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Inverse Problems and Imaging (02624)

Week 11

Linearization of nonlinear Inverse Problems

$$\begin{aligned} &\text{Inverse Problem} \qquad K(x) = y \\ &\text{with nonlinear} \qquad K : X \supset D(K) \to Y. \end{aligned}$$

Fréchet derivative of K at $x_0 \in D(K)$ is the operator $dK[x_0] \in B(X, Y)$ with

$$\lim_{\|h\|_X \to 0} \frac{\|K(x_0 + h) - K(x_0) - dK[x_0]h\|_Y}{\|h\|_X} = 0.$$

Substitute $x = x_0 + h$ into

$$dK[x_0]h \approx K(x_0 + h) - K(x_0) = y - K(x_0)$$

to get the linearized problem

$$dK[x_0]h = y - K(x_0).$$

Solving for h gives us the reconstruction $x = x_0 + h$.

Local ill-posedness

The inverse problem is called locally ill-posed at $x^\dagger \in D(K)$, if for any $r > 0$ there is a sequence $x_n \in D(K) \cap B(x^\dagger, r)$, such that

$$K(x_n) \to K(x^\dagger)$$

$$\text{but} \quad x_n \to x^\dagger \quad \text{fails.}$$

If $K(x) = y$ is locally ill-posed at x_0, then the linear problem

$$dK[x_0]h = y - K(x_0)$$

is ill-posed and requires regularization.

Autoconvolution

For $K \in B(L^2(0,1))$

$$K(x)(t) = \int_0^t x(t-s)x(s)\,\mathrm{d}s,$$

we find

$$dK[x_0]h(t) = 2\int_0^t x_0(t-s)h(s)\,\mathrm{d}s.$$

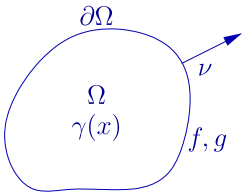
Electrical Impedance Tomography

Smooth bounded domain Ω; unknown conductivity coefficient γ.

Experiment: A boundary current flux g generate in Ω the voltage potential u satisfying the PDE

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \qquad \gamma \partial_\nu u|_{\partial\Omega} = g, \ \text{(strong)}$$

$$\Leftrightarrow \quad \int_\Omega \gamma \nabla u \cdot \nabla v \, \mathrm{d}x = \int_{\partial\Omega} g v \, \mathrm{d}S, \quad v \in H^1(\Omega) \ \text{(weak)}$$



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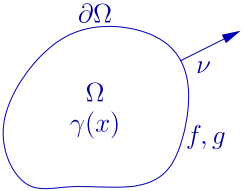
Measure voltage potential at the boundary: $f = u|_{\partial\Omega}$.

Repeating the experiment for many g gives the Neumann to Dirichlet (current to voltage) map

$$\Lambda_\gamma : L^2_\diamond(\Omega) \to L^2_\diamond(\Omega)$$
$$g \mapsto f.$$



Linearized EIT

Nonlinear forward problem:

$$K : \gamma \mapsto \Lambda_\gamma$$

Linearization around constant $\gamma_0 = 1$ denoted by $dK[1] = L$ acting on conductivity function (difference) h to obtain $Lh \in B(L^2_\diamond(\partial\Omega))$ via

$$((Lh)g_1, g_2)_{L^2(\partial\Omega)} = \int_\Omega h(x) \nabla v_1(x) \cdot \nabla \overline{v_2(x)} \,\mathrm{d}x$$

with

$$\Delta v_j = 0 \text{ in } \Omega \qquad \partial_\nu v_j = g_j \text{ on } \partial\Omega.$$

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Question: Are such $\nabla v_1 \cdot \nabla \overline{v_2}$ a rich enough family to get h?

Theorem: Products of gradients of harmonic functions are dense in $L^2(\Omega)$.

The radial problem

Suppose $\Omega = B(0,1)$ in 2D and $\gamma(x) = \gamma(|x|) = \gamma(r)$ (in polar coordinates).

Use $L^2(\partial\Omega)$ basis functions

$$g(\theta) = \phi_n(\theta) = (2\pi)^{-1/2} e^{in\theta}.$$

Then

$$(\Lambda_\gamma - \Lambda_1)\phi_n = \lambda_n \phi_n \quad \Rightarrow \quad \langle (\Lambda_\gamma - \Lambda_1)\phi_n, \phi_n \rangle = \lambda_n \qquad n = 1, 2, \ldots.$$

The linearized problem can then be posed as

$$((Lh)\phi_n, \phi_n)_{L^2(\partial\Omega)} = -\int_\Omega h |\nabla v_n|^2 \,\mathrm{d}x = \lambda_n \qquad n = 1, 2, \ldots$$

with

$$\Delta v_n = 0 \text{ in } \Omega, \quad \partial_r v_n = \phi_n \text{ on } \partial\Omega.$$