1 Independance to dominated references

1.1 Hypothesis

Let $F = \{f_1, ..., f_k, ..., f_q\}$ be a set of criteria.
Let R be a set of r reference profiles, $A = \{a_1, ..., a_i, a_j, ...a_n\}$ a set of alternatives and $R_i = R \cup \{a_i\}$.
We consider the following inequality :

$$\phi_{R_i}(a_i) \geq \phi_{R_j}(a_j)$$

. Lets consider $R_i^{'} = R_i \cup \{r_h\} \mid f_k(r_h) < min(x)$ with $x \in R_i \cup A$

1.2 Thesis

$\phi^{'}_{R_i}(a_i) \geq \phi^{'}_{R_j}(a_j)$ $\forall$ i≠j

1.3 Demonstration

$f(a_i) > f(r_h)$
$\iff d(a_i, r_h) > 0$
$\iff P(a_i, r_h) \geq 0$
and $\phi^{'}_{R_i}(a_i) = \frac{1}{r+1} \sum\limits_{x \in R_i} \sum\limits_{k=1}^{q} w_k(P_k(a_i, x) - P_k(x, a_i)) + \sum\limits_{k=1}^{q} w_k(P_k(a_i, r_h) - P_k(r_h, a_i))$

Though, $\phi_{R_i}(a_i) = \frac{1}{r} \sum\limits_{x \in R_i} \sum\limits_{k=1}^{q} w_k(P_k(a_i, x) - P_k(x, a_i))$

Therefore, $\phi^{'}_{R_i}(a_i) = \frac{r}{r+1}\phi_{R_i}(a_i) + \sum\limits_{k=1}^{q} w_k(P_k(a_i, r_h) - P_k(r_h, a_i))$
where $P_k(r_h, a_i) = 0$

Considering alternative a_j, we come to the same conclusion. We thus have the 2 following expressions :

$\phi^{'}_{R_i}(a_i) = \frac{r}{r+1}\phi_{R_i}(a_i) + \sum\limits_{k=1}^{q} w_k P_k(a_i, r_h)$
$\phi^{'}_{R_j}(a_j) = \frac{r}{r+1}\phi_{R_j}(a_j) + \sum\limits_{k=1}^{q} w_k P_k(a_j, r_h)$
Furthermore, $\phi_{R_i}(a_i) \geq \phi_{R_j}(a_j)$ (by hypothesis). The inequality between $\phi^{'}_{R_i}(a_i)$ and $\phi^{'}_{R_j}(a_j)$ is thus determined by the inequality between $P_k(a_i, r_h)$ and $P_k(a_j, r_h)$.

2 situations have to be considered :

1.3.1 $f_k(a_i) \geq f_k(a_j)$:

In this case : $P_k(a_i, r_h) \geq P_k(a_j, r_h)$ $\forall k$, thus $\phi'_{R_i}(a_i) \geq \phi'_{R_j}(a_j)$ $\forall$ i$\neq$j

1.3.2 $\exists$ *at least one* k : $f_k(a_i) < f_k(a_j)$:

Here there are 3 sub-situations to consider because the inequality $P_k(a_i, r_h) > P_k(a_j, r_h)$ is not verified $\forall k$:

a) : $\sum_{k=1}^{q} w_k P_k(a_i, r_h) > \sum_{k=1}^{q} w_k P_k(a_j, r_h) \iff \phi'_{R_i}(a_i) > \phi'_{R_j}(a_j)$

b) : $\sum_{k=1}^{q} w_k P_k(a_i, r_h) < \sum_{k=1}^{q} w_k P_k(a_j, r_h) \iff \phi'_{R_i}(a_i) < \phi'_{R_j}(a_j)$

The thesis is thus invalidated in this case

c) : $\sum_{k=1}^{q} w_k P_k(a_i, r_h) = \sum_{k=1}^{q} w_k P_k(a_j, r_h) \iff \phi'_{R_i}(a_i) = \phi'_{R_j}(a_j)$

1.4 Conclusion

We can see that independance to dominated reference is guaranteed for all cases excepted the 1.3.2 b).