: \quad C_b(a_i) \trianglerighteq C_\phi(a_i) \trianglerighteq C_w(a_i).$$

To give an easily understandable visualization of the assignments, we can represent the $\phi^-_{\mathcal{R}_i}$ and $\phi^+_{\mathcal{R}_i}$ flows of all the actions of $\mathcal{R}_i$ in a $[\phi^-, \phi^+]$-flow space. In this space, given a category C_h, the points $(\phi^-_{\mathcal{R}_i}(r_h), \phi^+_{\mathcal{R}_i}(r_h))$ and $(\phi^-_{\mathcal{R}_i}(r_{h+1}), \phi^+_{\mathcal{R}_i}(r_{h+1}))$ naturally define a rectangle. The flows of the action a_i to be assigned also defines a point $(\phi^-_{\mathcal{R}_i}(a_i), \phi^+_{\mathcal{R}_i}(a_i))$ in this space. If this point lies in the rectangle of category C_h (e.g., action 2 in Fig. 1), then a_i is obviously assigned to C_h (with both the positive and negative flow rules). However, the assignments obtained on the basis of the negative and positive flow may also differ, as is the case of actions 1 and 3 in Fig. 1. These actions are then assigned to a set of consecutive categories.

Let us now emphasize some properties of the assignment rules defined above. If an action a_i dominates another action a_j, then a_i can not be assigned to a higher (worse) category than action a_j. This is stated in the following proposition.

[1] $C \trianglerighteq \tilde{C}$ means that either $C \triangleright \tilde{C}$ or $C = \tilde{C}$.

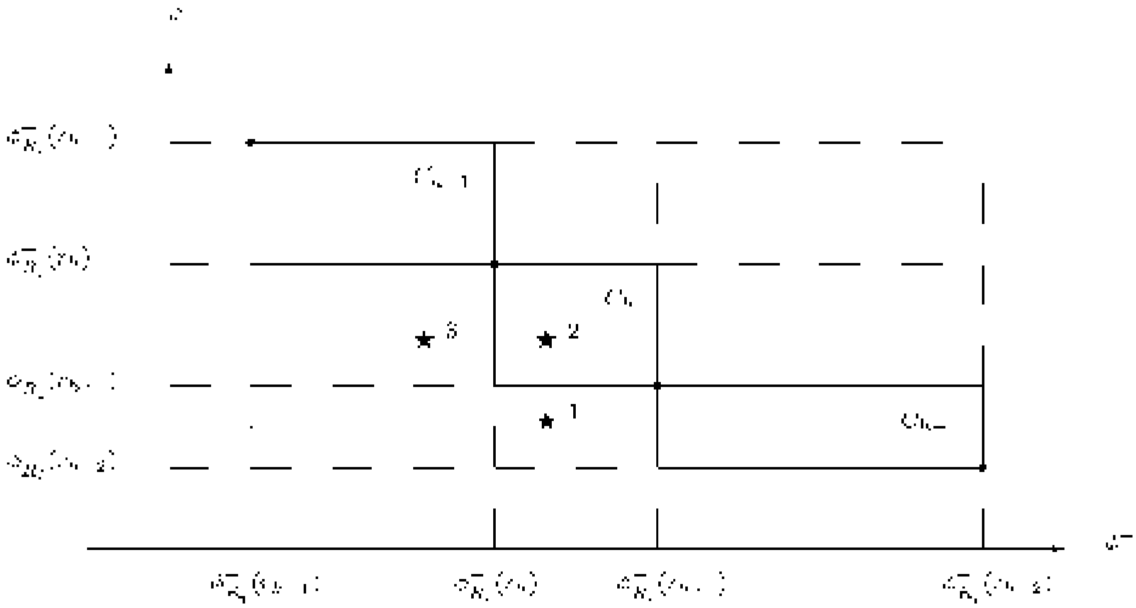

Fig. 1 A flow and category representation with limiting profiles

Proposition 3 *$\forall a_i, a_j$, if*

$$\forall k \in \{1, \ldots, q\}: \quad g_k(a_i) \geq g_k(a_j),$$

then

$$C_{\phi^+}(a_i) \trianglerighteq C_{\phi^+}(a_j) \quad \textit{and} \quad C_{\phi^-}(a_i) \trianglerighteq C_{\phi^-}(a_j).$$

So far, we simply supposed that $\pi(r_h, r_l) > 0$ with $h < l$. We can strengthen this condition by accepting that the upper profile r_j of a category C_j is "strongly preferred" to the lower profile r_{j+1}. This may be formalized as follows.

Condition 7 *$\forall r_h, r_l \in \mathcal{R}$ such that $h < l$: $\pi(r_h, r_l) = 1$.*

This condition on the reference profiles might at first sight be considered as too strong. In fact, it just ensures that the dominance between reference profiles is a strong preference according to the decision maker. Let us recall that in Electre-Tri a similar condition is imposed on the reference profiles (page 29 in Yu 1992). In case the reference profiles verify 7, we have moreover the following property:

Proposition 4 *$\forall a_i$, if $C_{\phi^-}(a_i) = C_l$ and $C_{\phi^+}(a_i) = C_h$, then $l - h \leq 0$.*

One consequence of this proposition is that not every category combination $(C_{\phi^-}(a_i), C_{\phi^+}(a_i))$ is possible. In fact, under the assumption that 7 is verified, the category $C_{\phi^-}(a_i)$ is always as least as good as the category $C_{\phi^+}(a_i)$.

3.2 Central profiles

We assume throughout this section that a set of central profiles $\tilde{\mathcal{R}}$ has been defined and that preference degrees have been computed between the actions of $\tilde{\mathcal{R}}_i$.

When assigning an action to a category, we compare its flows with the centroids' flows and assign an action to the category whose centroid has similar flows. More formally, the assignment rules based on either the positive or negative flow is defined as follows:

$$\tilde{C}_{\phi^+}(a_i) = \tilde{C}_h \quad \text{if } \frac{\phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_h) + \phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h+1})}{2} < \phi^+_{\tilde{\mathcal{R}}_i}(a_i) \le \frac{\phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_h) + \phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h-1})}{2},$$

$$\tilde{C}_{\phi^-}(a_i) = \tilde{C}_h \quad \text{if } \frac{\phi^-_{\tilde{\mathcal{R}}_i}(\tilde{r}_h) + \phi^-_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h+1})}{2} \ge \phi^-_{\tilde{\mathcal{R}}_i}(a_i) > \frac{\phi^-_{\tilde{\mathcal{R}}_i}(\tilde{r}_h) + \phi^-_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h-1})}{2}.$$

The philosophy of the assignment rule is the same as in the case of limiting profiles: an action is assigned, according to the first rule, to the category whose centroid has the same "preferred character" over the remaining reference actions. On the other hand, the second rule is based on the "being preferred character" of that action. Based on the two different assignment rules, two possible categories $\tilde{C}_{\phi^-}(a_i)$ and $\tilde{C}_{\phi^+}(a_i)$ can be obtained. As in the case of reference profiles, let us denote by $\tilde{C}_b$ the best and by $\tilde{C}_w$ the worst of these two categories.

If an action has to be assigned to one unique category, we can naturally define an assignment rule based on the net flows:

$$\tilde{C}_{\phi}(a_i) = C_h \quad \text{if } \frac{\phi_{\tilde{\mathcal{R}}_i}(\tilde{r}_h) + \phi_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h+1})}{2} < \phi^+_{\tilde{\mathcal{R}}_i}(a_i) \le \frac{\phi_{\tilde{\mathcal{R}}_i}(\tilde{r}_h) + \phi_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h-1})}{2}.$$

As in the case of limiting profiles, it can be proven that the category determined using the net-flow assignment rule is in between the categories determined by the positive and negative net-flow rules, respectively:

$$\forall a_i \in \mathcal{A}, \quad \tilde{C}_b(a_i) \trianglerighteq \tilde{C}_{\phi}(a_i) \trianglerighteq \tilde{C}_w(a_i).$$

Proposition 2 thus also holds when working with central profiles.

It is obvious that if an action a_i has the same performances as a centroid $\tilde{r}_h$, it is assigned to category $\tilde{C}_h$, since the flows of a_i and $\tilde{r}_h$ are the same.

