ON THE ANISOTROPIC CALDERÓN'S PROBLEM

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ABSTRACT. We prove that the Riemannian metric on a compact manifold of dimension $n \geq 3$ with smooth boundary can be uniquely determined, up to an isometry fixing the boundary, by the Dirichlet-to-Neumann map associated to the Laplace-Beltrami operator.

1. Introduction

Let Ω be a bounded domain in \mathbb{R}^n with smooth boundary $\partial\Omega$. Consider the equation

$$(1) -\nabla \cdot (\gamma \nabla u) = 0,$$

with $\gamma(x) = (\gamma^{ij}(x))_{i,j=1}^n$, where the functions $\gamma^{ij} \in C^{\infty}(\overline{\Omega})$. We assume that, for each x, $\gamma(x)$ is a positive definite symmetric matrix. If $\gamma(x) = \sigma(x)I$ for some scalar function σ , we say that γ is isotropic, otherwise it is anisotropic.

The Calderón's problem [2] asks if one can recover γ from the so-called Dirichlet-to-Neumann (DtN) map. The DtN is defined as the map $\Lambda_{\gamma}: f \mapsto \gamma \nabla u \cdot \nu$, where u solves the boundary value problem

$$-\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \Omega, \qquad u|_{\partial \Omega} = f,$$

and ν is the outer unit normal vector on $\partial\Omega$. If Ω represents an inhomogeneous body with conductivity γ , and u represents the electrical potential, then the DtN map encodes all possible voltage-current corresponding pairs at the surface of the body.

When γ is isotropic, the problem has many important results. The uniqueness of piecewise analytic γ was proved in [16, 17]. For $\gamma \in C^{\infty}(\overline{\Omega})$ and $n \geq 3$, the uniqueness is proved in [24]. For uniqueness results for lower regular conductivities, we refer to [12, 3, 11] and the references therein. A reconstruction formula is given in [20]. In the two-dimensional case n=2, the uniqueness of $\gamma \in C^{\infty}$ was established in [21], and $\gamma \in L^{\infty}$ in [1]. The anisotropic problem in dimension n=2 can be reduced to the isotropic case. However, in dimension $n \geq 3$, the anisotropic problem is still widely open and has only been studied under very special geometrical structures [5, 7].

The anisotropic Calderón problem has a geometric nature and thus can be reformulated as follows. Let $(\overline{\Omega}, g)$ be a compact Riemannian manifold of dimension $n \geq 3$ with smooth boundary $\partial\Omega$. Consider the Laplace-Beltrami operator, which, in local coordinates, can be written as

(2)
$$-\Delta_g u = -|g|^{1/2} \partial_j (|g|^{1/2} g^{jk} \partial_k u),$$

where $|g| = \det(g_{jk})$ is the determinant of $g = (g_{jk})$, and $g^{-1} = (g^{jk})$ is the inverse matrix of g.

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For the boundary value problem

(3)
$$\begin{cases} -\Delta_g u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

the inverse problems is to determine the metric g from the Dirichlet-to-Neumann map Λ_g defined as

$$\Lambda_g: f \mapsto \partial_{\nu} u|_{\partial\Omega},$$

where u is the unique solution to (3), and ν is the outer unit normal vector of Ω , i.e., $\nu \in N^* \partial \Omega$ and $|\nu|_g = 1$. More precisely

$$\partial_{\nu} u = \langle \nabla u, \nu \rangle_q |_{\partial \Omega}.$$

By standard elliptic theory, the DtN map Λ_g is a bounded linear operator $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$. In dimension $n \geq 3$, the above two inverse problems are equivalent [19] if Ω is a domain in \mathbb{R}^n . And the second problem is a more general one. Note here that there is a natural obstruction to uniqueness for the determination of g.

Lemma 1. If $\Phi: \overline{\Omega} \to \overline{\Omega}$ is a diffeomorphism and $\Phi|_{\partial\Omega} = \operatorname{Id}$, then

$$\Lambda_{\Phi^*q} = \Lambda_q$$

Here Φ^*g is the pullback of g, defined in local coordinates by

$$\Phi^* g(x) = D\Phi(x)^t g(\Phi(x)) D\Phi(x).$$

For $n \geq 3$, it is conjectured that this is the only obstruction to uniqueness. For n = 2 there is an extra conformal invariance and the uniqueness up to these invariances was proved in [18]. For higher dimensional cases, uniqueness has only been established for metrics in a fixed conformal class of a transversally anisotropic manifold where the geodesic ray transform on the transversal manifold is injective [5, 7]. In this article we prove the result for the general case.

Theorem 1. Let $(\overline{\Omega}, g_1)$ and $(\overline{\Omega}, g_2)$ be compact Riemannian manifolds of dimension $n \geq 3$ with smooth boundary $\partial \Omega$. If $\Lambda_{g_1} = \Lambda_{g_2}$, then there exists a diffeomorphism $\Phi : \overline{\Omega} \to \overline{\Omega}$ with $\Phi|_{\partial\Omega} = \operatorname{Id}$ such that $g_2 = \Phi^* g_1$.

The method of proof resembles that of the fractional Calderón's problem on closed Riemannian manifolds [6], which relates the elliptic problems to dynamical problems. For recent work on fractional Calderón's problems, we refer to [9, 8, 10, 6, 23, 4] and the references therein.

2. Preliminaries

We first extend the manifold $\overline{\Omega}$ to a smooth closed, compact, connected manifold M, such that $M \setminus \overline{\Omega}$ is connected (one can just take the double of the manifold $\overline{\Omega}$, cf., for example, [22, Lemma 3.1.8]). We have $\partial\Omega = \partial(M \setminus \overline{\Omega})$. Also we extend g to be a Riemannian metric on M. On the closed Riemannian manifold (M, g). We can consider the Laplace-Belmatri operator. It is a self-adjoint operator on $L^2(M) = L^2(M, \mathrm{d}V_g)$ equiped with inner product

$$(u,v)_{L^2(M)} = \int_M u\overline{v} dV_g,$$

where, in local coordinates

$$dV_q = |g|^{1/2} dx^1 \wedge \cdots \wedge dx^n.$$

We choose a non-zero function $V \in C_c^{\infty}(M \setminus \overline{\Omega})$ such that $V \geq 0$ on M. Then the operator $L_g := -\Delta_g + V$ is an unbounded positive definite self-adjoint operator on $L^2(M)$ with domain

 $H^2(\Omega)$. In particular, 0 is not an eigenvalue of L_g and L_g is invertible with L_g^{-1} a bounded operator on $L^2(M)$. To see this, we argue by contradiction. Assume that there exists $u \in H^2(M)$, $u \neq 0$ such that

$$-\Delta_q u + V u = 0.$$

Integrating by parts, we obtain

$$\int_{M} (|\nabla u|_g^2 + Vu^2) dV_g = 0.$$

Then $\nabla u = 0$ on M, and therefore $u \equiv c$ is a constant. Then $c \int_M V dV_g = 0$. Since $V \geq 0$ is not identically zero, this leads to a contradiction.

