Probabilistic Algorithms
Homework 2

P1 (100 points)

The Newton-Raphson methods may be also applied to find the zeroes of a vector-valued function $g(\mathbf{d})$. In this case, the sequence of successive estimates is given by

$$\hat{\mathbf{d}}_{k+1} = \hat{\mathbf{d}}_k - J(\hat{\mathbf{d}}_k)^{-1} g(\hat{\mathbf{d}}_k), k = 1, 2, ...$$

where $J(\mathbf{d})$ is the Jacobian of the $g(\cdot)$ function - more precisely, if $\mathbf{d} = [d_1, \ldots, d_n]'$ and $g : R^n \to R^n$, $g(\mathbf{d}) = [g_1(\mathbf{d}), \ldots, g_n(\mathbf{d})]'$, then

$$J(\mathbf{d}) = \begin{pmatrix} \frac{\partial g_1}{\partial d_1} & \cdots & \frac{\partial g_1}{\partial d_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial g_n}{\partial d_1} & \cdots & \frac{\partial g_n}{\partial d_n} \end{pmatrix}$$

Example: For $g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x^2 + xy + y^2 \end{bmatrix}$, we have

$$J\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} \frac{\partial(2x+3y)}{\partial x} & \frac{\partial(2x+3y)}{\partial y} \\ \frac{\partial(x^2+xy+y^2)}{\partial x} & \frac{\partial(x^2+xy+y^2)}{\partial y} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2x + y & x + 2y \end{pmatrix}$$

Consider the function $g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^3 - 3xy^2 - 1 \\ 3x^2y - y^3 \end{bmatrix}$. The zeroes of g are $z_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $z_2 = \begin{bmatrix} -0.5 \\ \sqrt{3}/2 \end{bmatrix}$ and $z_3 = \begin{bmatrix} -0.5 \\ -\sqrt{3}/2 \end{bmatrix}$. Depending of the initial starting point $\hat{\mathbf{d}}_0$, the sequence of estimates $\hat{\mathbf{d}}_k$ will converge toward one of these three points in R^2.

- Write a Matlab function `root_index(v)` which, for a given starting point $\hat{\mathbf{d}}_0 = v$ (a vector $[x; y]$), returns 1 if the sequence $\hat{\mathbf{d}}_k$ converges to z_1, 2 if it converges to z_2 and 3 if it converges to z_3.

Note 1: The inverse of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Note 2: The sequence $\hat{\mathbf{d}}_0, ..., \hat{\mathbf{d}}_k$ is considered as convergent to z_i if the euclidean distance between $\hat{\mathbf{d}}_k$ and z_i is less than $\varepsilon = 0.01$. You may use the MatLab function *norm(x)* which calculates the norm of a vector $x = (x_1, \ldots, x_n)$ as $\sqrt{\sum_{i=1}^{n} x_i^2}$.

P2 (50 points bonus)

Consider the grid covering the square centered in $(0, 0)$, with the side length equal m, having the distance between two horizontal/vertical lines equal h. Write a Matlab function `show(m, h)` to create a 2D plot where each point of the grid (x, y) is colored in blue (respectively red, green) if the sequence $\hat{\mathbf{d}}_k$ starting in $\hat{\mathbf{d}}_0 = \begin{bmatrix} x \\ y \end{bmatrix}$ converges toward z_1 (respectively z_2, z_3).