Let us finally note that if an action a_i dominates another action a_j (i.e., $\forall k \in \{1, \ldots, q\}$: $g_k(a_i) \ge g_k(a_j)$), then the action a_i cannot be assigned to a worse category than the action a_j (see Proposition 3).

The different positive and negative flows of the centroids and the action to be sorted can again be represented in the $[\phi^-, \phi^+]$ flow-space (Fig. 2). Let us remark that the "rectangles" representing the ordered categories are now obtained in a slightly different way. The upper-left point of the rectangle representing the category $\tilde{C}_h$ is the average between the flows values of $\tilde{r}_h$ and $\tilde{r}_{h-1}$, whereas the lower-right point is the average between the flows values of $\tilde{r}_h$ and $\tilde{r}_{h+1}$. If the point $(\phi^-_{\tilde{\mathcal{R}}_i}(a_i), \phi^+_{\tilde{\mathcal{R}}_i}(a_i))$ lies in the rectangle of the category $\tilde{C}_h$, then a_i is obviously assigned to $\tilde{C}_h$ (both with the positive and negative flow rules).

As in the case of limiting profiles, we could accept that two successive centroids are "strongly preferred". Formally, they respect the following condition:

Condition 8 $\quad$ *$\forall \tilde{r}_h, \tilde{r}_l \in \tilde{\mathcal{R}}$ such that $h < l$: $\pi(\tilde{r}_h, \tilde{r}_l) = 1$.*

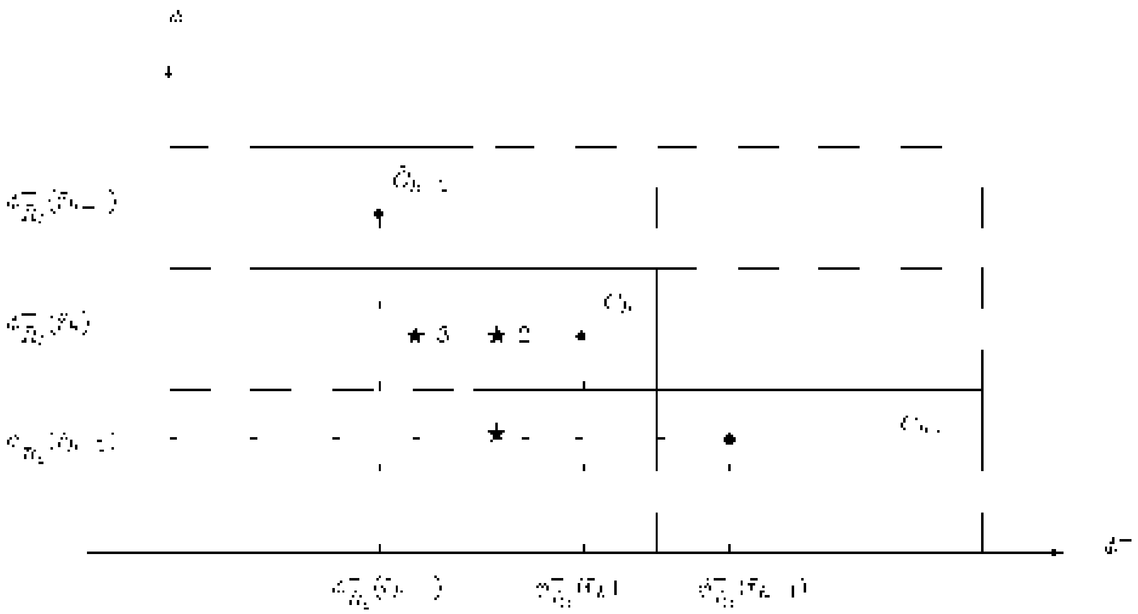

Fig. 2 A flow and category representation with central profiles

If, additionally, the centroids satisfy Condition 8, we can show that a property similar to Proposition 5 is verified: $\forall a_i$, if $\tilde{C}_{\phi^-}(a_i) = \tilde{C}_l$ and $\tilde{C}_{\phi^+}(a_i) = \tilde{C}_h$, then $l - h \leq 0$.

3.3 Relationship between the assignments with limiting profiles and central profiles

Let us recall that we denote the set of limiting profiles by $\mathcal{R} = \{r_1, \ldots, r_{K+1}\}$ and the set of centroids by $\tilde{\mathcal{R}} = \{\tilde{r}_1, \ldots, \tilde{r}_K\}$. We furthermore assume that $\mathcal{R}$ verifies 7 and that $\tilde{\mathcal{R}}$ verifies 8. This leads respectively to the assignments $C_{\phi^+}(a_i)$ and $\tilde{C}_{\phi^+}(a_i)$ and to the assignments $C_{\phi^-}(a_i)$ and $\tilde{C}_{\phi^-}(a_i)$.

In this section, we compare the assignment of an action when the same set of ordered categories $C_1 \rhd C_2 \rhd \cdots \rhd C_K$ is either defined by limiting profiles or by centroids. For this purpose, we suppose that the centroid of a category is in between the limiting profiles of that same category: $\forall h \in \{1, \ldots, K\}$, $\forall j \in \{1, \ldots, q\}$:

$$g_j(r_h) \geq g_j(\tilde{r}_h) \geq g_j(r_{h+1}).$$

Let us note that $\tilde{r}_h$ is not necessarily the "midpoint" or the "average" of r_h and r_{h+1}.

In such a situation, there exists a relationship between the categories to which an action will be assigned according to the limiting profiles or the centroids:

Proposition 5 *Let* $C_{\phi^+}(a_i) = C_h$, $C_{\phi^-}(a_i) = C_l$, $\tilde{C}_{\phi^+}(a_i) = C_{\tilde{h}}$ *and* $\tilde{C}_{\phi^-}(a_i) = C_{\tilde{l}}$. *Then*:

$$|h - \tilde{h}| \leq 1,$$
$$|l - \tilde{l}| \leq 1.$$

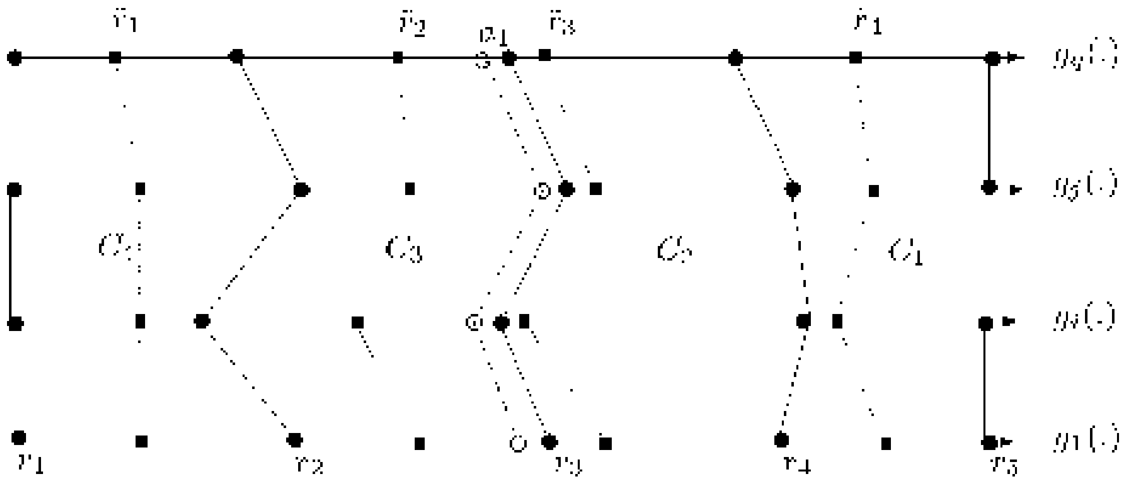

Fig. 3 Illustration of the relationship between the assignments with limiting profiles and centroids: case I