We denote $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ the eigenvalues (counted with multiplicities) of $-\Delta_g + V$ on M and $\varphi_1, \varphi_2, \varphi_3, \cdots$ be the associated orthonormal eigenfunctions. Then $\{\phi_j\}_{j=1}^{\infty}$ is an orthonormal basis for $L^2(M)$. For any function $f \in L^2(M)$, we have the eigen-expansion of f as

$$f = \sum_{j=1}^{\infty} (f, \varphi_j)_{L^2(M)} \varphi_j.$$

Therefore,

$$L_g u = \sum_{j=1}^{\infty} \lambda_j (f, \varphi_j)_{L^2(M)} \varphi_j.$$

Let $u \in C^{\infty}(M)$. For any $\alpha \in \mathbb{R}$, by functional calculus

$$L_g^{\alpha} u = \sum_{j=1}^{\infty} \lambda_j^{\alpha} (u, \varphi_j)_{L^2(M)} \varphi_j,$$

where L_g^{α} can be considered as an bounded or unbounded operator on $L^2(M)$. The domain of L_g^{α} is actually $H^{2\alpha} = \{u \in L^2(M) : \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |(u, \varphi_k)|^2 < +\infty\}$. Then one can verify that

$$(L_g^{\alpha})^{\beta}u = L_g^{\alpha\beta}u, \quad (L_g^{\alpha})^{-1}u = (L_g^{-1})^{\alpha}u,$$

for arbitrary real numbers α, β . In particular, we will use the identity

$$\sqrt{L_g^2} := (L_g^2)^{1/2} = L_g.$$

For any $f \in L^2(M)$, the equation

$$L_g u := -\Delta_g u + V u = f \quad \text{in } M$$

has a unique solution $u = u^f \in H^2(M)$. It can be represented as

$$u(x) = L_g^{-1} f(x) = \int_M G(x, y) f(y) dV_g(y),$$

where G(x, y) is the Green's function of $-\Delta_g + V$ on M, that is,

$$L_gG(x,\cdot) := (-\Delta_g + V)G(x,\cdot) = \delta_x,$$

or rigorously, for any $\varphi \in C^{\infty}(M)$.

$$\int_{M} G(x, y) L_{g} \varphi(y) dV_{g}(y) = \varphi(x).$$

We have the following bound for the Green's function [14]

$$|G(x,y)| \le \frac{C}{d_g(x,y)^{n-2}}, \quad x \ne y,$$

for some constants C > 0.

Take $\varphi \in C^{\infty}(M)$. By Green's formula, we have that, for any $\epsilon > 0$ sufficiently small

$$\int_{M \setminus B_{\epsilon}(x)} G(x, y) L_{g} \varphi(y) dV_{g}(y)
= \int_{M \setminus B_{\epsilon}(x)} G(x, y) (-\Delta_{g} + V(y)) \varphi(y) dV_{g}(y) - \int_{M \setminus B_{\epsilon}(x)} (-\Delta_{g, y} + V(y)) G(x, y) \varphi(y) dV_{g}(y)
= \int_{\partial B_{\epsilon}(x)} G(x, y) \frac{\partial \varphi(y)}{\partial \nu_{g}(y)} dS_{g}(y) - \int_{\partial B_{\epsilon}(x)} \frac{\partial G(x, y)}{\partial \nu_{g}(y)} \varphi(y) dS_{g}(y),$$

where $B_{\epsilon}(x)$ is a ball of radius ϵ centered at x. Letting $\epsilon \to 0$, since

$$\lim_{\epsilon \to 0} \int_{\partial B_{\epsilon}(x)} G(x, y) \frac{\partial \varphi(y)}{\partial \nu_g(y)} dS_g(y) = 0$$

by (4), we obtain

$$\lim_{\epsilon \to 0} \int_{\partial B_{\epsilon}(x)} \frac{\partial G(x, y)}{\partial \nu_g(y)} \varphi(y) dS_g(y) = -\varphi(x).$$

3. From Boundary to exterior measurements

Let g_i , i = 1, 2 be the Riemannian metrics on $\overline{\Omega}$ such that $\Lambda_{g_1} = \Lambda_{g_2}$. First, we recall the results for the boundary determination in [19] as follows.

Proposition 1. Let $(\overline{\Omega}, g_1)$ and $(\overline{\Omega}, g_2)$ be compact manifolds with smooth boundary, with dimension $n \geq 3$. If $\Lambda_{g_1} = \Lambda_{g_2}$, then the Taylor series of g_1 and g_2 in boundary normal coordinates are equal at each point on $\partial\Omega$.

By the above result, there exists a diffeomorphism Ψ fixing the boundary such that the jets of Ψ^*g_1 and g_2 at $\partial\Omega$ are the same. We can then extend g_2 smoothly to M such that (M,g_2) is a closed Riemannian manifold. Denote $\tilde{g}_1 = \begin{cases} \Psi^*g_1 & \text{on } \Omega \\ g_2 & \text{on } M \setminus \overline{\Omega} \end{cases}$. Then (M,\tilde{g}_1) is also a Riemannian manifold. Then $\Lambda_{\tilde{g}_1} = \Lambda_{g_1} = \Lambda_{g_2}$, and consequently without loss of generally we can assume that $g = g_2 = g_1$ on $M \setminus \overline{\Omega}$. For k = 1, 2, denote G_k to be the Green's function of $L_k := -\Delta_{g_k} + V$.

Proposition 2. If $\Lambda_{g_1} = \Lambda_{g_2}$, then $G_1(x,y) = G_2(x,y)$ for any $(x,y) \in (M \setminus \overline{\Omega}) \times (M \setminus \overline{\Omega}) \setminus \{x = y\}$. Proof. For arbitrary $f \in C^{\infty}(M)$ such that $f \in C_c^{\infty}(M \setminus \overline{\Omega})$, we take

(5)
$$u(x) = \int_{M} G_1(x, y) f(y) dV_{g_1} = \int_{M \setminus \overline{\Omega}} G_1(x, y) f(y) dV_g, \quad x \in M.$$

Then $L_{g_1}u=(-\Delta_{g_1}+V)u=f$ in M and so $u\in C^{\infty}(M)$. In particular, $-\Delta_{g_1}u=0$ in Ω . Let v be the solution to $L_2v:=-\Delta_{g_2}v=0$ in Ω with $v|_{\partial\Omega}=u|_{\partial\Omega}$. Applying the Green's formula to

 $G_2(x,y)$ and (u-v)(y) on $\Omega \setminus B_{\epsilon}(x)$ for $x \in \Omega$ and $\epsilon > 0$ sufficiently small,

$$\begin{split} &\int_{\Omega \backslash B_{\epsilon}(x)} G_{2}(x,y) \Delta_{g_{2}}(u-v)(y) \mathrm{d}V_{g_{2}}(y) \\ &= \int_{\Omega \backslash B_{\epsilon}(x)} G_{2}(x,y) \Delta_{g_{2}}(u-v)(y) \mathrm{d}V_{g_{2}}(y) - \int_{\Omega \backslash B_{\epsilon}(x)} \Delta_{g_{2},y} G_{2}(x,y)(u-v)(y) \mathrm{d}V_{g_{2}}(y) \\ &= \int_{\partial \Omega} G_{2}(x,y) \frac{\partial (u-v)(y)}{\partial \nu_{g}(y)} \mathrm{d}S_{g}(y) - \int_{\partial \Omega} \frac{\partial G_{2}(x,y)}{\partial \nu_{g}(y)} (u-v)(y) \mathrm{d}S_{g}(y) \\ &- \int_{\partial B_{\epsilon}(x)} G_{2}(x,y) \frac{\partial (u-v)(y)}{\partial \nu_{g}(y)} \mathrm{d}S_{g}(y) + \int_{\partial B_{\epsilon}(x)} \frac{\partial G_{2}(x,y)}{\partial \nu_{g}(y)} (u-v)(y) \mathrm{d}S_{g}(y). \end{split}$$

Here we have used the fact that $g_1 = g_2 = g$ on $\partial \Omega$, and ν_g is the unit outer normal of $\partial \Omega$. Letting $\epsilon \to 0$, we obtain

$$\int_{\partial\Omega} G_2(x,y) \frac{\partial(u-v)(y)}{\partial\nu_g(y)} dS_g(y)$$

$$= u(x) - v(x) - \int_{\Omega} G_2(x,y) L_2(u-v)(y) dV_{g_2}(y)$$

$$= u(x) - v(x) + \int_{\Omega} G_2(x,y) (L_1 - L_2) u(y) dV_{g_2}(y)$$

for $x \in \Omega$.

