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The factorization method for cracks in electrical impedance tomography

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Abstract
The inverse problem we are dealing with is to recover the inclusion of cracks in electrical impedance tomography from boundary measurements of current and voltage. Both the cases of Neumann and impedance boundary conditions posed on the cracks are considered. Assuming the anisotropic background conductivity is known a priori, we prove the factorization method can be applied to reconstruct the shape and location of the cracks. The numerical examples are shown to illustrate the correctness and effectiveness of the proposed method. This work is an extension of the study investigated by Brühl et al. (ESAIM Math Model Numer Anal 35:595–605, 2001) where an insulating crack embedded in homogeneously conducting object is considered.

Keywords Electrical impedance tomography · Crack · Anisotropic background conductivity · The factorization method

Mathematics Subject Classification 35R30 · 35Q60

1 Introduction

The inverse problem of electrical impedance tomography (EIT) is to recover spatial properties of the interior of a conducting object from current–voltage measurements taken on

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its boundary. Taking the medical imaging for example, the electrical properties of a living organism can be characterized by measuring the ratio of an externally applied current to the resulting voltage. The portability and the low cost of electronic devices capable of producing such data makes it an ideal tool for medical imaging. Various practically important imaging problems consider locating inhomogeneities inside objects with known background conductivities. For example, detection of cracks in some building material and air bubbles in distinguishing cancerous tissue from healthy background fall into this category of problems. In this paper, we cope with the determination of a crack type inclusion buried in a known anisotropic background electrical conductor.

We use factorization method to solve the inverse problem of under consideration. This qualitative and also non-iterative method is introduced by Brühl et al. (2001) to detect an insulating crack inside homogeneous material. We will extend their work in two aspects. The first one is that the background conductivity is anisotropic and the second is that the impedance boundary condition is posed on the crack. The anisotropic background conductivity gives rise to difficulties in the analysis under the framework of the factorization method. On the other hand, the impedance boundary condition requires a non intuitive choice of the solution space. Moreover, it is more harder to investigate the properties of the operators associated with the factorization form of the Neumann-to-Dirichlet (NtD) map in such a case. The similar situation has been considered by Boukari and Haddar (2013) in acoustic scattering, where the factorization method is used to retrieve cracks with impedance boundary conditions from far fields corresponding to incident plane waves at fixed frequency. But the analysis technique used here for EIT is different from theirs as we will see in the sequel context.

Historically, the factorization method is adopted initially by Kirsch and Ritter (2000) for inverse scattering problem of reconstructing cracks from far field pattern of scattering data. Brühl et al. (2001) develop this method in EIT to detect insulating cracks within a homogeneously conducting medium and give a modified version which is much simpler and different in spirit from Kirsch's. Applying an asymptotic expansion of the electrostatic potential on the boundary of the body as the thickness of the inclusions tends to zero, Griesmaier (2010) establishes an asymptotic characterization of these thin inclusions by using a generalization of the factorization method. A short survey on uniqueness results, stability results and reconstruction methods concerning crack identification using impedance imaging technique can be found in Bryan and Vogelius (2004) until 2002. We refer the reader to Alvarez et al. (2009) for using a level-set strategy and to Karhunen et al. (2014) for employing adaptive meshing approach to detect cracks in EIT. We also mention a recent work (Guo et al. 2015) of using near field factorization method to determine a crack with impedance boundary conditions due to the complex conjugate of point sources in inverse scattering problem.

The factorization method is introduced first by Kirsch in Kirsch (1998) for inverse scattering problems and then carried over by Brühl in Brühl (2001) to the inverse problems in EIT. Since then, it has been intensively studied in the mathematical literature for inverse problems in acoustic, electromagnetic and elastic scattering and in EIT. Overviews on this method for EIT have been given by Hanke and Bruhl (2003), Harrach (2013) and in the monographs (Kirsch and Grinberg 2008; Scherzer 2011). This method is proved to be very successful for electrical impedance tomography in the applications of finding inclusions in a known background. The case of complex and anisotropic conductivities is considered in Kirsch (2005), the complete electrode model has been covered in Lechleiter et al. (2008), the mixed inclusions with different types are treated in Schmitt (2009), the case of inclusions in realistic inhomogeneous three-dimensional background medium is discussed in Chaulet et al. (2014) and Harrach and Seo (2009), based on frequency-different measurements but without reference measurements, a new variant of the factorization method is established to locate inclusions.

The rest of the paper is organized as follows. In the next section, we briefly present two forward problems and show the results on variational solutions to these problems. Then we introduce the two inverse problems considered in this paper. In Sect. 3, following the idea in Kirsch (2005), the theoretical justification of the factorization method for the identification of the insulating crack within anisotropic background medium is proved. In Sect. 4, by a more detailed analysis of the mapping properties of the related operators, the same program is carried over in non-usual way to the case of impedance boundary condition imposed on both sides of the crack. A numerical experiment approach for the inverse problems is given in Sect. 5.

2 The formulation of the direct and inverse problems

Let $B \subset \mathbb{R}^n$, $n = 2$ or 3 denote a bounded connected domain with smooth boundary ∂B. Let $\gamma : B \to \mathbb{C}^{n\times n}$ be a matrix-valued complex function and we set γ be of the form $\gamma = \sigma - i\omega\varepsilon$ with conductivity σ, frequency ω and permittivity ε. We further assume that γ has the form

$$\gamma(x) = \begin{cases} \gamma_0, & x \in B\backslash\Omega, \\ \gamma_\Omega(x), & x \in \Omega, \end{cases}$$

where γ_0 is a constant Hermitian and $\gamma_\Omega \in L^\infty(\Omega)$ is a complex valued Hermitian such that there exists $c_0 > 0$ with

$$Re[z^*\gamma(x)z] \geq c_0|z|^2 \tag{2.1}$$

for all nonzero vector $z \in \mathbb{C}^n$ and $x \in B$. Here $\Omega \subset \mathbb{R}^n$ represents a bounded domain with smooth boundary $\partial\Omega$ such that $\overline{\Omega} \subset B$ and $B\backslash\overline{\Omega}$ is connected. The admittivity function γ models the anisotropic conductive object B.

Define the functional spaces $L^2_\diamond(B)$, $H^1_\diamond(B)$, $H^{\frac{1}{2}}_\diamond(\partial B)$ and $H^{-\frac{1}{2}}_\diamond(\partial B)$ as the subspaces of respectively $L^2(B)$, $H^1(B)$, $H^{\frac{1}{2}}(\partial B)$ and $H^{-\frac{1}{2}}(\partial B)$ of functions with zero mean values on ∂B. Moreover, for a smooth open crack Γ locating in Ω such that Γ can be extended to an arbitrary smooth, simply connected, closed curve Σ enclosing a bounded domain $D \subset \Omega$, the normal vector ν on Γ coincides with the outward normal vector on ∂D. We introduce the following spaces on Γ:

$$\begin{aligned} &H^{\frac{1}{2}}(\Gamma) = \{u|_\Gamma : \ u \in H^{\frac{1}{2}}(\partial D)\}, \\ &\widetilde{H}^{\frac{1}{2}}(\Gamma) = \{u \in H^{\frac{1}{2}}(\partial D) : \ \mathrm{supp} u \subseteq \overline{\Gamma}\}, \\ &H^{-\frac{1}{2}}(\Gamma) = (\widetilde{H}^{\frac{1}{2}}(\Gamma))', \ \text{the dual space of } \widetilde{H}^{\frac{1}{2}}(\Gamma), \\ &\widetilde{H}^{-\frac{1}{2}}(\Gamma) = (H^{\frac{1}{2}}(\Gamma))', \ \text{the dual space of } H^{\frac{1}{2}}(\Gamma). \end{aligned}$$

Then the spaces $H^{\frac{1}{2}}_\diamond(\Gamma)$, $H^{-\frac{1}{2}}_\diamond(\Gamma)$, $\widetilde{H}^{\frac{1}{2}}_\diamond(\Gamma)$ and $\widetilde{H}^{-\frac{1}{2}}_\diamond(\Gamma)$ possess exact definitions. We also need the following Sobolev spaces:

$$H^1_{\diamond,\partial B}(B\backslash\overline{\Gamma}) = \left\{u \in H^1(B\backslash\overline{\Gamma}) : \int_{\partial B} u\mathrm{d}s = 0\right\},$$

$$H^1_{\diamond,\Gamma}(B\backslash\overline{\Gamma}) = \left\{u \in H^1(B\backslash\overline{\Gamma}) : \int_\Gamma u\mathrm{d}s = 0\right\}.$$

In the absence of crack, for the injection of a current f on the boundary ∂B, the induced electric potential u_0 inside B satisfies the boundary value problem

$$\begin{cases} \nabla\cdot(\gamma\nabla u_0) = 0, & \text{in } B, \\ \gamma\dfrac{\partial u_0}{\partial\nu} = f, & \text{on } \partial B, \end{cases} \tag{2.2}$$

where ν denotes the outward normal to ∂B. It is well known that whenever $f \in H_\diamond^{-\frac{1}{2}}(\partial B)$, problem (2.2) has a unique solution $u_0 \in H^1_\diamond(B)$. Then using the trace theorem, we can define the so-called Neumann-to-Dirichlet (NtD) operator

$$\Lambda_0 : H_\diamond^{-\frac{1}{2}}(\partial B) \to H_\diamond^{\frac{1}{2}}(\partial B), \quad f \mapsto u_0|_{\partial B},$$

where u_0 solves (2.2).