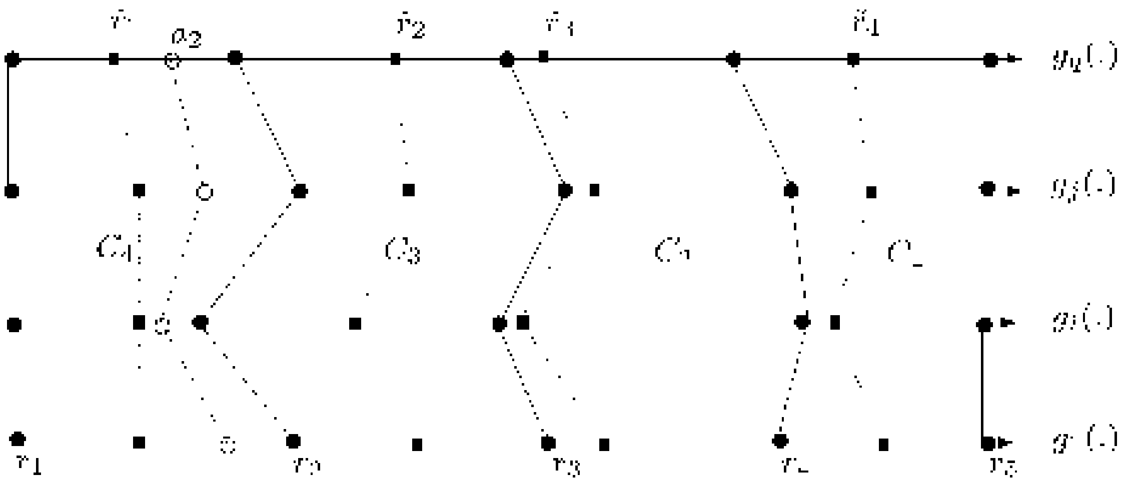

Fig. 4 Illustration of the relationship between the assignments with limiting profiles and centroids: case II

In other words, the difference in rank between the category assigned when working with strongly preferred limiting profiles or centroids (respecting the previous conditions) is at most one. Let us remark that this proposition does not necessarily hold when Conditions 7 and 8 are not satisfied. This result is also illustrated in Figs. 3 and 4, where the bold lines represent limiting profiles and the dotted lines centroids. In the Fig. 3, we can remark that action a_1 (thin line) is very close to r_3. It is obvious that $C_\phi(a_i) = C_2$, whereas $\tilde{C}_\phi(a_i) = C_3$. Hence the assignment differs by one category. In Fig. 4, we have that $C_\phi(a_2) = \tilde{C}_\phi(a_2) = C_4$.

3.4 Illustrative example

In order to illustrate the assignment rules in the case with limiting profiles, let us consider the following example. Suppose that we have 4 categories defined by 5 limiting profiles. Each profile is evaluated on the basis of 4 quantitative criteria which have to be maximized. The corresponding performances are given in Table 1 and illustrated in Fig. 5. To compute the preference degree we have used the Promethee methodol-

Table 1 The performances of the reference profiles

$\mathcal{R}$	g_1	g_2	g_3	g_4	g_5
r_1	100	100	100	100	100
r_2	75	75	75	75	75
r_3	50	50	50	50	50
r_4	25	25	25	25	25
r_5	0	0	0	0	0

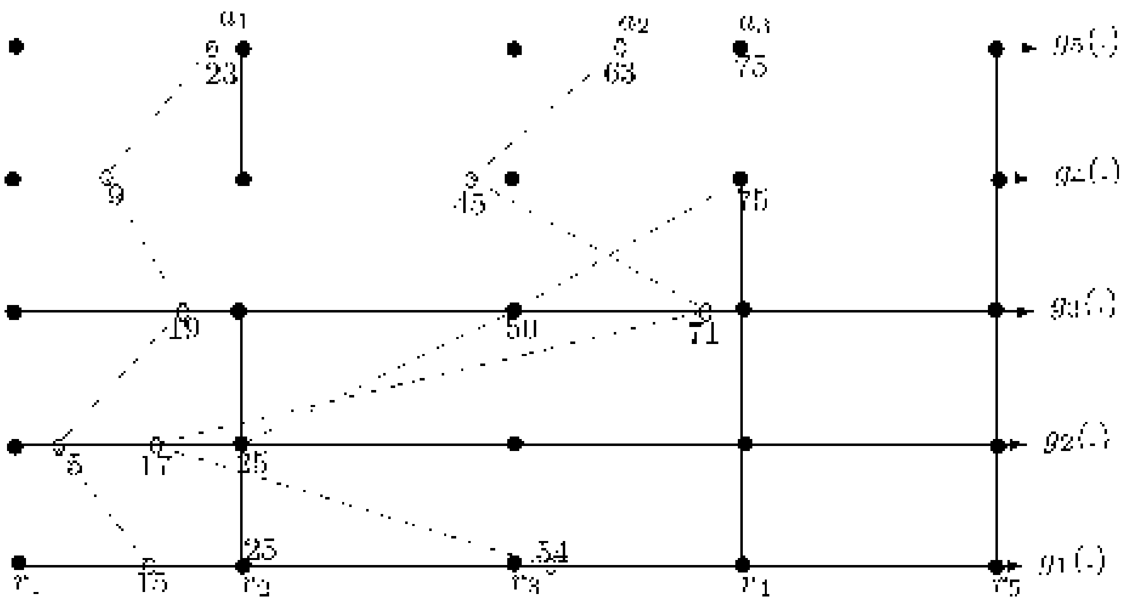

Fig. 5 Representation of the limiting profiles and the actions to be assigned

Table 2 The different thresholds and weights

	g_1	g_2	g_3	g_4	g_5
q_k	0	0	0	0	0
p_k	0	0	0	0	0
w_k	1	1	1	1	1

ogy. For each criterion, we have fixed the indifference and preference threshold equal to 0. Moreover, all criteria have the same weights and are given in Table 2.

Suppose now that we want to sort the following set of actions whose performances on the different criteria are as given in Table 3 and illustrated in Fig. 5. The preference degrees of a_1, a_2, and a_3 with respect to the reference profiles are given below (see Table 4). This allows us to compute the flows and determine their assignment (see Table 5).

We can conclude that action a_1 should be assigned to category 2 considering the positive flows and to category 3 considering the negative flows. In fact, action a_1 behaves analogously to reference profile r_3 in regards to the other profiles. Action a_2 can be unambiguously assigned to category 4. Moreover, action a_3 should be assigned

Table 3 The performances of the actions to be sorted

$\mathcal{A}$	g_1	g_2	g_3	g_4	g_5
a_1	54	17	71	45	63
a_2	15	5	19	9	23
a_3	25	25	50	75	75

Table 4 The preference degrees between the reference profiles and the actions

	r_1	r_2	r_3	r_4	r_5
$\pi(a_1, r_j)$	0	0	0.6	0.8	1
$\pi(r_j, a_1)$	1	1	0.4	0.2	0
$\pi(a_2, r_j)$	0	0	0	0	1
$\pi(r_j, a_2)$	1	1	1	1	0
$\pi(a_3, r_j)$	0	0	0.4	0.6	1
$\pi(r_j, a_3)$	1	0.6	0.4	0	0

Table 5 Computation of the different flow values

		r_1	r_2	r_3	r_4	r_5	a_i
$\mathcal{R}_1$	$\phi^+_{\mathcal{R}_1}$	1	0.8	0.48	0.24	0	0.48
	$\phi^-_{\mathcal{R}_1}$	0	0.2	0.52	0.76	1	0.52
	$\phi_{\mathcal{R}_1}$	1	0.6	−0.14	−0.48	−1	−0.14
$\mathcal{R}_2$	$\phi^+_{\mathcal{R}_2}$	1	0.8	0.6	0.4	0	0.2
	$\phi^-_{\mathcal{R}_2}$	0	0.2	0.4	0.6	1	0.8
	$\phi_{\mathcal{R}_2}$	1	0.6	0.2	−0.2	−1	−0.6
$\mathcal{R}_3$	$\phi^+_{\mathcal{R}_3}$	1	0.72	0.48	0.2	0	0.4
	$\phi^-_{\mathcal{R}_3}$	0	0.2	0.48	0.72	1	0.4
	$\phi_{\mathcal{R}_3}$	1	0.52	0	−0.52	−1	0

considering respectively the positive and negative flows to categories 2 and 3. The assignment of action a_3 is graphically represented in Fig. 6.