For $w \in L^2(M)$, we denote

$$L_k^{-1}w(x) = \int_M G_k(x, y)w(y)dV_{g_k}(y),$$

k=1,2. We will use the resolvent identity

$$L_2^{-1} - L_1^{-1} = L_2^{-1}(L_1 - L_2)L_1^{-1}.$$

In integral form, the above identity can be written as

$$\int_{M} G_{2}(x,y)w(y)dV_{g_{2}}(y) - \int_{M} G_{1}(x,y)w(y)dV_{g_{1}}(y)$$

$$= \int_{M} G_{2}(x,z) \int_{M} (L_{1,z} - L_{2,z})G_{1}(z,y)w(y)dV_{g_{1}}(y)dV_{g_{2}}(z).$$

Taking $w \in C_c^{\infty}(M \setminus \overline{\Omega})$, and using the fact that $L_1 = L_2$ $(g_1 = g_2 = g)$ on $M \setminus \overline{\Omega}$, we have

$$\int_{M\setminus\overline{\Omega}} (G_2(x,y) - G_1(x,y))w(y)dV_g(y)$$

$$= \int_{\Omega} G_2(x,z) \int_{M\setminus\overline{\Omega}} (L_{1,z} - L_{2,z})G_1(z,y)w(y)dV_g(y)dV_{g_2}(z).$$

Therefore for any $y \in M \setminus \overline{\Omega}$, $x \in M$, $x \neq y$,

(7)
$$G_2(x,y) - G_1(x,y) = \int_{\Omega} G_2(x,z)(L_{1,z} - L_{2,z})G_1(z,y)dV_{g_2}(z).$$

Using the formula (5), we rewrite (6) as

$$\int_{\partial\Omega} G_2(x,y) \frac{\partial (u-v)(y)}{\partial \nu_g(y)} dS_g(y)$$
(8)
$$= u(x) - v(x) + \int_{\Omega} G_2(x,z) (L_{1,z} - L_{2,z}) \int_{M \setminus \overline{\Omega}} G_1(z,y) f(y) dV_g(y) dV_{g_2}(z)$$

$$= u(x) - v(x) + \int_{M \setminus \overline{\Omega}} (G_2(x,y) - G_1(x,y)) f(y) dV_g(y).$$

Note that

$$\frac{\partial (u-v)}{\partial \nu_g} = (\Lambda_1 - \Lambda_2)u = 0.$$

Restricting (8) to the boundary, we have

$$\int_{M\setminus\overline{\Omega}} (G_2(x,y) - G_1(x,y)) f(y) dV_g(y) = 0.$$

for $x \in \partial \Omega$. This shows that $G_2(x,y) - G_1(x,y) = 0$ for $x \in \partial \Omega$ and $y \in M \setminus \overline{\Omega}$. Now fixing $y \in M \setminus \overline{\Omega}$, we have

$$(-\Delta_g + V)(G_1 - G_2)(\cdot, y) = 0$$
 in $M \setminus \overline{\Omega}$, $(G_1 - G_2)(\cdot, y)|_{\partial\Omega} = 0$.

By the uniqueness of the above boundary value problem, we have

(9)
$$G_1(x,y) = G_2(x,y), \quad x,y \in M \setminus \overline{\Omega}, x \neq y.$$

This completes the proof.

Therefore, we are left to show that (9) implies that there exists a diffeomorphism $\Phi: \overline{\Omega} \to \overline{\Omega}$ with $\Phi|_{\partial\Omega} = \text{Id}$ such that $g_2 = \Phi^* g_1$.

4. Proof of the main result

We start with the simple relation $L_g^{-1} = ((L_g)^2)^{-1/2}$, or more explicitly,

$$(-\Delta_g + V)^{-1} = ((-\Delta_g + V)^2)^{-1/2}.$$

Recall that the Gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha - 1} dt,$$

for $\alpha > 0$. We use the above identity with $\alpha = \frac{1}{2}$ and get

(10)
$$a^{-1/2} = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty e^{-at} \frac{1}{t^{1/2}} dt.$$

Applying the functional calculus to the positive definite operator L_q^2 , we have

(11)
$$L_g^{-1}v = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty e^{-tL_g^2} v \frac{1}{t^{1/2}} dt,$$

where $v \in L^2(M)$. To be more rigorous, we write v as

$$v = \sum_{j=1}^{\infty} (v, \varphi_j)_{L^2(M)} \varphi_j,$$

where (λ_j, φ_j) are the eigen-pairs of L_g . Then

$$L_g^{-1}v = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} (v, \varphi_j)_{L^2(M)} \varphi_j,$$

and

$$e^{-tL_g^2}v = \sum_{j=1}^{\infty} e^{-t\lambda_j^2}(v,\varphi_j)_{L^2(M)}\varphi_j.$$

Thus, using (10),

$$\int_{0}^{\infty} e^{-tL_{g}^{2}} v \frac{1}{t^{1/2}} dt = \int_{0}^{\infty} \sum_{j=1}^{\infty} e^{-t\lambda_{j}^{2}} (v, \varphi_{j})_{L^{2}(M)} \varphi_{j} \frac{1}{t^{1/2}} dt = \sum_{j=1}^{\infty} \left(\int_{0}^{\infty} e^{-t\lambda_{j}^{2}} \frac{1}{t^{1/2}} dt \right) (v, \varphi_{j})_{L^{2}(M)} \varphi_{j}$$

$$= \Gamma\left(\frac{1}{2}\right) \sum_{j=1}^{\infty} \frac{1}{\lambda_{j}} (v, \varphi_{j})_{L^{2}(M)} \varphi_{j} = \Gamma\left(\frac{1}{2}\right) L_{g}^{-1} v,$$

which is the desired identity (11).

Let $\omega_1 \subset\subset M\setminus\overline{\Omega}$ be an open nonempty set. Assume that $d_g(\omega_1,\partial\Omega)=\delta$, and we take $\omega_2=\{x\in M\setminus\overline{\Omega},d_g(x,\partial\Omega)<\frac{\delta}{2})\}$. Then $\overline{\omega_1}\cap\overline{\omega_2}=\emptyset$ and $\partial\Omega\subset\overline{\omega_2}$. Let $f\in C_0^\infty(\omega_1)$. For $g_1=g_2=g$ on $M\setminus\overline{\Omega}$, we have

$$L_1^{2m} f = L_2^{2m} f = L^{2m} f$$
 on ω_1 ,

for $m = 0, 1, \dots$. Since $G_1(x, y) = G_2(x, y)$ for any $x, y \in M \setminus \overline{\Omega}$, we have

$$L_1^{-1}L^{2m}f|_{M\setminus\overline{\Omega}} = L_2^{-1}L^{2m}f|_{M\setminus\overline{\Omega}}.$$

Using (11), we get

$$\int_0^{+\infty} ((e^{-tL_1^2} - e^{-tL_2^2})L^{2m}f)(x)\frac{1}{t^{1/2}}dt = 0,$$

for $x \in M \setminus \overline{\Omega}$ and $m = 0, 1, \cdots$.

Using the fact

$$(e^{-tL_j^2}L^{2m}f)(x) = \partial_t^m(e^{-tL_j^2}f)(x)$$

for $x \in M \setminus \overline{\Omega}$, we obtain

(12)
$$\int_0^{+\infty} \partial_t^m ((e^{-tL_1^2} - e^{-tL_2^2})f)(x) \frac{1}{t^{1/2}} dt = 0,$$

for $x \in M \setminus \overline{\Omega}$ and $m = 0, 1, \cdots$.

For $\ell = 0, \dots, m-1$ and $x \in \omega_2$, we have

$$\begin{split} \partial_t^\ell \left((e^{-tL_1^2} - e^{-tL_2^2}) f \right)(x) &= \left(e^{-tL_1^2} - e^{-tL_2^2} \right) L^{2\ell} f \right) f(x) \\ &= \int_{\omega_1} (e^{-tL_1^2}(x,y) - e^{-tL_2^2}(x,y)) L^{2\ell} f(y) \mathrm{d}V_g(y), \end{split}$$

where $e^{-tL_k^2}(x,y)$ is the kernel of $e^{-tL_k^2}$, i.e., the "heat" kernel associated with the fourth order parabolic equation

$$\partial_t u + (-\Delta_{g_k} + V)^2 u = 0$$
 in M .