In the presence of an insulating crack, the same Neumann boundary condition on ∂B yields a potential u_1 which solves the diffraction problem: the Neumann-crack problem (**NCP**)

$$\begin{cases} \nabla\cdot(\gamma\nabla u_1) = 0, & \text{in } B\backslash\overline{\Gamma}, \\ \gamma\dfrac{\partial u_1}{\partial\nu} = f, & \text{on } \partial B, \\ \gamma\dfrac{\partial u_1}{\partial\nu} = 0, & \text{on } \Gamma. \end{cases} \tag{2.3}$$

In this paper, we also consider the case when the boundary condition on Γ is set as impedance boundary condition. Then we have the boundary valued problem for elliptic equation as below: the mixed-crack problem (**MCP**)

$$\begin{cases} \nabla\cdot(\gamma\nabla u_2) = 0, & \text{in } B\backslash\overline{\Gamma}, \\ \gamma\dfrac{\partial u_2}{\partial\nu} = f, & \text{on } \partial B, \\ \left[\gamma\dfrac{\partial u_2}{\partial\nu}\right] + \lambda_+ u_{2,+} - \lambda_- u_{2,-} = 0, & \text{on } \Gamma, \\ \gamma\dfrac{\partial u_{2,-}}{\partial\nu} + \lambda_- u_{2,-} = 0, & \text{on } \Gamma. \end{cases} \tag{2.4}$$

Here the coefficients $\lambda_\pm$ are real constants and satisfy $\lambda_+ \geq 0, \lambda_- \leq 0$, the following notations are understood as $u_{2,\pm}(x) = \lim_{h\to 0^+} u_2(x \pm h\nu)$, $\frac{\partial u_{2,\pm}}{\partial\nu} = \lim_{h\to 0^+} \nu\cdot\nabla u_2(x \pm h\nu)$ and $\left[\frac{\partial u_2}{\partial\nu}\right] = \frac{\partial u_{2,+}}{\partial\nu} - \frac{\partial u_{2,-}}{\partial\nu}$ for $x \in \Gamma$.

Remark 2.1 The boundary condition of problem (2.4) on Γ is essentially the impedance-type, we design it into this form in order to study our inverse problem. This treatment is used by Boukari and Haddar in Boukari and Haddar (2013) to establish the factorization method for recovering cracks with impedance boundary conditions. But the justification of the factorization method in EIT is different from inverse scattering problem. Particularly, different techniques are employed to study NtD operator. Several similar boundary value problems should be considered in Sects. 3 and 4, where we will give necessary exposition in view of Theorem 2.1.

To study the related inverse problems described in the end of this section, we give a concise discussion on the well-posedness of the forward problems and state the result as follows.

Theorem 2.1 *Let $f \in H_\diamond^{-\frac{1}{2}}(\partial B)$, then under **Assumption** 2.1 given below, both of the problems **NCP** and **MCP** have a unique solution u_1 and u_2 in $H^1_{\diamond,\partial B}(B\backslash\overline{\Gamma})$, respectively, and satisfy*

$$\|u_{1,2}\|_{H^1(B\backslash\overline{\Gamma})} \le c\|f\|_{H^{-\frac{1}{2}}(\partial B)},$$

where the constant c depends on the domain B and the admittivity function γ.

Proof Problems (2.3) and (2.4) can be reformulated as the following variational formulas:

$$\int_{B\backslash\overline{\Gamma}} \nabla\psi^*\gamma\nabla u_1 \mathrm{d}x = \int_{\partial B} f\overline{\psi}\mathrm{d}s, \tag{2.5}$$

and

$$\int_{B\backslash\overline{\Gamma}} \nabla\psi^*\gamma\nabla u_2 \mathrm{d}x - \int_{\Gamma} \lambda_+ u_{2,+}\overline{\psi}_+ \mathrm{d}s + \int_{\Gamma} \lambda_- u_{2,-}\overline{\psi}_- \mathrm{d}s = \int_{\partial B} f\overline{\psi}\mathrm{d}s \tag{2.6}$$

for $\psi \in H^1_{\diamond,\partial B}(B\backslash\overline{\Gamma})$, respectively. It is easy to see that problem (2.5) has a unique solution by using the Lax–Milgram theorem.

We denote by a sesquilinear form $A(u_2, \psi)$ associated with the left hand side of (2.6) and by $L(\psi)$ the antilinear form associated with the right hand side. Since

$$\begin{aligned}
|A(u_2,\psi)| \le\ & \|\gamma\|_{L^\infty(B)}\|u_2\|_{L^2(B\backslash\overline{\Gamma})}\|\psi\|_{L^2(B\backslash\overline{\Gamma})} \\
&+|\lambda_+|\|u_{2,+}\|_{H^{\frac{1}{2}}(\Gamma)}\|\psi_+\|_{H^{\frac{1}{2}}(\Gamma)} \\
&+|\lambda_-|\|u_{2,-}\|_{H^{\frac{1}{2}}(\Gamma)}\|\psi_-\|_{H^{\frac{1}{2}}(\Gamma)} \\
\le\ & c\|u_2\|_{H^1(B\backslash\overline{\Gamma})}\|\psi\|_{H^1(B\backslash\overline{\Gamma})}
\end{aligned}$$

and

$$|L(\psi)| \le \|f\|_{H^{-\frac{1}{2}}(\partial B)}\|\psi\|_{H^{\frac{1}{2}}(\partial B)} \le c\|f\|_{H^{-\frac{1}{2}}(\partial B)}\|\psi\|_{H^1(B\backslash\overline{\Gamma})}$$

by the trace theorem, we obtain that $A(\cdot,\cdot)$ is a bounded sesquilinear form on $H^1_{\diamond,\partial B}(B\backslash\overline{\Gamma}) \times H^1_{\diamond,\partial B}(B\backslash\overline{\Gamma})$ and L is a bounded conjugate linear functional on $H^1_{\diamond,\partial B}(B\backslash\overline{\Gamma})$.

Decompose the sesquilinear form $A(\cdot,\cdot)$ into a coercive part

$$A_0(u_2,\psi) = \int_{B\backslash\overline{\Gamma}} \nabla\psi^*\gamma\nabla u_2 \mathrm{d}x$$

and a compact one

$$A_c(u_2,\psi) = -\int_{\Gamma} \lambda_+ u_{2,+}\overline{\psi}_+ \mathrm{d}s + \int_{\Gamma} \lambda_- u_{2,-}\overline{\psi}_- \mathrm{d}s.$$

Indeed, using the properties (2.1) and the Poincaré inequality, we obtain that

$$Re A_0(\psi,\psi) \ge c_0\|\nabla\psi\|^2_{L^2(B\backslash\overline{\Gamma})} \ge c\|\psi\|^2_{H^1(B\backslash\overline{\Gamma})},$$

which shows the coercivity of A_0. The compactness of A_c follows from the trace theorem and the Rellich compact embedding theorem. Thus $A(\cdot,\cdot)$ is a Fredholm operator of index 0.

We next show the injectivity of this operator. Note that the following holds:

$$
\begin{aligned}
&Re\Big(\int_{B\backslash\overline{\Gamma}} \nabla u_2{}^*\gamma\nabla u_2 \mathrm{d}x - \int_\Gamma \lambda_+ u_{2,+}\overline{u_2}_+ \mathrm{d}s + \int_\Gamma \lambda_- u_{2,-}\overline{u_2}_- \mathrm{d}s\Big)\\
&\geq c\|u_2\|^2_{H^1(B\backslash\overline{\Gamma})} - (\lambda_+ - \lambda_-)\|u_2\|^2_{L^2(\Gamma)}\\
&\geq c\|u_2\|^2_{H^1(B\backslash\overline{\Gamma})} - c_1(\lambda_+ - \lambda_-)\|u_2\|^2_{H^{\frac{1}{2}}(\Gamma)}\\
&\geq c\|u_2\|^2_{H^1(B\backslash\overline{\Gamma})} - c_2(\lambda_+ - \lambda_-)\|u_2\|^2_{H^1(B\backslash\overline{\Gamma})},
\end{aligned}
$$

where the first inequality is obtained by the properties (2.1) and the Poincaré inequality, the second inequality is due to the imbedding theorem and the third inequality is owing to the trace theorem. In order to prove the injectivity of A, we make the following assumption given by

Assumption 2.1 Assume that $c - c_2(\lambda_+ - \lambda_-) > 0$.

Remark 2.2 It seems that the inequality is rough, since we do not derive the exact expressions of the constants c and c_2. Here, we are not going to commit ourself to such a issue. We note that if **Assumption** 2.1 does not hold, then there may exists eigenvalues for **MCP**.