Let us note that, considering the same parameters, Electre-Tri assigns action a_1 according to the optimistic and pessimistic procedures to category 2 with λ-threshold equal to 0.5. With $\lambda = 0.85$, the optimistic and pessimistic procedures give respectively category 2 and 3. Moreover, Electre-Tri assigns unambiguously action a_2 to category 4. Finally, if $\lambda > 0.6$, the optimistic and pessimistic procedures assign a_3 respectively to category 2 and 3, and if $\lambda \leq 0.6$, unambiguously to category 3. All the results are summarized in Table 6.

Let us now consider the case where the 4 categories are defined by central profiles. We assign the same actions as in the previous section. The performances of the central profiles are given in Table 7. The preference thresholds and the weights are given in Table 8.

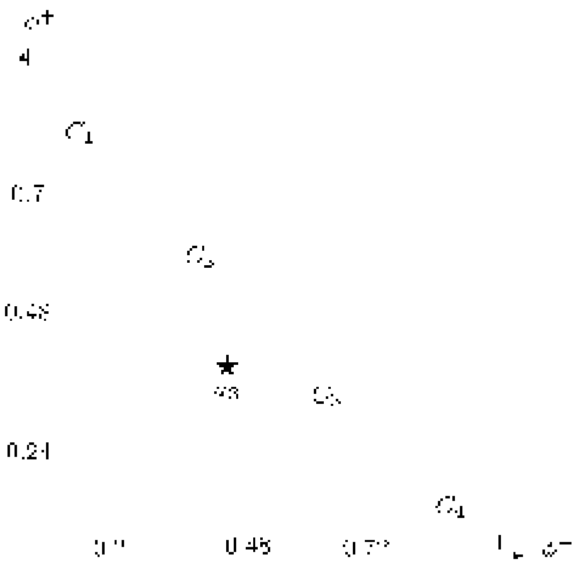

Fig. 6 Flow-diagram for a_3

Table 6 The assignments of the actions

a_i	λ	C_{pess}	C_{opt}	$C_{\phi+}$	$C_{\phi-}$	C_{ϕ}
a_1	[0.5; 0.6]	2	2			
	]0.6; 0.8]	3	2	2	3	3
	]0.8; 1]	4	2			
a_2	[0.5; 1]	4	4	4	4	4
a_3	[0.5; 0.6]	3	3	2	3	3
	]0.6; 1]	3	2			

Table 7 The performances of the reference profiles

$\tilde{\mathcal{R}}$	g_1	g_2	g_3	g_4	g_5
$\tilde{r}_1$	80	90	90	75	82
$\tilde{r}_2$	56	62	58	61	52
$\tilde{r}_3$	40	35	35	30	33
$\tilde{r}_4$	10	5	10	10	15

Table 8 The different thresholds and weights

	g_1	g_2	g_3	g_4	g_5
q_k	4	6	0	2	3
p_k	8	13	7	5	12
w_k	1	1	1	1	1

The preferences degrees of a_1, a_2, and a_3 with respect to the reference profiles are given in Table 9. This allows us to compute the flows which determine their assignment (see Table 10).

Hence, a_1 is assigned to C_2 in all three assignment rules. Action a_2 is unambiguously assigned to C_4, and action a_3 to C_2.

Table 9 The preference degrees between the reference profiles and the alternatives

	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$
$\pi(a_1, \tilde{r}_j)$	0	0.38	0.8	0.97
$\pi(\tilde{r}_j, a_1)$	1	0.4	0.2	0
$\pi(a_2, \tilde{r}_j)$	0	0	0	0.36
$\pi(\tilde{r}_j, a_2)$	1	1	0.95	0
$\pi(a_3, \tilde{r}_j)$	0	0.4	0.6	1
$\pi(\tilde{r}_j, a_3)$	0.69	0.6	0.31	0

Table 10 The flow-values of the alternatives

		$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$	a_i
$\tilde{\mathcal{R}}_1$	$\phi^+_{\tilde{\mathcal{R}}_1}$	1	0.6	0.3	0	0.54
	$\phi^-_{\tilde{\mathcal{R}}_1}$	0	0.34	0.7	0.99	0.4
	$\phi_{\tilde{\mathcal{R}}_1}$	1.0	0.25	−0.4	−0.99	0.14
$\tilde{\mathcal{R}}_2$	$\phi^+_{\tilde{\mathcal{R}}_2}$	1	0.75	0.49	0	0.09
	$\phi^-_{\tilde{\mathcal{R}}_2}$	0	0.25	0.5	0.84	0.74
	$\phi_{\tilde{\mathcal{R}}_2}$	1	0.5	−0.01	−0.84	−0.65
$\tilde{\mathcal{R}}_3$	$\phi^+_{\tilde{\mathcal{R}}_3}$	0.92	0.65	0.33	0	0.5
	$\phi^-_{\tilde{\mathcal{R}}_3}$	0	0.35	0.65	1	0.40
	$\phi_{\tilde{\mathcal{R}}_3}$	0.92	0.3	−0.32	−1	0.1

4 Empirical comparison with Electre-Tri

In order to analyze the assignments given by $\mathcal{F}$low$\mathcal{S}$ort, we compare the results to an existing sorting method, namely Electre-Tri (Yu 1992). As in the $\mathcal{F}$low$\mathcal{S}$ort method, Electre-Tri also uses limiting profiles and proposes two different assignment rules, an optimistic and pessimistic one. The simulations show us to what extent the assignments found by $\mathcal{F}$low$\mathcal{S}$ort are similar (or not) to the ones obtained with Electre-Tri. We assume that the preference degrees in the simulations have been obtained, as in PROMETHEE, using linear preference functions.

For this purpose, we have generated randomly a set of reference profiles, either verifying 7 or not, and an action to assign as follows. First, we compute a $(K+1)\times q$-dimensional performance matrix of the $K+1$ reference profiles evaluated on the q criteria. This performance matrix is obtained by sorting each column of a random matrix generated with a uniform distribution for each element. To obtain reference profiles verifying 7, the preference threshold for each criterion is chosen as the smallest difference between any two consecutive evaluations on that criterion. Alternatively, in order to generate nonstrongly preferred reference profiles, the preference threshold is chosen as the largest difference between two consecutive evaluations. In both situations, the indifference threshold is sampled from a uniform distribution between

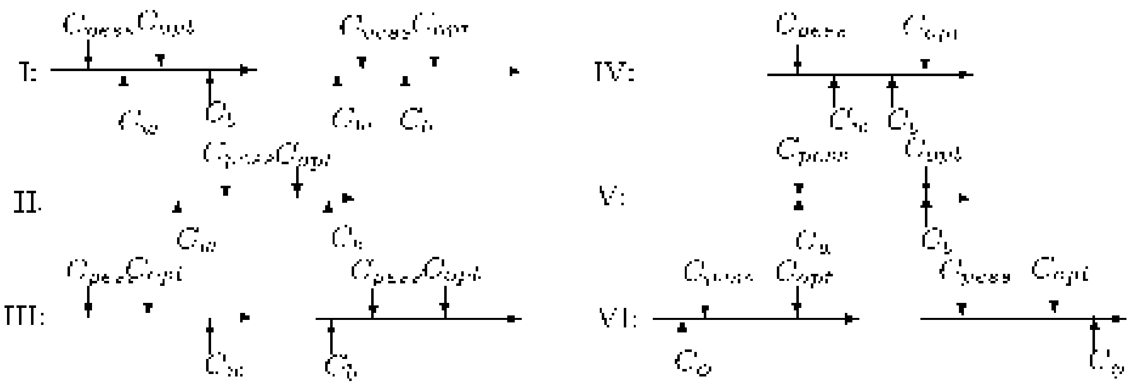

Fig. 7 Comparison of $\mathcal{F}$low$\mathcal{S}$ort and Electre-Tri: different scenarios

0 and the chosen preference threshold. Finally, q weights are generated uniformly on [0, 1], as well as the q evaluations of the action to assign. The same weights and thresholds are used in the Electre-Tri model. Furthermore, the λ-threshold is fixed at 0.85 and no veto has been used.