It follows that for t > 0 and $x \in \omega_2$,

$$\left| \partial_t^\ell \left((e^{-tL_1^2} - e^{-tL_2^2}) f \right)(x) \right| \leq \|e^{-tL_1^2}(\cdot, \cdot) - e^{-tL_2^2}(\cdot, \cdot)\|_{L^\infty(\omega_2 \times \omega_1)} \|L^{2\ell} f\|_{L^2(\omega_1)},$$

 $\ell = 0, \dots, m-1$. We need the pointwise estimate for the heat kernel:

Lemma 2. For t > 0 sufficiently small

(13)
$$|e^{-tL_k^2}(x,y)| \le Ct^{-n/4}e^{-\frac{cd_{g_k}^{4/3}(x,y)}{t^{1/3}}}, \quad x,y \in M.$$

Proof. We refer to [13, Lemma 7.1] for the $V \equiv 0$ case. For the self-containedness of this article, we include a proof for the $V \not\equiv 0$ case in the appendix.

Using the above estimate, we have that for t>0 sufficiently small, $x\in\omega_2$

(14)
$$\left| \partial_t^{\ell} \left((e^{-tL_1^2} - e^{-tL_2^2}) f \right) (x) \right| \le C e^{-\frac{\tilde{c}}{t^{1/2}}} \|L_g^{2\ell} f\|_{L^2(\omega_1)},$$

with some constant $\tilde{c} > 0$, for $\ell = 0, \dots, m-1$. Therefore

$$\lim_{t \to 0^+} \partial_t^{\ell} \left((e^{-tL_1^2} - e^{-tL_2^2}) f \right) (x) \frac{1}{t^{m-\ell-1/2}} = 0.$$

For t > 0 large enough and fix $m \in \mathbb{Z}^+$ large enough. Assume that $\lambda_{k,1} \geq a > 0$, where $\lambda_{k,1}$ is the smallest eigenvalue of L_k . Then

$$||e^{-tL_k^2}f||_{L^2(M)} \le Ce^{-a^2t}||f||_{L^2(M)},$$

and

$$\|(-\Delta_{g_k} + V)^j e^{-tL_k^2} f\|_{L^2(M)} \le \frac{C_j}{t^j} \|f\|_{L^2(\omega_1)},$$

for any $j = 1, 2, \dots, m$, which implies that

$$||e^{-tL_k^2}f||_{H^{2m}} \le \frac{C_m}{t^m}||f||_{L^2(\omega_1)},$$

with some constant C_m depending on m, Using Sobolev interpolation, we have

$$||e^{-tL_k^2}f||_{L^{\infty}(M)} \le \frac{C}{tK}||f||_{L^2(\omega_1)}$$

with some K > 0 sufficiently large.

For t > 1 and $x \in \omega_2$, the above estimate yields

(15)
$$\left| \partial_t^{\ell} \left((e^{-tL_1^2} - e^{-tL_2^2}) f \right) (x) \right| \le C t^{-K} \| L_g^{2\ell} f \|_{L^2(\omega_1)}.$$

Then, for $\ell = 0, \dots, m-1$,

$$\lim_{t \to +\infty} \partial_t^{\ell} \left((e^{-tL_1^2} - e^{-tL_2^2}) f \right) (x) \frac{1}{t^{m-\ell-1/2}} = 0,$$

where $x \in \omega_2$ and $m = 0, 1, \cdots$.

Then we can apply integration by parts to (12) and obtain

$$\int_0^{+\infty} ((e^{-tL_1^2} - e^{-tL_2^2})f)(x) \frac{1}{t^{m+1/2}} dt = 0,$$

for $x \in \omega_2$ and $m = 0, 1, \cdots$.

Making the change of variables $s = \frac{1}{t}$, we get

(16)
$$\int_0^{+\infty} \varphi(s) s^{m-1} \mathrm{d}s = 0,$$

where

$$\varphi(s) = \frac{\left(\left(e^{-\frac{1}{s}L_1^2} - e^{-\frac{1}{s}L_2^2}\right)f\right)(x)}{s^{1/2}}, \quad x \in \omega_2,$$

for $m = 1, 2, \cdots$. Using the estimates (28) and (29), we have

$$|\varphi(s)| \le C \frac{e^{-cs}}{s^{1/2}}$$

with c > 0. It follows that the Fourier transform of $1_{[0,\infty)}\varphi$,

$$\mathcal{F}(1_{[0,\infty)}\varphi)(\xi) = \int_0^{+\infty} \varphi(s)e^{-\mathrm{i}\xi s} \mathrm{d}s$$

is holomorphic in $\{\Im \xi < c\}$. Then all derivatives of $\mathcal{F}(1_{[0,\infty)}\varphi)$ vanishes at 0 by (16). Therefore $\varphi(s) = 0$ for s > 0. Thus

$$e^{-tL_1^2}f(x) = e^{-tL_2^2}f(x),$$

for t > 0 and $x \in \omega_2$. By the choice of ω_2 , we have $e^{-tL_1^2}f(x) = e^{-tL_2^2}f(x)$ for $x \in M \setminus \overline{\Omega}$ in a neighborhood of $\partial\Omega$.

Note that the function

$$z(t,x) = (e^{-tL_1^2} - e^{-tL_2^2})f(x)|_{(0,+\infty)\times (M\backslash \overline{\Omega})} \in C^{\infty}((0,+\infty)\times (M\setminus \overline{\Omega}))$$

satisfies the fourth order parabolic equation

(17)
$$(\partial_t + L_q^2)z(t,x) := (\partial_t + (-\Delta_g + V)^2)z(t,x) = 0, \quad \text{in } (0,+\infty) \times (M \setminus \overline{\Omega}).$$

We have that

(18)
$$z = \Delta_g z = 0 \quad \text{on } \partial \Omega, \quad z(0, x) = 0 \quad \text{for } x \in M \setminus \overline{\Omega}.$$

By the uniqueness of the initial boundary value problem for (17) (18) in $(0, +\infty) \times (M \setminus \overline{\Omega})$, we have

$$e^{-tL_1^2}f(x) = e^{-tL_2^2}f(x),$$

for t > 0 and $x \in M \setminus \overline{\Omega}$. Recalling that $\omega_1 \subset\subset M \setminus \overline{\Omega}$ is arbitrary, we conclude that

(19)
$$e^{-tL_1^2}(x,y) = e^{-tL_2^2}(x,y), \quad x,y \in M \setminus \overline{\Omega}, \quad t > 0.$$

Now the problem has be reduced to proving that (19) implies $g_2 = \Phi^* g_1$ on Ω for some diffeomorphism $\Phi : \overline{\Omega} \to \overline{\Omega}$ fixing the boundary $\partial \Omega$. We will relate this problem to an inverse problem for the wave equation.

Using the transmutation formula of Kannai (cf. [6]),

$$e^{-tL_k^2}v = \frac{1}{4\pi^{1/2}t^{3/2}} \int_0^{+\infty} e^{-\frac{\tau}{4t}} \frac{\sin(\sqrt{\tau}L_k)}{L_k} v d\tau, \quad t > 0.$$

for $v \in C^{\infty}(M)$. Let $f \in C_c^{\infty}(M \setminus \overline{\Omega})$. Then (19) implies that

$$\int_0^{+\infty} e^{-\tau t} \left(\frac{\sin(\sqrt{\tau} L_1)}{L_1} f \right) (x) d\tau = \int_0^{+\infty} e^{-\tau t} \left(\frac{\sin(\sqrt{\tau} L_2)}{L_2} f \right) (x) d\tau,$$

for t > 0 and $x \in M \setminus \overline{\Omega}$. Inverting the Laplace transform, we get

(20)
$$\left(\frac{\sin(\sigma L_1)}{L_1}f\right)(x) = \left(\frac{\sin(\sigma L_2)}{L_2}f\right)(x),$$

for $\sigma > 0$ and $x \in M \setminus \overline{\Omega}$. Differentiating (20) in σ , we get

$$(\cos(\sigma L_1)f)(x) = (\cos(\sigma L_2)f)(x), \quad \sigma > 0, \quad x \in M \setminus \overline{\Omega}.$$

Applying the local operator $L=L_1=L_2$ on $M\setminus\overline{\Omega}$ to (20), we obtain

$$(\sin(\sigma L_1)f)(x) = (\sin(\sigma L_2)f)(x), \quad \sigma > 0, \quad x \in M \setminus \overline{\Omega}.$$