Taking the real part of $A(u_2, u_2) = 0$, which implies, using the **Assumption** 2.1

$$
(c - c_2(\lambda_+ - \lambda_-))\|u_2\|^2_{H^1(B\backslash\overline{\Gamma})} \leq ReA(u_2, u_2) = 0.
$$

Therefore, we see that $\nabla u_2 = 0$ in $B\backslash\overline{\Gamma}$ from the properties (2.1). Since u_2 belongs to $H^1_{\diamond,\partial B}(B\backslash\overline{\Gamma})$ we have that $u_2 = 0$ in $B\backslash\overline{\Gamma}$. Hence, there exists a unique solution to (2.6) by the theorem of Riesz Fredholm. The proof is then completed. □

The well-posedness of the **NCP** defines the NtD operator Λ_1 by

$$
\Lambda_1 : H_\diamond^{-\frac{1}{2}}(\partial B) \to H_\diamond^{\frac{1}{2}}(\partial B), \quad f \mapsto u_1|_{\partial B},
$$

where u_1 is the unique solution to (2.3) for $f \in H_\diamond^{-\frac{1}{2}}(\partial B)$, and we denote by Λ_2 the NtD map corresponding to the forward problem **MCP**

Let the admittivity γ of the background object B be known in advance, the undetermined crack is buried in $\Omega \subset B$. The inverse problems that we deal with are: (**INCP**) reconstruct the Neumann-crack from the knowledge of the NtD map Λ_1 and (**IMCP**) determine the mixed-crack from the the knowledge of NtD map Λ_2, due to the injection of a current f on the boundary ∂B.

Remark 2.3 We aim at applying the factorization method to solve the above mentioned inverse problems, which can be seen as an extension of Brühl's work (Brühl et al. 2001).

3 The factorization method for INCP

This section is devoted to the inverse problem of determining the Neumann-crack using the factorization method, which based on an appropriate decomposition of the difference $\Lambda_1 - \Lambda_0$ between the two Neumann-to-Dirichlet maps Λ_0 and Λ_1 such that the range identity theorem (Kirsch and Grinberg 2008) is suitable for it.

We make the following assumption in this section:

Assumption 3.1 assume that there is nontrivial potential function $u \in H^1_\diamond(B)$ satisfies $\nabla \cdot (\gamma \nabla u) = 0$ in B such that $\gamma \frac{\partial u}{\partial \nu}|_\Gamma = 0$.

Consider the operator

$$G : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}_\diamond(\partial B), \quad \phi \mapsto u|_{\partial B}.$$

Here $u \in H^1_{\diamond,\partial B}(B \backslash \overline{\Gamma})$ solves the following boundary value problem with $\phi \in H^{-\frac{1}{2}}(\Gamma)$ (The unique solvability can be obtained by the same argument as the proof process of problem (2.3)).

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0, & \text{in } B \backslash \overline{\Gamma}, \\ \gamma \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial B, \\ \gamma \dfrac{\partial u}{\partial \nu} = \phi, & \text{on } \Gamma. \end{cases} \tag{3.1}$$

Let u_0 be the solution of problem (2.2) with boundary data $f \in H_\diamond^{-\frac{1}{2}}(\partial B)$, and we define the operator

$$F : H_\diamond^{-\frac{1}{2}}(\partial B) \to H^{-\frac{1}{2}}(\Gamma), \quad f \mapsto \gamma \frac{\partial u_0}{\partial \nu}\Big|_\Gamma.$$

Additionally, the following operator will appear:

$$T : \widetilde{H}^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma), \quad \varphi \mapsto \gamma \frac{\partial v}{\partial \nu}\Big|_\Gamma.$$

Here $v \in H^1_{\diamond,\partial B}(B \backslash \overline{\Gamma})$ is the solution to the transmission boundary value problem with $\varphi \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$:

$$\begin{cases} \nabla \cdot (\gamma \nabla v) = 0, & \text{in } B \backslash \overline{\Gamma}, \\ \gamma \dfrac{\partial v}{\partial \nu} = 0, & \text{on } \partial B, \\ \left[\gamma \dfrac{\partial v}{\partial \nu}\right] = 0, & \text{on } \Gamma, \\ [v] = \varphi, & \text{on } \Gamma. \end{cases} \tag{3.2}$$

Note that this problem is solvable by using the variational approach. See for example (Guo et al. 2016), where an analogous problem related to inverse scattering is discussed.

Since $u_1 - u_0$ satisfies problem (3.1) with $\phi = -\gamma \frac{\partial u_0}{\partial \nu}$, recall the definitions of the NtD map Λ_0 and Λ_1 and we deduce that

$$(\Lambda_1 - \Lambda_0)f = (u_1 - u_0)|_{\partial B} = -G\left(\gamma \frac{\partial u_0}{\partial \nu}\Big|_\Gamma\right) = -G(Ff). \tag{3.3}$$

In what follows, we proceed to explore the properties of the operators G, F and T and present a critical relationship between them.

Let v be the solution to problem (3.2), we conclude that the adjoint operator F^* of F is given by

$$F^* : \widetilde{H}^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}_\diamond(\partial B), \quad \varphi \mapsto v|_{\partial B}.$$

In fact, applying Green's formula to u_0 and v in the domain $B\backslash\overline{\Gamma}$ and noting that γ is a Hermitian and the boundary conditions satisfied by u_0 and v, we have

$$\begin{aligned}
\langle Ff, \varphi\rangle &= \int_\Gamma \gamma \frac{\partial u_0}{\partial \nu}\overline{\varphi}\mathrm{d}s = \int_\Gamma \gamma \frac{\partial u_0}{\partial \nu}(\overline{v}_+ - \overline{v}_-)\mathrm{d}s \\
&= \int_{\partial B} \gamma \frac{\partial u_0}{\partial \nu}\overline{v}\mathrm{d}s - \int_{B\backslash\overline{\Gamma}} (\gamma\nabla u_0)\cdot\nabla\overline{v}\mathrm{d}x \\
&= \int_{\partial B} \gamma \frac{\partial u_0}{\partial \nu}\overline{v}\mathrm{d}s - \int_{B\backslash\overline{\Gamma}} (\nabla u_0)\cdot(\gamma^\top\nabla\overline{v})\mathrm{d}x \\
&= \langle f, v\rangle - \int_{\partial B} \gamma^\top \frac{\partial \overline{v}}{\partial \nu}u_0\mathrm{d}s + \int_\Gamma \left[\gamma^\top \frac{\partial \overline{v}}{\partial \nu}\right]u_0\mathrm{d}s \\
&= \langle f, v\rangle = \langle f, F^*\varphi\rangle.
\end{aligned}$$

Here the expression $\langle\cdot,\cdot\rangle$ denotes the dual pairing between the corresponding dual spaces. The sixth identity is obtained since u_0 has no jump across Γ.

Next we show that $T : \widetilde{H}^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ is self-adjoint. To this end, let w be the solution of problem (3.2) for φ instead of $\eta \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$. Then by Green's formula, we have

$$\begin{aligned}
&\langle T\varphi, \eta\rangle - \overline{\langle \gamma\frac{\partial w}{\partial \nu}, \varphi\rangle} = \int_\Gamma \gamma\frac{\partial v}{\partial \nu}(\overline{w}_+ - \overline{w}_-)\mathrm{d}s - \int_\Gamma \gamma^\top\frac{\partial \overline{w}}{\partial \nu}(v_+ - v_-)\mathrm{d}s \\
&= \int_\Gamma \left(\gamma\frac{\partial v}{\partial \nu}\overline{w}_+ - v_+\gamma^\top\frac{\partial \overline{w}}{\partial \nu}\right)\mathrm{d}s - \int_\Gamma \left(\gamma\frac{\partial v}{\partial \nu}\overline{w}_- - v_-\gamma^\top\frac{\partial \overline{w}}{\partial \nu}\right)\mathrm{d}s \\
&= \int_{\partial B} \left(\gamma\frac{\partial v}{\partial \nu}\overline{w} - v\gamma^\top\frac{\partial \overline{w}}{\partial \nu}\right)\mathrm{d}s = 0,
\end{aligned}$$

which proves the desired assertion.

Since the solution v, uniquely determined by problem (3.2), satisfies problem (3.1) with $\phi = \gamma\frac{\partial v}{\partial \nu}|_\Gamma$; hence, $v|_{\partial B} = G(\gamma\frac{\partial v}{\partial \nu}|_\Gamma)$. By the definition of T^* we obtain that $v|_{\partial B} = G(T^*\varphi)$. On the other hand, the definition of F^* shows us that $v|_{\partial B} = F^*\varphi$. Therefore, we conclude that $F^* = GT^*$, which combined with (3.3) yields

$$\Lambda_1 - \Lambda_0 = G(-T)G^*. \tag{3.4}$$

As we will see later, the range of G depends in a very specific way on the crack Γ, or rather, the test function χ_z parameterized by $z \in R^n$ belongs to the range of G if and only of $z \in \Gamma$. The aim of the factorization method is to verify that the range of G equals to the available operator $\Lambda_1 - \Lambda_0$. This achievement can be accomplished by applying the range identity theorem to the decomposition formula (3.4). It is perfectly feasible provided the operators G and T satisfy appropriate properties.

Lemma 3.1 *The operator T is an isomorphism from $\widetilde{H}^{\frac{1}{2}}(\Gamma)$ to $H^{-\frac{1}{2}}(\Gamma)$. Assume that γ satisfies* (2.1)*, then we have*

$$-Re\langle T\varphi, \varphi\rangle \geq c\|\varphi\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)} \quad \text{and} \quad -Im\langle T\varphi, \varphi\rangle = 0, \textit{ for all } \varphi \in \widetilde{H}^{\frac{1}{2}}(\Gamma).$$

Proof We first demonstrate the surjectivity of T. To this end, let V be the solution of problem (3.1) with the given function $\phi \in H^{-\frac{1}{2}}(\Gamma)$. Then V satisfies problem (3.2) with $\varphi = [V]|_\Gamma \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$. The definition of T implies that $T\varphi = \phi$, hence T is surjective.