When comparing the assignments of Electre-Tri optimistic and pessimistic (respectively noted as C_{opt} and C_{pess}) with C_b and C_w, we distinguish 5 different situations (see also Fig. 7):

1. $C_b \rhd C_{\text{opt}} \unrhd C_w \unrhd C_{\text{pess}}$ or $C_{\text{opt}} \unrhd C_b \unrhd C_{\text{pess}} \rhd C_w$
2. $C_b \rhd C_{\text{opt}} \unrhd C_{\text{pess}} \rhd C_w$
3. $C_b \unrhd C_w \rhd C_{\text{opt}} \unrhd C_{\text{pess}}$ or $C_{\text{opt}} \unrhd C_{\text{pess}} \rhd C_b \unrhd C_w$
4. $C_{\text{opt}} \rhd C_b \unrhd C_w \rhd C_{\text{pess}}$
5. $C_{\text{opt}} = C_b \unrhd C_w = C_{\text{pess}}$

Finally, another point of interest is to count the cases where the unique assignment C_ϕ is inconsistent with the assignments of Electre-Tri:

6. $C_\phi \rhd C_{\text{opt}} \unrhd C_{\text{pess}}$ or $C_{\text{opt}} \unrhd C_{\text{pess}} \rhd C_\phi$

For simulations, we have fixed successively the number of categories K at 2, 4, 6, 8, 10 and the number of criteria at 3, 6, 9. For each case, we have generated 1000 different reference data sets. The results are given in Tables 11, 12, and 13.

First we analyze Table 11. The assignments given by $\mathcal{F}low\mathcal{S}ort$ can be seen as completely consistent with Electre-Tri in situations 4 and 5. These two situations occur in a large majority of cases. In general, these frequencies rise when the number of categories increases. Moreover, we can also notice that situation 6 does not happen very often, which means that the net-flow assignment procedure usually gives results compatible with Electre-Tri. Finally, when imposing Condition 7 (strongly preferred limiting profiles), situations 4 and 5 happen even more often and situation 6 even less often. This can be explained by the fact that Condition 7 ensures that classes are "well defined," which may lead to less ambiguous assignments.

Concerning Tables 12 and 13, when the number of classes is increasing, we can notice that the assignment-interval is becoming larger. Nevertheless, we can conclude that the assignment-interval is significantly smaller in $\mathcal{F}$low$\mathcal{S}$ort than in Electre-Tri. This can be explained by the fact that in a large majority the incoming-flow assignment coincides with the leaving-flow assignment. Finally, the last two columns in

Table 11 Comparison between $\mathcal{F}$low$\mathcal{S}$ort and Electre-Tri in the case of limiting profiles

K	q	Preferred limiting profiles (%)						Strongly preferred limiting profiles (%)					
		Sit-1	Sit-2	Sit-3	Sit-4	Sit-5	Sit-6	Sit-1	Sit-2	Sit-3	Sit-4	Sit-5	Sit-6
2	3	5	0	0.4	29.7	64.9	2.2	5	0.1	1.8	28.6	64.5	4
	6	2.8	0	0	49.6	47.6	1.1	2.8	0	0.4	44.8	52	1.1
	9	1.7	0	0	62.3	36	0.5	1.4	0	0	50	48.6	0.4
4	3	6.9	0	1.4	55.1	36.6	3.8	0.6	0	0.6	71.7	27.1	0.9
	6	1.7	0	0.2	83.2	14.9	0.9	0.3	0	0	90.1	9.6	0
	9	0.3	0	0	90	9.7	0	0	0	0	94.3	5.7	0
6	3	5.6	0.1	1.3	67.4	25.6	3.7	0.2	0	0.4	82.9	16.5	0.6
	6	1	0	0.1	91.4	7.5	0.5	0	0	0	96.8	3.2	0
	9	0.5	0	0	96.8	2.7	0	0	0	0	99.1	0.9	0
8	3	5.4	0.2	1	74	19, 4	2.2	0.2	0	0.7	87.6	11.5	0.7
	6	0.8	0	0	95.3	3.9	0.1	0	0	0	99	1	0
	9	0.1	0	0	99	0.9	0	0	0	0	100	0	0
10	3	3.7	0	1.8	82.3	12.2	3.3	0.4	0	0.6	89.2	9.8	0.7
	6	0.9	0	0.3	96, 3	2.5	0.5	0.1	0	0	99.4	0.5	0
	9	0.1	0	0	99.4	0.5	0	0	0	0	99.8	0.2	0

Table 12 Comparison between $\mathcal{F}$low$\mathcal{S}$ort and Electre-Tri in the case of limiting profiles: analysis of the assignments

K	q	Preferred limiting profiles				
		Mean of $C_{\text{opt}} - C_{\text{pess}}$	Mean of $\lvert C_{\phi^+} - C_{\phi^-}\rvert$	$C_{\phi^+} = C_{\phi^-}$ (%)	$\lvert C_{\phi^+} - C_{\phi}\rvert <$ $\lvert C_{\phi} - C_{\phi^-}\rvert$(%)	$\lvert C_{\phi^+} - C_{\phi}\rvert >$ $\lvert C_{\phi} - C_{\phi^-}\rvert$(%)
2	3	0.366	0.119	88.1	5.2	6.7
	6	0.608	0.14	86	5.9	8.1
	9	0.735	0.129	87.1	4.7	8.2
4	3	1.097	0.283	72.5	15.9	11.6
	6	1.714	0.234	77.4	11.1	11.5
	9	1.86	0.23	77.3	11.2	11.5
6	3	1.81	0.312	71.3	14	14.7
	6	2.719	0.288	72.8	13.9	13.3
	9	2.948	0.242	76.4	11.8	11.8
8	3	2.582	0.352	67.7	16.5	15.8
	6	3.896	0.304	71.4	15.1	13.5
	9	4.289	0.294	71	14.4	14.6
10	3	3.422	0.356	67.6	17.5	14.9
	6	4.89	0.319	69.2	16.2	14.6
	9	5.365	0.275	73.1	13.4	13.5

Table 13 Comparison between $\mathcal{F}$low$\mathcal{S}$ort and Electre-Tri in the case of limiting profiles: analysis of the assignments

K	q	Strongly preferred limiting profiles				
		Mean of $C_{\text{opt}} - C_{\text{pess}}$	Mean of $\lvert C_{\phi^+} - C_{\phi^-}\rvert$	$C_{\phi^+} = C_{\phi^-}$ (%)	$\lvert C_{\phi^+} - C_\phi\rvert <$ $\lvert C_\phi - C_{\phi^-}\rvert$(%)	$\lvert C_{\phi^+} - C_\phi\rvert >$ $\lvert C_\phi - C_{\phi^-}\rvert$(%)
2	3	0.373	0.146	85.4	8.3	6.3
	6	0.628	0.208	79.2	10.2	10.6
	9	0.741	0.255	74.5	13.8	11.7
4	3	1.338	0.124	87.6	6	6.4
	6	1.93	0.153	84.7	7.7	7.6
	9	2.13	0.191	80.9	9.1	10
6	3	2.195	0.119	88.1	6	5.9
	6	3.159	0.162	83.8	7.3	8.9
	9	3.448	0.149	85.1	8.3	6.6
8	3	3.061	0.108	89.2	5.4	5.4
	6	4.348	0.127	87.3	6.3	6.4
	9	4.779	0.124	87.6	6.2	6.2
10	3	3.906	0.1	90	5.2	4.8
	6	5.516	0.095	90.5	4.8	4.7
	9	5.937	0.098	90.2	6.1	3.7

Tables 12 and 13, which describe symmetric situations, are almost equal as the data is randomly generated.

It is well known that Electre-Tri can be used with vetoes. This aspect has not been considered in our simulations as vetoes cannot be easily taken into account in the $\mathcal{F}$low$\mathcal{S}$ort methodology. Nevertheless, working with vetoes in Electre-Tri will in general further increase the difference $C_{\text{opt}} - C_{\text{pess}}$.

5 Conclusions

In this paper, we developed a new multicriteria sorting method inspired by the PROMETHEE methodology. We analyzed the cases where the ordered categories are defined either by limiting profiles or by central profiles. Furthermore, we showed that the results obtained by the two approaches are closely linked.

We also presented some empirical comparisons with Electre-Tri. Although performing empirical comparisons between two multicriteria approaches can always be criticized, we concluded that, in our context, the assignments between both approaches are consistent. However, it seems that the difference between the categories obtained with the positive and negative flows is smaller than the difference between the categories obtained with the optimistic and pessimistic Electre-Tri rules. In order to distinguish more fundamentally $\mathcal{FlowSort}$ from other existing sorting methods, some further theoretical investigation needs to be done. More particularly, a future direction of research will be to axiomatically characterize $\mathcal{F}low\mathcal{S}ort$.

