Kim Knudsen, Technical University of Denmark

Inverse Problems and Imaging (02624)

Week 10

Nonlinear Inverse Problems

$$
\begin{aligned} K : D(K) &\to Y \\ x &\mapsto y \end{aligned} \tag{1}
$$

with K nonlinear; $D(K) \subset X$.

Hadamard criteria for well-posedness:

1. Existence: For what $y \in Y$ is there an $x \in D(K)$ with $K(x) = y$?
2. Uniqueness: Does $K(x_1) = K(x_2)$ imply $x_1 = x_2$?
3. Stability: Can we estimate $\|x_1 - x_2\|$ in terms of $\|K(x_1) - K(x_2)\|$?

For linear problems with compact K, these things boil down to analysis of subspaces $\mathrm{Ran} K, \ker K$ and $\dim(\mathrm{Ran} K)$.

For nonlinear problems this is more nasty.

Local ill-posedness

The equation (1) is called locally ill-posed at $x^\dagger \in D(K)$, if for any $r > 0$ there is a sequence $x_n \in D(K) \cap B(x^\dagger, r)$, such that

$$K(x_n) \to K(x^\dagger)$$

$$\text{but} \quad x_n \to x^\dagger \quad \text{fails.}$$

Reconstruction methods

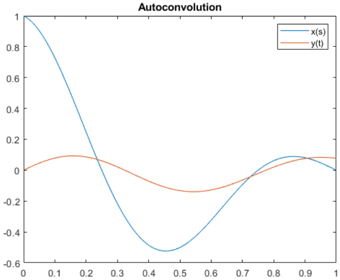
1. Direct methods
2. Linearization
3. Optimization
4. Statistical methods (Bayes)

Model problem 1: Autoconvolution

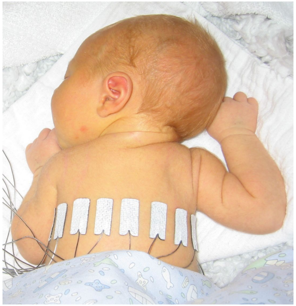
K in $L^2(0,1)$ defined by

$$y(t) = K(x)(t) = \int_0^t x(t-s)x(s)\,\mathrm{d}s, \;\; t \in (0,1).$$

Applications include Appearance Potential Optics, Laser Optics, Spectroscopy.



Model problem 2: EIT



Should EIT work for medical imaging?

- Fricke and Morse, *The electric capacity of tumours of the breast* (1926)
- Barber-Brown (1989):

Tissue	Conductivity (mS/cm)
Blood	6.7
Liver	2.8
Skeletal muscle	8.0 (long.), 0.6 (trans.)
Cardiac muscle	6.3(long.), 2.3 (trans.)
Lung (expiration-inspiration)	1.0 - 0.4
Fat	0.36
Bone	0.06

Modeling Electrical Impedance Tomography

Smooth bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$; conductivity coefficient γ.

A voltage potential u in Ω generated by boundary current flux g

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \qquad \gamma \partial_\nu u|_{\partial\Omega} = g,$$

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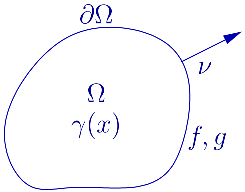

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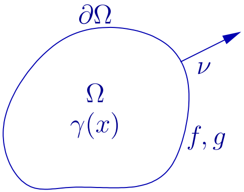
Voltage potential at the boundary:

$$f = u|_{\partial\Omega}.$$

Neumann to Dirichlet (current to voltage) map

$$\Lambda_\gamma : g \mapsto f.$$



The Calderón problem

Forward problem:

$$\Lambda \colon \gamma \mapsto \Lambda_\gamma .$$

$$\text{Conductivity} \rightarrow \text{Data}$$

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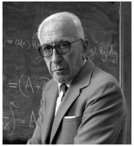
Inverse problem (Alberto Calderón, 1980):



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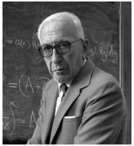

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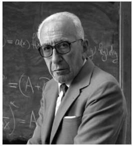

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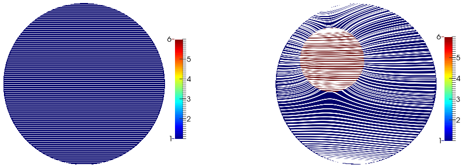
Features

- Non-linear
- Ill-posed

Data with $f = \cos(\theta)$

Current flow $J = \sigma \nabla u$:



Homogeneous conductivity | Perturbed conductivity

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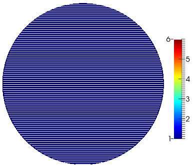

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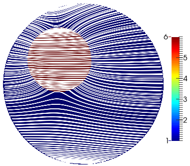
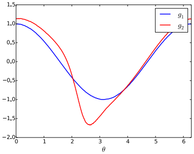
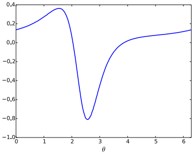


Perturbed conductivity

Boundary normal current $g = \sigma \partial_\nu u$:



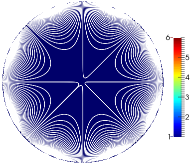
Boundary currents



Difference

Data with $f = \cos(4\theta)$

Current flow $J = \sigma \nabla u$:



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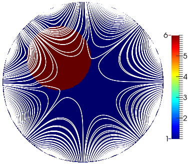

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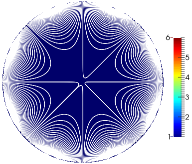

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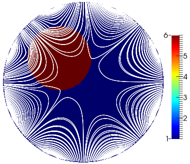

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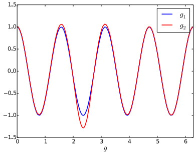

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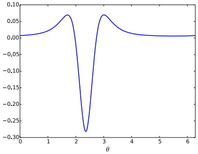

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