To show the injectivity of T, let $T\varphi = \gamma\frac{\partial v}{\partial \nu}|_\Gamma = 0$. Then the solution function v of problem (3.2) satisfies (3.1) with homogeneous boundary conditions. The uniqueness result indicates that $v = 0$ in the domain $B\backslash\overline{\Gamma}$, thus $\varphi = [v]|_\Gamma = 0$, which yields that T is injective. Therefor T is an isomorphism.

By an analogous argument, we can obtain that the map $P : \widetilde{H}^{\frac{1}{2}}(\Gamma) \to H^1_{\diamond,\partial B}(B\backslash\overline{\Gamma})$ given by $\varphi \mapsto v$ is bijective with a bounded inverse. Then by an application of the Green's formula, we have

$$
\begin{aligned}
-Re\langle T\varphi, \varphi\rangle &= -Re \int_\Gamma \gamma \frac{\partial v}{\partial \nu}(\overline{v}_+ - \overline{v}_-)\mathrm{d}s \\
&= -Re \left(\int_{\partial B} \gamma \frac{\partial v}{\partial \nu}\overline{v}\mathrm{d}s - \int_{B\backslash\overline{\Gamma}} (\gamma\nabla v)\cdot\nabla\overline{v}\mathrm{d}x \right) \\
&= Re \int_{B\backslash\overline{\Gamma}} \nabla v^* \gamma \nabla v \mathrm{d}x \geq c_0 \|\nabla v\|^2_{L^2(B\backslash\overline{\Gamma})} \geq c_1 \|v\|^2_{H^1(B\backslash\overline{\Gamma})} \\
&\geq c\|\varphi\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma)},
\end{aligned}
$$

where the condition (2.1) and the Poincaré inequality have been used. It is easy to observe that $-Im\langle T\varphi, \varphi\rangle = 0$ and the proof is then completed. □

Next we turn our attention to the operator G and consider the following problem with data $\psi \in H_\diamond^{-\frac{1}{2}}(\partial B)$:

$$
\begin{cases}
\nabla\cdot(\gamma\nabla w) = 0, & \text{in } B\backslash\overline{\Gamma}, \\
\gamma\dfrac{\partial w}{\partial \nu} = \psi, & \text{on } \partial B, \\
\gamma\dfrac{\partial w}{\partial \nu} = 0, & \text{on } \Gamma.
\end{cases}
\tag{3.5}
$$

One can see that $G^* : H_\diamond^{-\frac{1}{2}}(\partial B) \to \widetilde{H}^{\frac{1}{2}}(\Gamma)$ is given by $G^*\psi = -[w]|_\Gamma$. Indeed, recall the definition of G and make use of the Green's formula for u (the solution for problem (3.1)) and w, we have

$$
\begin{aligned}
\langle G\phi, \psi\rangle &= \int_{\partial B} u\gamma^\top \frac{\partial\overline{w}}{\partial\nu}\mathrm{d}s = \int_\Gamma [u]\gamma^\top\frac{\partial\overline{w}}{\partial\nu}\mathrm{d}s + \int_{B\backslash\overline{\Gamma}} (\nabla u)\cdot\gamma^\top\nabla\overline{w}\mathrm{d}x \\
&= \int_{B\backslash\overline{\Gamma}} (\nabla\overline{w})\cdot\gamma\nabla u\mathrm{d}x = \int_{\partial B} \overline{w}\gamma\frac{\partial u}{\partial\nu}\mathrm{d}s - \int_\Gamma [\overline{w}]\gamma\frac{\partial u}{\partial\nu}\mathrm{d}s \\
&= -\langle\phi, [w]_\Gamma\rangle = \langle\phi, G^*\psi\rangle.
\end{aligned}
$$

From this, we have $N(G^*) = \{\psi \in H_\diamond^{-\frac{1}{2}}(\partial B) : \text{such that } w \text{ satisfies (3.5) with } [w]|_\Gamma = 0\}$, then we conclude that $[\gamma\frac{\partial w}{\partial\nu}]|_\Gamma = 0$ and $[w]|_\Gamma = 0$ if $\gamma\frac{\partial w}{\partial\nu}|_{\partial B} \in N(G^*)$, hence $w \in H^1_\diamond(B)$. Thereby, if $N(G^*) = \{0\}$, in other words, except for zero potential there is no other potential function $w \in H^1_\diamond(B)$ satisfying

$$
\begin{cases}
\nabla\cdot(\gamma\nabla w) = 0, & \text{in } B, \\
\gamma\dfrac{\partial w}{\partial \nu} = \psi, & \text{on } \partial B,
\end{cases}
$$

for $\psi \in H_\diamond^{-\frac{1}{2}}(\partial B)$ such that $\gamma\frac{\partial w}{\partial\nu}|_\Gamma = 0$, then $\overline{R(G)} = N(G^*)^\perp = H_\diamond^{\frac{1}{2}}(\partial B)$. According to **Assumption** 3.1, this conclusion is valid.

The foregoing argument results in:

Lemma 3.2 *If **Assumption** 3.1 is satisfied, then the operator $G : H^{-\frac{1}{2}}(\Gamma) \to H_\diamond^{\frac{1}{2}}(\partial B)$ is compact and injective with dense range.*

Proof The compactness of G can be obtained by the interior regularity of elliptic equations and the injectivity follows from the Holmgren's uniqueness theorem, we omit them for brevity. □

Let $G(\cdot, z)$ be the Green function of the background electrical conductor, i.e., $G(\cdot, z)$ satisfies

$$\begin{cases} \nabla \cdot (\gamma \nabla G(\cdot, z)) = 0, & \text{in } B\backslash\{z\}, \\ \gamma \dfrac{\partial G(\cdot, z)}{\partial \nu} = 0, & \text{on } \partial B. \end{cases} \tag{3.6}$$

We introduce the test function χ_{Γ_0} by

$$\chi_{\Gamma_0}(x) := \int_{\Gamma_0} \frac{\partial G(x, z)}{\partial \nu(z)} \alpha(z) \mathrm{d}s(z), \quad x \in B,$$

where Γ_0 is an open smooth arc located at Ω, and the density α belongs to $\widetilde{H}^{\frac{1}{2}}(\Gamma_0)$. Then we have

Lemma 3.3 *The test function $\chi_{\Gamma_0}|_{\partial B}$ is in the range of G if and only if $\Gamma_0 \subset \Gamma$.*

Proof Assume first that $\Gamma_0 \subset \Gamma$. Then χ_{Γ_0} satisfies problem (3.1) with $\phi = \gamma \frac{\partial \chi_{\Gamma_0}}{\partial \nu}|_\Gamma$, thus $\chi_{\Gamma_0}|_{\partial B} \in R(G)$.

If $\Gamma_0 \not\subset \Gamma$ and assume that $\chi_{\Gamma_0}|_{\partial B} \in R(G)$, then there exists $\phi \in H^{-\frac{1}{2}}(\Gamma)$ such that $G\phi = \chi_{\Gamma_0}|_{\partial B}$. This implies that χ_{Γ_0} and the solution function u to problem (3.1) corresponding to ϕ have the same Dirichlet and Neumann values on ∂B, hence $u = \chi_{\Gamma_0}$ in $B\backslash(\overline{\Gamma} \cup \overline{\Gamma}_0)$ by the unique continuation principle.

Without lose of generality, take a point $x_0 \in \Gamma_0$ such that there exists a small neighborhood $B(x_0)$ centered at x_0 and not intersecting Γ. Then $\|u\|_{H^1(B(x_0))}$ is bounded, however χ_{Γ_0} does not belong to $H^1(B(x_0))$ since the double-layer potential has a jump across Γ_0. This contradicts the fact $u = \chi_{\Gamma_0}$ in $B(x_0)\backslash\overline{\Gamma}_0$. The proof is thus completed. □

Now we state the main result for the solution to **INCP** by combining the previous results of this section.

Theorem 3.4 *Assume that inequality (2.1) and **Assumption** 3.1 are satisfied. Then we have following result for the **INCP***

$$\Gamma_0 \subset \Gamma \Longleftrightarrow \chi_{\Gamma_0} \in R(\Lambda_1 - \Lambda_0)_\sharp^{1/2}), \tag{3.7}$$

and consequently

$$\Gamma_0 \subset \Gamma \Longleftrightarrow \sum_{j=1}^{\infty} \frac{|\langle \chi_{\Gamma_0}, \psi_j \rangle_{L^2(\partial B)}|^2}{|\lambda_j|} < \infty, \tag{3.8}$$

where (λ_j, ψ_j) is an eigensystem of the operator $(\Lambda_1 - \Lambda_0)_\sharp := |Re(\Lambda_1 - \Lambda_0)| + |Im(\Lambda_1 - \Lambda_0)|$. In other words, the sign of the function

$$W_n(\Gamma_0) = \left[\sum_{j=1}^{\infty} \frac{|\langle \chi_{\Gamma_0}, \psi_j \rangle_{L^2(\partial B)}|^2}{|\lambda_j|} \right]^{-1}$$

is just the characteristic function of Γ.