It seems natural to extend the model when there is possibly more than one central profile per category. $\mathcal{F}low\mathcal{S}ort$ needs to be adapted to handle this kind of situations. More generally, some effort should be spend on facilitating the determination of the parameters of the model such as, for instance, the reference profiles or the weights. Finally, it should be interesting to analyze the cases where there are some uncertainties in the parameters like, for example, in the thresholds, in the weights, and in the evaluation of the profiles or the actions.

Appendix

Proof of Proposition 1 Condition 1 requires that the reference profiles dominate each other:

$$g_j(r_h) \geq g_j(r_{h+1}) \quad \forall j \in \{1, \ldots, q\},\ \forall h \in \left\{1, \ldots, |\mathcal{R}| - 1\right\}.$$

Combining this with Condition 5, we obtain that:

$$\forall r_j \in \mathcal{R}, \quad \pi(r_h, r_j) \geq \pi(r_{h+1}, r_j).$$

Condition 6 requires that $\pi(r_h, r_{h+1}) > 0$, whereas Condition 4 tells us that $\pi(r_{h+1}, r_{h+1}) = 0$. That is why we can conclude that:

$$\pi(r_h, r_{h+1}) > \pi(r_{h+1}, r_{h+1}).$$

These last two observations tell us that:

$$\phi^-_{\mathcal{R}}(r_h) = \sum_{r_j \in \mathcal{R}} \pi(r_h, r_j) > \sum_{r_j \in \mathcal{R}} \pi(r_{h+1}, r_j) = \phi^-_{\mathcal{R}}(r_{h+1}).$$

This completes the first part of the proof.

In a similar way, the fact that the reference profiles dominate each other, i.e.,

$$g_j(r_h) \geq g_j(r_{h+1}) \quad \forall j \in \{1, \ldots, q\},\ \forall h \in \left\{1, \ldots, |\mathcal{R}| - 1\right\},$$

combined with Condition 5 implies that:

$$\forall a_i \in \mathcal{A}, \quad \pi(r_h, a_i) \geq \pi(r_{h+1}, a_i).$$

Since

- $\phi^-_{\mathcal{R}_i}(r_h) = \phi^-_R(r_h) + \pi(r_h, a_i)$,
- $\phi^-_{\mathcal{R}_i}(r_{h+1}) = \phi^-_R(r_{h+1}) + \pi(r_{h+1}, a_i)$,
- $\phi^-_{\mathcal{R}}(r_h) > \phi^-_{\mathcal{R}}(r_{h+1})$ (see the first part of the proof),

we can conclude that:

$$\phi^-_{\mathcal{R}_i}(r_h) > \phi^-_{\mathcal{R}_i}(r_{h+1}).$$

This completes the second part of the proof. The proofs for the incoming and net flows are similar. □

Proof of Proposition 2 We suppose that $C_{\phi^-}(a_i) = C_k$ and $C_{\phi^+}(a_i) = C_l$, which, by construction of the assignment rules, implies that:

$$\phi^-_{\mathcal{R}_i}(r_k) \leq \phi^-_{\mathcal{R}_i}(a_i) < \phi^-_{\mathcal{R}_i}(r_{k+1}), \tag{4}$$

$$\phi^+_{\mathcal{R}_i}(r_l) \geq \phi^+_{\mathcal{R}_i}(a_i) > \phi^+_{\mathcal{R}_i}(r_{l+1}). \tag{5}$$

By subtracting 4 from 5 we obtain:

$$\phi^+_{\mathcal{R}_i}(r_l) - \phi^-_{\mathcal{R}_i}(r_k) \geq \phi^+_{\mathcal{R}_i}(a_i) - \phi^-_{\mathcal{R}_i}(a_i) > \phi^+_{\mathcal{R}_i}(r_{l+1}) - \phi^-_{\mathcal{R}_i}(r_{k+1}). \tag{6}$$

If $k = l$: $\phi^+_{\mathcal{R}_i}(r_k) - \phi^-_{\mathcal{R}_i}(r_k) \geq \phi^+_{\mathcal{R}_i}(a_i) - \phi^-_{\mathcal{R}_i}(a_i) > \phi^+_{\mathcal{R}_i}(r_{k+1}) - \phi^-_{\mathcal{R}_i}(r_{k+1})$. In this case, the net flow rule assigns alternative a_i to class C_k:

$$C_k = C_l = C_\phi(a_i).$$

If $k < l$: From Proposition 1 we know that:

$$\phi^+_{\mathcal{R}_i}(r_k) > \phi^+_{\mathcal{R}_i}(r_l), \tag{7}$$

$$\phi^-_{\mathcal{R}_i}(r_{k+1}) > \phi^-_{\mathcal{R}_i}(r_{l+1}). \tag{8}$$

Combining (6) with (7) and (8), we obtain that:

$$\phi^+_{\mathcal{R}_i}(r_k) - \phi^-_{\mathcal{R}_i}(r_k) \geq \phi^+_{\mathcal{R}_i}(a_i) - \phi^-_{\mathcal{R}_i}(a_i) > \phi^+_{\mathcal{R}_i}(r_{l+1}) - \phi^-_{\mathcal{R}_i}(r_{l+1}).$$

In terms of flows, this is equivalent to:

$$\phi_{\mathcal{R}_i}(r_k) \geq \phi_{\mathcal{R}_i}(a_i) > \phi_{\mathcal{R}_i}(r_{l+1}).$$

Consequently, the net flow assignment must lie between the classes C_k and C_l:

$$C_k \trianglerighteq C_\phi(a_i) \trianglerighteq C_l.$$

If $k > l$: Similar to the case $k < l$. □

Proof of Proposition 3 Let us suppose that a_j is assigned to category C_h using the outgoing flows. In such a case, by construction of the assignment rule, we must have that:

$$\phi^+_{\mathcal{R}_j}(a_j) > \phi^+_{\mathcal{R}_j}(r_{h+1}). \tag{9}$$

Since

$$\forall k \in \{1, \ldots, q\}: \quad g_k(a_i) \geq g_k(a_j),$$

Condition 5 ensures us that

$$\pi(r_{h+1}, a_j) \geq \pi(r_{h+1}, a_i).$$

That is why we have that:

$$\big(|\mathcal{R}| - 1\big) \cdot \phi^+_{\mathcal{R}}(r_{h+1}) + \pi(r_{h+1}, a_j) \geq (|\mathcal{R}| - 1) \cdot \phi^+_{\mathcal{R}}(r_{h+1}) + \pi(r_{h+1}, a_i).$$

This implies that:

$$\phi^+_{\mathcal{R}_j}(r_{h+1}) \geq \phi^+_{\mathcal{R}_i}(r_{h+1}). \tag{10}$$

Similarly, since

$$\forall k \in \{1, \dots, q\}: \quad g_k(a_i) \geq g_k(a_j),$$

Condition 5 ensures us that

$$\forall r \in \mathcal{R}, \quad \pi(a_i, r) \geq \pi(a_j, r).$$

By Condition 4 we have that $\pi(a_i, a_i) = 0$ and $\pi(a_j, a_j) = 0$ and thus we can conclude that:

$$\sum_{r \in \mathcal{R}_i} \pi(a_i, r) \geq \sum_{r \in \mathcal{R}_j} \pi(a_j, r).$$

In terms of flows, this means that:

$$\phi^+_{\mathcal{R}_i}(a_i) \geq \phi^+_{\mathcal{R}_j}(a_j). \tag{11}$$

From (9), (10), and (11) we can conclude that:

$$\phi^+_{\mathcal{R}_i}(a_i) \geq \phi^+_{\mathcal{R}_j}(a_j) > \phi^+_{\mathcal{R}_j}(r_{h+1}) \geq \phi^+_{\mathcal{R}_i}(r_{h+1}).$$

Hence, we have that:

$$C_{\phi^+}(a_i) \trianglerighteq C_h = C_{\phi^+}(a_j).$$

This proves the proposition considering the outgoing flow assignment. The proof for the incoming flow assignment is analogous. □

Proof of Proposition 4 Since the reference profiles respect Condition 7, we have the following nonnormalized flow values:

- $(|\mathcal{R}_i| - 1) \cdot \phi^+_{\mathcal{R}_i}(r_h) = K + 1 - h + \pi(r_h, a_i)$,
- $(|\mathcal{R}_i| - 1) \cdot \phi^-_{\mathcal{R}_i}(r_l) = l - 1 + \pi(a_i, r_l)$.