For $\sigma < 0$, we can do even extension for the cosine function

$$(\cos(\sigma L_k)f)(x) = (\cos(-\sigma L_k)f)(x),$$

and odd extension for the sine function

$$(\sin(\sigma L_k)f)(x) = -(\sin(-\sigma L_k)f)(x).$$

So we have

$$(\cos(\sigma L_1)f)(x) = (\cos(\sigma L_2)f)(x), \quad (\sin(\sigma L_1)f)(x) = (\sin(\sigma L_2)f)(x), \quad \sigma \in \mathbb{R}, \quad x \in M \setminus \overline{\Omega}.$$

Therefore we have

$$(e^{i\sigma L_1}f)(x) = (e^{i\sigma L_2}f)(x), \quad \sigma \in \mathbb{R}, \quad x \in M \setminus \overline{\Omega}.$$

For any $\varphi \in \mathcal{S}(\mathbb{R})$, we have

$$(\varphi(L_k)f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (e^{i\sigma L_k} f)(x) \hat{\varphi}(\sigma) d\sigma,$$

where $\hat{\varphi}$ is the Fourier transform of φ . For $\varphi \in C_0(\mathbb{R}) := \{ f \in C(\mathbb{R}), \lim_{|x| \to \infty} f(x) = 0 \}$, we can take a sequence $\{ \varphi_j \}_{j=1}^{\infty} \subset \mathcal{S}$ such that $\lim_{j \to +\infty} \varphi_j = \varphi$ uniformly. In particular we take $\varphi(\sigma) = \frac{\sin(t\sqrt{|\sigma|})}{\sqrt{|\sigma|}}$ and get

(21)
$$\left(\frac{\sin(t\sqrt{L_1})}{\sqrt{L_1}}f\right)(x) = \left(\frac{\sin(t\sqrt{L_2})}{\sqrt{L_2}}f\right)(x),$$

for t > 0 and $x \in M \setminus \overline{\Omega}$.

Consider the wave equation

(22)
$$\begin{cases} (\partial_t^2 - \Delta_{g_k} + V)u_k = F(t, x), & (t, x) \in (0, +\infty) \times M, \\ u_k(0, x) = \partial_t u_k(0, x) = 0, & x \in M. \end{cases}$$

For any $F \in C_c^{\infty}([0, +\infty) \times (M \setminus \overline{\Omega}))$, the initial value problem (22) has a unique solution $u_k = u_k^F \in C^{\infty}([0, +\infty) \times M)$. The solution to (22) can be written as

$$u_k^F(t,x) = \int_0^t \frac{\sin((t-s)\sqrt{L_k})}{\sqrt{L_k}} F(s,x) ds.$$

So the local source-to-solution map $L_{q_k,M\backslash\overline{\Omega}}$ defined by

$$L_{q_k,M\setminus\overline{\Omega}}(F) = u_k^F|_{M\setminus\overline{\Omega}}.$$

By (21), we have

$$L_{g_1,M\backslash\overline{\Omega}}(F)=L_{g_2,M\backslash\overline{\Omega}}(F)$$

for any $F \in C_c^{\infty}([0, +\infty) \times (M \setminus \overline{\Omega}))$. By the proof of [15, Theorem 2], there exists a diffeomorphism $\Phi : M \to M$ such that $g_2 = \Phi^* g_1$ and $\Phi(x) = x$ for $x \in M \setminus \overline{\Omega}$. Considering Φ as a diffeomorphism on $\overline{\Omega}$ fixing the boundary $\partial \Omega$, we have finished the proof of the main theorem.

APPENDIX A. PROOF OF LEMMA 2

In this appendix we establish the Gaussian estimate for the heat kernel of the following second order parabolic equation.

(23)
$$\partial_t u = -L_q^2 u = -(-\Delta_g + V)^2 u \quad \text{in } M,$$

where (M, g) is a closed Riemannian manifold of dimension n and $V \in C^{\infty}(M)$. We basically follow the lines of arguments in [13]. The heat kernel b(x, y, t) is a smooth function on $M \times M \times (0, +\infty)$ such that

$$e^{-tL_g^2}f(x) = \int_M b(x, y, t)f(y)dy$$

for any $f \in L^2(M)$. In the following, we drop the subscript g to simplify the use of notations. For example, $|\cdot| = |\cdot|_g$, $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_g$, $\Delta = \Delta_g$.

For a fixed point $p \in M$, let r(x) be the distance from x to p. Since M is a closed manifold, by [13, Lemma 2.6] and the proof of [13, Lemma 7.1], we can construct a distance-like function $f = f_p$ on M such that

$$(24) cr(x) \le f(x) \le Cr(x), |\nabla f(x)| \le C$$

for all $x \in M \setminus \{p\}$, and

$$|\Delta f(x)| \le \frac{C}{r(x)},$$

where the constants 0 < c < 1, C > 1 depends on (M, g).

Denote

$$D_r := \{ f < r \}$$

for any r > 0. Then we have

$$B(p, C^{-1}r) \subset D_r \subset B(p, c^{-1}r).$$

Lemma 3. Let u be a solution of (23) on $D_R \times [0,T]$, then for R > 0 sufficiently small, $T = R^4$, $0 < t \le T$, $m = 0, 1, 2, \dots$, we have

(26)
$$\int_{D_{R/2}} |\Delta^m u|^2(x,t) \le C \frac{1}{t^{m+1}} \int_0^t \int_{D_R} u^2,$$

with some constant C > 0 depending on M, V, m.

Proof. For R > 0 small enough, let 0 < s < l < R. The distance function f chosen above satisfies

$$|\Delta f| \le \frac{C}{f(x)}.$$

Take the cut-off function ϕ as

$$\phi(x) = \eta^k (1 + (l - s)^{-1} (f(x) - s)),$$

where $\eta(r) = 1$ when $r \le 1$, $0 < \eta(r) < 1$ when 1 < r < 2 and $\eta(r) = 0$ when $r \ge 2$, and $-2 < \eta' \le 0$, $|\eta''| \le 10$. Then ϕ satisfies

$$\phi(x) = \begin{cases} 1, & x \in D_s; \\ \in [0, 1], & x \in D_l \setminus D_s; \\ 0, & x \in M \setminus D_l. \end{cases}$$

One can verify that

$$|\nabla \phi| \le (l-s)^{-1} k \eta^{k-1} \eta' |\nabla f| \le \frac{Ck}{l-s} \phi^{1-1/k}$$

and

$$\begin{split} |\Delta\phi| &= \left| (l-s)^{-2}k(k-1)\eta^{k-2}(\eta')^2 |\nabla f|^2 + (l-s)^{-2}k\eta^{k-1}\eta'' |\nabla f|^2 + (l-s)^{-1}k\eta^{k-1}\eta'\Delta f \right| \\ &\leq \max\left\{ \frac{C}{s(l-s)}\phi^{1-2/k}, \frac{C}{(l-s)^2}\phi^{1-2/k} \right\} \end{split}$$

with C depends on M and k, using (24) and (25).

Now we calculate

$$\frac{\mathrm{d}}{\mathrm{d}t} \int |\Delta^m u|^2 \phi^2$$

$$= 2 \int \Delta^m u \Delta^m u_t \phi^2$$

$$= -2 \int \Delta^m u \Delta^m (-\Delta + V)^2 u \phi^2$$

$$= -2 \int \Delta^m u \Delta^{m+2} u \phi^2 - \Delta^m u \Delta^{m+1} (V u) \phi^2 - \Delta^m \Delta^m (V \Delta u) \phi^2 + \Delta^m \Delta^m (V^2 u) \phi^2.$$

We estimate

$$\int -\Delta^{m} u \Delta^{m+1}(Vu) \phi^{2} - \Delta^{m} u \Delta^{m}(V\Delta u) \phi^{2} + \Delta^{m} u \Delta^{m}(V^{2}u) \phi^{2}$$

$$\leq C \int |\Delta^{m} u| |\Delta^{m+1} u| \phi^{2} + |\Delta^{m} u| |\nabla \Delta^{m} u| \phi^{2} + |\Delta^{m} u| |\Delta^{m} u| \phi^{2} + \dots + |\Delta^{m} u| |u| \phi^{2}$$

$$\leq \epsilon \int |\Delta^{m+1} u|^{2} \phi^{2} + \int |\nabla \Delta^{m} u|^{2} \phi^{2} + C\epsilon^{-1} \int |\Delta^{m} u|^{2} \phi^{2} + C \int |\nabla \Delta^{m-1} u|^{2} \phi^{2} + \dots + \int |u|^{2} \phi^{2},$$

where the constant C > 0 depends on M, m and V.