Proof Lemma (3.1) and (3.2) show that the range identity theorem is valid for the decomposition form (3.4), then we have

$$R(\Lambda_1 - \Lambda_0)_\sharp^{1/2}) = R(G).$$

Combining this relationship and Lemma (3.3) yields the assertion (3.7). The assertion (3.8) is a consequence of Picard's range criterion. □

4 The factorization method for IMCP

This section focuses on establishing the factorization method for solving the **IMCP** by the same process as former problem. However, the analysis here is more complex due to the geometric features of the impedance crack. We just consider the case where the injection current f belongs to $L^2_\diamond(\partial B)$. We note that the forward problem **MCP** is uniquely solvable in such a situation and hope that the factorization method can be extended to the more general anisotropic background medium rather than just for the admittivity γ being a Hermitian.

The solvability of problems (2.2) and (2.4) defines the NtD operators Λ_0 and Λ_2, respectively, by

$$\Lambda_0 : L^2_\diamond(\partial B) \to L^2_\diamond(\partial B), \quad f \mapsto u_0|_{\partial B}$$

and

$$\Lambda_2 : L^2_\diamond(\partial B) \to L^2_\diamond(\partial B), \quad f \mapsto u_2|_{\partial B}.$$

We start with the definitions of some operators concerning the decomposition of the difference of NtD map $\Lambda_2 - \Lambda_0$. The notations may be the same as previous section but with different meanings.

For the solution u_0 to problem (2.2) and u_2 to problem (2.4), let $u = u_2 - u_0$, then u satisfies

$$\begin{cases} \nabla\cdot(\gamma\nabla u) = 0, & \text{in } B\backslash\overline{\Gamma}, \\ \gamma\dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial B, \\ \left[\gamma\dfrac{\partial u}{\partial \nu}\right] + \lambda_+ u_+ - \lambda_- u_- = g, & \text{on } \Gamma, \\ \gamma\dfrac{\partial u_-}{\partial \nu} + \lambda_- u_- = h, & \text{on } \Gamma \end{cases} \tag{4.1}$$

with $g = -(\lambda_+ - \lambda_-)u_0|_\Gamma$ and $h = -\gamma\frac{\partial u_0}{\partial \nu}|_\Gamma - \lambda_- u_0|_\Gamma$. Inspired by the paper (Boukari and Haddar 2013), we restrict ourselves to the case $(g, h) \in L^2(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ for the requirement by the inverse problem. The equivalent variational formula of this problem is similar to (2.6), which is given by

$$\int_{B\backslash\overline{\Gamma}} \nabla\psi^*\gamma\nabla u \mathrm{d}x - \int_\Gamma \lambda_+ u_+ \overline{\psi}_+ \mathrm{d}s + \int_\Gamma \lambda_- u_- \overline{\psi}_- \mathrm{d}s = -\int_\Gamma g\overline{\psi}_+ \mathrm{d}s - \int_\Gamma h[\overline{\psi}]\mathrm{d}s,$$

for all $\psi \in H^1_{\diamond,\partial B}(B\backslash\overline{\Gamma})$. Obviously, there exists a unique solution $u \in H^1_{\diamond,\partial B}(B\backslash\overline{\Gamma})$ for this variational problem, which defines an operator

$$G : L^2(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \to L^2_\diamond(\partial B), \quad (g, h) \mapsto u|_{\partial B}.$$

For the solution u_0 to problem (2.2) with boundary data $f \in L^2_\diamond(\partial B)$, we define the operator

$$F : L^2_\diamond(\partial B) \to L^2(\Gamma) \times H^{-\frac{1}{2}}(\Gamma), \quad f \mapsto \left((\lambda_+ - \lambda_-)u_0|_\Gamma, \gamma \frac{\partial u_0}{\partial \nu}\Big|_\Gamma + \lambda_- u_0|_\Gamma \right).$$

Then by the linearity and superposition, we have that

$$\Lambda_2 - \Lambda_0 = -GF. \tag{4.2}$$

Consider the following problem with data $a \in L^2(\Gamma)$ and $b \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$:

$$\begin{cases} \nabla \cdot (\gamma \nabla v) = 0, & \text{in } B \backslash \overline{\Gamma}, \\ \gamma \dfrac{\partial v}{\partial \nu} = 0, & \text{on } \partial B, \\ \left[\gamma \dfrac{\partial v}{\partial \nu} \right] + \lambda_- [v] = -a, & \text{on } \Gamma, \\ (\lambda_+ - \lambda_-)[v] = b, & \text{on } \Gamma. \end{cases} \tag{4.3}$$

We claim that the adjoint operator $F^* : L^2(\Gamma) \times \widetilde{H}^{\frac{1}{2}}(\Gamma) \to L^2_\diamond(\partial B)$ of F is given by

$$F^*(a, b) = (\lambda_+ - \lambda_-) v|_{\partial B}.$$

Indeed, this result can be obtained by applying the Green's formula for u_0 and v via the following calculation:

$$\begin{aligned} \langle Ff, (a, b) \rangle &= \int_\Gamma (\lambda_+ - \lambda_-) u_0 \overline{a} \mathrm{d}s + \int_\Gamma \left(\gamma \frac{\partial u_0}{\partial \nu} + \lambda_- u_0 \right) \overline{b} \mathrm{d}s \\ &= - \int_\Gamma (\lambda_+ - \lambda_-) u_0 \left[\gamma^\top \frac{\partial \overline{v}}{\partial \nu} \right] \mathrm{d}s - \int_\Gamma (\lambda_+ - \lambda_-) \lambda_- u_0 [\overline{v}] \mathrm{d}s \\ &\quad + \int_\Gamma (\lambda_+ - \lambda_-) \gamma \frac{\partial u_0}{\partial \nu} [\overline{v}] \mathrm{d}s + \int_\Gamma (\lambda_+ - \lambda_-) \lambda_- u_0 [\overline{v}] \mathrm{d}s \\ &= \int_{\partial B} (\lambda_+ - \lambda_-) \gamma \frac{\partial u_0}{\partial \nu} \overline{v} \mathrm{d}s - \int_{\partial B} (\lambda_+ - \lambda_-) u_0 \gamma^\top \frac{\partial \overline{v}}{\partial \nu} \mathrm{d}s \\ &= \langle f, (\lambda_+ - \lambda_-) v \rangle. \end{aligned}$$

Introduce the operator $T : L^2(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma) \times \widetilde{H}^{\frac{1}{2}}(\Gamma)$ by

$$T(g, h) = \left(- \left[\gamma \frac{\partial u}{\partial \nu} \right] \Big|_\Gamma - \lambda_- [u]|_\Gamma, (\lambda_+ - \lambda_-)[u]|_\Gamma \right),$$

where u is the solution of problem (4.1) with boundary data (g, h). Then we get $G = \frac{1}{\lambda_+ - \lambda_-} F^* T$ by the definitions of related operators. Hence, the identity (4.2) becomes

$$(\lambda_+ - \lambda_-)(\Lambda_2 - \Lambda_0) = F^*(-T)F. \tag{4.4}$$

We next study the operators T and F.

Lemma 4.1 *Let γ satisfy (2.1), then we have*

$$-Im \langle T(g, h), (g, h) \rangle = 0, \quad \textit{for all } (g, h) \in L^2(\Gamma) \times H^{-\frac{1}{2}}(\Gamma).$$

Proof We denote by $\alpha = - \left[\gamma \frac{\partial u}{\partial \nu} \right] |_\Gamma - \lambda_- [u]|_\Gamma$, $\beta = (\lambda_+ - \lambda_-)[u]|_\Gamma$. Using Green's formula for u and $\overline{u}$, we see that

$$
\begin{aligned}
&\langle (g,h), T(g,h)\rangle \\
&= \int_\Gamma \left(\left[\gamma \frac{\partial u}{\partial \nu}\right] + \lambda_+ u_+ - \lambda_- u_- \right) \overline{\alpha} \mathrm{d}s + \int_\Gamma \left(\gamma \frac{\partial u_-}{\partial \nu} + \lambda_- u_- \right) \overline{\beta} \mathrm{d}s \\
&= \int_\Gamma (-\alpha + \beta + (\lambda_+ - \lambda_-) u_-) \overline{\alpha} \mathrm{d}s + \int_\Gamma \left(\gamma \frac{\partial u_+}{\partial \nu} + \lambda_- u_+ + \alpha \right) \overline{\beta} \mathrm{d}s \\
&= \int_\Gamma (\beta \overline{\alpha} + \alpha \overline{\beta} - \alpha \overline{\alpha}) \mathrm{d}s - \int_\Gamma (\lambda_+ - \lambda_-) u_- \left(\left[\gamma^\top \frac{\partial \overline{u}}{\partial \nu} \right] + \lambda_- [\overline{u}] \right) \mathrm{d}s \\
&\quad + \int_\Gamma \left(\gamma \frac{\partial u_+}{\partial \nu} + \lambda_- u_+ \right) (\lambda_+ - \lambda_-) [\overline{u}] \mathrm{d}s \\
&= (\lambda_+ - \lambda_-) \left\{ \int_\Gamma \left(u_- \gamma^\top \frac{\partial \overline{u}_-}{\partial \nu} + \lambda_- u_- \overline{u}_- \right) \mathrm{d}s + \int_\Gamma \left(\gamma \frac{\partial u_+}{\partial \nu} \overline{u}_+ + \lambda_- u_+ \overline{u}_+ \right) \mathrm{d}s \right\} \\
&\quad - (\lambda_+ - \lambda_-) \left\{ \int_\Gamma \left(u_- \gamma^\top \frac{\partial \overline{u}_+}{\partial \nu} + \lambda_- u_- \overline{u}_+ \right) \mathrm{d}s + \int_\Gamma \left(\gamma \frac{\partial u_+}{\partial \nu} \overline{u}_- + \lambda_- u_+ \overline{u}_- \right) \mathrm{d}s \right\} \\
&\quad + \int_\Gamma (\beta \overline{\alpha} + \alpha \overline{\beta} - \alpha \overline{\alpha}) \mathrm{d}s := I_1 + I_2 + I_3.
\end{aligned}
$$