Since we suppose that $C_{\phi^+}(a_i) = C_h$ and $C_{\phi^-}(a_i) = C_l$, the definition of the assignment rules imply that:

1. $(|\mathcal{R}_i| - 1) \cdot \phi^+(r_{h+1}) = K + 1 - (h + 1) + \pi(r_{h+1}, a_i) < (|\mathcal{R}_i| - 1) \cdot \phi^+_{\mathcal{R}_i}(a_i)$,
2. $(|\mathcal{R}_i| - 1) \cdot \phi^-(r_l) = l - 1 + \pi(a_i, r_l) \leq (|\mathcal{R}_i| - 1) \cdot \phi^-_{\mathcal{R}_i}(a_i)$.

Adding these two inequalities together, we obtain:

$$K+1-(h+1)+\pi(r_{h+1},a_i)+l-1+\pi(a_i,r_l)<\big(|\mathcal{R}_i|-1\big)\cdot\big(\phi^+_{\mathcal{R}_i}(a_i)+\phi^-_{\mathcal{R}_i}(a_i)\big).$$

Since

$$\big(|\mathcal{R}_i|-1\big)\cdot\phi^+_{\mathcal{R}_i}(a_i)=\sum_{r\in\mathcal{R}_i}\pi(a_i,r)$$

and

$$\big(|\mathcal{R}_i|-1\big)\cdot\phi^-_{\mathcal{R}_i}(a_i)=\sum_{r\in\mathcal{R}_i}\pi(r,a_i),$$

this can be rewritten as follows:

$$K+l-h+1+\pi(r_{h+1},a_i)+\pi(a_i,r_l)<\sum_{r\in\mathcal{R}_i}\big(\pi(a_i,r)+\pi(r,a_i)\big). \tag{12}$$

Moreover, Condition 3 tells us that

$$\forall a_i\in\mathcal{A},\ \forall r\in\mathcal{R},\quad \pi(a_i,r)+\pi(r,a_i)\le 1.$$

Since Condition 4 requires that $\pi(a_i,a_i)+\pi(a_i,a_i)=0$, we can conclude that:

$$\forall a_i\in\mathcal{A},\quad \sum_{r\in\mathcal{R}_i}(\pi(a_i,r)+\pi(r,a_i))\le(|\mathcal{R}_i|-1)=K.$$

Combining this observation with (12), we obtain that:

$$K+l-h-1+\pi(r_{h+1},a_i)+\pi(a_i,r_l)<K.$$

Since $\pi(r_{h+1},a_i)\ge 0$ and $\pi(a_i,r_l)\ge 0$, we have that:

$$K+l-h-1+<K.$$

In other words:

$$l-h<1.$$

Since l and h are integers, this implies that $l-h\le 0$. □

Proof of Proposition 5 We recall that $\mathcal{R}=\{r_1,\dots,r_{K+1}\}$ denotes the set of limiting profiles, whereas $\tilde{\mathcal{R}}=\{\tilde{r}_1,\dots,\tilde{r}_{K+1}\}$ denotes the set of central profiles. The proof of Proposition 5 is based on the following lemma:

Lemma 1

$$\big(|\mathcal{R}_i|-1\big)\cdot\phi^+_{\mathcal{R}_i}(a_i)\ge\big(|\tilde{\mathcal{R}}_i|-1\big)\cdot\phi^+_{\tilde{\mathcal{R}}_i}(a_i)\ge\big(|\mathcal{R}_i|-1\big)\cdot\phi^+_{\mathcal{R}_i}(a_i)-\pi(a_i,r_{K+1}).$$

Proof of Lemma 1 We assume that:

$$\forall h \in \{1, \ldots, K\},\ \forall j \in \{1, \ldots, q\}, \quad g_j(r_h) \geq g_j(\tilde{r}_h) \geq g_j(r_{h+1}).$$

Condition 6 requests that:

$$\forall a_i \in \mathcal{A}, \forall h \in \{1, \ldots, K\}, \quad \pi(a_i, r_{h+1}) \geq \pi(a_i, \tilde{r}_h) \geq \pi(a_i, r_h).$$

This implies that:

$$\sum_{h=1}^{K} \pi(a_i, r_{h+1}) \geq \sum_{h=1}^{K} \pi(a_i, \tilde{r}_h) \geq \sum_{h=1}^{K} \pi(a_i, r_h).$$

In terms of flows, this can be rewritten as follows:

$$\begin{aligned} &\big(|\mathcal{R}_i| - 1\big) \cdot \phi^+_{\mathcal{R}_i}(a_i) - \pi(a_i, r_1) \\ &\quad \geq \big(|\tilde{\mathcal{R}}_i| - 1\big) \cdot \phi^+_{\tilde{\mathcal{R}}_i}(a_i) \geq \big(|\mathcal{R}_i| - 1\big) \cdot \phi^+_{\mathcal{R}_i}(a_i) - \pi(a_i, r_{K+1}). \end{aligned}$$

Since we assumed that $\pi(a_i, r_1) = 0$, we must have that:

$$\big(|\mathcal{R}_i| - 1\big) \cdot \phi^+_{\mathcal{R}_i}(a_i) \geq \big(|\tilde{\mathcal{R}}_i| - 1\big) \cdot \phi^+_{\tilde{\mathcal{R}}_i}(a_i) \geq \big(|\mathcal{R}_i| - 1\big) \cdot \phi^+_{\mathcal{R}_i}(a_i) - \pi(a_i, r_{K+1}).$$

This completes the proof of the lemma. $\square$

We recall that the assignment rule in the limit case is such that

$$C_{\phi^+}(a_i) = C_h \quad \text{if } \phi^+_{\mathcal{R}_i}(r_{h+1}) < \phi^+_{\mathcal{R}_i}(a_i) \leq \phi^+_{\mathcal{R}_i}(r_h).$$

The assignment rule in the centroid case is such that

$$\tilde{C}_{\phi^+}(a_i) = C_{\tilde{h}} \quad \text{if } \frac{\phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{\tilde{h}}) + \phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{\tilde{h}+1})}{2} < \phi^+_{\tilde{\mathcal{R}}_i}(a_i) \leq \frac{\phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{\tilde{h}}) + \phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{\tilde{h}-1})}{2}.$$

Since the reference profiles respect Condition 7, we have the following nonnormalized flow values:

$$\text{LF:} \quad \forall h \in \{1, \ldots, K+1\}, \quad (|\mathcal{R}_i| - 1) \cdot \phi^+_{\mathcal{R}_i}(r_h) = K + 1 - h + \pi(r_h, a_i),$$

$$\text{CF:} \quad \forall h \in \{1, \ldots, K\}, \quad (|\tilde{\mathcal{R}}_i| - 1) \cdot \phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_h) = K - h + \pi(\tilde{r}_h, a_i).$$

We are going to show that (1) $\tilde{h} \geq h - 1$ and that (2) $\tilde{h} \leq h + 1$, which will complete the proof. We want to show that $\tilde{h} \geq h - 1$. If $h \leq 2$, then it is trivial that $\tilde{h} \geq h - 1$. Let us from now on suppose that $h \geq 3$. We assume that:

$$\forall h \in \{3, \ldots, K\}, \forall j \in \{1, \ldots, q\}, \quad g_j(\tilde{r}_{h-2}) \geq g_j(\tilde{r}_{h-1}) \geq g_j(r_h).$$

Condition 5 requests that:

$$\forall a_i \in \mathcal{A}, \forall h \in \{3, \ldots, K\}, \quad \pi(r_h, a_i) \geq \pi(\tilde{r}_{h-1}, a_i) \geq \pi(\tilde{r}_{h-2}, a_i). \tag{13}$$