For $j = 0, \dots, m - 1$,

$$\begin{split} &\int |\nabla \Delta^j u|^2 \phi^2 \\ &= \int -\Delta^j u \Delta^{j+1} u \phi^2 - 2\phi \langle \nabla \phi, \nabla \Delta^j u \rangle \Delta^j u \\ &\leq &\frac{1}{2} \int |\Delta^j u|^2 \phi^2 + |\Delta^{j+1} u|^2 \phi^2 + \frac{1}{2} \int |\nabla \Delta^j u|^2 \phi^2 + C \int |\nabla \phi|^2 |\Delta^j u|^2. \end{split}$$

Using the estimate for $|\nabla \phi|$, we obtain

(29)
$$\int |\nabla \Delta^j u|^2 \phi^2 \le \int |\Delta^j u|^2 \phi^2 + |\Delta^{j+1} u|^2 \phi^2 + \frac{C}{(l-s)^2} \int |\Delta^j u|^2 \phi^{2-2/k}.$$

with C > 0 depends on M and k. Similarly, we have

(30)
$$\int |\nabla \Delta^m u|^2 \phi^2 \le \frac{C}{\epsilon} \int |\Delta^m u|^2 \phi^2 + \epsilon \int |\Delta^{m+1} u|^2 \phi^2 + \frac{C}{(l-s)^2} \int |\Delta^m u|^2 \phi^{2-2/k}$$

with for any $\epsilon > 0$.

Combining (28), (29) and (30), we get

$$\int -\Delta^m u \Delta^{m+1}(Vu) \phi^2 - \Delta^m u \Delta^m (V\Delta u) \phi^2 + \Delta^m u \Delta^m (V^2 u) \phi^2$$

$$\leq 2\epsilon |\Delta^{m+1} u|^2 \phi^2 + \frac{C}{\epsilon} \int |\Delta^m u|^2 \phi^2 + \frac{C}{(l-s)^2} \sum_{j=0}^m \int |\Delta^j u|^2 \phi^{2-2/k} + C \sum_{j=1}^{m-1} \int |\Delta^j u|^2 \phi^2.$$

Taking $\epsilon = \frac{1}{4}$, we end up with

(31)
$$\int -\Delta^m u \Delta^{m+1}(Vu)\phi^2 - \Delta^m u \Delta^m (V\Delta u)\phi^2 + \Delta^m u \Delta^m (V^2 u)\phi^2$$

$$\leq \frac{1}{2} \int |\Delta^{m+1} u|^2 \phi^2 + \frac{C}{(l-s)^2} \sum_{j=0}^m \int |\Delta^j u|^2 \phi^{2-2/k}$$

with some constant C > 0 depending on M, V, m, k. We also have

$$(32) -2\int \Delta^{m}u\Delta^{m+2}u\phi^{2}$$

$$=\int 2\langle \nabla\Delta^{m}u, \nabla\Delta^{m+1}u\rangle\phi^{2} + 4\phi\Delta^{m}u\langle \nabla\Delta^{m+1}u, \nabla\phi\rangle$$

$$=-\int 2|\Delta^{m+1}u|^{2}\phi^{2} + 8\phi\Delta^{m+1}u\langle \nabla\Delta^{m}u, \nabla\phi\rangle + 4\Delta^{m}u\Delta^{m+1}u(\phi\Delta\phi + |\nabla\phi|^{2})$$

$$\leq -(2-3\epsilon)\int |\Delta^{m+1}u|^{2}\phi^{2} + \epsilon^{-1}\int 16|\nabla\Delta^{m}u|^{2}|\nabla\phi|^{2} + 4|\Delta^{m}u|^{2}(|\Delta\phi|^{2} + |\nabla\phi|^{4}\phi^{-2}).$$

and

$$\begin{split} &\int |\nabla \Delta^m u|^2 |\nabla \phi|^2 \\ \leq &\frac{C}{(l-s)^2} \int |\nabla \Delta^m u|^2 \phi^{2-2/k} \\ = &\frac{C}{(l-s)^2} \int -\Delta^m u \Delta^{m+1} u \phi^{2-2/k} - (2-2/k) \phi^{1-2/k} \Delta^m u \langle \nabla \Delta^m u, \nabla \phi \rangle \\ \leq &\frac{1}{2} \epsilon^2 \int |\Delta^{m+1} u|^2 \phi^2 + \frac{C}{(l-s)^4} \int |\Delta^m u|^2 \phi^{2-4/k} + \frac{1}{2} \int |\nabla \Delta^m u|^2 |\nabla \phi|^2, \end{split}$$

for $\epsilon > 0$ sufficiently small. So

(33)
$$\int |\nabla \Delta^m u|^2 |\nabla \phi|^2 \le \epsilon^2 \int |\Delta^{m+1} u|^2 \phi^2 + \frac{C}{(l-s)^4} \int |\Delta^m u|^2 \phi^{2-4/k}.$$

Taking ϵ small enough, and combining (27), (31), (32) and (33), we obtain

(34)
$$\frac{\mathrm{d}}{\mathrm{d}t} \int |\Delta^m u|^2 \phi^2 \le -\int |\Delta^{m+1} u|^2 \phi^2 + C \left(\frac{1}{(l-s)^2 s^2} + \frac{1}{(l-s)^4} \right) \int |\Delta^m u|^2 \phi^{2-4/k}$$

$$+ \frac{C}{(l-s)^2} \sum_{i=0}^m \int |\Delta^j u|^2 \phi^{2-4/k}.$$

For $i = 0, 1, \dots, m$, let ϕ_i be the cut-off function constructed above with $l = (1 - \frac{i}{2m+2})R$ and $l = (1 - \frac{i+1}{2m+2})R$. Then (34) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \int |\Delta^i u|^2 \phi_i^2 \le -\int |\Delta^{i+1} u|^2 \phi_i^2 + \frac{C}{R^4} \int |\Delta^i u|^2 \phi_i^{2-4/k} + \frac{C}{R^2} \sum_{j=0}^{i-1} \int |\Delta^j u|^2 \phi_i^{2-4/k}.$$

with some constant C > 0 depending on M, V, m, k. Now we define

$$F_m(t) = \sum_{i=0}^m a_i t^{i+1} \int |\Delta^i u|^2 \phi_i^2, \quad t \in [0, T].$$

Note that $\phi_j \leq \phi_{j-1}$. We calculate, for t, R small enough,

$$\frac{\mathrm{d}}{\mathrm{d}t}F_{m}(t)$$

$$\leq \sum_{i=1}^{m} \left(-a_{i-1}t^{i} \int |\Delta^{i}u|^{2}\phi_{i-1}^{2} + a_{i}(i+1)t^{i} \int |\Delta^{i}u|^{2}\phi_{i}^{2} + \frac{C}{R^{4}}a_{i}t^{i+1} \int |\Delta^{i}u|^{2}\phi_{i}^{2-4/k} \right) + \frac{C}{R^{2}}a_{i}t^{i+1} \sum_{j=1}^{i-1} \int |\Delta^{j}u|^{2}\phi_{i}^{2-4/k} \right) - a_{m}t^{m+1} \int |\Delta^{m+1}u|^{2}\phi_{m}^{2} + a_{0} \int u^{2}\phi_{0}^{2} + \frac{C}{R^{4}}a_{0}t \int u^{2}\phi_{0}^{2-4/k} \\
\leq \sum_{i=1}^{m} \left(-a_{i-1}t^{i} \int |\Delta^{i}u|^{2}\phi_{i-1}^{2} + a_{i}(i+1)t^{i} \int |\Delta^{i}u|^{2}\phi_{i}^{2} + \frac{C}{R^{4}}a_{i}t^{i+1} \int |\Delta^{i}u|^{2}\phi_{i}^{2-4/k} \right) \\
- a_{m}t^{m+1} \int |\Delta^{m+1}u|^{2}\phi_{m}^{2} + a_{0} \int u^{2}\phi_{0}^{2} + \frac{C}{R^{4}}a_{0}t \int u^{2}\phi_{0}^{2-4/k} .$$

Take

$$a_m = 1$$
, $a_{i=1} = (C\frac{T}{R^4} + i + 1)a_i$, $i = m, m - 1, \dots, 1$.