The terms I_2 and I_3 are real due to the conjugate relationship. A further calculation yields that

$$
\begin{aligned}
I_1 &= (\lambda_+ - \lambda_-) \left\{ \int_\Gamma \left(u_- \gamma^\top \frac{\partial \overline{u}_-}{\partial \nu} + \gamma \frac{\partial u_-}{\partial \nu} \overline{u}_- \right) \mathrm{d}s + \int_\Gamma (\lambda_- |u_-|^2 + \lambda_- |u_+|^2) \mathrm{d}s \right\} \\
&\quad - (\lambda_+ - \lambda_-) \int_{B \setminus \overline{\Gamma}} \nabla \overline{u} \cdot \gamma \nabla u \mathrm{d}x.
\end{aligned}
$$

Consequently, since γ is a Hermitian, taking the imaginary part shows

$$
-Im\langle T(g,h), (g,h)\rangle = Im\langle (g,h), T(g,h)\rangle = (\lambda_- - \lambda_+) Im \int_{B\setminus\overline{\Gamma}} \nabla \overline{u} \cdot \gamma \nabla u \mathrm{d}x = 0,
$$

which proves this lemma. □

Let $v_0 \in H^1_{\diamond,\partial B}(B\setminus\overline{\Gamma})$ solve the following problem for $(g,h) \in L^2(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$

$$
\begin{cases}
\nabla \cdot (\gamma \nabla v_0) = 0, & \text{in } B\setminus\overline{\Gamma}, \\
\gamma \dfrac{\partial v_0}{\partial \nu} = 0, & \text{on } \partial B, \\
\left[\gamma \dfrac{\partial v_0}{\partial \nu} \right] = g, & \text{on } \Gamma, \\
\gamma \dfrac{\partial v_{0,-}}{\partial \nu} = h, & \text{on } \Gamma.
\end{cases}
\tag{4.5}
$$

Then $w = u - v_0$ satisfies

$$
\begin{cases}
\nabla \cdot (\gamma \nabla w) = 0, & \text{in } B\setminus\overline{\Gamma}, \\
\gamma \dfrac{\partial w}{\partial \nu} = 0, & \text{on } \partial B, \\
\left[\gamma \dfrac{\partial w}{\partial \nu} \right] = -\lambda_+ u_+ + \lambda_- u_-, & \text{on } \Gamma, \\
\gamma \dfrac{\partial w_-}{\partial \nu} = -\lambda_- u_-, & \text{on } \Gamma.
\end{cases}
\tag{4.6}
$$

Define $T_0 : L^2(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma) \times \widetilde{H}^{\frac{1}{2}}(\Gamma)$ by

$$T_0(g, h) = (g - (\lambda_+ - \lambda_-)v_{0,+}|_\Gamma, -(\lambda_+ - \lambda_-)[v_0]|_\Gamma)$$

and let $T_c : L^2(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma) \times \widetilde{H}^{\frac{1}{2}}(\Gamma)$ be given as

$$T_c(g, h) = (\lambda_+ - \lambda_-)(-w_+|_\Gamma, -[w]|_\Gamma),$$

then we have the result as below.

Lemma 4.2 *The operator $-T$ can be decomposed as $-T = T_c + T_0$, where T_c and T_0 are given as above. Moreover, T_c is a compact operator and $Re(T_0)$ satisfies the coercivity property.*

Proof It is easy to deduce the decomposition $-T = T_c + T_0$. The trace theorem and the Rellich compact embedding theorem show us that T_c is compact. We next prove the coercivity of $Re(T_0)$.

The variational solution of problem (4.5) satisfies

$$\int_{B\setminus\overline{\Gamma}} \nabla\psi^*\gamma\nabla v_0 \mathrm{d}x + \int_\Gamma h[\overline{\psi}]\mathrm{d}s + \int_\Gamma g\overline{\psi}_+\mathrm{d}s = 0,$$

for all $\psi \in H^1_{\diamond,\partial B}(B\setminus\overline{\Gamma})$. Hence, replacing ψ by v_0 we obtain

$$-\int_\Gamma (\lambda_+ - \lambda_-)\overline{v}_{0,+}g\mathrm{d}s - \int_\Gamma (\lambda_+ - \lambda_-)[\overline{v}_0]h\mathrm{d}s = (\lambda_+ - \lambda_-)\int_{B\setminus\overline{\Gamma}} \nabla v_0^*\gamma\nabla v_0\mathrm{d}x.$$

On the other hand,

$$\langle (g, h), T_0(g, h)\rangle = \int_\Gamma |g|^2\mathrm{d}s - \int_\Gamma (\lambda_+ - \lambda_-)\overline{v}_{0,+}g\mathrm{d}s - \int_\Gamma (\lambda_+ - \lambda_-)[\overline{v}_0]h\mathrm{d}s.$$

From the relation (2.1), taking the real part and using the Poincaré inequality we have

$$Re\langle (g, h), T_0(g, h)\rangle \geq \|g\|^2_{L^2(\Gamma)} + \widetilde{c}_0\|\nabla v_0\|^2_{L^2(B\setminus\overline{\Gamma})} \geq \widetilde{c}(\|g\|^2_{L^2(\Gamma)} + \|v_0\|^2_{H^1(B\setminus\overline{\Gamma})}).$$

Therefore, the trace theorem shows us that

$$Re\langle (g, h), T_0(g, h)\rangle \geq c(\|g\|^2_{L^2(\Gamma)} + \|h\|^2_{H^{-\frac{1}{2}}(\Gamma)}),$$

which completes the proof of this lemma. □

Lemma 4.3 *The operator F^* is compact and injective with dense range.*

Proof The compactness of F^* is obvious. Since the injectivity of F implies the denseness of F^*, we need only prove the injectivity of F^* and F.

Let us go back to problem (4.3) and assume that $F^*(a, b) = (\lambda_+ - \lambda_-)v|_{\partial B} = 0$. Then $v = \gamma\frac{\partial v}{\partial \nu} = 0$ on ∂B, Holmgren's uniqueness theorem indicates that $v = 0$ in $B\setminus\overline{\Gamma}$. Hence, we know $a = b = 0$ from the boundary conditions, which shows that F^* is injective. The injectivity of F can be obtained by a similar argument and the proof is completed. □

For the Green function $G(\cdot, z)$ of the background medium we introduce the test function χ_{Γ_0} by

$$\chi_{\Gamma_0}(x) := \int_{\Gamma_0} \left\{ \frac{\partial G(x, z)}{\partial \nu(z)}\alpha(z) + G(x, z)\beta(z) \right\} \mathrm{d}s(z), \quad x \in B,$$

where Γ_0 is an open smooth arc located at Ω, and the densities α belongs to $\widetilde{H}^{\frac{1}{2}}(\Gamma_0)$, β belongs to $L^2(\Gamma_0)$. Following the mapping properties of the layer potentials presented in Kirsch (1989), we have

$$\left[\gamma \frac{\partial \chi_{\Gamma_0}}{\partial \nu}\right] + \lambda_-[\chi_{\Gamma_0}] = \beta + \lambda_-\alpha \in L^2(\Gamma_0)$$

and

$$(\lambda_+ - \lambda_-)[\chi_{\Gamma_0}] = (\lambda_+ - \lambda_-)\alpha \in \widetilde{H}^{\frac{1}{2}}(\Gamma_0).$$

Then reasoning by the same way as Lemma 3.3 we have the following conclusion

Lemma 4.4 *The test function* $(\lambda_+ - \lambda_-)\chi_{\Gamma_0}|_{\partial B}$ *is in the range of* F^* *if and only if* $\Gamma_0 \subset \Gamma$.

Combining Lemma 4.1–Lemma 4.4 leads to the solvability of the **IMCP** using the factorization method.

Theorem 4.5 *Assume that the inequality* (2.1)*, then we have the following result for the* ***IMCP***

$$\Gamma_0 \subset \Gamma \Longleftrightarrow \chi_{\Gamma_0} \in R(\Lambda_2 - \Lambda_0)_\sharp^{1/2}), \tag{4.7}$$

and consequently

$$\Gamma_0 \subset \Gamma \Longleftrightarrow \sum_{j=1}^{\infty} \frac{|\langle \chi_{\Gamma_0}, \psi_j \rangle_{L^2(\partial B)}|^2}{|\lambda_j|} < \infty, \tag{4.8}$$

where (λ_j, ψ_j) *is an eigensystem of the operator* $(\Lambda_2 - \Lambda_0)_\sharp := |Re(\Lambda_2 - \Lambda_0)| + |Im(\Lambda_2 - \Lambda_0)|$. *In other words, the sign of the function*

$$W_m(\Gamma_0) = \left[\sum_{j=1}^{\infty} \frac{|\langle \chi_{\Gamma_0}, \psi_j \rangle_{L^2(\partial B)}|^2}{|\lambda_j|}\right]^{-1}$$

is just the characteristic function of Γ.