Given LF, (13), and the fact that $\pi(\tilde{r}_{h-1}, a_i) \leq 1$, we obtain that:

$$\begin{aligned}(|\mathcal{R}_i| - 1) \cdot \phi^+_{\mathcal{R}_i}(r_h) &= K + 1 - h + \pi(r_h, a_i)\\ &\leq K + \frac{3}{2} - h + \frac{\pi(\tilde{r}_{h-1}, a_i) + \pi(\tilde{r}_{h-2}, a_i)}{2}. \qquad (14)\end{aligned}$$

Using CF, the reader can check that

$$\begin{aligned}K + \frac{3}{2} - h + \frac{\pi(\tilde{r}_{h-1}, a_i) + \pi(\tilde{r}_{h-2}, a_i)}{2}\\ = \frac{(|\tilde{\mathcal{R}}_i| - 1) \cdot \phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h-1}) + (|\tilde{\mathcal{R}}_i| - 1) \cdot \phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h-2})}{2}. \qquad (15)\end{aligned}$$

Combining (14) and (15), we obtain that:

$$(|\mathcal{R}_i| - 1) \cdot \phi^+_{\mathcal{R}_i}(r_h) \leq \frac{(|\tilde{\mathcal{R}}_i| - 1) \cdot \phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h-1}) + (|\tilde{\mathcal{R}}_i| - 1) \cdot \phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h-2})}{2}. \qquad (16)$$

Given the assignment rule in the limit case, we know that

$$\phi^+_{\mathcal{R}_i}(a_i) \leq \phi^+_{\mathcal{R}_i}(r_h),$$

which implies that

$$(|\mathcal{R}_i| - 1) \cdot \phi^+_{\mathcal{R}_i}(a_i) \leq (|\mathcal{R}_i| - 1) \cdot \phi^+_{\mathcal{R}_i}(r_h).$$

Given Lemma 1, we know that:

$$(|\tilde{\mathcal{R}}_i| - 1) \cdot \phi^+_{\tilde{\mathcal{R}}_i}(a_i) \leq (|\mathcal{R}_i| - 1) \cdot \phi^+_{\mathcal{R}_i}(a_i).$$

Combining the two last observations with (15), we finally obtain that:

$$\phi^+_{\tilde{\mathcal{R}}_i}(a_i) \leq \frac{\phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h-1}) + \phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h-2})}{2}.$$

This means that $\tilde{C}_{h-1} \trianglerighteq \tilde{C}_{\phi^+}(a_i)$, or, in other words, that $\tilde{h} \geq h - 1$. This concludes the first part of the proof.

We want to show that $\tilde{h} \leq h + 1$. If $h \geq K - 1$, then it is trivial that $\tilde{h} \leq h + 1$. Let us from now on suppose that $h \leq K - 2$. We assume that:

$$\forall h \in \{1, \ldots, K-2\}, \forall j \in \{1, \ldots, q\}, \quad g_j(r_h) \geq g_j(\tilde{r}_{h+1}) \geq g_j(\tilde{r}_{h+2}).$$

Condition (5) requests that:

$$\forall a_i \in \mathcal{A}, \forall h \in \{1, \ldots, K-2\}, \quad \pi(r_h, a_i) \geq \pi(\tilde{r}_{h+1}, a_i) \geq \pi(\tilde{r}_{h+2}, a_i). \qquad (17)$$

Combining LF with (17), we obtain that:

$$\begin{aligned}&\big(|\mathcal{R}_i|-1\big)\cdot\phi^+_{\mathcal{R}_i}(r_{h+1})-\pi(a_i,r_{K+1})\\&\quad\geq K-h-\frac{3}{2}+\frac{\pi(\tilde{r}_{h+1},a_i)+\pi(\tilde{r}_{h+2},a_i)}{2}.\end{aligned}\tag{18}$$

Using CF, the reader can check that:

$$\begin{aligned}&K-h-\frac{3}{2}+\frac{\pi(\tilde{r}_{h+1},a_i)+\pi(\tilde{r}_{h+2},a_i)}{2}\\&\quad=\frac{(|\tilde{\mathcal{R}}_i|-1)\cdot\phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h+1})+(|\tilde{\mathcal{R}}_i|-1)\phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h+2})}{2}.\end{aligned}\tag{19}$$

Combining (18) and (19), we obtain that:

$$\begin{aligned}&\big(|\mathcal{R}_i|-1\big)\cdot\phi^+_{\mathcal{R}_i}(r_{h+1})-\pi(a_i,r_{K+1})\\&\quad\geq\frac{(|\tilde{\mathcal{R}}_i|-1)\cdot\phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h+1})+(|\tilde{\mathcal{R}}_i|-1)\phi^+_{\tilde{\mathcal{R}}_i}(\tilde{r}_{h+2})}{2}.\end{aligned}\tag{20}$$

Given the assignment rule in the limit case, we know that

$$\big(|\mathcal{R}_i|-1\big)\cdot\phi^+_{\mathcal{R}_i}(a_i)>\big(|\mathcal{R}_i|-1\big)\cdot\phi^+_{\mathcal{R}_i}(r_{h+1}).$$

Given Lemma 1, we know that:

$$\big(|\tilde{\mathcal{R}}_i|-1\big)\cdot\phi^+_{\tilde{\mathcal{R}}_i}(a_i)\geq\big(|\mathcal{R}_i|-1\big)\cdot\phi^+_{\mathcal{R}_i}(a_i)-\pi(a_i,r_{K+1}).$$

Combining these two observations with 20, we obtain that:

$$\phi^+_{\tilde{\mathcal{R}}_i}(a_i)\geq\frac{\phi^+_{\mathcal{R}_i}(\tilde{r}_{h+1})+\phi^+_{\mathcal{R}_i}(\tilde{r}_{h+2})}{2}.\tag{21}$$

This means that $\tilde{C}_{\phi^+}(a_i)\trianglerighteq\tilde{C}_{h+1}$, or, in other words, that $\tilde{h}\leq h+1$. This concludes the second part of the proof.

The proof for the incoming flows is similar. □

Acknowledgements We wish to thank Mr. Yves De Smet, Mr. Laurent Huenaerts, and Mrs. Catherinel Turcanu for their helpful comments.

References

Araz C, Ozkarahan I (2007) Supplier evaluation and management system for strategic sourcing based on a new multicriteria sorting procedure. Int J Prod Econ 106:585–606

Brans JP, Vincke Ph (1985) A preference ranking organization method: the Promethee method for multiple criteria decision making. Manag Sci 31:647–656

Devaud J, Groussaud G, Jacquet-Lagrèze E (1980) Une méthode de construction de fonctions d'utilité additives redandant compte de jugements globaux. European Working Group on Multicriteria Decision Aid, Bochum

Dias L, Mousseau V (2006) Inferring electre's veto-related parameters from outranking examples. Eur J Oper Res 170:172–191

Doumpos M, Zopounidis C (2004a) Developing sorting models using preference disaggregation analysis: an experimental investigation. Eur J Oper Res 154:585–598

Doumpos M, Zopounidis C (2004b) A multicriteria classification approach based on pairwise comparisons. Eur J Oper Res 158:378–389

Figueira J, De Smet Y, Brans J (2004) MCDA methods for sorting and clustering problems: Promethee TRI and Promethee CLUSTER, VUB. Tech Rep

Greco S, Matarazzo B, Slovinski R (2001) Rough sets theory for multicriteria decision analysis. Eur J Oper Res 129:1–47

Huenaerts L, Nemery P (2007) Détermination des profils de référence dans la problématique de tri, Internet draft. [Online]. Available: http://code.ulb.ac.be

Ngo The A, Mousseau V (2002) Using assignment examples to infer category limits for the ELECTRE TRI method. JMCDA 11:29–43

Roy B, Bouyssou D (1993) Aide multicritère à la décision: méthodes et cas. Economica, Paris

Yu W (1992) ELECTRE TRI: Aspects méthodologiques et manuel d'utilisation. Document du Lamsade, Université Paris-Dauphine, 74

Zopounidis C, Doumpos M (2002a) Business failure predictions using the UTADIS multicriteria analysis method. J Oper Res Soc 50:1138–1148

Zopounidis C, Doumpos M (2002b) Multicriteria classification and sorting methods: a literature review. Eur J Oper Res 138:229–246