We then have

$$\frac{\mathrm{d}}{\mathrm{d}t}F_m(t) \le a_0 \int u^2 \phi_0^2 + \frac{C}{R^4} a_0 t \int u^2 \phi_0^{2-4/k}$$

Since $F_m(0) = 0$ and ϕ_0 is supported on D_R ,

$$F_m(t) \le (1 + CTR^{-4})a_0 \int_0^t \int_{D_R} |u(x,t)|^2.$$

Fixing k = 4, then for $T = R^4$ with R sufficiently small, one can prove (26) inductively.

We rewrite the above estimate (26) as

$$\int_{D_{R/2}} |(\sqrt{t}\Delta)^m u|^2(x,t) \le \frac{C}{t} \int_0^t \int_{D_R} u^2.$$

Taking $t = R^4$ and using [13, Lemma 3.6], we get the mean-value inequality

(35)
$$|u(p,t)| \le \frac{C}{R^{2+n/2}} \left(\int_0^t \int_{D_R} |u|^2 \right)^{1/2}$$

similar to [13, Lemma 3.7].

Lemma 4. Let $\xi = \xi(f(x), t)$ be a C^1 function whose Laplacian exists a.e. and let G(x, t) be a Lipschitz function satisfying

$$|\nabla \xi|^2(x,t) \le G(x,t),$$

then for any solution u of (23), we have

$$\partial_t \int_M u^2 e^{\xi} \le \int_M u^2 e^{\xi} (\partial_t \xi + CG^2 + CG + C|\nabla G|^2 G^{-1} + C|\Delta \xi|^2),$$

where C is some positive constant depending on M and V.

Proof. We calculate

$$\begin{split} &\frac{\mathrm{d}}{\mathrm{d}t} \int u^2 e^{\xi} - \int u^2 e^{\xi} \partial_t \xi \\ &= 2 \int u u_t e^{\xi} \\ &= -2 \int u (-\Delta + V)^2 u e^{\xi} \\ &= -2 \int \langle \nabla u, \nabla (-\Delta u + V u) \rangle e^{\xi} - 2 \int \langle \nabla \xi, \nabla (-\Delta u + V u) u e^{\xi} - 2 \int V u (-\Delta u + V u) e^{\xi} \\ &= -2 \int (-\Delta u + V u)^2 e^{\xi} + 4 \int \langle \nabla u, \nabla \xi \rangle (-\Delta u + V u) e^{\xi} + 2 \int \Delta \xi (-\Delta u + V u) u e^{\xi} \\ &+ 2 \int |\nabla \xi|^2 (-\Delta u + V u) e^{\xi}. \end{split}$$

Applying Cauchy-Schwarz inequality to each term containing $-\Delta u + Vu$, we get

$$\begin{split} &\frac{\mathrm{d}}{\mathrm{d}t} \int u^2 e^{\xi} - \int u^2 e^{\xi} \partial_t \xi \\ &\leq -2 \int (-\Delta u + Vu)^2 e^{\xi} + 2\epsilon \int (-\Delta u + Vu)^2 e^{\xi} + 2\epsilon^{-1} \int |\nabla u|^2 |\nabla \xi|^2 e^{\xi} \\ &+ \epsilon \int (-\Delta u + Vu)^2 e^{\xi} + \epsilon^{-1} \int |\Delta \xi|^2 u^2 e^{\xi} + \epsilon \int (-\Delta u + Vu)^2 e^{\xi} + \epsilon^{-1} \int |\nabla \xi|^4 u^2 e^{\xi} \\ &= (-2 + 4\epsilon) \int (-\Delta u + Vu)^2 e^{\xi} + 2\epsilon^{-1} \int |\nabla u|^2 |\nabla \xi|^2 e^{\xi} + \epsilon^{-1} \int |\Delta \xi|^2 u^2 e^{\xi} + \epsilon^{-1} \int |\nabla \xi|^4 u^2 e^{\xi} \end{split}$$

Taking $\epsilon = \frac{1}{4}$, we get

(36)
$$\frac{\frac{\mathrm{d}}{\mathrm{d}t} \int u^{2}e^{\xi} - \int u^{2}e^{\xi} \partial_{t} \xi}{\leq -\int (-\Delta u + Vu)^{2}e^{\xi} + 8\int |\nabla u|^{2}|\nabla \xi|^{2}e^{\xi} + 4\int |\Delta \xi|^{2}u^{2}e^{\xi} + 4\int |\nabla \xi|^{4}u^{2}e^{\xi}.}$$

Now note that by the assumption of the lemma

$$\int |\nabla u|^2 |\nabla \xi|^2 e^{\xi} \le \int |\nabla u|^2 G e^{\xi}.$$

Then we estimate

$$\int |\nabla u|^2 G e^{\xi} = -\int u(\Delta u - Vu) G e^{\xi} - \int Vuu G e^{\xi} - \int \langle \nabla u, \nabla G \rangle u e^{\xi} - \int \langle \nabla u, \nabla \xi \rangle G u e^{\xi}$$

$$\leq \epsilon \int (-\Delta u + Vu)^2 e^{\xi} + \frac{1}{4\epsilon} \int u^2 G^2 e^{\xi} + \epsilon \int |\nabla u|^2 G e^{\xi} + \frac{1}{4\epsilon} \int u^2 |\nabla G|^2 G^{-1} e^{\xi}$$

$$+ \epsilon \int |\nabla u|^2 G e^{\xi} + \frac{1}{4\epsilon} \int u^2 |\nabla \xi|^2 G e^{\xi} + C \int u^2 G e^{\xi}$$

$$\leq \epsilon \int (-\Delta u + Vu)^2 e^{\xi} + 2\epsilon \int |\nabla u|^2 G e^{\xi} + C \int u^2 G e^{\xi} + C(\epsilon) \int u^2 G^2 e^{\xi}$$

$$+ C(\epsilon) \int u^2 |\nabla G|^2 G^{-1} e^{\xi} + C(\epsilon) \int u^2 |\nabla \xi|^2 G e^{\xi}.$$

Taking ϵ small enough, we obtain

(37)
$$8 \int |\nabla u|^2 |\nabla \xi|^2 e^{\xi} \le 8 \int |\nabla u|^2 G e^{\xi} \le \int (-\Delta u + V u)^2 e^{\xi} + C \int u^2 G e^{\xi} + C \int u^2 G^2 e^{\xi} + C \int u^2 |\nabla G|^2 G^{-1} e^{\xi} + C \int u^2 |\nabla \xi|^2 G e^{\xi}$$

for some constant C > 0. Combining (36) and (37), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int u^{2}e^{\xi} - \int u^{2}e^{\xi} \partial_{t}\xi$$

$$\leq C \int u^{2}Ge^{\xi} + C \int u^{2}G^{2}e^{\xi} + C \int u^{2}|\nabla G|^{2}G^{-1}e^{\xi} + C \int u^{2}|\nabla \xi|^{2}Ge^{\xi}$$

$$+ C \int |\Delta \xi|^{2}u^{2}e^{\xi} + C \int |\nabla \xi|^{4}u^{2}e^{\xi}$$

$$\leq C \int u^{2}Ge^{\xi} + C \int u^{2}G^{2}e^{\xi} + C \int u^{2}|\nabla G|^{2}G^{-1}e^{\xi} + C \int |\Delta \xi|^{2}u^{2}e^{\xi}.$$

This completes the proof.