5 On the numerical experiment

In this section, we give a numerical experiment approach on the inverse problem **INCP** under the condition that γ is the identity matrix. The procedure of numerical implementation is similar to Brühl et al. (2001); Kirsch and Ritter (2000), where the measured data are obtained by artificial experiments and the boundary integral equation method is used to solve the direct problem.

To construct the solutions of problems (2.2) and (2.3), let u_0 be the double layer potential in the Sobolev space $H^1(B)$

$$u_0(x) = \int_{\partial B} \frac{\partial}{\partial \nu(y)} \Phi(x, y)\varphi(y)\mathrm{d}s(y), \quad x \in B \tag{5.1}$$

with a density $\varphi \in H^{\frac{1}{2}}(\partial B)$ satisfying $\frac{\partial u_0}{\partial \nu} = f$ on ∂B. Where $\Phi(x, y)$ is the fundamental solution given by

$$\Phi(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|}, \quad x \neq y.$$

Then there exists a constant c_0 such that $u_0 - c_0 \in H^1_\diamond(B)$ solving problem (2.2). Further consider the potential $u_1 \in H^1(B\backslash\overline{\Gamma})$ given by

$$u_1(x) = \int_{\partial B} \frac{\partial}{\partial \nu(y)}\Phi(x,y)\chi(y)\mathrm{d}s(y) + \int_\Gamma \frac{\partial}{\partial \nu(y)}\Phi(x,y)\psi(y)\mathrm{d}s(y),\ \ x \in B\backslash\overline{\Gamma} \quad (5.2)$$

with densities $\chi \in H^{\frac{1}{2}}(\partial B)$, $\psi \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$ satisfying $\frac{\partial u_1}{\partial \nu} = f$ on ∂B and $\frac{\partial u_1}{\partial \nu} = 0$ on Γ. We conclude that $u_1 - c_1 \in H^1_\diamond(B\backslash\overline{\Gamma})$ is the solution of problem (2.3), here c_1 is a constant.

Then the NtD operator $\Lambda_1 - \Lambda_0 : H_\diamond^{-\frac{1}{2}}(\partial B) \to H_\diamond^{\frac{1}{2}}(\partial B)$ can be described by the map

$$\Lambda_1 - \Lambda_0 : f \mapsto (u_1 - u_0 - c_1 + c_0)|_{\partial B}. \quad (5.3)$$

Next we will deduce the specific expression of the NtD operator $\Lambda_1 - \Lambda_0$ and in the subsequent analysis, we denote by Γ_1 the crack Γ and use Γ_2 to represent the boundary ∂B to facilitate the expression of mathematical notations. For the numerical solution we use the boundary integral equation method to solve the direct problem which is based on the collocation method (Kress 1998). To this end we introduce the double layer operator K_{jl} $(j, l = 1, 2)$ with

$$(K_{jl}\phi)(x) := \int_{\Gamma_j} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\phi(y)\mathrm{d}s(y) \quad x \in \Gamma_l$$

and the normal derivative operator T_{jl} $(j, l = 1, 2)$ with

$$(T_{jl}\phi)(x) := \frac{\partial}{\partial \nu(x)}\int_{\Gamma_j} \frac{\partial \phi(x,y)}{\partial \nu(y)}\varphi(y)\mathrm{d}s(y) \quad x \in \Gamma_l.$$

The boundary condition for the potential u_0 yields that

$$T_{22}\varphi = f, \quad (5.4)$$

and the boundary conditions for the potential u_1 give that

$$M\begin{pmatrix}\chi\\ \psi\end{pmatrix} = \begin{pmatrix}f\\ 0\end{pmatrix} \ \text{ with } \ M := \begin{pmatrix}T_{22} & T_{12}\\ T_{21} & T_{11}\end{pmatrix}. \quad (5.5)$$

By the Lax-Milgram theorem, one can show that T_{22} and M are invertible with bounded inverse operators T_{22}^{-1} and M^{-1}, respectively. We refer the reader to the work (Mclean 2000) for detail discussion on the operator T_{jl}.

Therefore, the operator Λ_0 can be rewritten as

$$\Lambda_0 f = K_{22}\varphi - \frac{1}{2}\varphi - c_0 = (K_{22} - \frac{I}{2})T_{22}^{-1}f - c_0,$$

where I is the identity operator and c_0 is determined by

$$c_0 = \int_{\partial B}(K_{22} - \frac{I}{2})T_{22}^{-1}f\mathrm{d}s.$$

The operator Λ_1 has the representation

$$\Lambda_1 f = (K_{22} - \frac{I}{2}, K_{12})M^{-1}\begin{pmatrix}f\\ 0\end{pmatrix} - c_1,$$

where it holds that

$$c_1 = \int_{\partial B}(K_{22} - \frac{I}{2}, K_{12})M^{-1}\begin{pmatrix}f\\ 0\end{pmatrix}\mathrm{d}s.$$

Let $(P_1, P_2) = (K_{22} - \frac{I}{2}, K_{12})M^{-1}$, then the NtD map $\Lambda_1 - \Lambda_0$ has the following expression

$$(\Lambda_1 - \Lambda_0)f = \left\{P_1 - \left(K_{22} - \frac{I}{2}\right)T_{22}^{-1}\right\} f - c_1 + c_0. \tag{5.6}$$

In the numerical treatment of the integral equations, if $2p$ quadrature points on Γ and ∂B, respectively, are used, we then obtain that through discretization

$$c_0 = \frac{\pi}{p}E\left(K_{22} - \frac{I}{2}\right)T_{22}^{-1}f \quad \text{and} \quad c_1 = \frac{\pi}{p}EP_1 f,$$

where $E \in \mathbb{R}^{2p\times 2p}$ with each element 1.

We assume that the smooth curves Γ_1 and Γ_2 have parametric representations

$$\Gamma_1 = \{z_1(t) : t \in (0, \pi)\}, \quad \Gamma_2 = \{z_2(t) : t \in [0, 2\pi]\}.$$

Setting $\omega_l = \varphi \circ z_l, l = 1, 2$ and $a^\perp = (a_2, -a_1)$ for any vector $a = (a_1, a_2)$. For $\omega_1(t) \in \widetilde{H}^{\frac{1}{2}}((0, \pi))$ we still denote by $\omega_1(t)$ the zero extension of $\omega_1(t)$ to the whole boundary $\partial D := \{z_1(t) : t \in [0, 2\pi]\}$. Then the parameterized form of double layer operator is given by

$$(K_{ll}\omega_l)(t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{[z_l'(\tau)]^\perp \cdot [z_l(t) - z_l(\tau)]}{|z_l(t) - z_l(\tau)|^2}\omega_l(\tau)\mathrm{d}\tau,$$

the parametric form of the normal derivative operator is

$$\begin{aligned}(T_{ll}\omega_l)(t) &= \frac{1}{2\pi|z_l'(t)|}\int_0^{2\pi}\left\{\frac{z_l'(t)\cdot z_l'(\tau)}{|z_l(t) - z_l(\tau)|^2}\right.\\ &\left.\quad +2\frac{([z_l'(\tau)]^\perp \cdot [z_l(t) - z_l(\tau)])([z_l'(t)]^\perp \cdot [z_l(\tau) - z_l(t)])}{|z_l(t) - z_l(\tau)|^4}\right\}\omega_l(\tau)d\tau\\ &= \frac{1}{2\pi|z_l'(t)|}\int_0^{2\pi}(A_{l1} + A_{l2})\omega_l(\tau)\mathrm{d}\tau.\end{aligned}$$

The operator K_{ll} has smooth kernel with the diagonal value given by

$$\lim_{\tau\to t}\frac{[z_l'(\tau)]^\perp \cdot [z_l(t) - z_l(\tau)]}{|z_l(t) - z_l(\tau)|^2} = \lim_{\tau\to t}\frac{[z_l'(t)]^\perp \cdot [z_l(\tau) - z_l(t)]}{|z_l(t) - z_l(\tau)|^2} = \frac{[z_l'(t)]^\perp \cdot z_l''(t)}{2|z_l'(t)|^2}.$$

The kernel A_{l2} of the operator T_{ll} is smooth with diagonal value

$$\lim_{\tau\to t} 2\frac{([z_l'(\tau)]^\perp \cdot [z_l(t) - z_l(\tau)])([z_l'(t)]^\perp \cdot [z_l(\tau) - z_l(t)])}{|z_l(t) - z_l(\tau)|^4} = \frac{\left([z_l'(t)]^\perp \cdot z_l''(t)\right)^2}{2|z_l'(t)|^4}.$$

The kernel A_{l1} of operator T_{ll} is hypersingular and according to the analysis in the paper (Kieser et al. 1992) we can split A_{l1} into

$$\begin{aligned}\frac{z_l'(t)\cdot z_l'(\tau)}{|z_l(t) - z_l(\tau)|^2} &= \left(\frac{2(1-\cos(t-\tau))z_l'(t)\cdot z_l'(\tau) - \cos(t-\tau)|z_l(t) - z_l(\tau)|^2}{2(1-\cos(t-\tau))|z_l(t) - z_l(\tau)|^2} - \frac{1}{2}\right)\\ &\quad + \frac{1}{4\sin^2\frac{t-\tau}{2}}.\end{aligned}$$

The first term is smooth with diagonal value

$$\frac{\frac{1}{6}z_l'(t)\cdot z_l'''(t) - \frac{1}{4}|z_l''(t)|^2 + \frac{5}{12}|z_l'(t)|^2}{|z_l'(t)|^2} - \frac{1}{2}.$$

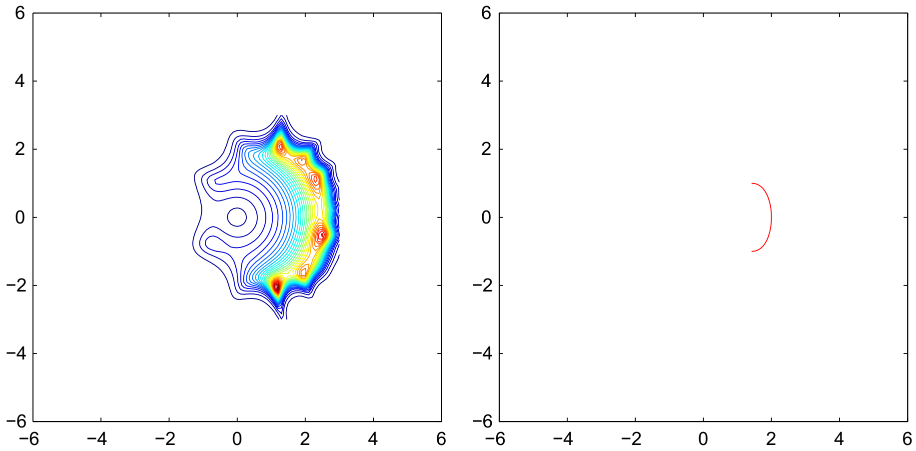

Fig. 1 The reconstruction (left) with noise 3% and the true object (right)

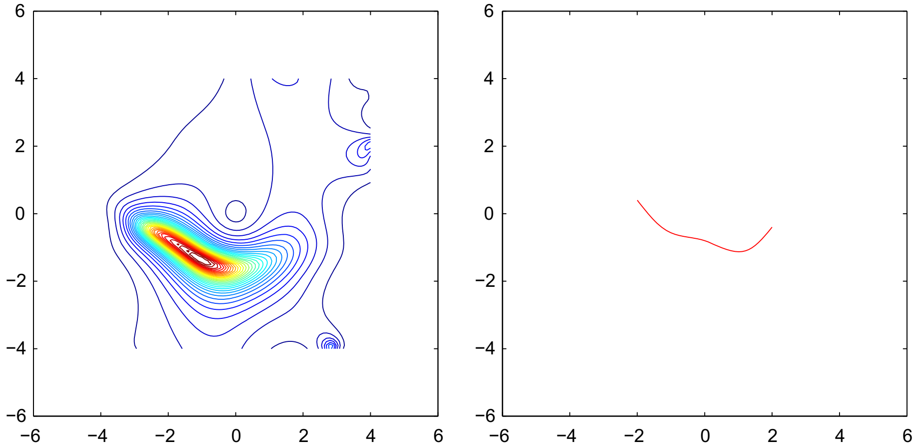

Fig. 2 The reconstruction (left) and the true object (right)

Using partial integration one can see that

$$\int_0^{2\pi} \frac{1}{4\sin^2\frac{t-\tau}{2}}\omega(\tau)\mathrm{d}\tau = \frac{1}{2}\int_0^{2\pi} \cot\frac{t-\tau}{2}\omega'(\tau)\mathrm{d}\tau,$$

its numerical integration can be calculated by combining quadrature and collocation methods (Kress 1998).

In the next numerical examples, 64 quadrature points on Γ and ∂B, respectively, are used, and we solve the direct problem by the boundary integral equation method to generate the data $\Lambda_1 - \Lambda_0$ given by (5.6), which are stored in a matrix of $\mathbb{C}^{64\times 64}$. In the reconstructions, we used a grid of 121×121 equally spaced sampling points on the rectangle $[-6, 6] \times [-6, 6]$ such that the conductive object B is included in. Let (λ_j, ψ_j), $j = 1, \ldots, 64$ be an eigensystem of the operator $(\Lambda_1 - \Lambda_0)_\sharp := |Re(\Lambda_1 - \Lambda_0)| + |Im(\Lambda_1 - \Lambda_0)|$. Then we calculate the characteristic function for every sampling point z by

$$W_n(z) = \left[\sum_{j=1}^{64} \frac{|\langle \phi_z, \psi_j \rangle_{L^2(\partial B)}|^2}{|\lambda_j|} \right]^{-1},$$

where we choose the Green function $\phi_z = G(\cdot, z)$ given by (3.6) as the test function instead of χ_{Γ_0}. We think it will not affect the numerical experiment because the numerical integration on the test curve needs to be discretized into the values at some points. For example, this approach has been adopted by Cakoni and Colton (2003) to recovery cracks in acoustic scattering.

In Fig. 1, we consider curve described by

$$\Gamma = \left\{ \left(2\sin\frac{t}{2}, \sin t \right),\ t \in \left(\frac{\pi}{2}, \frac{3\pi}{2} \right) \right\}.$$

The measured data are obtained on the circle ∂B with center (0, 0) and radius 4.

In Fig. 2, the curve is parameterized by

$$\Gamma = \left\{ (2s, -\cos\frac{\pi s}{2} - 0.4\sin\frac{\pi s}{2} + 0.2\cos\frac{3\pi s}{2}),\ s \in (-1, 1) \right\}$$

and assume ∂B is the circle with center (0, 0) and radius 6.

6 Conclusion

In this paper, we consider the direct and inverse problems in electrical impedance tomography for cracks buried in anisotropic background electrical conductor. Both of the Neumann and impedance boundary conditions are investigated. We show that the direct problems are well-posedness by using the variational approach. The factorization method is justified suitable for the two inverse problems under some assumptions. The inverse problem for crack with Neumann boundary condition is proved by a standard way. However, the function spaces setting for the impedance boundary condition is unusual, and the decomposition technique of the NtD operator is rather difficult. Using the boundary integral equation method we give two numerical examples to show the the correctness and effectiveness of the factorization method.

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References

Alvarez D, Dorn O, Irishina N, Moscoso M (2009) Crack reconstruction using a level-set strategy. J Comput Phys 228:5710–5721

Boukari Y, Haddar H (2013) The Factorization method applied to cracks with impedance boundary conditions. Inverse Probl Imaging 4:1123–1138

Brühl M (2001) Explicit characterization of inclusions in electrical impedance tomography. SIAM J Math Anal 32:1327–1341

Brühl M, Hanke M, Pidcock M (2001) Crack detection using electostatic measurements. ESAIM Math Model Numer Anal 35:595–605

Bryan K, Vogelius MS (2004) A review of selected works on crack identification, chapter of geometric methods in inverse problems and PDE control. IMA Vol Math Appl 137:25–46

Cakoni F, Colton D (2003) The linear sampling method for cracks. Inverse Probl 19:279–295
Chaulet N, Arridge S, Betcke T, Holder D (2014) The factorization method for three dimensional electrical impedance tomography. Inverse Probl 30:045005
Griesmaier R (2010) Reconstruction of thin tubular inclusions in three-dimensional domain using electrical impedance tomography. SIAM J Imaging Sci 3:340–362
Guo J, Hu J, Yan G (2016) Application of the factorization method to retrieve a crack from near field data. J Inverse Ill Posed Probl 24:527–541
Guo J, Wu Q, Yan G (2016) The linear sampling method for a mixed scattering problem. Inverse Probl Sci Eng 24:2–21
Hanke M, Bruhl M (2003) Recent progress in electrical impedance tomography. Inverse Probl 19:S65–S90
Harrach B (2013) Recent progress on the factorization method for electrical impedance tomography. Comput Math Methods Med 2013:425184
Harrach B, Seo JK (2009) Detecting inclusions in electrical impedance tomography without reference measurements. SIAM J Appl Math 69:1662–1681
Karhunen K, Seppanen A, Kaipio JP (2014) Adaptive meshing approach to identification of cracks with electrical impedance tomography. Inverse Probl Imaging 8:127–148
Kieser R, Kleemann B, Rathsfeld A (1992) On a full discretization scheme for a hypersingular boundary integral equation over smooth curves. Z Anal Anwend 11:385–396
Kirsch A (1989) Surface gradients and continuity properties for some integral operators in classical scattering theory. Math Methods Appl Sci 11:789–804
Kirsch A (1998) Characterization of the shape of a scattering obstacle using the spectral data of the far field operator. Inverse Probl 14:1489–1512
Kirsch A (2005) The factorization method for a class of inverse elliptic problems. Math Nachr 278:258–277
Kirsch A, Grinberg N (2008) Th factorization method for inverse problems. Oxford University Press, Oxford
Kirsch A, Ritter S (2000) A linear sampling method for inverse scattering from an open arc. Inverse Probl 16:89–105
Kress R (1998) Integral equations, 2nd edn. Springer, New York
Lechleiter A, Hyvönen N, Hakula H (2008) The factorization method applied to the complete electrode model of impedance tomography. SIAM J Appl Math 68:1097–1121
Mclean W (2000) Strongly elliptic systems and boundary integral equation. Cambridge University Press, Cambridge
Scherzer O (2011) Handbook of mathematical methods in imaging. Springer
Schmitt S (2009) Th factorization method for EIT in the case of mixed inclusions. Inverse Probl 25:065012