Now we are ready to prove the estimate (13) for the biharmonic heat equation. Let $f = f_p$ be the distance function. Choose R, S small enough and let $0 \le t < T$. Define

(38)
$$\xi = \begin{cases} 0 & \text{on } D_R; \\ -\frac{S^{-8/3}(f-R)^4[1-\frac{8}{3}S^{-1}(f-R-S)]}{A(T-t)^{1/3}} & \text{on } D_{R+S} \setminus D_R; \\ -\frac{(f-R)^{4/3}}{A(T-t)^{1/3}} & \text{on } M \setminus D_{R+S}, \end{cases}$$

and

(39)
$$G = \begin{cases} 0 & \text{on } D_R; \\ \frac{C^2 S^{-16/3} (f-R)^6 [4 - \frac{32}{3} S^{-1} (f-R-S) - \frac{8}{3} S^{-1} (f-R)]^2}{A^2 (T-t)^{2/3}} & \text{on } D_{R+S} \setminus D_R; \\ \frac{16 C^2 (f-R)^{2/3}}{9 A^2 (T-t)^{2/3}} & \text{on } M \setminus D_{R+S}, \end{cases}$$

with some constant C appearing in (24). One can check that ξ is in C^1 , G is Lipschitz and $|\nabla \xi|^2 \leq G$.

On $D_{R+S} \setminus D_R$, the terms in the brackets in the definitions of ξ and G are bounded by uniform positive constants from both above and below, and their gradients are bounded by CS^{-1} . By direct calculation, we find

$$\mathcal{N} = \partial_t \xi + CG^2 + CG + C|\nabla G|^2 G^{-1} + C|\Delta \xi|^2
\leq -\frac{S^{-8/3} (f - R)^4}{3A(T - t)^{4/3}} + \frac{CS^{-32/3} (f - R)^{12}}{A^4 (T - t)^{4/3}} + \frac{CS^{-16/3} (f - R)^6}{A^2 (T - t)^{2/3}}
+ \frac{CS^{-16/3} (f - R)^4}{A^2 (T - t)^{2/3}} + \frac{CS^{-16/3} (f - R)^6}{A^2 R^2 (T - t)^{2/3}}
= \frac{S^{-8/3} (f - R)^4}{3A(T - t)^{4/3}} \Big[-1 + \frac{3CS^{-8} (f - R)^8}{A^3} + \frac{3CS^{-8/3} (f - R)^2 (T - t)^{2/3}}{A}
+ \frac{3CS^{-8/3} (T - t)^{2/3}}{A} \Big] + \frac{3CS^{-8/3} (f - R)^2 (T - t)^{2/3}}{AR^2} \Big]$$

Here we have used the estimates (24) and (25). Using the fact that $0 \le f - R \le S$ on $D_{R+S} \setminus D_R$, we can take A large enough and choose

$$(40) T \le \min\{S^4, R^3S\},$$

we have $\mathcal{N} \leq 0$.

On $M \setminus D_{R+S}$, we have

$$\begin{split} \mathcal{N} & \leq -\frac{(f-R)^{4/3}}{3A(T-t)^{4/3}} + \frac{C(f-R)^{4/3}}{A^4(T-t)^{4/3}} + \frac{C(f-R)^{2/3}}{A^2(T-t)^{2/3}} + \frac{C(f-R)^{-4/3}}{A^2(T-t)^{2/3}} + \frac{C(f-R)^{2/3}}{A^2(T-t)^{2/3}} + \frac{C(f-R)^{2/3}}{A^2(T-t)^{2/3}f^2} \\ & \leq \frac{(f-R)^{4/3}}{3A(T-t)^{4/3}} \Big[-1 + \frac{C}{A^3} + \frac{C(T-t)^{2/3}S^{-2/3}}{A} + \frac{C(T-t)^{2/3}S^{-8/3}}{A} + \frac{C(S+R)^{-2}S^{-2/3}(T-t)^{2/3}}{A} \Big]. \end{split}$$

Here we have used the fact that $f - R \ge S$ on $M \setminus D_{R+S}$. Then we can make $\mathcal{N} \le 0$ by choosing A large enough and

$$(41) T \le S^4.$$

Then for any solution u of (23) that is in L^2 , by Lemma 4, we get

$$\partial_t \int u^2 e^{\xi} \le 0.$$

for $t \in (0,T)$. Now for t > 0 sufficiently small, we take $R = S = t^{1/4}$. Then

$$\int u^2 e^{\xi}(t) \le \int u^2 e^{\xi}(0).$$

By the mean value inequality (35), we have

$$(42) u(p,t)^2 \le \frac{C}{R^{4+n}} \int_0^t \int_{D_R} u^2 \le \frac{C}{R^{4+n}} \int_0^t \int_M u^2 e^{\xi} \le \frac{Ct}{R^{4+n}} \int_M u(x,0)^2 e^{\xi(x,0)}.$$

Now, we take a particular solution of (23)

(43)
$$u(x,s) = \int_{M} b(x,y,s)b(p,y,t)e^{-\xi(y,0)}dy.$$

By the properties of the fundamental solution,

$$u(x,0) = b(x, p, t)e^{-\xi(x,0)}$$

which is in L^2 . Then by (43) and (42),

$$\left(\int b(p,y,t)^2 e^{-\xi(y,0)} \mathrm{d}y \right)^2 = u(p,t)^2 \le \frac{Ct}{R^{n+4}} \int_M b(x,p,t)^2 e^{-\xi(x,0)} \mathrm{d}x.$$

By the symmetry of b(x, y, t), we have

$$\int_{M} b(p, y, t)^{2} e^{-\xi(y, 0)} dy \le \frac{Ct}{R^{n+4}},$$

which can be rewritten as

$$E_p(t) := \int_M b(p, y, t)^2 e^{\eta_p(y, t)} dy \le \frac{Ct}{R^{n+4}},$$

where

$$\eta_p = \begin{cases} 0 & \text{on } D_R; \\ -\frac{S^{-8/3}(f-R)^4[1-\frac{8}{3}S^{-1}(f_p-R-S)]}{At^{1/3}} & \text{on } D_{R+S} \setminus D_R; \\ -\frac{(f-R)^{4/3}}{At^{1/3}} & \text{on } M \setminus D_{R+S}. \end{cases}$$

By the semigroup property of the biharmonic heat equation, we have

$$b(p,q,t) = \int_M b(p,x,\frac{t}{2})b(x,q,\frac{t}{2})\mathrm{d}x.$$

By the triangle inequality

$$d(p,q) \le d(p,y) + d(q,y) \le f_p(y) + f_q(y),$$

we get

$$(d(p,q) - 2R - 2S)_{+} \le (f_p(y) - R - S)_{+} + (f_q(y) - R - S)_{+}, \quad \forall y \in M.$$

We can choose c > 0 small enough such that

$$\frac{c(d(p,y) - 2R - 2S)_{+}^{4/3}}{t^{1/3}} \le \eta_p(y,t) + \eta_q(y,t), \quad \forall y \in M.$$

By Hölder's inequality,

$$|b(p,q,t)| \le \int |b(p,x,t/2)| e^{\eta_p(x,t/2)} |b(q,x,t/2)| e^{\eta_1(x,t/2)} e^{-\frac{c(d(p,q)-2R-2S)_+^{4/3}}{t^{1/3}}} dx$$

$$\le \sqrt{E_p(t/2)E_q(t/2)} e^{-\frac{c(d(p,q)-2R-2S)_+^{4/3}}{t^{1/3}}}.$$

By the choice of S, R, t, and using the fact

$$(d(p,q) - 2R - 2S)_{+}^{4/3} + (2R)^{4/3} + (2S)^{4/3} \ge c_0((d(p,q) - 2R - 2S)_{+} + 2R + 2S)^{4/3} \ge c_0d(p,q)^{4/3}$$

for some constant $c_0 > 0$, we conclude

$$|b(p,q,t)| \le \frac{C}{t^{n/4}} e^{-\frac{cd(p,q)^{4/3}}{t^{1/3}}},$$

as desired.

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